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Stability for the deep Bénard problem

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Abstract. Existence and uniqueness of a local in time strong solution for the unsteady deep Bénard convection problem is proved through use of semigroup theory. The bifurcation problem is also dealt with.

1. Introduction

As is well known, when a layer of a heavy homogeneous and viscous fluid is heated from below, convective instabilities set in when the vertical temperature gradient exceeds a certain critical value; motion occurs because hotter fluid is less dense and therefore tends to rise. This instability known as the Rayleigh-Bénard problem has been studied within the scheme of the Oberbeck-Boussinesq approximation [6]-[10]-[19] and under the assumption of a shallow layer of fluid.

Nevertheless, in most applications such as oceanography, geophysics and astrophysics, the convective zone is deep and one has to take into account the influence of the layer's depth. In this connection, R. Zeytounian [37] derived a model called the deep Bénard convection problem as opposed to the classical (or shallow) Rayleigh-Bénard problem. This model has been obtained rigorously through use of a perturbation technique and leads to the appearance of a new parameter linked to the depth of the fluid's layer, which in turn implies that the viscous dissipation term in the conservation of energy equation is no more negligible.

In this paper, we will investigate the wellposedness of the deep Bénard convection equations. The main tools used are the general existence and uniqueness theorems elaborated in [15]-[31]. The solution found is thus only

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local in time. We will also extend some well known results for the Rayleigh-Bénard problem to the deep Bénard convection problem. This article is organized as follows : the relevant equations will be given in Section 2. The functional framework and the notations will be made precise in Section 3. Existence and uniqueness of a local in time strong solution to the deep Bénard problem will be established in Section 4. Section 5 is devoted to the analysis of the linear and nonlinear bifurcation problems. At last, the possibility of existence of periodic and quasi-periodic solutions to the deep Bénard problem is analyzed in Section 6.

2. Statement of the problem

We consider a deep layer of a heavy homogeneous viscous fluid heated from below. We assume that the fluid is confined between the horizontal planes $z = 0$ and $z = 1$ in $\mathbb{R}^3$ and that the lateral boundaries are sufficiently far away so that the layer may be considered as having an infinite horizontal extension. Then the fluid motion is governed by the following dimensionless deep Bénard convection equations [37]

$$(2.1)_1 \quad \partial_t \vec{u} + (\vec{u}.\nabla)\vec{u} + \sigma\nabla p - \sigma\vartheta\vec{e} - \sigma \bigtriangleup \vec{u} = \vec{0} \qquad \text{in } \Omega_0 \text{ X}(0,T),$$

$$(2.1)_2 \quad \partial_t\vartheta + \vec{u}.\nabla\vartheta - \mathrm{R}\ \vec{u}.\vec{e} - \frac{\bigtriangleup\ \vartheta}{\chi(z)} - \frac{2\ \delta}{\chi(z)} D(\vec{u}) : D(\vec{u}) = 0 \qquad \text{in } \Omega_0 \text{ X } (0,T),$$

$$(2.1)_3 \quad \text{div } \vec{u} = 0 \qquad \text{in } \Omega_0\text{X}(0,T),$$

$T > 0$ and $x = (x_1, x_2, z) \in \Omega_0 = \mathbb{R}^2\text{X}(0,1)$.

Here the unknowns are the vector field $\vec{u}(x,t) = (u_1, u_2, u_3)$ which represents the velocity, and the scalar fields $p(x,t)$ and $\vartheta(x,t)$ denoting respectively the disturbances of the pressure and the temperature. $\sigma > 0$ is the Prandtl number, $R \geq 0$ is the Rayleigh number, $\delta \in [0,1]$ is the Zeytounian's depth parameter, $\chi(z) = 1 + \delta(1-z)$ where $z \in (0,1)$ is the vertical coordinate. $\vec{e}$ is the unit vector pointing upward, $D(\vec{u}) = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})^t]$ denotes the rate of deformation tensor and

$$D(\vec{u}) : D(\vec{u}) = \frac{1}{4}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2.$$

We supplement the system (2.1) with the following boundary conditions for rigid horizontal planes

$$\vec{u} = \vec{0}, \vartheta = 0 \text{ on } z = 0,\ z = 1, t \in (0, T), \tag{2.2}$$

and initial conditions

$$\vec{u}(x, 0) = \vec{u}^0(x), \vartheta(x, 0) = \vartheta^0(x), x \in \Omega_0. \tag{2.3}$$

3. Functional framework

The domain Ω_0 is not compact and the linearized equations have an infinite dimensional kernel for every value of the control parameter R above a certain threshold. To overcome this difficulty, we restrict attention to spatially periodic solutions; in this way, we achieve a compact domain and introduce a hidden symmetry in the problem.

3.1. Transformations and invariances

As is easily seen, the equations (2.1)–(2.2) are equivariant with respect to rigid motion in the plane. We will assume that, at the onset of convection, the pattern is doubly periodic in the horizontal plane, and that the pattern in each cell is repeated so as to tile the whole plane. In order to deal with these cellular solutions, we denote by T_1 the group generated by the translations

$$x_1 \to x_1 + \frac{2\pi}{\alpha}, x_2 \to x_2 + \frac{2\pi}{\beta},\ \alpha^2 + \beta^2 \neq 0.$$

Then

$$\begin{gathered} \vec{u}(\gamma x, t) = \vec{u}(x, t), \\ \vartheta(\gamma x, t) = \vartheta(x, t), \\ p(\gamma x, t) = p(x, t), \forall \gamma \in T_1, (x, t) \in \Omega_0 X (0, T). \end{gathered}$$

The fundamental region of periodicity of T_1 is $\Omega = [0, \frac{2\pi}{\alpha})\ X\ [0, \frac{2\pi}{\beta})\ X\ (0, 1)$, the layer is then generated by T_1.

Let $0(3)$ denote the 3-dimensional orthogonal group [32] and let T_2 be the subgroup of $0(3)$ generated by the rotations

$$T_\omega = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \omega \in (0, 2\pi). \tag{3.1$_1$}$$

We define the action of T_2 by

$$\vec{u}(\gamma x, t) = \gamma \vec{u}(x, t), \tag{3.1$_2$}$$

$$\vartheta(\gamma x, t) = \vartheta(x, t), \tag{3.1$_3$}$$

$$p(\gamma x, t) = p(x, t), \qquad \forall \gamma \in T_2, \forall (x, t) \in \Omega_0 X(0, T). \tag{3.1$_4$}$$

The equations (2.1) are equivariant with respect to these transformations. Solutions with various cell patterns may solve the deep Bénard convection problem. If the cell patterns exist, the triple α, β and ω must satisfy the following compatibility relations :

LEMMA 3.1. *A necessary condition for the existence of nontrivial, spatially periodic and invariant solutions (under T_2) of the problem (2.1)–(2.2) is*

$$\omega = \frac{2\pi}{k}, k \in \left\{1, 2, 3, 4, 6\right\}. \tag{3.2}$$

COROLLARY 3.1. *Let ω satisfy (3.2), then the only possible combinations of k, α, β, are*

$$\begin{array}{ll}
k = 1,\ \alpha \geq 0, \beta \geq 0 & : \textit{no cell structure}; \\
k = 2 \begin{cases} \alpha > 0,\ \beta = 0 \\ \alpha > 0,\ \beta > 0 \end{cases} & \begin{array}{l} : \textit{rolls}; \\ : \textit{rectangles}; \end{array} \\
k = 3,\ \alpha = \dfrac{\beta}{\sqrt{3}},\ \beta > 0 & : \textit{hexagons}; \\
k = 4, \alpha = \beta,\ \beta > 0 : & : \textit{squares}; \\
k = 6,\ \alpha = \dfrac{\beta}{\sqrt{3}},\ \beta > 0 & : \textit{triangles}.
\end{array}$$

For the proof of the above Lemma and Corollary, see [25].

3.2. Mathematical preliminaries

As usual, $L^p(\Omega), 1 \leq p \leq \infty, H^s(\Omega), s \in \mathbb{R}$, and $W^{s,p}(\Omega)$ will denote the classical Lebesgue and Sobolev Spaces [1]-[2]-[28]-[29]-[30]; their respective norms will be denoted by $|.|_p$, $\|.\|_s$, $\|.\|_{s,p}$. The Hilbert spaces $L^2(\Omega)$ and $H_0^m(\Omega)$ are respectively endowed with the inner products and norms :

$$(f,g) = \int_\Omega f(x).g(x)\, dx, \qquad \forall\, f,g \in L^2(\Omega), \qquad |f|^2 = (f,f);$$

$$((f,g))_m = \sum_{|\alpha|=m} (D^\alpha f, D^\alpha g), \qquad \forall\, f,g \in H_0^m(\Omega), \qquad \|f\|_m^2 = ((f,f))_m.$$

For convenience, we set $\|.\|_1 \equiv \|.\|$, $((.,.))_1 \equiv ((.,.))$.

To simplify the notations, if X is any functional space then we set $\mathbb{X} = X^n, n = 3$ or $n = 4$; we agree to use the same notation for the norm (or the inner product) in X or $\mathbb{X}$.

In order to accommodate the divergence free condition, we choose our working spaces to be solenoidal, we will also impose a periodicity condition. We will say simply that a function $f \equiv (\vec{u}, \vartheta, p)$ is periodic if f is periodic in the direction $x_i, i = 1, 2$.

Let $\infty > p > 3/2$ and $s \in \mathbb{N}$, then we set:

$$\mathcal{W}^{s,p} = \Big\{ f \in W^{s,p}(\Omega), f = 0 \text{ on } z = 0, z = 1, f \text{ is periodic} \Big\},$$

$$\mathcal{V}^{s,p} = \Big\{ \vec{u} \in \Big[W^{s,p}(\Omega) \Big]^3 : \text{div}\, \vec{u} = 0, \vec{u} = \vec{0}$$

$$\text{on } z = 0, z = 1, \vec{u} \text{ is periodic} \Big\},$$

$$\mathcal{U}^{s,p} = \mathcal{V}^{s,p}\ X\ \mathcal{W}^{s,p}.$$

These spaces are Banach spaces with naturally induced norms.

By the Hodge decomposition theorem [4], there is a bounded operator

$$P : \mathbb{W}^{s,p}(\Omega) \longrightarrow \mathcal{V}^{s,p}.$$

For $s = 0$, $p = 2$, P is an orthogonal projector [27]–[35]; on the other hand $\mathcal{W}^{s,2}$, $\mathcal{V}^{s,2}$ and $\mathcal{U}^{s,2}$ are Hilbert spaces with naturally induced inner products and norms. We shall set for convenience $H \equiv \mathcal{V}^{0,2}$, $V \equiv \mathcal{V}^{1,2}$.

3.3. Operators

Using the projector P we can write the system (2.1)–(2.2)–(2.3) in the following abstract form in $\mathcal{U}^{0,p}$

$$(3.3)_1 \qquad \frac{d\overrightarrow{u}}{d\,t} = \sigma A_{0p}\overrightarrow{u} + \sigma\vartheta\overrightarrow{e} + B_{0p}(\overrightarrow{u},\overrightarrow{u});$$

$$(3.3)_2 \qquad \frac{d\vartheta}{d\,t} = A_{1p}\vartheta + R\overrightarrow{u}.\overrightarrow{e} + B_{1p}(\overrightarrow{u},\vartheta) + B_{3p}(\overrightarrow{u},\overrightarrow{u});$$

$$(3.3)_3 \qquad \overrightarrow{u}(0) = \overrightarrow{u}^0, \qquad \vartheta(0) = \vartheta^0.$$

Here A_{0p} and A_{1p} are the linear operators defined by

$$(3.4) \quad \begin{cases} D(A_{0p}) = \mathbb{W}^{2,p}(\Omega) \cap \mathcal{V}^{1,p} \\ A_{0p}\overrightarrow{u} = P \bigtriangleup \overrightarrow{u}; \end{cases} \quad ; \quad \begin{cases} D(A_{1p}) = W^{2,p}(\Omega) \cap \mathcal{W}^{1,p} \\ A_{1p}\vartheta = \dfrac{\bigtriangleup \vartheta}{\chi(z)}. \end{cases}$$

$$B_{0p}(\overrightarrow{u},\overrightarrow{v}) = -P(\overrightarrow{u}.\nabla)\overrightarrow{v}, \forall \overrightarrow{u},\overrightarrow{v} \in D(A_{0p});$$

$$B_{1p}(\overrightarrow{u},\vartheta) = -\overrightarrow{u}.\nabla\vartheta, \forall(\overrightarrow{u},\vartheta) \in D(A_{0p})XD(A_{1p});$$

$$B_{3p}(\overrightarrow{u},\overrightarrow{v}) = \frac{2\delta}{\chi(z)} D(\overrightarrow{u}) : D(\overrightarrow{v}), \forall \overrightarrow{u},\overrightarrow{v} \in D(A_{0p}).$$

Let us set
$\varphi = (\overrightarrow{u},\vartheta)$, $M_p\varphi = (\sigma\ \vartheta\overrightarrow{e},\ R\ \overrightarrow{u}.\overrightarrow{e})$, $A_p\varphi = (\sigma A_{0p}\overrightarrow{u},\ A_{1p}\vartheta)$ with $D(A_p) = D(A_{0p})X\ D(A_{1p})$ and $\forall\varphi,\psi \in D(A_p)$, we put, for $\psi = (\overrightarrow{v},\tilde{T})$, $B_p(\varphi,\psi) = (B_{0p}(\overrightarrow{u},\overrightarrow{v}), B_{1p}(\overrightarrow{u},\tilde{T}) + B_{3p}(\overrightarrow{u},\overrightarrow{v})), B_p(\varphi) = B_p(\varphi,\varphi)$.

With these notations, the equations (3.3) become

$$(3.5)_1 \qquad \frac{d\varphi}{dt} = A_p\varphi + M_p\varphi + B_p(\varphi),$$

$$(3.5)_2 \qquad \varphi(0) = \varphi^0.$$

4. Existence and uniqueness of solutions

First, let us note that A_p is the infinitesimal generator of an analytic semigroup on $\mathbb{L}^p(\Omega)$ and $0 \in \rho(-A_p)$ (where $\rho(A)$ is the resolvent set of A). One can define fractional powers of $-A_p$ [14]. Let us set $D((-A_p)^\alpha) = X_p^\alpha, 0 < \alpha \le 1,\ p \in (3/2, \infty)$ and let X_p^α be endowed with the graph norm of $(-A_p)^\alpha$, denoted by $\|.\|_\alpha$.

Now, let us recall some embeddings [15, pp. 39–40]–[31, p. 243] :

$$(4.1)_1 \qquad X_p^\alpha \subset \mathbb{W}^{k,q}(\Omega) \text{ for } k - \frac{3}{q} < 2\alpha - \frac{3}{p},\ q \ge p;$$

$$(4.1)_2 \qquad X_p^\alpha \subset \mathbb{L}^q(\Omega) \text{ for } \frac{1}{q} > \frac{1}{p} - \frac{2\alpha}{3},\ q \ge p;$$

$$(4.1)_3 \qquad X_p^\alpha \subset C^\nu(\bar{\Omega}) \text{ for } 0 \le \nu \le 2\alpha - \frac{3\,;}{p}$$

with continuous injections.

Choosing p, α such that

$$(4.2) \qquad p \in (3/2, \infty), \alpha > \frac{3}{4p} + \frac{1}{2},$$

we infer from (4.1) that

$$(4.3) \qquad X_p^\alpha \subset \mathbb{W}^{1,q}(\Omega) \cap \mathbb{L}^\infty(\Omega),\ q \ge p,\ 1 - \frac{3}{q} < 2\alpha - \frac{3}{p}.$$

Remark 4.1. Assume that α, p has been chosen so that (4.2) is satisfied; then one can find q, q' such that

$$(4.4) \qquad \frac{1}{q} + \frac{1}{q'} = \frac{1}{p},\ 1 - \frac{3}{q} < 2\alpha - \frac{3}{p},\ 1 - \frac{3}{q'} < 2\alpha - \frac{3}{p},$$
$$q, q' \ge p.$$

In order to appeal to the general existence and uniqueness theorems in [15]–[31], we need the following technical result.

Lemma 4.1. *Let $F_p(\varphi, \psi) = M_p\,\varphi + B_p(\varphi, \psi)$ and assume that (4.2) is satisfied. Then F_p is well defined on X_p^α and F_p is locally Lipschitzian.*

PROOF. First note that

$$(4.5)_1 \qquad B_p(\varphi) - B_p(\psi) = B_p(\varphi - \psi, \varphi) + B_p(\psi, \varphi - \psi).$$

(i) F is well defined on X_p^α

Let α, p satisfy (4.2), then $\forall \varphi, \psi \in X_p^\alpha$

$$|F_p(\varphi,\psi)|_p \leq |M_p\varphi|_p + |B_p(\varphi,\psi)|_p.$$

According to remark 4.1, $\exists q, q'$ such that (4.4) holds. Thus

$$|B_p(\varphi,\psi)|_p \leq c_0'\Big(|\vec{u}|_\infty.|\nabla\psi|_p + |\nabla\vec{u}|_q.|\nabla\vec{v}|_{q'}\Big) \leq c_0\|\varphi\|_\alpha.\|\psi\|_\alpha;$$

where c_0, c_0' are strictly positive constants.

Since $\exists\, c_1 > 0 : |M\varphi|_p < c_1\|\varphi\|_\alpha$, we finally deduce that $\exists\, c > 0$:

$$(4.5)_2 \qquad |F_p(\varphi,\psi)|_p < c\Big(\|\varphi\|_\alpha + \|\varphi\|_\alpha.\|\psi\|_\alpha\Big).$$

(ii) F is locally Lipschitzian

$$|F_p(\varphi) - F_p(\psi)|_p \leq c\,\|\varphi-\psi\|_\alpha\Big(1 + \|\varphi\|_\alpha.\|\psi\|_\alpha\Big)$$

(by virtue of $(4.5)_1 - (4.5)_2$). Hence $\exists\, c_2 = c_2(\|\varphi\|_\alpha, \|\psi\|_\alpha) > 0$:

$$(4.5)_3 \qquad |F_p(\varphi) - F_p(\psi)|_p \leq c_2\|\varphi-\psi\|_\alpha. \ \square$$

From [15; theorem 3.3.3, p.54] or [31 ; theorem 6.3.1, p.196] we infer the following

THEOREM 4.1. *Assume that α, p has been chosen so that (4.2) holds. Then $\forall\varphi^0 \in X_p^\alpha, \exists\, T = T(\varphi^0) > 0$ such that the initial value problem (3.5) has a unique local strong solution in $\mathcal{U}^{0,p}$.*

Thus the system (3.5) generates a local semiflow ϕ_p in X_p^α [3].

Well known regularity results and embedding theorems finally tell us that every solution of (3.5) is a classical solution for $p > \frac{3}{2}$. In fact, for $t > 0$,

$$\varphi(t;\varphi^0) \in D(A_p),$$
$$t \longrightarrow \frac{d\varphi}{dt} \in X_p^\alpha \text{ is locally Hölder continuous}$$
$$\text{[15; theorem 3.5.2, p. 71]; consequently,}$$
$$(t,x) \longrightarrow \varphi(x,t;\varphi^0),$$
$$(t,x) \longrightarrow \frac{\partial\varphi}{\partial t}(x,t;\varphi^0),$$

are continuous on $x \in \bar{\Omega},\ 0 < t < T$.

Let $\varphi \in D(A_p)$, then $\nabla\varphi \in \mathbb{W}^{1,p}(\Omega)$ and

(i) for $p > 3, B_p(\varphi) \in \mathbb{W}^{1,p}(\Omega)$; thus $A_p\varphi \in \mathbb{W}^{1,p}(\Omega) \subset C^{0,\lambda}(\bar{\Omega}),\ \lambda \in (0, 1-3/p]$ which finally gives $\varphi \in C^{2,\lambda}(\bar{\Omega})$;

(ii) for $p \in (\frac{3}{2}, 3), \nabla\varphi \in \mathbb{L}^q(\Omega), \frac{1}{q} = \frac{1}{p} - \frac{1}{3}$; thus $B_p(\varphi) \in \mathbb{L}^{q/2}(\Omega) \Rightarrow A_p\varphi \in \mathbb{L}^{q/2}(\Omega) \Rightarrow \varphi(t) \in \mathbb{W}^{2,q/2}(\Omega)$.

In conclusion, if $\varphi \in \mathbb{W}^{2,p_i}(\Omega) \Rightarrow \varphi \in \mathbb{W}^{2,p_{i+1}}(\Omega)$, where $p_0 = p$ and

$$\frac{1}{p_{i+1}} = 2\left(\frac{1}{p_i} - \frac{1}{3}\right), i \in \mathbb{N}(< \frac{1}{p_i}\ for\ p_i > \frac{3}{2} \Rightarrow p_{i+1} > p_i).$$

At last, one can find $p_{i+1} = p_j > 3$ for some $i \in \mathbb{N}$, and as in (i) $B_p(\varphi) \in C^{0,\mu}(\bar{\Omega}), \mu \in (0, 1 - \frac{3}{p_j}]$ and $\varphi(t) \in C^{2,\mu}(\bar{\Omega})$.

(iii) for $p = 3$, it is easy to check that $\varphi \in C^{2,\mu}(\bar{\Omega})$, for some $\mu \in (0,1)$ owing to the compact Sobolev embedding $W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [1,\infty)$.

Hence, for $t > 0,\ p > 3/2,\ \varphi(t,x;\varphi^0)$ is a classical solution of the problem (3.5).

5. Eigenvalues and bifurcations

In this Section, we shall mainly study the cellular bifurcation from the equilibrium state $\hat{\varphi} = 0,\ p =$constant, for $\varphi^0 = 0$.

5.1. Bifurcation in the steady problem

In the steady case, the nonlinear deep Bénard convection equations become

$$(5.1)_1 \quad \frac{1}{\sigma\lambda}(\vec{v}.\nabla)\vec{v} + \lambda \nabla p - \lambda\vartheta\vec{e} - \triangle \vec{v} = \vec{0} \qquad \text{in } \Omega;$$

$$(5.1)_2 \quad \frac{\chi(z)}{\lambda}\vec{v}.\nabla\vartheta - \lambda\,\chi(z)\vec{v}.\vec{e} - \triangle\,\vartheta - \frac{2\delta}{\lambda^2}D(\vec{v}) : D(\vec{v}) = 0 \qquad \text{in } \Omega;$$

$$(5.1)_3 \quad \mathrm{div}\vec{v} = 0 \qquad \text{in } \Omega;$$

$$(5.1)_4 \quad \vec{v} = \vec{0},\ \vartheta = 0 \qquad \text{on } z = 0,\ z = 1.$$

Here $\vec{v} = \lambda\vec{u}, \lambda = \sqrt{R} \neq 0$. Now, let us introduce the following operators

$$\begin{cases} D(A_{0m}) = \mathbb{H}^{m+2}(\Omega) \cap V, \\ A_{0m}\vec{v} = -P \triangle \vec{v}. \end{cases} \qquad \begin{cases} D(A_{1m}) = H^{m+2}(\Omega) \cap \mathcal{W}^{1,2}, \\ A_{1m}\vartheta = -\triangle\,\vartheta. \end{cases}$$

$A_m\varphi = (A_{0m}\vec{v}, A_{1m}\vartheta)$; $M_m\varphi = -(\vartheta\vec{e},\ \chi\,\vec{v}.\vec{e})$, where $\varphi = (\vec{v}, \vartheta)$;

$B_m(\varphi,\psi) = (\frac{1}{\sigma\lambda}(\vec{v}.\nabla)\vec{u}, \frac{\chi(z)}{\lambda}\vec{v}.\nabla T - \frac{2\delta}{\lambda^2}\,D(\vec{v}) : D(\vec{u}))$ for $\psi = (\vec{u}, T)$ and $B_m(\varphi) = B_m(\varphi,\varphi)$. With these notations the equations (5.1) are easily written

$$(5.2) \qquad A_m\varphi + \lambda M_m\varphi + B_m(\varphi) = 0.$$

Note that A_m is surjective and that A_m^{-1} is a compact operator in $\mathcal{U}^{m,2}$. Next, observe that $\exists\, c, c' > 0$:

$$(5.3)_1 \quad \|M_m\varphi\|_{m+1} \leq c\|\varphi\|_{m+2} \qquad \forall\varphi \in D(A_m),$$

$$(5.3)_2 \quad \|M_mA_m^{-1}\varphi\|_{m+1} \leq c\|A_m^{-1}\varphi\|_{m+2} \leq c'\|\varphi\|_m \qquad \forall\,\varphi \in \mathcal{U}^{m,2}.$$

The abstract equation (5.2) is equivalent to

$$(5.4) \qquad \psi + \lambda K_m\psi + B_m(A_m^{-1}\psi) = 0,$$

with $\varphi = A_m^{-1}\psi$ and $K_m = M_mA_m^{-1}$.

Then as shown in [25, theorem 4.2], using a theorem of Krasnosel'skii [26], we have

Theorem 5.1. *Let $m > 3/2$, then for every eigenvalue $(-\lambda_j)^{-1}$ of $K_m, \lambda_j \neq 0$, of odd multiplicity, $(\lambda_j, 0)$ is a bifurcation point of (5.2).*

5.2. The linearized problem

Let us set p = constant + q, $\varphi = (\vec{v}, \vartheta)$, then the unsteady linearized equations of the system (2.1)–(2.2) around the equilibrium state are

$$(5.5)_1 \qquad \frac{1}{\sigma}\,\partial_t \vec{v} + \lambda \nabla q - \lambda \vartheta \vec{e} - \triangle\, \vec{v} = \vec{0};$$

$$(5.5)_2 \qquad \chi(z)\Big(\partial_t \vartheta - \lambda \vec{v}.\vec{e}\Big) - \triangle\, \vartheta = 0;$$

$$(5.5)_3 \qquad \mathrm{div}\,\vec{v} = 0;$$

$$(5.5)_4 \qquad \vec{v} = \vec{0}, \vartheta = 0 \qquad \text{on } z = 0,\ z = 1.$$

Throughout this section, we will assume that there is a unit cell which repeats itself regularly and that the walls of the unit cell are vertical and are surfaces of symmetry. We will also assume that, on the cell walls, the normal gradient of the vertical velocity vanishes. The cell walls are supposed to be perfectly thermal insulators.

Thus the solution must satisfy, besides the periodicity and the invariance with repect to the rotations, some particular conditions. Here, we introduce the evenness and oddness conditions after Iudovich [18].

$$(5.6)_1 \qquad u(x_1, x_2, z) = -u(-x_1, x_2, z) = u(x_1, -x_2, z),$$

$$(5.6)_2 \qquad v(x_1, x_2, z) = v(-x_1, x_2, z) = -v(x_1, -x_2, z),$$

$$(5.6)_3 \qquad w(x_1, x_2, z) = w(-x_1, x_2, z) = w(x_1, -x_2, z),$$

$$(5.6)_4 \qquad \vartheta(x_1, x_2, z) = \vartheta(-x_1, x_2, z) = \vartheta(x_1, -x_2, z),$$

$$(5.6)_5 \qquad q(x_1, x_2, z) = q(-x_1, x_2, z) = q(x_1, -x_2, z),$$

where $\vec{v} = (u, v, w)$.

As is easily verified, the system (5.5) preserves these invariance conditions. Furthermore, by virtue of Lemma 3.1, ω, α, β must satisfy to a certain relation. Since we are considering only invariant and periodic solutions of (5.5), we will restrict the form of $\varphi = (\vec{v}, \vartheta)$ as follows

$$(5.7)_1 \qquad \varphi = \sum_{m,n=-\infty}^{\infty} \varphi_{mn}(z)\ \exp\Big(i\,m\,\alpha\,x_1 + i\,n\,\beta\,x_2\Big)e^{-\mu t \sigma}, \mu \in \mathbb{C},$$

thus

$$(5.7)_2 \qquad \triangle_\perp\, \varphi = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\varphi = -a^2\varphi,\ a^2 = m^2\alpha^2 + n^2\beta^2.$$

5.3. Principle of exchange of stabilities

In this subsection, we will prove that the system (5.5) gives no rise to oscillatory instabilities and thus that the principle of exchange of stabilities holds for the deep Bénard convection problem. In order to eliminate the pressure term in $(5.5)_1$, let us take $\overrightarrow{e}.$ curl curl $(5.5)_1$, then we obtain the following system

$$(5.8)_1 \qquad \frac{1}{\sigma}\,\partial_t \bigtriangleup w - \lambda \bigtriangleup_\perp \vartheta = \bigtriangleup^2 w,$$

$$(5.8)_2 \qquad \chi(z)\Big(\partial_t \vartheta - \lambda w\Big) = \bigtriangleup \vartheta$$

$$(5.8)_3 \qquad \vartheta = 0, w = 0, \frac{\partial w}{\partial z} = 0 \text{ on } z = 0, z = 1.$$

Taking into account (5.7), we get

$$(5.9)_1 \qquad \Big(D^2 - a^2\Big)^2 V + \lambda a T + \mu\Big(D^2 - a^2\Big) V = 0, \qquad \text{in } (0,1);$$

$$(5.9)_2 \qquad a\,\lambda\,\chi\,V - \Big(D^2 - a^2\Big) T - \mu\,\sigma\,\chi\,T = 0, \qquad \text{in } (0,1);$$

$$(5.9)_3 \qquad T = 0, V = 0, DV = 0 \text{ on } z = 0, z = 1.$$

$$\text{Here} \quad D = \frac{d}{dz}, V = \frac{W_{mn}(z)}{a}, T = -\vartheta_{mn}(z), a \neq 0.$$

The system (5.9) may be written

$$(5.10) \qquad (L - \mu M_0)\phi = 0,\ \phi = (V, T),\ L = L_s + \tilde{B},$$

$$M_0 = diag(-(D^2 - a^2), \sigma\,\chi\,), L_s = diag\Big((D^2 - a^2)^2, -(D^2 - a^2)\Big),$$

$$\tilde{B} = \begin{pmatrix} 0 & a\,\lambda \\ a\,\lambda\,\chi & 0 \end{pmatrix}.$$

Let us introduce the following Hilbert space H_{M_0} defined by $H_{M_0} = H_0^1(0,1) X L^2(0,1)$. H_{M_0} is endowed with the following inner product and norm

$$\left[\varphi, \psi\right] = \int_0^1 \left(D\varphi_1.\overline{D\psi_1} + a^2\varphi_1\overline{\psi_1} + \sigma\chi\varphi_2\overline{\psi_2}\right)dz,$$
$$\|\varphi\|_M = \left[\varphi, \varphi\right]^{1/2};$$

where $\varphi = (\varphi_1, \varphi_2), \psi = (\psi_1, \psi_2)$ and the overbar denotes the complex conjugation.

Then, as is easily checked, all the assumptions in a theorem due to R. C. Diprima & G. Habetler [9] are satisfied. So we infer that the eigenvalues of (L, M_0) lie within circles of radii $\|M_0^{-1}\ \tilde{B}\|_M$ about those of (L_s, M_0), where the notation μ is an eigenvalue of (C, F) means $\exists\ \varphi \neq 0 : C\ \varphi = \mu\ F\ \varphi$. Let C be an arbitrary linear operator, then we denote the spectrum of C by $\sigma(C)$ and its point spectrum by $P\sigma(C)$. Let G^{-1} be the inverse of the selfadjoint extension of $M_0^{-1}\ L_s$ in H_{M_0}, then $\sigma(G) = P\sigma(M_0^{-1}\ L_s)$ and $\sigma(T) = P\sigma(M_0^{-1}\ L)$, where $T = G + M_0^{-1}\ \tilde{B}$. By virtue of [24, theorem 3.16, p.212], if η is an isolated element of multiplicity one of $P\sigma(L_s, M_0)$ such that $\mathrm{dist}(\eta, P\sigma(L_s, M_0) - \{\lambda\}) > 2\ \|M_0^{-1}\ \tilde{B}\|_M$ and Γ a circle of radius $\|M_0^{-1}\tilde{B}\|_M$ centered about η, then Γ contains exactly one eigenvalue of (L, M_0) and it has multiplicity one.

The first conclusion we draw is that the eigenvectors of problem (5.10) span the Hilbert space H_{M_0} in a certain sense (see [9, p.220]).

To prove that the principle of exchange of stabilities holds for the deep Bénard convection problem, we need to show that the eigenvalue with least real part is necessarily real when its real part vanishes [8]–[36]. To this end, let us set

$$(5.11)\quad L_{s_1} = \begin{pmatrix} (D^2 - a^2)^2 & a\ \lambda \\ a\ \lambda & -(D^2 - a^2) \end{pmatrix} \quad , \quad \tilde{B}_1 = \begin{pmatrix} 0 & 0 \\ a\ \lambda\ \delta(1 - z) & 0 \end{pmatrix}$$

It is clear that $\|M_0^{-1}\ \tilde{B}_1\|_M$ is a continuous function of δ and that

$$\|M_0^{-1}\ \tilde{B}_1\|_M \longrightarrow 0 \ \text{ as } \delta \longrightarrow 0.$$

For $\delta = 0$, it is easy to show that the eigenvalues are real and can be ordered as follows

$0 < \mu_1 < \mu_2 < \mu_3 < \ldots\ldots$ Then the principle of exchange of stabilities holds for the deep Bénard convection problem $\forall\ \delta \in [0,1]$ such that

$$\|M_0^{-1}\ \tilde{B}\|_M \leq \frac{|\mu_1 - \mu_2|}{2}. \tag{5.12}$$

REMARK 5.1. A similar argument leads to the same conclusions if we replace the rigid upper and lower plates by a top free surface and a lower rigid plate or by a top and bottom free surfaces. In the free-free case the proof is sraightforward (see [11])

Before we close this subsection, we give an energy inequality which will be used later; to this end, let us take the scalar product of (5.5) with $(\vec{v}, \frac{\vartheta}{\chi})$ in $\mathbb{L}^2(\Omega)$, then we obtain

$$(5.13)_1 \qquad \frac{1}{2\sigma}\ \frac{d}{dt}|\vec{v}|^2 - \lambda(\vartheta\vec{e}, \vec{v}) + \|\vec{v}\|^2 = 0\ \ ;$$

$$(5.13)_2 \qquad \frac{1}{2}\ \frac{d}{dt}\ |\vartheta|^2 - \lambda(\vec{v}.\vec{e}, \vartheta) - (\triangle\ \vartheta,\ \frac{\vartheta}{\chi}) = 0.$$

Observing that : $\forall\ \vartheta \in \mathcal{W}^{1,2}$,

$$(5.14)_1 \quad \exists\ C(\Omega,\delta) > 0 : (-\ \triangle\ \vartheta, \frac{\vartheta}{\chi}) = (\nabla\vartheta, \frac{\nabla\vartheta}{\chi}) + \left(\nabla\vartheta, \frac{\delta\vartheta\vec{e}}{\chi^2}\right) \leq C\|\vartheta\|^2,$$

$$(5.14)_2 \quad \exists\ k > 0, k = \frac{1}{1+\delta} - \frac{\delta^2}{\pi^2} : -(\triangle\ \vartheta, \frac{\vartheta}{\chi}) \geq k\|\vartheta\|^2 \ \ \text{(see [13])};$$

from (5.13), we get

$$\frac{1}{2\sigma}\ \frac{d}{dt}|\vec{v}|^2 + \frac{1}{2}\ \frac{d}{dt}\ |\vartheta|^2 + k\|\vartheta\|^2 + \|\vec{v}\|^2 \leq 2\lambda(\vartheta\ \vec{e}, \vec{v}). \tag{5.15}$$

5.4. Variational formulation

Let us choose $\delta \in [0,1]$ such that the condition (5.12) is satisfied; since for such δ's the principle of exchange of stabilities holds, to study the linear stability of the equilibrium state, we have only to consider the steady linearized problem, i.e.

$$(5.16)_1 \qquad -\triangle \vec{v} - \lambda\,\vartheta\,\vec{e} + \lambda\,\nabla q = \vec{0} \qquad \text{in } \Omega;$$

$$(5.16)_2 \qquad -\triangle\ \vartheta - \lambda\,\chi \vec{v}.\vec{e} = 0 \qquad \text{in } \Omega;$$

$$(5.16)_3 \qquad \operatorname{div} \vec{v} = 0 \qquad \text{in } \Omega;$$

$$(5.16)_4 \qquad \vec{v} = \vec{0},\ \vartheta = 0 \qquad \text{on } \ z = 0,\ z = 1;$$

or equivalently

$$(5.17)_1 \qquad A\,\varphi - \lambda\,M\,\varphi = 0 = \tilde{A}(\lambda)\,\varphi\,;$$

with

$$\begin{cases} D(A) = \mathcal{U}^{1,2} \cap \mathbb{H}^2(\Omega) \\ A\,\varphi = (-P\,\triangle\ \vec{v},\ -\triangle\ \vartheta), \qquad \varphi = (\vec{v}, \vartheta); \end{cases}$$

$$M\,\varphi = (\vartheta\,\vec{e}, \chi(z)\,\vec{v}.\vec{e}).$$

As is easily verified, the adjoint problem to $(5.17)_1$ is

$$(5.17)_2 \qquad A\,\varphi^+ - \lambda\,M^+\,\varphi^+ = 0,$$

where $\varphi^+ = (\vec{v}^+,\ \vartheta^+)$ and $M^+\,\varphi^+ = (\chi(z)\,\vartheta^+\,\vec{e}, \vec{v}^+.\vec{e})$.

Hence problem $(5.17)_1$ is not self adjoint. We will then use an adjoint variational principle [6]–[12]. First, define the functional $I(\varphi, \varphi^+)$ on $\tilde{\mathcal{U}}$ by

$$(5.18) \qquad I(\varphi, \varphi^+) = \frac{(M\,\varphi, \varphi^+)}{(A\,\varphi, \varphi^+)},$$

where $\tilde{\mathcal{U}} = \left\{ (\varphi, \varphi^+) \in \mathcal{U}^{1,2} X\ \mathcal{U}^{1,2}\ : (A\,\varphi, \varphi^+) \neq 0 \right\}$.

Then consider the following variational problem

Variational problem. Maximize the functional $I(\varphi, \varphi^+)$ for $(\varphi, \varphi^+) \in \tilde{\mathcal{U}}$.

Then as is easily verified, the vanishing of the first variation yields the problem $(5.17)_1$ and its adjoint. Let us replace φ^+ by φ in (5.18), this is possible since the adjoint boundary conditions are the same as $(5.16)_4$. Then we have

LEMMA 5.1. *$\exists\ \hat{\varphi} \in \mathcal{U}^{1,2}$ such that $I(\hat{\varphi}) = I(\hat{\varphi}, \hat{\varphi}) = \max_{\varphi \in \mathcal{U}^{1,2} \setminus \{0\}} I(\varphi)$.*

PROOF. First observe that

$$I(\varphi) = \frac{((1+\chi)\vec{v}.\vec{e}, \vartheta)}{\|\vec{v}\|^2 + \|\vartheta\|^2} = \frac{((1+\chi)\vec{v}.\vec{e}, \vartheta)}{\|\varphi\|^2}. \tag{5.19}$$

If $\varphi_m \in \mathcal{U}^{1,2}$ converges weakly to φ in $\mathcal{U}^{1,2}$, then, by compactness of the injection $\mathcal{U}^{1,2} \subset \mathcal{U}^{0,2}$, we can extract a sequence, again denoted by φ_m, such that

$$\varphi_m \text{ converges to } \varphi \text{ in } \mathcal{U}^{0,2}.$$

Therefore

$$\lim_{m\to\infty} \inf \|\varphi_m\| \geq \|\varphi\|,$$
$$\lim_{m\to\infty} ((1+\chi)\vec{v}_m.\vec{e}, \vartheta_m) = ((1+\chi)\vec{v}.\vec{e}, \vartheta).$$

Hence

$$\lim_{m\to\infty} \sup I(\varphi_m) \leq I(\varphi).$$

Thus I is weakly upper semicontinuous, and attains its maximum in $\mathcal{U}^{1,2}$. $\square$

5.5. Bifurcations

In this subsection, we will study the bifurcation from the eigenvalue 0 of $\tilde{A}(\lambda)$ with respect to the eigenvalues λ_i of $(5.17)_1$. Note that by the elliptic nature of the problem (5.16), we expect an increasing sequence of $\left\{\lambda_i\right\}, 0 < \lambda_1 \leq \lambda_2 \leq \dots$ Moreover, each λ_i will be of finite mutiplicity. If we require the solutions of (5.16) to be cellular and set $L_i = (D^2 - a^2)^i$, $i = 1, 2$, then we get as before

$$(5.20)_1 \qquad L_2\, W \;-\; \lambda\theta a^2 = 0 \qquad \text{in } (0,1)\ ;$$

$$(5.20)_2 \qquad -L_1\, \theta \;-\lambda\, \chi\, W = 0 \qquad \text{in } (0,1)\ ;$$

$$(5.20)_3 \qquad W = 0,\ DW = 0,\ \theta = 0 \qquad \text{on } z = 0, z = 1\ ;$$

where $\vartheta_{mn} = \theta$, $w_{mn} = W$. Note that the system (5.20) is equivalent to (5.16).

Following Iudovich [17], let $G_1(z,z'), G_2(z,z')$ be the Green's operators for the differential operators -L_1 and L_2 subject to the boundary conditions $(5.20)_3$. Both G_1 and G_2 are positive compact integral operators with positive symmetric oscillatory kernels [23]. Then the system (5.20) is equivalent to

$$(5.21)_1 \qquad W \;=\; \lambda a^2 G_2\, \theta, \theta = \lambda\, G_1 \chi W,$$

or

$$(5.21)_2 \qquad W \;=\; \lambda^2 a^2 G_2 G_1 \chi W, \theta = \lambda^2 a^2 G_1 \chi G_2 \theta.$$

Since $\chi > 0$, the operator $G = G_2\ G_1\ \chi$ is a compact operator with positive oscillatory kernel and analytic with respect to α, β. Therefore the spectrum of G is a sequence of positive and simple eigenvalues

$$0 < \mu_1(a)\ < \mu_2(a)\ < \mu_3(a)\ < \ldots\ldots < \mu_n(a)\ < \ldots$$

and $\mu(a) = a^2\lambda^2$ is analytic with respect to $a > 0$.

Thus the spectrum of (5.20) consists of the positive eigenvalues

$$(5.22) \qquad \Lambda_{n,k,m} = \sqrt{\frac{\mu_n(a)}{a^2}} \quad (\text{since } \lambda \geq 0),\ a^2 = (k\alpha)^2 + (m\beta)^2, \quad (m,k,n) \in \mathbb{N}^3.$$

Since $\mu_1(0) > 0$, it follows that

$$(5.23)_1 \qquad \lim_{a \to 0} \Lambda_{1,k,m} \longrightarrow \infty.$$

On the other hand, from (5.20), we get

$$|D^2\,W|^2\ +2a^2|DW|^2\ +a^4|W|^2\ =a^2\Lambda_{1,k,m}(W,\theta),$$
$$|D\theta|^2\ +\ a^2|\theta|^2=\Lambda_{1,k,m}(\chi\,W,\theta),$$

where $(.,.)$ and $|.|$ are $L^2(0,1)$ inner product and norm. Using Cauchy-Schwarz's inequality we obtain

$$a^2\,|W|^2\leq\Lambda_{1,k,m}|W|.|\theta| \qquad \text{and}$$
$$a^2|\theta|^2\leq(1+\delta)\ \Lambda_{1,k,m}|W|.|\theta|.$$

Thus

$$(5.23)_2 \qquad \Lambda_{1,k,m}\geq\frac{a^2}{\sqrt{1+\delta}}\longrightarrow\infty \text{ as } a=\left[(k\alpha)^2+(m\beta)^2\right]^{1/2}\longrightarrow\infty.$$

To study the simplicity of the eigenvalues of problem (5.20), let us define the function

$$(5.24) \qquad \psi(\alpha,\beta)=\Lambda_{n_1,k_1,m_1}-\Lambda_{n_2,k_2,m_2}.$$

The function ψ is real analytic in α,β since μ_n and $\Lambda_{n,k,m}$ are analytic functions in a, $a>0$. Proceeding as in Iudovich [17], we conclude that the zeros of ψ are denumerable and so the set

$$\Sigma=\left\{(\alpha,\beta)\ :\ \psi(\alpha,\beta)=0,(n_1,k_1,m_1)\neq(n_2,k_2,m_2)\right\} \text{ is denumerable.}$$

Thus we have proved

LEMMA 5.2. *The system (5.20) has a sequence of positive and simple eigenvalues for all* $(\alpha>0,\ \beta>0)$ *except at most countably many points in the* (α,β)*-plane.*

5.6. Linear stability

Setting $\varphi^+ = (\vec{v}, \frac{\vartheta}{\chi})$ in (5.18) yields

$$I(\varphi, \varphi^+) = \frac{2(\vartheta\, \vec{e}, \vec{v})}{\|\vec{v}\|^2 + (-\bigtriangleup \vartheta, \frac{\vartheta}{\chi})} \leq \frac{2(\vartheta\, \vec{e}, \vec{v})}{\|\vec{v}\|^2 + k\|\vartheta\|^2} \equiv \frac{1}{\lambda_c} \tag{5.25}$$

(by virtue of $(5.14)_2$). The calculations of R_c (and hence of λ_c) has been done in [11] for various boundary conditions.

From (5.15), if $0 \leq \lambda < \lambda_c$, we infer

$$\frac{1}{2\sigma}\frac{d}{dt}|\vec{v}|^2 + \frac{1}{2}\frac{d}{dt}|\vartheta|^2 + k\|\vartheta\|^2 + \|\vec{v}\|^2 \leq \frac{\lambda}{\lambda_c}\Big(k\,\|\vartheta\|^2 + \|\vec{v}\|^2\Big).$$

Hence

$$\frac{1}{2\sigma}\frac{d}{dt}|\vec{v}|^2 + \frac{1}{2}\frac{d}{dt}|\vartheta|^2 + c\Big[k\,|\vartheta|^2 + |\vec{v}|^2\Big]\Big(1 - \frac{\lambda}{\lambda_c}\Big) \leq 0, \tag{5.26}$$

where $c > 0$.

Thus $|\vec{v}(t)|$, $|\vartheta(t)| \longrightarrow 0$ as $t \longrightarrow \infty$.

Summarizing, we have obtained the following

Lemma 5.3. *Let λ_c be given by (5.25). If $0 \leq \lambda < \lambda_c$, then $\varphi = (\vec{v}, \vartheta) = 0$ is the unique solution of (5.5).*

Lemma 5.3 may also be derived using the following argument : it is matter of routine to write the Eqs. (5.5) as an abstract evolution equation. To this end, let us introduce the following operators

$$\begin{cases} A_\lambda : D(A_\lambda) \longrightarrow H\ X\ \mathcal{W}^{0,2} = \mathcal{H}, D(A_\lambda) = \mathbb{H}^2(\Omega) \cap (V\ X\ \mathcal{W}^{1,2}) \\ \qquad\qquad\qquad\qquad\qquad\qquad = \mathbb{H}^2(\Omega) \cap \mathcal{V}, \\ A_\lambda\, \varphi = -(P\ \bigtriangleup \vec{v}, \frac{\bigtriangleup \vartheta}{\chi}),\ \varphi = (\vec{v}, \vartheta)\ , M_\lambda\, \varphi = -\lambda(\vartheta\, \vec{e}, \vec{v}.\vec{e}). \end{cases}$$

With these notations, the system (5.5) becomes

$$\frac{d\,\varphi}{d\,t} + A_\lambda \varphi + M_\lambda \varphi = 0, \tag{5.27$_1$}$$

$$\varphi(0) = \varphi_0. \tag{5.27$_2$}$$

As is easily seen, for $\lambda \in [0, \lambda_c[$, the operator $A_\lambda + M_\lambda = \tilde{A}_\lambda$ is accretive in $\mathcal{H}$. From the Hille-Yosida theorem (cf. e.g [5]), we infer that the spectrum of $\tilde{A}_\lambda$ lies in the positive complex half plane and we have the following

LEMMA 5.4. *$\forall\, \varphi_0 \in D(A_\lambda)$, problem (5.27) admits a unique solution*

$$\varphi \in C^1([0,\infty);\mathcal{H}) \cap C([0,\infty);D(A_\lambda)).$$

Moreover

$$\forall\, t \geq 0, |\varphi(t)| \leq |\varphi_0| \text{ and } |\frac{d\,\varphi}{d\,t}(t)| = |\tilde{A}_\lambda\, \varphi(t)| \leq |\tilde{A}_\lambda\, \varphi_0|. \tag{5.28}$$

For $\varphi_0 = 0$, from (5.28), we get $\forall\, t \geq 0, |\varphi(t)| = 0$.

We are now in position to analyse the stability of the bifurcating solutions of the system (5.5). Recall that the trivial solution will lose stability for some $\lambda_c \in [0, \lambda_1]$. Note also that when the principle of exchange of stabilities holds, $\lambda_1 = \lambda_c$ (where λ_c is given by (5.25)). For those values of δ such that the condition (5.12) holds, all the assumptions in [33, theorem 4.2] are satisfied as is easily checked, then we infer the following

THEOREM 5.2. *For the deep Bénard problem, solutions bifurcating above criticality are stable, while subcritical bifurcating solutions are unstable.*

6. Some remarks on the existence of periodic and quasiperiodic solutions

6.1. Introduction

According to the Landau-Hopf scenario, the transition to turbulence may be described as repeated branching of quasi-periodic solutions into quasi-periodic solutions with more frequencies as the control parameter is increased. The most important assumption in this scenario is that quasi-modes do not interact or at least only weakly, i.e. each quasi-mode retains its individuality.

In this Section, we will begin by investigating the bifurcation of a periodic solution from a steady solution. Next, we study the bifurcation of a quasi-periodic solution from a time periodic solution.

6.2. Bifurcation of periodic solutions for the deep Bénard problem

Assume that for some $R = R_0 > R_c$ a steady solution $\tilde{\varphi}$ loses its stability and is replaced by a time periodic solution $\hat{\varphi}$ with period $\frac{2\pi}{\omega_0(R)}$. Setting $\varphi = \tilde{\varphi}(x,R) + \hat{\varphi}(x,t,R)$ in $(2.1) - (2.2)_1$ $(\tilde{\varphi} = (\overrightarrow{u}_s, \vartheta_s), \hat{\varphi} = (\overrightarrow{u}_p, \vartheta_p))$, we obtain

$(6.1)_1$ $\quad \partial_t \hat{\varphi} + \mathcal{L}\hat{\varphi} + N(\hat{\varphi}) + \sigma\,(\nabla p, 0) = 0\,;$

$(6.1)_2$ $\quad \operatorname{div} \overrightarrow{u}_p = 0\,;$

$(6.1)_3$ $\quad \hat{\varphi} = 0 \qquad \text{on } z = 0,\ z = 1\,;$

$(6.1)_4$ $\quad \hat{\varphi}(x, t + \dfrac{2\pi}{\omega_0}, R) = \hat{\varphi}(x,t,R)\,;$

where $\mathcal{L}\hat{\varphi} = (-\sigma \vartriangle \overrightarrow{u}_p - \sigma\vartheta_p \overrightarrow{e} + (\overrightarrow{u}_p.\nabla)\overrightarrow{u}_s + (\overrightarrow{u}_s.\nabla)\overrightarrow{u}_p, -\frac{\vartriangle\vartheta_p}{\chi} + \overrightarrow{u}_s.\nabla\,\vartheta_p + \overrightarrow{u}_p.\nabla\,\vartheta_s - R\,\overrightarrow{u}_p.\overrightarrow{e} - \frac{4\delta}{\chi}D(\overrightarrow{u}_p) : D(\overrightarrow{u}_s)), N(\hat{\varphi}) = ((\overrightarrow{u}_p.\nabla)\overrightarrow{u}_p, \overrightarrow{u}_p.\nabla\vartheta_p - \frac{2\delta}{\chi}D(\overrightarrow{u}_p) : D(\overrightarrow{u}_p))$.

Now, we assume that $\mathcal{L}$ has a simple complexe eigenvalue $\mu = \xi + i\,\eta$, i.e. $\exists\ \zeta \neq 0 : \mathcal{L}\ \zeta = \mu\ \zeta$, and all the eigenvalues of $\mathcal{L}$ have negative real parts. Note that $\bar{\mu}$ is also an eigenvalue of $\mathcal{L}$. We will also assume that ζ and μ vary continuously with R and that

$$\xi(R) > 0 \text{ for } R < R_0, \xi(R_0) = 0,\ \xi(R) < 0 \text{ for } R > R_0.$$

We can then carry a similar argument to that of [34] to obtain the following

Theorem 6.1. *[34] Assume that the domain Ω is of class $C^{2,2\alpha}$, and that $\mathcal{L}$ has a simple eigenvalue $\mu(R)$ such that $\mu(R)$ crosses to the left half-plane as R increases across R_0 and that $\mathcal{L}$ has no eigenvalue of the form $\mp i\ k\ \omega(R_0), k \in \mathbb{N}$, for $R = R_0$. Then there exists a family of periodic solutions of the deep Bénard convection problem of period $T(R)$ such that $T(R) \longrightarrow \frac{2\pi}{\eta(R_0)}$ as $R \longrightarrow R_0$ and the amplitude of oscillations tends to zero. The periodic solutions are of class $C^{2,2\alpha;1,\alpha}$.*

(The reader is referred to [7]-[34] for the definition and properties of the spaces $C^{l,\alpha;m,\beta}(\bar{\Omega}\ X[0,T])$, $T > 0$, $0 < \alpha,\ \beta < 1, l,\ m \in \mathbb{N}$).

In a similar manner, we can derive for the deep Bénard convection problem all the results found in [22]. In particular, we have the following

THEOREM 6.2. *For the deep Bénard convection problem, subcritical periodic motions are unstable, while supercritical periodic motions are stable in the linearized theory.*

Note that the equations (6.1) fit in the frame studied in [16]. Thus under the same assumptions as in [16], all the results of [16] concerning the existence and stability of periodic solutions are valid for the deep Bénard convection problem.

6.3. Bifurcation of quasi-periodic solutions bifurcating from periodic solutions of the deep Bénard problem

It is a matter of routine to rewrite (6.1) as an abstract evolution equation

$$\frac{d\,\hat{\varphi}}{d\,t} = F(R,\hat{\varphi}). \tag{6.2}$$

As R is increased further, we assume that $\hat{\varphi}$ loses stability when $R = R_{01}$. Let ψ be a small disturbance of $\hat{\varphi}$, then ψ satisfies the linearized equation

$$\frac{d\,\psi}{d\,t} = \Big[D_\varphi F(R,\hat{\varphi})\Big]\psi, \tag{6.3}$$

where $D_\varphi F$ is the Fréchet derivative of F.

Setting

$$\psi = e^{\mu\, t}\zeta(t), \qquad \text{where } \zeta(t) \text{ is } \frac{2\pi}{\omega_0} - \text{periodic, we get}$$

$$\mu\,\zeta = -\frac{d\zeta}{dt} + D_\varphi F(R,\hat{\varphi})\zeta \equiv L\,\zeta, \quad \mu = \xi + i\,\omega_1 \text{ and} \tag{6.4}$$

$$\xi(R_{01}) = 0,\ \xi(R) > 0(\text{resp.} < 0) \text{ for } R > R_{01} \text{ (resp.} < R_{01}).$$

We again assume that $i\,\omega_1(R_{01})$ is a simple eigenvalue of L and that the rest of the eigenvalues of L have negative real parts.

Using a similar argument as in [20]-[21], we arrive at the following conclusions :

Case (i)

If $\frac{\omega_1(R_{01})}{\omega_0(R_{01})} = \frac{m}{n}$, $0 \le \frac{m}{n} < 1, (m,n) \in \mathbb{N}^2$, $n \neq 0$, then we have for

(α) $n = 1,2,3,4$, subharmonic periodic solutions bifurcate;

(β) $n > 4$, under some conditions, a torus T^2 of asymptotically doubly periodic solutions bifurcate.

Case (ii)

If $\frac{\omega_1(R_{01})}{\omega_0(R_{01})}$ is irrational, then a torus T^2 bifurcates. Solutions on T^2 are asymptotically doubly periodic with two frequencies ω_0 and ω_1.

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