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Formal solutions with Gevrey type estimates of nonlinear partial differential equations

By Sunao Ōuchi

Abstract. Let $L(u) = L(z, \partial^\alpha u; |\alpha| \le m)$ be a nonlinear partial differential operator defined in a neighbourhood Ω of $z = 0$ in $\boldsymbol{C}^{n+1}$, where $z = (z_0, z') \in \boldsymbol{C} \times \boldsymbol{C}^n$. $L(u)$ is a polynomial of the unknown and its derivatives $\{\partial^\alpha u; |\alpha| \le m\}$ with degree M. In this paper we consider a nonlinear partial differential equation $L(u) = g(z)$. The main purpose of this paper is to find a formal solution $u(z)$ of $L(u) = g(z)$ with the form

$$u(z) = z_0^q (\sum_{n=0}^{+\infty} u_n(z') z_0^{q_n}) \qquad u_0(z') \not\equiv 0,$$

where $q \in \boldsymbol{R}$ and $0 = q_0 < q_1 < \ldots < q_n < \ldots \to +\infty$, and to obtain estimates of coefficients $\{u_n(z'); n \ge 0\}$. It is shown under some conditions that we can construct formal solutions with

$$|u_n(z')| \le AB^{q_n} \Gamma(\frac{q_n}{\gamma_*} + 1) \qquad 0 < \gamma_* \le +\infty,$$

which we often call the Gevrey type estimate.

0. Contents

In §1 we give notations, the form of $L(u)$ treated in the present paper and some definitions. We state some of the main results (Theorems 1.6-1.9), which follow from the results in §2. In §2 we study nonlinear equations satisfying some assumptions. For such equations we study more precisely the notions introduced in §1 and give the existence theorem (Theorem 2.4) and the estimate of formal solutions (Theorem 2.6), which are the core of

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this paper. We apply them to the equations considered in §1 and show Theorems given in §1. It is also shown that we have the possibility of the improvement of the estimates of formal solutions (Theorems 2.7, 2.8 and 2.11), which are also main results. All the proofs of the results in §2 except Theorem 2.6 are given there. The proof of Theorem 2.6 requires some preliminaries. In §3 we prepare majorant functions to estimate the coefficients of the formal solutions. In §4 we estimate them and complete the proof of Theorem 2.6.

1. Notations, definitions and some of results

Firstly we give usual notations and definitions: $z = (z_0, z_1, \dots, z_n) = (z_0, z_1, z'') = (z_0, z')$ is the coordinates of $\boldsymbol{C}^{n+1}$. $|z| = \max\{|z_i|;\ 0 \le i \le n\}$ and $\partial = (\partial_0, \partial_1, \dots, \partial_n) = (\partial_0, \partial')$, $\partial_i = \partial/\partial z_i$. The set of all non-negative integers (integers) is denoted by $\boldsymbol{N}$ (resp. $\boldsymbol{Z}$). For multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) = (\alpha_0, \alpha')$, $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n = \alpha_0 + |\alpha'|$. $\partial^\alpha = \partial_0^{\alpha_0}\partial^{\alpha'} = \partial_0^{\alpha_0}\partial'^{\alpha'} = \prod_{i=1}^n \partial_i^{\alpha_i}$ and $z^\alpha = z_0^{\alpha_0} z_1^{\alpha_1} \dots z_n^{\alpha_n}$. Now we introduce notations for products of multi-indices. Let $A \in (\boldsymbol{N}^{n+1})^s$, $A = (A_1, A_2, \dots, A_s), A_i = (A_{i,0}, A_i') \in \boldsymbol{N} \times \boldsymbol{N}^n$. Put $s = s_A$, $k_A = \max\{|A_i|;\ 1 \le i \le s_A\}$, $k'_A = \max\{|A'_i|;\ 1 \le i \le s_A\}$, $|A| = \sum_{i=1}^{s_A} |A_i|$, and $l_A = \sum_{i=1}^{s_A} |A'_i|$. Let $A, B \in (\boldsymbol{N}^{n+1})^s$. If some rearrangement of the components A_i's coincides with B, we identify A with B. We denote by $\mathcal{N}^S$ the set of all different elements of $(\boldsymbol{N}^{n+1})^s, 1 \le s \le S$. For a real number a, $[a]$ means the integral part of a. For an open set W in $\boldsymbol{C}^N$, $\mathcal{O}(W)$ is the set of all holomorphic functions on W. Put $\Omega = \{z \in \boldsymbol{C}^{n+1};\ |z| \le R\}$.

In this section firstly we introduce nonlinear operators with formal series coefficients and some definitions. Secondly we treat formal nonlinear equations and give results concerning the existence of formal series solutions and the estimate of their coefficients. We appply them to *not formal* nonlinear operators. The results in this section follow from Theorems in §2, which are more precise concerning the estimates than those in this section.

Definition 1.1. $\mathcal{F}$ is the set of all formal series $f(z) = \sum_{n=0}^{+\infty} f_n(z') z_0^{r_n}$, where $f_n(z') \in \mathcal{O}(\omega)$, ω is a neighbourhood of $z' = 0$ in $\boldsymbol{C}^n$ depending on $f(z)$, and $r_0 < r_1 < \dots < r_n <\to +\infty$.

Definition 1.2. For $f(z) \in \mathcal{F}$, $\min\{r_n;\ f_n(z') \not\equiv 0\}$ is said to be the formal evaluation of $f(z)$. If $f_n(z') \equiv 0$ for all n, the formal evaluation of

$f(z)$ is $+\infty$. $\mathcal{F}_+$ is the set of all $f(z) \in \mathcal{F}$ with the nonnegative evaluation.

Let $L(u)$ be an operator of the form,

$$(1.1)\qquad \begin{aligned} L(u) &= L(z, \partial^\alpha u;\ |\alpha| \le m) \\ &= \sum_{A \in \mathcal{N}^M} a_A(z) \prod_{i=1}^{s_A} (\partial^{A_i} u), \end{aligned}$$

where

$$(1.2)\qquad a_A(z) = z_0^{j_{A,L}} b_A(z), \quad b_A(z) \in \mathcal{F}_+,$$

$j_{A,L}$ is the formal evaluation of $a_A(z)$. $L(u)$ is a polynomial of $\{\partial^\alpha u;\ |\alpha| \le m\}$ with coefficients in $\mathcal{F}$ and degree M, which we call *a formal nonlinear operator*. Recalling $|A| = \sum_{i=1}^{s_A} |A_i|$, $l_A = \sum_{i=1}^{s_A} |A_i'|$ and putting $d_{A,L} = l_A + j_{A,L}$, we have

$$(1.3)\qquad L(u) = \sum_{A \in \mathcal{N}^M} z_0^{d_{A,L} - |A|} b_A(z) \prod_{i=1}^{s_A} (z_0^{A_{i,0}} \partial_0^{A_{i,0}} \partial^{A_i'} u).$$

We denote the linear part of $L(u)$ by $\mathcal{L} = \mathcal{L}(z, \partial)$. Let us define the linearization of $L(u)$. Put for $v = v(z) \in \mathcal{F}$

$$(1.4)\qquad L^v(u) = L(u + v(z)) - L(v(z)).$$

Definition 1.3. The linear part of $L^v(u)$ is called the linearization of $L(u)$ at $v = v(z)$ and is denoted by $L_{lin}(v; z, \partial)$ or shortly $\mathcal{L}^v$.

Put for $r \in \boldsymbol{R}$

$$(1.5)\qquad \begin{cases} e_{k,L}(r) = \min\{s_A r + d_{A,L} - |A|;\ A \in \mathcal{N}^M \ \mathit{with}\ k_A = k\} \\ e_L(r) = \min\{e_{k,L}(r);\ 0 \le k \le m\} \\ k_L(r) = \max\{k_A;\ s_A r + d_{A,L} - |A| = e_L(r)\}, \end{cases}$$

$$(1.6)\qquad \Delta_L(r) = \{A \in \mathcal{N}^M;\ s_A r + d_{A,L} - |A| = e_L(r)\},$$

$$(1.7)\qquad \begin{cases} e_{k,\mathcal{L}} = \min\{d_{A,L} - |A|;\ A \in \mathcal{N}^M \ \mathit{with}\ s_A = 1 \ \mathit{and}\ k_A = k\} \\ e_{\mathcal{L}} = \min\{e_{k,\mathcal{L}};\ 0 \le k \le m\} \\ k_{\mathcal{L}} = \max\{k_A;\ A \in \mathcal{N}^M \ \mathit{with}\ s_A = 1 \ \mathit{and}\ d_{A,L} - |A| = e_{\mathcal{L}}\} \end{cases}$$

and

$$(1.8)\qquad \Delta_{\mathcal{L}} = \{A \in \mathcal{N}^M \ \mathit{with}\ s_A = 1\,; d_{A,L} - |A| = e_{\mathcal{L}}\}.$$

Definition 1.4. (1) The minimal irregularity $\gamma_{\min,L}(r)$ of $L(u)$ for the evaluation r is defined by

(1.9)
$$\gamma_{\min,L}(r) = \min\{\frac{s_A r + d_{A,L} - |A| - e_L(r)}{k_A - k_L(r)};\ A \in \mathcal{N}^M \ \ \mathit{with}\ k_A > k_L(r)\}.$$

(2) The minimal irregularity $\gamma_{\min,\mathcal{L}}$ of $\mathcal{L}(z,\partial)$ is defined by

(1.10)
$$\gamma_{\min,\mathcal{L}} = \min\{\frac{d_{A,L} - |A| - e_{\mathcal{L}}}{k_A - k_{\mathcal{L}}};\ A \in \mathcal{N}^M \ \ \mathit{with}\ s_A = 1 \ \mathit{and}\ k_A > k_{\mathcal{L}}\}.$$

If the set in the right hand side in (1.9) ((1.10)) is void, we put $\gamma_{\min,L}(r) = +\infty$ (resp. $\gamma_{\min,\mathcal{L}} = +\infty$). The minimal irregularities are used in the following Theorems of this paper to estimate formal solutions of nonlinear partial differential equations, that is, to show Gevrey type estimates of them.

We restrict the coefficients of $L(u)$ and study equations. So let us introduce a subclass of $\mathcal{F}$.

Definition 1.5. Let $\mathcal{S}$ be a finitely generated additive semi-group, $\mathcal{S} = \{q_i; i \in \boldsymbol{N}\}$, $0 = q_0 < q_1 < \cdots < q_i <\to +\infty$. $\mathcal{F}_{\mathcal{S}}$ is the set of all formal series $f(z) = \sum_{n=0}^{+\infty} f_n(z')z_0^{q_n} \in \mathcal{F}_+$.

Now suppose that the coefficients $b_A(z) \in \mathcal{F}_S$ in (1.3). So we have $b_A(z) = \sum_{n=0}^{+\infty} b_{A,n}(z')z_0^{q_n}$. Consider

$$(1.11)\qquad \begin{cases} L(u) = g(z) \\ g(z) = z_0^r \displaystyle\sum_{n=0}^{+\infty} g_n(z')z_0^{q_n} \in \mathcal{F}. \end{cases}$$

For given $q \in \boldsymbol{R}$ we try to find a formal solution $u(z)$ of (1.11)

$$u(z) = z_0^q(\sum_{n=0}^{+\infty} u_n(z')z_0^{q_n}), \quad u_0(z') \not\equiv 0, \tag{1.12}$$

that is, a formal solution with the formal evaluation q. For a given $q \in \boldsymbol{R}$, put as defined in (1.5) and (1.6)

$$q* = e_L(q) = \min\{s_A q + d_{A,L} - |A|; A \in \mathcal{N}^M\}, \tag{1.13}$$

$$\Delta_L(q) = \{A \in \mathcal{N}^M; s_A q + d_{A,L} - |A| = q^*\}. \tag{1.14}$$

In order to give conditions for the existence of formal solutions which have a Gevrey type estimate we define some operators. Put for $A \in \mathcal{N}^M$

$$\mathfrak{L}_{0,A}(z', \mu, p) = b_{A,0}(z') \prod_{i=1}^{s_A} \mu(\mu-1)\dots(\mu - A_{i,0}+1)p_{A'_i} \tag{1.15}$$

and

(1.16)

$$\begin{aligned}&\mathfrak{L}_{1,A}(z', \lambda, \mu, p, \partial') \\ &\quad = b_{A,0}(z')\{\sum_{i=1}^{s_A}(\prod_{h\neq i} \mu(\mu-1)\dots(\mu - A_{h,0}+1)p_{A'_h}) \\ &\qquad \times \lambda(\lambda-1)\dots(\lambda - A_{i,0}+1)\partial^{A'_i}\},\end{aligned}$$

where $p = (p_{\alpha'};\ \alpha' \in \boldsymbol{N}^n)$ and λ, μ are parameters. $\mathfrak{L}_{1,A}(z', \lambda, \mu, p, \partial')$ is a linear partial differential operator with order $k'_A = \max\{|A'_i|; 1 \le i \le s_A\}$ and a polynomial of λ and ∂' with degree $k_A = \max\{|A_i|; 1 \le i \le s_A\}$. Define

$$\begin{cases} \mathfrak{L}_0(z', \mu, p) = \sum_{A \in \Delta_L(q)} \mathfrak{L}_{0,A}(z', \mu, p) \\ \mathfrak{L}_1(z', \lambda, \mu, p, \partial') = \sum_{A \in \Delta_L(q)} \mathfrak{L}_{1,A}(z', \lambda, \mu, p, \partial'). \end{cases} \tag{1.17}$$

$\mathfrak{L}_1(z', \lambda, \mu, p, \partial')$ is a linear partial differential operator with order $k'_L(q) = \max\{k'_A; A \in \Delta_L(q)\}$ and a polynomial of λ and ∂' with degree $k_L(q) = \max\{k_A; A \in \Delta_L(q)\}$.

Now we give several conditions to state results.

Condition 0. *$\mathcal{S} \supset \{s_A q + d_{A,L} - |A| - q^*; A \in \mathcal{N}^M\}$ and $g(z) = z_0^{q*} \sum_{n=0}^{+\infty} g_n(z') z_0^{q_n}$, that is, $r = q^*$ in (1.11).*

Condition 1. *There is a solution $u_0(z') \not\equiv 0$ of*

$$\mathfrak{L}_0(z', q, \partial^{\alpha'} u_0(z')) = g_0(z'), \tag{1.18}$$

which is holomorphic in a neighbourhood ω of $z' = 0$.

Assume Conditions 0 and 1. Using $u_0(z')$ in Condition 1, define

$$\mathfrak{L}_1(z', \lambda, \partial') = \mathfrak{L}_1(z', \lambda, q, \partial^{\alpha'} u_0(z'), \partial'). \tag{1.19}$$

Let $m_{\mathfrak{L}_1}$ be the order of $\mathfrak{L}_1(z', \lambda, \partial')$. Let $P.S.\mathfrak{L}_1(z', \lambda, \xi')$ be the principal symbol of $\mathfrak{L}_1(z', \lambda, \partial')$ and $\overset{\circ}{k}_{\mathfrak{L}_1}$ be its degree as a polynomial of (λ, ξ').

Condition 2. *$P.S.\mathfrak{L}_1(0, \lambda, \hat{\xi}')$, $\hat{\xi}' = (1, 0, \dots, 0)$, is a polynomial of λ with degree $\overset{\circ}{k}_{\mathfrak{L}_1} - m_{\mathfrak{L}_1}$ and does not vanish for $\lambda = q + q_n$, $n = 1, 2, \dots$.*

As for the existence of formal solutions with the formal evaluation q, we have

Theorem 1.6. *Suppose that Conditions 0-2 hold. Then there exists uniquely $u(z) = z_0^q(\sum_{n=0}^{+\infty} u_n(z') z_0^{q_n}) \in \mathcal{F}$ satisfing $L(u) = g(z)$ formally and $\partial_1^h u(0, z'') = 0$ $(n \geq 1)$ for $0 \leq h \leq m_{\mathfrak{L}_1} - 1$.*

Conditions 0-1 assure the existence of the non zero initial term $u_0(z')$. We can determine $u_n(z')$ successively by Condition 2. We note that $m_{\mathfrak{L}_1} \leq k'_L(q)$, $P.S.\mathfrak{L}_1(z', \lambda, \xi')$ is a polynomial of λ and ξ' with degree $\overset{\circ}{k}_{\mathfrak{L}_1} \leq k_L(q)$ and homogeneous in ξ' with degree $m_{\mathfrak{L}_1}$. We give a condition in order that the constructed formal solution has a Gevrey type estimate.

Condition 3. *$P.S.\mathfrak{L}_1(0, \lambda, \hat{\xi}')$ is a polynomial of λ with degree $k_L(q) - m_{\mathfrak{L}_1}$.*

We have

THEOREM 1.7. *Put $\gamma_* = \gamma_{\min,L}(q)$. Suppose that Conditions 0–3 and*

$$|b_{A,n}(z')|,\ |g_n(z')| \le B_1^{q_n+1}\Gamma(q_n/\gamma_* + 1) \tag{1.20}$$

hold. Then the coefficients $u_n(z')$ of the formal solution $u(z)$ in Theorem 1.6 have the estimate

$$|u_n(z')| \le AB^{q_n}\Gamma(\frac{q_n}{\gamma_*} + 1) \tag{1.21}$$

for some constants A and B.

In the preceding of this section we have treated operators with coefficients in formal series. Hereafter we assume that $a_A(z)$ and $g(z)$ are holomorphic in Ω. So $L(u)$ is *not a formal* operator. Put $\overset{\circ}{\mathcal{S}}(q) = \{(s_A q + d_{A,L} - |A|) - q^*;\ A \in \mathcal{N}^M\} \cup \{1, [q^* + 1] - q^*\}$ and let $\mathcal{S}(q)$ be the additive semigroup generatesd by $\overset{\circ}{\mathcal{S}}(q)$. Put $\mathcal{S} = \mathcal{S}(q)$. In this case Condition 0 is replaced by

CONDITION 0' $\partial_0^k g(0, z') = 0$ *for $k < q^*$.*

If $q^* < 0$, then Condition 0' has no meaning. Suppose $g(z)$ satisfies Condition 0'. We have $g(z) = z_0^{q^*} \sum_{k \ge q*} \partial_0^k g(0, z') z_0^{k-q^*}/k!$. So

$$g(z) = z_0^{q^*} \sum_{n=0}^{+\infty} g_n(z') z_0^{q_n}, \tag{1.22}$$

where if $q_n = k - q^*$ for $k \in \boldsymbol{N}$, $g_n(z') = \partial_0^k g(0, z') z_0^{k-q^*}/k!$, and otherwise $g_n(z') = 0$. We have as an easy consequence of Theorem 1.7

THEOREM 1.8. *Assume that the coefficints $a_A(z)$ of $L(u)$ and $g(z)$ are holomorphic in Ω. Put $\gamma_* = \gamma_{\min,L}(q)$ and suppose that Condition 0' and Conditions 1-3 hold. Then there exists uniquely $u(z) = z_0^q(\sum_{n=0}^{+\infty} u_n(z') z_0^{q_n})$ $\in \mathcal{F}$ satisfing $L(u) = g(z)$ formally with $\partial_1^h u_n(0, z'') = 0$ $(n \ge 1)$ for $0 \le h \le m_{\mathfrak{L}_1} - 1$ and the coefficients have the estimate (1.21).*

The constructed solutions are formal. But they converge in some case. We have

Theorem 1.9. *Suppose that the conditions in Theorem 1.8 hold. If $k_L(q) = m$, then the formal series $u(z)$ in Theorem 1.8 converges in $\{0 < |z_0| < r\} \times \omega$ for some $r > 0$ and it is a genuine solution of $L(u) = g(z)$.*

Remark 1.10. (1) The assumption that the operator $L(z, \partial^\alpha u)$ is a polynomial of $\partial^\alpha u$ is superfluous, if we consider formal solutions with the formal evaluation $q > q_L$, q_L being a constant with $q_L \leq m$ depending on L.
(2) Theorem 1.9 is a typical case in Ishii [2] and the similar result is also obtained in Leichtnam [4], where they constructed convergent solutions. In [2] genuine solutions represented with the series of not only z_0^a but also $(\log z_0)^b$ are constructed. In this paper we don't use $(\log z_0)^b$, because it is not easy to give the meaning of formal solutions. In this paper we treat formal series with respect to one variale z_0. In Gérard and Tahara [1], the similar problem is considered for solutions of formal power series with respect to multi-variable $t \in \boldsymbol{C}^d$, $u(x,t) = \sum_{\{k \in \boldsymbol{N}^d; |k| \geq 1\}} u_k(x) t^k$, of some class of nonlinear partial differntial equations. $u_k(x)$ is determined by solving linear algebraic equations for the equation studied in [1], that is, $C(x,k)u_k(x) = \{\, \textit{the terms determined by}\, u_i(x)\, (0 \leq |i| \leq k-1)\}$. They obtained a Gevrey type estimate for $u(x,t)$ under the Poincaré's condition for functions $C(x,k)$'s. If t is one variable (d=1), the estimate is coincident with that in Theorem 1.8.
(3) We investigate in §2 the Gevrey index γ_* of the formal solution $u(z)$, give it more precisely than that in Theorems 1.7 and 1.8 and determine the best one in some sense (see Theorems 2.7, 2.8 and 2.11).
(4) We construct formal solutions of nonlinear partial differentail equations in this paper. We will investigate the relations between the formal solutions and genuine solutions in the forthcomming paper. For this purpose the best determination of the Gevrey index γ_* will be available.

2. Constrution of formal solutions and improvement of estimates

In §1 we have introduced nonlinear operators with formal series coefficients. Let us write again $L(u)$ considered for the convenience. Let $\mathcal{S}$ be a finitely generated additive semi-group, $\mathcal{S} = \{q_i;\ i \in \boldsymbol{N}\}$, $0 = q_0 < q_1 <$

$\cdots < q_i <\to +\infty$. Let $L(u)$ be a formal nonlinear operator

$$
\begin{aligned}
L(u) &= L(z, \partial^\alpha u;\ |\alpha| \le m) \\
&= \sum_{A\in\mathcal{N}^M} a_A(z) \prod_{i=1}^{s_A} (\partial^{A_i} u) \\
&= \sum_{A\in\mathcal{N}^M} z_0^{d_{A,L}-|A|} b_A(z) \prod_{i=1}^{s_A} (z_0^{A_{i,0}} \partial_0^{A_{i,0}} \partial^{A'_i} u),
\end{aligned} \tag{2.1}
$$

where

$$
a_A(z) = z_0^{j_{A,L}} b_A(z), \quad b_A(z) = \sum_{n=0}^{+\infty} b_{A,n}(z') z_0^{q_n} \in \mathcal{F}_S \tag{2.2}
$$

and $j_{A,L}$ is the formal evaluation of $a_A(z)$.

We treat $L(u)$ under some assumptions. Firstly assume that $L(u)$ satisfies the following Assumption 0.

Assumption 0. $e_L(0) = \min\{d_{A,L} - |A|;\ A \in \mathcal{N}^M\} = 0$ and $e_{\mathcal{L}} = 0$.

Assumption 0 means that $d_{A,L} - |A| = 0$ is attained by some of the linear terms of $L(u)$. We have easily from Assumption 0 and (1.5)-(1.8)

Lemma 2.1. *Suppose that $L(u)$ satisfies Assumption 0. Let $r > 0$. Then $e_L(r) = r$, $\Delta_L(r) = \Delta_{\mathcal{L}}$ and $k_L(r) = k_{\mathcal{L}}$.*

Hence if $r > 0$, $\Delta_L(r)$ and $k_L(r)$ is independent of r and

$$
\begin{cases}
\gamma_{\min,L}(r) = \min\{\dfrac{(s_A-1)r + d_{A,L} - |A|}{k_A - k_{\mathcal{L}}};\ A \in \mathcal{N}^M \ \ \textit{with}\ k_A > k_{\mathcal{L}}\}, \\
((s_A-1)r + d_{A,L} - |A|) \ge \gamma_{\min,L}(r)(k_A - k_{\mathcal{L}}) \quad \textit{for all}\ A \in \mathcal{N}^M
\end{cases} \tag{2.3}
$$

holds. We have by (2.3)

Proposition 2.2. *Suppose that $L(u)$ satisfies Assumption 0. Then $0 < \gamma_{\min,L}(r) \le \gamma_{\min,\mathcal{L}} \le +\infty$ for $r > 0$ and $\gamma_{\min,L}(r) \le \gamma_{\min,L}(r')$ for $0 < r \le r'$.*

Now let us investigate the minimal irregularities of $L(u)$, its linear part $\mathcal{L}(z, \partial)$ and $L^v(u)$. Put $v = v(z')z_0^r$. Then $L^v(u) = L(v(z')z_0^r + u) -$

$L(v(z')z_0^r)$. Let us calculate it. Let $A = (A_1, A_2, \dots, A_{s_A}) \in \mathcal{N}^M$, $\mathcal{I}$ be a subset of $\{1, 2, \dots s_A\}$ and $|\mathcal{I}|$ be the cardinal number of the set $\mathcal{I}$. Put

$$\text{(2.4)} \quad \begin{aligned} L_A^v(u) = z_0^{d_{A,L}-|A|} b_A(z) \sum_{\{\mathcal{I};\, |\mathcal{I}|\geq 1\}} z_0^{(s_A-|\mathcal{I}|)r} \times \\ \{(\prod_{h\notin\mathcal{I}} r(r-1)\dots(r-A_{h,0}+1)\partial^{A'_h} v(z')) \prod_{i\in\mathcal{I}} z_0^{A_{i,0}} \partial^{A_i} u\}. \end{aligned}$$

Then we have

$$\text{(2.5)} \qquad L^v(u) = \sum_A L_A^v(u).$$

Let $\gamma_{\min,L^v}(r')$ be the minimal irregularity for the evaluation r' of the operator of $L^v(u)$ and $\mathcal{L}^v$ be the linear part of $L^v(u)$. Then

Proposition 2.3. *Suppose that $L(u)$ satisfies Assumption 0. Let $v = v(z')z_0^r$, $r > 0$.*

(1) *$e_{L^v}(0) = e_{\mathcal{L}^v} = 0$, that is, L^v satisfies Assumption 0 and $\Delta_{L^v}(0) = \Delta_L(0)$.*
(2) *Let $r' > 0$. Then $e_{L^v}(r') = r'$, $\Delta_{L^v}(r') = \Delta_{\mathcal{L}}$ and $k_{L^v}(r') = k_{\mathcal{L}}$.*
(3) *$\gamma_{\min,L}(r) \leq \gamma_{\min,L^v}(r')$ for $r \leq r'$.*
(4) *Let $w = w(z')z_0^{r'}$ $(r < r')$. If $\gamma_{\min,\mathcal{L}^v} = \gamma_{\min,\, L}(r)$, then $\gamma_{\min,\, \mathcal{L}^{v+w}} = \gamma_{\min,\, L^v}(r') = \gamma_{\min,\mathcal{L}^v}$*

Proof. Let us note the expression of $L_A^v(u)$ by (2.4). We have $(d_{A,L} - |A|) + (s_A - |\mathcal{I}|)r \geq 0$ and the equality holds if and only if $A \in \Delta_L(0)$ and $|\mathcal{I}| = s_A$. Hence (1) is valid. We have, if $s_A > 1$ in (2.4),

$$|\mathcal{I}|r' + (d_{A,L} - |A|) + (s_A - |\mathcal{I}|)r \geq (s_A - |\mathcal{I}|)r + |\mathcal{I}|r' > r'.$$

If $s_A = 1$ in (2.4),

$$|\mathcal{I}|r' + (d_{A,L} - |A|) + (s_A - |\mathcal{I}|)r = d_{A,L} - |A| + r' \geq r'$$

and the equality holds only for $A \in \Delta_{\mathcal{L}}$. Hence we have (2). We have from (2), if $\max\{|A_i|; i \in \mathcal{I}\} > k_{\mathcal{L}}$,

$$\begin{aligned} &(|\mathcal{I}|r' + (d_{A,L} - |A|) + (s_A - |\mathcal{I}|)r - e_{L^v}(r'))/(\max\{|A_i|;\ i \in \mathcal{I}\} - k_{\mathcal{L}}) \\ \geq &((s_A r + d_{A,L} - |A| - r) + (|\mathcal{I}| - 1)(r' - r))/(k_A - k_{\mathcal{L}}) \geq \gamma_{\min,L}(r), \end{aligned}$$

which implies (3). Let us show (4). We note that $(L^v)^w = L^{v+w}$ and $e_{\mathcal{L}} = e_{\mathcal{L}^v} = e_{\mathcal{L}^{v+w}} = 0$. Put

$$\begin{aligned}\mathcal{L}^v_A &= z_0^{d_{A,L}-|A|} b_A(z) \sum_{i=1}^{s_A} z_0^{(s_A-1)r} \\ &\quad \times \{(\prod_{h\neq i} r(r-1)\dots(r-A_{h,0}+1)\partial^{A'_h} v(z')) z_0^{A_{i,0}} \partial^{A_i}\}\end{aligned} \tag{2.6}$$

and

(2.7)

$$\begin{aligned}\mathcal{L}^{v+w}_A =& z_0^{d_{A,L}-|A|} b_A(z) \sum_{i=1}^{s_A} z_0^{(s_A-1)r} [\{\prod_{h\neq i} (r(r-1)\dots(r-A_{h,0}+1)\partial^{A'_h} v(z') \\ &+ r'(r'-1)\dots(r'-A_{h,0}+1) z_0^{r'-r} \partial^{A'_h} w(z'))\} z_0^{A_{i,0}} \partial^{A_i}] \\ =& \mathcal{L}^v_A + \tilde{\mathcal{L}}_A,\end{aligned}$$

where

$$\tilde{\mathcal{L}}_A = \sum_{\alpha} l_{A,\alpha}(z) (z_0\partial_0)^{\alpha_0} \partial'^{\alpha'} \tag{2.8}$$

and *the formal evaluation of* $l_{A,\alpha}(z) \geq d_{A,L} - |A| + (s_A - 1)r + r' - r$. We have $\mathcal{L}^v = \sum_A \mathcal{L}^v_A$ and $\mathcal{L}^{v+w} = \sum_A \mathcal{L}^{v+w}_A = \mathcal{L}^v + \tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}} = \sum_A \tilde{\mathcal{L}}_A$. If $|\alpha| > k_{\mathcal{L}}$ in (2.8), we have from the assumption and $r' > r$

$$\begin{aligned}&(d_{A,L} - |A| + (s_A-1)r + r' - r)/(|\alpha| - k_{\mathcal{L}}) \\ \geq&(d_{A,L} - |A| + (s_A-1)r + r' - r)/(k_A - k_{\mathcal{L}}) \\ >&(d_{A,L} - |A| + (s_A-1)r)/(k_A - k_{\mathcal{L}}) \geq \gamma_{\min L}(r) = \gamma_{\min \mathcal{L}^v}.\end{aligned}$$

Hence it follows from$\mathcal{L}^{v+w} = \mathcal{L}^v + \tilde{\mathcal{L}}$ that $\gamma_{\min,\mathcal{L}^{v+w}}$ is determined by the term of $\mathcal{L}^v$ and $\gamma_{\min,\mathcal{L}^{v+w}} = \gamma_{\min,\mathcal{L}^v}$. On the other hand, from Proposition 2.2 and (3), $\gamma_{\min,\mathcal{L}^v} \geq \gamma_{\min,L^v}(r') \geq \gamma_{\min,L}(r)$. Thus we have (4). □

Now let us consider the equation

$$L(u) = g(z), \quad g(z) = \sum_{n=n'}^{+\infty} g_n(z') z_0^{q_n} \in \mathcal{F}_S \quad n' \geq 1. \tag{2.9}$$

We try to find $u(z)=\sum_{n=n'}^{+\infty}u_n(z')z_0^{q_n}\in\mathcal{F}_S$ which formally satisfies (2.9). Put

(2.10)
$$\begin{aligned}\mathfrak{L}(z',\lambda,\partial')&=\sum_{\alpha\in\Delta_{\mathcal{L}}}b_{\alpha,0}(z')\lambda(\lambda-1)\dots(\lambda-\alpha_0+1)\partial^{\alpha'}\\&=\sum_{|\alpha'|\le m_{\mathfrak{L}}}\mathfrak{L}_{\alpha'}(z',\lambda)\partial^{\alpha'}.\end{aligned}$$

$\mathfrak{L}(z',\lambda,\partial')$ is determined by some terms of the linear part of $L(u)$ and its order is $m_{\mathfrak{L}}$. Let $P.S.\mathfrak{L}(z',\lambda,\xi')=\sum_{|\alpha'|=m_{\mathfrak{L}}}\mathfrak{L}_{\alpha'}(z',\lambda)\xi'^{\alpha'}$ be the principal symbol of $\mathfrak{L}(z',\lambda,\partial')$. It is a polynomial of (λ,ξ') with degree $\mathring{k}_{\mathfrak{L}}\le k_{\mathcal{L}}$. We further assume the following Assumptions 1 and 2:

Assumption 1. $\mathcal{S}\supset\{d_{A,L}-|A|;\ A\in\mathcal{N}^M$ *with* $a_A(z)\not\equiv 0\}$.

Assumption 2. $P.S.\mathfrak{L}(0,\lambda,\hat{\xi}')$, $\hat{\xi}'=(1,0.\dots,0)$, is a polynomial of λ with degree $\mathring{k}_{\mathfrak{L}}-m_{\mathfrak{L}}$ and does not vanish for all $\lambda=q_n,\ n\ge n'$.

Assumption 2 means that $|\mathfrak{L}_{\alpha'}(z',\lambda)\mathfrak{L}_{\hat{\alpha}'}(z',\lambda)|^{-1}$ ($|\alpha'|=m_{\mathfrak{L}},\hat{\alpha}'=(m_{\mathfrak{L}},0,\cdots,0)$) are bounded in a neighbourhood of $z'=0$ for $\lambda=q_n,n\ge n'$.

We have

Theorem 2.4. *Suppose that Assumptions 0–2 hold. Then there exists uniquely* $u(z)=(\sum_{n=n'}^{+\infty}u_n(z')z_0^{q_n})\in\mathcal{F}_S$ *satifying* $L(u)=g(z)$ *formally, where* $\partial_1^h u_n(0,z'')=0$ $(n\ge n')$ *for* $0\le h\le m_{\mathfrak{L}}-1$.

Before the proof we give simple formulae for later calculations. We have for a multi-index $\alpha\in\boldsymbol{N}^{n+1}$

$$\left\{\begin{aligned}&\partial^\alpha(z_0^\lambda w(z'))=(M_\alpha(\lambda,\partial')w(z'))z_0^{\lambda-\alpha_0},\\&M_\alpha(\lambda,\partial')=\lambda(\lambda-1)\cdots(\lambda-\alpha_0+1)\partial^{\alpha'}.\end{aligned}\right.\tag{2.11}$$

So for a formal series $u(z)=\sum_{n=0}^{+\infty}u_n(z')z_0^{q_n}$,

$$\partial^\alpha u(z)=\sum_{n=0}^{+\infty}M_\alpha(q_n,\partial')u_n(z')z_0^{q_n-\alpha_0}.\tag{2.12}$$

Let $A = (A_1, A_2, \cdots, A_s) \in (\boldsymbol{N}^{n+1})^s$. We have

$$\prod_{i=1}^{s_A}(\partial^{A_i}u(z)) = z_0^{l_A-|A|} \{\prod_{i=1}^{s_A}(\sum_{n_i=0}^{+\infty} M_{A_i}(q_{n_i}, \partial')u_{n_i}(z')z_0^{q_{n_i}})\}. \tag{2.13}$$

Expanding $a_A(z)$

$$a_A(z) = z_0^{j_{A,L}} b_A(z) = z_0^{j_{A,L}}(\sum_{n_0=0}^{+\infty} b_{A,n_0}(z')z_0^{q_{n_0}}), \tag{2.14}$$

we have

$$a_A(z)\prod_{i=1}^{s_A}(\partial^{A_i}u(z)) = z_0^{d_{A,L}-|A|}(\sum_{n_0=0}^{+\infty} b_{A,n_0}(z')z_0^{q_{n_0}}) \{\prod_{i=1}^{s_A}(\sum_{n_i=0}^{+\infty} M_{A_i}(q_{n_i}, \partial')u_{n_i}(z')z_0^{q_{n_i}})\}. \tag{2.15}$$

It follows from Assumption 1 that

$$a_A(z)\prod_{i=1}^{s_A}(\partial^{A_i}u(z)) = \sum_{n=0}^{+\infty} L_{A,n}(u)z_0^{q_n}, \tag{2.16}$$

where

$$L_{A,n}(u) = \sum_{\left\{\substack{q_{n_0}+\cdots+q_{n_{s_A}} \\ +d_{A,L}-|A|=q_n}\right\}} b_{A,n_0}(z')(\prod_{i=1}^{s_A} M_{A_i}(q_{n_i}, \partial')u_{n_i}(z')). \tag{2.17}$$

Proof of Theorem 2.4. Now let us return to (2.9). Substituting a formal series $u(z) = \sum_{n=n'}^{+\infty} u_n(z')z_0^{q_n}$ into $L(u)$, we have

$$\begin{cases} L(u) = \sum_{n=0}^{+\infty} L_n(u)z_0^{q_n}, \\ L_n(u) = \sum_A L_{A,n}(u). \end{cases} \tag{2.18}$$

Therefore, in order to satisfy $L(u) = g(z)$ formally, we put the coefficients of $z_0^{q_n}$ in $\{L(u) - g(z)\}$ equal to 0 and we have, by Lemma 2.1,

$$\mathfrak{L}(z', q_n, \partial')u_n(z') + \mathcal{M}_n(u_j(z'); j < n) = g_n(z') \quad (n \geq n'), \tag{2.19}$$

where $\mathcal{M}_n(u_j(z'); j < n)$ is a term determined by $u_j(z')$ $(j < n)$:

$$\mathcal{M}_n(u_j(z'); j < n) = \sum_{\left\{\substack{(q_{n_0}, q_{n_1}, \ldots, q_{n_s}, A);\ ,\ A \in \mathcal{N}^M \\ q_{n_0} + q_{n_1} + \cdots + q_{n_s} + d_{A,L} - |A| = q_n \\ n_i < n \ for \ all \ i \geq 1}\right\}} b_{A,n_0}(z')(\prod_{i=1}^{s_A} M_{A_i}(q_{n_i}, \partial')u_{n_i}(z')). \tag{2.20}$$

So we determine $u_n(z')$ $(n \geq n')$ so that they satisfy

$$\begin{cases} \mathfrak{L}(z', q_n, \partial')u_n(z') + \mathcal{M}_n(u_j(z'); j < n) = g_n(z'), \\ \quad \partial_1^h u_n(z')|_{z_1=0} = 0 \quad for \quad 0 \leq h \leq m_{\mathfrak{L}} - 1. \end{cases} \tag{2.21}$$

It follows from Assumption 2 and the remark following it that $u_n(z')$ $(n \geq n')$ are successively determined by Cauchy Kowalevskja's Theorem, which are holomorphic in a neighbourhood of ω' of $z' = 0$. Thus we have Theorem 2.4. □

Remark 2.5. For the simplicity we put the zero initial conditions to determine $u_n(z')$ in (2.19), but it is obvious that we can give non-zero initial conditions. If $\mathfrak{L}(z, \lambda, \partial')$ is degenerate, by imposing suitable solvability condition on it, we have a formal solution.

Let us proceed to obtain an estimate of the coefficients $u_n(z')$ of $u(z)$ in Theorem 2.4. In order to do so, in addition to Assumptions 0-2, we put the following assumption:

Assumption 3. $P.S.\mathfrak{L}(0, \lambda, \hat{\xi}')$ is a polynomial of λ with degree $k_{\mathcal{L}} - m_{\mathfrak{L}}$.

Assumption 3 means that $P.S.\mathfrak{L}(0,\lambda,\xi')$ is a polynomial of (λ,ξ') with degree $k_{\mathcal{L}}$, that is, $\overset{\circ}{k}_{\mathfrak{L}} = k_{\mathcal{L}}$. We assume that the coefficients $b_{A,n}(z')$ of $b_A(z)$ and $g_n(z')$ of $g(z)$ have a Gevrey type estimate,

$$\left\{ \begin{array}{l} |b_{A,n}(z')| \le B_1^{q_n+1}\Gamma(q_n/\gamma+1) \\ \quad |g_n(z')| \le B_1^{q_n-q_{n'}+1}\Gamma((q_n-q_{n'})/\gamma+1) \quad (n \ge n'), \end{array} \right. \tag{2.22}$$

where $0<\gamma\le+\infty$. If $\gamma=+\infty$, then $b_A(z)$ and $g(z)$ converge in $\{0<|z_0|<r\}$ for some $r>0$.

THEOREM 2.6. *Suppose that Assumptions 0-3 hold and* $\gamma = \gamma_{\min,L}(q_{n'})$ *in (2.22). Then for the coefficients* $u_n(z')$ $(n \ge n')$ *of the formal solution* $u(z) \in \mathcal{F}_S$ *of* $L(u) = g(z)$ *in Theorem 2.4, it holds similarly to (2.22) that*

$$|u_n(z')| \le AB^{q_n-q_{n'}}\Gamma(\frac{q_n-q_{n'}}{\gamma_{\min,L}(q_{n'})}+1) \tag{2.23}$$

for some constants A *and* B.

The proof of Theorem 2.6 is given in §4. Here we show that the estimate of $u_n(z')$ will be improved. Let $u(z)$ be the solution of $L(u)=g(z)$ given in Theorem 2.4. By using the coefficients $\{u_n(z');\ n\ge n'\}$ of $u(z)$, define

$$\left\{ \begin{array}{l} v_l(z) = \displaystyle\sum_{n=n'}^{n'+l-1} u_n(z')z_0^{q_n} \quad \textit{for } l\ge 1, \quad v_0(z)=0, \\ w_l(z) = \displaystyle\sum_{n=n'+l}^{+\infty} u_n(z')z_0^{q_n} \end{array} \right. \tag{2.24}$$

and

$$\left\{ \begin{array}{l} L(v_l;w) = L^{v_l}(w) = L(w+v_l(z)) - L(v_l(z)), \\ \quad h_l(z) = g(z) - L(v_l(z)) = \displaystyle\sum_{n=n'+l}^{+\infty} h_{l,n}(z')z_0^{q_n}. \end{array} \right. \tag{2.25}$$

Now consider the equation of w

$$L(v_l;w) = h_l(z). \tag{2.26}$$

$w(z) = w_l(z)$ is a unique formal solution satisfying $\partial_1^h u_n(0, z'') = 0$ $(n \geq n' + l)$ for $0 \leq h \leq m_{\mathfrak{L}} - 1$. It is not difficult to show by Propsition 2.3 and $L^{v+w} = (L^v)^w$ that

$$(2.27) \qquad \begin{cases} e_{L^{v_l}}(0) = e_{\mathcal{L}^{v_l}} = 0 \\ \Delta_{\mathcal{L}} = \Delta_{\mathcal{L}^{v_l}} \\ \mathfrak{L}(z', \lambda, \partial') = \mathfrak{L}(v_l; z', \lambda, \partial'), \end{cases}$$

where $\mathfrak{L}(v_l; z', \lambda, \partial')$ is that defined by (2.10) for $L(v_l; \cdot\ )$, and $L(v_l; \cdot\ )$ satisfies Assumptions 0-3. By applying Theorem 2.6 to the equation (2.26), we have

Theorem 2.7. *Suppose that Assumptions 0-3 hold and* $\gamma = \gamma_{\min, L^{v_l}}(q_{n'+l})$ *in (2.22). Then for the coefficients* $u_n(z')$ $(n \geq n')$ *of the formal solution* $u(z) \in \mathcal{F}_S$ *of* $L(u) = g(z)$ *in Theorem 2.4, it holds that*

$$(2.28) \qquad |u_n(z')| \leq AB^{q_n - q_{n'}} \Gamma(\frac{q_n - q_{n'}}{\gamma_{\min, L^{v_l}}(q_{n'+l})} + 1)$$

for other constants A *and* B.

We have from the above Theorem

Theorem 2.8. *Suppose that Assumptions 0-3 hold and* $\gamma = +\infty$ *in (2.22). Let* $u_n(z')$ $(n \geq n')$ *be the coefficients of the formal solution* $u(z) \in \mathcal{F}_S$ *of* $L(u) = g(z)$ *in Theorem 2.4.*

(1) For each $l \in \boldsymbol{N}$ *there are* A_l *and* B_l *such that*

$$(2.29) \qquad |u_n(z')| \leq A_l B_l^{q_n - q_{n'}} \Gamma(\frac{q_n - q_{n'}}{\gamma_{\min, L^{v_l}}(q_{n'+l})} + 1).$$

(2) If there is an $N \in \boldsymbol{N}$ *such that* $\gamma_{\min, L^{v_N}}(q_{n'+N}) = \gamma_{\min, \mathcal{L}^{v_{N+1}}}$, *then there are* A *and* B *such that*

$$(2.30) \qquad |u_n(z')| \leq AB^{q_n - q_{n'}} \Gamma(\frac{q_n - q_{n'}}{\gamma_{\min, \mathcal{L}^{v_{N+1}}}} + 1).$$

Suppose that the assumption in (2) in Theorem 2.8 holds. Put $R = L^{v_N}$, $v = u_{n'+N}(z') z_0^{q_{n'+N}}$ and $w = u_{n'+N+1}(z') z_0^{q_{n'+N+1}}$. The assumption means $\gamma_{\min,R}(q_{n'+N}) = \gamma_{\min,\mathcal{R}^v}$. Then it follows from Propsition 2.3 that $\gamma_{\min,R^v}(q_{n'+N+1}) = \gamma_{\min,\mathcal{R}^{v+w}} = \gamma_{\min,\mathcal{R}^v}$, which means $\gamma_{\min,L^{v_{N+1}}}(q_{n'+N+1}) = \gamma_{\min,\mathcal{L}^{v_{N+2}}} = \gamma_{\min,\mathcal{L}^{v_{N+1}}}$. By the induction we have $\gamma_{\min,L^{v_n}}(q_{n'+n}) = \gamma_{\min,\mathcal{L}^{v_{n+1}}} = \gamma_{\min,\mathcal{L}^{v_{N+1}}}$ for $n \geq N$. So the estimate (2.30) may be considered to be best in some sense.

Let us apply Theorems in this section to show Theorems in §1. Let $L(u)$ be a nonlinear operator defined by (1.1) and (1.2) in §1. Define

$$P(u) = z_0^{-q^*}\{L((u+u_0)z_0^q) - L(u_0 z_0^q)\} = z_0^{-q^*} L^v(z_0^q u), \quad v = u_0 z_0^q. \tag{2.31}$$

Let us calculate $P(u)$. Let $A = (A_1, A_2, \dots, A_{s_A}) \in \mathcal{N}^M$, $\mathcal{I}$ be a subset of $\{1, 2, \dots, s_A\}$ and $|\mathcal{I}|$ be the cardinal number of the set $\mathcal{I}$. Put

$$\begin{aligned} P_A(u) &= z_0^{d_{A,L}-|A|-q^*} b_A(z) \sum_{\{\mathcal{I};\, |\mathcal{I}|\geq 1\}} z_0^{(s_A-|\mathcal{I}|)q} \\ &\times\{(\prod_{h\notin\mathcal{I}} q(q-1)\dots(q-A_{h,0}+1)\partial^{A'_h} u_0(z')) \prod_{i\in\mathcal{I}} z_0^{A_{i,0}} \partial_0^{A_{i,0}} \partial^{A'_i}(z_0^q u(z))\}. \end{aligned} \tag{2.32}$$

Then we have

$$P(u) = \sum_A P_A(u). \tag{2.33}$$

$\mathcal{P}(z,\partial)$, the linear part of $P(u)$, is

$$\begin{aligned} \mathcal{P}(z,\partial) &= \sum_A z_0^{d_{A,L}-|A|-q^*+(s_A-1)q} b_A(z) \\ &\times \{\sum_{i=1}^{s_A} (\prod_{h\neq i} q(q-1)\dots(q-A_{h,0}+1)\partial^{A'_h} u_0(z')) z_0^{A_{i,0}} \partial_0^{A_{i,0}} \partial^{A'_i}(z_0^q\ \cdot\ )\}. \end{aligned} \tag{2.34}$$

We note that

$$\gamma_{\min,L^v}(q+r) = \gamma_{\min,P}(r), \quad \gamma_{\min,\mathcal{L}^v} = \gamma_{\min,\mathcal{P}} \quad (v = u_0(z') {z_0}^q). \tag{2.35}$$

Now suppose that $L(u)$ satisfies Conditions 0-2 in §1 and let $u_0(z')$ be that in Condition 1. Put

(2.36)
$$\mathcal{P}_0(z,\partial)=\sum_{A\in\Delta_L(q)} z_0^{-q}b_{A,0}(z')\times\{\sum_{i=1}^{s_A}(\prod_{h\neq i} q(q-1)\dots(q-A_{h,0}+1)\partial^{A'_h}u_0(z'))z_0^{A_{i,0}}\partial_0^{A_{i,0}}\partial^{A'_i}(z_0^q\ \cdot\ )\}$$

and

(2.37)
$$\mathfrak{P}(z',\lambda,\partial')=\sum_{A\in\Delta_L(q)} b_{A,0}(z')\{\sum_{i=1}^{s_A}(\prod_{h\neq i} q(q-1)\dots(q-A_{h,0}+1)\partial^{A'_h}u_0(z'))\times(\lambda+q)(\lambda+q-1)\cdots(\lambda+q-A_{i,0}+1)\partial^{A'_i}\}.$$

Let $m_{\mathfrak{P}}$ be the order of $\mathfrak{P}(z',\lambda,\partial')$. We have from (1.16), (1.17) and (1.19)

$$\mathfrak{P}(z',\lambda,\partial')=\mathfrak{L}_1(z',\lambda+q,\partial'). \tag{2.38}$$

So $m_{\mathfrak{P}}=m_{\mathfrak{L}_1}$. Before the proofs of Theorems 1.6 and 1.7 we give Propositions.

Proposition 2.9. *Suppose that $L(u)$ satisfies Conditions 0-2. Then $P(u)$ satisfies Assumptions 0-2. Moreover if $L(u)$ satisfies Condition 3, then $P(u)$ satisfies Assumption 3.*

Proof. Firstly we note that Condition 2 means $\mathcal{P}_0(z,\partial)\not\equiv 0$. So $\mathcal{P}(z,\partial)\not\equiv 0$. We show Assumption 0. We have $e_P(0)\geq 0$ by $s_Aq+d_{A,L}-|A|-q^*\geq 0$ and the above remark implies $e_{\mathcal{P}}=0$. Condition 0 means Assumption 1. $\mathfrak{P}(z',\lambda,\partial')$ is nothing but defined by formula (2.10) by replacing $L(u)$ by $P(u)$. Condition 2 and (2.38) mean Assumption 2. We have $m_{\mathfrak{P}}=m_{\mathfrak{L}_1}\leq k_{\mathcal{P}}\leq k_L(q)$. If Condition 3 holds, $k_{\mathcal{P}}=k_L(q)$ and Assumption 3 holds. □

Proposition 2.10. *Suppose that $L(u)$ satisfies Conditions 0-3. Let $v=u_0(z')z_0^q$ and $r>0$. Then the following holds:*

(1) $\gamma_{\min,L}(q)\leq\gamma_{\min,P}(r)$.
(2) *If* $\gamma_{\min,L}(q)=\gamma_{\min,\mathcal{L}^v}$, *then* $\gamma_{\min,P}(r)=\gamma_{\min,\mathcal{P}}$.

Proof. By Lemma 2.1 and the proof of Proposition 2.9 $e_P(r) = r$ and $k_{\mathcal{P}} = k_P(r) = k_L(q)$. If $\max\{|A_i|; i \in \mathcal{I}\} > k_P(r)$ in (2.32),

$$(|\mathcal{I}|r + s_A q + d_{A,L} - |A| - q^* - e_P(r))/(\max\{|A_i|; i \in \mathcal{I}\} - k_P(r))$$
$$\geq (s_A q + d_{A,L} - |A| - q^*)/(k_A - k_L(q)) \geq \gamma_{\min,L}(q).$$

Hence $\gamma_{\min,P}(r) \geq \gamma_{\min,L}(q)$. We show (2). By the definition of $P(u)$, we have $\gamma_{\min,\mathcal{P}} = \gamma_{\min,\mathcal{L}^v}$. By Proposition 2.2 $\gamma_{\min,P}(r) \leq \gamma_{\min,\mathcal{P}}$. Since $\gamma_{\min,L}(q)$
$\leq \gamma_{\min,P}(r) \leq \gamma_{\min,\mathcal{P}} = \gamma_{\min,\mathcal{L}^v}$, we have (2). □

Proof of Theorems 1.6 and 1.7. $L(u) = g(z)$ is transformed to $P(w) = f(z)$, where by Condition 0

$$f(z) = z_0^{-q^*}(g(z) - L(u_0(z')z_0^q)) = \sum_{n=1}^{+\infty} f_n(z')z_0^{q_n}. \tag{2.39}$$

So consider $P(w) = f(z)$. It follows from Theorem 2.4 that there exists a formal solution $w(z) = \sum_{n=1}^{+\infty} u_n(z')z_0^{q_n}$ of $P(w) = f(z)$. Hence $u(z) = z_0^q(u_0(z') + w(z))$ is a desired solution, which shows Theorem 1.6. If Condition 3 holds, it follows from Theorem 2.6 that

$$|u_n(z')| \leq AB^{q_n - q_1}\Gamma(\frac{q_n - q_1}{\gamma_{\min,P}(q_1)} + 1). \tag{2.40}$$

Since $\gamma_{\min,L}(q) \leq \gamma_{\min,P}(q_1)$ by Proposition 2.10-(1), we have Theorem 1.7. □

The estimate in Theorem 1.7 is improved by Theorem 2.8. Define for a formal series $u(z) = z_0^q(\sum_{n=0}^{+\infty} u_n(z')z_0^{q_n})$ with the formal evaluation q

$$\begin{cases} v_0(z) \equiv 0, \quad v_l(z) = \displaystyle\sum_{n=1}^{l} u_n(z')z_0^{q_n}, \\ u_{-1}^*(z) \equiv 0 \quad u_l^*(z) = z_0^q(u_0(z) + v_l(z)) \quad l \geq 0. \end{cases} \tag{2.41}$$

Theorem 2.11. *Suppose that Conditions 0-3 in §1 hold and let $a_A(z)$ and $g(z)$ be holomorphic in Ω. Furthermore assume that there is an $N \in \boldsymbol{N}$*

such that $\gamma_{\min, L^{u^*_{N-1}}}(q+q_N)=\gamma_{\min,\mathcal{L}^{u^*_N}}$. *Then it holds for the coefficients* $u_n(z')$ $(n\geq 0)$ *of the formal solution* $u(z)\in\mathcal{F}_S$ *of* $L(u)=g(z)$ *in Theorem 1.6 that*

$$|u_n(z')|\leq AB^{q_n}\Gamma(\frac{q_n}{\gamma_{\min,\mathcal{L}^{u^*_N}}}+1). \tag{2.42}$$

Proof. We note that if $l\geq 0$, $\gamma_{\min,L^{u^*_l}}(q+q_{l+1})=\gamma_{\min,P^{v_l}}(q_{l+1})$ and $\gamma_{\min,\mathcal{L}^{u^*_l}}=\gamma_{\min,\mathcal{P}^{v_l}}$. Assume $N\geq 1$. Then

$$\gamma_{\min,P^{v_{N-1}}}(q_N)=\gamma_{\min,L^{u^*_{N-1}}}(q+q_N)=\gamma_{\min,\mathcal{L}^{u^*_N}}=\gamma_{\min,\mathcal{P}^{v_N}}.$$

We apply Thorem 2.8-(2) to $P(w)=f(z)$. By replacing N by $N-1$ and putting $n'=1$, we have

$$|u_n(z')|\leq AB^{q_n-q_1}\Gamma(\frac{q_n-q_1}{\gamma_{\min,\mathcal{P}^{v_N}}})=AB^{q_n-q_1}\Gamma(\frac{q_n-q_1}{\gamma_{\min,\mathcal{L}^{u^*_N}}}).$$

Assume $N=0$. Then the assumption means $\gamma_{\min,L}(q)=\gamma_{\min,\mathcal{L}^{u^*_0}}$. We have from Proposition 2.10-(2) $\gamma_{\min,L}(q)=\gamma_{\min,\mathcal{L}^{u^*_0}}=\gamma_{\min,\mathcal{P}}=\gamma_{\min,P}(q_1)$. Hence the assertion follows. □

We can say under the conditions of Theorem 2.11 that the class, so called Gevrey class, to which the formal solution $u(z)$ of $L(u)=g(z)$ belongs is determined by $\gamma=\gamma_{\min,\mathcal{L}^{u^*_N}}$ which is minimal irregularity of the linearization of $L(u)$ at $u^*_N(z)$.

Definition 2.12. Suppose that Condition 0-3 holds. Let $u(z)=z_0^q(\sum_{n=0}^{+\infty}u_n(z')z_0^{q_n})$ be a formal series of solution of $L(u)=g(z)$ with the formal evaluation q. Put $n_*=\min\{l\in\boldsymbol{N};\gamma_{\min,L^{u^*_{l-1}}}(q+q_l)=\gamma_{\min,\mathcal{L}^{u^*_l}}\}$. Then we call $q+q_{n_*}$ $(+\infty$, if $n_*=+\infty)$ the true evalution of formal solution $u(z)$. If $n_*>0$, the formal evaluation q is called apparent.

3. Majorant functions

In order to obtain estimates of $u_n(z')(n\geq 0)$ constructed in §2, that is, to show Theorem 2.6 we prepare majorant functions. For formal series of s-variables $w=(w_1,\dots,w_s)$ $A(w)=\sum_{\alpha\in\boldsymbol{N}^s}A_\alpha w^\alpha$ and $B(w)=$

$\sum_{\alpha\in \boldsymbol{N}^s} B_\alpha w^\alpha$, $A(w) \ll B(w)$ means $|A_\alpha| \le B_\alpha$ for all $\alpha \in \boldsymbol{N}^s$ and $A(w) \gg 0$ means $A_\alpha \ge 0$ for all $\alpha \in \boldsymbol{N}^s$.

Put

$$\phi(t) = c\sum_{n=0}^{+\infty} t^n/(n+1)^2, \tag{3.1}$$

which was used in Lax [3] and Wagschal [5] and the following Lemma 3.1 is stated there.

Lemma 3.1. *There is a positive constant c such that $\phi(t)\phi(t) \ll \phi(t)$.*

Proof. We have

$$\phi(t)\phi(t) = c^2\sum_{n=0}^{+\infty}\Big(\sum_{l+m=n}\frac{1}{(l+1)^2(m+1)^2}\Big)t^n$$

and

$$\sum_{l+m=n}\frac{1}{(l+1)^2(m+1)^2} \le \frac{A}{(n+1)^2} \quad \text{for some } A > 0.$$

So if $cA \le 1$, we have $\phi(t)\phi(t) \ll \phi(t)$. □

Fix $c > 0$ so that $\phi(t)^2 \ll \phi(t)$. Put $\phi_R(t) = \phi(t/R)$ for $R > 0$. We have

Lemma 3.2. *The following majorant estimates hold:*

$$\phi_R(t)^2 \ll \phi_R(t), \tag{3.2}$$

$$\phi_{R_1}(t) \ll \phi_{R_2}(t) \quad \textit{for} \quad 0 < R_2 < R_1, \tag{3.3}$$

$$\frac{d}{dt}\phi_R(t) \ll \frac{1}{R-t}\phi_R(t), \tag{3.4}$$

$$\frac{1}{R'-t} \ll C(R,R')\phi_R(t) \quad \textit{for} \quad 0 < R < R', \tag{3.5}$$

$$\frac{d}{dt}\phi_R(t) \ll C(R,r)\phi_r(t) \quad \textit{for} \quad 0 < r < R. \tag{3.6}$$

Proof. The estimates (3.2) and (3.3) are obvious. We show (3.4). We have

$$\frac{d}{dt}\phi_R(t) = \frac{c}{R}\sum_{n=0}^{+\infty}\frac{(n+1)}{(n+2)^2}\frac{t^n}{R^n}$$

and

$$(R-t)^{-1}\phi_R(t) = \frac{c}{R}\sum_{n=0}^{+\infty}(\sum_{k=0}^{n}\frac{1}{(k+1)^2})\frac{t^n}{R^n}.$$

Since

$$\sum_{k=0}^{n}\frac{1}{(k+1)^2} \geq \frac{1}{n+1} > \frac{n+1}{(n+2)^2},$$

we have (3.4). We show (3.5). Since $(R'-t)^{-1} = (\sum_{n=0}^{+\infty}(t/R')^n)/R'$ and $(1/R')^{n+1} \leq C/R^n(n+1)^2$ for some $C = C(R,R') > 0$, we have (3.5). We have from (3.3), (3.4) and (3.5)

$$\frac{d}{dt}\phi_R(t) \ll \frac{\phi_R(t)}{R-t} \ll \frac{\phi_r(t)}{R-t} \ll C(R,r)\phi_r(t)\phi_r(t) \ll C(R,r)\phi_r(t),$$

which implies (3.6). □

Now let $\{r_k > 0;\ -k_0 \leq k \leq 0\}$ and $R > 0$ such that $0 < r = r_0 < r_{-1} < \ldots < r_{-k_0} < R < 1$, where $k_0 \in \boldsymbol{N}$ will be concretely chosen in §4 and be fixed. Put

$$(3.7)\qquad \begin{cases} \theta_k(t) = \phi(\dfrac{t}{r_k}) \quad for\ -k_0 \leq k \leq 0 \\ \theta_k(t) = \dfrac{1}{(r-t)^k}\phi(\dfrac{t}{r}) \quad for\ k > 0. \end{cases}$$

We have

$$(3.8)\qquad \begin{cases} \theta_k(t) \ll \theta_l(t) \quad for\ k \leq l \\ \theta_k(t)\theta_l(t) \ll \theta_{k+l}(t) \quad for\ k,l \geq 0 \end{cases}$$

and

Proposition 3.3. *The following majorant estimates hold:*

$$(3.9)\qquad (r-t)\theta_k(t) \gg 0 \quad for\ k > 0,$$

$$(3.10)\qquad k\theta_{k+1}(t) \ll \frac{d}{dt}\theta_k(t) \ll (k+1)\theta_{k+1}(t) \quad for \quad k \geq 0,$$

$$(3.11)\qquad \frac{d}{dt}\theta_k(t) \ll C\theta_{k+1}(t) \quad for\ -k_0 \leq k \leq 0,$$

$$(3.12)\qquad \theta_k(t)\theta_l(t) \ll \theta_s(t),\ s = \max\{k,l,k+l\}.$$

PROOF. The estimate (3.9) is obvious. Let us show (3.10). By (3.4) we have for $k \geq 0$

$$\begin{aligned}\frac{d}{dt}\theta_k(t) &= \frac{k}{(r-t)^{k+1}}\phi_r(t) + \frac{1}{(r-t)^k}\frac{d}{dt}\phi_r(t)\\ &\ll k\theta_{k+1}(t) + \frac{1}{(r-t)^{(k+1)}}\phi_r(t)\\ &= (k+1)\theta_k(t)\end{aligned}$$

and

$$\frac{d}{dt}\theta_k(t) \gg k\theta_{k+1}(t).$$

We have (3.11) from (3.6). We show (3.12). If $k, l \geq 0$, it follows from (3.8). Suppose $k \geq 0 > l$. Then we have

$$\theta_k(t)\theta_l(t) = \frac{1}{(r-t)^k}\phi(\frac{t}{r})\phi(\frac{t}{r_l}) \ll \frac{1}{(r-t)^k}\phi(\frac{t}{r})\phi(\frac{t}{r}) \ll \frac{1}{(r-t)^k}\phi(\frac{t}{r}).$$

When $k, l < 0$, we have (3.12) easily from (3.3). □

4. Estimates

The aim of §4 is to show Theorem 2.6, that is, to give estimates of $\{u_n(z');\ n \geq n'\}$ constructed in §2. Put $a = \max\{2,\ m/q_1\}$ and $-k_0 = [-am]$ in the definition of $\{\theta_k(t);\ -k_0 \leq k < +\infty\}$ in (3.7). We assume Assumptions 0-3 given in §2. So we have $\gamma_{\min,L}(q_{n'}) = \min\{((s_A-1)q_{n'} + d_{A,L} - |A|)/(k_A - k_{\mathcal{L}});\ A \textit{ with } k_A > k_{\mathcal{L}}\}$ (see (2.3)). Let $\{e_i > 0;\ 1 \leq i \leq l\}$ be the generators of $\mathcal{S}$. Put $b = m + l + 1$. In this section we assume $b_{A,n}(z')$ and $g_n(z')$ are holomorphic on $\{z'; |z'| \leq R\}$ and have the following bounds (see (2.22)):

$$(4.1)\qquad \begin{cases} |b_{A,n}(z')| \leq \dfrac{B_1^{q_n+1}}{(q_n+1)^b}\Gamma(q_n/\gamma+1) \\ |g_n(z')| \leq \dfrac{B_1^{q_n-q_{n'}+1}}{(q_n+1)^b}\Gamma((q_n-q_{n'})/\gamma+1) \quad \textit{for } n \geq n', \end{cases}$$

where $\gamma = \gamma_{\min,L}(q_{n'})$.

Theorem 4.1. *There are positive constants C and B such that*

$$u_n(z') \ll \frac{CB^{q_n-q_{n'}}}{(q_n+1)^b}\Gamma(\frac{q_n-q_{n'}}{\gamma}+1)\theta_{[a(q_n-m)]}(t) \quad for \quad n \geq n' \tag{4.2}$$

holds in a neighbourhood of ω of $z'=0$, where $t=\rho z_1+z_2+\cdots+z_n$ for some $\rho>1$.

Since $|\theta_n(t)| \leq C^{n+1}$ $(n\geq 0)$ for $|t|\leq r/2$, the estimate in Theorem 2.6 immediately follows from Theorem 4.1.

In order to show Theorem 4.1, we need some lemmas. In the following $\rho>1$ is some constant. Put for $n\geq n'$

$$\Theta_n(t) = CB^{q_n-q_{n'}}(q_n+1)^{-b}\Gamma(\frac{q_n-q_{n'}}{\gamma}+1)\theta_{[a(q_n-m)]}(t), \tag{4.3}$$

where $t=\rho z_1+z_2+\cdots+z_n$.

Lemma 4.2. *The following majorant estimate holds:*

$$M_\alpha(q_n,\partial')\theta_{[a(q_n-m)]}(t) \ll C_0(q_n+1)^{|\alpha|}\theta_{[a(q_n-m)]+|\alpha|}(t), \tag{4.4}$$

where C_0 depends only on ρ.

Proof. We have, by (2.11) and Proposition 3.3,

$$\begin{aligned}&M_\alpha(q_n,\partial')\theta_{[a(q_n-m)]}(t)\\&=\lambda(\lambda-1)\dots(\lambda-\alpha_0+1)\partial^{\alpha'}\theta_{[a(q_n-m]}(t)|_{\lambda=q_n}\\&\ll C_0(q_n+1)^{|\alpha|}\theta_{[a(q_n-m)]+|\alpha|}(t). \quad \square\end{aligned}$$

By Lemma 4.2 we have

Lemma 4.3. *Let $A=(A_1,A_2,\dots,A_{s_A})\in\mathcal{N}^M$. Then*

(4.5)

$$\begin{aligned}&\prod_{i=1}^{s_A} M_{A_i}(q_{n_i},\partial')\Theta_{n_i}(t) \ll (CC_0)^{s_A}B^{q'}\\&\quad\times\prod_{i=1}^{s_A}(q_{n_i}+1)^{|A_i|-b}\prod_{i=1}^{s_A}\Gamma(\frac{q_{n_i}-q_{n'}}{\gamma}+1)(\prod_{i=1}^{s_A}\theta_{[a(q_{n_i}-m)]+|A_i|}(t)),\end{aligned}$$

where $q' = \sum_{i=1}^{s_A}(q_{n_i} - q_{n'})$.

LEMMA 4.4. *The following inequalities hold:*

$$\Gamma(x+1)\Gamma(y+1) \le \Gamma(x+y+1) \quad \textit{for } x, y \ge 0, \tag{4.6}$$

$$\Gamma(x-y+1) \le K\frac{\Gamma(x+1)}{(x-y+1)^y} \quad \textit{for } x \ge y \ge 0, \tag{4.7}$$

where $K > 0$ *is a constant independent of* x *and* y.

PROOF. If $y = 0$, then the inequalities are trivial. We assume that $x \ge y > 0$. It holds that

$$\begin{aligned}\Gamma(x+1)\Gamma(y+1) &= B(x+1, y+1)\Gamma(x+y+2)\\ &= (x+y+1)B(x+1, y+1)\Gamma(x+y+1),\end{aligned}$$

where

$$B(x+1, y+1) = \int_0^1 t^x(1-t)^y dt.$$

We have

$$B(x+1, y+1) = \frac{y}{x+y+1}\int_0^1 t^x(1-t)^{y-1}dt \le \frac{1}{x+y+1},$$

from which (4.6) follows. We show (4.7). It follows from Stirling's formula that there exists constant $K > 0$ such that

$$(x+1)^\delta\Gamma(x+1) \le K\Gamma(x+\delta+1) \quad \textit{for} \quad x \ge 0 \quad \textit{and} \quad 0 \le \delta < 1.$$

Hence we have

$$\begin{aligned}\Gamma(x+1) &= x(x-1)\dots(x-[y]+1)\Gamma(x-[y]+1)\\ &\ge K^{-1}x(x-1)\dots(x-[y]+1)(x-y+1)^{y-[y]}\Gamma(x-y+1)\\ &\ge K^{-1}(x-y+1)^y\Gamma(x-y+1). \ \square\end{aligned}$$

Lemma 4.5. *Suppose that* $d_{A,L} - |A| + q_{n_0} + \sum_{i=1}^{s_A} q_{n_i} = q_n$. *Then*

$$\Gamma(\frac{q_{n_0}}{\gamma}+1)\prod_{i=1}^{s_A}\Gamma(\frac{q_{n_i}-q_{n'}}{\gamma}+1) \le C'\Gamma((q_n-q_{n'})/\gamma+1)(q_n+1)^{k_{\mathcal{L}}-k_A} \tag{4.8}$$

for some $C' > 0$.

Proof. It holds that $s_A q_{n'} + d_{A,L} - |A| - q_{n'} \ge \gamma(k_A - k_{\mathcal{L}})$, $\gamma = \gamma_{\min,L}(q_{n'})$ (see (2.3)). So $q_n - q_{n'} - \gamma(k_A - k_{\mathcal{L}}) \ge q_{n_0} + \sum_{i=1}^{s_A}(q_{n_i} - q_{n'})$. If $k_A \ge k_{\mathcal{L}}$, by Lemma 4.4

$$\begin{aligned}\Gamma(\frac{q_{n_0}}{\gamma}+1)\prod_{i=1}^{s_A}\Gamma(\frac{q_{n_i}-q_{n'}}{\gamma}+1) &\le C''\Gamma((q_n-q_{n'})/\gamma-k_A+k_{\mathcal{L}}+1)\\ &\le C'\Gamma((q_n-q_{n'})/\gamma+1)(q_n+1)^{k_{\mathcal{L}}-k_A}.\end{aligned}$$

If $k_A < k_{\mathcal{L}}$, it is easy to show the assertion. □

Lemma 4.6. *Suppose that* $d_{A,L} - |A| + q_{n_0} + \sum_{i=1}^{s_A} q_{n_i} = q_n$ *and* $q_n > q_{n_i}$ *for all* $i > 0$. *Put* $q' = \sum_{i=1}^{s_A}(q_{n_i} - q_{n'})$. *Then*

$$q' \le (q_n - q_{n'}) - q_{n_0}/2 - (d_{A,L} - |A|)/2 - q_1/2. \tag{4.9}$$

Proof. We have $q' = q_n - s_A q_{n'} - (q_{n_0} + d_{A,L} - |A|)$. Suppose $d_{A,L} - |A| > 0$. Then $d_{A,L} - |A| \ge q_1$ and $q' \le q_n - q_{n'} - q_{n_0} - (d_{A,L} - |A|)/2 - q_1/2$. Suppose $d_{A,L} - |A| = 0$. Then $q' = q_n - s_A q_{n'} - q_{n_0}$. If $s_A = 1$, $q_{n_0} \ge q_1$ and $q' \le q_n - q_{n'} - q_{n_0}/2 - q_1/2$. If $s_A \ge 2$, $q' \le q_n - 2q_{n'} - q_{n_0} \le q_n - q_{n'} - q_1/2 - q_{n_0}/2$. This completes the proof. □

Lemma 4.7. *Suppose that* $q_n > m$, $d_{A,L} - |A| + q_{n_0} + \sum_{i=1}^{s_A} q_{n_i} = q_n$ *and* $q_n > q_{n_i}$ *for all* $i > 0$. *Then*

$$\prod_{i=1}^{s_A}\theta_{[a(q_{n_i}-m)]+m}(t) \ll \theta_{[a(q_n-m)]}(t). \tag{4.10}$$

PROOF. Put $N_1 = \{i > 0; a(q_{n_i} - m) + m > 0\}$ and $N_2 = \{i > 0; a(q_{n_i} - m) + m \le 0\}$. Put $s_1 = \#N_1$ and $s_2 = \#N_2$. We have $s_A = s_1 + s_2$. If $s_1 = 0$, since $q_n > m$, the assertion is obvious. Suppose that $s_1 \ge 1$. Then by Proposition 3.3

$$\begin{aligned}\prod_{i=1}^{s_A} \theta_{[a(q_{n_i}-m)]+m}(t) &\ll \prod_{i\in N_1} \theta_{[a(q_{n_i}-m)]+m}(t) \\ &\ll \theta_{\sum_{i\in N_1}([a(q_{n_i}-m)]+m)}(t). \end{aligned} \tag{4.11}$$

We have

$$\begin{aligned}\sum_{i\in N_1} ([a(q_{n_i} - m)] + m) &\le \sum_{i\in N_1} a(q_{n_i} - m) + s_1 m \\ = aq_n - a(q_{n_0} + d_{A,L} - |A|) &- a\sum_{i\in N_2} q_{n_i} + s_1 m(1-a).\end{aligned}$$

We show

$$\begin{aligned}&aq_n - a(q_{n_0} + d_{A,L} - |A|) - a\sum_{i\in N_2} q_{n_i} + s_1 m(1-a) \\ &\le a(q_n - m).\end{aligned} \tag{4.12}$$

If the inequality (4.12) holds, the majorant estimate (4.10) easily follows from Proposition 3.3. We note $a = \max\{2, m/q_1\}$. The above inequality is equivalent to

$$a \ge \frac{s_1 m}{(s_1 - 1)m + q_{n_0} + d_{A,L} - |A| + \sum_{i\in N_2} q_{n_i}}.$$

If $s_1 \ge 2$,

$$\frac{s_1 m}{(s_1 - 1)m + q_{n_0} + d_{A,L} - |A| + \sum_{i\in N_2} q_{n_i}} \le \frac{s_1 m}{(s_1 - 1)m} \le 2 \le a.$$

If $s_1 = 1$, say $N_1 = \{1\}$, we have $q_{n_0} + d_{A,L} - |A| + \sum_{i\in N_2} q_{n_i} = q_n - q_{n_1} > 0$. Hence $q_{n_0} + d_{A,L} - |A| + \sum_{i\in N_2} q_{n_i} \ge q_1$ and

$$\frac{m}{q_{n_0} + d_{A,L} - |A| + \sum_{i\in N_2} q_{n_i}} \le \frac{m}{q_1} \le a.$$

Thus (4.10) is valid. □

Lemma 4.8. *There exists $C_1 > 0$ such that*

$$\sum_{\left\{ \substack{(q_{n_0}, q_{n_1}, \dots, q_{n_s}) \\ q_{n_0}+q_{n_1}+\cdots+q_{n_s}=d} \right\}} \prod_{i=0}^{s} \frac{1}{(q_{n_i}+1)^{b-k_A}} \le \frac{C_1^s}{(d+1)^{b-k_A}}. \tag{4.13}$$

Proof. Let $\{e_i > 0; 1 \le i \le l\}$ be generators of $\mathcal{S}$. Hence $q_n = a_{1,n}e_1 + a_{2,n}e_2 + \ldots + a_{l,n}e_l$ for some $a_{i,n} \in \boldsymbol{N}$. We have $b = m + l + 1 \ge k_A + l + 1$. Put $e_0 = \min\{e_i; 1 \le i \le l\}$ and $b' = b - k_A$. Firstly we show

$$\sum_{k=0}^{+\infty} \frac{1}{(q_k+1)^{b'}} < +\infty. \tag{4.14}$$

We have

$$\begin{aligned}\sum_{k=0}^{+\infty} \frac{1}{(q_k+1)^{b'}} &= \sum_{k=0}^{+\infty} \frac{1}{(a_{1,k}e_1 + a_{2,k}e_2 + \ldots + a_{l,k}e_l + 1)^{b'}} \\ &\le \sum_{k=0}^{+\infty} \frac{e_0^{-b'}}{(a_{1,k} + a_{2,k} + \ldots + a_{l,k} + 1)^{b'}} \\ &\le e_0^{-b'} \sum_{a_1=0}^{+\infty} \sum_{a_2=0}^{+\infty} \cdots \sum_{a_l=0}^{+\infty} \frac{1}{(a_1 + a_2 + \ldots + a_l + 1)^{l+1}} < +\infty.\end{aligned}$$

Now we show this lemma by induction on s. When $s = 0$, it is obvious. Suppose $s = 1$. We have by (4.14)

$$\begin{aligned}\sum_{q_{n_0}+q_{n_1}=d} \frac{1}{(q_{n_0}+1)^{b'}(q_{n_1}+1)^{b'}} &\le \sum_{0 \le q_k \le d} \frac{C'}{(q_k+1)^{b'}(d-q_k+1)^{b'}} \\ &\le \frac{C_1}{(d+1)^{b'}}.\end{aligned}$$

Assume that the assertion is vaild for $0 \le s \le S$. Then

$$\sum_{q_{n_0}+q_{n_1}+\cdots+q_{n_{S+1}}=d} \prod_{i=0}^{S+1} \frac{1}{(q_{n_i}+1)^{b'}}$$
$$\le \sum_{0\le q_{n_{S+1}}\le d} \frac{C_1^S}{(q_{n_{S+1}}+1)^{b'}(d-q_{n_{S+1}}+1)^{b'}} \le \frac{C_1^{S+1}}{(d+1)^{b'}}. \quad \square$$

We have from the preceding Lemmas

Proposition 4.9. *Suppose that $q_n > m$, $d_{A,L}-|A|+q_{n_0}+\sum_{i=1}^{s_A} q_{n_i} = q_n$ and $q_n > q_{n_i}$ for all $i > 0$. Then following estimate holds:*

$$\begin{aligned} & b_{A,q_{n_0}}(z')\prod_{i=1}^{s_A} M_{A_i}(q_{n_i},\partial')\Theta_{n_i}(t) \\ & \ll B'(CC_0)^{s_A} 2^{-d_{A,L}+|A|} B^{q_n-q_{n'}-q_1/2}(q_n+1)^{k_{\mathcal{L}}-k_A} \\ & \times \Gamma(\frac{q_n-q_{n'}}{\gamma}+1)(\prod_{i=0}^{s_A}(q_{n_i}+1)^{-b+k_A})\theta_{[a(q_n-m)]}(t) \end{aligned} \tag{4.15}$$

for a large constant B and some B'.

Proof. It follows from the assumption on $b_{A,n}(z')$, Lemmas 4.3, 4.5 and 4.7 that there is a constant B_2 such that

$$\begin{aligned} & b_{A,q_{n_0}}(z')\prod_{i=1}^{s_A} M_{A_i}(q_{n_i},\partial')\Theta_{n_i}(t) \\ & \ll B_2^{q_{n_0}+1}(CC_0)^{s_A} B^{q'}(q_n+1)^{k_{\mathcal{L}}-k_A}\Gamma(\frac{q_n-q_{n'}}{\gamma}+1) \\ & \cdot (\prod_{i=0}^{s_A}(q_{n_i}+1)^{-b+k_A})\theta_{[a(q_n-m)]}(t), \end{aligned}$$

where $q' = \sum(q_{n_i}-q_{n'})$. Choose $B \ge 4$ so large that $B_2 B^{-1/2} \le 1/2$. Then, by Lemma 4.6, it holds that $B^{q'} \le 2^{-(d_{A,L}-|A|)} B^{q_n-q_{n'}-q_1/2}$ and this lemma follows. $\square$

Proposition 4.10. *Suppose that $q_n > m$. Then the following majorant estimate holds:*

(4.16)

$$I(n) = \sum_{\left\{\substack{(q_{n_0}, q_{n_1}, \dots, q_{n_s}, A);\ A \in \mathcal{N}^M \\ q_{n_0} + q_{n_1} + \cdots + q_{n_s} \\ + d_{A,L} - |A| = q_n,\ s = s_A \\ q_n > q_{n_i}\ for\ i > 0}\right\}} b_{A, q_{n_0}}(z') \prod_{i=1}^{s_A} M_{A_i}(q_{n_i}, \partial')\Theta_{n_i}(t)$$

$$\ll C^* B^{q_n - q_{n'} - q_1/2}(q_n + 1)^{k_{\mathcal{L}} - b}\Gamma(\frac{q_n - q_{n'}}{\gamma} + 1)\, \theta_{[a(q_n - m)]}(t)$$

for some constant C^*.

Proof. Put

$$I(n, A) = \sum_{\left\{\substack{(q_{n_0}, q_{n_1}, \dots, q_{n_s}); \\ q_{n_0} + q_{n_1} + \cdots + q_{n_s} \\ = q_n - d_{A,L} + |A|,\ s = s_A \\ q_n > q_{n_i}\ for\ i > 0}\right\}} b_{A, q_{n_0}}(z') \prod_{i=1}^{s_A} M_{A_i}(q_{n_i}, \partial')\Theta_{n_i}(t).$$

Then we have, by Lemma 4.8 and Propsition 4.9,

$$I(n, A) \ll B'(CC_0C_1)^{s_A} B^{q_n - q_{n'} - q_1/2}(q_n + 1)^{k_{\mathcal{L}} - k_A}$$
$$\times 2^{-(d_{A,L} - |A|)}(q_n - d_{A,L} + |A| + 1)^{k_A - b}\Gamma(\frac{q_n - q_{n'}}{\gamma} + 1)\theta_{[a(q_n - m)]}(t).$$

Hence we have, by putting $\tilde{C} = B'(CC_0C_1)^M$,

$$I(n) = \sum_{\{A;\ 0 \le d_{A,L} - |A| \le q_n\}} I(n, A)$$
$$\ll \tilde{C} B^{q_n - q_{n'} - q_1/2}\Gamma(\frac{q_n - q_{n'}}{\gamma} + 1)\theta_{[a(q_n - m)]}(t)$$
$$\times \Big(\sum_{\{A;\ 0 \le d_{A,L} - |A| \le q_n\}} 2^{-d_{A,L} + |A|}$$
$$\cdot (q_n - d_{A,L} + |A| + 1)^{k_A - b}(q_n + 1)^{k_{\mathcal{L}} - k_A}\Big)$$
$$\ll \tilde{C} C_3 B^{q_n - q_{n'} - q_1/2}(q_n + 1)^{k_{\mathcal{L}} - b}\, \Gamma(\frac{q_n - q_{n'}}{\gamma} + 1)\, \theta_{[a(q_n - m)]}(t).$$

Thus we have the majorant estimate, putting $C^* = \tilde{C}C_3$. □

In order to complete the proof of Theorm 4.1 we need an estimate of solutions of Cauchy Problems of partial differential equations. Let us consider

$$(4.17)\qquad \begin{cases} A_{\hat{\alpha}'}(z')\partial_1^{m^*}u(z') + \displaystyle\sum_{\{\alpha';|\alpha'|\le m^*,\ \alpha_1<m^*\}} A_{\alpha'}(z')\partial^{\alpha'}u(z') = g(z') \\ \partial_1^h u(0,z'') = 0 \quad for \quad 0 \le h \le m^*-1. \end{cases}$$

where $\hat{\alpha}' = (m^*, 0, \cdots, 0)$, $A_{\alpha'}(z')$ and $g(z')$ are holomorphic in $\{z' \in \boldsymbol{C}^n;\ |z'| \le R\}$. We note that $z' = (z_1, z'')$, $\alpha' = (\alpha_1, \alpha'')$ and $t = \rho z_1 + z_2 + \ldots + z_n$.

LEMMA 4.11. *Let s be a positive integer. Assume that $|A_{\alpha'}(z')| \le D_0(1+s)^{k^*-|\alpha'|}$ for $\alpha' \ne \hat{\alpha'}$ and $|A_{\hat{\alpha}'}(z')| \ge D_0^{-1}(1+s)^{k^*-m^*}$. Let $u(z')$ be a unique solution of (4.17). Then there exists $\rho > 1$ such that if*

$$(4.18)\qquad g(z') \ll (1+s)^{k^*}\theta_{s+m^*}(t),$$

the following estimate for $u(z')$ holds:

$$(4.19)\qquad u(z') \ll D\theta_s(t),$$

where constant $D > 0$ depends on ρ and D_0, but is independent of s.

PROOF. It follows from Propositoin 3.3 that

$$\sum_{\{\alpha';|\alpha'|\le m^*,\ \alpha_1<m^*\}} A_{\hat{\alpha}'}(z')^{-1}A_{\alpha'}(z')\partial^{\alpha'}\theta_s(t) \ll D_2\rho^{m^*-1}(1+s)^{m^*}\theta_{s+m^*}(t),$$

$$A_{\hat{\alpha}'}(z')^{-1}g(z') \ll D_2(1+s)^{m^*}\theta_{s+m^*}(t)$$

and

$$\rho^{m^*}s(s+1)\ldots(s+m^*-1)\theta_{s+m^*}(t) \ll \partial_1^{m^*}\theta_s.$$

We take $\rho > 1$ and $D > 0$ so that

$$D_2(1 + D\rho^{m^*-1})(1+s)^{m^*} \le D\rho^{m^*}s(s+1)\ldots(s+m^*-1).$$

Since initial values $\partial_1^h u(0,z'')$ $(0 \le h \le m^*-1)$ are zero, we have $u(z') \ll D\theta_s(t)$. □

Proof of Theorem 4.1. Firstly we summerize what we need concerning $\mathfrak{L}(z',\lambda,\partial')$:

$$\mathfrak{L}(z',\lambda,\partial') = \sum_{|\alpha'|\le m_{\mathfrak{L}}} \mathfrak{L}_{\alpha'}(z',\lambda)\partial^{\alpha'}, \tag{4.20}$$

where

$$\mathfrak{L}_{\alpha'}(z',\lambda) = \sum_{\{\alpha=(\alpha_0,\alpha')\in\Delta_{\mathcal{L}}\}} b_{\alpha,0}(z')\lambda(\lambda-1)\dots(\lambda-\alpha_0+1). \tag{4.21}$$

It follows from Assumptions 2 and 3 that

$$|\mathfrak{L}_{\alpha'}(z',\lambda)| \le C(1+|\lambda|)^{k_{\mathcal{L}}-|\alpha'|} \tag{4.22}$$

and

$$|\mathfrak{L}_{\hat{\alpha}'}(z',\lambda)| \ge C^{-1}(1+|\lambda|)^{k_{\mathcal{L}}-m_{\mathfrak{L}}} \quad for\ \lambda = q_n\ (n \ge n'). \tag{4.23}$$

Now fix $\rho > 1$ so that Lemma 4.11 can be applied for $\mathfrak{L}(z',\lambda,\partial')$ (see (2.19)). Choose $C > 0$ so large that for $q_{n'} \le q_n \le \max\{q_{n'}, m+1\}$

$$u_n(z') \ll \frac{C}{(q_n+1)^b}\Gamma(\frac{q_n-q_{n'}}{\gamma}+1)\theta_{[a(q_n-m)]}(t)$$

holds. Suppose that (4.2) is valid for n with $n' \le n < N$ such that $q_N \ge \max\{q_{n'}, m+1\}$. Then Proposition 4.10 means that

$$\begin{aligned}&\mathcal{M}_N(u_j(z');\ j<N)) \\ &<<C^* B^{q_N-q_{n'}-q_1/2}(q_N+1)^{k_{\mathcal{L}}-b}\Gamma(\frac{q_N-q_{n'}}{\gamma}+1)\theta_{[a(q_N-m)]}(t).\end{aligned}$$

We apply Lemma 4.11 to $u_N(z')$, which is determined by (2.19). Put $g(z') = g_N(z') - \mathcal{M}_N(u_j(z');\ j<N)$, $k^* = k_{\mathcal{L}}$, $m^* = m_{\mathfrak{L}}$ and $s = [a(q_N-m)]$. Then it follows from the above inequalities that $\mathfrak{L}(z', q_n, \partial')$ satisfies the conditions in Lemma 4.11 and

$$\begin{aligned}g(z') &<< (C^* B^{q_N-q_{n'}-q_1/2} + B_1^{q_N-q_{n'}+1}) \\ &\times(q_N+1)^{k_{\mathcal{L}}-b}\Gamma(\frac{q_N-q_{n'}}{\gamma}+1)\theta_{[a(q_N-m)]+m_{\mathfrak{L}}}(t).\end{aligned}$$

Hence we have by Lemma 4.11,

$$u_N(z') << D(C^* B^{q_N - q_{n'} - q_1/2} + B_1^{q_N - q_{n'} + 1}) \\ \times (q_N + 1)^{-b} \Gamma(\frac{q_N - q_{n'}}{\gamma} + 1) \theta_{[a(q_N - m)]}(t).$$

By choosing B so large that $(DC^* B^{-q_1/2} + B_1 (B_1/B)^{q_N - q_{n'}}) < C$, where C^* is independent of B, we have (4.2) for $n = N$. Thus the proof of Theorem 4.1 is completed.

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Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku
Tokyo 102
Japan