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On some differential inclusions and their applications

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Abstract. The existence of a solution of the evolution inclusion

$$u' + \partial\phi(t,u) + g(t,u) - F(u) \ni 0 \text{ on } (0,T), u(0) = \xi$$

is established. For each t in [0,T], ϕ (t, $\cdot$) is a proper l.s.c. convex function from H to [0, ∞] and F is an upper hemicontinuous set-valued mapping of $L^2(0,T;H)$ into its closed convex subsets.

The time periodic problem

$$u' + \partial\phi(t,u) - F(u) \ni 0 \text{ on } (0,T), \quad u(0) = u(T)$$

is studied. Applications to the heat equation with mixed boundary conditions and to the coupled Navier Stokes and heat equations with convection, dissipation and control terms in non-cylindrical domains are given.

Introduction

In this paper we consider the initial value problem for the nonlinear evolution inclusion

$$(0.1) \qquad u_t + \partial\phi(t,u) + g(t,u) - F(u) \ni 0 \text{ on } (0,T), \quad u(0) = \xi,$$

in a real Hilbert space H. For each $t \in [0,T]$, $\phi(t,\cdot)$ is a proper lower semicontinuous convex function from H to $[0, +\infty]$, and F is an upper hemicontinuous set-valued mapping of $L^2(0,T;H)$ into its closed and convex subsets.

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The time periodic problem

$$u_t + \partial\phi(t,u) - F(u) \ni 0 \quad \text{on} \quad (0,T), \quad u(0) = u(T), \tag{0.2}$$

is also studied.

Abstract evolution equations as in this paper, with $F(u(t)) = f(t)$, where f is an element of $L^2(0,T;H)$, have been studied by Brezis [B], then by Attouch and Damlamian [AD], Kenmochi [K], Yamada [Y1],[Y2], Watanabe [W], and others.

In the works of Kenmochi [K] and of Yamada [Y1],[Y2], the lower semicontinuous convex function ϕ depends on t, and this allows, in particular, to have a unified approach to the study of parabolic equations in both cylindrical and noncylindrical domains.

In this paper we shall extend the above cited works to the case when F is a set-valued mapping. In Section 1 the notations and the basic assumptions of the paper are given. The initial value problem for (0.1) is studied in Section 2. The existence of periodic solutions of problem (0.2) is established in Section 3. Applications are given in Section 4, first to the coupled Navier-Stokes and heat equations with convection, dissipation and control terms, then to the heat equation with mixed boundary conditions.

1. Notations and basic assumptions

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We denote by $L^2(0,T;H)$ the space of strongly measurable functions from $(0,T)$ to H, with the norm

$$\|v\|_{L^2(0,T;H)} = \left\{\int_0^T \|v(t)\|_H^2 dt\right\}^{1/2},$$

and the obvious inner product.

$C(0,T;H)$ is the space of continuous functions from $[0,T]$ to H, with the usual norm. For each $t \in [0,T]$, let $\phi(t,\cdot)$ be a proper lower semicontinuous convex function from H to $[0,+\infty]$, with

$$D(\phi(t,\cdot)) = \{u \in H : \quad 0 \leq \phi(t,u) < \infty\}.$$

The subdifferential of $\phi(t,\cdot)$ at u is the set

$$\partial\phi(t,u) = \{f \in H : \quad \phi(t,v) - \phi(t,u) \geq (f, v-u) \text{ for all } v \in D(\phi(t,\cdot))\}.$$

The domain of the subdifferential is

$$D(\partial\phi(t,\cdot)) = \{u \in D(\phi(t,\cdot)) : \quad \partial\phi(t,u) \neq \emptyset\}.$$

It is known that $A(t,\cdot) = \partial\phi(t,\cdot)$ is maximal monotone in H. The images of $A(t,\cdot)$ are closed and convex subsets of H, and hence, for each $t \in [0,T]$ and $u \in D(A(t,\cdot))$ there exists a unique element $m[A(t,u)]$ in the set $A(t,u)$, with minimal H-norm.

ASSUMPTION I.1. *For each $t \in [0,T]$ and $c > 0$*

$$K_c(t) = \{u \in H : \quad 0 \leq \phi(t,u) \leq c\}$$

is a non-empty and compact subset of H, with $\phi(t,0) = 0$.

Let $S_c(T)$ be the set

$$S_c(T) = \{u \in L^2(0,T;H) : \ \|u_t\|^2_{L^2(0,T;H)} + \sup_{0\leq t\leq T} \phi(t,u) + \|m[A(t,u)]\|^2_{L^2(0,T;H)} \leq c\},$$

and denote by $X_c(T)$ the closure in $L^2(0,T;H)$ of the convex hull of $S_c(T)$. The set $S_c(T)$ is not empty as it contains zero.

ASSUMPTION I.2. *$\phi(t,\cdot)$ is a proper lower semicontinuous convex function from H to $[0,+\infty]$ such that for some positive constants τ_0, C_1 and C_2, each $t_0 \in [0,T]$ and each $v_0 \in D(\phi(t_0,\cdot))$, there exists an H-valued function v on the interval $I(t_0) = [\max\{t_0 - \tau_0, 0\}, \min\{t_0 + \tau_0, T\}]$ with*

$$\|v(t) - v_0\| \leq C_1|t - t_0|\phi^{1/2}(t_0, v_0) \tag{1.1}$$

and

$$\phi(t, v(t)) \leq \phi(t_0, v_0) + C_2|t - t_0|\phi^{1/2}(t_0, v_0), \tag{1.2}$$

for all $t \in I(t_0)$.

Assumption I.3. *For given $\lambda > 0$, F is a set-valued mapping of $X_\lambda(T) \subset L^2(0,T;H)$ into subsets of $L^2(0,T;H)$, such that*

1) *F is upper hemicontinuous,*
2) *For each $u \in X_\lambda(T), F(u)$ is a closed and convex subset of $L^2(0,T;H)$,*
3) *For each $\epsilon > 0$, there exists $C(\epsilon) > 0$, independent of λ, such that*
$$\sup\{\|y(t)\|^2 : \quad y \in F(u)\} \le \epsilon\lambda + C(\epsilon)(1 + \|u(t)\|^2),$$

for all $u \in X_\lambda(T)$, and almost all $t \in [0,T]$.

Remark. Alternatively, one can write "semicontinuous" instead of "hemicontinuous", and "compact" instead of "closed", in conditions 1) and 2), respectively, of Assumption I.3.

Definition. F is upper semi-continuous at $x_0 \in L^2(0,T;H)$ if for any open N containing $F(x_0)$ there exists a neighborhood M of x_0 such that $F(M) \subset N$.

In the next two sections we shall use the following lemma (cf. [Y1], [$\hat{O}$Y]):

Lemma 1.1. *Let ϕ be a proper lower semicontinuous convex function from H to $[0,+\infty]$. Suppose that Assumption I.2 is satisfied, and let $u \in H^1(0,T;H)$, with $u(t) \in D(\partial\phi(t,\cdot))$ for almost all $t \in [0,T]$.*

Suppose further that the function $t \to \phi(t,u(t))$ is absolutely continuous on $[0,T]$.

Then there exist positive constants C_3, C_4 such that

$$\left|\frac{d}{dt}\phi(t,u(t)) - (a(t,u), \frac{d}{dt}u(t))\right| \le C_3\phi(t,u(t)) + C_4\phi^{1/2}(t,u(t))\|a(t,u)\|,$$

for $a(t,u) \in A(t,u)$ and for almost all $t \in (0,T)$.

2. Initial value problems for inclusions

The main result of this section is

Theorem 2.1. *Let Assumptions I.1–I.3 be satisfied, and let for some $p \in [2,+\infty)$, $C\|u\|^p \le \phi(t,u)$ for all $u \in D(\phi(t,\cdot)), t \in [0,T]$. For an arbitrary $\lambda > 0$, let $u \to g(\cdot,u(\cdot))$ be a single-valued mapping of $X_\lambda(T)$ into $L^2(0,T;H)$. Suppose that:*

1) For any given $\epsilon > 0$, there exist $C(\epsilon) > 0$, independent of λ, and $r \geq 0$ with

$$\|g(t,u)\| \leq \epsilon\lambda + C(\epsilon)\{1 + \phi^r(t,u)\} \quad a.e. \text{ in } \quad t \in [0,T],$$

for all $u \in X_\lambda(T)$,

2) If $u_n \in X_\lambda(T)$ and $u_n \to u$ in $L^2(0,T;H)$, then for a subsequence $\{u_\mu\}$, $g(t,u_\mu) \to g(t,u)$ weakly in $L^2(0,T;H)$.

Then, for each $\xi \in D(\phi(0,\cdot))$ there exist:

(i) a non-empty interval $(0,T_\star)$, with $T_\star = T$ if $0 \leq r \leq 1/2$,

(ii) a solution $u \in C(0,T_\star;H)$ of the differential inclusion

$$u_t + g(t,u) \in -A(t,u) + F(u) \quad on \quad (0,T_\star), \quad u(0) = \xi. \tag{2.1}$$

Furthermore, u_t and $A(t,u)$ are in $L^2(0,T_\star;H)$ and $\phi(t,u(t))$ is absolutely continuous on $[0,T_\star]$.

For $c > 0$, the set-valued mapping $\xi \to u_\xi$, where u_ξ is a solution of (2.1), is upper semicontinuous from $K_c(0) \subset H$ into $L^2(0,T_\star;H)$, with $T_\star = T_\star(c)$.

Theorem 2.1 extends earlier results of Attouch and Damlamian [AD], where $g(t,u) = g(t), F(u) = 0$ and of Ōeda [Ō], Yamada [Y1], where $F(u) = f(t)$.

Let v be an element of $X_\lambda(T)$ and consider the initial value problem

$$u_t \in -A(t,u) + f(v) - g(t,v) \quad \text{on} \quad (0,T), \quad u(0) = \xi. \tag{2.2}$$

Lemma 2.1. *Suppose that all the hypotheses of Theorem 2.1 are satisfied, and let, for $\lambda > 0, f$ be a continuous mapping of $X_\lambda(T) \subset L^2(0,T;H)$ into $L^2(0,T;H)$. Suppose that for $\epsilon > 0$ given, there exists a constant $C(\epsilon) > 0$, independent of λ, with*

$$\|f(v(t))\|^2 \leq \epsilon\lambda + C(\epsilon)\{1 + \|v(t)\|^2\} \quad a.e. \quad in \quad t \in [0,T],$$

for all $v \in X_\lambda(T)$.

Then there exists a unique solution u of (2.2). Moreover,

$$\begin{aligned} \|u_t\|^2_{L^2(0,t;H)} + \phi(t,u(t)) + \|m[A(\cdot,u(\cdot)]\|^2_{L^2(0,t;H)} \\ \leq M\{\phi(0,\xi) + \epsilon\lambda t + C(\epsilon) + \int_0^t (\phi(s,v)) + \phi^{2r}(s,v)ds\}, \end{aligned} \tag{2.3}$$

where M is a constant independent of t, ξ, v, and λ.

Proof. The existence of a unique solution $u \in L^2(0,T;H)$ with u_t and $A(t,u)$ in $L^2(0,T;H)$ is known. Moreover, $\phi(t,u(t))$ is absolutely continuous on $[0,T]$ (cf. Yamada [Y1], [ÔY]).

Let $a(t,u)$ be an element of $A(t,u)$. From (2.2) we have

$$(\frac{d}{dt}u, a(t,u)) + \|a(t,u)\|^2 \le \frac{1}{2}\|a(t,u)\|^2 + \|g(t,v)\|^2 + \|f(v)\|^2. \tag{2.4}$$

By Lemma 1.1

$$\frac{d}{dt}\phi(t,u) + \frac{1}{2}\|a(t,u)\|^2 \le C_3\phi(t,u) + C_4\phi^{1/2}(t,u)\|a(t,u)\| + \|g(t,v)\|^2 + \|f(v)\|^2.$$

Thus

$$\frac{d}{dt}\phi(t,u) + \frac{1}{4}\|a(t,u)\|^2 \le C_5\phi(t,u) + C\{\epsilon\lambda + C(\epsilon)(1 + \|v\|^2 + \phi^{2r}(t,v))\}. \tag{2.5}$$

Hence

$$\phi(t,u) + \frac{1}{4}\|a(t,u)\|^2_{L^2(0,t;H)} \le \phi(0,\xi) + Ct[\epsilon\lambda + C(\epsilon)] + C_5\int_0^t \phi(s,u)ds + C_6\int_0^t (\phi(s,v) + \phi^{2r}(s,v))ds.$$

The Gronwall lemma gives

$$\phi(t,u) \le C_7\{\phi(0,\xi) + t(\epsilon\lambda + C(\epsilon)) + \int_0^t (\phi(s,v) + \phi^{2r}(s,v))ds\}. \tag{2.6}$$

Therefore

$$\|a(t,u)\|^2_{L^2(0,t;H)} \le C_8\{\phi(0,\xi) + t(\epsilon\lambda + C(\epsilon)) + \int_0^t (\phi(s,v) + \phi^{2r}(s,v))ds\}. \tag{2.7}$$

It then follows from (2.2), (2.7) and from the hypotheses on f, g that

$$\|u_t\|^2_{L^2(0,t;H)} \le C_9\{\phi(0,\xi) + t(\epsilon\lambda + C(\epsilon)) + \int_0^t (\phi(s,v) + \phi^{2r}(s,v))ds\}. \tag{2.8}$$

The different constants C are independent of λ, t, v. The lemma is proved. □

LAMMA 2.2. *Suppose that all the hypotheses of Lemma 2.1 are satisfied. Then, for sufficiently large λ, there exists $T_\star(\lambda) > 0$ such that if v is in $X_\lambda(T_\star)$ then the unique solution u of (2.2) is in $S_\lambda(T_\star)$.*

PROOF. For λ large we take $T_\star = \min(T, \bar{T})$, where

$$\bar{T} = \{\lambda - M\phi(0,\xi)\}/\{M(\epsilon\lambda + C(\epsilon) + \lambda + \lambda^{2r})\}.$$

Then it follows from (2.3) that u is in $S_\lambda(T_\star)$. $\square$

LEMMA 2.3. *Suppose that all the hypotheses of Lemma 2.2 are satisfied and let $T_\star$ be as in Lemma 2.2. Then there exists a solution $u \in C(0, T_\star; H)$, with $u_t, A(t,u)$ in $L^2(0, T_\star; H)$, of the inclusion*

$$u_t \in -A(t,u) - g(t,u) + f(u) \quad \text{on} \quad (0, T_\star), \quad u(0) = \xi. \tag{2.9}$$

Moreover, u is in $S_\lambda(T_\star)$ for some $\lambda > 0$.

PROOF. Let L be the mapping of $X_\lambda(T_\star)$, considered as a subset of $L^2(0, T_\star; H)$, into $L^2(0, T_\star; H)$, given by $L(v) = u$, where u is the unique solution of (2.2).

From Lemma 2.2 we know that L maps $X_\lambda(T_\star)$ into $X_\lambda(T_\star)$. It is clear that $X_\lambda(T_\star)$ is a closed and convex subset of $L^2(0, T_\star; H)$. Since $K_\lambda(t)$ is a compact subset of H for each $t \in [0, T]$ by Assumption I.1, it follows from the Arzela-Ascoli theorem that $S_\lambda(T_\star)$ is a compact subset of $L^2(0, T_\star; H)$, and hence $X_\lambda(T_\star)$ is also compact in $L^2(0, T_\star; H)$.

To show that L has a fixed point we apply Schauder's theorem and thus, it suffices to prove the continuity of L.

Suppose that $\{v_n\} \subset X_\lambda(T_\star)$ and that $v_n \to v$ in $L^2(0, T_\star; H)$. Since $X_\lambda(T_\star)$ is closed, $v \in X_\lambda(T_\star)$. Let $u_n = L(v_n)$ be the solution of the inclusion

$$(u_n)_t + g(t, v_n) - f(v_n) \in -A(t, u_n) \quad \text{on} \quad (0, T_\star), \quad u(0) = \xi. \tag{2.10}$$

Since f is continuous in $L^2(0, T_\star; H)$, $f(v_n) \to f(v)$ in $L^2(0, T_\star; H)$. With our hypotheses on g we have $g(t, u_\mu) \to g(t, v)$ weakly in $L^2(0, T_\star; H)$. Since $\{u_m u\} \subset S_\lambda(T_\star)$ we may assume that $u_\mu \to u$ in $L^2(0, T_\star; H)$, $(u_\mu)_t \to u_t$

weakly in $L^2(0,T_\star;H)$ and $a(t,u_\mu)\to\psi$ weakly in $L^2(0,T_\star;H)$. From the maximal monotonicity of $A(\cdot,\cdot)$ we obtain $\psi=a(t,u)$. By (2.10)

$$u_t+g(t,v)-f(v)\in -A(t,u)\quad\text{on}\quad (0,T_\star),\quad u(0)=\xi. \tag{2.11}$$

Since the inclusion (2.11) has a unique solution, it follows that the sequence $\{u_n\}$ itself and not just a subsequence of it converges to u in $L^2(0,T_\star;H)$. The lemma is proved. □

Lemma 2.4. *Suppose that all the hypotheses of Lemma 2.1 are satisfied and let $0\le r\le 1/2$, where r is as in Theorem 2.1. Then $T_\star=T$.*

Proof. Let u be a solution of (2.9) given by Lemma 2.3. With $v=u$ in (2.6) we get

$$\phi(t,u)\le C_7\{\phi(0,\xi)+t(\epsilon\lambda+C(\epsilon))+\int_0^t(\phi(s,u)+1)ds\}$$

From (2.7) we obtain

$$\|m[A(\cdot,u(\cdot))]\|^2_{L^2(0,t;H)}\le C_8\{\phi(0,\xi)+t(\epsilon\lambda+C(\epsilon))+\int_0^t(\phi(s,u)+1)ds\}.$$

It follows from the Gronwall lemma that

$$\phi(t,u(t))+\|m[A(\cdot,u(\cdot))]\|^2_{L^2(0,t;H)}\le C_9\phi(0,\xi)\exp(C_{10}t).$$

C is independent of t and $T_\star$. By continuation we get $T_\star=T$. □

Proof of Theorem 2.1. Since F is an upper semicontinuous map of $X_\lambda(T_\star)\subset L^2(0,T_\star;H)$ into closed and convex subsets of $L^2(0,T_\star;H)$, it follows from the approximate selection theorem that there exists a sequence $\{f_n\}$ of a single-valued continuous mappings $X_\lambda(T_\star)\subset L^2(0,T_\star;H)\to L^2(0,T_\star;H)$ such that for all $n\in N$:

(i) the range of f_n is contained in the convex hull of the range of F,
(ii) $\mathrm{graph}(f_n)\subset\mathrm{graph}(F)+(1/n)$ (unit ball about the graph of F).

cf. Aubin and Cellina [AC. p. 84]

1) Let u_n be a solution of the inclusion

$$(2.12)\qquad (u_n)_t \in -A(t,u_n) - g(t,u_n) + f_n(u_n) \quad \text{on} \quad (0,T_\star), \quad u_n(0)=\xi.$$

From Lemma 2.2 we know that $T_\star$ is independent of n and it follows from Lemma 2.3 that u_n is in $S_\lambda(T_\star)$. There exists a subsequence $\{\mu\}$ of integers such that: $u_\mu \to u$ in $L^2(0,T_\star;H)$, $(u_\mu)_t \to u_t$ weakly in $L^2(0,T_\star;H)$, and $m[A(t,u_\mu)] \to \psi$ weakly in $L^2(0,T_\star;H)$. From the maximal monotonicity of $A(\cdot,\cdot)$ we deduce that $\psi = m[A(t,u)]$. With our hypotheses on g and F we may assume that $g(t,u_\mu) \to g(t,u)$ and $f_\mu(u_\mu) \to h$, both weakly in $L^2(0,T_\star;H)$. On the other hand there exist $y_\mu \in F(u_\mu)$ such that $\|y_\mu - f_\mu(u_\mu)\|_{L^2(0,T_\star;H)} \leq 1/\mu$. Thus, $y_\mu \to h$ weakly in $L^2(0,T_\star;H)$. By Assumption I.3 the graph of F is strongly-weakly closed, hence $h \in F(u)$, and we have

$$(2.13)\qquad u_t \in -A(t,u) - g(t,u) + F(u) \quad \text{on} \quad (0,T_\star), \quad u(0)=\xi.$$

2) We shall show that the mapping $L : \xi \to u_\xi$ of $K_c(0) \subset H$ into $L^2(0,T_\star;H), T_\star = T_\star(c)$, is upper semicontinuous. Here u_ξ is a solution of (2.13).

Since $S_\lambda(T_\star)$ is, by Assumption I.1, a compact subset of $L^2(0,T_\star;H)$, to prove that L is upper semicontinuous it suffices to show that its graph is closed. Suppose that $\xi_n \to \xi$ in H and that $u_n \to u$ in $L^2(0,T_\star;H), u_n \in L(\xi_n)$. We have to show that $u \in L(\xi)$. Since $S_\lambda(T_\star)$ is closed, it contains u. As in the previous part we show that (2.13) holds. Thus $u \in L(\xi)$. The theorem is proved. □

It is clear that when L is single-valued, i.e. when (2.13) has a unique solution for a given ξ, then upper semicontinuity is equivalent to continuity and we have the continuous dependence of the solution on the initial data.

3. Periodic solutions of evolution inclusions

In this section we consider the inclusion problem

$$(3.1)\qquad u_t \in -A(t,u) + F(u) \quad \text{on} \quad (0,T), \quad u(0)=u(T).$$

The main result is the following

Theorem 3.1. *Let ϕ be as in Theorem 2.1, and suppose that Assumptions I.1 and I.2 are satisfied, with $\phi(0,w) = \phi(T,w)$. Assume moreover that A is strictly maximal monotone. Let F be an upper hemicontinuous set-valued mapping of $L^2(0,T;H)$ into its closed and convex subsets, with*

$$\sup\{\|y(t)\| : y \in F(u)\} \leq C\{1 + \|u(t)\|^\alpha\} \quad \textit{a.e. in} \quad t \in [0,T],$$

for all $u \in L^2(0,T;H)$ and for some $\alpha \in [0,1)$.

Then there exists $u \in C(0,T;H)$, solution of (3.1). Moreover, u_t, $A(t,u)$ are in $L^2(0,T;H)$, and $\phi(t,u(t))$ is absolutely continuous on $[0,T]$.

In [Y2] Yamada proved the existence of a periodic solution of (3.1) when F is a single-valued mapping with $F(u(t)) = f(t)$.

First we consider the differential inclusion

$$u_t \in -A(t,u) + f(v) \quad \text{on} \quad (0,T), \quad u(0) = u(T). \tag{3.2}$$

Lemma 3.1. *Let f be a single-valued continuous mapping in $L^2(0,T;H)$ such that*

$$\|f(w(t))\| \leq C\{1 + \|w(t)\|^\alpha\} \quad \textit{a.e.} \quad \textit{in} \quad t \in [0,T],$$

for all $w \in L^2(0,T;H)$ with some $\alpha \in [0,1)$. Suppose that all the hypotheses of Theorem 3.1 are satisfied.

Then for each $v \in L^2(0,T;H)$ there exists a unique solution $u \in C(0,T;H)$ of (3.2), with u_t, $A(t,u)$ in $L^2(0,T;H)$, and $\phi(t,u(t))$ is absolutely continuous on [0,T].

Proof. To establish the existence of a solution of (3.2) we use the Poincaré method and study the single-valued mapping $\xi \to u(T)$, where $u = u(t)$ is the unique solution of the initial value problem

$$u_t \in -A(t,u) + f(v) \quad \text{on} \quad (0,T), \quad u(0) = \xi.$$

A standard argument shows that for large λ the above mapping takes the compact convex set $K_\lambda(0) = \{w : \phi(0,w) \leq \lambda\}$ into itself. Moreover, the mapping is continuous. Thus the Schauder fixed point theorem gives the stated result. Since the estimates needed in order to show that the mapping takes $K_\lambda(0)$ into itself are similar to the ones used in the next lemma, we shall not reproduce them.

Suppose that u_1, u_2 are two solutions of (3.2). Then, for $a(t,u_j) \in A(t,u_j), j = 1,2,$

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|^2 + (a(t,u_1) - a(t,u_2), u_1 - u_2) = 0.$$

So

$$\int_0^T (a(t,u_1) - a(t,u_2), u_1 - u_2)dt = 0.$$

Hence $u_1 = u_2$. □

LEMMA 3.2. *Suppose that all the hypotheses of Theorem 3.1 and of Lemma 3.1 are satisfied. Let u be a solution of the inclusion*

$$u_t \in -A(t,u) + \mu f(u) \quad \textit{on} \quad (0,T), \quad u(0) = u(T). \tag{3.3}$$

Then there exists a constant M independent of $\mu, 0 \le \mu \le 1,$ such that

$$\sup_{0 \le t \le T} \phi(t,u) + \|u_t\|_{L^2(0,T;H)} + \|m[A(\cdot, u(\cdot))]\|_{L^2(0,T;H)} \le M.$$

PROOF. 1) We have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + C\|u\|^p \le \|u\|\|f(u)\| \le C\|u\| + C\|u\|^{1+\alpha} + C.$$

Thus, with $0 \le \alpha < 1$ and $p \ge 2$, we get

$$\frac{d}{dt}\|u\|^2 + C\|u\|^2 \le C_1.$$

Hence, $\|u(t)\|^2 \le \|u(0)\|^2 \exp(-Ct) + C_1/c$. It follows that $\|u(T)\|^2 \le C_1/c(1 - \exp(-CT))$. Thus

$$\|u(t)\|^2 \le C_1\{\exp(-Ct)/c(1 - \exp(-CT)) + 1\} \tag{3.4}$$

2) As in (2.4) let $a(t,u)$ be an element of $A(t,u)$, we have

$$\frac{d}{dt}\phi(t,u)+\frac{1}{2}\|a(t,u)\|^2 \le C_3\phi(t,u)+C_4\phi^{1/2}(t,u)\|a(t,u)\|+\frac{1}{2}\|f(u)\|^2,$$

so that

$$\frac{d}{dt}\phi(t,u)+\frac{1}{4}\|a(t,u)\|^2 \le C_5\phi(t,u)+C_6(1+\|u\|^2).$$

On the other hand $\phi(t,u)\le (a(t,u),u)\le \|a(t,u)\|\|u\|$. It follows that

$$\frac{d}{dt}\phi(t,u)+\frac{1}{4}\|a(t,u)\|^2 \le \frac{1}{8}\|a(t,u)\|^2+C_7(1+\|u\|^2).$$

Hence, taking into account (3.4), we have

$$\frac{d}{dt}\phi(t,u)+\frac{1}{8}\|a(t,u)\|^2 \le C_8. \tag{3.5}$$

Therefore

$$\frac{d}{dt}\phi(t,u)+\frac{C}{4}\phi(t,u)\le C_9,$$

and

$$\phi(t,u)\le \phi(0,u(0))\exp(-(C/4)t)+C_{10}. \tag{3.6}$$

With $\phi(T,u(T))=\phi(0,u(0))$ we obtain $\phi(0,u(0))\le C_{10}/(1-\exp(-CT/4))$. Hence

$$\phi(t,u(t))\le C_{10}\{\exp(-C/4)t)/(1-\exp(-(C/4)T)+1\}. \tag{3.7}$$

3) It follows from (3.5)-(3.7) that

$$\|m[A(t,u)]\|^2_{L^2(0,t;H)} \le C_{11}\{\phi(0,u(0))+1\}\le C_{12}. \tag{3.8}$$

From the equation we obtain $\|u_t\|^2_{L^2(0,t;H)}\le C_{13}$. The different constants C are all independent of t and of μ, $0\le\mu\le 1$. $\square$

REMARK. If $C\|u\|^p \leq \phi(t,u)$ for all $u \in D(\phi(t,\cdot))$ and some $p \in (2,+\infty)$ then we may take $\alpha = 1$ in Lemma 3.2.

Let L be the single-valued mapping of $[0,1]\times L^2(0,T;H)$ into $L^2(0,T;H)$ defined by

$$L(\mu, v) = u, \tag{3.9}$$

where u is the unique solution of

$$u_t \in -A(t,u) + \mu f(v) \quad \text{on} \quad (0,T), \quad u(0) = u(T), \tag{3.10}$$

given by Lemma 3.1.

LEMMA 3.3. *The mapping L is continuous and compact.*

PROOF. 1) We show that L is continuous. Suppose that $\mu_n \to \mu$ and that $v_n \to v$ in $L^2(0,T;H)$, with $u_n = L(\mu_n, v_n)$. We have

$$(u_n)_t \in -A(t,u_n) + \mu_n f(v_n) \quad \text{on} \quad (0,T), \quad u_n(0) = u_n(T).$$

A proof as that in Lemma 3.2 gives

$$\|(u_n)_t\|_{L^2(0,T;H)} + \sup_{0\leq t\leq T} \phi(t,u_n(t)) + \|m[A(\cdot,u_n(\cdot))]\|_{L^2(0,T;H)} \leq M,$$

with M independent of n.

From Assumption I.1 and the Arzela-Ascoli theorem as well as the weak compactness of the unit ball in a Hilbert space we obtain, for a subsequence $\{\nu\} : u_\nu \to u$ in $L^2(0,T;H)$, $(u_\nu)_t \to u_t$ weakly in $L^2(0,T;H), m[A(\cdot,u_\nu(\cdot))] \to \psi$ weakly in $L^2(0,T;H)$. Moreover, $\sup_{0\leq t\leq T} \phi(t,u(t)) \leq M$.

The maximal monotonicity of $A(\cdot,\cdot)$ and the strong convergence of $\{u_\nu\}$ yield $\psi = m[A(\cdot,u(\cdot))]$. Moreover

$$u_t \in -A(t,u) + \mu f(v) \quad \text{on} \quad (0,T), \quad u(0) = u(T). \tag{3.11}$$

Thus, $L(\mu_\nu, v_\nu) = u_\nu \to u = L(\mu, v)$ in $L^2(0,T;H)$. Since the problem has a unique solution, we may take the sequence $\{u_n\}$ itself instead of a subsequence, i.e. $L(\mu_n, v_n) \to L(\mu, v)$ in $L^2(0,T;H)$.

2) We now show that L is compact. Suppose that $0 \leq \mu_n \leq 1, \|v_n\|_{L^2(0,T;H)} \leq M$, and set $u_n = L(\mu_n, v_n)$. As above, $\{u_n\}$ stays in a compact subset of $L^2(0,T;H)$, i.e. L is compact. This completes the proof of the lemma. □

Lemma 3.4. *1)* $L(0,v)=0$ *for all* $v \in L^2(0,T;H)$.

2) If $u = L(\mu, u)$ *then there exists* C*, independent of* μ*, such that* $\|u\|_{L^2(0,T;H)} < C$.

Proof. The first assertion is trivial to check. The lemma is now an immediate consequence of Lemma 3.2. □

Proof of Theorem 3.1. 1) The existence of a periodic solution of the differential inclusion

$$u_t \in -A(t,u) + f(u) \quad \text{on} \quad (0,T), \quad u(0) = u(T)$$

is a consequence of the Leray-Schauder fixed point theorem. Indeed, it follows from Lemmas 3.3 and 3.4 that $L(\mu, \cdot)$ given by (3.9) satisfies all the conditions of the Leray-Schauder theorem. Thus, the equation $u = L(1,u)$ has a solution.

2) Since F is upper hemicontinuous in $L^2(0,T;H)$, with convex and weakly compact images, we may apply the approximate selection theorem just as in Section 2. This completes the proof of Theorem 3.1. □

4. Applications

Let Ω_t be a bounded open set of R^n with boundary Γ_t and set $\Omega = \cup_{0<t<T}(\Omega_t \times \{t\}), \Gamma = \cup_{0<t<T}(\Gamma_t \times \{t\})$. We shall make the following assumptions on Ω.

Assumption IV. *1) There exist* $k \in N$ *and* $\epsilon_0 > 0$ *such that for each* $t \in [0,T], \Gamma_t$ *consists of* k *closed hypersurfaces* Γ_t^j *of class* C^3*, and* $dist(\Gamma_t^j, \Gamma_t^i) \geq \epsilon_0 > 0$ *for* $j \neq i$.

2) Let $\Omega_s^t = \cup_{s<r<t}(\Omega_r \times \{r\})$*. Then the domain* Ω *is covered by* N *slices* $\Omega_{t_j}^{\delta_j+t_j}, \delta_j > 0$ *and* $j = 1, \cdots, N$*. For each* $j, \Omega_{t_j}^{t_j+\delta_j}$ *is mapped onto a cylindrical domain* $\Omega_{t_j} \times (t_j, t_j+\delta_j)$ *by a diffeomorphism of class* C^4 *up to the boundary which preserves the time variable.*

Let G be an open ball in R^n with $\bar{\Omega}_t \subset G$ for all $t \in [0,T]$. By $W^{k,p}(G), W_0^{k,p}(G)$ we denote the usual Sobolev spaces, and by $W^{-1,2}(G)$ the dual of $W_0^{1,2}(G)$.

1. Navier-Stokes equations coupled with a heat equation involving convection and dissipation terms. We shall take $n = 3$ and denote by $H_\sigma(G), H^1_\sigma(G)$ the closure of the set

$$D_\sigma(G) = \{w = (w_1, w_2, w_3) \in C_0^\infty(G) : \quad div(w) = 0\}$$

with respect to the $L^2(G)$ and to the $W^{1,2}(G)$ norms, respectively.

By P we denote the orthogonal projection of $L^2(G)$ onto $H_\sigma(G)$.

Consider the initial boundary value problem in (u, θ):

$$(4.1) \qquad \begin{aligned} &u_t - \nu \triangle u + (u \cdot \nabla)u + \nabla p \in F_1(\theta), \quad div(u) = 0 \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \Gamma, \quad u(x,0) = u_0 \quad \text{in} \quad \Omega_0, \end{aligned}$$

with

$$(4.2) \qquad \begin{aligned} &\theta_t - \mu\triangle\theta + u \cdot \nabla\theta - \frac{\nu}{2}\sum_{j,k=1}^{n}(u_{k,x_j} + u_{j,x_k})^2 \in F_2(\theta) \quad \text{in} \quad \Omega, \\ &\theta = 0 \quad \text{on} \quad \Gamma, \quad \theta(x,0) = \theta_0 \quad \text{in} \quad \Omega_0. \end{aligned}$$

We shall apply Theorem 2.1 to problem (4.1)-(4.2), and establish the existence of its local solution. Let $U = (u, \theta)$ and let $\bar{\phi}$ be the proper lower semicontinuous convex function on $H = H_\sigma(G) \times W^{-1,2}(G)$ given by

$$(4.3) \qquad \begin{aligned} \bar{\phi}(U) &= \frac{\nu}{2}\|\nabla u\|^2_{L^2(G)} + \frac{\mu}{2}\|\theta\|^2_{L^2(G)}, \quad \text{if} \quad U \in H^1_\sigma(G) \times L^2(G); \\ &= +\infty, \quad \text{otherwise.} \end{aligned}$$

The canonical isomorphism of $W^{1,2}_0(G)$ onto $W^{-1,2}(G)$ is $\Lambda w = -\triangle w$, for $w \in W^{1,2}_0(G)$. The inner product in $W^{-1,2}(G)$ is given by $(f, g)_{W^{-1,2}(G)} = < \Lambda^{-1}f, g >$, where $< \cdot, \cdot >$ is the pairing between $W^{1,2}_0(G)$ and its dual.

Since $\bar{\phi}$ is a lower semicontinuous convex function on H, its subdifferential $\partial\bar{\phi}$ is maximal monotone in H.

Let $\bar{\phi}_1$ be the lower semicontinuous convex function on $W^{-1,2}(G)$ given by

$$(4.4) \qquad \begin{aligned} \bar{\phi}_1(\theta) &= \frac{\mu}{2}\|\theta\|^2_{L^2(G)}, \quad \text{if} \quad \theta \in L^2(G); \\ &= +\infty, \quad \text{otherwise.} \end{aligned}$$

Lemma 4.1. *Let $\bar\phi_1$ be as in (4.4). Then $\partial\bar\phi_1 = \mu\Lambda$.*

Proof. If $f \in \partial\bar\phi_1(\theta)$ then

$$\bar\phi_1(\sigma) - \bar\phi_1(\theta) \geq (f, \sigma - \theta)_{W^{-1,2}(G)}, \quad \text{for all} \quad \sigma \in D(\bar\phi_1).$$

First we shall show that $W_0^{1,2}(G) \subset D(\partial\bar\phi_1)$, and that $-\mu\triangle \subset \partial\bar\phi_1$, with $-\mu\triangle$ defined on $W_0^{1,2}(G)$. In fact, it is easy to check that if $\theta \in W_0^{1,2}(G)$ and $f = -\mu\triangle\theta$ then the above inequality reduces to

$$\int_G \theta\sigma \leq \int_G \frac{1}{2}(\theta^2 + \sigma^2) \quad \text{for all} \quad \sigma \in L^2(G).$$

To show that $-\mu\triangle = \partial\bar\phi_1$ with $D(\partial\bar\phi_1) = W_0^{1,2}(G)$ it suffices to prove that $-\mu\triangle$ is maximal monotone. Monotonicity is evident, moreover $R(-\mu\triangle + I) = W^{-1,2}(G)$. This completes the proof. □

Lemma 4.2. *Let $\bar\phi$ be as in (4.3). Then $\partial\bar\phi(U) = (-\nu P(\triangle u), \mu\Lambda)$.*

Proof. It follows from Lemma 4.1 and results in [ÔY]. □

Let $K(t)$ be the closed convex set

$$K(t) = \{U \in H : \quad U = 0 \quad \text{in} \quad G - \Omega_t\},$$

and let

$$\phi(t, U) = \bar\phi(U) + I_{K(t)}(U), \tag{4.5}$$

where $I_{K(t)}$ is the indicator of the set $K(t)$. We have

$$D(\phi(t,\cdot)) = \{U \in H : \ U\,|_{\Omega_t} \in H^1_\sigma(\Omega_t) \times L^2(\Omega_t), \ U = 0 \ \text{ on } \ G - \Omega_t\},$$

and

$$D(\partial\phi(t,\cdot)) = \{U \in H : U\,|_{\Omega_t} \in (H^1_\sigma(\Omega_t) \cap W^{2,2}(\Omega_t)) \times W_0^{1,2}(\Omega_t), \quad U = 0 \text{ on } G - \Omega_t\}. \tag{4.6}$$

Lemma 4.3. *If $\phi(t,\cdot)$ is as in (4.5) then Assumptions I.1 and I.2 are satisfied. Moreover $X_\lambda(T) = co(S_\lambda(T)) = co(S_\lambda(T)) = S_\lambda(T)$.*

Proof. For each $t \in [0,T]$ the set $K_\lambda(t) = \{U \in H : \quad \phi(t,U) \leq \lambda\}$ is a compact convex subset of H. Indeed, a straightforward application of the Sobolev imbedding theorem gives the stated assertion.

We now show that Assumption I.2 is verified. Let V_0 be in $D(\phi(t_0,\cdot))$. The existence of $V = (v,\theta) \in D(\phi(t,\cdot))$ with all the properties stated in Assumption I.2 follows from Lemma 3.2 of $\hat{O}$tani and Yamada [$\hat{O}$Y] for v, and from Yamada [Y1], p.119 for θ.

Moreover,

$$\begin{aligned}\partial\phi(t,U) = \{(f_1,f_2) \in H : \quad & P(\Omega_t)(f_1 \,|_{\Omega_t}) = -\nu P(\Omega_t)(\triangle(u|_{\Omega_t})),\\ & f_2 = -\mu\Lambda(\theta), \quad U = (u,\theta) \in D(\partial\phi(t,\cdot))\},\end{aligned}$$

whence the last statement of the lemma. □

Lemma 4.4. *Let g_1 be the mapping of $H_0^1(G) \cap W^{2.2}(G)$ into $W^{-1,2}(G)$, given by*

$$g_1(u) = \frac{\nu}{2}\sum_{j,k=1}^{3}(u_{k,x_j} + u_{j,x_k})^2.$$

Then

$$\|g_1(u)\|_{W^{-1,2}(G)} \leq \epsilon\|P(\triangle u)\|_{H_\sigma(G)} + C(\epsilon)\|u\|^3_{H^1_\sigma(G)}.$$

Proof. With G being a bounded subset of R^3, the Sobolev imbedding theorem gives

$$\|u\|_{L^6(G)} \leq C\|u\|_{W_0^{1,2}(G)}, \quad \|\nabla u\|_{L^3(G)} \leq \|\nabla u\|^{1/2}_{W^{1,2}(G)}\|\nabla u\|^{1/2}_{L^2(G)}.$$

Thus

$$\begin{aligned}|(g_1(u),v)| &\leq C\|u\|_{W_0^{1,2}(G)}\|\nabla u\|_{L^3(G)}\|v\|_{L^6(G)}\\ &\leq C\|u\|^{3/2}_{W^{1,2}(G)}\|u\|^{1/2}_{W^{2,2}(G)}\|v\|_{W_0^{1,2}(G)}.\end{aligned}$$

Moreover, for u as in the lemma, $\|u\|_{W^{2,2}(G)} \leq C_3\|P(\triangle u)\|_{H_\sigma(G)}$, and now the desired inequality easily follows. □

LEMMA 4.5. *Let $g_2(u,\theta)$ be the mapping of $H^1_\sigma(G)\times W^{1,2}_0$ into $W^{-1,2}(G)$, given by*

$$< g_2(u,\theta), v >= (u\cdot\nabla\theta, v), \quad \textit{for all} \quad v\in W^{1,2}_0(G).$$

Then

$$\|g_2(u,\theta)\|_{W^{-1,2}(G)} \le \epsilon\|\triangle\theta\|_{W^{-1,2}(G)} + C(\epsilon)(\|u\|^4_{H^1_\sigma(G)} + \|\theta\|^2_{L^2(G)}).$$

PROOF. With u, v, and θ as in the lemma we have

$$|(u\cdot\nabla\theta, v)| = |(u\cdot\nabla v,\theta)| \le C\|u\|_{L^6(G)}\|v\|_{W^{1,2}_0(G)}\|\theta\|_{L^3(G)}.$$

Moreover,

$$\|\theta\|_{L^3(G)} \le C\|\theta\|^{1/2}_{W^{1,2}_0(G)}\|\theta\|^{1/2}_{L^2(G)}, \quad \|\theta\|_{W^{1,2}_0(G)} \le C\|\triangle\theta\|_{W^{-1,2}(G)},$$

whence the lemma follows. □

THEOREM 4.1. *Let $(u_0,\theta_0)\in H^1_\sigma(\Omega_0)\times L^2(\Omega_0)$, and let $F_j, j=1,2$ be set-valued mappings of $L^2(0,T;W^{1,2}_0(G)) \subset L^2(0,T;W^{-1,2}(G))$ into $L^2(0,T;W^{-1,2}(G))$. Suppose that*

1) F_j are upper hemicontinuous,
2) For each $\theta, F_j(\theta)$ is a closed convex subset of $L^2(0,T;W^{-1,2}(G))$,
3) For each $\epsilon>0$ there exists $C(\epsilon)$ such that

$$\begin{aligned}&\sup\{\|f(\theta(t))\|^2_{W^{-1,2}(G)} : \quad f(\theta)\in F_j(\theta)\}\\ &\qquad \le \epsilon\|\theta(t)\|^2_{W^{1,2}_0(G)} + C(\epsilon)\{1+\|\theta(t)\|^2_{W^{-1,2}(G)}\},\end{aligned}$$

for all $\theta\in L^2(0,T;W^{1,2}_0(G))$ and for almost all $t\in[0,T]$.

Then there exists a non-empty interval $(0,T_\star)$ and a pair of functions

$$(u,\theta)\in (L^2(0,T_\star;H^1_\sigma(\Omega_t))\cap W^{2,2}(\Omega_t))\times L^2(0,T_\star;W^{1,2}_0(\Omega_t)),$$

such that

$$(u_t,\theta_t)\in L^2(0,T_\star;H_\sigma(G))\times L^2(0,T_\star;W^{-1,2}(G)),$$

satisfying system (4.1)-(4.2), with $T=T_\star$.

The mapping $(u_0,\theta_0)\to(u,\theta)$ of $K_c(0)\subset H$ into $L^2(0,T_\star;H)$, where $H=H_\sigma(G)\times W^{-1,2}(G)$, is upper semicontinuous.

PROOF. Let $U = (u, \theta)$ and let $\phi(t, U)$ be as in (4.5). Set

$$g(U) = (g_0(u), g_1(u) + g_2(u, \theta)),$$

where $g_0(u) = P((u \cdot \nabla)u)$,

$$F(U) = (F_1(\theta), F_2(\theta)), \quad U(0) = (u_0, \theta_0),$$

and $A(t, U) = \partial\phi(t, U)$.

Using Lemmas 4.4 and 4.5 as well as earlier results (cf. [$\hat{O}$Y], [$\bar{O}$], [KF]) it is easy to check that

$$\|g(U)\|_H \leq \epsilon \|A(t, U)\|_H + C(\epsilon)(1 + \phi^2(t, U)).$$

A direct application of Theorem 2.1 gives the stated result. □

2. Mixed boundary problems for evolution inclusions. Let Ω_t be as before and let G be a bounded, open and simply connected subset of R^n with a smooth boundary. We assume that Ω_t is a subset of G and that for each $t \in [0, T], \gamma_t = \partial G \cap \Gamma_t$ is a non-empty closed surface. Set $\gamma = \cup_{0<t<T} \gamma_t$ and let

$$H(G) = \{u \in W^{1,2}(G) : \quad u = 0 \quad \text{on} \quad \partial G - \gamma\}.$$

Let j be a proper lower semicontinuous convex function from R to $[0, +\infty]$ with $j(0) = 0$ and let $\beta = \partial j$. We shall consider the initial boundary value problem

$$\begin{aligned} &u_t - \triangle u \in F(u) \quad \text{in} \quad \Omega, \quad -\frac{\partial}{\partial n} u \in \beta(u) \ \text{ on } \ \gamma, \\ &u = 0 \quad \text{on} \quad \Gamma - \gamma, \quad u(x, 0) = \xi \quad \text{in} \quad \Omega_0; \end{aligned} \tag{4.7}$$

as well as the time periodic problem

$$\begin{aligned} &u_t - \triangle u \in F(u) \quad \text{in} \quad \Omega, \quad -\frac{\partial}{\partial n} u \in \beta(u) \ \text{ on } \ \gamma, \\ &u = 0 \quad \text{on} \quad \Gamma - \gamma, \quad u(x, 0) = u(x, T) \ \text{ in } \ \Omega_0 = \Omega_T. \end{aligned} \tag{4.8}$$

Let $H = L^2(G)$ and $\bar{\phi}$ be defined by

$$\begin{aligned} \bar{\phi}(u) &= \frac{1}{2} \int_G |\nabla u|^2 dx + \int_\gamma j(u) d\sigma, \\ &\qquad \text{if} \quad u \in H(G) \quad \text{and} \quad j(u) \in L^1(\gamma); \\ &= +\infty, \quad \text{otherwise.} \end{aligned} \tag{4.9}$$

Lemma 4.6. *Let $\bar{\phi}$ be as in (4.9). Then 1) $\partial\bar{\phi}(u) = -\triangle u$, 2) $D(\partial\bar{\phi}(u)) = \{u \in H(G) : \quad \triangle u \in H, \quad -\frac{\partial}{\partial n}u \in \beta(u) \quad on \quad \gamma\}$.*

Proof. For $u \in W^{1,2}(G)$ and $\triangle u \in L^2(G)$, $\frac{\partial}{\partial n}u \in W^{-1/2,2}(\partial G)$ (cf. Lions and Magenes [LM]). Let $Au = -\triangle u$, with

$$D(A) = \{u \in H(G); \quad \triangle u \in L^2(G), \quad -\frac{\partial}{\partial n}u \in \beta(u) \quad \text{on} \quad \gamma\}.$$

We shall show that A is maximal monotone in H and that $A \subset \partial\bar{\phi}$.

1) Clearly A is monotone. For $u \in D(A)$, and $v \in D(\bar{\phi})$ we have

$$-\int_G \triangle u(v-u)dx = \int_G \nabla u \nabla(v-u)dx - < \frac{\partial}{\partial n}u, v-u >,$$

where $< \cdot, \cdot >$ is the pairing between $W^{-1/2,2}(\gamma)$ and $W^{1/2,2}(\gamma)$. Thus

$$-\int_G \triangle u(v-u)dx \leq \bar{\phi}(v) - \bar{\phi}(u),$$

whence $A \subset \partial\bar{\phi}$.

2) To show that A is maximal monotone in H it suffices to prove that $I+A$ is onto. Since β is maximal monotone, its resolvent operator $(I+\lambda\beta)^{-1}$ is non-expansive for all $\lambda > 0$.

Consider the elliptic boundary value problem

$$-\triangle u_\lambda + u_\lambda = f \quad \text{in} \quad G, \quad u_\lambda = 0 \quad \text{on} \quad \partial G - \gamma,$$
$$u_\lambda + \lambda\frac{\partial}{\partial n}u_\lambda = (I+\lambda\beta)^{-1}v \quad \text{on} \quad \gamma. \tag{4.10}$$

For $(f, v) \in L^2(G) \times L^2(\gamma)$ there exists a unique solution $u_\lambda \in W^{1,2}(G)$ of (4.10). Let L be the mapping of $L^2(\gamma)$ into itself, defined by

$$Lv = u_\lambda \mid_\gamma. \tag{4.11}$$

3) We now show that L is a contraction. Let $Lv^k = u^k_\lambda \mid_\gamma$. Then

$$\|u^1_\lambda - u^2_\lambda\|^2_{W^{1,2}(G)} - < \frac{\partial}{\partial n}(u^1_\lambda - u^2_\lambda), u^1_\lambda - u^2_\lambda >= 0.$$

Hence

$$\|u_\lambda^1 - u_\lambda^2\|_{W^{1,2}(G)}^2 + \lambda^{-1}\|u_\lambda^1 - u_\lambda^2\|_{L^2(\gamma)}^2$$
$$= \lambda^{-1}((I+\lambda\beta)^{-1}v^1 - (I+\lambda\beta)^{-1}v^2, u_\lambda^1 - u_\lambda^2).$$

In particular, with $0 < \alpha = 1/(\lambda c + 1) < 1$,

$$\|u_\lambda^1 - u_\lambda^2\|_{L^2(\gamma)} \le \alpha\|v^1 - v^2\|_{L^2(\gamma)}.$$

It follows that L has a fixed point, i.e. $Lu_\lambda = u_\lambda$.

The constant c is given by $\|w\|_{L^2(\gamma)} \le c^{-1/2}\|w\|_{w^{1,2}(G)}$ for all w in $W^{1,2}(G)$

4) We have

$$\|u_\lambda\|_{W^{1,2}(G)}^2 - < \frac{\partial}{\partial n}u_\lambda, u_\lambda >= (f, u_\lambda) \le \|f\|_{L^2(G)}\|u_\lambda\|_{L^2(G)},$$

so, using the boundary conditions we obtain

$$\frac{1}{2}\|u_\lambda\|_{W^{1,2}(G)}^2 + \frac{1}{\lambda}(u_\lambda, u_\lambda) \le \frac{1}{2}\|f\|_{L^2(G)}^2 + \frac{1}{\lambda}\|u_\lambda\|_{L^2(\gamma)}\|(I+\lambda\beta)^{-1}u_\lambda\|_{L^2(\gamma)}.$$

Since $0 \in \beta(0)$ and $(I+\lambda\beta)^{-1}$ is non-expansive, we have

$$\|(I+\lambda\beta)^{-1}u_\lambda\|_{L^2(\gamma)} \le \|u_\lambda\|_{L^2(\gamma)}.$$

Thus,

$$\|u_\lambda\|_{W^{1,2}(G)} \le \|f\|_{L^2(G)}.$$

Let $\lambda \to 0^+$, and we obtain by taking subsequences: $u_\lambda \to u$ weakly in $W^{1,2}(G)$ and strongly in $W^{r,2}(G)$ for $0 \le r < 1$. It is clear that $u = 0$ on $\partial G - \gamma$. On the other hand

$$-\frac{\partial}{\partial n}u_\lambda = \beta_\lambda(u_\lambda) \to -\frac{\partial}{\partial n}u$$

weakly in $W^{-1/2,2}(\gamma)$.

The Yosida approximation β_λ has the property: $\beta_\lambda(u_\lambda) \in \beta((I+\lambda\beta)^{-1}u_\lambda)$. Since $(I+\lambda\beta)^{-1}u_\lambda \to u$ in $L^2(\gamma)$, it follows from the maximal monotonicity of β that $-\frac{\partial}{\partial n}u \in \beta(u)$. The lemma is proved. □

Let $K(t) = \{u \in L^2(G);\ \ u = 0 \ \text{ a.e. in } \ G - \Omega_t\}$, and let

$$\phi(t,u) = \bar{\phi}(t,u) + I_{K(t)}(u), \tag{4.12}$$

where $\bar{\phi}$ is as in (4.9) and $I_{K(t)}$ is the indicator of the set $K(t)$. Then

$$\begin{aligned} D(\partial\phi(t,\cdot)) = \{u \in L^2(G), \triangle u \in L^2(G), u\,|_{\Omega_t} \in W^{1,2}(\Omega_t),& \\ u = 0 \quad \text{on} \ \ G - \Omega_t, \quad -\frac{\partial}{\partial n}u \in \beta(u) \text{ on } \gamma_t\}.& \end{aligned}$$

Lemma 4.7. *Let $\phi(t,\cdot)$ be as in (4.12). Then Assumptions I.1 and I.2 are satisfied.*

Proof. As in Lemma 4.3. □

For the initial value problem (4.7) we have the following result.

Theorem 4.2. *Let F be an upper hemicontinuous set-valued map in $L^2(0,T;L^2(G))$. Suppose that for each u, $F(u)$ is a closed and convex subset of $L^2(0,T;L^2(G))$. Suppose further that*

$$\sup\{\|y(t)\|^2_{L^2(G)} : \quad y \in F(u)\} \le C\{1 + \|u(t)\|^{2\alpha}_{L^2(G)}\},$$

for some $\alpha \in [0,1]$, all $u \in L^2(0,T;L^2(G))$ and almost all $t \in [0,T]$.

Then, for any given $\xi \in W^{1,2}(\Omega_0)$, with $\xi = 0$ on $\Gamma_0 - \gamma_0$ and $-\frac{\partial}{\partial n}\xi \in \beta(\xi)$ on γ_0, there exists a solution u of (4.7) such that $u \in C(0,T;L^2(G))$, u_t and $\triangle u$ are in $L^2(0,T;L^2(G))$, $u \in L^2(0,T;W^{1,2}(\Omega_t))$.

The set-valued mapping $\xi \to u_\xi$ of $K_c(0)$ into $L^2(0,T;L^2(G))$ is upper semicontinuous.

Proof. In view of Lemmas 4.6 and 4.7 the stated result is an immediate consequence of Theorem 2.1. □

For time-periodic solutions, by applying Theorem 3.1 we obtain the following

Theorem 4.3. *Suppose all the hypotheses of Theorem 4.2 are satisfied and suppose further that $\Omega_0 = \Omega_T$ and that $0 \le \alpha < 1$. Then there exists a solution u of (4.8) with the same regularity properties as in Theorem 4.2.*

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