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On the number of rational maps between varieties of general type

By T. Bandman* and D. Markushevich**

Abstract. Let X, Y be two complex projective varieties with only canonical singularities and big and nef canonical line bundles K_X, K_Y. Then the set $R(X,Y)$ of all dominant rational maps $f : X \to Y$ is finite. We prove that the number $\#R(X,Y)$ of this maps has the upper estimate, which depends only on the dimension $dimX = n$, selfintersection K_X^n and product $r = r_X r_Y$ of indices r_X and r_Y of varieties X and Y.

§1. Introduction

In this paper the set $R(X,Y)$ of all the dominant rational mappings $f : X$ - - $> Y$ between two complex algebraic varieties X, Y is considered. It is well known that if the image variety Y is hyperbolic in a sense (which will be specified later) the set $R(X,Y)$ is finite. One of the first results in this field was the de Franchis Theorem ([Fr]. It has two parts:

1) For any Riemann surface X and hyperbolic Riemann surface Y the set $R(X,Y)$ is finite;

2) the number $\#R(X,Y)$ of mappings $f \in R(X,Y)$ has an estimate $c(X)$ depending on the surface X only.

The first statement of this Theorem as well as the notion of hyperbolicity itself was lately generalized for the higher dimensional situations. Sh. Kobayashi and T.Ochiai ([Ko-O]) proved that the set $R(X,Y)$ is finite for two complex projective varieties X, Y provided that Y is of general type. In the review ([Ko]) the question was raised whether the second part of the

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de Franchis Theorem is valid for multidimensional varieties as well. It appeared to be the case for nonsingular projective varieties X, Y with ample canonical line bundles ([Ba1]). Moreover in this situation there exists an upper estimate for $\#R(X, Y)$ depending on the "almost topological" invariants of the variety only, namely on $n = \dim X$ and $(c_1(X))^n$ ([Ba2]).

The subject of this paper is the generalization of this fact to a wider class of varieties.

THEOREM 1. *There exists such an integer function σ of three variables, that for any pair of projective varieties X, Y with only canonical singularities and nef and big canonical line bundles K_X and K_Y, the number*

$$\#R(X, Y) \leq \sigma(n, k, r)$$

where $n = \dim X, k = K_X^n$ and $r = r(X) \times r(Y)$ is the product of indices of varieties X, Y.

Note that the bound σ depends once more not on the variety X itself but on its topological invariants. If X and Y are nonsingular, $r = 1$ and σ becomes a function of $n = \dim X$ and $K_X^n = (-1)^n (c_1(X))^n$, where $c_1(X)$ denotes the first Chern class of variety X. The proof of Theorem 1 is similar to one used already in [Ba1] but the modification of technical details enabled us to obtain a more general result.

For the two dimensional case, Theorem 1 provides the following

COROLLARY 1. *There is an integer function σ_2 of two variables, such that for any two surfaces of general type the number*

$$\#R(X, Y) \leq \sigma_2(h^1, h^2)$$

where $h^1 = \dim H^1(X, R), h^2 = \dim H^2(X, R)$.

This Corollary was stated in [Ba1] but there was an error in the proof. Now it easily follows from the existence of the minimal model for surfaces and the Noether formula for K_X^2.

By the "variety" we always mean an irreducible quasi-projective variety over the complex numbers. This does not concern the term "Chow variety" which may be reducible. We will not distinguish between divisors, divisor classes and line bundles when no confusion may arise. Further on, the following notations will be used:

K_X—the canonical class of variety X;
$|D|$—the linear system of a divisor D;
$D^n, D_1 \cdots D_n$—the intersection indices of $\mathbb{Q}$-Cartier divisors on an n-dimensional variety;
α—the map of X - - > $\mathbb{P}^n = \Gamma(sK_X)^V$ defined by the linear system $|sK_X|$.

2. Estimate depending on degrees and dimensions

Let $X \subseteq \mathbb{P}^N$, and $Y \subseteq \mathbb{P}^M$ be projective varieties. Denote by $R^k(X, Y)$ the set of all the dominant rational maps $f : X$ - - > Y whose graph $\Gamma_f \subseteq X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^M$ has degree $\leq k$ with respect to the standard hyperplane sections of $\mathbb{P}^N \times \mathbb{P}^M \subseteq \mathbb{P}^{N+M+NM}$.

PROPOSITION 1. *Let M, N, k, n, m, d_1, d_2 be positive integers. Then the graphs of all the maps $f \in \mathbb{R}^k(X, Y)$ for all $X \subseteq \mathbb{P}^N, Y \subseteq \mathbb{P}^M$ with dim $X = n, dimY = m$, $\deg X = d_1, degY = d_2$ form a bounded algebraic family of subvarieties of $\mathbb{P}^N \times \mathbb{P}^M$. More precisely let T (resp. U) be the Chow variety parameterizing all the n-dimensional (resp. m-dimensional) projective varieties $X_t \subseteq \mathbb{P}^N$ (resp. $Y_u \subseteq \mathbb{P}^M$) with degrees $\deg X_t \leq d_1$, $\deg Y_u \leq d_2$, and let V be the Chow variety of all the n-dimensional subvarieties $Z_v \subseteq X_t \times Y_u$ of $\deg Z_v \leq k$ in $\mathbb{P}^N \times \mathbb{P}^M$. Then there exists a closed algebraic subset $V' \subseteq V$ whose points correspond to the graphs of all the maps $f \in \mathbb{R}^k(X_t \times X_u)$ for all $t \in T$ and $u \in U$.*

PROOF. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be the universal families of projective varieties over T, U, V. There is a commutative diagram of natural morphisms.

$$\begin{array}{ccccccccc}
\mathcal{X} \leftarrow & \tilde{\mathcal{X}} = \mathcal{X} \times_V V & \overset{\pi_{\tilde{\mathcal{X}}}}{\leftarrow} & \mathcal{Z} & \overset{\pi_{\tilde{\mathcal{Y}}}}{\rightarrow} & \tilde{\mathcal{Y}} = \mathcal{Y} \times_V V & \rightarrow \mathcal{Y} \\
p_{\mathcal{X}} \searrow & p_{\tilde{\mathcal{X}}} \searrow & & p_{\mathcal{Z}} \downarrow & & \swarrow p_{\tilde{\mathcal{Y}}} & \swarrow p_{\mathcal{Y}} \\
& T & \overset{\pi_T}{\leftarrow} & V & \overset{\pi_V}{\rightarrow} & U &
\end{array}$$

The morphisms $p_{\tilde{\mathcal{X}}}, p_{\tilde{\mathcal{Y}}}, p_{\tilde{\mathcal{Z}}}, \pi_{\mathcal{X}}, \pi_{\mathcal{Y}}$ are projective ($\pi_{\tilde{\mathcal{X}}}$, resp. $\pi_{\tilde{\mathcal{Y}}}$ being induced by the projection of subvariety $Z_v \subseteq X_t \times Y_u, \pi_T(V) = t, \pi_U(V) = u$ to the first resp. to the second factor). Hence the image $\mathcal{X}_0 = \pi_{\tilde{\mathcal{X}}}(\mathcal{Z}), \mathcal{Y}_0 = \pi_{\tilde{\mathcal{Y}}}(\mathcal{Z})$ are closed and, hence, projective over V. By the semicontinuity of the dimension of fibers of a proper morphism ([S1], Corollary of Theorem 7, Ch. 1, n.6) the following two subsets of V are closed:

$$\begin{aligned} V_1 =& \{v \in V \,|\, \dim(p_{\tilde{\mathcal{X}}}|\mathcal{X}_0)^{-1}(v) \geq n\} \\ V_2 =& \{v \in V \,|\, \dim(p_{\tilde{\mathcal{Y}}}|\mathcal{Y}_0)^{-1}(v) \geq m\} \end{aligned}$$

Hence $V_0 = V_1 \cap V_2$ is closed. The points $v \in V_0$ parameterize all the subvarieties $Z_v \subseteq X_t \times Y_u$ whose projection to X_t and Y_u are dominant (and hence surjective). To be a graph of a rational map $f : X \to Y$ it is sufficient for Z_v that the projection of it onto X_t was birational.

Let $\mathcal{X}'$ be the union of such components $\mathcal{X}_\alpha$ of $p_{\tilde{\mathcal{X}}}^{-1}(\mathcal{X}_\alpha)$ that $\pi_{\tilde{\mathcal{X}}} \,|\, \pi_{\tilde{\mathcal{X}}}^{-1}(\mathcal{X}_\alpha)$ is birational isomorphism and $V' = p_{\tilde{\mathcal{X}}}(\mathcal{X}')$. Since the fibers of $p_{\tilde{\mathcal{X}}}$ are irreducible varieties the inverse image $p_{\tilde{\mathcal{X}}}^{-1}(V_0')$ of any irreducible component $V_0' \subseteq V$ is irreducible ([S1], Theorem 8, n.6, Ch.1). Hence V' is a closed subset of V_0.. For the generic point $v \subseteq V'$ this map is generically finite. By the Zarisky Theorem ([Ha], Corollary 11.4, Ch.3) this means that the fibers of $\pi_{\tilde{\mathcal{X}}}$ are connected over every normal point of $\mathcal{X}'$. In fact, it is true for every point of $\mathcal{X}'$. Indeed, let us consider the normalization $\mathcal{X}_n'$ and V_n' of $\mathcal{X}'$ and V' respectively. Since $p_{\tilde{\mathcal{X}}}$ is smooth in the generic points of its fibers, the lift $p_n : \mathcal{X}_n' \to V_n'$ has the same generic points of fibers as $p_{\tilde{\mathcal{X}}}$ itself.

So, for every point $v \in V'$ the restriction $\pi_{\tilde{\mathcal{X}}} \,|\, p_{\tilde{\mathcal{X}}}^{-1}(v)$ is a birational isomorphism. Hence, V' is the desired set. □

Corollary. *There exists a function $\tau = \tau(N, M, k, n, m, d_1, d_2)$, such that for every pair of projective varieties $X \subseteq \mathbb{P}^N, \dim X = n$, and $Y \subseteq \mathbb{P}^M, \dim Y = m$, of degrees d_1 and d_2 respectively, the inequality*

$$\# R^k(X, Y) \leq \tau(N, M, k, n, m, d_1, d_2)$$

holds.

Proof. Let $p = \pi_T \times \pi_U \,|\, V' : V' \to T \times U$. Then $p^{-1}(t, u)$ parameterizes the graphs of all the maps from $R(X_t, Y_u)$. By construction V' is the quasiprojective scheme over $T \times U$ and we may choose τ to be the maximum of the projective degrees of fibers of p. □

3. Estimates for pluricanonical mappings

Proposition 2. *Let X, Y be projective n- and m-dimensional varieties with canonical singularities of indices $r(X)$ and $r(Y)$; K_X and K_Y their canonical divisor classes, $r = r(X) \cdot r(Y)$. Assume that $R(X,Y) \neq \emptyset, K_X$ and K_Y are nef and big and $\alpha_X : X$ - - $> X' \subseteq \mathbb{P}^N, \alpha_Y : Y$ - - $> Y' \subseteq \mathbb{P}^M$ be birational maps defined by the systems $|srK_X|$ and $|srK_Y|$ for some $s \geq 1$. Then we have*

1) $M \leq N \leq \dfrac{s^n r^n K_X^n}{n!} + q(s)$;
2) $\deg Y' \leq \deg X' \leq s^n r^n K_X^n$;
3) *for every $f \in R(X,Y)$ the map $f' = \alpha_X f \alpha_Y^{-1} \in R^k(X', Y')$ for some $k \leq 2^n s^n r^n K_X^n$.*

Here $q(s)$ is a known polynomial of degree $n-1$, coefficients of which depend on K_X^n and r.

Proof.

1) By the Kawamata Base Free Theorem ([Ka], [Sh]) divisors K_X and K_Y are semiample Cartier divisors on X and Y respectively, and by the Kollar-Matsusaka Theorem ([K-M]) we have:

$$\dim H^0(X, \nu r K_X) \leq \frac{\nu^n r^n K_X^n}{n!} + q(\nu, r^n K_X^n, r^n K_X^n) \tag{1}$$

where $q(\nu, \alpha, \beta)$ is a known polynomial in ν of degree $n-1$, coefficients of which depend on α and β. Hence for $\nu = s$ we obtain the estimate for N.

Now consider the resolution of singularities of X, Y and mappings α_X, α_Y such that the following diagram is commutative:

$$\begin{array}{ccccc}
 & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \\
\sigma_X \swarrow & |\varphi_X & & |\varphi_Y & \searrow \sigma_Y \\
X - - - & | - - \overset{f}{-} - - & & | - - - > & Y \\
\alpha_X \searrow & \downarrow & & \downarrow & \swarrow \alpha_Y
\end{array} \tag{2}$$

$$X' \xrightarrow{f'} Y'$$

in which $\tilde{f}$ is a surjective morphism σ_X, σ_Y, φ_X, φ_Y are birational morphisms and $\tilde{X}$, $\tilde{Y}$ are nonsingular projective varieties. For every μ and $\omega \in H^0(Y, \mu K_Y)$ the inverse image $f^*\sigma_Y^*\omega \in H^0(\tilde{X}, \mu K_{\tilde{X}})$. By definition of canonical singularities $\sigma_{X^*}\mathcal{O}_{\tilde{X}}(\mu K_{\tilde{X}}) = \mathcal{O}_X(\mu K_X)$, hence

$$\dim H^0(Y, \mu K_Y) \leq \dim H^0(X, \mu K_X). \tag{3}$$

For $\mu = rs$ the required estimate of the number M follows from (3).

2) to prove the estimates for $\deg X'$ and $\deg Y'$ we need the following

LEMMA 1. *Let V be an n-dimensional normal projective variety; H_i, E_i-Cartier divisors on V such that H_i, $H_i + E_i$ are nef and E_i are effective $(i = 1, \ldots, n)$. Then*

$$H_i \cdots H_n \leq (H_1 + E_1) \cdots (H_n + E_n).$$

PROOF. Split one factor $H_n + E_n$ on the right hand side of the inequality and observe that

$$(H_1 + E_1) \cdots (H_{n-1} + E_{n-1})E_n \geq 0$$

as all $H_i + E_i$ are nef ([Kl]).

Hence

$$(H_1 + E_1) \cdots (H_{n-1} + E_{n-1})H_n \leq (H_1 + E_1) \cdots (H_n + E_n).$$

We proceed by induction on the number of $E_i \neq 0$ to conclude the proof. □

Now we return to the proof of the Proposition 2. We have $\sigma_X^*(srK_X) = \varphi_X^* H'_X + E$, where E is the fixed part, $\varphi_X^* H'_X$ is the movable part of the system $\sigma_X^*(srK_X)$ and H'_X is a hyperplane section of X'. By Lemma 1

$$\deg X' = (H'_X)^n = (\varphi_X^* H'_X)^n \leq \sigma_X^*(srK_X)^n = s^n r^n K_X^n.$$

The inequality $\deg Y' \leq \deg X'$ follows from the fact that $f' : X' \dashrightarrow Y'$ is the restriction on X' of a linear projection map $\mathbb{P}^N \dashrightarrow \mathbb{P}^M = |srK_Y|^V$ and from [F], example 8.4.6.

3) To prove the last inequality of the statement of Proposition 2 we introduce the projections

$$\tilde{X} \xleftarrow{\tilde{\pi}_X} \tilde{X} \times \tilde{Y} \xrightarrow{\tilde{\pi}_Y} \tilde{Y}$$

$$X' \xleftarrow{\pi'_X} X' \times Y' \xrightarrow{\pi'_Y} Y'$$

and the morphism $q = \varphi_X \times \varphi_Y : \tilde{X} \times \tilde{Y} \to X' \times Y'$. Consider the divisors $H'_X, H'_Y, \tilde{H}_X = \varphi_X^* H'_X, \tilde{H}_Y = \varphi_Y^* H'_Y, H' = \pi_X^{'*} H'_X + \pi_Y^{'*} H'_Y, \tilde{H} = q^* H'$ on $X', Y', \tilde{X}, \tilde{Y}, X' \times Y', \tilde{X} \times \tilde{Y}$ respectively, where H'_X, H'_Y are hyperplane sections of $X' \subset \mathbb{P}^N, Y' \subset \mathbb{P}^M$. We have

$$\deg \Gamma_{f'} = \Gamma_{f'} \cdot H^{'n} = \sum \binom{n}{i} (\pi_X^{'*} H'_X)^i (\pi_Y^{'*} H'_Y)^{n-i}. \tag{4}$$

Since $q^* \pi_X^{'*} H'_X = \tilde{\pi}_X^* \tilde{H}_X, q^* \pi_Y^{'*} H'_Y = \tilde{\pi}_Y^* \tilde{H}_Y$, and $q_* \Gamma_{\tilde{f}} = \Gamma_{f'}$, by projection the formula for q we have

$$\Gamma_{f'} (\pi_X^{'*} H'_X)^i (\pi_Y^{'*} H'_Y)^{n-i} = \Gamma_{\tilde{f}} (\tilde{\pi}_X^* \tilde{H}_X)^i (\tilde{\pi}_Y^* \tilde{H}_Y)^{n-i} = H_X^i \tilde{f}^* \tilde{H}_Y^{n-i}$$

by the definition of $\tilde{f}^*$. Now recall that by our construction, $\tilde{H}_Y$ is the movable part of $|\sigma_Y^* srK_Y|$, hence $\sigma_Y^* srK_Y = \tilde{H}_Y + E$ for some effective divisor E, and $\tilde{f}^* \tilde{H}_Y = sr\tilde{f}^* \sigma_Y^* K_Y - \tilde{f}^* E$. Similarly, $\tilde{H}_X = sr\sigma_X^* K_X^* - F$ with an effective divisor F. As $\tilde{H}_X, \tilde{f}^* \sigma_Y^* K_Y, \tilde{f}^* \tilde{H}_Y$ and $\sigma_X^* K_X$ are nef, the Lemma yields the inequality

$$\tilde{H}_X^i \tilde{f}^* \tilde{H}_Y^{n-i} \leq s^n r^n (\sigma_X^* K_X)^i (\tilde{f}^* \sigma_Y^* K_Y)^{n-i}. \tag{5}$$

By the Kawamata Base Point Free Theorem ([Ka]) there is such an integer s_0 that the linear systems $|s_0 K_X|, |s_0 K_Y|$ are base point free. By Bertini's Theorem [S2] applied i times we can find a nonsingular $(n-i)$-dimensional subvariety $\tilde{L} \subseteq X$ representing the cycle $s_0^i (\sigma_X^* K_X)^i$. We have $\tilde{L} = \sigma_X^{-1}(L)$

and by [Rei], (1.14), L has only canonical singularities. Let $\sigma = \sigma_X \mid \tilde{L}, D = s_0 \tilde{f}^* \sigma_Y^* K_Y \mid \tilde{L}$. We have

$$\begin{aligned} K_L = (s_0+1)K_X \mid L, K_{\tilde{L}} &= (s_0 \sigma_X^* K_X + K_{\tilde{X}}) \mid \tilde{L} \\ &= (s_0+1)\sigma_X^* K_X \mid \tilde{L} + \sum a_j E_j \end{aligned}$$

where E_j are the exceptional divisors of σ and $a_j \geq 0$ are the j discrepancies ([Rei]). As above it is possible to pull back the pluricanonical differentials from Y to $\tilde{X}$, hence $K_X = \tilde{f}^* \sigma_Y^* K_Y + R$ with an effective divisor R. Upon the restriction to $\tilde{L}$, we have for any $k \geq 0$,

$$\begin{aligned} H^0(\tilde{L}, k(s_0+1)D) &\subseteq H^0(\tilde{L}, k(s_0+1)s_0 K_{\tilde{X}} \mid \tilde{L}) \\ &= H^0(\tilde{L}, ks_0 \sigma^* K_L + \sum k(s_0+1)s_0 a_j E_j). \end{aligned}$$

Now we use the following fact: if $\sigma : \tilde{L} \to L$ is a resolution of canonical singularities, then

$$\sigma_* \mathcal{O}_{\tilde{L}}(\nu \sigma^* K_L + \sum b_j E_j) = \mathcal{O}_L(\nu K_L)$$

for any $N \geq 0, b_j \geq 0$ ([Rei]). This implies that

$$H^0(\tilde{L}, ks_0 \sigma^* K_L + \sum k(s_0+1)s_0 a_j E_j) = H^0(\tilde{L}, ks_0 \sigma^* K_L). \tag{7}$$

Now, combining (6), (7) and applying once more the Kollar-Matsusaka Theorem([K-M]) mentioned above, we obtain

$$\begin{aligned} ((s_0+1)D)^{n-i} &= \lim_{k \to \infty} \frac{\dim H^0(\tilde{L}, k(s_0+1)D)(n-i)!}{k^{n-i}} \\ &\leq \lim_{k \to \infty} \frac{\dim H^0(L, k(s_0+1)s_0 K_X \mid L)(n-i)!}{k^{n-i}} = s_0^{n-i} K_L^{n-i}, \end{aligned}$$

which implies

$$(\sigma^* K_X)^i (\tilde{f}^* \sigma_Y^* K_Y)^{n-i} \leq \frac{D^{n-i}}{s_0^{n-i} s_0^i} \leq \frac{s_0^{n-i} K_L^{n-i}}{s_0^n (s_0+1)^{n-i}} = K_X^n.$$

Now from (4) and (5) it follows that

$$\deg \Gamma_{f'} \leq \sum \binom{n}{i} \tilde{H}_X^i f^* H_Y^{n-i} \leq \sum \binom{n}{i} r^n s^n K_X^n \leq 2^n r^n s^n K_X^n. \ \square$$

4. Proof of the Theorem 1

Let X, Y be as in the hypothesis of Theorem 1, $K_X^n = k$ and $R(X,Y) \neq \emptyset$. Let $(\tilde{X}, \sigma_X)$, $(\tilde{Y}, \sigma_Y)$ be the desingularizations of X and Y. Then the divisors $r\sigma_X^* K_X$ and $r\sigma_Y^* K_Y$ are semiample Cartier divisors on $\tilde{X}$ and $\tilde{Y}$ respectively. For every integer $t \geq 1$,

$$\begin{aligned}(-1)^n \chi(\tilde{Y}, -rt\sigma_Y^* K_Y) &= \sum (-1)^i \dim H^i(\tilde{Y}, -tr\sigma_Y^* K_Y) \\ &= \sum (-1)^j \dim H^j(\tilde{Y}, K_{\tilde{Y}} + tr\sigma_Y^* K_Y) \\ &= \dim H^0(\tilde{Y}, (tr+1)\sigma_Y^* K_Y) \\ &\leq \dim H^0(\tilde{X}, (tr+1)\sigma_X^* K_X) \\ &\leq (t+1)^n r^n K_X^n + n - 1\end{aligned}$$

([K-M], Lemma 5.1). In particular, for $1 \leq t \leq m+2$

$$|\, \chi(\tilde{Y}, -rt\sigma_Y^* K_Y) \,| \leq (m+3)^n r^n K_X^n + n - 1.$$

It is obvious that there may be only finite set $Q_1, \ldots Q_\nu$ of polynomials which have integer values in integer points and fulfill this condition. On the other hand, by the Kollar-Matsusaka Lemma ([K-M]) there may be only finite set $P_1, \ldots, P_\mu$ of polynomials $\chi(\tilde{X}, tr\sigma_X^* K_X)$ with given value of $r^n K_X^n = r^n k$. For each polynomial $Q_1, \ldots, Q_\nu$ $P_1, \ldots, P_\mu$ there is a number s_Y (s_X) such that the linear system $|s_Y tr\sigma_Y^* K_Y|$ $(|s_X tr\sigma_X^* K_X|)$ defines the birational map for $t \geq 1$ ([Luo], Theorem 1.2). Choosing s to be the product of all the s_X and s_Y for all the polynomials $P_1, \ldots, P_\mu$, $Q_1, \ldots, Q_\nu$, we obtain the birational maps $\alpha_X : X \text{ - - > } X'$ and $\alpha_Y : Y \text{ - - > } Y'$, defined by the linear systems $|sr\sigma_X^* K_X|$ and $|sr\sigma_Y^* K_Y|$ for all pairs of varieties X, Y with $R(X,Y) \neq \emptyset$ and with given parameters k, r, n.

By Proposition 2 there are estimates N', d' for all parameters $N, M, d_1 = \deg X', d_2 = \deg Y'$ depending only on $r, k = K_X^n, s$, and the latter is also defined by k. Moreover $R(X,Y) \approx R^q(X', Y')$ for $q = 2^n r^n s^n k$.

Now by Proposition 1,

$$\# R^q(X', Y') \leq \tau(N, M, q, n, m, d_1, d_2).$$

Assuming

$$\sigma(n,k,r) \\ = \max\{\tau(N,M,q,n,m,d_1,d_2)\,|\,M \le N \le N',\ d_1 \le d_2 \le d',m \le n\},$$

we obtain the desired function. □

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T. Bandman
Department of Mathematics
Bar-Ilan University
Ramat-Gan 52900
Israel

D. Markushevich
URA, 746, bat. 101
Mathématiques
Université Claude Bernard
(Lyon 1) 43 bd du
11 novembre 1918
69622 Villeurbanne Cedex
France