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Transformations and contiguity relations for Gelfand's hypergeometric functions

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Abstract. The contiguity relations and transformation fromulae are studied for the hypergeometric functions on the Grassmannian. They are clarified by the action of Lie algebra of GL(n) and generalize the classical results for Gauss' hypergeometric function.

§1. Introduction

I.M.Gelfand et al. [2, 3] (see also papers in Gelfand's collected works vol. 3) introduced a generalization of hypergeometric function which is essentially defined on the Grassmannian $G_{k,n}$ of k planes in an n space. We take $k \times n$ independent variables $v = (v_{ip})_{i=1,2,\dots,k,\ j=1,2,\dots,n}$ (real or complex) and define the differential operators

$$
\begin{aligned}
Z_{ij} &= \sum_{p=1}^{n} v_{ip} \frac{\partial}{\partial v_{jp}}, \qquad i, j = 1, 2, \dots, k, \\
L_p &= \sum_{i=1}^{k} v_{ip} \frac{\partial}{\partial v_{ip}}, \qquad p = 1, 2, \dots, n, \\
\Box_{ip,jq} &= \frac{\partial^2}{\partial v_{ip} \partial v_{jq}} - \frac{\partial^2}{\partial v_{iq} \partial v_{jp}}, \\
&\qquad i, j = 1, 2, \dots, k,\ p, q = 1, 2, \dots, n.
\end{aligned}
$$

Then consider the following system of differential equations for an unknown

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function $\Phi(v)$:

$$Z_{ij}\Phi = -\delta_{ij}\Phi, \tag{1}$$

$$L_p\Phi = (\alpha_p - 1)\Phi, \tag{2}$$

$$\square_{ip,jq}\Phi = 0, \tag{3}$$

where the α_p are the constants satisfying

$$\sum_{p=1}^{n} \alpha_p = n - k,$$

which are supposed to be in general position, and δ denotes Kronecker's delta.

The equations (1) mean that if we take $h \in \mathrm{GL}(k)$ then

$$\Phi(h \cdot v) = \det(h)^{-1}\Phi(v), \tag{4}$$

while the equations (2) determine an action of $(\mathbf{R}^*)^n$ (or $(\mathbf{C}^*)^n$) on Φ by

$$\Phi((t_j v_{ij})) = \left(\prod_j t_j^{\alpha_j - 1}\right)\Phi(v). \tag{5}$$

This system is holonomic, and its solution sheaf at a general point is of rank $\binom{n-2}{k-1}$ [3].

In this paper, we study transformation formulae and contiguity relations for these equations, which generalize the classical results for Gauss' hypergeometric function, and Appell's F_1. We can, in particular, derive very explicit formulae for Lauricella's F_D. These symmetries are very clear from the viewpoint of Gelfand's equation and we can translate the result to the case of the classical functions. In §§2 and 3 we state some general aspects on these equations. §4 is devoted to the transformation formulae for F_D. In §§5, 6, we study contiguity relations, which is applied to F_D in §7. In Appendix A, we prove the equivalence of some reduction, of which we could not find an appropriate reference. In Appendix B, we show that Lauricella's F_A and F_B are birationally equivalent to each other. Although this fact was known to Lauricella himself [6, pp.133–134], we include it here to show the naturality of the present point of view.

There are many works on contiguity of hypergeometric functions, starting with Gauss. Miller [7, 8, 9] studied such operators for various hypergeometric functions. More recently, Sasaki [11] studied the contiguity relations from a viewpoint which is close to ours. In particular, our infinitesimal operators $\pi(E_{ij})$ in §3 are noted in [11]. We hope the present paper is still worth being published because of the following points: 1. We clarify more direct connection with Lie algebras, and prove the invariance of the system of differential equaitons; 2. We can explain transformation formulae in terms of the Weyl group as well. We refer the reader to [13, 14] for related results.

The author would like to thank Professor Israel M. Gelfand for introducing him to this rich area of study at the occasion of his visits to Japan in 1989. He also thanks A. V. Zelevinsky, K. Okamoto and M. Yoshida for valuable discussion.

§2. Reduction of the number of variables

By the homogeneity (4) and (5), we can reduce the number of variables of the equation. We suppose that the first $k \times k$ minor $\det(v_{ip})_{i,p=1,2,\ldots,k}$ does not vanish. Then, by (4), we may assume $v_{ip} = \delta_{ip}$ for $1 \le i,\ p \le k$. To be precise, let $w = (w_{ip})_{1\le i\le k, k+1\le p\le n}$, and define $\varphi(w)$ to be $\Phi((1_k w))$, where 1_k is the identity matrix of size k. By (1), Φ and φ are related by the formula

$$\det(h)\Phi(v) = \varphi(w),$$

where $h = (v_{ip})_{1\le i,p\le k}$ and $w = h^{-1}v$, and $\varphi(w)$ satisfies

$$(6) \qquad \sum_i w_{ip}\frac{\partial}{\partial w_{ip}}\varphi = (\alpha_p - 1)\varphi, \qquad p = k+1,\ldots,n,$$

$$(7) \qquad \sum_p w_{ip}\frac{\partial}{\partial w_{ip}}\varphi = (-\alpha_i)\varphi, \qquad i = 1,\ldots,k,$$

$$(8) \qquad \left(\frac{\partial^2}{\partial w_{ip}\partial w_{jq}} - \frac{\partial^2}{\partial w_{iq}\partial w_{jp}}\right)\varphi = 0, \qquad i,j = 1,\ldots,k,\ p,q = k+1,\ldots,n.$$

Proposition 1. *The system of equations* (1)–(3) *for* Φ *is equivalent to* (6)–(8) *for* φ.

For the sake of completeness we give a proof in Appendix A.

We set $l = n - k$. We also set $\beta_{k+p} = 1 - \alpha_{k+p}$ for $p = 1, \ldots, l$. By the homogeneity (6) and (7), we can normalize $w_{1,k+1}, w_{1,k+2}, \ldots, w_{1,k+l}$, $w_{2,k+1}, \ldots, w_{l,k+1}$ to 1. In fact, in view of

$$\varphi((s_i t_p w_{i,k+p})) = \left(\prod_i s_i^{-\alpha_i} \prod_p t_p^{-\beta_{k+p}}\right) \varphi(w),$$

we set

$$\begin{aligned} s_i &= 1/w_{i,k+1}, & i &= 1, \ldots, k, \\ t_p &= w_{1,k+1}/w_{1,k+p}, & p &= 1, \ldots, l, \\ x_{i,k+p} &= w_{1,k+1} w_{i,k+p}/w_{i,k+1} w_{1,k+p}, & & \\ & & i &= 2, \ldots, k,\ p = 2, \ldots, l. \end{aligned} \tag{9}$$

Then

$$\varphi((w_{i,k+p})) = \rho \Psi((x_{i,k+p})), \quad \rho = w_{1,k+1}^{\gamma_0} \prod_{i \geq 2} w_{i,k+1}^{-\alpha_i} \prod_{p \geq 2} w_{1,k+p}^{-\beta_{k+p}},$$

where $\gamma_0 = -\alpha_1 + \sum_{p \geq 2} \beta_{k+p} = \sum_{i \geq 2} \alpha_i - \beta_{k+1}$, and Ψ denotes the restriction of φ to the subset defined by $w_{i,k+1} = w_{1,k+p} = 1$ for $i = 1, \ldots, k,\ p = 1, \ldots, l$. By virtue of (6) and (7), we have

$$\begin{aligned} \frac{\partial}{\partial w_{1,k+1}} \varphi &= \frac{\rho}{w_{1,k+1}} \left(\sum_{j,q \geq 2} x_{j,k+q} \frac{\partial}{\partial x_{j,k+q}} + \gamma_0\right) \Psi, \\ \frac{\partial}{\partial w_{i,k+1}} \varphi &= \frac{\rho}{w_{i,k+1}} \left(-\sum_{q \geq 2} x_{i,k+q} \frac{\partial}{\partial x_{i,k+q}} - \alpha_i\right) \Psi, \qquad i \geq 2, \\ \frac{\partial}{\partial w_{1,k+p}} \varphi &= \frac{\rho}{w_{1,k+p}} \left(-\sum_{j \geq 2} x_{j,k+p} \frac{\partial}{\partial x_{j,k+p}} - \beta_{k+p}\right) \Psi, \qquad p \geq 2, \\ \frac{\partial}{\partial w_{i,k+p}} \varphi &= \frac{\rho}{w_{i,k+p}} x_{i,k+p} \frac{\partial}{\partial x_{i,k+p}} \Psi, \qquad i, p \geq 2. \end{aligned}$$

Then the equations (8) imply

(10)
$$\partial_{i,k+p}\left(\sum_{j,q}\theta_{j,k+q}+\gamma_0\right)\Psi=\left(\sum_q\theta_{i,k+q}+\alpha_i\right)\left(\sum_j\theta_{j,k+p}+\beta_{k+p}\right)\Psi$$
$$i,p\geq 2,$$

(11)
$$\partial_{i,k+p}\partial_{i',k+p'}\Psi=\partial_{i',k+p}\partial_{i,k+p'}\Psi,\qquad i,i',p,p'\geq 2,$$

where
$$\partial_{i,k+p}=\frac{\partial}{\partial x_{i,k+p}},\qquad \theta_{i,k+p}=x_{i,k+p}\frac{\partial}{\partial x_{i,k+p}}.$$

More precisely, let D_{ip} be the differntial operator such that $\dfrac{\partial\varphi}{\partial w_{i,k+p}}=(\rho/w_{i,k+p})D_{ip}\Psi$, then the equations (8) are expressed as

$$\frac{1}{w_{i,k+p}w_{i',k+p'}}D_{ip}D_{i'p'}\Psi=\frac{1}{w_{i,k+p'}w_{i',k+p}}D_{ip'}D_{i'p}\Psi,$$
$$\text{for } i,i'=1,\dots,k,\ p=1,\dots,l.$$

The equations (10) and (11) are special case of these, and other equations can be derived from (10) and (11) as integrability conditions. This is similar to the case of 2×2 minors of a usual matrix. We note that, although the $\dfrac{\partial}{\partial w_{i,k+p}}$ are commutative to each other, the D_{ip} are not. From (10), it easily follows that we have a formal power series solution

(12)
$$\sum_{m_{ip}\geq 0}\frac{\prod_i(\alpha_i;\sum_q m_{iq})\prod_p(\beta_{k+p};\sum_j m_{jp})}{(\gamma_0+1;\sum_{j,q}m_{jp})}\frac{\prod_{i,p}x_{i,k+p}^{m_{ip}}}{\prod_{i,p}m_{ip}!},$$

where $(\alpha;m)=\alpha(\alpha+1)\dots(\alpha+m-1)=\dfrac{\Gamma(\alpha+m)}{\Gamma(\alpha)}$. This power series converges for $|x_{i,k+p}|$ sufficiently small, and (11) automatically holds.

§3. General structure of symmetry

Let g be an element of $\mathrm{GL}(n)$, which maps v to $\tilde{v} = vg$, i.e., $\tilde{v}_{i\bar{p}} = \sum_p v_{ip} g_{p\bar{p}}$. We consider the function $\Phi^g(v) = \Phi(vg)$. Then

$$\frac{\partial \Phi^g}{\partial v_{ip}}(v) = \sum_{\bar{p}} \frac{\partial \Phi}{\partial v_{i\bar{p}}}(vg) g_{p\bar{p}}$$

and

$$\frac{\partial^2 \Phi^g}{\partial v_{i\bar{p}} \partial v_{j\bar{q}}}(v) = \sum_{\bar{p},\bar{q}} \frac{\partial^2 \Phi}{\partial v_{ip} \partial v_{jq}}(vg) g_{p\bar{p}} g_{q\bar{q}}.$$

It follows that, if Φ satisfies (3), then Φ^g also satisfies (3). On the other hand, Φ^g satisfies the equations (1), because one has

$$\Phi^g(hv) = \Phi(hvg) = \det(h)^{-1}\Phi(vg) = \det(h)^{-1}\Phi^g(v).$$

Equations (2) are equivalent to

$$\begin{aligned} \Phi(vt) &= \chi(t)\Phi(v), \\ \chi(t) &= \prod_p t_p^{\alpha_p - 1} \qquad \text{for } t = \mathrm{diag.}\,[t_1, \ldots, t_n]. \end{aligned}$$

Note that

$$\Phi^g(vt) = \Phi(vtg) = \Phi(vg \cdot g^{-1}tg).$$

Therefore, if g normalizes the diagonal group, then we have

$$\Phi^g(vt) = \chi(g^{-1}tg)\Phi^g(v).$$

This explains how the Weyl group, i.e., the symmetric group $\mathfrak{S}_n$, acts on the space of solutions. In paricular, for $k = 2$, this gives the transformation formulae for Lauricella's F_D in $n - 3$ variables (see §4).

To obtain contiguity relations, we consider 1-parameter family $g(\lambda)$ of elememts of $\mathrm{GL}(n)$ with $g(0) = 1$. For simplicity of notation, we introduce the following symbol:

$$D_\lambda f = \left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=0}.$$

We set

$$X = D_\lambda g(\lambda),$$

and define the action $\pi(X) : \Phi \mapsto \pi(X)\Phi$ by

$$\pi(X)\Phi(v) = D_\lambda(\Phi^{g(\lambda)}(v)),$$

which depends on k. It is easily checked that the right hand side depends only on X, and that $\pi(X)\Phi(v)$ satisfies equations (1) and (3). As to (2), we have

$$\begin{aligned}\pi(X)\Phi(vt) &= D_\lambda\Phi(vtg(\lambda)) \\ &= \chi(t)D_\lambda\Phi(vtg(\lambda)t^{-1}) \\ &= \chi(t)\pi(\mathrm{ad}(t)X)\Phi(v), \\ &\qquad \text{where } \mathrm{ad}(t)X = tXt^{-1}.\end{aligned} \tag{13}$$

In particular, let $X = E_{ij}$ be the matrix element i.e., its (i, j)-component is 1, while the others are 0. Then $\mathrm{ad}(t)E_{ij} = t_i t_j^{-1} E_{ij}$, and hence

$$\pi(E_{ij})\Phi(vt) = \chi(t)t_i t_j^{-1}\pi(E_{ij})\Phi(v).$$

That is, $\pi(E_{ij})\Phi(v)$ satisfies the same type of equations with α_i and α_j being replaced by $\alpha_i + 1$ and $\alpha_j - 1$, respectively. This generalizes the so-called contiguity relations for Gauss' hypergeometric function.

THEOREM 1. *We have* $[\pi(X), \pi(Y)] = \pi([X, Y])$ *for any* $X, Y \in \mathfrak{gl}(n)$.

PROOF. This follows from a standard calculation of exponential maps on Lie algebras. □

§4. Transformations of F_D

In this section, we write down the transformations of F_D from the view point of generalized hypergeometric functions. We set $k = 2$, and let

$$\Phi(\alpha_1, \ldots, \alpha_n; (v_{ip})) = \Phi_\alpha((v_{ip}))$$

be a solution of (1)–(3). Then (10) becomes

$$\left[\partial_p(\textstyle\sum_q \theta_q+\gamma_0)-(\sum_q \theta_q+\alpha_2)(\theta_p+\beta_p)\right]\Psi=0 \qquad p=2,\dots,l, \tag{14}$$

and the power series (12) is

$$\sum_{m_2\geq 0,\dots,m_l\geq 0}\frac{(\alpha_2;\sum m_q)\prod_p(\beta_p;m_p)}{(\gamma_0+1;\sum m_q)}\frac{x_2^{m_2}\dots x_l^{m_l}}{m_2!\dots m_l!}. \tag{15}$$

Therefore

$$\Psi(x_4,x_5,\dots,x_n)=\Phi_\alpha\left(\begin{pmatrix}1 & 0 & 1 & 1 & 1 & \dots & 1\\ 0 & 1 & 1 & x_4 & x_5 & \dots & x_n\end{pmatrix}\right)$$

satisfies the same differential equations as

$$F_D(\alpha_2;1-\alpha_4,\dots,1-\alpha_n;\alpha_2+\alpha_3;x_4,\dots,x_n)$$

(see [5, 3.3.1]). Conversely, Φ is reconstructed from Ψ as

$$\Phi_\alpha(v)=\rho\Psi(x_4,\dots,x_n),$$

where

$$\rho=(21)^{\alpha_1+\alpha_2-1}(31)^{-\alpha_2}(23)^{\alpha_2+\alpha_3-1}\prod_{j\geq 4}(2j)^{\alpha_j-1},$$

$$x_j=\frac{(1j)(23)}{(2j)(13)}, \qquad j\geq 4,$$

$$(ij)=v_{1i}v_{2j}-v_{1j}v_{2i}.$$

It is clear that, for any permuation σ,

$$\Phi(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(n)};v^\sigma), \quad v^\sigma=(v_{i\sigma(p)})$$

satisfies the same type of equations. This proves the following theorem.

THEOREM 2. *For any permutaion σ of $1,2,\dots,n$,*

$$\rho_\sigma F_D(\alpha_{\sigma(2)};1-\alpha_{\sigma(4)},\dots,1-\alpha_{\sigma(n)};\alpha_{\sigma(2)}+\alpha_{\sigma(3)};x_4^\sigma,\dots,x_n^\sigma)$$

satisfies the same differential equations as

$$F_D(\alpha_2; 1-\alpha_4, \dots, 1-\alpha_n; \alpha_2+\alpha_3; x_4, \dots, x_n),$$

where

$$\begin{aligned} \rho_\sigma &= (x_{\sigma(1)}-x_{\sigma(2)})^{\alpha_{\sigma(1)}+\alpha_{\sigma(2)}-1}(x_{\sigma(1)}-x_{\sigma(3)})^{-\alpha_{\sigma(2)}} \\ &\quad \cdot (x_{\sigma(2)}-x_{\sigma(3)})^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}-1}\prod_{j\geq 4}(x_{\sigma(j)}-x_{\sigma(2)})^{\alpha_{\sigma(j)}-1}, \\ x_j^\sigma &= \frac{x_{\sigma(j)}-x_{\sigma(1)}}{x_{\sigma(j)}-x_{\sigma(2)}} \Big/ \frac{x_{\sigma(3)}-x_{\sigma(1)}}{x_{\sigma(3)}-x_{\sigma(2)}} \qquad j \geq 4, \end{aligned}$$

with the following conventions:

$$\begin{aligned} &x_1 = 0, \quad x_2 = \infty, \quad x_3 = 1, \\ &x_2 - x_j = 1 \; \left(= \det \begin{pmatrix} 1 & 0 \\ x_j & 1 \end{pmatrix} \right). \end{aligned}$$

In the case of $n = 4$, these transformations are the famous 24 transformations of Kummer for Gauss hypergeometric functions (see [1, p.6], [12,pp.284–285]). In the case of $n = 5$, Appell-Kampé de Fériet described 60(=5!/2) transformations for F_1 ([1, pp.62–64]). They ignore the transposition of the variables x, y i.e., x_4, x_5 in our notation.

We want to discuss the transformations of Lauricella's F_C in a future paper.

§5. Explicit formulae for the action of Lie algebra

In this section, we give explicit form of the contiguity relations described in §3. Recall that φ is related to Φ by

$$\varphi(w) = \Phi((1w)),$$

where $w = (w_{ip})_{1\leq i\leq k, k+1\leq p\leq n}$ is a $k \times (n-k)$-matrix, and 1 denotes the identity matrix of size k. For $g \in \mathrm{GL}(n)$, we define

$$\varphi^g(w) = \Phi^g((1w)) = \Phi((1w)g).$$

In other words, let us divide the matrix g as

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

where A consists of the first k rows and k columns. Then, the above equation can be written as

$$\varphi^g(w) = \det(A + wB)^{-1}\varphi((A + wB)^{-1}(C + wD)).$$

Next we suppose that $g = g(\lambda)$ depends on a parameter λ with $g(0) = 1$, and let $X = D_\lambda g$ be the corresponding element of the Lie algebra $\mathfrak{gl}(n)$. We set

$$\pi(X)\varphi(w) = \pi(X)\Phi((1w)).$$

By direct calculations, we obtain the following formulae. We fix the indexing as

$$i, j \in [1, k],\ \ p, q \in [k + 1, n].$$

The action of $\mathfrak{gl}(n)$ on the space of functions φ is given as follows.

$$\begin{aligned}
\pi(E_{ip})\varphi &= \frac{\partial\varphi}{\partial w_{ip}}, \\
\pi(E_{pi})\varphi &= -\left(w_{ip} + \sum_{j,q} w_{jp}w_{iq}\frac{\partial}{\partial w_{jq}}\right)\varphi, \\
\pi(E_{ij})\varphi &= -\delta_{ij}\varphi - \sum_{p} w_{jp}\frac{\partial\varphi}{\partial w_{ip}}, \\
\pi(E_{pq})\varphi &= \sum_{i} w_{ip}\frac{\partial\varphi}{\partial w_{iq}}.
\end{aligned}$$

In particular,

$$\pi(E_{ii})\varphi = (\alpha_i - 1)\varphi, \qquad \pi(E_{pp})\varphi = (\alpha_p - 1)\varphi.$$

As an example, we shall carry out the calculation for E_{pi}. We set $g(\lambda) = 1 + \lambda E_{pi}$. Then

$$(1w)g(\lambda) = \begin{pmatrix} 1 & \dots & \lambda w_{1p} & \dots & 0 & w_{1\,k+1} & \dots & w_{1n} \\ \vdots & \ddots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 1+\lambda w_{ip} & \dots & 0 & w_{i\,k+1} & \dots & w_{in} \\ \vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \lambda w_{kp} & \dots & 1 & w_{k\,k+1} & \dots & w_{kn}, \end{pmatrix}$$

which we write $(w_0 w)$. It follows

$$w_0^{-1} \equiv \begin{pmatrix} 1 & & -\lambda w_{1p} & & \\ & \ddots & \vdots & & \\ & & 1-\lambda w_{ip} & & \\ & & \vdots & \ddots & \\ & & -\lambda w_{kp} & & 1 \end{pmatrix} \mod \lambda^2.$$

Hence

$$w_0^{-1} w_1 \equiv \begin{pmatrix} w_{1,k+1} - \lambda w_{1p} w_{i,k+1} & \dots & w_{1n} - \lambda w_{1p} w_{in} \\ \vdots & \ddots & \vdots \\ w_{i,k+1} - \lambda w_{ip} w_{i,k+1} & \dots & w_{in} - \lambda w_{ip} w_{in} \\ \vdots & \ddots & \vdots \\ w_{k,k+1} - \lambda w_{kp} w_{i,k+1} & \dots & w_{kn} - \lambda w_{kp} w_{in} \end{pmatrix} \mod \lambda^2.$$

Since $\det(w_0) \equiv 1 + \lambda w_{ip}$, we obtain

$$\Phi((w_0 w_1)) \equiv (1 - \lambda w_{ip}) \Phi((1\, \tilde{w}_1)) \mod \lambda^2,$$
$$\tilde{w}_1 = w_0^{-1} w_1$$

Our formula follows from this.

§6. Relation with classical hypergeometric functions

We introduce $(k-1)(n-1)$ variables $(x_{ip})_{2\le i\le k, k+2\le p\le n}$. We set

$$\tilde{x} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & x_{2,k+2} & \dots & x_{2n} \\ . & . & \dots & . \\ 1 & x_{k,k+2} & \dots & x_{kn} \end{pmatrix},$$

and define

$$\Psi(x) = \varphi(\tilde{x}).$$

Then, by the homogeneity (4) (5), we have

$$\varphi((w_{ip})) = \rho\Psi((x_{ip})), \qquad \rho = w_{1,k+1}^{\gamma_0} \prod_{i\ge 2} w_{i,k+1}^{-\alpha_i} \prod_{p\ge k+2} w_{1p}^{\alpha_p - 1},$$

where

$$\gamma_0 = -\alpha_1 + \sum_{p=k+2}^{n} (1-\alpha_p) = \sum_{i=2}^{k+1} \alpha_i - 1,$$
$$x_{ip} = \frac{w_{1,k+1} w_{ip}}{w_{i,k+1} w_{1p}}, \qquad i = 2, \dots, k,\ p = k+2, \dots, n.$$

We define

$$\pi(X)\Psi(x) = \pi(X)\varphi(\tilde{x}).$$

By direct caluculations, we obtain the following expressions, where the summations are all extended over $j,\ l \in [2,k]$ or $q \in [k+2, n]$, and $i, j \in [1,k],\ p, q \in [k+1, n]$ as before:

$$\pi(E_{ip})\Psi = \frac{\partial \Psi}{\partial x_{ip}}, \quad i \ne 1,\ p \ne k+1,$$
$$\pi(E_{i,k+1})\Psi = -\left(\sum_q x_{iq} \frac{\partial}{\partial x_{iq}} + \alpha_i\right)\Psi, \quad i \ne 1,$$
$$\pi(E_{1p})\Psi = -\left(\sum_j x_{jp} \frac{\partial}{\partial x_{jp}} + 1 - \alpha_p\right)\Psi, \quad p \ne k+1,$$

$$\pi(E_{1,k+1})\Psi = \left(\sum_{j,q} x_{jq}\frac{\partial}{\partial x_{jq}} + \gamma_0\right)\Psi,$$

$$\pi(E_{pi})\Psi = -\Bigg[x_{ip} + \sum_{j,q}\left(x_{jp}x_{iq} + x_{jq}(1 - x_{jp} - x_{iq})\right)\frac{\partial}{\partial x_{jq}}$$

$$- \sum_q (1-\alpha_q)x_{iq} - \sum_j \alpha_j x_{jp} + \gamma_0\Bigg]\Psi, \quad i \neq 1, p \neq k{+}1,$$

$$\pi(E_{k+1,i})\Psi = -\Bigg[1 + \sum_{j,q} x_{iq}(1 - x_{jq})\frac{\partial}{\partial x_{jq}}$$

$$- \sum_q (1-\alpha_q)x_{iq} - \sum_j \alpha_j + \gamma_0\Bigg]\Psi, \quad i \neq 1,$$

$$\pi(E_{p1})\Psi = -\Bigg[\sum_{j,q}\left(x_{jp}(1 - x_{jq})\right)\frac{\partial}{\partial x_{jq}}$$

$$- \sum_q (1-\alpha_q) - \sum_j \alpha_j x_{jp} + \gamma_0 + 1\Bigg]\Psi, \quad p \neq k{+}1,$$

$$\pi(E_{k+1,1})\Psi = -\Bigg[\sum_{j,q}\left(1 - x_{jq}\right)\frac{\partial}{\partial x_{jq}}$$

$$- \sum_q (1-\alpha_q) - \sum_j \alpha_j + \gamma_0 + 1\Bigg]\Psi,$$

$$\pi(E_{ij})\Psi = \left[\sum_q (x_{iq} - x_{jq})\frac{\partial}{\partial x_{iq}} + \alpha_i - \delta_{ij}\right]\Psi, \quad i \neq 1,\ j \neq 1,$$

$$\pi(E_{i1})\Psi = \left[\sum_q (x_{iq} - 1)\frac{\partial}{\partial x_{iq}} + \alpha_i\right]\Psi, \quad i \neq 1,$$

$$\pi(E_{1j})\Psi = \left[\sum_{q,l} x_{lq}(x_{jq} - 1)\frac{\partial}{\partial x_{lq}} + \sum_q (1-\alpha_q)x_{jq} - \gamma_0\right]\Psi, \quad j \neq 1$$

$$\pi(E_{11})\Psi = (\alpha_1 - 1)\Psi,$$

$$\pi(E_{pq})\Psi = \left[\sum_j (x_{jp} - x_{jq}) \frac{\partial}{\partial x_{jq}} - (1-\alpha_q)\right]\Psi, \qquad p \neq k+1,\ q \neq k+1,$$

$$\pi(E_{k+1,q})\Psi = \left[\sum_j (1 - x_{jq}) \frac{\partial}{\partial x_{jq}} - (1-\alpha_q)\right]\Psi, \quad q \neq k+1,$$

$$\pi(E_{p,k+1})\Psi = \left[\sum_{j,q} x_{jq}(1 - x_{jp}) \frac{\partial}{\partial x_{jq}} - \sum_j \alpha_j x_{jp} + \gamma_0\right]\Psi, \qquad p \neq k+1,$$

$$\pi(E_{k+1,k+1})\Psi = (\alpha_{k+1} - 1)\Psi.$$

§7. Contiguity for Lauricella's F_D

In this section we give the formulae of the contiguity relations for Lauricella's F_D. This is a special case of the result of §6 where $k = 2$. The function φ with parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ corresponds to $F = F_D(\alpha; \beta_4, \ldots, \beta_n; \gamma; x_4, \ldots x_n)$ in the classical notation. Here

$$x_j = x_{2j} \qquad (j \geq 4),$$

and the parameters are related as

$$\begin{aligned} &\alpha = \alpha_2, \\ &\gamma = \alpha_2 + \alpha_3 = 2 - \alpha_1 + \sum_j \beta_j = \gamma_0 + 1, \\ &\beta_j = 1 - \alpha_j \qquad (j \geq 4). \end{aligned} \tag{16}$$

The action of $\mathfrak{gl}(n)$ on F is given by the following formulae.

$$\pi(E_{2p})F = \frac{\partial F}{\partial x_p} \qquad (p \geq 4),$$

$$\pi(E_{23})F = -\left(\sum_j x_j \frac{\partial}{\partial x_j} + \alpha\right)F,$$

$$\pi(E_{1p})F = -\left(x_p\frac{\partial}{\partial x_p} + \beta_p\right)F \qquad (p \geq 4),$$

$$\pi(E_{13})F = \left(\sum_p x_p\frac{\partial}{\partial x_p} + \gamma - 1\right)F,$$

$$\pi(E_{p2})F = \Bigg[x_p + \sum_q x_q(1-x_q)\frac{\partial}{\partial x_q} - \sum_q \beta_q x_q - \alpha x_p + \gamma - 1\Bigg]F \qquad (p \geq 4),$$

$$\pi(E_{32})F = \left[\sum_q x_q(1-x_q)\frac{\partial}{\partial x_q} - \sum_q \beta_q x_q - \alpha + \gamma\right]F,$$

$$\pi(E_{p1})F = -\left[\sum_q x_p(1-x_q)\frac{\partial}{\partial x_q} - \sum_q \beta_q - \alpha x_p + \gamma\right]F \qquad (p \geq 4),$$

$$\pi(E_{31})F = -\left[\sum_q (1-x_q)\frac{\partial}{\partial x_q} - \sum_q \beta_q - \alpha + \gamma\right]F,$$

$$\pi(E_{21})F = \left[\sum_q (x_q - 1)\frac{\partial}{\partial x_q} + \alpha\right]F,$$

$$\pi(E_{22})F = (\alpha - 1)F,$$

$$\pi(E_{11})F = (1 + \sum_j \beta_j - \gamma)F,$$

$$\pi(E_{12})F = \left[\sum_q x_q(x_q - 1)\frac{\partial}{\partial x_q} + \sum_q \beta_q x_q - \gamma + 1\right]F,$$

$$\pi(E_{pq})F = \left[(x_p - x_q)\frac{\partial}{\partial x_q} - \beta_q\right]F \qquad (p, q \geq 4),$$

$$\pi(E_{3q})F = \left[(1 - x_q)\frac{\partial}{\partial x_q} - \beta_q\right]F,$$

$$\pi(E_{p3})F = \left[\sum_q x_q(1-x_p)\frac{\partial}{\partial x_q} - \alpha x_p + \gamma - 1\right]F \qquad (p \geq 4),$$

$$\pi(E_{33})F = -\beta_3 F.$$

In the notation of Miller [7], our operators are in the following correspondence.

$$\pi(E_{2p}) \leftrightarrow E_{\alpha\beta_p\gamma}, \quad \pi(E_{23}) \leftrightarrow -E_{\alpha}, \quad \pi(E_{1p}) \leftrightarrow -E_{\beta_p},$$
$$\pi(E_{13}) \leftrightarrow E_{-\gamma}, \quad \pi(E_{32}) \leftrightarrow -E_{-\alpha}, \quad \pi(E_{p1}) \leftrightarrow -E_{-\beta_p}, \text{etc.}$$

Here, for example, $E_{\alpha\beta_p\gamma}$ is an operator which raises α, β_p, γ and $E_{-\alpha}$ lowers α. These unevenness reflects the change of parameters (16).

Appendix A. Proof of Proposition 1

We write $v = (v\,v')$, where v is a $k \times k$-matrix and v' is a $k \times (n-k)$-matrix. We further introduce the new variables $u = (u_{ij}), 1 \leq i, j \leq k$ by $u_{ij} = v_{ij}$. Then we have

$$(v\,v') = u \cdot (1\,w),$$

i.e.,

$$\begin{cases} v_{ij} = u_{ij}, \\ v'_{ip} = \sum_j u_{ij} w_{jp}. \end{cases} \tag{17}$$

By definition, Φ and φ are related to each other by the formula

$$\Phi(v\,v') = \det(u)^{-1}\varphi(w).$$

LEMMA 1. *Set* $h(u) = \det(u)^{-1}$. *Then*

$$\frac{\partial^2 h}{\partial u_{ij}\partial u_{\bar{i}\bar{j}}} = \frac{\partial^2 h}{\partial u_{i\bar{j}}\partial u_{\bar{i}j}}. \tag{18}$$

PROOF. Let Δ_{ij} be the cofactor of u_{ij} in $\det(u)$. Then

$$\frac{\partial h}{\partial u_{ij}} = -h(u)^2 \Delta_{ij}.$$

It follows that

$$\frac{\partial^2 h}{\partial u_{ij}\partial u_{\bar{i}\bar{j}}} = h(u)^3\{2\Delta_{ij}\Delta_{\bar{i}\bar{j}} - \det(u)\Delta_{ij,\bar{i}\bar{j}}\},$$

where $\Delta_{ij,\bar{i}\bar{j}}$ denotes the coefficient for $u_{ij}u_{\bar{i}\bar{j}}$ in $\det(u)$. Since $\Delta_{ij,\bar{i}\bar{j}} = -\Delta_{i\bar{j},\bar{i}j}$, the equation (18) is equivalent to

$$\det(u)\Delta_{ij,\bar{i}\bar{j}} = \Delta_{ij}\Delta_{\bar{i}\bar{j}} - \Delta_{i\bar{j}}\Delta_{\bar{i}j}.$$

This is known as Jacobi's formula (see e.g. [10], p.78). □

We regard (17) as a coordinate change from (u, w) to (v, v'). We easily obtain

$$\frac{\partial}{\partial u_{ij}} = \frac{\partial}{\partial v_{ij}} + \sum_q w_{jq}\frac{\partial}{\partial v'_{iq}},$$
$$\frac{\partial}{\partial w_{jp}} = \sum_{\bar{i}} v_{\bar{i}j}\frac{\partial}{\partial v'_{\bar{i}p}}.$$

We also have

$$\sum_j u_{ij}\frac{\partial w_{jp}}{\partial v'_{kp}} = \delta_{ik}.$$

It follows that

$$\sum_j u_{\bar{i}j}\frac{\partial h}{\partial u_{ij}}\cdot\varphi = \sum_j v_{\bar{i}j}\frac{\partial \Phi}{\partial v_{ij}} + \sum_p v'_{\bar{i}p}\frac{\partial \Phi}{\partial v'_{ip}},$$
$$h\sum_j w_{jp}\frac{\partial \varphi}{\partial w_{jp}} = \sum_{\bar{i}} v'_{\bar{i}p}\frac{\partial \Phi}{\partial v'_{\bar{i}p}},$$
$$h\sum_p w_{jp}\frac{\partial \varphi}{\partial w_{jp}} = \sum_{\bar{i},p} u_{\bar{i}j}w_{jp}\frac{\partial \Phi}{\partial v'_{\bar{i}p}} = \sum_{\bar{i}} u_{\bar{i}j}\left(\frac{\partial}{\partial u_{\bar{i}j}} - \frac{\partial}{\partial v_{\bar{i}j}}\right)\Phi$$
$$= -\Phi - \sum_{\bar{i}} v_{\bar{i}j}\frac{\partial \Phi}{\partial v_{\bar{i}j}}.$$

These prove the equivalence of (2) and (6)(7).

Next we have

$$h\frac{\partial^2\varphi}{\partial w_{ip}\partial w_{jq}} = \sum_{\bar{i},\bar{j}} u_{\bar{i}i}u_{\bar{j}j}\frac{\partial^2\Phi}{\partial v'_{\bar{i}p}\partial v'_{\bar{j}q}}.$$

This implies that (3) for $p, q \geq k+1$ is equivalent to (8). It remains to show that (6)–(8) imply (3) for the cases $p \leq k$ or $q \leq k$. For this, note that

$$\begin{aligned}
\sum_j \left(\frac{\partial}{\partial w_{jp}}\right)\left(\frac{\partial}{\partial u_{ij}}\right)\Phi &= \sum_{\bar{i},j} v_{\bar{i}j}\left(\frac{\partial}{\partial v_{ij}}\right)\left(\frac{\partial}{\partial v'_{\bar{i}p}}\right)\Phi \\
&\quad + \sum_{q,j} v_{\bar{i}j} w_{jq}\left(\frac{\partial}{\partial v'_{iq}}\right)\left(\frac{\partial}{\partial v'_{\bar{i}p}}\right)\Phi + \sum_{\bar{i},j,q} u_{\bar{i}j}\frac{\partial w_{jq}}{\partial v'_{\bar{i}p}}\frac{\partial \Phi}{\partial v'_{iq}} \\
&= \sum_{\bar{i}} \frac{\partial}{\partial v'_{\bar{i}p}}\left(\sum_j v_{\bar{i}j}\frac{\partial}{\partial v_{ij}} + \sum_q v'_{\bar{i}q}\frac{\partial}{\partial v'_{iq}}\right)\Phi \\
&= -\frac{\partial \Phi}{\partial v'_{ip}}.
\end{aligned}$$

Then we obtain, for $j \leq k$, $p \geq k+1$,

$$\begin{aligned}
\frac{\partial^2 \Phi}{\partial v'_{ip}\partial v_{\bar{i}j}} &= -\frac{\partial}{\partial v_{\bar{i}j}}\sum_l \left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial w_{lp}}\right)\Phi \\
&= -\sum_l \left(\frac{\partial}{\partial u_{\bar{i}j}} - \sum_q w_{jq}\frac{\partial}{\partial v'_{\bar{i}q}}\right)\left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial w_{lp}}\right)\Phi \\
&= -\sum_l \left[\frac{\partial}{\partial u_{\bar{i}j}} + \sum_q w_{jq}\sum_{\bar{l}}\left(\frac{\partial}{\partial u_{\bar{i}\bar{l}}}\right)\left(\frac{\partial}{\partial w_{\bar{l}q}}\right)\right]\left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial w_{lp}}\right)\Phi \\
&= -\left[\sum_l \left(\frac{\partial}{\partial u_{\bar{i}j}}\right)\left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial w_{lp}}\right)\right. \\
&\qquad \left. + \sum_{l,\bar{l},q} w_{jq}\left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial u_{\bar{i}\bar{l}}}\right)\left(\frac{\partial^2}{\partial w_{lp}\partial w_{\bar{l}q}}\right)\right]\Phi.
\end{aligned}$$

By Lemma 1 the last expression is invariant under $i \leftrightarrow \bar{i}$.

Finally, from

$$\frac{\partial \Phi}{\partial v_{ij}} = \left[\frac{\partial}{\partial u_{ij}} - \sum_{q,l} w_{jq}\left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial w_{lq}}\right)\right]\Phi,$$

we obtain

$$\frac{\partial^2 \Phi}{\partial v_{ij}\partial v_{\bar{i}\bar{j}}} = \Bigg[\frac{\partial^2}{\partial u_{ij}\partial u_{\bar{i}\bar{j}}} - \sum_{\bar{q},\bar{l}} w_{\bar{j}\bar{q}} \left(\frac{\partial}{\partial u_{ij}}\right)\left(\frac{\partial}{\partial u_{\bar{i}\bar{l}}}\right)\left(\frac{\partial}{\partial w_{\bar{l}\bar{q}}}\right)$$
$$- \sum_{q,l} w_{jq} \left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial u_{\bar{i}\bar{j}}}\right)\left(\frac{\partial}{\partial w_{lq}}\right)$$
$$- \sum_{q,\bar{q},l,\bar{l}} w_{jq}w_{\bar{j}\bar{q}} \left(\frac{\partial}{\partial u_{il}}\right)\left(\frac{\partial}{\partial u_{\bar{i}\bar{l}}}\right)\left(\frac{\partial}{\partial w_{lq}}\right)\left(\frac{\partial}{\partial w_{\bar{l}\bar{q}}}\right)$$
$$- \sum_{q,\bar{l}} w_{jq} \left(\frac{\partial}{\partial u_{i\bar{j}}}\right)\left(\frac{\partial}{\partial u_{\bar{i}\bar{l}}}\right)\left(\frac{\partial}{\partial w_{\bar{l}q}}\right)\Bigg]\Phi.$$

On the right hand side, the 1st, 3rd and 4th terms are invariant under $i \leftrightarrow \bar{i}$ by Lemma 1 and (8), while the 2nd and 5th terms are interchanged. Hence the equation (3) for Φ is established. Proposition 1 is proved.

Appendix B. Lauricella's functions F_A and F_B

In this appendix, we study the restrictions of the generalized hypergeometric functions to some strata and another normalization. We see how F_A and F_B appear in our context. We also show that these two functions are birationally transformed to each other.

Suppose that $n = 2k$, and $x_{i,k+p} = 0$ for $i \neq p$. We set $x_i = x_{i,k+i}$ and use the notation $\partial_i = \dfrac{\partial}{\partial x_i}$ and $\theta_i = x_i \dfrac{\partial}{\partial x_i}$ and write β_i in place of β_{k+i}. Then the power series (12) is reduced to

$$\sum_{m_p \geq 0} \frac{\prod_i(\alpha_i; m_i)\prod_i(\beta_i; m_i)}{(\gamma_0 + 1; \sum_i m_i)} \frac{x_2^{m_2} \dots x_l^{m_l}}{m_2! \dots m_l!},$$

which satisfies the equations

$$[\partial_i(\textstyle\sum \theta_j + \gamma_0) - (\theta_i + \alpha_i)(\theta_i + \beta_i)]\,\Psi = 0, \qquad i = 2, \dots, l. \tag{19}$$

These are nothing but Lauricella's F_B and its differential equations.

To obtain F_A, we assume $n = 2k$ and consider another normalization:

$$w_{i,k+1} = 1, \quad i = 1, \dots, k, \qquad w_{i,k+i} = 1, \quad i = 2, \dots, k.$$

This is done by choosing

$$s_i = w_{i,k+1}^{-1}, \quad i = 1, \dots, k, \qquad t_p = w_{p,k+1}/w_{p,k+p}, \quad p = 2, \dots, k.$$

The new coordinates are

$$y_{i,k+p} = w_{i,k+p} w_{p,k+1}/w_{i,k+1} w_{p,k+p}, \qquad i \neq p, \ p \neq 1.$$

By a calculation similar to the above, we obtain the following equations:

$$\begin{aligned} &\Big[\partial_{1,k+p}(-\textstyle\sum_{q\neq 1,p} \theta_{p,k+q} + \sum_{j\neq p} \theta_{j,k+p} + \delta_p) \\ &\quad -(\textstyle\sum_{q\neq 1} \theta_{1,k+q} + \delta_1)(\sum_{j\neq p} \theta_{j,k+p} + \gamma_p)\Big] \Psi = 0, \qquad p \neq 1, \\ &\Big[\partial_{i,k+p}(-\textstyle\sum_{q\neq 1,p} \theta_{p,k+q} + \sum_{j\neq p} \theta_{j,k+p} + \delta_p) \\ &\quad -(\textstyle\sum_{q\neq 1,i} \theta_{i,k+q} - \sum_{j\neq i} \theta_{j,k+i} - \delta_i)(\sum_{j\neq p} \theta_{j,k+p} + \gamma_p)\Big] \Psi = 0 \\ &\qquad i \neq p, \ i \geq 2, \end{aligned} \tag{20}$$

where

$$\begin{aligned} &\gamma_1 = -\alpha_1, \ \gamma_p = \beta_{k+p} \ (p \geq 2), \\ &\delta_i = \alpha_i + \beta_{k+i} \ (i \geq 2). \end{aligned}$$

We obtain a power series solution

$$\sum \frac{(\gamma_1; \sum_{q\neq 1} m_{1q}) \prod_{p\geq 2}(\gamma_p; \sum_{j\neq p} m_{jp})}{\prod_{p\geq 2}(\delta_p + 1; \sum_{j\neq p} m_{jp} - \sum_{q\neq 1,p} m_{pq})} \frac{\prod y_{i,k+p}^{m_{ip}}}{\prod m_{ip}!}.$$

If we set $y_{i,k+p} = 0$ for $i \neq 1$, then we obtain

$$\sum_{m_p \geq 0} \frac{(\gamma_1; \sum m_p) \prod(\gamma_p; m_p)}{\prod(\delta_p + 1; m_p)} \frac{y_2^{m_2} \dots y_l^{m_l}}{m_2! \dots m_l!},$$

where we set $y_p = y_{1,k+p} = w_{1,k+p}w_{p,k+1}/w_{1,k+1}w_{p,k+p}$. This is Lauricella's F_A. The equations for F_A is obtained from (20) as follows:

$$[\partial_p(\theta_p + \delta_p) - (\sum \theta_q + \gamma_1)(\theta_p + \gamma_p)]\,\Psi = 0, \qquad p = 2, \ldots, l, \tag{21}$$

where $\partial_i = \dfrac{\partial}{\partial y_i}$ etc.

From these facts we readily infer that F_A and F_B are transformed to each other. To be more precise, we let

$$\Psi_B(\alpha_2, \ldots, \alpha_l; \beta_2, \ldots, \beta_l; \gamma_0; x_2, \ldots, x_l)$$

denote a solution of (19), and introduce new variables

$$y_p = 1/x_p, \qquad p = 2, \ldots, l.$$

Let

$$\varphi(y_2, \ldots, y_l) = \rho\,\Psi_B(1/y_2, \ldots, 1/y_l),$$

where

$$\rho = \prod_p y_p^{-\beta_p}.$$

Then

$$\frac{\partial}{\partial x_p}\Psi_B = \rho^{-1}\left(-y_p\frac{\partial}{\partial y_p} - \beta_p\right)\varphi.$$

It follows that (19) can be written as

$$\left[y_i^{-1}(-\vartheta_i - \beta_i)(-\sum \vartheta_j - \sum \beta_j + \gamma_0) - (-\vartheta_i - \beta_i + \alpha_i)(-\vartheta_i)\right]\varphi = 0,$$

where,

$$\vartheta_i = y_i\frac{\partial}{\partial y_i}.$$

Changing the order, and cancelling y_i^{-1}, we obtain

$$\left[\frac{\partial}{\partial y_i}(\vartheta_i - \alpha_i + \beta_i) - (\vartheta_i + \beta_i)(\sum \vartheta_j + \sum \beta_j - \gamma_0)\right]\varphi = 0.$$

This coincides with (21), provided that we take

$$\delta_i = \beta_i - \alpha_i,$$
$$\gamma_1 = \sum \beta_j - \gamma_0, \qquad \gamma_i = \beta_i \quad (i \geq 2).$$

This type of relation between Appell's $F_2(= F_A$ of 2 variables) and $F_3(= F_B$ of 2 variables) is noted in [4, 5.2]. The general case was already known to Lauricella.

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