**Contractions and flips for varieties with group action of small complexity**

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**Abstract.** We consider projective, normal algebraic varieties $X$ equipped with the action of a reductive algebraic group $G$. We assume that a Borel subgroup of $G$ has an orbit of codimension at most one in $X$ (i.e. the complexity of the $G$-variety $X$ is at most one) and that $X$ is unirational. Then we prove that the cone of effective one-cycles $\text{NE}(X)$ is finitely generated, and that each face of $\text{NE}(X)$ can be contracted. Moreover, flips exist when $X$ is $\mathbb{Q}$-factorial, and any sequence of directed flips terminates. Finally, we prove that any homogeneous space of complexity at most one admits an equivariant completion whose anticanonical divisor is ample.

**Introduction**

Consider a projective, normal algebraic variety $X$ over an algebraically closed field. In the study of morphisms $\varphi : X \to X'$ where $X'$ is another projective, normal variety, a fundamental role is played by the “cone of effective one-cycles” $\text{NE}(X)$. Namely, the curves contracted by $\varphi$ define a face $F$ of $\text{NE}(X)$; moreover, $\varphi$ can be recovered from $F$, provided that $\varphi$ has connected fibers (then $\varphi$ is the contraction of $F$). But it may happen that some faces of $\text{NE}(X)$ do not arise from morphisms; and the geometry of $\text{NE}(X)$ can be quite complicated, see e.g. [2] §4.

In the present paper, we prove that everything is fine for a class of varieties with group actions. More precisely, we consider a connected reductive group $G$ acting on a projective, normal variety $X$. We assume that $X$ is unirational, and that the complexity of the action is at most one, i.e. that

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a Borel subgroup of \( G \) has an orbit of codimension at most one in \( X \). Then we prove that the convex cone \( NE(X) \) is finitely generated, and that each of its faces can be contracted (1.3). Moreover, if \( X \) is \( \mathbb{Q} \)-factorial, then we can always flip bad contractions (1.4) and every sequence of directed flips is finite (2.5). It follows that for any closed subgroup \( H \) of \( G \) such that the complexity of \( G/H \) is at most one, there exists an equivariant completion \( X \) of \( G/H \) such that the opposite of the canonical divisor is ample (2.5). It is tempting to conjecture that the assumption on the complexity of \( G/H \) is not necessary.

Our results generalize work of the first author (see [1]) which concern spherical varieties, i.e. varieties of complexity zero. We also mention related work of L. Moser-Jauslin and T. Nakano on threefolds where the group \( \text{SL}(2) \) acts with a dense orbit (see [7] and [8]); these examples have complexity one.

Our proofs are based on two finiteness results. The first one asserts that the algebra of regular functions \( \Gamma(X, \mathcal{O}_X) \) is finitely generated, whenever \( X \) is a normal, unirational \( G \)-variety of complexity at most one; see [5]. For the second one, we consider a normal \( G \)-variety \( X \) of complexity at most one, and we prove that \( X \) has only finitely many equivariant completions \( \overline{X} \), if we prescribe the valuations associated to all prime divisors in \( X \setminus \overline{X} \); see 2.1-2.4.

**Notation and terminology.** We consider algebraic varieties and groups which are defined over a fixed algebraically closed field \( k \). The field of rational functions on a variety \( X \) is denoted by \( k(X) \). We denote by \( G \) a connected reductive group; we choose a Borel subgroup \( B \) of \( G \), and a maximal torus \( T \) of \( G \). A \( G \)-variety \( X \) is a variety endowed with an action of \( G \); then the complexity of \( X \) is the minimal codimension of a \( B \)-orbit in \( X \); see [10]. The complexity of \( X \) is equal to the transcendence degree of \( k(X)^B \) over \( k \), where \( k(X)^B \) denotes the subfield of \( B \)-invariants in \( k(X) \).

Consider two varieties \( X \) and \( S \), and a proper morphism \( f : X \to S \). For any line bundle \( \mathcal{L} \) over \( X \), and for any (reduced and irreducible) complete curve \( C \) in \( X \), we denote by \( (\mathcal{L} \cdot C) \) the degree of the restriction of \( \mathcal{L} \) to \( C \). Denote by \( Z_1(X/S) \) the free abelian group generated by all closed curves \( C \) in \( X \) such that \( f(C) \) is a point; denote by \( \text{Pic}(X/S) \) the quotient of \( \text{Pic}(X) \) by \( f^*\text{Pic}(S) \). Then the assignment \( (\mathcal{L}, C) \to (\mathcal{L} \cdot C) \) defines a bilinear
form

\[ \text{Pic}(X/S) \times \mathbb{Z}_1(X/S) \to \mathbb{Z}. \]

Dividing by the kernels and tensoring by \( \mathbb{Q} \), we obtain a non-degenerate pairing

\[ N^1(X/S) \times N_1(X/S) \to \mathbb{Q} \]

where \( N^1(X/S) \) (resp. \( N_1(X/S) \)) is the space of relative line bundles (resp. one-cycles), with rational coefficients, modulo numerical equivalence. We denote by \( NE(X/S) \) the convex cone of \( N_1(X/S) \) which is generated by the classes of closed curves \( C \) in \( X \), such that \( f(C) \) is a point.

Let \( \mathcal{L} \) be a line bundle over \( X \). Then \( \mathcal{L} \) is called \( f \)-nef if \( (\mathcal{L} \cdot C) \geq 0 \) for any curve \( C \) in \( X \) such that \( f(C) \) is a point. Equivalently, the linear form on \( N_1(X/S) \) defined by \( \mathcal{L} \) is non-negative on \( NE(X/S) \). On the other hand, \( \mathcal{L} \) is called \( f \)-semi-ample if there exists an integer \( n > 0 \) such that the natural homomorphism \( f^*f_*(\mathcal{L}^n) \to \mathcal{L}^n \) is surjective. Observe that any \( f \)-semi-ample line bundle is \( f \)-nef. The converse is not true in general, but it holds whenever \( X \) is unirational and has complexity at most one; see 1.2.

1. Existence of contractions and of flips

1.1. For later purpose, we need the following characterization of semi-ample divisors among nef divisors, which may be of independant interest.

**Proposition.** Consider a projective morphism \( f : X \to S \) between normal varieties, and a \( f \)-nef line bundle \( \mathcal{L} \) over \( X \). Then the following conditions are equivalent:

(i) \( \mathcal{L} \) is \( f \)-semi-ample.

(ii) For any \( f \)-ample line bundle \( \mathcal{M} \) over \( X \), the sheaf of algebras

\[ A(\mathcal{L}, \mathcal{M}) := \bigoplus_{l,m \geq 0} f_*(\mathcal{L}^l \otimes \mathcal{M}^m) \]

is finitely generated over \( \mathcal{O}_S \).

(iii) There exists a \( f \)-ample line bundle \( \mathcal{M} \) over \( X \), such that \( A(\mathcal{L}, \mathcal{M}) \) is finitely generated over \( \mathcal{O}_S \).
Proof. (i) ⇒ (ii) Denote by $\mathcal{L}$ (resp. $\mathcal{M}$) the total space of the dual bundle of $\mathcal{L}$ (resp. $\mathcal{M}$). Consider the vector bundle $\mathcal{L} \oplus \mathcal{M}$ over $X$, and the associated projective bundle $\pi : P \to X$. Set $g = f \circ \pi$. We have the tautological line bundle $\mathcal{O}_P(1)$ over $P$, such that $\pi_* \mathcal{O}_P(1) = \mathcal{L} \oplus \mathcal{M}$. So for any integer $n \geq 0$, we have:

$$g_* \mathcal{O}_P(n) = \bigoplus_{0 \leq l \leq n} f_*(\mathcal{L}^\otimes l \otimes \mathcal{M}^{\otimes (n-l)})$$

and therefore:

$$A(\mathcal{L}, \mathcal{M}) = \bigoplus_{n=0}^{\infty} g_* \mathcal{O}_P(n).$$

By a version of a theorem of Zariski [12], the $\mathcal{O}_S$-algebra $A(\mathcal{L}, \mathcal{M})$ is finitely generated if the line bundle $\mathcal{O}_P(1)$ is $g$-semi-ample. But this follows from the $f$-semi-ampleness of $\mathcal{L}$, and the $f$-ampleness of $\mathcal{M}$.

(iii) ⇒ (i) We may assume that $S$ is affine: then we have to show that $\mathcal{L}$ is semi-ample. Choose an arbitrary point $x \in X$. We show that the restriction map $\mathcal{L}^\otimes l \to \mathcal{L}^\otimes l |_x$ is surjective for $l$ large. The $\mathbb{N}^2$-graded algebra

$$\bigoplus_{l,m \geq 0} \Gamma(\{x\}, \mathcal{L}^\otimes l \otimes \mathcal{M}^{\otimes m})$$

can be identified with the polynomial algebra $k[u, v]$ where the degree of $u$ (resp. $v$) is $(1, 0)$ (resp. $(0, 1)$). The evaluation at $x$ defines a morphism of $\mathbb{N}^2$-graded algebras

$$e_x : A(\mathcal{L}, \mathcal{M}) \to k[u, v].$$

Because the algebra $A(\mathcal{L}, \mathcal{M})$ is finitely generated, the set of all degrees occurring in $e_x(A(\mathcal{L}, \mathcal{M}))$ is a finitely generated semigroup. Choose non-zero generators $(l_1, m_1), \ldots, (l_t, m_t)$ of this semigroup with $l_i m_{i+1} - l_{i+1} m_i \geq 0$ for $1 \leq i \leq t - 1$. If $m_1 \neq 0$ then $e_x(A(\mathcal{L}, \mathcal{M}))_{l, m} = 0$ for any $(l, m)$ such that $lm_1 - l_1 m > 0$. Choose such a couple $(l, m)$ with $m > 0$. Then the line bundle $\mathcal{L}^\otimes l \otimes \mathcal{M}^{\otimes m}$ is ample (because $\mathcal{L}$ is nef and $\mathcal{M}$ is ample), but all sections of all powers of this line bundle vanish at $x$, a contradiction. So $m_1 = 0$, and $\mathcal{L}^\otimes l_1$ has global sections which do not vanish at $x$. □

1.2. Theorem. Let $f : X \to S$ be a proper $G$-morphism between normal $G$-varieties. Assume that $X$ is unirational and of complexity at most one. Then every $f$-nef line bundle over $X$ is $f$-semi-ample.
Proof. By standard reductions based on [9] Theorem 4.9, we may assume that the morphism $f$ is projective. Let $L$ be a $f$-nef line bundle over $X$. By replacing $L$ with some positive power, we may assume that $L$ is $G$-linearized. Choose a $G$-linearized, $f$-ample line bundle $M$ over $X$.

By [4] §2, we can cover $S$ by translates of $B$-stable affine open subsets. Choose such a subset $S_0$. We have to show that the algebra

$$\bigoplus_{l,m \geq 0} \Gamma(S_0, f_*(L^\otimes l \otimes M^\otimes m))$$

is finitely generated. For this, we may assume that $S = G \cdot S_0$. Then $D := S \setminus S_0$ is a Cartier divisor of $S$; see [6] Lemma 2.2. There exists a positive integer $N$ such that the line bundle $\mathcal{O}_S(ND)$ is $G$-linearized. Set $\mathcal{N} := f^*\mathcal{O}_S(ND)$. Then the group $\hat{G} := G \times (\mathbb{G}_m)^3$ acts on the variety

$$\hat{X} := \text{Spec}_{\mathcal{O}_X} \bigoplus_{l,m,n \geq 0} L^\otimes l \otimes M^\otimes m \otimes N^\otimes n.$$ 

Moreover, $\hat{X}$ is a normal, unirational $\hat{G}$-variety of complexity at most one. By [5], the algebra $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ is finitely generated. Therefore, the algebra

$$\bigoplus_{l,m,n \geq 0} \Gamma(S, f_*(L^\otimes l \otimes M^\otimes m) \otimes \mathcal{O}_S(nND))$$

is, too. So the same holds for the algebra

$$\bigoplus_{l,m \geq 0} \Gamma(S_0, f_*(L^\otimes l \otimes M^\otimes m)) = \bigcup_{n \geq 0} \bigoplus_{l,m \geq 0} \Gamma(S, f_*(L^\otimes l \otimes M^\otimes m) \otimes \mathcal{O}_S(nND)).$$

We conclude by 1.1. □

1.3. Theorem. Let $f : X \to S$ be a projective $G$-morphism between normal $G$-varieties. Assume that $X$ is unirational and of complexity at most one.

(i) The cone $NE(X/S)$ is polyhedral, and each of its extremal rays is generated by the class of a $B$-stable, rational curve.

(ii) For any face $F$ of $NE(X/S)$, there exists a unique normal $G$-variety $X_F$, projective over $S$, and a unique $G$-morphism $\text{cont}_F : X \to X_F$ with
connected fibers, such that $F = NE(X/X_F)$. Moreover, $F$ generates the kernel of $(\text{cont}_F)_*: N_1(X/S) \to N_1(X_F/S)$; and the space $N^1(X_F/S)$ is identified with the orthogonal of $F$ in $N^1(X/S)$.

(iii) If $\varphi: X \to X'$ is any morphism to a projective variety over $S$, such that $F$ is contained in $NE(X/X')$, then $\varphi$ factorizes through $\text{cont}_F$.

Proof. (i) It follows from [7] Lemma 6.1 that any effective cycle of $X$ which is contracted by $f$, is rationally equivalent to a $B$-stable effective cycle which is contracted by $f$. Therefore, it is enough to show that the $B$-stable irreducible curves of $X$ which are contracted by $f$ are rational, and that their images in $NE(X/S)$ generate only finitely many half-lines. Let $C$ be such a curve. If $B$ acts non-trivially on $C$, then $C$ is obviously rational. Moreover, $C^T$ consists in exactly 2 points, and the image of the half-line $Q^+C$ in $NE(X/S)$ only depends on the connected components of $X^T$ which meet $C$(see [1] 1.6). On the other hand, if $B$ acts trivially on $C$, then there exists a unique parabolic subgroup $P$ containing $B$ which is opposite to the isotropy subgroups of all points in a non-empty open subset of $C$. By [4] 1.2, we can choose a $P$-stable open affine subset $X_0$ of $X$ meeting $C$, such that the quotient $\pi: X_0 \to X_0/P^u$ exists. Therefore, the restriction of $\pi$ to $C \cap X_0$ is injective. We set: $L := P/P^u$ and $\Sigma := X_0/P^u$. Observe that $\Sigma$ is an affine, unirational $L$-variety of complexity at most one; hence its (Mumford) quotient $\Sigma/L$ is a point or a rational, irreducible curve. But $\pi(C \cup X_0)$ is a curve in $\Sigma^L$; moreover, the composition $\Sigma^L \to \Sigma \to \Sigma/L$ is injective. Therefore, the composition $\pi(C \cap X_0) \to \Sigma/L$ is bijective. It follows that $\pi(C \cap X_0)$ is rational, and that $C$ is rational, too.

(ii) and (iii) are formal consequences of 1.2 (see [3] 3.2.5, [1] 3.1).

1.4. Let $X$ be a $\mathbb{Q}$-factorial, unirational $G$-variety of complexity at most one. Let $f: X \to S$ be a projective $G$-morphism; let $R$ be an extremal ray $R$ of $NE(X/S)$. By 1.3, the contraction of $R$ exists; denote it by $\varphi: X \to X'$. We assume that $\varphi$ is birational, and an isomorphism in codimension one.

Proposition. Under the assumptions above, there exists a unique $\mathbb{Q}$-factorial $G$-variety $X^+$, projective over $S$, and a unique birational $G$-morphism $\varphi^+: X^+ \to X'$ such that:

(i) $\varphi^+$ is the contraction of an extremal ray $R^+$ of $NE(X^+/S)$.
(ii) $\varphi^+$ is an isomorphism in codimension one.
(iii) If the spaces \( N^1(X/S) \) and \( N^1(X^+ / S) \) are identified via \( \varphi^+ \circ \varphi^{-1} \), then the half-lines \( R \) and \( R^+ \) are opposite in \( N_1(X/S) = \text{Hom}(N^1(X/S), \mathbb{Q}) \).

We call \( \varphi^+ : X \to X^+ \) the flip of \( \varphi \).

**Proof.** By [3] Proposition 5.1.11, the statement is a consequence of the following assertion, whose proof (analogous to 1.2) is left to the reader: For any line bundle \( L \) on \( X \), the sheaf of algebras \( \bigoplus_{n=0}^{\infty} \varphi^*(L^\otimes n) \) is finitely generated over \( S \). □

## 2. Termination of flips

### 2.1. Let \( X \) be a homogeneous \( G \)-variety. Denote by \( \mathcal{V} \) the set of all \( G \)-invariant \( k \)-valuations of the field \( k(X) \) with values in \( \mathbb{Q} \). For any equivariant normal embedding \( \overline{X} \) of \( X \), denote by \( D(\overline{X}) \) the set of all \( G \)-stable prime divisors in \( \overline{X} \). We identify a prime divisor \( D \subset \overline{X} \) and the associated (normalized) valuation \( v_D \) of \( k(\overline{X}) = k(X) \), so \( D(\overline{X}) \) is a finite subset of \( \mathcal{V} \).

**Theorem.** Let \( X \) be a homogeneous \( G \)-variety of complexity at most one. Let \( D \) be a finite subset of \( \mathcal{V} \). Then there exist only finitely many complete normal embeddings \( \overline{X} \) with \( D(\overline{X}) = D \).

### 2.2. Before we enter the proof we need some preparation. Denote by \( \mathcal{F} \) the set of all \( B \)-stable prime divisors in \( X \). For any \( G \)-stable subvariety \( Y \) in \( \overline{X} \) define

\[
\mathcal{V}_Y(\overline{X}) := \{ D \in D(\overline{X}) \mid Y \subset D \} ;
\]

\[
\mathcal{F}_Y(\overline{X}) := \{ D \in \mathcal{F} \mid Y \subset D \} ;
\]

\[
\mathbf{F}_Y(\overline{X}) := \mathcal{V}_Y(\overline{X}) \times \mathcal{F}_Y(\overline{X}) .
\]

So the pair \( \mathbf{F}_Y(\overline{X}) \) describes the set of \( B \)-stable divisors of \( \overline{X} \) which contain \( Y \). We recall that the embedding \( \overline{X} \) is uniquely determined by

\[
\mathbf{F}(\overline{X}) := \{ \mathbf{F}_Y(\overline{X}) \mid Y \subset \overline{X} \text{ closed orbit} \}.
\]
(see [4] 3.8). This immediately implies Theorem 2.1 when \( c(X) = 0 \), because \( \mathcal{F} \) is finite in this case. Therefore, we assume from now on that \( c(X) = 1 \), i.e. that the transcendence degree of \( k(X)^B \) over \( k \) is one.

Let \( C \) be the smooth projective curve with \( k(C) = k(X)^B \). The points of \( C \) can be identified with the equivalence classes of non-trivial valuations of \( k(X)^B \). Let \( v_0 \) be the trivial valuation. Then we can break up \( \mathcal{V} \) and \( \mathcal{F} \) into pieces, as follows. For any \( c \in C \cup \{ o \} \), we set (with \( 0v_c := v_0 \)):

\[
\mathcal{V}_c := \{ v \in \mathcal{V} \mid v|_{k(C)} \in \mathbb{Q}_{\geq 0} v_c \}; \\
\mathcal{F}_c := \{ D \in \mathcal{F} \mid v_D|_{k(C)} \in \mathbb{Q}_{\geq 0} v_c \}.
\]

Observe that \( \mathcal{V}_c \cap \mathcal{V}_d = \mathcal{V}_o \) for any distinct \( c, d \) in \( C \cup \{ 0 \} \). Let \( \mathcal{O}_c \) be the valuation ring of \( v_c \) in \( k(C) \). Consider the \( \mathbb{Q} \)-vector space

\[
\mathbb{Q}_c := \text{Hom}(k(X)^{(B)}/k(C)^x, \mathbb{Q}).
\]

Then \( \mathbb{Q}_c \) is finite-dimensional (see [4] §5). Moreover, \( \mathbb{Q}_0 \) is a hyperplane in \( \mathbb{Q}_c \) for \( c \neq 0 \). Restriction to \( k(X)^{(B)} \) defines maps

\[
\mathcal{V}_c \to \mathbb{Q}_c; \quad \rho : \mathcal{F}_c \to \mathbb{Q}_c.
\]

The first one is injective ([4] 3.6) and we will identify \( \mathcal{V}_c \) and its image in \( \mathbb{Q}_c \).

**Lemma.** Let \( c \in C \cup \{ o \} \).

a) The set \( \mathcal{V}_c \) is a finitely generated convex cone.
b) If \( c \neq o \) then \( \mathcal{V}_o \) is a 1-codimensional face of \( \mathcal{V}_c \).
c) The set \( \mathcal{F}_c \) is finite.
d) There is a non-empty open subset \( C^0 \) of \( C \) such that \( \mathcal{F}_d \) consists in exactly one divisor \( D_d \) whenever \( d \in C^0 \).
e) There exists a non-empty open subset \( C^1 \) of \( C^0 \) such that \( \mathcal{V}_d \) is contained in the convex cone generated by \( \rho(D_d) \) and \( \mathcal{V}_o \) whenever \( d \in C^1 \).
Varieties with group action

Proof. For a) and b) see [4] 6.5. We may choose a non-empty, $B$-stable open subset $X_0$ of $X$, such that the orbit space $X_0/B$ exists, with quotient map $\pi$. Moreover, we may identify $X_0/B$ with an open subset $C^0$ of $C$. If $D \in \mathcal{F}_c$ meets $X_0$ then $D$ is the closure of $\pi^{-1}(c)$; denote it by $D_c$. Otherwise, $D$ is one of the finitely many components of $X \setminus X_0$. This implies c) and d).

To prove e), we construct a certain embedding of $X$. Because $X$ is homogeneous, $C$ is unirational. By L"uroth’s theorem, there exists $t \in k(C)$ such that $k(C) = k(t)$. The choice of $t$ identifies $C$ with $\mathbb{P}^1$. Denote by $D_0$ the divisor on $X$ $(t)_\infty + \sum_{D \in \mathcal{F}_0} D$

Set $L := \mathcal{O}_X(D_0)$, and denote by $\sigma_0$ the canonical section of $\mathcal{L}$. Then $\sigma_1 := t\sigma_0$ is a section as well. By replacing $G$ with a finite cover we may assume that $\mathcal{L}$ is $G$-linearized. Let $M$ be the $G$-submodule of $\Gamma(X, \mathcal{L})$ generated by $\sigma_0$ and $\sigma_1$. Let $\overline{X}$ be an equivariant normal, complete embedding such that $L$ extends to $\overline{X}$ and that the linear system $M$ has no base point in $\overline{X}$.

Set $\overline{X}_0 := \{ x \in \overline{X} \mid \sigma_0(x) \neq 0 \}$. Then $t = \sigma_1/\sigma_0$ defines a $B$-invariant morphism $\tau : \overline{X}_0 \to \mathbb{A}^1 \subset \mathbb{P}^1 = C$. The generic fiber of $\tau$ is connected because $k(t) = k(X)^B$ is algebraically closed in $k(X)$. Now let $C^1$ be the set of all $c \in C^0 \cap \mathbb{A}^1$ such that $\tau^{-1}(c)$ is non-empty and irreducible, and meets $X$.

We check that the lemma holds for $C^1$. Let $c \in C^1$. Then $\overline{\tau^{-1}(c)}$ is an irreducible divisor, stable by $B$ but not by $G$. Hence $\tau^{-1}(c)$ is equal to $D_c$. Now choose $v \in \mathcal{V}_c$. Then $c \in \mathbb{A}^1$ means $v(t) \geq 0$ and this implies $v(M/\sigma_0) \geq 0$ by [4] 3.3. Let $Z$ be the center of $v$ in $\overline{X}$. Because $M$ is base point free, $\sigma_0$ cannot vanish on $Z$, i.e. $Z$ meets $\overline{X}_0$. Moreover, $v \in \mathcal{V}_c$ implies $\tau(Z \cap \overline{X}_0) = \{c\}$. Therefore, $D_c$ is the only $B$-stable prime divisor which contains $Z$ and which is not mapped dominantly to $C$ by $\tau$. Hence we get $\mathcal{V}_Z(\overline{X}) \subset \mathcal{V}_0$ and $\mathcal{F}_Z(\overline{X}) = \{D_c\}$ because, by definition of $D_0$, no $D \in \mathcal{F}_o$ meets $\overline{X}_0$.

Assume that $v$ is not in the convex cone generated by $\rho(D_c)$ and $\mathcal{V}_0$. Then there exists $f \in k(X)^{(B)}$ such that $v(f) < 0$ but $v_D(f) \geq 0$ for any $B$-stable prime divisor $D$ which contains $Z$. But this contradicts the fact that $Z$ is the center of $v$. $\square$
2.3. Proof of Theorem 2.1. Define a map \( \zeta : \mathcal{V} \to C \cup \{o\} \) by \( \zeta(\mathcal{V}_o) = \{o\} \) and \( \zeta(\mathcal{V}_c \setminus \mathcal{V}_o) = \{c\} \). Choose \( C^1 \subset C \) as in Lemma 2.2 and set

\[
C^2 := C^1 \setminus \zeta(\mathcal{D}), \quad S := C \setminus (C^2 \cup \{o\}) \quad \text{and} \quad \mathcal{F} := \bigcup_{c \in S} \mathcal{F}_c.
\]

Observe that \( \mathcal{F}' \) is finite. We consider sets of couples \( (V, F) \) such that \( V \subset \mathcal{V} \) and \( F \subset \mathcal{F} \). We call such a set \( \mathcal{D} \)-admissible if it is the union of sets which appear in the following list:

A) \( \{(V, F)\} \) for some \( V \subset \mathcal{D} \) and \( \mathcal{F} \setminus \mathcal{F}' \subset F \subset \mathcal{F} \).

B) \( \{(V, F)\} \) for some \( V \subset \mathcal{D} \) and \( F \subset \mathcal{F}' \).

C) \( \{(V, F' \cup \{D_c\}) \mid c \in C^2\} \) for some \( V \subset \mathcal{D} \cap \mathcal{V}_o \) and \( F' \subset \mathcal{F}_o \).

The admissible sets of types A and B consist of a single element, while those of type C are infinite. Observe that there are only finitely many \( \mathcal{D} \)-admissible subsets for prescribed \( \mathcal{D} \), due to the fact that \( \mathcal{D}, \mathcal{F}' \) and \( \mathcal{F}_o \) are finite. Now Theorem 2.1 results from the following

Lemma. Let \( X \subset \overline{X} \) be a complete normal embedding with \( \mathcal{D}(X) = \mathcal{D} \). Then the set \( F(X) \) is \( \mathcal{D} \)-admissible.

Proof of the lemma. Let \( Y \) be a closed \( G \)-orbit in \( \overline{X} \). Let \( P \) be the parabolic subgroup of \( G \) containing \( B \) which is opposite to some isotropy subgroup of \( G \) in \( Y \). By [4] 1.2, there exists a \( P \)-stable open affine subset \( \overline{X}_0 \) of \( \overline{X} \) meeting \( Y \), such that the quotient \( \pi : \overline{X}_0 \to \overline{X}_0/P^u \) exists. It follows that \( \pi(\overline{X}_0 \cap Y) \) is a point, which we denote by \( y \). Moreover, \( y \) is a fixed point of \( P \) in \( \overline{X}_0/P^u := \Sigma \).

The equality \( k(\Sigma)^B = k(X)^B = k(C) \) induces a \( B \)-invariant rational map \( f : \Sigma - \to C \). Denote by \( \Sigma' \) the normalization of the closure of the graph of \( f \). Then we have a morphism \( f' : \Sigma' \to C \) and a proper, birational morphism \( p : \Sigma' \to \Sigma \) such that \( f' = f \circ p \).

If \( f \) is not defined at \( y \) then \( f' \) maps \( p^{-1}(y) \) onto \( C \). For any \( c \in C \) choose a component \( \Sigma_c \) of \( p(f'^{-1}(c)) \) containing \( y \). Then \( \overline{X}_c := \pi^{-1}(\Sigma_c) \) is a \( B \)-stable divisor of \( \overline{X} \) containing \( Y \). It induces on \( k(C) \) a valuation which is equivalent to \( v_c \). If \( \overline{X}_c \) is \( G \)-stable, then \( c \in \zeta(\mathcal{D}) \). Hence \( c \in C^2 \) implies \( \overline{X}_c \in \mathcal{F}_c = \{D_c\} \), i.e. \( F_Y(\overline{X}) \) is of type A.

If \( f \) is defined at \( y \), then we set \( c := f(y) \). Let \( D \) be a \( B \)-stable prime divisor in \( \Sigma \) containing \( y \). Then either \( f(D) \) is dense in \( C \), or \( f(D) = \{c\} \).
This implies $\mathcal{F}_Y(\overline{X}) \subset \mathcal{F}_o \cup \mathcal{F}_c$. If moreover $c \notin C^2$ then the set $\mathbf{F}_Y(\overline{X})$ is of type B, and we are done.

Assume from now on that $c \in C^2$. Then $c \notin \zeta(\mathcal{D})$ implies that no component of $(f \circ \pi)^{-1}(c)$ is $G$-stable. Because $\mathcal{F}_c = \{D_c\}$, we have $\mathcal{V}_Y(\overline{X}) \subset \mathcal{D} \cap \mathcal{V}_o$ and $\mathcal{F}_Y(\overline{X}) = F \cup \{D_c\}$ for some $F \subset \mathcal{F}_o$. Therefore, $\mathbf{F}_Y(\overline{X})$ is an element of a $\mathcal{D}$-admissible set of type $C$. We have to prove that all other elements of this set are in $\mathbf{F}(\overline{X})$.

First we claim that $f$ is actually $P$-invariant. Namely, let $d \neq c$ be in the image of $f$. Then $D := f^{-1}(d)$ is a $B$-stable divisor with $y \notin D$. Because $P/B$ is complete, $PD$ is closed in $\Sigma$. Moreover, $y \notin PD$. For dimension reasons, this implies $PD = D$ and the claim.

Let $\mathcal{C} \subset \mathcal{Q}_c$ be the convex cone spanned by $\mathcal{V}_Y(\overline{X})$ and $\rho(\mathcal{F}_Y(\overline{X}))$. Set $\mathcal{C}_o := \mathcal{C} \cap \mathcal{Q}_o$. Then $\mathcal{C}$ is generated by $\rho(D_c)$ and $\mathcal{C}_o$. Choose a valuation $v \in \mathcal{V}$ with center $Y$. We can write $v = ap(D_c) + v_o$ with $a \in \mathbf{Q}^{>0}$ and $v_o \in \mathcal{C}_o$. Then Lemma 2.2 e) implies that $v_o \in \mathcal{V}_o$. Let $Z \subset \overline{X}$ be the center of $v_o$. Then $v_o \in \mathcal{C}$ implies $v_o(\mathcal{O}_{\overline{X},Y}) \geq 0$ and therefore $Y \subset Z$ by [4] 3.7.

Set $Q := \pi(Z \cap \overline{X}_0)$, and $W := Q \cap f^{-1}(c)$; then $y \in W$. We claim that $W = \{y\}$. Otherwise, there exists $h \in k[\Sigma](B)$ with $h(y) \geq 0$ and $h|_W \neq 0$ ([4] 2.2 applied to the action of $P/P_u$ on $\Sigma$). But this implies $v_o(h) > 0$ which is absurd. Because $v_o \in \mathcal{V}_o$, the restriction $f|_Q$ is dominant. By the claim, $f|_Q$ is quasifinite. Therefore $Q \subset \Sigma^L$. So we have $k(Z)(B) = k(Q) = k(Z)^B$, which implies that all $G$-orbits in $Z$ are closed ([4] 8.5). Moreover, we have $\mathcal{V}_Z(\overline{X}) = \mathcal{V}_Y(\overline{X})$ and $\mathcal{F}_Z(\overline{X}) = F = \mathcal{F}_Y(\overline{X}) \setminus \{D_c\}$.

The restriction of $f \circ \pi$ to $Y_0$ induces a rational $G$-invariant map $Z - \rightarrow C$ which is regular on the normalization $\tilde{Z}$ of $Z$ (observe that all $G$-orbits in $Z$ are closed of codimension one). Because $\overline{X}$ is complete, the induced map $\tilde{Z} - \rightarrow C$ is surjective and its fibres are exactly the $G$-orbits. For $d \in C^2$ let $Y_d$ be the image in $Z$ of the orbit over $d$. Now the discussion above with $Y$ replaced by $Y_d$ shows that $\mathcal{F}_{Y_d}(\overline{X})$ is an element of a set of type $C$, and hence $\mathcal{F}_{Y_d}(\overline{X}) = (V, F \cup \{D_d\})$ with $V = \mathcal{V}_Z(\overline{X})$ and $F = \mathcal{F}_Z(\overline{X})$ independant of $d$. This ends the proof of Lemma 2.3. □

2.4. There is a generalization to the case where $X$ is any normal $G$-variety of complexity one. An equivariant model of $X$ is a normal $G$-variety $\overline{X}$ together with a birational equivariant map $X - \rightarrow \overline{X}$.
**Theorem.** Let $X$ be a normal $G$-variety of complexity at most one. Let $\mathcal{D}$ be a subset of $\mathcal{V}$. Then there exist only finitely many complete normal equivariant models $\overline{X}$ of $X$ with $\mathcal{D}(\overline{X}) = \mathcal{D}$.

**Proof.** We may assume that $X$ does not contain a dense $G$-orbit. We will only sketch the proof because it goes along the same lines of that of Theorem 2.1 with the roles of $\mathcal{V}$ and $\mathcal{F}$ being exchanged. Here $\mathcal{F}$ is the set of $B$-stable prime divisors of $X$ which are not $G$-stable. So $\mathcal{F}$ depends only on the birational class of $X$. Then the definitions of $\mathcal{V}_Y(X)$, $\mathcal{D}_Y(X)$, $\mathcal{F}_Y(X)$ go through, and $\overline{X}$ is uniquely determined by the collection of all $\mathcal{F}_Y(\overline{X})$.

By assumption we have $k(X)^G \neq k$. It follows that $k(X)^G = k(X)^B = k(C)$ where $C$ is a uniquely defined smooth, projective curve. The definitions of $\mathcal{V}_c, \mathcal{F}_c$ and $\mathcal{Q}_c$ for $c \in C \cup \{0\}$ are the same as in the homogeneous case, and parts a) and b) of Lemma 2.2 hold verbatim.

By making $X$ smaller, we may assume that the rational map $f : X \rightarrow C$ is regular, and that the fibers of $f$ are $G$-orbits. Set $C^0 := f(X)$. Then for every $c \in C^0$ the fiber $f^{-1}(c)$ is a prime divisor which induces a normalized valuation $v_c \in \mathcal{V}_c$. This also shows that $\mathcal{F}_c$ is empty unless $c = o$ in which case it is finite. Now Lemma 2.2 e) has the following analogue with a similar proof.

**Lemma.** There is a non-empty open subset $C^1 \subset C^0$ such that $\mathcal{V}_c$ is the convex cone spanned by $\mathcal{V}_o$ and $v_c$ for every $c \in C^1$.

Now let $\overline{X}$ be any complete equivariant model of $X$ with $\mathcal{D}(\overline{X}) = \mathcal{D}$. Then the rational map $\overline{f} : \overline{X} \rightarrow C$ is defined on a $G$-stable open subset which contains $X$. Therefore, the sets $\mathcal{D}$ and $\{v_c \mid c \in C^0\}$ coincide up to a finite set. For $c \in C \cup \{0\}$ let $\mathcal{D}_c := \mathcal{D} \cap \zeta^{-1}(c)$. This set is finite. We define

$$C^2 := \{c \in C^1 \mid \mathcal{D}_c = \{v_c\}\}, \ S := C \setminus (C^2 \cup \{0\}), \ \mathcal{D}' := \bigcup_{c \in S} \mathcal{D}_c.$$  

Observe that $\mathcal{D}'$ is finite. We define a $\mathcal{D}$-admissible set as a set of pairs $(\mathcal{V}, F)$ which is the union of sets appearing in the following list:

A) $\{(V, F)\}$ for some $\mathcal{D} \setminus \mathcal{D}' \subset V \subset \mathcal{D}$ and $F \subset \mathcal{F}$.
B) $\{(V, F)\}$ for some $V \subset \mathcal{D}'$ and $F \subset \mathcal{F}$.
C) \{ (V \cup \{v_c\}, F) \mid c \in C^2 \} for some \( V \subset \mathcal{D}_o \) and \( F \subset \mathcal{F}_o \).

Again, for a given \( \mathcal{D} \), there exist only finitely many \( \mathcal{D} \)-admissible sets. One proves in the same way as in 2.3 that \( \mathbf{F}(\overline{X}) \) is \( \mathcal{D} \)-admissible. This ends the proof of Theorem 2.4.

2.5. Consider a \( \mathbb{Q} \)-factorial, projective variety \( X \). Assume that \( G \) acts on \( X \) with an open orbit of complexity at most one. Let \( \varphi : X \rightarrow X' \) be the contraction of an extremal ray \( R \) of \( NE(X) \). We assume that \( \varphi \) is birational, and an isomorphism in codimension one; we denote by \( \varphi^+ : X^+ \rightarrow X' \) the flip of \( \varphi \) (see 1.4). We call this flip direct (resp. inverse) if \( K_X < 0 \) (resp. \( K_X > 0 \)) on \( R \setminus \{0\} \).

**Theorem.** Under the assumptions above, every sequence of direct flips is finite, and every sequence of inverse flips as well.

**Proof.** By 2.2, there are only finitely many isomorphism classes of \( G \)-varieties which are obtained from \( X \) by a sequence of flips. This implies our statement, by using [3] Proposition 5.1.11 (3); see [1] 4.7 for details. □

**Corollary.** Let \( X \) be a \( \mathbb{Q} \)-factorial, projective \( G \)-variety of complexity at most one. Assume that the morphism \( G \rightarrow X : g \mapsto g \cdot x \) is dominant and separable for some \( x \in X \). Then there exists a projective, \( \mathbb{Q} \)-factorial \( G \)-variety \( X' \) and a birational, \( G \)-equivariant map \( \varphi : X \rightarrow X' \) such that:

i) \( \varphi \) factors through inverse flips and divisorial contractions of positive extremal rays.

ii) \( -K_{X'} \) is semi-ample.

Moreover, there exists a projective, \( \mathbb{Q} \)-Gorenstein \( G \)-variety \( X'' \) and a birational \( G \)-morphism \( \varphi' : X' \rightarrow X'' \) such that \( -K_{X''} \) is ample.

**Proof.** Observe that the contraction \( \varphi \) of a non-negative extremal ray of \( NE(X) \) is always birational. Namely, let \( C \) be an irreducible curve in \( X \) such that \( \varphi(C) \) is a point. We show that \( C \) dose not meet the open \( G \)-orbit in \( X \). Otherwise, we may choose \( \zeta_1, \ldots, \zeta_d \) in \( \text{Lie}(G) \), and \( x \in C \), such that \( C \) is smooth at \( x \), the orbit \( G \cdot x \) is open in \( X \), and that the vectors \( \zeta_1 \cdot x, \ldots, \zeta_d \cdot x \) form a basis of the tangent space of \( X \) at \( x \). Then \( s := \zeta_1 \wedge \cdots \wedge \zeta_d \) is a global section of \( -K_X \), which does not vanish at \( x \). Therefore, we have: \( (-K_X \cdot C) \geq 0 \). But \( (K_X \cdot C) \geq 0 \) by assumption. So
(\(K_X \cdot C\)) = 0 and \(s\) has no zero on \(C\). It follows that \(C\) is contained in the open \(G\)-orbit. Then the isotropy group \(G_x\) is infinite, and its connected component \(G^0_x\) is not normal in \(G\) (otherwise \(G \cdot x\) is affine; but \(G \cdot x\) contains a projective curve). Now we can choose \(\zeta_1, \ldots, \zeta_d\) as before, such that \(\zeta_1 \in Lie(G_y)\) for some \(y \in C\). Then \(s\) vanishes at \(y\), a contradiction.

Now the proof of the corollary is the same as [1] 4.7 Corollaire. □

Remark. The separability assumption cannot be removed in the corollary. Namely, in every characteristic \(p > 0\), we construct an example of a projective homogeneous variety \(X\) of complexity zero such that \(K_X\) and \(-K_X\) are not semi-ample. Consider the group \(G := SL(3, k)\). The Frobenius endomorphism \(F\) of \(k\) extends to an endomorphism of \(G\). We denote by \(V\) the \(G\)-module \(k^3\), by \(V^*\) the dual \(G\)-module, and by \(P(V), P(V^*)\) the associated projective spaces. We let \(G\) act on \(V \times V^*\) by \(g \cdot (v, f) = ((F^2g) \cdot v, g \cdot f)\). This defines a \(G\)-action on \(P := P(V) \times P(V^*)\). Clearly, \(B\) has a unique fixed point \(x\) in \(P\) and its isotropy group \(G_x\) is exactly \(B\). Therefore, \(G\) has a unique closed orbit \(X = G \cdot x\) in \(P\) and the map \(G/B \to X\) is bijective. In particular, \(X\) is nonsingular, of complexity zero.

For any non-zero integer \(n\), we claim that \(nK_X\) has no global section. Namely, \(X\) is a hypersurface in \(P \simeq P^2 \times P^2\) of bidegree \((1, p^2)\) (the homogeneous equation of \(X\) is \((F^2f)(v) = 0\)). Therefore, we have \(\omega_X = (\mathcal{O}_{P^2}(-2) \otimes \mathcal{O}_{P^2}(p^2 - 3))|_X\). Moreover, for any \(v \in V\) with coordinates in the prime field, the image in \(P\) of the set \(v \times (v = 0) \subset V \times V^*\) is a curve \(C_v \subset X\), and \((K_X \cdot C_v) = p^2 - 3 > 0\). Similarly, for any \(f \in V^*\) with coordinates in the prime field, we have a curve \(C_f \subset X\) with \((K_X \cdot C_f) < 0\). This implies our claim.

Analogous considerations hold more generally for quotients of semisimple groups by non-reduced parabolic subgroup-schemes; see [11].

References


Varieties with group action


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