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On the Lie Algebra $\Theta(X)$ of Vector Fields on a Singularity

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Abstract. To any germ X of a complex analytic variety with local ring $\mathcal{O}_X$ one associates the topological Lie algebra $\Theta(X) = \operatorname{Der} \mathcal{O}_X$ of vector fields on X. We show that isolated hypersurface singularities X of dimension at least 3 are uniquely determined up to isomorphism by the topological Lie algebra $\Theta(X)$.

1. Introduction

Let X be the germ of a complex analytic variety with local ring $\mathcal{O}_X$. Consider the Lie algebra and $\mathcal{O}_X$-module $\Theta(X) = \operatorname{Der} \mathcal{O}_X$ of vector fields on X. The Zariski-Lipman conjecture asserts that X is smooth if and only if $\Theta(X)$ is a free module. Jordan [J] and Siebert [Si] prove that X is smooth if and only if $\Theta(X)$ is a simple Lie algebra. In the present paper the Lie algebra structure of $\Theta(X)$ is related to X in the singular case:

THEOREM. *Isolated hypersurface singularities X of dimension at least 3 are uniquely determined up to isomorphism by the abstract topological Lie algebra $\Theta(X)$.*

Actually, it will be shown that any bicontinuous isomorphism $\Phi : \Theta(Y) \to \Theta(X)$ is induced by a unique analytic isomorphism $\varphi : X \to Y$. The topology on $\Theta(X)$ is the one induced by the weak topology on $\mathcal{O}_X$. We have no counter-examples for varieties which do not satisfy the assumptions of the theorem. However, the method of proof does not extend to more general cases without substantial modification. There are three main

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ingredients: Subalgebras $\Theta_Z(X)$ of vector fields tangent to subvarieties Z of X are characterized in purely Lie algebra theoretic terms. The $\mathcal{O}_X$-module generated by a Hamiltonian vector field H is expressed as an intersection of such $\Theta_Z(X)$. Thirdly, the map $\mathrm{Twist}_{\Phi,h} : \Theta(X) \to \Theta(X)$ introduced in [HM] is exploited to construct the map φ. The proof follows the pattern given in [HM], the arguments being more involved due to the absence of vanishing vector fields.

In the affine algebraic case, the corresponding result has been proven by Siebert [Si] for normal varieties.

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2. Characterization of $\Theta_Z(X)$

The germ X is assumed to be reduced and irreducible. For an embedding $X \subseteq (\mathbb{C}^n, 0)$ let $\mathcal{O}_n = \mathcal{O}_{(\mathbb{C}^n,0)}$ and $\mathcal{O}_X = \mathcal{O}_n/I_X$ so that $\Theta(X) \simeq \mathbb{D}_X/I_X \cdot \mathbb{D}$ where $\mathbb{D} = \mathrm{Der}\, \mathcal{O}_n$ and $\mathbb{D}_X = \{D \in \mathbb{D},\ D(I_X) \subseteq I_X\}$. For a reduced but possibly reducible subvariety Z of X with ideal $I_Z \subseteq \mathcal{O}_X$ let

$$\Theta_Z(X) = \{D \in \Theta(X),\ D(I_Z) \subseteq I_Z\}$$

be the *tangent algebra* of Z relative to X. Given any subalgebra A of $\Theta(X)$, the subvariety X_A of X defined by the radical of the ideal

$$I_A = \{g \in \mathcal{O}_X,\ g \cdot \Theta(X) \subseteq A\}$$

is called the *integral variety* of A relative to $\Theta(X)$. Similarly as in [HM] one has $A \subseteq \Theta_{X_A}(X)$.

PROPOSITION 1. *Let $Z \subseteq X$ and $A = \Theta_Z(X)$.*
(a) *If $A \subsetneq \Theta(X)$ then $Z = X_A$.*
(b) *If $A = \Theta(X)$ then $Z = X$ or $Z \subseteq \mathrm{Sing}\, X$.*
(c) *Let $Y \subsetneq X$ with $Y \not\subseteq \mathrm{Sing}\, X$ and assume Y and Z irreducible. If $A \subseteq \Theta_Y(X)$ then $Y = Z$ or $Y \subseteq \mathrm{Sing}\, Z$.*

PROOF. (a) and (b) being analogous to [HM] we only show (c). Set $B = \Theta_Y(X)$. Then $I_Z \subseteq I_A \subseteq I_B$. Thus $B \subsetneq \Theta(X)$ and

$$Y = X_B \subseteq X_A \subseteq Z.$$

We prove that $Y \not\subseteq \operatorname{Sing} Z$ implies $\dim Y = \dim Z$ and hence $Y = Z$. Embed X in some $(\mathbb{C}^n, 0)$ and let k, resp. c denote the codimension of Z, resp. X in $(\mathbb{C}^n, 0)$. Choose points $p \in Y$ arbitrarily close to 0 such that $p \notin \operatorname{Sing} Z \cup \operatorname{Sing} X$. If $f_1, \dots, f_m \in \mathcal{O}_n$ define Z in $(\mathbb{C}^n, 0)$, the corresponding Jacobian matrix $(\partial_i f_j)$ has rank $\leq k$ on Z, and $\operatorname{Sing} Z$ is defined by the vanishing of its k-minors. Applying this in turn to X and Z one may choose f_i and coordinates such that $f_1, \dots, f_c$ vanish on X and the upper left k-minor of $(\partial_i f_j)$ does not vanish in p. Consider the $n-k$ vector fields $(i = k+1, \dots, n)$

$$D_i = \begin{vmatrix} \partial_1 & \partial_1 f_1 & \dots & \partial_1 f_k \\ \vdots & \vdots & & \vdots \\ \partial_k & \partial_k f_1 & \dots & \partial_k f_k \\ \partial_i & \partial_i f_1 & \dots & \partial_i f_k \end{vmatrix}$$

given by the cofactor expansion along the first column. If f vanishes on Z the functions $D_i(f)$ vanish on Z because the resulting $(k+1)\times(k+1)$-matrix has rank $\leq k$ on Z. And if f vanishes on X the matrix

$$\begin{pmatrix} \partial_1 f & \partial_1 f_1 & \dots & \partial_1 f_c \\ \vdots & \vdots & & \vdots \\ \partial_k f & \partial_k f_1 & \dots & \partial_k f_c \\ \partial_i f & \partial_i f_1 & \dots & \partial_i f_c \end{pmatrix}$$

has rank $\leq c$ on X, hence the $D_i(f)$ vanish on X. Therefore the vector fields D_i are tangent to Z and X. From $\Theta_Z(X) \subseteq \Theta_Y(X)$ we conclude that they are tangent to Y. As they are linearly independent in p a theorem of Rossi [R] implies that the dimension of Y in p must be at least $n-k$. But p was arbitrarily close to 0. We obtain $\dim Y \geq n-k = \dim Z$, proving the Proposition. $\square$

For any inclusion $A \subseteq B$ of Lie algebras we define a decreasing series $A^{[i]}$ of subalgebras of A by

$$A^{[1]} = \{D \in A,\ [D, B] \subseteq A\}, \qquad A^{[i]} = (A^{[i-1]})^{[1]}.$$

Moreover set $A^{[\infty]} = \bigcap A^{[i]}$. This is the largest ideal of B contained in A.

PROPOSITION 2. *Let $Z \subseteq X$ be irreducible with $A = \Theta_Z(X) \subsetneq \Theta(X)$. Then $A^{[\infty]}$ consists of the vector fields vanishing on the minimal $V \supseteq Z$ with $\Theta_V(X) = \Theta(X)$.*

PROOF. Let $B = \{D \in \Theta(X),\ D(\mathcal{O}_X) \subseteq I_Z\}$. By [Si, 3.31] one has

$$B^{[\infty]} = \{D \in \Theta(X),\ D(\mathcal{O}_X) \subseteq I_V\}.$$

Clearly $B^{[\infty]} \subseteq A^{[\infty]}$. Conversely, take $D \in A^{[\infty]}$ and arbitrary $E_1, \dots, E_k \in \Theta(X)$. Then $D' = [\dots[D, E_1], \dots, E_k] \in A$, in fact $D' \in A^{[\infty]}$. For all $g \in \mathcal{O}_X$ and $E \in \Theta(X)$ we have

$$D'g \cdot E = [D', gE] - g \cdot [D', E] \in A,$$

hence $D'g \in I_A$. Proposition 1 implies $I_A \subseteq \sqrt{I_A} = I_Z$. Thus $D'g$ vanishes on Z and $D' \in B$. This means $D \in B^{[\infty]}$. □

PROPOSITION 3. (a) *Let $Z \subseteq X$ and $A = \Theta_Z(X)$. Then $A^{[2]} \neq 0$. If $Z \subsetneq X$ is irreducible and $Z \not\subseteq \operatorname{Sing} X$ then $A^{[\infty]} = 0$.*
(b) *Let $A \subseteq \Theta(X)$ be a subalgebra and $Z = X_A$. If $A^{[2]} \neq 0$ then $Z \subsetneq X$. If $A^{[\infty]} = 0$ then $Z \not\subseteq \operatorname{Sing} X$.*

PROOF. (a) For the first assertion, take $g \in I_Z$ and $D \in A$ and use the structural equation of the preceding proof to show that $g^2 D$ is contained in $A^{[2]}$. The second follows from Propositions 1 and 2.
(b) The first part is similar to [HM, Proposition I.6.2.(a)]. For the second, let $S = \operatorname{Sing} X$. Then $I_S^k \cdot \Theta(X)$ is a non-zero ideal of $\Theta(X)$ for all k. If $Z \subseteq S$ then $I_S^k \cdot \Theta(X) \subseteq A$ for some k. □

We say that a subalgebra A of a Lie algebra B is *balanced* if $A^{[2]} \neq 0$ and $A^{[\infty]} = 0$.

THEOREM 1. *For an irreducible germ X of an analytic variety the map*

$$Z \mapsto \Theta_Z(X)$$

defines a bijection between the set of irreducible subvarieties $Z \subsetneq X$ with $Z \not\subseteq \operatorname{Sing} X$ but $\operatorname{Sing} Z \subseteq \operatorname{Sing} X$ and the set of maximal balanced subalgebras of $\Theta(X)$. In particular, every maximal balanced subalgebra of $\Theta(X)$ is an $\mathcal{O}_X$-submodule.

PROOF. (a) Let Z be a subvariety of X as in the statement of the Theorem. By Proposition 3(a) the subalgebra $A = \Theta_Z(X)$ is balanced in $\Theta(X)$. To prove maximality let $B \subseteq \Theta(X)$ be balanced with $\Theta_Z(X) \subseteq B$. By Proposition 3(b) we can choose a component Y of X_B with $Y \not\subseteq \operatorname{Sing} X$. Moreover $Y \subsetneq X$. By Seidenberg [Se] any vector field tangent to a variety is tangent to its components. Hence

$$\Theta_Z(X) \subseteq B \subseteq \Theta_{X_B}(X) \subseteq \Theta_Y(X).$$

As $Y \not\subseteq \operatorname{Sing} X$ we conclude by Proposition 1(c) that $Y = Z$ and $\Theta_Z(X) = B$.

(b) Let $A \subseteq \Theta(X)$ be maximal balanced. Proposition 3(b) allows to choose a component Z of X_A with $Z \not\subseteq \operatorname{Sing} X$ and $Z \subsetneq X$. By Proposition 3(a) the subalgebra $\Theta_Z(X)$ is balanced in $\Theta(X)$. From $A \subseteq \Theta_{X_A}(X) \subseteq \Theta_Z(X)$ and maximality of A follows $A = \Theta_Z(X)$. Proposition 1(a) shows that Z is uniquely determined as $Z = X_A$. If we had $\operatorname{Sing} Z \not\subseteq \operatorname{Sing} X$ there were a component Y of $\operatorname{Sing} Z$ with $Y \not\subseteq \operatorname{Sing} X$. As

$$A = \Theta_Z(X) \subseteq \Theta_{\operatorname{Sing} Z}(X) \subseteq \Theta_Y(X)$$

the same argument as above gives $A = \Theta_Y(X)$ and $Y = Z$, contradiction. □

Propositions 1 and 3 also yield information on the ideals of $\Theta(X)$:

COROLLARY. (a) ([J], [Si, 3.56]) *X is smooth if and only if $\Theta(X)$ is a simple Lie algebra.*

(b) *If X has an isolated singularity then every ideal of $\Theta(X)$ has finite codimension.*

PROOF. We remarked above that $I_S \cdot \Theta(X)$ with $S = \operatorname{Sing} X$ is a non-zero ideal of $\Theta(X)$ which is clearly different from $\Theta(X)$ if X is not smooth. Conversely, let A be a non-trivial ideal of $\Theta(X)$. It is easy to show that every vector field on X is tangent to $Z = X_A$. Hence $Z = X$ or $Z \subseteq \operatorname{Sing} X$. But $A^{[2]} \neq 0$ gives $Z \subsetneq X$. And $A \neq \Theta(X)$ implies $Z \neq \emptyset$. In the smooth case we are done. If X has an isolated singularity we conclude $Z = 0$. Then the assertion follows from [HM, Proposition I.4.2]. □

REMARK. The results and proofs of this section hold true for affine algebraic varieties X and the Lie algebra $\Theta(X)$ of derivations of their coordinate ring. One obtains in particular a one to one correspondence between the non-singular points of an affine algebraic variety X and the maximal balanced subalgebras of $\Theta(X)$ having finite codimension. Similarly as in [HM, part II, sec. 6] one deduces that normal affine algebraic varieties X are uniquely determined up to isomorphism by the abstract Lie algebra $\Theta(X)$, see also [Si, Theorem 3].

3. Isomorphic varieties

Equip $\mathcal{O}_X$ with the weak topology, i.e. the initial topology with respect to the natural maps $\mathcal{O}_X \to \mathcal{O}_X/m_X^k$ where m_X denotes the maximal ideal of $\mathcal{O}_X$. With the induced topology $\Theta(X)$ becomes a topological Lie algebra. Any isomorphism $\varphi : X \to Y$ induces a continuous Lie algebra isomorphism $\varphi^\# : \Theta(Y) \to \Theta(X)$.

THEOREM 2. *Let X and Y be isolated hypersurface singularities of dimension ≥ 3. For every isomorphism $\Phi : \Theta(Y) \to \Theta(X)$ of topological Lie algebras there is a unique isomorphism $\varphi : X \to Y$ such that $\Phi = \varphi^\#$.*

Given an isomorphism $\Phi : \Theta(Y) \to \Theta(X)$ and an element $h \in \mathcal{O}_Y$ define a $\mathbb{C}$-linear map

$$\mathrm{Twist}_{\Phi,h} : \Theta(X) \to \Theta(X) : D \mapsto \Phi(h \cdot \Phi^{-1}(D)).$$

If $\Phi = \varphi^\#$ for some isomorphism $\varphi : X \to Y$ one has

$$\mathrm{Twist}_{\Phi,h}(D) = \varphi^*(h) \cdot D$$

for all $h \in \mathcal{O}_Y$ and $D \in \Theta(X)$. This shows that there can be only one φ inducing Φ (at least if X is irreducible).

In the sequel X denotes an isolated hypersurface singularity of dimension at least 3, defined in $(\mathbb{C}^n, 0)$ by some $f \in \mathcal{O}_n$. Fix coordinates on $(\mathbb{C}^n, 0)$ with induced partial derivatives $\partial_1, \ldots, \partial_n$. Consider the $\mathcal{O}_X$-submodule $\mathbb{H}(X)$ of $\Theta(X)$ generated by the Hamiltonian vector fields

$$D_{ij} = \partial_j f \cdot \partial_i - \partial_i f \cdot \partial_j.$$

Proposition 4. (cf. [J], [Si]) $\mathbb{H}(X)$ *is an ideal of* $\Theta(X)$ *independent of the choice of* f *and the coordinates.*

Proof. As X has an isolated singularity the module of relations between the partials is generated by the trivial relations. Hence $\mathbb{H}(X)$ is obtained from $\{D \in \mathbb{D},\ Df = 0\}$ by restriction to X. For $D, E \in \mathbb{D}$ with $Df = 0$ and $Ef = af$ for some $a \in \mathcal{O}_n$, there is $\Delta \in \mathbb{D}$ with $Da = \Delta f$. Thus $[D, E] - f \cdot \Delta$ annihilates f, which shows that $\mathbb{H}(X)$ is an ideal. The rest is standard. $\square$

Proposition 5. *Let* X *and* Y *be isolated hypersurface singularities of dimension* ≥ 3. *Let* $\Phi : \Theta(Y) \to \Theta(X)$ *be an isomorphism of topological Lie algebras. For every* $h \in \mathcal{O}_Y$ *there is an element* $\alpha(h) \in \mathcal{O}_X$ *such that for all* $D \in \mathbb{H}(X)$:

$$\mathrm{Twist}_{\Phi,h}(D) = \alpha(h) \cdot D.$$

Proof of Theorem 2. It is immediately seen that the map $\alpha : \mathcal{O}_Y \to \mathcal{O}_X$ is an injective algebra homomorphism. To prove surjectivity apply Proposition 5 to $\Phi^{-1} : \Theta(X) \to \Theta(Y)$. One obtains a map $\beta : \mathcal{O}_X \to \mathcal{O}_Y$ with

$$\Phi^{-1}(g \cdot \Phi(E)) = \beta(g) \cdot E$$

for all $E \in \mathbb{H}(Y)$. Fix $0 \neq E \in \mathbb{H}(Y)$. By the Corollary of section 2 the ideal $\mathbb{H}(X)$ has finite codimension in $\Theta(X)$. Hence there is $0 \neq g \in \mathcal{O}_X$ with $D := g \cdot \Phi(E) \in \mathbb{H}(X)$. Clearly $\Phi^{-1}(D) = \beta(g) \cdot E \in \mathbb{H}(Y)$. By computation one gets

$$h \cdot D = \alpha(\beta(h)) \cdot D$$

for all $h \in \mathcal{O}_X$, proving $\beta = \alpha^{-1}$. Finally, to show that

$$\Phi(E) \circ \alpha = \alpha \circ E$$

for all $E \in \Theta(Y)$ one has to repeat the calculations in part (c) of the proof of [HM, Proposition II.5.1.] using the fact that $\mathbb{H}(X)$ is an ideal of $\Theta(X)$. This proves the Theorem. $\square$

For the proof of Proposition 5, fix Φ and h and call $D \in \Theta(X)$ *appropriate* if $\mathrm{Twist}_{\Phi,h}(D)$ belongs to the $\mathcal{O}_X$-module (D).

LEMMA 1. *If $f, \partial_1 f, \ldots, \partial_{n-1} f$ form a regular sequence in $\mathcal{O}_n$ then $u \cdot D_{ij}$ is appropriate for $i, j < n$ and any unit $u \in \mathcal{O}_n^*$.*

PROOF. (a) We consider $D = D_{12}$. By definition of Twist it is enough to show that Φ^{-1} maps the $\mathcal{O}_X$-module (D) onto an $\mathcal{O}_Y$-submodule of $\Theta(Y)$. Define $\mathbf{H}$ as the set of irreducible subvarieties $Z \subsetneq X$ of dimension ≥ 1 with at worst an isolated singularity at 0 and such that D is tangent to Z. By Theorem 1 the Lie algebra isomorphism Φ^{-1} maps the tangent algebra $\Theta_Z(X)$ onto an $\mathcal{O}_Y$-submodule of $\Theta(Y)$ for any $Z \in \mathbf{H}$. The Lemma then follows from

$$(D) = \bigcap_{Z \in \mathbf{H}} \Theta_Z(X).$$

(b) To prove this equality let E be an element of the intersection and $\tilde{E} = \sum_i a_i \, \partial_i \in \mathbb{D}_X$ an extension of E to $(\mathbb{C}^n, 0)$. In a first step we show that $a_3, \ldots, a_n$ vanish on X. Let $3 \leq i \leq n-1$ and set $y = x_i, z = x_n$. For $c \in \mathbb{C} - \{0\}$ and $k \in \mathbb{N}$ consider the hypersurface section Z_{kc} of X given by $g = z + cy^k$. Its singular locus is defined by

$$\partial_1 f, \ldots, \widehat{\partial_i f}, \ldots, \partial_{n-1} f \quad \text{and} \quad \partial_i f - cky^{k-1} \, \partial_n f.$$

By assumption $J = (f, \partial_1 f, \ldots, \partial_{n-1} f)$ is an m-primary ideal, m the maximal ideal of $\mathcal{O}_n$. Hence $m^{k-1} \subseteq m \cdot J$ for large k. For such k one has

$$(f, \partial_1 f, \ldots, \widehat{\partial_i f}, \ldots, \partial_{n-1} f, \partial_i f - cky^{k-1} \, \partial_n f) = J$$

and consequently Z_{kc} has an isolated singularity at 0. Since $n \geq 4$ and g defines a smooth variety, Z_{kc} is an isolated hypersurface singularity of dimension ≥ 2, in particular irreducible. Moreover, as $i, n \geq 3$, D is tangent to Z_{kc}. Thus $Z_{kc} \in \mathbf{H}$ and E is tangent to Z_{kc}. Now

$$cy^{k-1}(ka_i z - a_n y) = cka_i y^{k-1} z + a_n z - a_n(z + cy^k) = z\tilde{E}g - a_n g$$

vanishes on Z_{kc}. Varying c and k this implies that a_i and a_n vanish on X.
(c) We can now assume that $\tilde{E} = a_1 \, \partial_1 + a_2 \, \partial_2 \in \mathbb{D}_X$. As $\tilde{E} f = af$ for some a and $f, \partial_1 f, \partial_2 f$ is a regular sequence, $\tilde{E}$ must be a multiple of $\partial_2 f \cdot \partial_1 - \partial_1 f \cdot \partial_2$ modulo $f \cdot \mathbb{D}$. This ends the proof of Lemma 1. $\square$

LEMMA 2. *Let $b \in \mathcal{O}_n$ not depend on the last two variables and write x' for the first $n-2$ variables. There is a coordinate change $\psi(x) = (\psi'(x'), x_{n-1}, x_n)$ such that, setting $\delta_i = \psi^\#(\partial_i)$:*
(a) $D_{n-1,1} + b \cdot D_{n-1,2} = \psi^\#(D_{n-1,1})$.
(b) *$f, \delta_1 f, \dots, \delta_{n-1} f$ generate the same ideal as $f, \partial_1 f, \dots, \partial_{n-1} f$.*
(c) *If $f, \partial_1 f, \dots, \partial_{n-1} f$ form a regular sequence then $u \cdot \psi^\#(D_{n-1,1})$ are appropriate for all $u \in \mathcal{O}_n^*$.*

PROOF. As $\partial_1 + b\,\partial_2$ is a non-singular vector field on $(\mathbb{C}^{n-2}, 0)$ there is a coordinate change as in the Lemma such that $\psi^\#(\partial_1) = \partial_1 + b\,\partial_2$. Then $\delta_i f \in (\partial_1 f, \dots, \partial_{n-1} f)$ for $i < n$ and (b) follows by symmetry. Moreover $\delta_{n-1} = \partial_{n-1}$ and

$$D_{n-1,1} + b \cdot D_{n-1,2} = (\partial_1 + b\,\partial_2) f \cdot \partial_{n-1} - \partial_{n-1} f \cdot (\partial_1 + b\,\partial_2) = \psi^\#(D_{n-1,1}).$$

Part (c) follows from (a), (b) and Lemma 1. □

PROOF OF PROPOSITION 5. Using the fact that for generic coordinates $f, \partial_1 f, \dots, \widehat{\partial_i f}, \dots, \partial_n f$ form a regular sequence for all i we can write

$$\mathrm{Twist}_{\Phi,h}(u \cdot D_{ij}) = c_{ij}(u) \cdot u \cdot D_{ij}$$

with some $c_{ij}(u) \in \mathcal{O}_X$. For fixed unit u the $c_{ij}(u)$ are independent of i and j. To see this, let $j \neq k$ so that $u \cdot (D_{ij} + D_{ik})$ is appropriate by Lemma 2. Hence

$$c_{ij}(u) \cdot u \cdot D_{ij} + c_{ik}(u) \cdot u \cdot D_{ik} = \mathrm{Twist}_{\Phi,h}(u \cdot (D_{ij} + D_{ik})) = c \cdot u \cdot (D_{ij} + D_{ik})$$

for some $c \in \mathcal{O}_X$. As X is an isolated singularity, D_{ij} and D_{ik} are linearly independent over $\mathcal{O}_X$. Therefore $c_{ij}(u) - c = 0 = c - c_{ik}(u)$. Next, write $c(u) = c_{ij}(u)$. For a unit u not depending on the last two variables the vector field $D_{n-1,1} + u \cdot D_{n-1,2}$ is appropriate and the argument from above gives $c(u) = c(1)$. Since $\mathrm{Twist}_{\Phi,h}$ is continuous and the units are dense in $\mathcal{O}_n$ we conclude

$$\mathrm{Twist}_{\Phi,h}(a \cdot D_{ij}) = c(1) \cdot a \cdot D_{ij}$$

for those $a \in \mathcal{O}_n$ which do not depend on at least two variables. To show this for all $a \in \mathcal{O}_n$ we may assume a monomial, $i = n-1$ and $j = 2$. Write

$a = b \cdot a_0$ where b is a monomial in the first $n-2$ variables and a_0 is a monomial in the last 2 variables. Then

$$a \cdot D_{n-1,2} = a_0 \cdot D - a_0 \cdot D_{n-1,1}$$

with $D = D_{n-1,1} + b \cdot D_{n-1,2}$. By Lemma 2 we have $D = \delta_1 f \cdot \delta_{n-1} - \delta_{n-1} f \cdot \delta_1$. Therefore $\mathrm{Twist}_{\Phi,h}(a \cdot D_{n-1,2}) = c(1) \cdot a \cdot D_{n-1,2}$ proving the Proposition. □

4. Finite generation

It is desirable to reveal the structure of the infinite dimensional Lie algebras $\Theta(X)$. Here we only show that in many cases they are finitely generated as topological Lie algebras. By this we mean that $\Theta(X)$ is the topological closure of a subalgebra which is finitely generated as an abstract Lie algebra. Given an *Euler vector field* $E = \sum_i \lambda_i x_i \,\partial_i$ we say that $D \in \mathbb{D}$ is *homogeneous* w.r.t. E of *degree* $\deg D$ if $[E, D] = \deg D \cdot D$.

PROPOSITION 6. *Let $A \subseteq \mathbb{D}$ be a Lie submodule. Assume that A contains an Euler vector field $E = \sum_i \lambda_i x_i \,\partial_i$ with positive integers λ_i and that A is generated as an $\mathcal{O}_n$-module by homogeneous elements. Then A is finitely generated as a topological Lie algebra.*

PROOF. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ let $(\alpha, \lambda) = \sum_i \alpha_i \lambda_i$. One calculates

$$[x_j E, x^\alpha E] = ((\alpha, \lambda) - \lambda_j) x_j x^\alpha E.$$

Thus starting with $E, x_1 E, \ldots, x_n E$ and the finitely many $x_j x^\alpha E$ with $(\alpha, \lambda) = \lambda_j$ one can generate all monomial multiples of E. Therefore the module (E) is finitely generated as a topological Lie algebra. For a homogeneous $D \in A$ and all $g \in \mathcal{O}_n$ we have

$$\deg D \cdot g \cdot D = Dg \cdot E - [D, gE].$$

If $\deg D \neq 0$ then the module (D, E) is generated as a Lie algebra by D and (E). If $\deg D = 0$ it is enough to add the generators $x_j D$ since $[E, x_j D] = \lambda_j x_j D$, i.e. $x_j D$ has degree $\lambda_j \neq 0$. □

COROLLARY. *If the isolated hypersurface singularity X is weighted homogeneous then $\mathbb{D}_X$ and $\Theta(X)$ are finitely generated as topological Lie algebras.*

PROOF. Let f be a weighted homogeneous generator of I_X, say $Ef = df$ with $E = \sum_i \lambda_i x_i \, \partial_i$. It is well known and easily proven that the module $\mathbb{D}_X$ is generated by E and the Hamiltonian vector fields $D_{ij} = \partial_j f \cdot \partial_i - \partial_i f \cdot \partial_j$. As

$$[E, \partial_j f \cdot \partial_i] = E(\partial_j f) \cdot \partial_i + \partial_j f \cdot [E, \partial_i] = (d - \lambda_j) \cdot \partial_j f \cdot \partial_i - \partial_j f \cdot \lambda_i \cdot \partial_i$$

we see that D_{ij} is homogeneous of degree $d - \lambda_i - \lambda_j$ and Proposition 6 applies. □

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