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Divisorial log terminal singularities

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Abstract. In this paper we prove that a singularity is Kawamata weak log terminal if and only if it is divisorial log terminal, and give a local characterisation for them. Then we prove the covering theorem and the adjunction theorem for this class of singularities.

One of the difficulties of higher dimensional birational geometry is that minimal models do not exist in the category of smooth varieties; therefore one has to work with singular varieties. The situation is especially bad in the "log category" where at the moment there are about a dozen different proposed versions of "log terminal". Two versions that appeared very useful are the notion of "weak log terminal" introduced by Kawamata in [3] (named as weak Kawamata log terminal in [2]) and "divisorial log terminal" introduced in [4]. Both definitions have the problem that they are not local (not even in the Zariski topology) and it is usually hard to check if they are satisfied or not.

The main result of this article gives a local characterization of these notions, which also shows that they are in fact equivalent. Also, the new definition can be checked on any given resolution very easily:

Divisorial Log Terminal Theorem. *Let X be an irreducible normal variety, and D be an effective Q-Weil divisor with coefficients at most 1. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then the following conditions are equivalent.*

(a) (X, D) *is divisorial log terminal.*
(b) (X, D) *is weakly Kawamata log terminal.*

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(c) *If E is a divisor in the function field of X centered on X with discrepancy $a(E,D) \leq -1$ then X is smooth and D has simple normal crossings at the generic point of center (E). It follows then that supp (E) at the generic point is the intersection of the reduced components of D passing through that point.*

(d) *There is a log resolution $\pi : Y \longrightarrow X$ such that π is a composition of blowing ups of smooth centers (in particular, π is projective), the exceptional set is a divisor, and for every exceptional divisor E of π the discrepancy $a(E,D) > -1$.*

(e) *If $\pi : Y \longrightarrow X$ is any log resolution which is isomorphism over the generic points of the intersection of any number of the reduced components of D, then the discrepancies $a(E,D) > -1$ for all exceptional divisors E of π.*

Remark. The referee has pointed out, that characterization (c) is easyly shown to be invariant under flips.

The new definition makes it possible to prove in all dimensions the covering theorem for this class of singularities. The earlier result of Corti was restricted to dimension three (16.13 in [2]).

Covering Theorem. *Let $f : Y \longrightarrow X$ be a finite morphism of normal irreducible varieties, and B be a Weil divisor on X. Assume that f is étale in codimension one, $K_X + B$ is $\mathbb{Q}$-Cartier, and let $f^*(K_X + B) = K_Y + D$. If (X,B) is divisorial log terminal, then (Y,D) is divisorial log terminal.*

Then we generalize the notion of divisorial log terminal to nonnormal varieties. The previously existing version, called divisorial semi log terminal (see 12.2 of [2]), were made for surfaces, and did not work very well in higher dimensions. With our definition we can prove the Adjunction Theorem in all dimensions. The earlier result of Corti (16.9 of [2]) was hard to formulate, and was restricted to dimension three.

Adjunction Theorem. *Let X be an irreducible variety, $S + B$ a Weil divisor. Assume that S is the reduced part of $S + B$ and $K_X + S + B$*

is divisorial log terminal. Then

$$\textit{divdiscrep}(S, \textit{Diff}(B)) \geq \textit{divdiscrep}(X, S+B)$$

In particular S is generalized divisorial log terminal.

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In this work variety means a (possible reducible) reduced separable scheme of finite type over a field of characteristic 0.

In the proof of the Divisorial Log Terminal Theorem we construct resolutions via the following lemma.

Resolution Lemma. *Let X be an irreducible variety over an algebraicly closed field of characteristic 0, D a pure codimension one reduced subvariety of X, and U an open subset of the smooth locus of X such that $U \bigcap D$ has smooth components crossing normally. Then there is a projective morphism $\pi : \tilde{X} \longrightarrow X$ which satisfies the following conditions:*

(a) *π is a composition of blowing ups of smooth subvarieties,*
(b) *π induces an isomorphism $\pi^{-1}(U) \longrightarrow U$*
(c) *$\tilde{X}$ is smooth, and*
(d) *$\pi^{-1}(D \bigcup (X \backslash U))$ is a divisor. Moreover, it has smooth components crossing normally.*

Proof. The proof is based on Hironaka's resolution paper [1]. We use more than just his Main Theorems, but our arguments do not depend on his proof. We recall, that in [1] the resolution datum of type $\Re_I^{N,n}$ on X is an object of the form $(E; V_1, V_2, \ldots, V_\beta; W)$ where E is a divisor on X with only simple normal crossing, and V_i and W are subvarieties of X. In this proof we use only resolution datum of this type with $\beta = 0$. First we apply (Main Theorem I*, [1]) to X, and get a morphism $f : \bar{X} \longrightarrow X$ which satisfies conditions (a), (b), (c) of the lemma. Let $\Re = (0;; f^{-1}(X \backslash U))$ be a resolution datum of $\bar{X}$. We apply (THEOREM $I_2^{N,n}$, [1]) to the resolution datum $(0;; f^{-1}(X \backslash U))$ on $\bar{X}$, and get a map $g : \underline{X} \longrightarrow \bar{X}$ which is an isomorphism over $f^{-1}(U)$. The exceptional set of g is a divisor, and $\underline{B}$, the strict transform of $\bar{X}\backslash f^{-1}(U)$, is smooth. Let $h : X_0 \longrightarrow \underline{X}$ be the monoidal transformation with center $\underline{B}$ and let $\pi_0 : X_0 \longrightarrow X$ be the composition of f, g, h. Then π_0 satisfy the conditions (a),(b),(c) of the lemma, and $\pi_0^{-1}(X\backslash U)$ is a divisor. Let $\check{D}$ be the reduced inverse image $\pi_0^{-1}(D\bigcup(X\backslash U))$, and let $\check{D} = \bigcup_{i=1}^{m} \check{D}_i$ be its decomposition into irreducible components. It is clear that $\check{D}$ is a divisor.

Let N be the dimension of X_0 and let $n = N-1$. We define on X_0 the resolution datum $\Re_0$ as follows. Set $\hat{D}_1 = \check{D}_1$, and let $\Re_0 = (\emptyset; ; \hat{D}_1)$. Let

$$U_0 = \left\{ x \in X_0 \,\middle|\, \begin{array}{c} \check{D} \text{ has only smooth components} \\ \text{crossing normally in a neighborhood of } x. \end{array} \right\}$$

Clearly $\pi_0^{-1}(U) \subseteq U_0, \check{D}\bigcap U_0$ is dense in $\check{D}$, and $(\Re_0, U_0 \bigcap \hat{D}_1)$ is a resolution datum with open restriction.

Then we define by induction for $i = 1, 2, \dots m$ the smooth varieties X_i, the maps $\pi_i : X_i \longrightarrow X$, the open subset $U_i \subset X_i$, the closed subsets $\hat{D}_{i+1} \subset X_i$, and the resolution datum $(\Re_i, U_i \bigcap \hat{D}_{i+1})$ on X_i with open restriction. We already defined these for $i = 0$. If they are defined for $i-1$, then we apply (THEOREM $\mathrm{I}_2^{N,n}$, [1]) to the resolution datum $(\Re_{i-1}, U_{i-1} \bigcap \hat{D}_i)$, and get a finite composition of monoidal transformations $p_i : X_i \longrightarrow X_{i-1}$ such that $p_i^{-1}(\Re_{i-1}) = (E_i; W_i)$ is resolved. Set $\pi_i = p_i \circ \pi_{i-1} : X_i \longrightarrow X$. Then we define $\Re_i = (E_i \bigcup W_i; ; \hat{D}_{i+1})$ where $\hat{D}_{i+1}$ is just the strict transform of $\check{D}_{i+1}$ by the composition map $q_i = p_1 \circ p_2 \circ \cdots p_i : X_i \longrightarrow X_0$. We set $U_i = q_i^{-1}(U_0)$.

Since $\check{D}_{i+1} \bigcap U_0$ is smooth, and dense in $\check{D}_{i+1}$, and $q_i : U_i \longrightarrow U_0$ is an isomorphism, we see that $(\Re_i, U_i \bigcap \hat{D}_{i+1})$ is a resolution datum with open restriction. Hence our inductive definition make sense. It is easy to see by induction that $E_i \bigcup W_i$ is just the reduced inverse image $q_i^{-1}(\bigcup_{j=1}^{i} \check{D}_j)$; hence $E_m \bigcup W_m$ is just the reduced inverse image $\pi_m^{-1}(D)$. It is also clear that $E_m \bigcup W_m$ has smooth components crossing normally. So we can take $\tilde{X} = X_m, \pi = \pi_m$. The Resolution Lemma is proved. □

We want to apply this Resolution Lemma to study birational geometry. For the basic definitions we refer to ([3], §0–2) and ([2], chapter 2). We note that the notion of weak log terminal in [3] is slightly more general than the notion of weakly Kawamata log terminal in [2], because on the resolution it allows transversal selfintersection of the components of the boundary and the exceptional divisor. We shall always use the definition in [2]. To begin with, we recall some results that we shall use later.

PROPOSITION. ([2], 2.16 Proposition.) *If (X, D) is weakly Kawamata log terminal then it is divisorial log terminal.*

PROPOSITION. ([2], proof of 20.3 Proposition.) *Let $h : U \longrightarrow V$ be a finite, dominant morphism between irreducible normal varieties. Let $B = \sum b_i B_i$ be a Weil divisor on V such that $\bigcup B_i$ contains the branch locus of h. (We allow $b_i = 0$.) Assume that $K_V + B$ is $\mathbb{Q}$-Cartier and let $h^*(K_V + B) = K_U + \bar{B}$. Let E be a divisor in the function field of V with center in V and let $f^*E = \sum e_j F_j$ where F_j are divisors in the function field of U with centers in U. Then $a_l(F_j, B) = e_j a_l(E, D)$*

DEFINITION-PROPOSITION ([2], 16.5 Prop.-Def. and 16.6 Prop.) *Let $(X, S+B)$ be log canonical in codimension one, where S is the reduced part of $S+B$, and assume that B is $\mathbb{Q}$-Cartier. Then S has normal crossings in codimension one. If $i : S \longrightarrow X$ is the embedding then $i^*(K_X + S + B) = K_S + Diff(B)$ for some (unique) effective $\mathbb{Q}$-Weil divisor $Diff(B)$ on S. We call this divisor the different of B. If X is smooth, then $Diff\,(B) = B|S$.*

Now we are ready to prove the Divisorial Log Terminal Theorem and the Covering Theorem.

PROOF OF THE DIVISORIAL LOG TERMINAL THEOREM. Assume that either (c) or (e) hold, and define

$$U = \left\{ x \in X \;\middle|\; \begin{array}{c} X \text{ is smooth and } D \text{ has simple} \\ \text{normal crossings in a neighborhood of } x \end{array} \right\}$$

The Resolution Lemma gives us a resolution satisfying (d). Therefore both (c) and (e) imply (d).

Assume now (a), and let E be a divisor of the function field of X such that $a(E, D) \leq -1$. Then (a) implies that $a(E, D) = -1$, and the support of E on Y is the intersection of some of the components of D_Y with discrepancy -1. These components cannot be π-exceptional; hence π is an isomorphism near the generic point of the support of E. This implies (c).

It is obvious that (d) implies (b) and (c) implies (e). Finally (b) implies (a) by [2], 2.15 Proposition. This proves the theorem. □

Proof of the Covering Theorem. By (20.3, [2]) if E is a divisor in the function field of X centering in X, and $f^*E = \sum e_i F_i$ where F_i are divisors in the function field of Y then $a_l(F_i, D) = e_i a_l(E, D)$. Assume that (X, B) is divisorial log terminal, and let F be a divisor in the function field of Y with log discrepancy $a_l(F, D) \leq 0$. Then there is a divisor E in the function field of X such that $f^*E = \sum e_i F_i$, and $F_1 = F$. Hence $a_l(E) \leq 0$, so X is smooth and B has simple normal crossings in a neighborhood U of the generic point of supp$E$. Since f is finite and U is smooth, the ramification locus of f in U must have pure codimension one. Since f is étale in codimension one, it is unramified over U. Hence $f^{-1}(U)$ is smooth, and D has simple normal crossings in $f^{-1}(U)$. Since the generic point of F is in $f^{-1}(U)$, we see that (Y, D) is divisorial log terminal. □

Remark. The above theorem is not true without the assumption that f is étale in codimension one. Let for example $X = \mathbb{C}^2$ with coordinates x, y, and let Y be the normalization of $\{z^n = x^a y^b\} \subset \mathbb{C}^3$ for integers n, a, b, and let f be the projection onto $\{z = 0\}$. Let $B = \{xy = 0\} \subset X$. If we choose n, a, b so that Y is singular, then (Y, D) will not be dlt.

The converse of the above theorem is false. It is possible that (Y, D) is dlt but (X, B) is not dlt. Let for example $X = \mathbb{C}^2/\mathbb{Z}_2(1,1)$ and $D = \{xy = 0\}/\mathbb{Z}_2(1,1)$.

In the Adjunction Theorem we want to compare the properties of X with the properties of a divisor on X. Since divisors are usually not irreducible, we need to generalize the notion dlt to nonnormal varieties. There is an existing notion of semi divisorial log terminal variety, and there is a variant of our result in (16.9 Proposition, [2]), but it is proved only for threefolds. The main difficulty comes from the exceptional nature of the high multiplicity normal crossing points. Instead of trying to work out a theorem with this notion, we rather choose a new way to resolve these singularities (generalized resolution), which reflects better (for this purpose) the higher dimensional geometry. Our definition is made for the class of dlt singularities, it does not work well for lc singularities.

The notion of divisorial discrepancy is a global numerical measure of how 'good' is our generalized divisorial log terminal variety. It is inspired by the versions of discrepancy introduced in (chapter 17, [2]). Our Adjunction Theorem could also be seen as a divisorial log terminal variant of (17.2 Theorem, [2]).

Definition. A proper morphism $f : Y \longrightarrow X$ is a *generalized resolution* if Y has smooth components intersecting transversally and f is an isomorphism over all generic points of the intersection of any number of components of X.

Let D be a Weil divisor on X. A morphism $Y \longrightarrow X$ is a *generalized good divisorial resolution* of (X, D) if it is a generalized resolution of X, the exceptional locus E of f is a Weil divisor on Y and $f_*^{-1}D + E$ is a simple normal crossing divisor meeting transversally with the singular locus of Y.

Let D be a boundary on X. We say that (X, D) is *generalized divisorial log terminal* or *gdlt* if there is a generalized good divisorial resolution $f : Y \longrightarrow X$ such that every exceptional divisor E of f has discrepancy $a(E, D) > -1$.

Definition. Let (X, D) be a reduced variety with a Weil divisor, and assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then we define the *divisorial discrepancy* by

$$\mathrm{divdiscrep}(X, D) =$$

$$\sup_Y \inf_E \left\{ a(E, X, D) \,\middle|\, \begin{matrix} Y \text{is a generalized good divisorial resolution of } (X, D) \\ E \text{ is exceptional on } Y \end{matrix} \right\}$$

This is $-\infty$ if there is no generalized good divisorial resolution and $+\infty$ if (X, D) is already resolved. If it is a finite value then the infinum is actually a minimum, and the supremum is actually a maximum.

Proof of the Adjunction Theorem. Choose any log resolution $f : Y \longrightarrow X$ that satisfies the definition of dlt. Let $S' = f_*^{-1}S$ be the birational transform of S on Y. Then $S' \longrightarrow S$ is a generalized good divisorial resolution. Every exceptional divisor E' on S' is a component of $E \bigcap S'$ for some exceptional divisor E of f on Y. To complete the proof of the theorem it is enough to show that $a(E', S, \mathrm{Diff}(B)) = a(E, X, S + B)$.

The rest of the proof was taken from the proof of (17.2 Theorem and 7.2.3 Lemma, [2]). Let $\{E_j\}$ be the exceptional divisors of f and let $\{F_j\}$ be the components of $f_*^{-1}(B)$. Let $S' = f_*^{-1}(S)$. Let $S' \bigcap E_j =$

$\sum_k C_{jk} + \sum_i D_{ji}$ where C_{jk} are the $(f|S')$-exceptional components and $f|D_{ji}$ is birational. Assume that $m(K_X + S + B)$ is Cartier. Then

$$mK_{S'} + m\left(f_*^{-1}(B)\bigcap S'\right) - m\sum a(E_j, X, S+B)(E_j|S') =$$
$$= m\Bigg(K_X + S' + f_*^{-1}B - \sum a(E_j, X, S+B)E_j\Bigg)\Bigg|S'$$
$$= f^*(m(K_X + S + B))|S' = (f|S')^*(m(K_X + S + B)|S)$$
$$= (f|S')^*(mK_S + m\mathrm{Diff}(B))$$
$$= mK_{S'} + m(f|S')_*^{-1}(\mathrm{Diff}(B)) - \sum_{j,k} a(C_{jk}, S, \mathrm{Diff}(B))C_{ji}$$

This implies that $(f|S')_*^{-1}(\mathrm{Diff}(B)) = (f_*^{-1}(B)\bigcap S') - \sum a(E_j, X, S+B)D_{ji}$ and therefore

$$a(D_{ji}, S, \mathrm{Diff}\ (B)) = a(C_{jk}, S, \mathrm{Diff}(B)) = a(E_j, X, S+B)$$

for all j, i, k. This completes the proof. □

Remark. It is proved in (17.5, [2]) that S is S_2 and seminormal, and if X is $\mathbb{Q}$-factorial, then the components of S are normal.

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