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The dynamics of a degenerate reaction diffusion equation

By Sigurd B. Angenent and Donald G. Aronson

Abstract. We consider the initial-boundary value problem for a degenerate reaction diffusion equation consisting of the porous medium operator plus a nonlinear reaction term. The structure of the set of equilibria depends on the length of the spatial domain. There are two critical lengths $0<L_0<L_1$ such that the equation possesses one equilibrium if $L\in(0,L_0)$, three equilibria for $L\in(L_0,L_1]$ or two plus a one or more parameter family of equilibria when $L>L_1$.

Using a topological argument we show existence of connecting orbits joining the unstable equilibrium with the two stable equilibria for $L\in(L_0,L_1]$, when there are three equilibria. By showing that the principle of linearized stability can sometimes be applied with succes to degenerate parabolic equations, these connections are found to be unique for $L_0<L<L_1$.

We also investigate the nature of the connecting orbits at the critical value L_1 of the length L, i.e. just before the unstable equilibrium bifurcates into a continuum of equilibria.

1. Introduction

In this paper we continue the investigation of the dynamics of

$$\begin{aligned}(1.1)\qquad \frac{\partial u}{\partial t} &= \frac{\partial^2 u^m}{\partial x^2} + f(u) \qquad (-L<x<L, t\geq 0),\\ u(\pm L,t) &= 0, \qquad u(x,0)=u_0(x)\end{aligned}$$

with $1<m<\infty$, $f(u)=u(1-u)(u-a)$ for some $0<a<1$, which was begun by Aronson, Crandall & Peletier [ACP], and extended by Langlais

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& Phillips [LP]. Here we investigate the existence of heteroclinic orbits connecting the unstable and stable equilibria which were found in [ACP].

The weak solutions of this initial value problem define a semiflow $\{\phi_t : t \geq 0\}$ on the space $X = \{u \in L_\infty(\Omega) : 0 \leq u \leq 1\}$, which is gradient-like with respect to the functional

$$(1.2) \qquad E(u) = \int_\Omega \{{}^1\!/_2((u^m)_x)^2 - F(u)\}\, dx, \qquad F(u) = \int_0^u f(s) ds^m.$$

The dynamics of (1.1) with $m = 1$ has been studied extensively (see [He, BF1, BF2]), in the sense that the equilibria and the complete orbits connecting them have been classified. It has been found that, for $0 < a < {}^1\!/_2$, there is a critical value $L_* > 0$ such that the zero solution is the only equilibrium of (1.1) when $0 < L < L_*$; in this case all solutions converge to the zero solution as $t \to \infty$. If $a \geq {}^1\!/_2$ then the same situation occurs for any value of $L > 0$. However, if $a < {}^1\!/_2$ and $L > L_*$, then there exist exactly two solutions, apart from the zero solution, which we denote by p and q. These solutions are ordered, and we assume that $0 < p < q$. It then turns out that q is asymptotically stable, while p is a hyperbolic fixed point for the semiflow. There are only two connecting orbits, one from p to q, and one from p to the zero solution [He].

The proof of these statements concerning the nondegenerate case $m = 1$ relies heavily on the fact that for $m = 1$ the PDE is a semilinear heat equation, which allows one to use standard linearization techniques [He] to study the solutions of (1.1) near its equilibria.

In the present paper we try to prove similar statements about the degenerate equation. The initial value problem and the set of equilibria have been studied by Aronson, Crandall and Peletier. They found that, just as in the semilinear case, (1.1) generates a semiflow for which (1.2) is a Lyapunov function, so that one can try to describe the dynamics in terms of the equilibria and their connecting orbits. They also found that the zero solution is again the only equilibrium if a is too large, i.e. if $a \geq (m + 1)/(m + 3)$, and that this solution attracts all orbits in this case. Similarly, if $a < (m+1)/(m+3)$ there turns out to be a critical value $L_0 = L_0(m, a)$ such that the zero solution is a global attractor if $L < L_0$.

Persuing the analogy with the semilinear case, one would expect two positive equilibria for $L > L_0(m, a)$, but because of the nonlinear degeneracy of equation (1.1) the situation turns out to be more complicated. There is a second threshold $L_1(m,a) > L_0(m,a)$ such that (1.1) does indeed have exactly two nonzero equilibria if $L_0 < L < L_1$, which are again ordered, and which we denote by $0 < p(L,\cdot) < q(L,\cdot)$. The complications which arise when $L > L_1(m,a)$ are caused by the fact that

$$p_*(x) = \begin{cases} p(L_1, x) & \text{for } |x| \leq L_1, \\ 0 & \text{for } x \in [-L, -L_1) \cup (L_1, L] \end{cases}$$

is a nonnegative compactly supported solution to

$$(p^m)_{xx} + f(p) = 0, \qquad x \in \mathbf{R}$$

If $L > L_1$, then any of the translates $p_\xi(x) = p_*(x-\xi)$ with $|\xi| \leq L - L_1$ is an equilibrium of (1.1). Thus for $L > L_1$ the set of equilibria contains a whole interval. In fact, when $L > 2L_1$ one can take two translates of p_* whose supports are disjoint subsets of $[-L, L]$, and the sum of these two functions will again be a steady state of (1.1). Throughout this paper we assume that

$$0 < a < \frac{m+1}{m+3}. \tag{1.3}$$

Our main results are about connecting orbits, i.e. weak solutions to (1.1) which are bounded, nonconstant, and defined for all $t \in \mathbf{R}$.

1.1. Main Result. *If*

$$L_0(a,m) < L \leq L_1(a,m)$$

there exists a connecting orbit from p to q and also a connecting orbit from p to the zero solution. In case

$$L_0(a,m) < L < L_1(a,m)$$

both of these connecting orbits are unique.

To prove this theorem we use the abstract theory of quasilinear initial value problems of parabolic type (see [Am, An1, Lu] and also the appendix.)

This theory allows us to show that sufficiently regular weak solutions that are close to one of the equilibria p or q can be described by a linearization procedure.

We begin by defining a "very weak solution" of (1.1) on $Q_T = [-L, L] \times [0, T]$ to be a bounded measurable function $u(x, t)$ which satisfies

$$\int_0^T \int_{-L}^L \{\sigma_t u + \sigma_{xx} u^m + f(u)\sigma\} dx\, dt = -\int_{-L}^L \sigma(x, 0) u_0(x)\, dx \tag{1.4}$$

for any test function $\sigma \in C^\infty(Q_T)$ which vanishes on $\{\pm L\} \times [0, T]$ and on $[-L, L] \times \{T\}$.

This definition is weaker than the one given in [ACP]: there a weak solution was defined to be a bounded measurable function which also satisfies $u \in C([0, T]; L_1(\Omega))$. We will prove:

1.2. THEOREM. *The initial value problem has exactly one "very weak solution" for each $u_0 \in X$. Moreover, it defines a continuous semiflow $\phi_t : X \to X$ with respect to the weak-star topology on X.*

This theorem implies that our "very weak solutions" actually coincide with the weak solutions of [ACP].

The fact that the semiflow generated by (1.1) is continuous with respect to the weak-star topology on X may be seen as a manifestation of the smoothing effect of the diffusion equation. It turns out to be a useful fact, since X is a compact metrizable space in this topology. Thus one can directly apply Conley's theory [Co] without any further modifications. The continuity with respect to the weak star topology also immediately gives the existence part of the main theorem for all values of $L \in (L_0(m, a), L_1(m, a)]$. However we are unable to prove uniqueness without the restriction $L < L_1(m, a)$. This is essentially because the linearization of (1.1) at $p(L_1, \cdot)$ has continuous spectrum intersecting the imaginary axis, while our uniqueness proof requires hyperbolicity of the equilibrium $p(L, \cdot)$.

In section 7 we show that for some parameter values (a, m) one can regard (1.1) as a perturbation of the semilinear case where $m = 1$. For the critical situation with $L = L_1(m, a)$ this implies that even though the linearization of the semiflow near the fixed point p_* is not hyperbolic, for small values of $m > 1$ it does have a separate positive eigenvalue, to which one can associate a one dimensional "fast unstable manifold." The two solutions $u_{\pm}(x,t)$ of (1.1) corresponding to this fast unstable manifold have a "waiting time at infinity." We refer to section 7 for a more precise statement.

The maximal invariant set

Given the semiflow ϕ_t on X, one can define

$$\mathcal{A} = \bigcap_{t>0} \phi_t(X),$$

which is an invariant set for the semiflow. It turns out to be the largest invariant set in X (i.e. it contains all others), and it has the property that any $u_0 \in \mathcal{A}$ lies on a complete orbit $\{u(t)\}_{t\in\mathbf{R}}$. Conversely, any complete orbit lies in $\mathcal{A}$. We refer the reader to [Ha], where similar maximal invariant sets are defined for a variety of dissipative systems, and for our system (1.1) in particular [Ha, p.154].

If we had a backward uniqueness theorem for weak solutions, then the semiflow restricted to $\mathcal{A}$ would be a flow, a one parameter group of homeomorphisms of $\mathcal{A}$.

Intuitively, one expects $\mathcal{A}$ to be a contractible set, but we have no proof of this. However, we can show that $\mathcal{A}$ is topologically trivial in the following sense.

1.3. Lemma. *All Cech cohomology groups* $\check{\mathrm{H}}^n(\mathcal{A})$ *of* $\mathcal{A}$ *vanish.*

Proof. Let X_t be the set $\phi_t(X)$. Then the map $\phi_t : X_t \to X_t$ is homotopic to the identity map on X_t, while it also factors as $\phi_t|X_t = \phi_t \circ j_t$, where $j_t : X_t \to X$ is the inclusion map. Since X is a closed convex subset of a (topological) vectorspace, it is contractible, and hence all its cohomology groups vanish. This forces the map $\phi_t|X_t$ to induce the zero map on cohomology, but since it is homotopic to the identity map this can only happen if all cohomology groups of X_t vanish.

The sets X_t decrease to $\mathcal{A}$, so since taking Cech cohomology groups is compatible with taking direct limits, we find that $\check{\mathrm{H}}^k(\mathcal{A}) = \lim_{t\to\infty} \check{\mathrm{H}}^k(X_t) = (0)$, as claimed. $\square$

It is easy to see that $E(u)$ defined by (1.2) is a *formal* Lyapunoff function, but it is not so easy to prove this rigorously. For the one dimensional case ACP showed that this function really is a Lyapunoff function for the semiflow ϕ_t. Thus we have the following result.

1.4. GRADIENT FLOW THEOREM ([ACP]). *The function $E(u)$ is strictly decreasing on orbits of ϕ_t, except at equilibrium states, where it must be constant.*

Any complete orbit $\{u(t)\}_{t\in\mathbf{R}} \subset \mathcal{A}$ has ω- and α- limit sets, both of which consist of equilibrium states for the semiflow ϕ_t.

Langlais and Phillips [LP] showed in a somewhat more general setting that the second part of this theorem is still true if the domain Ω is an open subset of $\mathbf{R}^n$ $(n \geq 1)$, even though it is not clear whether $E(u)$ is a Lyapunoff function or not.

The Equilibria

Denote the set of equilibria in X by $\mathcal{E}$. The following description of $\mathcal{E}$ was given in [ACP]. An equilibrium is a function $\varphi : [-L, L] \to [0, 1]$ which satisfies

$$(\varphi^m)_{xx} + g(\varphi)\varphi = 0 \tag{1.5}$$

in the sense of distributions, and which vanishes at $x = \pm L$. Here and throughout this paper $g(u) = (1 - u)(u - a)$.

If $\varphi \in \mathcal{E}$ then φ^m is actually C^2 on the closed interval $[-L, L]$, while φ is even a smooth solution of (1.5) on the set $\{x : \varphi(x) > 0\}$. If φ has an interior zero, i.e. $\varphi(x) = 0$ at some $x_0 \in (-L, L)$, then positivity of φ implies that $(\varphi^m)_x(x_0) = 0$.

Equation (1.5) can be integrated by putting $v = m\varphi^{m-1}\varphi_x$: one has

$${}^1\!/_2 v^2 + \varphi^{m+1} G_m(\varphi) = {}^1\!/_2 A^2 \tag{1.6}$$

where

$$G_m(u) = u^{-m-1} \int_0^u m s^m g(s)\,ds = -\frac{m}{m+3}u^2 + \frac{m(1+a)}{m+2}u - \frac{am}{m+1} \tag{1.7}$$

is again a quadratic polynomial.

Since $(u^{m+1}G_m(u))' = u^m g(u) = u^m(1-u)(u-a)$ the function $u^{m+1}G_m(u)$ has a local minimum at $u=a$, and a local maximum at $u=1$. If $m>1$ and

$$0 < a < \frac{m+1}{m+3}$$

then the local maximum $G_m(1)$ is positive. Let $\alpha_1(m,a) \in (0,1)$ be the unique root of $G_m(\alpha)=0$.

For $\alpha_1(m,a) < \alpha < 1$, the solutions $\varphi(\alpha,x)$ of

$$(\varphi^m)_{xx} + g(\varphi)\varphi = 0, \qquad \varphi(0) = \alpha, \varphi'(0) = 0,$$

are then decreasing for $x>0$, and become zero at some $x>0$, say at $x = L = L(m,\alpha)$. One has

$$L(m,\alpha) = \int_0^\alpha \frac{du^m}{v} = \frac{1}{\sqrt{2}} \int_0^\alpha \frac{mu^{m-1}du}{\sqrt{\alpha^{m+1}G_m(\alpha) - u^{m+1}G_m(u)}}.$$

It was shown in [ACP] that $L(m,\alpha)$ as a function of α, has a unique minimum in $[\alpha(m,a),1)$. We denote this minimum by L_0. It was also observed in [ACP] that $L_1 = L(m,\alpha_1(m,a)) > L_0$ is finite,

$$L_1 = m \int_0^{\alpha_1(m,a)} \left(-2G_m(u)\right)^{-1/2} u^{(m-3)/2} du,$$

while $\lim_{\alpha\uparrow 1} L(m,\alpha) = \infty$.

For $L < L_0$ there is only one equilibrium, the zero solution; in this case all solutions of (1.1) decay to zero.

There are exactly three equilibria when $L \in (L_0, L_1)$; they are the zero solution, and two other solutions, $p(x)$ and $q(x)$, which are ordered as $0 < p(x) < q(x)$. At $L = L_0$ the two positive solutions coincide: $p(x) \equiv q(x)$.

For $L_0 < L < L_1$ both solutions p and q have nonzero flux at the boundary, but as $L \uparrow L_1$, the flux at the boundary of the middle solution p tends to zero. By definition, the flux of a solution $u(x,t)$ of (1.1) is $(u^m)_x$.

The solution q can be continued for all parameter values L; both q and the zero solution are asymptotically stable for all L (see [LP]).

Connecting orbits

Fix some value $L > L_0$, and let $\mathcal{N}_0$ and $\mathcal{N}_q$ be the basins of attraction of 0 and q respectively. Put $\mathcal{K} = \mathcal{A} \setminus (\mathcal{N}_0 \cup \mathcal{N}_q)$. Since $\mathcal{A}$ is connected, $\mathcal{K}$ will always be nonempty; in fact, we know that it contains all equilibria except 0 and q. The connectedness of $\mathcal{A}$ also implies that the points $0 \in \mathcal{A}$ and $q \in \mathcal{A}$ are not isolated in $\mathcal{A}$, so that $\mathcal{N}_0 \setminus \{0\}$ and $\mathcal{N}_q \setminus \{q\}$ are not empty.

Let $\Gamma_0 \subset \mathcal{N}_q \setminus \{q\}$ be an orbit of the semiflow. Then its alpha-limit set consists of equilibria, none of which are attractors, so that $\alpha(\Gamma_q) \subset \mathcal{K}$. Thus we have shown:

1.5. Theorem. *If $L > L_0$, then $X \setminus \{0, q\}$ contains an isolated invariant set for the semiflow ϕ_t, and there exist connecting orbits from this set to* 0 *and to q.*

If $L_0 < L \leq L_1$, then $\mathcal{K} = \{p\}$, so that we have established the *existence* of connecting orbits from p to 0 and to q. One of the main results in this paper is the uniqueness of these connecting orbits for $L < L_1$, as well as a more precise description of there asymptotic behaviour at the boundary of Ω, and at $t = \pm\infty$.

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2. The Weak Semiflow

Throughout this section we assume that $f : [0, 1] \to [0, 1]$ is Lipschitz continuous with $|f(u) - f(v)| \leq M_f |u - v|$.

Uniqueness of the very weak solution

Existence of a weak solution was already shown in [ACP], so we only have to prove that the very weak solution (as defined above) is unique, and depends continuously on the initial value u_0. Our proof is only a small variation on the proof in [ACP].

Assume that $u, v \in L_\infty(Q_T)$ are two different solutions with initial values u_0 and v_0. Then for any test function $\sigma \in C^\infty(Q_T)$ which vanishes on

$\partial\Omega \times [0, T]$ and on $\Omega \times \{T\}$ we have

$$\int_\Omega (u_0(x) - v_0(x))\sigma(x,0)\,dx = -\int\int_{Q_T} [(u-v)(\sigma_t + a\sigma_{xx}) + (f(u) - f(v))\sigma]\,dx\,dt$$

where

$$0 \le a(x,t) \equiv \frac{u^m - v^m}{u - v} \in L_\infty(Q_T).$$

As in [ACP] we choose a sequence $a_n \in C^\infty(Q_T)$ which satisfies

$$a_n \ge \frac{1}{n},$$

and

$$\|\frac{a_n - a}{\sqrt{a_n}}\|_{L_2(Q_T)} \downarrow 0.$$

Let $\chi \in C^\infty(Q_T)$ be any given test function, and let $\sigma_n \in C^\infty(Q_T)$ be the solution of

$$\text{(2.1)} \qquad \begin{cases} \sigma_t + a_n\sigma_{xx} = \chi(x,t) \qquad (x \in \Omega, 0 \le t < T) \\ \sigma(x,T) = 0 \text{ for } |x| < L, \text{ and } \sigma(\pm L, t) = 0 \text{ for } 0 < t < T \end{cases}$$

The solution χ_n then satisfies

$$\text{(2.2)} \qquad -\frac{1}{2}\frac{d}{dt}\int_\Omega |\sigma_{n,x}|^2\,dx + \int_\Omega (\sqrt{a_n}\sigma_{n,xx})^2\,dx = -\int_\Omega \sigma_{n,x}\chi_{n,x}\,dx$$

which implies that

$$-\frac{d}{dt}\left(\|\sigma_{n,x}\|_{L_2(\Omega)}\right) \le \|\chi_x(\cdot,t)\|_{L_2(\Omega)},$$

and, by integrating (2.2)

$$\int_0^T \int_\Omega (\sqrt{a_n}\sigma_{n,xx})^2\,dx\,dt \le \text{Const}.$$

This means that the σ_n are bounded in $L_\infty([0,T];H^1(\Omega))$, and because of (2.1), also in $H^1([0,T];L_2(\Omega))$.

We can therefore extract a subsequence, which we shall again denote by σ_n, which converges in $L_\infty([0,T];H^1(\Omega))$ (in the weak-star topology), as well as in $H^1([0,T];L_2(\Omega))$ (weakly).

By the same argument as in [ACP, lemma 10] one now shows that the limit σ of this subsequence satisfies:

$$\begin{aligned}\tag{2.3} &\int_\Omega (u_0(x)-v_0(x))\sigma(x,0)\,dx \\ &\qquad = -\int\int_{Q_T}[(u-v)\chi(x,t)+(f(u)-f(v))\sigma]\,dx\,dt\end{aligned}$$

If we assume that $0\le\chi\le e^{-\lambda t}$, then, by the maximum principle, all approximating solutions σ_n, and hence their limit σ, will satisfy

$$0\le-\sigma\le\int_t^T e^{-\lambda\tau}d\tau\le\frac{1}{\lambda}e^{-\lambda t}.$$

Using this inequality, and taking all terms with σ in (2.3) to one side, and the term with χ on the other, we get:

$$\begin{aligned}&\int\int_{Q_T}(u-v)\chi\,dx\,dt\le\\ &\qquad\le\int_\Omega(u_0(x)-v_0(x))^+\left(-\sigma(x,0)\right)\,dx+\\ &\qquad\qquad+\int\int_{Q_T}|f(u)-f(v)|(-\sigma)\,dx\,dt\\ &\qquad\le\frac{1}{\lambda}\left\{\|(u_0-v_0)^+\|_{L_1(\Omega)}+\frac{M_f}{\lambda}\int\int_{Q_T}e^{-\lambda t}|u-v|\,dx\,dt\right\},\end{aligned}$$

where $M_f=\sup_{0\le s\le 1}|f'(s)|$.

This holds for all $\chi\in C^\infty(Q_T)$ which vanish on ∂Q_T, and satisfy $0\le\chi\le e^{-\lambda t}$. Let χ converge in measure to $e^{-\lambda t}\chi_{\{u>v\}}$, with $\chi_{\{u>v\}}$ the characteristic function of the set where $u>v$. Then one finds:

$$\|e^{-\lambda t}(u-v)^+\|_{L_1(Q_T)}\le\frac{1}{\lambda}\|(u_0-v_0)^+\|_{L_1(\Omega)}+\frac{M_f}{\lambda}\|e^{-\lambda t}(u-v)\|_{L_1(Q_T)}$$

Exchange u and v, and add the corresponding inequality to get:

$$\|e^{-\lambda t}(u-v)\|_{L_1(Q_T)} \le \frac{1}{\lambda}\|u_0 - v_0\|_{L_1(\Omega)} + \frac{2M_f}{\lambda}\|e^{-\lambda t}(u-v)\|_{L_1(Q_T)}$$

Hence, if $\lambda > 2M_f$, then

$$\|e^{-\lambda t}(u-v)\|_{L_1(Q_T)} \le \frac{1}{\lambda - 2M_f}\|u_0 - v_0\|_{L_1(\Omega)}$$

which implies the uniqueness of the very weak solution.

Continuous dependence of the weak solution on the initial data

To prove the continuous dependence of solutions on initial data, we first have to derive some a priori estimates for an arbitrary weak solution.

Define $\eta(s) = s^m$ and

$$\zeta(u) = \int_0^u \sqrt{\eta'(s)}\, ds = \frac{2\sqrt{m}}{m+1} u^{\frac{m+1}{2}}.$$

Then we have the following.

2.1. Lemma. *If $u \in L_\infty(Q_T)$ is a weak solution of* (1.1) *with* $0 \le u \le 1$*, then*

$$\int_0^T \int_{-L}^{L} (\zeta(u)_x)^2\, dx\, dt \le c,$$

$$\int_{-L}^{L} (\eta(u)_x)^2\, dx \le \frac{c}{t},$$

and

$$\int_\delta^T \int_{-L}^{L} (\zeta(u)_t)^2\, dx\, dt \le \frac{c}{\delta},$$

where c is some constant which only depends on m, L, T and f.

Proof. In [ACP] the weak solutions were constructed by replacing $\eta(u) = u^m$ with a smooth $\eta_\epsilon(u)$ (e.g. $(u+\epsilon)^m$), whose derivative is bounded away from zero for $u \geq 0$. The modified problem is no longer degenerate, and therefore has a global solution, u^ϵ, the limit of which is the weak solution of (1.1).

Assuming, for the moment, that $\eta'(u) \geq \epsilon$, one easily obtains

$$\begin{aligned} {}^1\!/_2 \int_{-L}^{L} u(T,x)^2\,dx + \int_0^T \int_{-L}^{L} (\zeta(u)_x)^2\,dx\,dt = {}^1\!/_2 \int_{-L}^{L} u(0,x)^2\,dx \\ + \int_0^T \int_{-L}^{L} f(u)u\,dx\,dt, \end{aligned}$$

(just multiply the equation with u and integrate), and also

$$E(u(\cdot,T)) + \int_\delta^T \int_{-L}^{L} \zeta(u)_t^2\,dx\,dt = E(u(\cdot,\delta))$$

where

$$E(u(\cdot,\tau)) = \left[\int_{-L}^{L} ({}^1\!/_2 \eta(u)_x^2 - F_\eta(u))\,dx\right]_{t=\tau}$$

and

$$F_\eta(u) = \int_0^u f(s)\eta'(s)\,ds.$$

The first identity implies that $\zeta(u)_x \in L_2(Q_T)$, with

$$\|\zeta(u)_x\|_{L_2}^2 \leq L + 2LT \sup\{uf(u) | 0 \leq u \leq 1\}.$$

The second identity tells us that for solutions to nondegenerate equations the quantity $E(u(\cdot,t))$ is nonincreasing in time.

Since $\eta(u)_x^2 = \eta'(u)^2 u_x^2 \leq m\eta'(u)u_x^2 = m\zeta(u)_x^2$, we already know that $\frac{1}{2}\eta(u)_x^2 - F_\eta(u)$ is integrable on Q_T, with

$$\int_0^T \int_{-L}^{L} \left\{{}^1\!/_2 \eta(u)_x^2 - F_\eta(u)\right\}\,dx\,dt \leq c.$$

Since $E(u(\cdot,t))$ is nonincreasing, this implies

$$\int_0^\delta E(u(\cdot,t))\,dt \geq \delta E(u(\cdot,\delta)),$$

so that $E(u(\cdot,\delta)) \le c/\delta$, which implies the second inequality of the lemma (for the nondegenerate case).

The second identity also implies that $\zeta(u)_t \in L_2([\delta,T]\times(-L,L))$ and $\|\zeta(u)_t\|_{L_2}^2 \le E(u(\cdot,\delta)) \le c/\delta$.

Therefore the inequalities in the lemma hold if the equation is nondegenerate; to get them for $\eta(u) = u^m$ one puts $\eta(u) = (u+\epsilon)^m$, $f(u) = f(u+\epsilon)$, as in [ACP], and lets $\epsilon \downarrow 0$. $\square$

To complete the proof of the theorem we turn to the proof of continuity of the semiflow, with respect to the weak-star topology. Since X is a complete metrisable space, we only have to verify sequential continuity.

Let $u_{0n} \in X$ be a weak-star convergent sequence with limit u_0, and let $u_n \in L_\infty(Q)$ be the corresponding weak solutions. Since the u_n are bounded in $L_\infty(Q)$, we may assume that they have a weak-star convergent subsequence, which, by an abuse of notation, we shall again denote by u_n.

The integral estimates of the previous lemma imply that the u_n are precompact in $L_2([\delta,T]\times(-L,L))$ (with respect to the norm topology), for any $\delta, T > 0$. This allows us to pass to another subsequence which converges almost everywhere on Q_T to some function $u_\infty \in L_\infty(Q)$.

Using the dominated convergence theorem one verifies that u_∞ is again a weak solution of (1.1), with initial value u_0. Since this solution is unique, we have shown that any convergent subsequence of the u_n's has u_∞ as its limit, in $L_\infty(Q)$ and in $L_2([\delta,T]\times(-L,L))$, for any $\delta, T > 0$. This implies that the entire sequence u_n converges to u_∞.

We have shown that

$$\hat{X} = \{u \in L_\infty(Q) | u \text{ is a weak solution of } (1.1)\}$$

is a weak-star closed subset of $L_\infty(Q)$, and that the map $u_0 \in X \to u(t;u_0) \in \hat{X}$ is continuous.

Since $\hat{X}$ is a closed subset of the unit ball of $L_\infty(Q)$, it is weak-star compact, and therefore the continuous, one-to-one map which sends the initial value u_0 to the corresponding weak solution $u(t;u_0)$ in $\hat{X}$ is a homeomorphism.

On $L_\infty(Q)$ we have the translation semigroup, Φ^t $(t \ge 0)$, defined by

$$(\Phi^t u)(s,x) = u(t+s,x) \qquad (t,s \ge 0, x \in \Omega).$$

It is continuous with respect to the weak-star topology, and it maps weak solutions to weak solutions; i.e. it leaves $\hat{X}$ invariant. The semiflow $\Phi^t|\hat{X}$ and the semiflow ϕ^t on X are conjugate by the homeomorphism $u_0 \mapsto u(\cdot; u_0)$, so that continuity of Φ^t implies continuity of ϕ^t. □

3. The Flux of Strictly Positive Solutions

The quantity

$$\Phi(x,t) = -\frac{\partial u^m}{\partial x}$$

is called the *flux* of the solution $u(x,t)$.

We shall call a function $u \in X$ *strictly positive* if it satisfies

1. *$u(x) > 0$ for all $x \in (-L, L)$.*
2. *There is an $\epsilon > 0$ such that*

$$u(x)^m \geq \epsilon \cos\left(\frac{\pi x}{2L}\right), \qquad -L < x < L.$$

We define $\mathcal{P} \subset X$ to be the set of all strictly positive states.

3.1. Strict Positivity Lemma. *$\mathcal{P}$ is positively invariant under the semiflow.*

Proof. Let $U(x)$ be a smooth function on the interval $[-L, L]$, for which $\eta(U)_{xx} = 0$ holds on $[-L, -L/2] \cup [L/2, L]$, for which $U(\pm L) = 0$, and which is positive in the interior of the interval. Define $k_1 = \sup_{0 \leq u \leq 1} |g(u)|$, and

$$k_2 = \sup_{-L<x<L} \frac{-(U^m)_{xx}}{U}$$

and observe that both are finite constants.

Then consider $v(x,t) = \epsilon(t)U(x)$, for some function $\epsilon(t)$. One has

$$v_t - (v^m)_{xx} - f(v) \leq \{\epsilon'(t) + k_2\epsilon(t)^m + k_1\epsilon(t)\}U(x),$$

so that v will be a subsolution if $\epsilon(t)$ satisfies

$$\epsilon'(t) + k_2\epsilon(t)^m + k_1\epsilon(t) = 0.$$

If one chooses $\epsilon(0) > 0$ small enough, then $v(x,0)$ will be less than $u(x,0)$, and by the comparison principle one has $v(x,t) \leq u(x,t)$ for all time. □

The main results of this section are the following.

3.2. Continuous Flux Theorem. *If the nonlinearity $f(u)$ in the initial value problem is a C^1 function, and $u \in C(Q_T)$ is a weak solution, which is strictly positive for all $0 < t \le T$, then the flux $-(u^m)_x$ is uniformly continuous on $\Omega \times [\tau, T]$ for any $\tau > 0$.*

3.3. Monotone Convergence Theorem. *Let u_n be a decreasing sequence of strictly positive solutions which satisfy*

$$\delta \le \frac{u_n(x,t)^m}{\cos(\pi x/2L)} \le \delta^{-1}.$$

Then u_n converges to a strictly positive solution U of (1.1), and in fact

$$\frac{u_n(x,t)^m}{\cos(\pi x/2L)} \longrightarrow \frac{U(x,t)^m}{\cos(\pi x/2L)}$$

uniformly on $(-L, L) \times [\tau, T]$, for any $0 < \tau < T$. In particular, the boundary fluxes $\Phi_{\pm,n}(t)$ of the u_n converge uniformly on $[\tau, T]$ to the boundary fluxes $\Phi_\pm(t)$ of the limit solution U.

The proof of these theorems will be given in several steps. We begin with a lemma which tells us that the flux at the boundary points is always finite.

3.4. Lemma. *There is a constant $0 < C < \infty$ such that any weak solution u of (1.1) satisfies*

$$u(x,t)^m \le \left(\frac{C}{t}\right)^{\frac{m}{m-1}} \cos\frac{\pi x}{2L}, \qquad |x| < L, 0 < t < 1.$$

Proof. The function $w(x,t) = u(x,t)^m$ is a continuous function; wherever it is positive it is also a smooth solution of

$$w_t = mw^{1-1/m} w_{xx} + mg(w^{1-1/m})w. \tag{3.1}$$

Since $0 \le w \le 1$ there is a $c < \infty$ such that $|mg(w^{1-1/m})| \le c$.

Let $\theta(x)$ be the solution of the ODE

$$\theta''(x) + \theta(x)^{1/m} = 0, \qquad \theta(0) = 1, \quad \theta'(0) = 0.$$

It is not hard to verify that there is an $x_0 > 0$ such that θ is defined and positive for $|x| < x_0$, while $\theta(x_0) = 0$; moreover, θ is a C^2 function on $[-x_0, x_0]$ with $\theta'(x_0) \neq 0$, and, in particular, there is a constant such that

$$c \le \frac{\theta(x)}{\cos(\pi x/2x_0)} \le \frac{1}{c}, \qquad |x| < x_0. \tag{3.2}$$

Now consider for small positive ϵ the function

$$W_\epsilon(x,t) = A(t)\theta\left(\frac{x}{L+\epsilon}x_0\right),$$

where $A(t)$ is to be determined.

Substitution in (3.1) shows that W_ϵ is a supersolution of (3.1) if

$$A'(t) \ge -m\left(\frac{x_0}{L+\epsilon}\right)^2 A(t)^{2-1/m} + mcA(t).$$

this inequality is satisfied for $0 < t \le 1$ by $A(t) = (C/t)^{m/(m-1)}$, if one chooses the constant C large enough – in fact, one can choose C independent of ϵ.

With this choice of $A(t)$ we therefore find that W_ϵ is a supersolution of (3.1) on $[-L, L] \times (0, 1]$. Since W_ϵ is strictly positive on the edges $\{\pm L\} \times (0, 1]$, and since w is a smooth solution of (3.1) wherever $w > 0$, we can apply the maximum principle. The conclusion is that $w \le W_\epsilon$ for all $\epsilon > 0$, and hence that $w \le W_0$ on $(-L, L) \times (0, 1]$. The lemma now follows from (3.2). $\square$

We shall need some estimates which are similar to the "Aronson–Benilan" estimates for solutions of the porous medium equation.

3.5. AB-Estimates. *Assume $0 \le u(x,t) \le 1$ is a weak solution of (1.1). Then*

$$(u^m)_t \ge -\frac{A}{t}u^m, \qquad (x,t) \in Q_T \tag{3.3}$$

and,

$$(u^m)_{xx} \ge -\frac{A}{t}u, \qquad (x,t) \in Q_T. \tag{3.4}$$

in which A is some constant depending only on T.

Proof. Consider $v(x,t) = u(x,t)^m$, and $w = v_t$. Then v is a solution of

$$v_t = mv^{1-1/m}v_{xx} + h(v)$$

where $h(v) = mg(v^{1/m})v$, while w satisfies

$$w_t = mv^{1-1/m}w_{xx} + \frac{m-1}{m}\frac{w^2}{v} + \left[mh'(v) - (m-1)h(v)\right]w, \tag{3.5}$$

Since $h(v) = mvg(v^{1/m})$ is a C^1 function of v, the coefficient of w in the last term is bounded.

It turns out that $w^*(x,t) = -\frac{A}{t}v(x,t)$ satisfies

$$\begin{aligned} w^*_t - mv^{1-1/m}w^*_{xx} + \frac{m-1}{m}\frac{(w^*)^2}{v} + \left[mh'(v) - (m-1)h(v)\right]w^* \ge \\ \ge \frac{Av}{t^2}\left(1 + C_f t - \frac{m-1}{m}A\right), \end{aligned}$$

in which C_f is a constant that only depends on nonlinearity f. Hence if A is chosen large enough, w^* is a subsolution. This would imply that $w \ge w^*$ on Q_T, if we could apply the maximum principle. Since we don't know any a priori bound for $w = (u^m)_t$ near the boundary we must use a slightly more involved argument.

Choose a sequence of smooth functions $u_{0,\epsilon}$ on $[-L, L]$ which converge to $u(x,0)$, and which satisfy $\epsilon \le u_{0,\epsilon} \le 1$. Let $u_\epsilon \in L_\infty(Q_T)$ be the solution of

$$u_t = (u^m)_{xx} + f(u), \qquad u(\pm L, t) \equiv \epsilon.$$

Then the u_ϵ are smooth, and the previous arguments show that for $A = A(f, T)$ large enough, the u_ϵ satisfy $(u_\epsilon^m)_t \geq -Au_\epsilon^m/t$; that they also satisfy (3.4) can be shown by using $u_t = (u^m)_{xx} + f(u)$. Now we let $\epsilon \downarrow 0$, and we obtain the same inequalities in the sense of distributions for u. □

By combining this with lemma 3.4 we get:

3.6. COROLLARY. *If u is a weak solution of (1.1) then*

$$(u^m)_t \geq -\left(\frac{A}{t}\right)^{2+\frac{1}{m-1}} \cos(\pi x/2L),$$

$$(u^m)_{xx} \geq -\left(\frac{A}{t}\right)^{1+\frac{1}{m-1}} (\cos(\pi x/2L))^{1/m}.$$

Let $u(x,t)$ be a weak solution to (1.1), which is strictly positive. Then since the PDE is nondegenerate at positive values of u, our solution will be C^∞ in the interior of Q_T. In particular $(u^m)_x$ is a continuous function, away from the boundary. The AB–estimates which we have just proved imply that the two limits

$$\Phi_\pm \equiv \mp \lim_{x \to \pm L} (u(\cdot, t)^m)_x$$

exist for any $t > 0$. By lemma 3.4 these limits are also finite.

3.7. CONTINUITY OF THE FLUX AT THE BOUNDARY. *The functions $\Phi_\pm(t)$ are continuous on $(0, T]$.*

This theorem implies that the flux is in fact continuous on $[-L, L] \times (0, T]$, i.e. theorem 3.2. Indeed, if we fix some $0 < \tau < T$, then the AB–estimates imply that there is a $C > 0$ such that $(u^m)_{xx} \geq -C$ for $\tau \leq t \leq T$. Hence $r(x,t) = (u^m)_x + Cx$ is a monotone function of x, for each fixed $t \in [\tau, T]$. Continuity of $\Phi_\pm(t)$ implies that $\lim_{n\to\infty} r(x, t_n) = r(x,t)$ pointwise, if $t_n \to t$. By Helly's selection lemma the monotonicity of the $r(\cdot, t_n)$ and the continuity of their limit $r(\cdot, t)$ implies that $r(x, t_n) \to r(x,t)$ uniformly in $x \in [-L, L]$. This implies that r is continuous on $[-L, L] \times [\tau, T]$; since τ was arbitrary, continuity on $[-L, L] \times (0, T]$ follows.

We turn to the continuity of $\Phi_\pm$. Since the functions

$$\Phi^x(t) \equiv \frac{\pi}{2L} \frac{v(x,t)}{\cos(\pi x/2L)}$$

are uniformly bounded for $0 < t < T$, and since they converge pointwise to $\Phi_\pm(t)$ as $x \to \pm L$, the AB-estimate for v_t implies that $\Phi_\pm(t)$ is of bounded variation on $[\tau, T]$ for any $\tau > 0$, and that

$$\Phi'_\pm(t) \geq -\left(\frac{A}{t}\right)^{2+\frac{1}{m-1}} \tag{3.6}$$

holds in the sense of distributions. From this one-sided bound we conclude the following:

$$\liminf_{t \downarrow t_0} \Phi_-(t) \geq \Phi_-(t_0),$$
$$\limsup_{t \uparrow t_0} \Phi_-(t) \leq \Phi_-(t_0).$$

From here on we shall ignore the right hand boundary, and only consider the left hand boundary. By symmetry this is no restriction.

3.8. Lemma. $\limsup_{t \downarrow t_0} \Phi_-(t) \leq \Phi_-(t_0)$

Proof. Since the function $g(u)$ is bounded for $0 \leq u \leq 1$, the function $\bar{v} = e^{C(t-t_0)}(x + L)$ is a supersolution for (3.1), if C is chosen large enough.

Let $\delta > 0$ be given; then for small enough $\epsilon > 0$ one has

$$v(x, t_0) \leq (\Phi_-(t_0) + \delta)(x + L) \qquad (0 \leq x + L \leq \epsilon)$$

and

$$v(x, t_0) \leq e^{C(t-t_0)}(\Phi_-(t_0) + \delta)\epsilon, \qquad (0 \leq t - t_0 \leq \epsilon)$$

by continuity of v.

The maximum principle then implies that $v \leq \bar{v}$ for $0 \leq x + L \leq \epsilon$ and $0 \leq t - t_0 \leq \epsilon$, and hence

$$\Phi_-(t) = v_x(-L, t) \leq e^{C(t-t_0)}(\Phi_-(t_0) + \delta)$$

so that $\limsup_{t\downarrow t_0} \Phi_-(t) \le \Phi_-(t_0) + \delta$. Since δ was arbitrary, this proves the lemma. □

To prove the continuity of $\Phi_-(t)$ at t_0 we therefore still have to show that

$$\liminf_{t\uparrow t_0} \Phi_-(t) \le \Phi_-(t_0).$$

To this end we consider the rescaled functions

$$V_\lambda(\xi,\tau) = \lambda^{-1} v(\lambda\xi - L, t_0 + \lambda^{\frac{m+1}{m}}\tau).$$

They are solutions of

$$V_\tau = V^{1-1/m} V_{\xi\xi} + \lambda^{1/m} h(\lambda V), \tag{3.7}$$

where $h(v) = vg(v^{1/m})$. Our estimates for v imply the following for the V_λ:

(i) $V_{\lambda,\xi}$ is uniformly bounded,

(ii) The V_λ (with $0 \le \lambda \le 1$) are equicontinuous functions (since v is continuous),

(iii) There is a $\delta > 0$ such that for $0 < \xi < L/\lambda$, and for all (relevant) τ one has
$$\delta\xi \le V_\lambda(\xi,\tau) \le \delta^{-1}\xi.$$

It follows from these bounds that any sequence $\lambda_n \downarrow 0$ has a subsequence λ_{n_i} for which the corresponding V_λ's converge uniformly on compact sets in $Q_- = \{(\xi,\tau) : \xi \ge 0, \tau \le 0\}$. Moreover, each V_λ is bounded away from zero on $Q_{a,-} = \{(\xi,\tau) : a \le \xi \le a^{-1}, \tau \le 0\}$, by (iii), so that its derivatives $V_\tau, V_{\xi\xi}$ will be uniformly Hölder continuous on each $Q_{a,-}$. The limit V_0 of the convergent subsequence of the V_λ's will therefore satisfy

$$V_\tau = V^{\frac{m-1}{m}} V_{\xi\xi} \qquad (\xi > 0, \tau < 0) \tag{3.8}$$

as well as (i), (ii) and (iii).

By definition of $\Phi_-(t_0)$ one has

$$V_0(\xi,0) = \lim_{\lambda\downarrow 0} V_\lambda(\xi,0) = \Phi_-(t_0)\xi.$$

Finally, the AB-estimates imply that

$$V_{\lambda,\tau} \geq -\left(\frac{A}{t_0 + \lambda^{1+1/m}\tau}\right)^{2+\frac{1}{m-1}} \sin\left(\frac{\pi\lambda}{2L}\xi\right),$$

which in the limit $\lambda \downarrow 0$ means that $V_{0,\tau} \geq 0$ on Q_-, i.e. that V_0 is a subsolution, and also that $V_0(\xi,\tau) \leq \Phi_-(t_0)\xi$ holds everywhere on Q_-.

But $V_1(\xi,\tau) = \Phi_-(t_0)\xi$ is a solution of (3.7), so that the strong maximum principle forces

$$V_0(\xi,\tau) \equiv \Phi_-(t_0)\xi.$$

Thus any subsequence V_{λ_j} which converges on $\{\xi \geq 0, \tau \leq 0\}$ must have the time independent and linear function V_1 as a limit, so that we have proved:

$$\lim_{\lambda\downarrow 0} V_\lambda(\xi,\tau) = \Phi_-(t_0)\xi, \text{ uniformly on compact subsets of } Q_-.$$

What we want is convergence of the derivative (i.e. the flux) on the boundary. To get this we introduce the following two parameter family of functions:

$$w_{a,b}(\xi) = a\xi + b\xi^{1+1/m},$$

and we define

$$W_{a,b,c}(\xi,\tau) = w_{a,b}(\xi + c\tau).$$

These functions are supposed to give us "travelling subsolutions," which we will place under one of the V_λ's; when the sharp edge of such a subsolution hits the boundary it will give us a lower estimate for the derivative of V_λ, and hence for v_x at the boundary.)

A straightforward, but tedious calculation shows that

$$\begin{aligned}-W_\tau + W^{\frac{m-1}{m}} W_{\xi\xi} + \lambda^{1/m} h(\lambda W) \geq \\ \frac{m+1}{m^2}ab - ac - \frac{m+1}{m}bc\bar{\xi}^{1/m} + \lambda^{1/m}h(\lambda W),\end{aligned}$$

where we have written $\bar{\xi}$ for $\xi + c\tau$.

Given $a, b > 0$ we can choose $c > 0$ so small that $W_{a,b,c}$ is a subsolution for (3.1) on $0 \leq \xi \leq 1$, for small enough $\lambda > 0$ (here we use that $h(v)$ is bounded).

Now let any $0 < a < \Phi_-(t_0)$ be given. Choose $b > 0$ so small that

$$w_{a,b}(\xi) \leq \frac{a + \Phi_-(t_0)}{2}\xi \qquad (0 < \xi < 1) \tag{3.9}$$

holds, and choose c so small that $W_{a,b,c}$ is a subsolution for (3.1), whenever $\lambda \leq \lambda(a, b, c)$.

We have shown that the V_λ converge uniformly on $\{0 \leq \xi \leq 2, -\frac{2}{c} \leq \tau \leq 0\}$ to $\Phi_-(t_0)\xi$, as $\lambda \downarrow 0$. Hence we can choose λ so small that

$$|V_\lambda(\xi, \tau) - \Phi_-(t_0)\xi| \leq \frac{\Phi_-(t_0) + a}{2} \tag{3.10}$$

for all $\xi \in [0, 2]$ and all $\tau \in [-\frac{2}{c}, 0]$.

Consider the strip

$$S_{\tau_0} = \left\{(\xi, \tau) : \tau_0 - c^{-1} \leq \tau \leq \tau_0, 0 < \xi + c(\tau - \tau_0) \leq 1\right\},$$

where $-1/c \leq \tau_0 \leq 0$. Then we have just verified that $W_0 = W_{a,b,c}$ is a subsolution for (3.6) on this strip S_{τ_0}, and the conditions (3.8,9) imply that $W_0 \leq V_\lambda$ on the parabolic boundary of the strip S_{τ_0}. By the maximum principle we therefore get $W_0 \leq V_\lambda$ on all of the strip, and in particular $V_\lambda(\xi, \tau) \geq a\xi + b\xi^{1+1/m}$ for $\xi \in [0, 1]$ and $-1/c \leq \tau \leq 0$. After rescaling this then implies that $\Phi_-(t) = v_x(-L, t) \geq a$ for all $t \leq t_0$ sufficiently close to t_0.

Since a was any number less than $\Phi_-(t_0)$, this completes the argument which shows that $\Phi_-(t)$ is continuous. □

Proof of the Monotone convergence theorem

Let $\epsilon > 0$ and $0 < \tau < T$ be given. Away from the edges $\{\pm L\} \times (0, T]$ the u^n are classical solutions of a nondegenerate parabolic equation, so they surely converge uniformly on any region of the form $[-x_0, x_0] \times [\tau, T]$, with $0 < x_0 < L$. Hence we consider regions of the form $[x_0, L] \times [\tau, T]$.

Since the flux $(U^m)_x$ is continuous on $[0, L] \times [\tau, T]$, there is an $x_0 \in (0, L)$ such that

$$|U(x, t)^m - \Phi(t)(L - x)| < \frac{\epsilon}{4}(L - x), \tag{3.11}$$

for $\tau \le t \le T$ and $x_0 \le x \le L$.

Moreover, the AB-estimates imply that $(u_n^m)_{xx} \ge -C_1$ on $[-L, L]\times[\tau, T]$ for some constant C_1. We assume that x_0 has been chosen so close to L that $L - x_0 < \epsilon/2C_1$.

Finally, the uniform convergence of the u_n to U in the interior implies that there is an n_ϵ such that

$$(u_n(x_0,t))^m \le U(x_0,t)^m + \frac{\epsilon}{4}(L - x_0)$$

for $\tau \le t \le T$, $n \ge n_\epsilon$.

Now, let $x_0 < x < L$, $\tau < t < T$, then $(u_n^m)_{xx} \ge -C_1$ implies

$$\begin{aligned}
u_n(x,t)^m &\le u_n(x_0,t)^m \frac{L-x}{L-x_0} + \frac{C_1}{2}(L-x)(x-x_0) \\
&\le u_n(x_0,t)^m \frac{L-x}{L-x_0} + \frac{\epsilon}{4}(L-x) \\
&\le U(x_0,t)^m \frac{L-x}{L-x_0} + \frac{\epsilon}{2}(L-x) \\
&\le \left(\Phi(t) + {}^3\!/_4\epsilon\right)(L-x) \qquad \text{(by (3.11) at } x_0) \\
&\le U(x,t)^m + \epsilon(L-x) \qquad \text{(by (3.11))}
\end{aligned}$$

holds for $n \ge n_\epsilon$.

Combined with the uniform convergence on $[0, x_0] \times [\tau, T]$, this implies that

$$\lim_{n\to\infty} \sup_{[0,L]\times[\tau,T]} \left| \frac{(u_n^m)}{L-x} - \frac{U^m}{L-x} \right| \le \epsilon$$

for any $\epsilon > 0$, which, in turn, implies the montone convergence theorem. $\square$

4. Further Regularity of the Flux

Let $\varphi(x)$ be an equilibrium state for our semiflow. We introduce a new independent variable, y, defined by

$$y(x) = \int_0^x \frac{d\xi}{\varphi(\xi)^{(m-1)/2}}. \tag{4.1}$$

Derivatives with respect to x and y are related via

$$\frac{\partial}{\partial y} = \varphi(x)^{\frac{m-1}{2}} \frac{\partial}{\partial x}. \tag{4.2}$$

As x varies from $-L$ to $+L$, the range of this new variable will be an interval $(-Y_\varphi, Y_\varphi)$, where Y_φ may be infinite, as happens if one chooses $\varphi = p$ and $L = L_1$. However, we shall assume in this section that φ is strictly positive, in which case

$$Y_\varphi = \int_0^L \frac{d\xi}{\varphi(\xi)^{(m-1)/2}} \tag{4.3}$$

is finite.

The coordinate transformation $x \mapsto y$ is everywhere smooth, except at the ends of the interval $(-L, L)$. Assuming that $(\varphi^m)'(L) = -A$, with $A > 0$, the following lemma tells us how x, y and φ^m are related near the end point $x = L$.

4.1. xy-CONVERSION LEMMA. *Writing $\xi = L - x$ and $\eta = Y_\varphi - y$, we have*

$$\begin{aligned}
\varphi^m &= A\xi \left(1 + \xi^{1+1/m} B_1(\xi^{1/m}, \xi^{1+1/m})\right) \\
\eta &= c_1 \xi^{\frac{m+1}{2m}} \left(1 + \xi^{1+1/m} B_2(\xi^{1/m}, \xi^{1+1/m})\right) \\
\xi &= c_2 \eta^{\frac{2m}{m+1}} \left(1 + \eta^2 B_3(\eta^{2/(m+1)}, \eta^2)\right) \\
\varphi &= c_3 \eta^{\frac{2}{m+1}} \left(1 + \eta^2 B_4(\eta^{2/(m+1)}, \eta^2)\right)
\end{aligned}$$

where c_1, c_2, c_3 are constants, and $B_1, \ldots, B_4$ are smooth functions on a neighborhood of the origin in $\mathbf{R}^2$.

PROOF. Once the first relation is established, the others follow by direct computation.

To prove the first relation, we recall that $(\varphi^m)_{xx} = -g(\varphi)\varphi$; using $(\varphi^m)_x(L) = -A$, we therefore get

$$\varphi^m(L - \xi) = A\xi - \int_0^\xi (\xi - \xi') g(\varphi(L - \xi'))\varphi(L - \xi') d\xi'.$$

Putting $\varphi^m(L-\xi) = A\xi(1+\alpha(\xi))$, and substituting $\xi' = \sigma\xi$, we find

$$\alpha(\xi) = -A^{1-1/m}\xi^{1+1/m} \cdot \int_0^1 (1-\sigma) g\left((A\sigma\xi)^{1/m}(1+\alpha(\sigma\xi))^{1/m}\right)\sigma^{1/m}(1+\alpha(\sigma\xi))^{1/m} d\sigma.$$

We now regard ξ as a small parameter, and consider $\beta_\xi(t) = \alpha(\xi t)$. Then $\beta_\xi \in C([0,1])$ satisfies:

$$\beta_\xi = -\xi^{1+1/m}\mathrm{T}(\xi^{1/m}, \beta_\xi),$$

where $\mathrm{T} : \mathbf{R} \times C([0,1]) \to C([0,1])$ is defined by

$$\mathrm{T}(\lambda, \beta)(t) = A^{1-1/m}t^{1+1/m} \cdot \int_0^1 (1-\sigma)\sigma^{1/m} g\left(\lambda(A\sigma t)^{1/m}(1+\beta(\sigma t))^{1/m}\right)(1+\beta(\sigma t))^{1/m} d\sigma.$$

Clearly T is a smooth Fréchet differentiable map: if the function $g(u)$ were C^k, then T would be C^k too – in our situation g is a polynomial.

By the implicit function theorem, the fixed point equation $\beta = -\mu\mathrm{T}(\lambda, \beta)$ has a unique solution for small enough λ and μ, and this solution will depend smoothly on the parameters λ and μ. Choosing $\lambda = \xi^{1/m}$ and $\mu = \xi^{1+1/m}$, we find that β_ξ is a smooth $C([0,1])$ valued function of $\xi^{1/m}$ and $\xi^{1+1/m}$. Using that β_ξ satisfies the fixed point equation, we even find that $\xi^{-1-1/m}\beta_\xi = -\mathrm{T}(\xi^{1/m}, \beta_\xi)$ is a smooth function of $\xi^{1/m}$ and $\xi^{1+1/m}$.

Hence $\beta_\xi(1) = \xi^{1+1/m}B(\xi)$, where $B(\xi)$ is a smooth function of $\xi^{1/m}$ and $\xi^{1+1/m}$; this is what is claimed in the first identity of the lemma. □

If $u(x,t)$ is a weak solution of (1.1), then

$$v(y,t) = \left(\frac{u(x(y),t)}{\varphi(x(y))}\right)^m$$

satisfies

$$v_t = mv^{1-1/m}\left\{\frac{\partial^2 v}{\partial y} + P(y)\frac{\partial v}{\partial y}\right\} + h(v,y), \tag{4.4}$$

where

$$P(y) = \frac{3m+1}{2}\frac{\varphi_y}{\varphi},$$

and h is given by

$$h(v, y) = m\left[g\left(\varphi v^{1/m}\right)v - g(\varphi)v^{2-1/m}\right], \qquad \varphi = \varphi(y).$$

The condition of strict positivity of the previous section is equivalent with

$$\delta \le v(y, t) \le \delta^{-1}, \qquad |y| < Y_\varphi, 0 < t < T,$$

for some $\delta > 0$. Hence if u is strictly positive, and if one only looks at the highest order terms, (4.4) is a nondegenerate parabolic PDE. However, the first order term, $P(y)v_y$, turns out to be singular at $y = \pm Y_\varphi$. Using the xy-Conversion Lemma we can determine the exact nature of the singularity of $P(y)$:

$$P(y) = \frac{3m+1}{m+1}\frac{1}{\eta} + \eta B_5(\eta^{2/m+1}, \eta^2) \tag{4.5}$$

where $\eta = y - Y_\varphi$ or $\eta = y + Y_\varphi$, and B_5 is again a smooth function near the origin of $\mathbf{R}^2$.

If we augment (4.5) with Neumann boundary conditions,

$$v_y(\pm Y_\varphi, t) = 0, \tag{4.6}$$

then it is shown in the appendix that we obtain a well posed initial value problem.

4.2. The Smooth Subflow. *The equations (4.4,6) generate an analytic local semiflow Φ_t on*

$$\mathcal{O}_\alpha = \left\{v \in h^\alpha\left([-Y_\varphi, Y_\varphi]\right) : \delta \le v \le \delta^{-1}, \textit{for some } \delta > 0\right\}$$

for each $0 < \alpha < 1$.

The main result of this section will be that this local semiflow is in fact global.

4.3. Regularity of Strictly Positive Solutions. *Let $u(x,t)$ be a strictly positive solution of (1.1), and let $v(y,t)$ be the associated solution of (4.4).*

Then $v(\cdot,t) \in h^{2,\alpha}([-Y_\varphi, Y_\varphi])$ for all $t > 0$, and $v(\cdot,t)$ satisfies the Neumann boundary conditions (4.6). In particular, $v(\cdot,t)$ constitutes an orbit of the smooth subflow Φ, and any orbit of this subflow exists for all $t > 0$.

The proof of this theorem will proceed along the following lines: First we establish some interior estimates for arbitrary positive solutions of (4.5) which allow us to verify that any such solution comes from a weak solution of our original PDE (1.1). We then obtain an a priori estimate for the L_2 norm of v_y, which only uses the continuity of the flux that was established in the previous section.

4.4. Interior v_y Bound. *If $v(y,t)$ is a solution of (4.6) on $-Y_\varphi < y < Y_\varphi, 0 < t < T$, which satisfies $\delta \le v \le \delta^{-1}$ for some $\delta > 0$, then*

$$\left|\frac{\partial v}{\partial y}\right| \le C \max\left(\frac{1}{|y \pm Y_\varphi|}, \frac{1}{\sqrt{T}}\right).$$

Here C is a constant which does not depend on v.

Proof. Given $-Y_\varphi < y_0 < Y_\varphi, 0 < t_0 < T$, we choose $\epsilon = \min(1, |y_0 \pm Y_\varphi|, \sqrt{t_0})$, and consider

$$v^\epsilon(y,t) = v(y_0 + \epsilon y, t_0, \epsilon^2 t).$$

Then v^ϵ satisfies

$$\frac{\partial v^\epsilon}{\partial t} = (v^\epsilon)^{1-1/m}\left\{\frac{\partial^2 v^\epsilon}{\partial y^2} + \epsilon P(y_0 + \epsilon y)\frac{\partial v^\epsilon}{\partial y}\right\} + \epsilon^2 h(y_0 + \epsilon y, v^\epsilon).$$

We have chosen ϵ so that v^ϵ is defined for $|y| < 1, -1 < t < 0$. From (4.5) one deduces that both coefficients $\epsilon P(y_0 + \epsilon y)$ and $\epsilon^2 h(y_0 + \epsilon y, v^\epsilon)$ are bounded on $|y| < {}^2/_3$, $-1 < t < 0$, independently of $\epsilon > 0$, i.e. independently of y_0, t_0.

Classical interior estimates for quasilinear parabolic PDE now imply that $v^\epsilon_y(0,0)$ is bounded by some constant that does not depend on (y_0, t_0). After rescaling this leads to the desired estimate for v_y. $\square$

4.5. BACK–SUBSTITUTION LEMMA. *Let v be a solution of (4.4) which is continuous on $(-Y_\varphi, Y_\varphi) \times [0,T]$, and which is bounded by $\delta \le v \le \delta^{-1}$ for some $\delta > 0$. Then $u(x,t) = \varphi(x) v(y(x),t)^{1/m}$ is a weak solution of (1.1) on Q_T.*

PROOF. Since v is smooth, u is a smooth solution of the PDE in (1.1), on the interior of Q_T. All we have to worry about is that the boundary terms which arise when we try to verify that u is indeed a weak solution vanish.

From

$$(u^m)_x = \varphi(x)^{-(m-1)/2} \left(\varphi^m v\right)_y = \varphi(x)^{(m+1)/2} \left(v_y + m \frac{\varphi_y}{\varphi} v \right),$$

the interior estimate for v_y, and (4.5) one finds that

$$|(u^m)_x| \le \frac{C}{\sqrt{t}}.$$

With this estimate one can justify all partial integrations that are necessary to show that u satisfies (1.4) for arbitrary test functions. $\square$

4.6. H^1-ESTIMATE. *Let $v(y,t)$ be a smooth solution of (4.4,6) defined for $0 < t < T$ with $\delta \le v \le \delta^{-1}$. Assume it satisfies*

$$\left| v(y,t)^{1-1/m} - v(\pm Y_\varphi, t)^{1-1/m} \right| \le \epsilon, \quad \text{for } |\pm Y_\varphi - y| \le \epsilon, 0 < t < T. \tag{4.7}$$

Then there is an $\epsilon_\delta > 0$ such that if $\epsilon < \epsilon_\delta$ one has

$$\int_{-Y_\varphi}^{Y_\varphi} \left(v_y(y,t)\right)^2 dy \le \frac{A}{t^2}, \qquad (0 < t < T),$$

for some $A = A(\delta, \epsilon, T) < \infty$.

PROOF. Let $X(t) = \int_{-Y_\varphi}^{Y_\varphi} v_y^2 dy$. Then integration by parts results in

$$\begin{aligned}\frac{dX}{dt} &= -2\int_{-Y_\varphi}^{Y_\varphi} v_t v_{yy} dy \\ &= -2m\int_{-Y_\varphi}^{Y_\varphi} v^{1-1/m}\left((v_{yy})^2 + P(y)v_y v_{yy}\right)dy - 2\int_{-Y_\varphi}^{Y_\varphi} v_{yy}h(y,v)dy\end{aligned}$$

The second integral may be estimated directly: It follows from the AB-estimates that $v_{yy}(y,t) \geq -\frac{C}{t}$ for some $C > 0$. Since v_y vanishes at the endpoints of the interval $(-Y_\varphi, Y_\varphi)$, this implies that

$$\int_{-Y_\varphi}^{Y_\varphi} |v_{yy}(y,t)|\, dy \leq \frac{2CY_\varphi}{t}. \tag{4.8}$$

Hence

$$\left|\int_{-Y_\varphi}^{Y_\varphi} v_{yy}h(y,v)dy\right| \leq \frac{C}{t},$$

for some $C < \infty$.

To estimate the first integral, we split it into three parts:

$$\begin{aligned}-2m\int_{-Y_\varphi}^{Y_\varphi} v^{1-1/m}\left((v_{yy}^2 + P(y)v_y v_{yy}\right)dy &= \int_{-Y_\varphi}^{-Y_\varphi+\epsilon} + \int_{-Y_\varphi+\epsilon}^{Y_\varphi-\epsilon} + \int_{Y_\varphi-\epsilon}^{Y_\varphi} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

The middle term satisfies

$$|I_2| \leq -\lambda\int_{-Y_\varphi+\epsilon}^{Y_\varphi-\epsilon}(v_{yy})^2 dy + C\int_{-Y_\varphi}^{Y_\varphi}(v_y)^2 dy \tag{4.9}$$

for some small $\lambda > 0$ and some finite $C < \infty$. From here on λ and C will denote generic small and large constants, respectively.

To estimate I_1 we use the hypothesis (4.7) to obtain

$$\begin{aligned}|I_3| \leq 2v(Y_\varphi,t)^{1-1/m}&\int_{Y_\varphi-\epsilon}^{Y_\varphi}\left(v_{yy}^2 + P(y)v_y v_{yy}\right)dy \\ &+ 2\epsilon\int_{Y_\varphi-\epsilon}^{Y_\varphi}\left(v_{yy}^2 + |P(y)v_y v_{yy}|\right)dy.\end{aligned}$$

It follows from (4.5) that for $0 < Y_\varphi - y < \epsilon$

$$P(y) \le \frac{C}{y - Y_\varphi} < 0, \qquad P'(y) \le \frac{-C}{(y - Y_\varphi)^2} \le -CP(y)^2 < 0,$$

provided ϵ has been chosen small enough.

Hence we have

$$-\int_{Y_\varphi-\epsilon}^{Y_\varphi} P(y)v_y v_{yy} dy = {}^1\!/_2 P(Y_\varphi - \epsilon) v_y(Y_\varphi - \epsilon, t)^2 + {}^1\!/_2 \int_{Y_\varphi-\epsilon}^{Y_\varphi} P'(y) v_y^2 dy$$
$$\le -C \int_{Y_\varphi-\epsilon}^{Y_\varphi} (P(y)v_y)^2 \, dy.$$

Using Cauchy-Schwarz we therefore find

$$\int_{Y_\varphi-\epsilon}^{Y_\varphi} P(y)v_y v_{yy} dy \ge 0$$

and

$$\int_{Y_\varphi-\epsilon}^{Y_\varphi} (P(y)v_y)^2 \, dy \le C \int_{Y_\varphi-\epsilon}^{Y_\varphi} (v_{yy})^2 \, dy.$$

Using these two inequalities we find that I_3 satisfies

$$|I_3| \le -\lambda \int_{Y_\varphi-\epsilon}^{Y_\varphi} (v_{yy})^2 dy + C \int_{-Y_\varphi}^{Y_\varphi} (v_y)^2 dy.$$

By the same arguments one finds that I_1 satisfies a similar bound, and hence we find

$$\frac{dX}{dt} \le -\lambda \int_{-Y_\varphi}^{Y_\varphi} (v_{yy})^2 \, dy + C_1 X + \frac{C_2}{t}$$

Denoting by $\|\cdot\|_p$ the $L_p(-Y_\varphi, Y_\varphi)$-norm we note that

$$\begin{aligned} \|v_y\|_2 &\le \|v_y\|_1^{1/2} \cdot \|v_y\|_\infty^{1/2} \\ &\le C\|v\|_1^{1/4} \cdot \|v_{yy}\|_1^{1/4} \cdot \|v_{yy}\|_2^{1/2} \\ &\le \frac{C}{t^{1/4}} \|v_{yy}\|_2^{1/2} \end{aligned}$$

which yields $-\|v_{yy}\|_2^2 \le -\lambda t \|v_y\|_2^4$, and thus

$$\frac{dX}{dt} \le -\lambda' t X^2 + C_1 X + \frac{C_2}{t}.$$

Direct substitution shows that $\bar{X}(t) = At^{-2}$ is a supersolution for this equation if A is chosen large enough. Therefore $X(t) \le A/t^2$ holds for some large A, as claimed. □

Proof of theorem 4.3

Let $v_0 \in \mathcal{O}_\alpha$ be given, and consider the maximal solution $v(y,t)$ of (4.4,6) which is provided by the smooth semiflow Φ_t on $\mathcal{O}_\alpha$. Assume that v exists for $0 \le t < T$; then we shall now show by contradiction that $T = \infty$.

Suppose that T is finite. Since solutions of the smooth semiflow all become singular at the same time, i.e. independent of α, we may assume that $0 < \alpha < {}^1\!/_2$.

Let $u(x,t)$ be the weak solution of (1.1), corresponding to $v(y,t)$. The strict positivity lemma 3.1 applied to u implies that $\delta \le v \le \delta^{-1}$, for some small $\delta > 0$. The continuous flux theorem then tells us that v is uniformly continuous on $[-Y_\varphi, Y_\varphi] \times [\tau, T)$ for any $\tau > 0$.Hence our H^1 bound applies, and, using the compact embedding of H^1 into h^α for $0 < \alpha < {}^1\!/_2$, we see that $\{v(\cdot,t) : \tau < t < T\}$ is precompact in $\mathcal{O}_\alpha$. Thus it cannot be a maximal orbit of the semiflow Φ_t. □

Next, let $u(x,t)$ be any strictly positive weak solution of (1.1), and define $v(y,t) = (u(x,t)/\varphi(x))^m$. We shall show that $v(\cdot,t) \in \mathcal{O}_\alpha$ for any small $t > 0$, and, in view of what we have just derived, that $\{v(\cdot,t) : \tau < t < \infty\}$ is an orbit of the smooth semiflow.

Let $t_0 > 0$ be given. Then $v(y, t_0/3) = (u(x, t_0/3)/\varphi(x))^m$ is continuous, by the continuous flux theorem again. Choose a decreasing sequence $v_n \in \mathcal{O}_\alpha$ with $v_n \downarrow v(\cdot, t_0/3)$, and let $v_n(y,t)$ be the corresponding solutions to (4.4,6) with $v_n(y, t_0/3) = v_n(y)$.

The monotone convergence theorem 3.3 now implies that $v_n(y,t)$ converges uniformly to $v(y,t)$ on $[-Y_\varphi, Y_\varphi] \times [{}^2\!/_3 t_0, 2t_0]$. The continuity condition (4.7) therefore applies to all v_n's, if one chooses ϵ small enough, and we get a uniform H^1-bound for the v_n:

$$\int_{-Y_\varphi}^{Y_\varphi} (v_{n,y})^2 \, dy \le \frac{C}{t - {}^2\!/_3 t_0}, \qquad {}^2\!/_3 t_0 < t < 2t_0$$

with C independent of n.

The conclusion we draw from this is that the limit $v(\cdot,t_0)$ of the $v_n(\cdot,t_0)$ must belong to H^1, and hence to $\mathcal{O}_\alpha$, as claimed. $\square$

5. Strict Positivity of the Connecting Orbits

We still assume that $L < L_1$. Let $u(t,x)$ $(-\infty < t < \infty, |x| \le L)$ be a weak solution of (1.1) which connects $p(x)$ either to $q(x)$ or to 0.

Then we have $p(x) = \lim_{t\to-\infty} u(t,x)$ uniformly in $|x| \le L$.

5.1. THEOREM. *$u(t,\cdot) \in \mathcal{P}$ for all $t \in \mathbf{R}$. If one defines $v = \left(\frac{u(t,x(y))}{p(x(y))}\right)^m$ then*

$$\lim_{t\to-\infty} v(t,y) = 1 \quad \textit{in } h^\alpha([-Y,Y]).$$

PROOF. We already know that $u(x,t)$ tends to $p(x)$ uniformly, so for every x there is a $t_x > -\infty$ with $u(t_x,x) > 0$. Since the support of $u(t,\cdot)$ does not shrink we conclude that $u(t,x) > 0$ for all $x \in (-L,L)$. We must therefore show that $(u^m)_x(\pm L,t) \neq 0$.

Let us consider $x = -L$, the argument for $x+L$ being similar. Let $A = (p^m)_x(-L)$, so that $p(x)^m = (A+o(1))(x+L)$. Consider

$$W(x,t) = w(x+t),$$

where

$$w(\xi) = a\xi + b\xi^{1+1/m}.$$

Then

$$\begin{aligned} W_t - W^{1-1/m}W_{xx} - h(W) = & \\ &= w' - w^{1-1/m}w'' - h(w) \\ &= a + \frac{m+1}{m}b\xi^{1+1/m} - b\left(a + b\xi^{1/m}\right)^{1-1/m}\frac{m+1}{m^2} \\ &\quad - h(a\xi + b\xi^{1+1/m}) \end{aligned}$$

where $\xi = x+t$.

Using $|h(v)| \leq Cv$ we find

$$W_t - W^{1-1/m}W_{xx} - h(W) \leq$$
$$\leq a + \frac{m+1}{m} b\xi^{1+1/m} - \left(\frac{m+1}{m^2}b - C\xi\right)\left(a + b\xi^{1/m}\right).$$

Choose $0 < a < A$ and $b = 2m^2/(m+1)$. Then there is a $\xi_0 > 0$ such that

$$W_t - W^{1-1/m}W_{xx} - h(W) \leq 0$$

for $0 < \xi < \xi_0$.

Since $p(x)^m = (A + o(1))(x + L)$, we can arrange by choosing ξ_0 smaller, if necessary, that $2\xi_0 < L$ and

$$\left(a + b\xi_0^{1/m}\right)\xi_0 < p(-L + \xi_0)^m.$$

Then $p(x) > \left(a + b\xi_0^{1/m}\right)\xi_0$ for $\xi_0 \leq x + L \leq 2\xi_0$. Since $u(x,t) \to p(x)$ uniformly as $t \to -\infty$, there is some $t_0 \in \mathbf{R}$ such that

$$u(x,t) > \left(a + b\xi_0^{1/m}\right)\xi_0$$

for $\xi_0 \leq x + L \leq 2\xi_0$ and $-\infty < t < t_0$.

We claim that $(u^m)_x(-L,t) \geq a$ for any $t \leq t_0$.

Indeed, let $t_1 < t_0$ be given and consider

$$W(x,t) = w(x + L + t - t_1)$$

on

$$\Omega = \left\{(x,t) : -\xi_0 \leq t - t_1 \leq 0, 0 \leq x + L + t - t_1 \leq \xi_0\right\}.$$

We have just shown that W is a subsolution of $W_t = W^{1-1/m}W_{xx} + h(W)$ on Ω while u^m is a solution of this equation. Furthermore our construction is such that $W \leq u^m$ on the parabolic boundary of Ω. By the maximum principle we therefore find that $W \leq u^m$ on $\bar{\Omega}$. In particular we find

$$u(x,t_1)^m \geq a(x+L) + b(x+L)^{1+1/m}, \tag{5.1}$$

whence $u_x^m(-L, t_1) \geq a$, as claimed.

So far we have shown $u(\cdot, t) \in \mathcal{P}$ for every $t \in \mathbf{R}$. Using (5.1) we can derive a uniform lower bound of the form

$$\left(\frac{u(x,t)}{p(x)}\right)^m \geq 1 - \delta, \qquad -\infty < t < t_\delta,$$

for any $\delta > 0$.

One can also derive a similar uniform upper bound,

$$\left(\frac{u(x,t)}{p(x)}\right)^m \leq 1 + \delta, \qquad -\infty < t < t_\delta,$$

by constructing traveling wave supersolution of the form $W(x,t) = a\xi + b\xi^{1+1/m}$, with $\xi = x - t$, and $a > A$, $b = -2m^2 a^{1/m}/(m+1)$, and applying the analogous argument.

The upper and lower bounds together imply that $(u/p)^m$ converges uniformly to 1 as $t \to -\infty$. The H^1 estimate 4.6 then implies that $v = (u/p)^m \to 1$ in $\mathcal{O}^\alpha$ for any $0 < \alpha < 1/2$.

Hence the orbit $u(\cdot, t)$ of the weak semiflow lies on the unstable manifold $W^u(p)$ of p in the smooth subflow. □

6. Linearization

We again assume that $L_0 < L < L_1$ and put $\varphi(x) = p(L, x)$; we define y, $\mathcal{O}^\alpha$ as in section 4.

Then $v \equiv 1$ is an equilibrium of the smooth subflow on $\mathcal{O}^\alpha$.

6.1. THEOREM. *$v \equiv 1$ is a hyperbolic fixed point of $\left\{\Phi^t | \mathcal{O}^\alpha\right\}_{t \geq 0}$ with a one dimensional unstable manifold.*

This theorem combined with theorem 5.1 directly imply the Main Theorem. For if $u(x,t)$ is a connecting orbit from $p(L,x)$ to either 0 or to $q(L,x)$, then by theorem 5.1 it must lie on the unstable manifold through $p(L,x)$ of the semiflow $\{\Phi_t\}$; by the theorem we are about to prove there are exactly two such orbits, one increasing and one decreasing.

PROOF. The linearized equation at $v = 1$ is obtained by substituting $v = 1 + \epsilon w(y,t)$ in equation (4.1) and discarding all terms involving ϵ^2 and higher powers of ϵ. The result is

$$\begin{aligned}&\frac{1}{m}\frac{\partial w}{\partial t} = w_{yy} + P(y)w_y + Q(y)w,\\&\frac{\partial w}{\partial y}(\pm Y_\varphi, t) = 0,\end{aligned}$$

where

$$P(y) = \frac{3m+1}{2}\frac{\varphi_y}{\varphi}, \tag{6.1}$$

$$mQ(y) = h_v(y,1) = (m-3)\varphi^2 - (m-2)\varphi + (m-1)a. \tag{6.2}$$

The associated eigenvalue problem is

$$\begin{aligned}&\frac{\lambda}{m}\psi = \psi'' + P(y)\psi' + Q(y)\psi,\\&\psi'(\pm Y_\varphi) = 0.\end{aligned}$$

It follows from the Sturm-Liouville theory that this eigenvalue problem has a sequence of eigenvalues $\lambda_0 > \lambda_1 > \cdots$ where each eigenvalue is simple, and the eigenfunction corresponding to λ_n has exactly n zeroes in the interval $(-Y_\varphi, Y_\varphi)$.

Since $P(y)$ is odd and $Q(y)$ is even in y, the ψ_{2n} are even functions of y, and the ψ_{2n+1} are odd.

What we must show is that $\lambda_0 > 0 > \lambda_1$.

We shall first show that $\lambda_1 < 0$. To do this we consider

$$v(\xi; y) = \left(\frac{p(x(y)+\xi)}{p(x(y))}\right)^m. \tag{6.3}$$

Each $v(\xi;\cdot)$ is a solution of

$$mv^{1-1/m}\left\{v'' + P(y)v'\right\} + h(v,y) = 0,$$

so that

$$\psi_*(y) = -\left(\frac{\partial v}{\partial \xi}\right)_{\xi=0} = -m\frac{p'(x(y))}{p(x(y))} \tag{6.4}$$

satisfies

$$\psi_*'' + P(y)\psi_*' + Q(y)\psi_* = 0 \tag{6.5}$$

on the interval $-Y < y < Y$. Moreover, $\psi_*(y) > 0$ on $(0, Y)$, and since $p(\pm L) = 0$, ψ_* is unbounded at $y = \pm L$, so that ψ_* is not one of the ψ_n.

Let $\psi_1(y)$ be the eigenfunction corresponding to λ_1, and assume that $\psi_1(y) > 0$ for $0 < y \le Y$. If λ_1 were nonnegative then ψ_1 would be a subsolution of (6.5). Define $c_0 = \sup\{c \ge 0 : c\psi_1 \le \psi_* \text{ on } (0, Y)\}$. Then $c_0\psi_1 \le \psi_*$, and either $c_0\psi_1(y) = \psi_*(y)$ for some $y \in (0, Y)$, or $c_0\psi_1'(0) = \psi_*'(0)$. Since ψ_* cannot be a multiple of ψ_1 both conditions lead to a contradiction: the first with the strong maximum principle, the second with the boundary point lemma. Hence we have shown that $\lambda_1 < 0$.

To show that $\lambda_0 > 0$ we recall that the middle solution $p(L, x)$ depends on the length L of the interval, and vary this length parameter L, i.e. we consider

$$\psi_\dagger(y) = \left(\frac{\partial}{\partial \xi}\right)_{\xi=0} \left(\frac{p(L+\xi, x(y))}{p(L, x(y))}\right)^m.$$

The analysis in [ACP] implies that $\psi_\dagger$ has a zero in the interval $0 < y < Y$. Since $\psi_\dagger(y)$ is an even function it therefore has at least two zeroes in the interval $(-Y, Y)$. Moreover, it is easily seen that $\psi_\dagger$ satisfies (6.5).

By the Sturmian oscillation theorem every solution of the eigenvalue equation $\frac{\lambda}{m}\psi = \psi'' + P(y)\psi' + Q(y)\psi$ with $\lambda \le 0$ must have at least one zero. Since the first eigenfunction ψ_0 has no zeroes we may therefore conclude that $\lambda_0 > 0$, as announced. □

7. The critical case

Transformation and Smooth Semiflow

Our study of the connecting orbits in the case $L_0 < L < L_1$ was based on the fact that by performing the transformations (4.1–3) and considering $v = (u/\varphi)^m$, the degenerate PDE (1.1) is transformed to a nondegenerate initial value problem (4.4, 6) whose associated local semiflow is smooth and hence can be linearized. In this section we assume $L = L_1(m, a)$, we put $\varphi = p_*$ and apply the same transformation (4.1–3), and investigate the resulting initial value problem.

A fundamental difference with the precritical case $L_0 < L < L_1$ immediately appears when one computes the range $(-Y_\varphi, Y_\varphi)$ of the new coordinate y: it turns out that $p_*(x) \sim C|x \pm L_1|^{2/(m-1)}$ at $x = \pm L_1$, so that the integral defining Y_φ diverges, i.e. the y-variable assumes all real values. The analog of the xy-conversion lemma says that

$$y = \begin{cases} -C \log |L_1 - x| + \cdots & \text{for } x \uparrow L_1, \\ C \log |L_1 + x| + \cdots & \text{for } x \downarrow -L_1, \end{cases} \tag{7.1}$$

in which $\cdots$ stands for lower order terms.

If $u(x,t)$ is a weak solution of (1.1) with $u(x,t) > 0$ everywhere, then $v = (u/\varphi)^m$ is again a solution of the quasilinear PDE (4.4)

$$v_t = mv^{1-1/m} \left\{ \frac{\partial^2 v}{\partial y^2} + P(y) \frac{\partial v}{\partial y} \right\} + h(v,y), \qquad y \in \mathbf{R}. \tag{4.4$'$}$$

Unlike in the precritical case the coefficients turn out not to be singular as $y \to \pm Y_\varphi$. In fact, both $P(y)$ and $h(y,v)$ are C^∞ in both y, v, as long as $v > 0$, with all derivatives uniformly bounded in regions of the form $\delta \le v \le \delta^{-1}$, $\delta > 0$, i.e.

$$\left| P^{(n)}(y) \right| + \left| \frac{\partial^{n+l} h}{\partial v^n \partial y^l} \right| \le C(\delta, n, l), \qquad \delta \le v \le \delta^{-1}, y \in \mathbf{R}. \tag{7.2}$$

Moreover, $P(y)$ and $h(y,v)$ have definite limits as $y \to \pm\infty$. One way to see this is by observing that one can explicitly compute φ as a function of y. Indeed, by (1.6) and (4.2) we have

$$(\varphi_y)^2 + \frac{2}{m} \varphi^2 G_m(\varphi) = 0, \tag{7.3}$$

(the constant A in (1.6) must vanish.) One can separate variables in this first order ODE which after integration leads to

$$\varphi(y) = \left(k_1(m,a) + k_2(m,a) \cosh \sqrt{\frac{2a}{m+1}} y \right)^{-1} \tag{7.4}$$

for certain positive coefficients $k_j(m,a)$. The derivative estimates for P and h are now easily verified.

The nice behaviour of φ at $y = \pm\infty$ implies that the Cauchy problem (4.4′) is well posed, and has a short time solution $\{v(y,t) : 0 < t < T\}$ for any continuous initial value $v(y,0)$ which is uniformly bounded and uniformly bounded away from zero. A straightforward application of theorem 8.6 in the appendix gives that (4.4′) generates a smooth local semiflow Φ_t on

$$\mathcal{O}^\alpha = \left\{v \in h^\alpha(\mathbf{R}) : \exists_{\delta>0}\delta \le v \le \delta^{-1}\right\}.$$

It follows from well known interior estimates for quasilinear parabolic equations that any solution $v(y,t)$ of the Cauchy problem (4.4′) with

$$\delta \le v(y,t) \le \delta^{-1}, \quad y \in \mathbf{R}, 0 < t < T,$$

is smooth and has derivative estimates of the form

$$\left|\frac{\partial^n v}{\partial y^n}\right| \le C(n,\delta,T)t^{-n/2}. \tag{7.5}$$

We leave to the reader the task of using these estimates to prove a "back-substitution lemma," i.e. of showing that any bounded solution of the Cauchy problem (4.4′) generates a weak solution to (1.1) by $u(x,t) = p_*(x)v(y(x),t)^{1/m}$.

The local semiflow Φ_t is truly local, since any orbit of Φ_t corresponds to a solution to (4.4′), and hence to a weak solution of (1.1) whose flux on the boundary vanishes, i.e. for which $(u^m)_x = 0$ at $x = \pm L_1$. This condition is not preserved by the weak semiflow ϕ_t. One could, for instance, consider an orbit $u(t) = \phi_t(u_0)$ whose initial value u_0 satisfies $p_* < u_0 < q(L_1,\cdot)$, and for which $v_0 = (u_0/\varphi)^m$ belongs to $\mathcal{O}^\alpha$. Then for a short time the orbit $u(t)$ will be given by the weak solution generated by the smooth local semiflow Φ_t on $\mathcal{O}^\alpha$, but as $t \to \infty$ the orbit must converge to $q(L_1,\cdot)$; in particular the flux at the boundary of $u(t)$ must be positive for large enough t. Hence $\Phi_t(v_0)$ does not exist for all $t > 0$.

Linearization at the equilibrium

As in the precritical case of the previous sections the transformation $v = (u/\varphi)^m$ sends the equilibrium $p_* = \varphi$ of the semiflow ϕ_t to the equilibrium $v_*(y) \equiv 1$ of the smooth local semiflow Φ_t. We can therefore linearize again and hope either to find that $v_* \equiv 1$ is hyperbolic, or else to understand at least something about the equilibrium $v_* \equiv 1$ by constructing invariant manifolds.

The linearization at $v_* \equiv 1$ of the local semiflow is a one parameter analytic semigroup $d\Phi_t(v_*) = e^{mAt}$, whose generator is mA, where A is given by

$$A = \left(\frac{\partial}{\partial y}\right)^2 + P(y)\frac{\partial}{\partial y} + Q(y), \qquad \mathrm{dom}A = h^{2,\alpha}(\mathbf{R}).$$

Here

$$P(y) = \frac{3m+1}{2}\frac{\varphi_y}{\varphi} = \pm\sqrt{-\frac{2}{m}G_m(\varphi)} \tag{7.6}$$

$$\begin{gathered} mQ(y) = h_v(y,1) = (m-3)\varphi^2 - (m-2)\varphi + (m-1)a, \\ \varphi = \varphi(y) \end{gathered} \tag{7.7}$$

are the same as in (6.1,2).

The coefficients P and Q have limits at $y = \infty$ given by

$$P(\infty) = -\frac{3m+1}{2}\sqrt{\frac{2a}{m+1}}, \qquad Q(\infty) = \frac{m-1}{m}a.$$

The limits at $y = -\infty$ are determined by the fact that P is an odd function and Q is an even function.

Since our operator is defined on an unbounded interval we cannot expect the spectrum to consist of pure point spectrum. In the example on p.140 of [He81] Dan Henry computes the *essential spectrum,* $E\sigma(A)$, of operators such as A, i.e. the set of $\lambda \in \mathbf{C}$ for which $\lambda - A : h^{2,\alpha} \to h^\alpha$ is not a Fredholm operator of index zero. This portion of the spectrum is of interest to us, since it basically is the part of the spectrum with which we do not know what to do. The remainder, $\mathbf{C} \setminus E\sigma(A)$, consists of eigenvalues of finite multiplicity, or of points in the resolvent set of A.

Henry considers operators on $L_p(\mathbf{R})$ rather than on the little Hölder space h^α we use here, but all his arguments can be carried over to the little Hölder spaces. His conclusion for our operator A is that its essential spectrum occupies a parabolic region given by

$$\Re\lambda \le a\frac{m-1}{m} - \left(\frac{\Im\lambda}{k}\right)^2, \qquad k^2 = \frac{a(3m+1)^2}{2(m+1)}.$$

Thus we see that the spectrum of A intersects the imaginary axis rather drastically, and that the fixed point $v_* \equiv 1$ is not hyperbolic. In fact, we cannot even expect there to be a finite dimensional centre manifold: the conclusion must be that analogies with finite dimensional ODE theory do not seem to shed much light on the nature of Φ_t near v_*.

A fast unstable manifold

Having determined the essential spectrum of A we now consider its point spectrum.

If we put $\mu(y) = \int_0^y P(\eta)d\eta$, then A is symmetric with respect to the inner product

$$\langle u, v\rangle_\mu = \int_{\mathbf{R}} e^{\mu(y)} u(y)v(y)dy.$$

Indeed, since $\mu(y) = P(\infty)|y| + O(1)$ at $y = \pm\infty$, and $P(\infty) < 0$, e^μ decays exponentially at $\pm\infty$, and one easily computes that for any $u, v \in h^{2,\alpha}(\mathbf{R})$ one has

$$\langle u, Av\rangle_\mu = \int_{\mathbf{R}} e^{\mu(y)} \left\{-u'v' + Qu(y)v(y)\right\} dy.$$

Hence all eigenvalues of A are real, and if there are any eigenvalues they form a countable sequence

$$\lambda_0 > \lambda_1 > \cdots \geq a\frac{m-1}{m}$$

which is either finite, or else accumulates at $a\frac{m-1}{m}$ (see fig. 7.1).

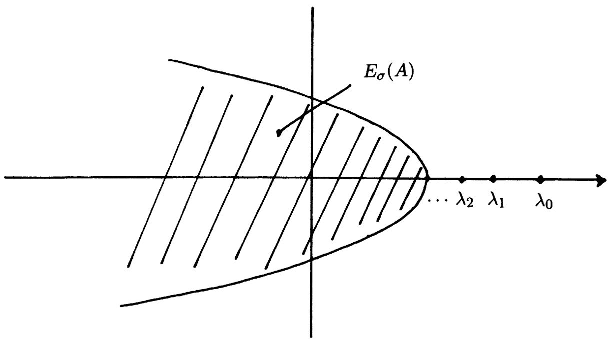

Fig. 7.1. The possible spectrum of A.

If the operator A does indeed have at least one eigenvalue, then the eigenfunction φ_0 corresponding to the largest eigenvalue λ_0 must be positive, and λ_0 must be a simple eigenvalue. We can then apply the invariant manifold theorem, and conclude the existence of a one dimensional fast unstable manifold through $v_* \equiv 1$. Thus there exist two orbits $w_\pm(y,t) \in \mathcal{O}^\alpha$ of Φ_t, which are defined for all $t \le 0$, and have the following asymptotics as $t \to -\infty$

$$w_\pm(y,t) = 1 \pm e^{(\lambda_0+o(1))t}\varphi_0(y) + \cdots, \qquad (t \to -\infty) \tag{7.8}$$

i.e.

$$\frac{w_\pm(y,t)-1}{\|w_\pm(y,t)-1\|_{h^\alpha}} \to \varphi_0 \text{ in } h^\alpha(\mathbf{R}) \text{ as } t \to -\infty.$$

The corresponding solutions $u_\pm(x,t)$ to (1.1) are then solutions which exist for all $t \le 0$. Using the weak semiflow we can extend them to $t > 0$ in a unique way. The resulting solutions will have zero flux at the boundaries $x = \pm L_1$ for $t \le 0$, but as we mentioned before this situation need not persist. As $t \to +\infty$ $u_+(x,t)$ will converge to $q(L_1,x)$, so that it will become strictly positive, while $u_-(x,t)$ will converge to 0.

There are many other ways of extending the solution u_+ to positive times. We mention one extension whose conjectured existence formed a part of the original motivation for the present work.

Since the solution u_+ has no flux at the boundaries $\pm L_1$, it can be extended to a solution of the Cauchy Problem for (1.1), simply by defining $u_+(x,t) \equiv 0$ for $|x| \ge L_1$, $t \le 0$. The Cauchy problem for (1.1) is well-posed so that we can also extend $u_+(x,t)$ for $t > 0$ by defining u_+ for $t > 0$ to be the unique weak solution of the Cauchy problem with initial data $u_+(x,0)$ for $|x| \le L_1$, and 0 for $|x| \ge L_1$. This way we obtain a solution $u_+(x,t)$ of the Cauchy problem, defined for all time, whose free boundaries remain fixed at $x = \pm L_1$ for an infinite time period. Such a solution has been called a solution with a waiting time at infinity, and our discussion shows that such a solution exists if the operator A has at least one eigenvalue outside of its essential spectrum.

Whether or not A has isolated eigenvalues turns out to depend on the values of the parameters a and m. Based on numerical calculations we conjecture that there is a monotone concave function $m_*(a)$ defined for $0 \le a \le {}^1\!/_2$, with $m_*(0) = 2$, $m_*({}^1\!/_2) = 1$ such that A has one isolated

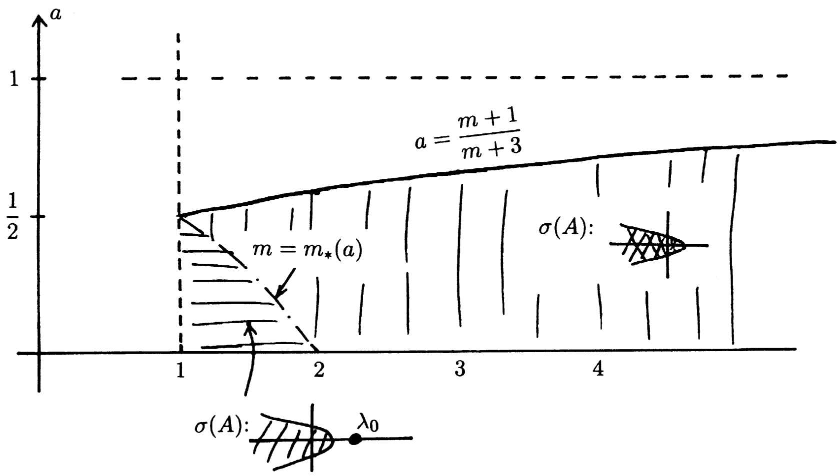

Fig. 7.2. The parameter domain (a, m).

eigenvalue for $1 < m < m_*(a)$ and no isolated eigenvalues otherwise (see fig 7.2). We can only prove the following.

7.1. Theorem. *If $m \geq 2$ then A has no isolated eigenvalues. For any $a \in (0, {}^1\!/_2)$ there is an $m_0(a) > 1$ such that A does have an isolated eigenvalue for $1 < m < m_0(a)$.*

Proof. Let $m \geq 2$. Then we claim that $mQ(y) \leq mQ(\infty)$. Indeed, from (7.7) and $\varphi(\infty) = 0$ it follows that for $2 \leq m \leq 3$ one has $mQ < a(m-1)$ wherever $\varphi(y) > 0$, i.e. everywhere, while for $m > 3$ one has $mQ(y) < a(m-1)$ wherever $0 < \varphi(y) < \frac{m-2}{m-3}$. Since $0 < \varphi(y) < 1$ for all y, this last condition is also fulfilled for all y.

Any eigenvalue λ must be real, and outside the essential spectrum, so if there is an eigenvalue with corresponding eigenfunction $\psi(y)$, then we must have $\lambda > a\frac{m-1}{m} = Q(\infty)$. An analysis of the differential equation

$$\psi''(y) + P(y)\psi'(y) + (Q(y) - \lambda)\,\psi(y) = 0, \qquad y \in \mathbf{R}, \tag{7.9}$$

satisfied by any eigenfunction ψ shows that $\psi(y)$ must decay exponentially at $y = \pm\infty$. In particular at some y_0 one must have $\psi(y_0) = \pm \sup_{\mathbf{R}} |\psi(y)|$, and one may assume that $\psi(y_0) > 0$. At y_0 one then also has $\psi'' \leq 0$ and $Q(y_0) - \lambda < 0$. Upon substitution in (7.9) this yields a contradiction. Hence there can be no eigenvalues if $m \geq 2$.

We turn to the existence part of the theorem.

The starting point of our argument is the observation that the operator A can actually be defined for all (a, m) with $m > 0$ and $0 < a < \frac{m+1}{m+3}$, instead of just for $m > 1$. Indeed, the coefficients of A depend only on $\varphi(y)$, and the explicit form (7.4) is defined for all $m > 1$ and $0 < a < \frac{m+1}{m+3}$.

When $m = 1$, the essential spectrum of the operator A occupies the parabolic region $4a\Re\lambda + (\Im\lambda)^2 \leq 0$. In particular, it is contained in the closed left half plane.

We consider the function ψ_* introduced in (6.1,2),

$$\psi_*(y) = -m\frac{\varphi_x}{\varphi} = -m\varphi(y)^{-\frac{m-1}{2}}\frac{\varphi_y}{\varphi}.$$

By (6.3) ψ_* satisfies the eigenvalue equation (7.9) with $\lambda = 0$. Since $\varphi(y)$ vanishes at $\pm\infty$, ψ_* is unbounded for $m > 1$, but when $m = 1$ we simply get $\psi_*(y) = -\varphi_y/\varphi$, so that in this case ψ_* is bounded, and converges to a finite positive limit as $y \to \infty$.

For any $\lambda > -P(\infty)^2/4$ the eigenvalue equation has a unique solution $\Psi(\lambda, y)$ with

$$\Psi(\lambda, y) = (1 + o(1))e^{\gamma y}, \quad (y \to \infty),$$
$$\gamma = \gamma(\lambda) = -P(\infty)/2 - \sqrt{P(\infty)^2/4 + \lambda}$$

For $\lambda = 0$ we have $\gamma = 0$, so $\Psi(\lambda, y)$ must be a multiple of ψ_*. Since the graph of φ is bell shaped we have $\Psi(0, y) > 0$ for $y > 0$, $\Psi(0, 0) = 0$ and $\Psi_y(0, 0) > 0$.

When $\lambda > \sup_{\mathbf{R}} Q(y)$, the asymptotics at $y = \infty$ imply $\Psi(\lambda, y) > 0$ and $\Psi_y(\lambda, y) < 0$ for large y. Since Ψ satisfies (7.9) Ψ must be decreasing for all $y \in \mathbf{R}$. For if there were some y_0 with $\Psi_y(\lambda, y_0) \geq 0$, then there would also be a largest y_1 with the same property; at this y_1 one then would have $\Psi_y = 0$, and, since Ψ is decreasing on (y_1, ∞), one would also have $\Psi(\lambda, y_1) > 0$; substitution in (7.9) shows that this would lead to a contradiction.

Thus we find that for $\lambda = 0$ we have $\Psi_y(\lambda, 0) > 0$, while for large λ the opposite inequality holds. There must then be some $\lambda_0 > 0$ for which $\Psi_y(\lambda_0, 0) = 0$. The coefficients P and Q are odd and even, respectively, so $\Psi_y(\lambda_0, 0) = 0$ implies that $\Psi(\lambda_0, y)$ is an even solution of (7.9), and hence is an eigenfunction of A, with $m = 1$.

To complete the proof we simply observe that this isolated eigenvalue λ_0 persists under small perturbations of the operator A, and thus that A also has an isolated eigenvalue for m close to 1. $\square$

8. Appendix

Smooth Local Semiflows

Recall that a local semiflow on a topological space $\mathcal{O}$ is a continuous map $\Phi : \mathcal{D} \to \mathcal{O}$ defined on an open subset $\mathcal{D} \subset \mathcal{O} \times [0, \infty)$ which satisfies

lsf$_1$ *$\mathcal{O} \times \{0\} \subset \mathcal{D}$ and $\Phi(x, 0) \equiv x$;*

lsf$_2$ *if $(x, t) \in \mathcal{D}$ then $(x, s) \in \mathcal{D}$ for all $s \in [0, t)$;*

lsf$_3$ *if $(x, t) \in \mathcal{D}$ and $(\Phi(x, t), s) \in \mathcal{D}$ then $(x, t+s) \in \mathcal{D}$ and $\Phi(x, t+s) = \Phi(\Phi(x, t), s)$.*

The domain of a local semiflow can always be represented as

$$\mathcal{D} = \{(x, t) : 0 \le t < T(x)\}$$

where $T : \mathcal{O} \to (0, \infty]$ is a lower semi continuous function. One calls $T(x)$ the life-span of the orbit starting at x. When Φ is generated by the initial value problem associated with some PDE then one says that a solution $x(t) = \Phi(x_0, t)$ "blows up in finite time" if $T(x_0) < \infty$. This can only occur if the orbit $\gamma(x_0) = \{\Phi(x_0, t) : 0 \le t < T(x_0)\}$ is not precompact in $\mathcal{O}$.

For a local semiflow $\Phi : \mathcal{D} \to \mathcal{O}$ we define $\mathcal{D}_t = \{x \in \mathcal{O} : T(x) > t\}$ and $\Phi_t : \mathcal{D}_t \to \mathcal{O}$, with $\Phi_t(x) = \Phi(x, t)$.

Assuming that $\mathcal{O}$ is an open subset of a Banach space E (or, more generally, a Banach manifold modelled on E), we shall say that a local semiflow $\Phi : \mathcal{D} \to \mathcal{O}$ is C^k smooth if

sm$_1$ *each $\Phi_t : \mathcal{D}_t \to \mathcal{O}$ is k times continuously Fréchet differentiable;*

sm$_2$ *the derivatives $d^j\Phi_t(x) \in \mathbf{L}_j(E, E)$ are strongly continuous in $(x, t) \in \mathcal{D}$.*

Here $\mathbf{L}_j(E,E)$ is the space of j-linear mappings from $E\times\cdots\times E$ to E, and strong continuity means that the map

$$\begin{aligned}\mathcal{D}\times E\times\cdots\times E &\to E\\ (x,t,\xi_1,\dots,\xi_j) &\mapsto d^j\Phi_t(x)\cdot(\xi_1,\dots,\xi_j)\end{aligned}$$

is continuous.

The first condition already implies that $d^j\Phi_t(x)$ depends continuously on x (even with respect to the norm topology on $\mathbf{L}_j(E,E)$), so the second condition is mainly an assumption about the way the derivatives $d^j\Phi_t(x)$ depend on time.

Stable and Unstable Manifolds

Let $x_0\in\mathcal{O}$ be a fixed point of the smooth local semiflow $\Phi:\mathcal{D}\to\mathcal{O}$, and consider the local stable and unstable manifolds of x_0 associated with some neighbourhood $\mathcal{U}\subset\mathcal{O}$ of x_0:

$$\begin{aligned}W^s(x_0,\mathcal{U}) &= \{x\in\mathcal{U}:\Phi(x,t)\in\mathcal{U}\text{ for all }t\ge 0\}\\ W^u(x_0,\mathcal{U}) &= \{x\in\mathcal{U}:x=\hat{x}(0)\text{ for some orbit }\hat{x}:(-\infty,0]\to\mathcal{U}\}\end{aligned}$$

Here $\hat{x}:(-\infty,0]\to\mathcal{O}$ is by definition an orbit if $\Phi(\hat{x}(t),s)=\hat{x}(t+s)$ for all $t\le 0, 0\le s\le -t$.

The smooth semigroup properties of Φ imply that $\{d\Phi_t(x_0):t\ge 0\}$ is a strongly continuous semigroup on E. By the Hille-Yosida theorem it may be written as $e^{tA}=\lim_{n\to\infty}(1-tA/n)^{-n}$ for some (possibly unbounded) operator A on E.

Recall that the fixed point if called hyperbolic if the spectrum of $d\Phi_t(x_0)$ is disjoint from the unit circle for any $t>0$. If x_0 is hyperbolic, then there exists a splitting $E=E^s\oplus E^u$ which is invariant under $d\Phi_t(x_0)$, and for which one has

$$\begin{aligned}\|d\Phi_t(x_0)\xi\| &\le Me^{-\delta t}\|\xi\|, &\quad \xi\in E^s,\\ \|d\Phi_t(x_0)\xi\| &\ge me^{\delta t}\|\xi\|, &\quad \xi\in E^u,\end{aligned}$$

for all $t>0$ and certain constants $m,M,\delta>0$.

If the semigroup $\{d\Phi_t(x_0):t\ge 0\}$ happens to analytic, then x_0 is hyperbolic if and only if the spectrum of the generator A is disjoint from the imaginary axis.

8.1. INVARIANT MANIFOLD THEOREM. *If x_0 is a hyperbolic fixed point, then the local stable and unstable manifolds in a small enough neighbourhood $\mathcal{U}$ of x_0 are smooth submanifolds of E near x_0. Their tangent spaces are given by*

$$T_{x_0}W^s(x_0,\mathcal{U}) = E^s, \qquad T_{x_0}W^u(x_0,\mathcal{U}) = E^u.$$

This follows immediately from the usual stable and unstable manifold theorem (see [HPS]; [Ch] gives a particularly transparent account) once one realizes that the stable manifold of the semiflow coincides with that of its time-t_0 map Φ_{t_0}.

Maximal regularity classes

Let $E_1 \subset E_0$ be Banach spaces, as above, and let $A : E_1 \to E_0$ be any bounded operator. One can regard such an operator as an unbounded operator in E_0, with domain E_1. The simplest typical example which the reader should keep in mind is:

8.2. *Example.* $E_0 = C(\mathbf{R}/\mathbf{Z})$, $E_1 = C^2(\mathbf{R}/\mathbf{Z})$ and $A = a(x)(d/dx)^2$, for some continuous periodic function $a(x)$.

Write E for the pair of Banach spaces (E_1, E_0). We define $\mathrm{Hol}(E)$ to be the set of all such linear operators which generate a holomorphic semigroup on E_0 (in the example one has $A \in \mathrm{Hol}(E)$ iff the function $a(x)$ is strictly positive). It turns out that $\mathrm{Hol}(E)$ is an open subset of $\mathbf{L}(E_1, E_0)$, the set of linear operators from E_1 to E_0, equipped with the norm topology.

If $A \in \mathbf{L}(E_1, E_0)$, then it is known that $A \in \mathrm{Hol}(E)$ if and only if there exist constants $M, C < \infty$ such that

(i) the resolvent of A, $R(\lambda, A) = (\lambda - A)^{-1} : E_0 \to E_1$ exists for all $\lambda \in \mathbf{C}$ with $\Re(\lambda) \geq M$,

(ii) for all $x \in E_0$, and all λ with $\Re(\lambda) \geq M$ one has $\|R(\lambda, A)x\|_{E_1} \leq C\|x\|_{E_0}$.

Let $0 < \rho \leq 1$ be given. For any given $A \in \mathrm{Hol}(E)$ we may consider the inhomogeneous linear initial value problem

$$\begin{cases} u'(t) = Au(t) + f(t) & (0 < t \leq T) \\ u(0) = 0. \end{cases} \tag{8.1}$$

If this initial value problem has a solution $u \in C^0((0,T];E_1) \cap C^1((0,T];E_0)$ for any $f \in C((0,T];E_0)$ with

$$\lim_{t \downarrow 0} t^{1-\rho} \|f(t)\|_{E_0} = 0,$$

and if the solution satisfies

$$\lim_{t \downarrow 0} t^{1-\rho} \{\|u(t)\|_{E_1} + \|u'(t)\|_{E_0}\} = 0,$$

then we say that the operator A belongs to the maximal regularity class $MR_\rho(E)$.

In general the solution to the inhomogeneous equation is given by the variation of constants formula, i.e., by

$$u(t) = \int_0^t e^{(t-s)A} f(s)\, ds.$$

If one tries to estimate the E_1 norm of this solution, then one encounters the following integral:

$$Au(t) = \int_0^t Ae^{(t-s)A} f(s)\, ds.$$

(The E_1 norm of x is essentially equivalent to $\|Ax\|_{E_0}$.) If one tries to estimate the E_0 norm of the above integral, using only the boundedness of $f(t)$, then one runs into trouble since the best possible bound for Ae^{tA} is

$$\|Ae^{tA}\|_{E_0 \to E_0} \le \frac{C}{t} \qquad (0 < t \le T).$$

In fact Baillon [Ba] has shown that the constant C can never be chosen smaller than $1/e$, unless $E_1 = E_0$, which never occurs for differential operators.

Thus, if $A \in \mathrm{Hol}(E)$ then one does not expect the solution $u(t)$ to be a continuous E_1 valued function, if one only knows that f is continuous with values in E_0. In the example this means that one does not know whether $u_{xx}(\cdot, t) \in C(\mathbf{R}/\mathbf{Z})$ or not, if u is the solution to the inhomogeneous heat equation $u_t = a(x)u_{xx} + f(x,t)$, where f is some continuous function.

In [DaPG] G. DaPrato and P. Grisvard observed that there is a large class of Banach pairs $E = (E_1, E_0)$ for which $MR_1(E)$ is nonempty; in [An1] their construction was generalized to the case $0 < \rho < 1$, in a very straightforward way. The precise result is the following:

8.3. THEOREM *Let $A \in \mathrm{Hol}(E)$ be given. For any $0 < \sigma < 1$ define E_σ to be the continuous interpolation space between E_1 and E_0 of exponent σ, $(E_1, E_0)_\sigma$. Let $E_{1+\sigma}$ be the space $\{x \in E_1 | Ax \in E_\sigma\}$. Then the operator*

$$A\Big|_{E_{1+\sigma}} : E_{1+\sigma} \to E_\sigma$$

belongs to $MR_\rho(E)$ for any $\rho \in (0, 1]$.

In what follows we shall only have to know what the "continuous interpolation space of exponent σ" is for a few examples. In general it can be described as the closure of E_1 in the perhaps more familiar real interpolation space (see [BL]) $(E_1, E_0)_{\sigma,\infty}$.

The examples which will be relevant for our purposes are the following. Recall that the little Hölder space of exponent $\mu \in (0, 1)$ is defined by

$$h^\mu([0, 1]) = \textit{Closure of } C^\infty([0, 1]) \textit{ in } C^\mu([0, 1]),$$

while the space $h^{2,\mu}([0, 1])$ is defined to consist of those $u \in h^\mu$ for which u' and u'' also belong to $h^\mu([0, 1])$.

Another, more direct description of h^μ is as follows: $u \in h^\mu([0, 1])$ iff

$$\lim_{\epsilon \downarrow 0} \sup_{|t-s| \le \epsilon} \epsilon^{-\mu} |u(t) - u(s)| = 0.$$

The norms on h^μ and $h^{2,\mu}$ are the usual Hölder norms.

If one now defines $E_0 = h^\mu([0, 1])$, and $E_1 = \{u \in h^{2,\mu} | u'(0) = u'(1) = 0\}$, then the continuous interpolation spaces of the pair E are given by the following lemma, at least when $\mu + 2\sigma$ is not an integer.

8.4. LEMMA. *If $\mu + 2\sigma < 1$ then $E_\sigma = h^{\mu+2\sigma}([0, 1])$.*
If $1 < \mu + 2\sigma < 2$ then $E_\sigma = \{u \in h^{1,\mu+2\sigma-1}([0, 1]) | u'(0) = u'(1) = 0\}$.
If $\mu + 2\sigma > 2$ then $E_\sigma = \{u \in h^{2,\mu+2\sigma-2}([0, 1]) | u'(0) = u'(1) = 0\}$.

See [DaPG] for a proof.

The operators which we shall encounter are second order differential operators of the form

$$A = a(x) \left\{ \left(\frac{d}{dx} \right)^2 + \frac{b(x)}{x(1-x)} \left(\frac{d}{dx} \right) \right\}, \tag{8.2}$$

where $a(x)$ and $b(x)$ belong to $h^\mu([0,1])$ and $a(x)$ is strictly positive.

Using the DaPrato-Grisvard construction, and the characterization of E_σ of the previous lemma, the following was proved in [An2].

8.5. Lemma. *If $b(0) > -1$ and $b(1) > -1$, then A belongs to the maximal regularity class $MR_\rho(E)$ for any $0 < \rho \leq 1$.*

Quasilinear initial value problems

Let (E_1, E_0) be a Banach couple, and let $\mathcal{O} \subset E_\theta$ be an open subset of the interpolation space of exponent θ. Fix some $0 < \theta < \sigma < 1$, and define $\mathcal{O}_\sigma = \mathcal{O} \cap E_\sigma$. Then we consider initial value problems of the type

$$\begin{aligned} u_t &= A(u)u + f(u) \\ u(0) &= u_0, \end{aligned} \tag{8.3}$$

where

(i) $A : \mathcal{O} \to \mathbf{L}(E_1, E_0)$ is a C^k smooth mapping,
(ii) $A(x) \in MR_\sigma(E)$ for every $x \in \mathcal{O}_\sigma$,
(iii) $f : \mathcal{O} \to E_0$ is a C^k smooth mapping.

We shall look for solutions $u \in Y_{\theta,T}$, where $Y_{\theta,T}$ consists of those $u \in C^1((0,T]; E_0) \cap C^0((0,T]; E_1)$ which satisfy

$$\lim_{t \downarrow 0} t^{1-\theta}\{\|u(t)\|_{E_1} + \|u'(t)\|_{E_0}\} = 0.$$

Equipped with the norm

$$[u]_\theta = \sup_{0<t\leq T} t^{1-\theta}\{\|u(t)\|_{E_1} + \|u'(t)\|_{E_0}\}$$

$Y_{\theta,T}$ becomes a Banach space.

The following theorem gives the main local existence and uniqueness theorem for (8.3). Its proof is given in [An1], and essentially involves nothing more complicated than a Picard iteration.

8.6. Theorem. *For any $u_0 \in \mathcal{O}_\sigma$ there is a $T = T(u_0) > 0$ such that (8.3) has a unique solution $u \in Y_{\theta,T}$.*

For $0 < t \le T$ the solution is a C^k function with values in E_1, and one actually has the following estimate:

$$\|u^{(j)}(t)\|_{E_1} \le \frac{C}{t^{j+1-\theta}} \qquad (j = 0, 1, 2, \dots, k).$$

For any $u_1 \in \mathcal{O}_\sigma$ near u_0, in the E_σ topology, (8.3) will also have a solution in $Y_{\theta,T}$, with the same $T = T(u_0)$. This solution depends C^k on its initial value u_1.

.

In other words, (8.3) generates a local C^k smooth semiflow on $\mathcal{O}_\sigma$.

Quasilinear equations with regular singular coefficients

We have seen that the degenerate parabolic PDE's in this paper can sometimes be transformed to the following kind of problem.

$$\text{(8.4)} \qquad \begin{cases} u_t = a(x,u)\left\{u_{xx} + \dfrac{b(x,u)}{x(1-x)}u_x\right\} + f(x,u), \\ \qquad (0 < x < 1, 0 < t \le T), \\ u_x(0,t) = u_x(1,t) = 0, \qquad (0 < t \le T), \\ u(x,0) = u_0(x). \end{cases}$$

Here we assume that the functions a, b and f are defined on $\{(x,u) : 0 \le x \le 1, u_{\min} < u < u_{\max}\}$, and that they satisfy the following conditions on this domain.

[1] *For all $s \in \mathbf{R}$ the functions $g(x,s), g_u(x,s)$ and $g_{uu}(x,s)$ are little Hölder continuous functions of x, of exponent $\theta \in (0,1)$. Here g stands for either a, b or f.*

[2] *(uniform parabolicity) $\delta \le a(x) \le \delta^{-1}$ for some $\delta > 0$.*

[3] *For any $s \in \mathbf{R}$ one has $b(0,s) > -1$ and $b(1,s) > -1$.*

Define the spaces E_j as in the previous section, let $\mathcal{O}$ be the set of $u \in E_\theta$ for which $u_{\min} < u(x) < u_{\max}$ for $x \in [0,1]$, and define the operators $A(u)$, and the map $f : \mathcal{O} \to E_0$ as follows:

$$A(u) = a(x,u(x))\left\{\left(\frac{d}{dx}\right)^2 + \frac{b(x,u(x))}{x(1-x)}\left(\frac{d}{dx}\right)\right\},$$

and

$$f(u)(x) = f(x, u(x)).$$

We use the same symbol f to denote the function of two real variables, and its corresponding substitution operator and hope that the reader will not find this abuse of notation too confusing.

It follows from our assumption [1] that $f : \mathcal{O} \to E_0$ is in fact C^1 Fréchet differentiable, and using the results that were quoted in the previous section, one shows that $A : \mathcal{O} \to \mathbf{L}(E_1, E_0)$ *is a* C^1 *mapping with values in* $MR_\rho(E)$, *for any* $0 < \rho \le 1$. Thus we may apply the local existence theorem to conclude that (8.4) generates a C^1 local semiflow on $\mathcal{O}_\sigma$ for any $\sigma \in (\theta, 1)$. In view of the description of the continuous interpolation spaces which we have, this gives us the following local existence and uniqueness theorem for (8.4).

8.7. Theorem. *If* a, b *and* f *satisfy* [1, 2, 3], *and if* $\theta < \mu < 1$, *then the initial value problem (8.4) generates a* C^1 *local semiflow on*

$$\mathcal{O}^\mu = \{u \in h^\mu | u_{\min} < u < u_{\max}\}.$$

If $u_0 \in \mathcal{O}^\mu$, *then the maximal solution* $u(x,t)$ *of (8.1) provided by the local semiflow is smooth, in the sense that for any* $t > 0$ *the functions* $u(\cdot, t)$ *and* $u_t(\cdot, t)$ *lie in* $h^{2,\theta}([0,1])$.

The second part of this statement follows from the estimates on $\|u^{(j)}\|_{E_1}$ which the abstract existence theorem provides, for $j = 0, 1$.

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Sigurd B. Angenent
University of Wisconsin-Madison
Department of Mathematics
Van Vleck Hall
480 Lincoln Drive
Madison, WI 53706-1388
U.S.A.

Donald G. Aronson
University of Minnesota
School of Mathematic
Institute of Technology
Minneapolis MN 55455-0488
U.S.A.

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