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On Hölder's transformation

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Abstract. This paper investigates the main properties of Hölder's transformation $\mathcal{H}_F$ of the contact space $\hat{M}$, generated by some function $F : \hat{M} \to \mathbb{R}$, and its relation to Legendre's transformation $\mathcal{L}_H$. The commuting diagram $\mathcal{L}_H \circ \mathcal{H}_F = \mathcal{H}_W \circ \mathcal{L}_F$ and the related global properties of $\mathcal{H}$ and $\mathcal{L}$ are of particular interest. Hölder's transformation can be used to transform Hamiltonian systems into Lie systems and Euler-Lagrange equations into Herglotz equations, this way establishing four equivalent pictures of the one-dimensional calculus of variations.

1. Introduction

In [8] E. Hölder gave another and quite geometric proof of Boerner's embedding theorem for extremals of multiple variational integrals into geodesic fields as defined by Carathéodory [1]. Hölder simplified Boerner's somewhat tedious computations considerably by using a suitable coordinate transformation suggested by the theory of contact transformations. Thereby he developed transformation formulas relating the variational calculus and the Hamilton–Jacobi formalism to Lie's theory of contact transformations. As Hölder's beautiful paper is at times somewhat brief[1], I have tried in [7] to elaborate his ideas and to develop them to their full extent. The underlying transformation theory needed for this purpose is described in the present paper. Some of the identities presented here can already be found either in the footnotes of [8] or in the first part of [1], but I thought it better to develop the whole formalism independently and ab ovo.

The main goal of this paper consists in obtaining results about the global invertibility of Hölder's and Haar's transformations $\mathcal{H}_F$ and $\mathcal{R}_F$. The key to the invertibility theorems 3.1 and 3.2 lies in the factorization $\mathcal{R}_F = \mathcal{L}_H \circ \mathcal{H}_F$ and in formula (2.26).

The definitions of $\mathcal{H}_F$ and $\mathcal{R}_F$ used in this paper are slightly different from those in [5] and [8] in as much as different signs are used. It should

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[1] see the comments by Carathéodory given in [2]

also be mentioned that Haar's transformation $\mathcal{R}_F$ can more or less explicitly be found in Lie's work, and it is certainly contained in Douglas's paper [3], while Hölder's transformation $\mathcal{H}_F$ is part of the beautiful transformation theory developed by Carathéodory [1]. Yet our terminology seems justified, as both Haar and Hölder realized the importance of $\mathcal{R}_F$ and $\mathcal{H}_F$ and gave relevant applications.

The last section describes the connection of the preceding discussion with the theory of contact transformations. The pertinent results are in principle well known; nevertheless the reader might possibly find our explanations useful. I should like to thank U. Dierkes, M. Giaquinta and C. Hamburger for several discussions concerning this paper. Moreover, I am very grateful to Tokyo University for giving me the possibility to complete this work; in particular I want to thank Professors T. Ochiai and H. Matano for their hospitality and support.

2. Hölder's transformation

Let $M = \mathbb{R}^n \times \mathbb{R}$ be the configuration space of points $Q = (x, z)$, $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, $z \in \mathbb{R}$, and let $\hat{M} = M \times \mathbb{R}^n$. We interpret the points p in the fibre $\mathbb{R}^n$ either as vectors or covectors on the base space $\underline{M} = \mathbb{R}^n$; since the following formalism will be the same in both cases, we assume $p = (p_1, \cdots, p_n)$ to be covectors on $\underline{M}$. Consider a domain G in $\hat{M}$ of the kind

$$G = \{(x, z, p) : (x, z) \in U, p \in B(x, z)\}$$

where U is a domain in M, and $B(x, z)$ are domains in $\mathbb{R}^n$. We say that G is a *normal domain of type* B, C, *or* S respectively if $B(x, z)$ contains the origin $0 \in \mathbb{R}^n$ and if $B(x, z)$ is a ball of radius $R(x, z)$ centered at 0, $0 < R(x, z) \leq \infty$, a convex set, or a star–shaped set with respect to $p = 0$.

Suppose that $F(x, z, p)$ is a function of class $C^2(G)$, and let $\Phi(x, z, p)$ be its *adjoint* defined by

$$\Phi(x, z, p) := p \cdot F_p(x, z, p) - F(x, z, p) \tag{2.1}$$

Then we define *Hölder's transformation* $\mathcal{H}_F$, generated by F, as a mapping $\mathcal{H}_F : \hat{M} \to \hat{M}$ given by

$$\begin{aligned} &\mathcal{H}_F(x, z, p) := (x, z, \mathbf{y}(x, z, p)), \\ &\mathbf{y}(x, z, p) := \frac{p}{F(x, y, z)} \end{aligned} \tag{2.2}$$

First we want to investigate when $\mathcal{H}_F$ is a local diffeomorphism on $\hat{M}$. To this end we introduce the *momentum tensor* $T = (T^i_k)$ of F by

$$T := p \otimes F_p - FI \tag{2.3}$$

where I is the identity, i.e. the components of T are given by

$$T^i_k = p_k F_{p_i} - F\delta^i_k.$$

LEMMA 2.1. *We have*

$$det\ T = (-1)^{n-1} F^{n-1} \Phi. \tag{2.4}$$

PROOF. Let $e_1, \cdots, e_n$ be the canonical base of $\mathbb{R}^n$, and write $e_1, \cdots e_n$, and F_p as columns. Then we obtain

$$det\ T = (-1)^n D$$

where

$$D := [\ Fe_1 - p_1 F_p, \ldots, Fe_n - p_n F_p\ ]$$

If $p_1 \neq 0$, we can write

$$D = [\ Fe_1 - p_1 F_p\ ,\ F(e_2 - \frac{p_2}{p_1} e_1)\ , \ldots,\ F(e_n - \frac{p_n}{p_1})\ ] = D_1 + D_2$$

where

$$\begin{aligned}
D_1 :=& \quad [\ Fe_1, Fe_2 - \frac{p_2}{p_1} e_1, \cdots, Fe_n - \frac{p_n}{p_1} e_1\ ] = F^n, \\
D_2 :=& \quad [-p_1 F_p\ ,\ F(e_2 - \frac{p_2}{p_1} e_1), \cdots, F(e_n - \frac{p_n}{p_1})\ ] \\
=& \quad -F^{n-1} [\ p_1 F_p\ ,\ e_2 - \frac{p_2}{p_1} e_1, \cdots, e_n - \frac{p_n}{p_1} e_1\ ] \\
=& \quad -F^{n-1} p \cdot F_p.
\end{aligned}$$

Therefore

$$D = F^n - F^{n-1}\ p \cdot F_p = -F^{n-1}\Phi$$

if $p_1 \neq 0$, and more generally if $p \neq 0$. However, this formula is trivially correct if $p = 0$, and (2.4) is proved. □

LEMMA 2.2. *The Jacobi matrix* $\mathbf{y}_p$ *of the mapping* $p \mapsto \mathbf{y}(x,z,p)$ *defined by* (2.2) *is given by*

$$\frac{\partial \mathbf{y}_k}{\partial p_i} = -F^{-2}T^i_k, \tag{2.5}$$

and its Jacobian is given by

$$det\ \mathbf{y}_p = -\Phi F^{-(n+1)}. \tag{2.6}$$

PROOF. Formula (2.5) follows by a straight–forward computation whence

$$det\ \mathbf{y}_p = (-1)^n F^{-2n}\ det\ T,$$

and by virtue of (2.4) we obtain (2.6). □

An immediate consequence of this result is

PROPOSITION 2.1. *If* F *and* Φ *are nowhere zero on* G*, then* $\mathcal{H}_F : G \to \hat{M}$ *is a local* C^2*–diffeomorphism.*

In order to have a clear–cut situation we formulate the following assumption to be required throughout the rest of this section if nothing else is said.

ASSUMPTION (A). *Suppose that*

$$F(x,z,p) \neq 0 \qquad and \qquad \Phi(x,z,p) \neq 0,$$

and that Hölder's transformation yields a diffeomorphism of G *onto* $G_* := \mathcal{H}_F(G)$.

Now we introduce the Hölder transform $H(x,z,y)$ of $F(x,z,p)$ by

$$H := \frac{1}{F \circ \mathcal{H}_F^{-1}} \tag{2.7}$$

Since $\mathcal{H}_F$ is of class C^2, we have $H \in C^2(G_*)$. If we assume that (x,z,p) and (x,z,y) are corresponding points in G and G_* respectively under $\mathcal{H}_F$, we can write (2.2) and (2.7) as

$$y = \frac{p}{F(x,z,p)}\ ,\ H(x,z,y) = \frac{1}{F(x,z,p)}\ ; \tag{2.8}$$

these formulas are clearly equivalent to

$$p = \frac{y}{H(x,z,y)} \quad , \quad F(x,z,p) = \frac{1}{H(x,z,y)} \; . \tag{2.9}$$

Therefore

$$\mathcal{H}_H = \mathcal{H}_F^{-1} \qquad \text{and} \qquad F = \; 1/H \circ \mathcal{H}_H^{-1}, \tag{2.10}$$

and we realize at once the involutory character of Hölder's transformation.

Let $\Psi(x,z,y)$ be the adjoint of $H(x,z,y)$, that is

$$\Psi(x,z,y) := y \cdot H_y(x,z,y) - H(x,z,y). \tag{2.11}$$

We claim that

$$\Psi = 1/\Phi \circ \mathcal{H}_F^{-1}. \tag{2.12}$$

More generally, we have

Proposition 2.2. *If* $(x,y,z) = \mathcal{H}_F(x,z,p)$, *then*

$$\begin{aligned} H_x(x,z,y) &= \frac{F_x(x,z,p)}{F(x,z,p)\Phi(x,z,p)}, \quad H_y(x,z,y) = \frac{F_p(x,z,p)}{\Phi(x,z,p)} \\ H_z(x,z,y) &= \frac{F_z(x,z,p)}{F(x,z,p)\Phi(x,z,p)}, \quad \Psi(x,z,y) = \frac{1}{\Phi(x,z,p)} \end{aligned} \tag{2.13}$$

Proof. Let us write $\mathcal{H}_H$ in the form

$$\mathcal{H}_H(x,z,y) = (x,z,\mathbf{p}(x,z,y)) \quad , \quad \mathbf{p}(x,z,y) = \frac{y}{H(x,z,y)},$$

and set $\hat{F} := F \circ \mathcal{H}_H$, i.e. $\hat{F}(x,z,y) = F(x,z,\mathbf{p}(x,z,y))$, etc. In order to prove $H_y = \hat{F}_p/\hat{\Phi}$, we fix x and z, i.e. we set $dx^i = 0$ and $dz = 0$ in the following differential forms. From $\mathbf{p}_k = y_k/H$ we infer that

$$d\mathbf{p}_k = H^{-1} \; dy_k - H^{-2} \; y_k \; H_{y_l} \; dy_l.$$

Since $\hat{F}H = 1$, we also have

$$\hat{F} H_{y_i} \; dy_i + H \hat{F}_{p_k} \; d\mathbf{p}_k = 0 \qquad , \quad \hat{F}_{p_k} := F_{p_k} \circ \mathcal{H}_H.$$

Then we infer that

$$0 = \hat{F}\{H_{y_i}\ dy_i + \hat{F}_{p_k}[H dy_k - y_k\, H_{y_l}\, dy_l]\}.$$

Since $\hat{F} \neq 0$, we obtain

$$H_{y_i} + H\hat{F}_{p_i} - y_l \hat{F}_{p_l} H_{y_i} = 0$$

and therefore

$$H\hat{F}_{p_i} = (y_l \hat{F}_{p_l} - 1) H_{y_i}$$

which is just

$$\hat{F}_{p_i} = H_{y_i}\hat{\Phi} \quad , \quad \text{i.e.} \quad H_y = \hat{F}_p/\hat{\Phi}.$$

This implies

$$\Psi = y \cdot H_y - H = \frac{\mathbf{p}\cdot\hat{F}_p}{\hat{F}\hat{\Phi}} - \frac{1}{\hat{F}} = \frac{1}{\hat{\Phi}}\,.$$

Finally we infer from $\mathbf{p} = y/H$ that

$$\mathbf{p}_z = -(H_z/H)\mathbf{p} \quad , \quad \mathbf{p}_x = -(H_x/H)\mathbf{p}.$$

From $H = 1/\hat{F}$ it follows that

$$H_z = -\hat{F}^{-2}(\hat{F}_z + \hat{F}_p \cdot \mathbf{p}_z) = -\hat{F}_z\hat{F}^{-2} + (\mathbf{p}\cdot\hat{F}_p)\ H_z\hat{F}^{-1}$$

whence

$$(\hat{F} - \mathbf{p}\cdot\hat{F}_p)\, H_z = -\hat{F}_z\hat{F}^{-1}.$$

On account of

$$H = \hat{F}^{-1} \quad , \quad \Psi = (\mathbf{p}\cdot\hat{F}_p - F)^{-1} = \hat{\Phi}^{-1}$$

we obtain

$$H_z = H\Psi\hat{F}_z = \hat{F}_z/(\hat{F}\hat{\Phi}),$$

and similarly

$$H_x = H\Psi\hat{F}_x = \hat{F}_x/(\hat{F}\hat{\Phi})$$

is proved. This completes the proof of Proposition 2.2. $\square$

Very sloppily, but quite instructively we write formulas (2.8) and (2.13) as

$$H = \frac{1}{F}\ ,\ \Psi = \frac{1}{\Phi}\ ,\ H_y = \frac{F_p}{\Phi}\ ,\ H_x = \frac{F_x}{F\Phi}\ ,\ H_z = \frac{F_z}{F\Phi}\,. \tag{2.14}$$

These identities are equivalent to

$$(2.14') \qquad F = \frac{1}{H}\ ,\ \ \Phi = \frac{1}{\Psi}\ ,\ \ F_p = \frac{H_y}{\Psi}\ ,\ \ F_x = \frac{H_x}{H\Psi}\ ,\ \ F_z = \frac{H_z}{H\Psi}.$$

Now we want to determine H_{yy}. The computations are considerably simplified if we use *Legendre's transformation* $\mathcal{L}_F$ generated by F.

Assumption (B). *Legendre's transformation $\mathcal{L}_F$ of G onto $G^* := \mathcal{L}_F(G)$, defined by*

$$(2.15) \qquad \mathcal{L}_F(x,z,p) := (x,z,\mathbf{g}(x,z,p)) \quad , \quad \mathbf{g}(x,z,p) := F_p(x,z,p),$$

is a diffeomorphism; in particular we have

$$(2.16) \qquad det\ F_{pp}(x,z,p) \neq 0 \qquad on \quad G.$$

Then we can define the Legendre transform $W(x,z,\xi)$ of $F(x,z,p)$ by

$$(2.17) \qquad W := \Phi \circ \mathcal{L}_F^{-1}.$$

If (x,z,p) and (x,z,ξ) are related by $(x,z,\xi) = \mathcal{L}_F(x,z,p)$, we have the following well-known formulas:

$$(2.18) \qquad \begin{aligned} &F(x,z,p) + W(x,z,\xi) = p\cdot\xi\ , \\ &\xi = F_p(x,z,p)\ \ ,\ \ p = W_\xi(x,z,\xi)\ , \\ &F_x(x,z,p) + W_x(x,z,\xi) = 0\ ,\ \ F_z(x,z,p) + W_z(x,z,\xi) = 0\ . \end{aligned}$$

In the spirit of (2.14) we write these relations in short-hand as

$$(2.19) \qquad \begin{aligned} &F + W = p\cdot\xi\ ,\ \ \xi = F_p\ ,\ \ p = W_\xi\ , \\ &F_x + W_x = 0\ ,\ \ F_z + W_z = 0. \end{aligned}$$

Let us introduce the adjoint M and the momentum tensor Γ of W by

$$(2.20) \qquad \mathrm{M} := \xi\cdot W_\xi - W\ ,\ \ \Gamma := \xi\otimes W_\xi - WI.$$

We immediately infer from (2.18), $F \in C^2$ and $\mathcal{L}_F^{-1} \in C^1$ that $W \in C^2$ and, moreover, that

$$(2.21) \qquad \mathcal{L}_W = \mathcal{L}_F^{-1}\ ,\ \ F = \mathrm{M}\circ\mathcal{L}_F.$$

The following result is well-known and easy to prove.

PROPOSITION 2.3. *Suppose that G is a normal domain of type C, and that $F_{pp}(x,z,p) > 0$ (or < 0) on G. Then $\mathcal{L}_F : G \to G^*$ is a diffeomorphism.*

Global invertibility of $\mathcal{H}_F$ will be discussed in Section 4. Let us also introduce the momentum tensor P of H by

$$\mathrm{P} := y \otimes H_y - HI \tag{2.22}$$

and the mapping $\mathcal{A}_F : G_* \to G^*$ by

$$\mathcal{A}_F := \mathcal{L}_F \circ \mathcal{H}_F^{-1} \tag{2.23}$$

LEMMA 2.3. *Suppose that assumptions (A) and (B) are satisfied. Then F_{pp} and H_{yy} are related by*

$$H_{y_i y_k} = [(-W^{-2}\Gamma^i_l) \circ \mathcal{A}_F](F_{p_l p_j} \circ \mathcal{H}_F^{-1})[-H^{-2}\mathrm{P}^k_j]. \tag{2.24}$$

PROOF. Let us express the mapping $\mathcal{A}_F$ by

$$\begin{aligned} &\mathcal{A}_F(x,z,y) = (x,z,\mathbf{x}(x,z,y)),\\ &\mathbf{x}(x,z,y) = F_p(x,z,\mathbf{p}(x,z,y)),\\ &\mathbf{p}(x,z,y) = \frac{y}{H(x,z,y)}. \end{aligned} \tag{2.25}$$

Moreover, we have

$$H_y = (F_p/\Phi) \circ \mathcal{H}_F^{-1} = (\xi/W) \circ \mathcal{A}_F \ ,$$

and therefore

$$H_y = \mathbf{x}/W(\ \cdot\ ,\ \ \cdot\ ,\ \ \mathbf{x}).$$

It follows that

$$H_{y_i y_k} = \ \frac{\partial}{\partial y_k}\ \frac{\mathbf{x}^i}{W(\ \cdot\ ,\ \ \cdot\ ,\mathbf{x})}\ \ = \ (\frac{\partial}{\partial \xi^l}\ \frac{\xi^i}{W}) \circ \mathcal{A}_F\ \frac{\partial \mathbf{x}^l}{\partial y_k}.$$

Moreover

$$\frac{\partial}{\partial \xi^l}\ \frac{\xi^i}{W} = -W^{-2}\ \Gamma^i_l$$

and

$$\frac{\partial \mathbf{x}^l}{\partial y_k} = F_{p_l p_j}(\ \cdot\ ,\ \cdot\ ,\ \mathbf{p})\ \frac{\partial \mathbf{p}^j}{\partial y_k}$$

Finally Lemma 2.2 yields

$$\frac{\partial \mathbf{p}_j}{\partial y_k} = -H^{-2}\ \mathrm{P}_j^k.$$

Combining these formulas we obtain (2.24).

In our usual short-hand (2.24) reads

$$H_{y_i y_k} = W^{-2}\ \Gamma_l^i\ F_{p_l p_j}\ H^{-2}\ \mathrm{P}_j^k \tag{2.26}$$

Moreover, we also have

$$\mathrm{M} = F = 1/H\ , \qquad W = \Phi = 1/\Psi\ , \tag{2.27}$$

(which means that

$$\mathrm{M}(x,z,\xi) = F(x,z,p) = 1/H(x,z,y), W(x,z,\xi) = \Phi(x,z,p) = 1/\Psi(x,z,y)$$

if $(x,z,\xi) \leftrightarrow (x,z,p) \leftrightarrow (x,z,y)$), and thus (2.26) can be rewritten into

$$H_{y_i y_k} = (F/\Phi)^2\ \Gamma_l^i\ F_{p_l p_j}\ \mathrm{P}_j^k. \tag{2.28}$$

□

Lemma 2.4. *Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be two vectors in $\mathbb{R}^n, \lambda \in \mathbb{R}$, and $\mu := a \cdot b - \lambda$. Then the matrix $T = (t_{ik})$ defined by*

$$t_{ik} = a_i b_k - \lambda \delta_{ik}\ \ ,\ \ 1 \le i, k \le n,$$

is invertible of both $\lambda \neq 0$ and $\mu \neq 0$, and its inverse $S = (s_{ik})$ is given by

$$s_{ik} = \frac{1}{\lambda\mu}(a_i b_k - \mu \delta_{ik})$$

Proof. A straight-forward computation shows that $s_{ik} t_{kl} = \delta_{il}$ if $\lambda \neq 0$ and $\mu \neq 0$. □

LEMMA 2.5. *If assumptions (A) and (B) are satisfied, then*

$$\mathrm{P} = T^{-1} \qquad \textit{and} \qquad \mathrm{P} = (F\Phi)^{-1}\ \Gamma^T. \tag{2.29}$$

PROOF. From

$$\mathbf{p}_y = \mathbf{y}_p^{-1}\ ,\ \ \mathbf{y}_p = -F^{-2}T\ ,\ \ \mathbf{p}_y = -H^{-2}\mathrm{P}$$

we infer that

$$H^{-2}\mathrm{P} = F^2T^{-1}$$

and $FH = 1$ yields $\mathrm{P} = T^{-1}$.

Now let $S = (S^i_k) := T^{-1}$. By Lemma 2.4 we have

$$S^i_k \ = \ \frac{1}{F\Phi}\ (p_k F_{p_i} - \Phi\delta^i_k).$$

Since $p_k = W_{\xi k}$, $F_{p_i} = \xi^i$, and $\Phi = W$, it follows that

$$T^{-1} = (F\Phi)^{-1}(W_\xi \otimes \xi - WI) = (F\Phi)^{-1}\ \Gamma^T,$$

and, finally, $\mathrm{P} = T^{-1}$ yields $\mathrm{P} = (F\Phi)^{-1}\ \Gamma^T$. This completes the proof of Lemma 2.5. □

PROPOSITION 2.4. *Suppose that assumptions (A) and (B) are satisfied. Then we have*

$$H_{yy} = (F^3/\Phi)\mathrm{P}^T F_{pp}\ \mathrm{P}. \tag{2.30}$$

PROOF. Relation (2.28) can be written as

$$H_{yy} = (F/\Phi)^2\,\Gamma\,F_{pp}\,\mathrm{P}.$$

By (2.29) it follows that

$$H_{yy} = F\,\Phi^{-3}\,\Gamma\,F_{pp}\,\Gamma^T,$$

and $\Gamma = F\,\Phi\,\mathrm{P}^T$ implies (2.30). □

COROLLARY 2.1. *Let $\epsilon = \pm 1$ be the sign of $F\Phi$. Then $F_{pp} > 0$ (< 0) implies that $\epsilon H_{yy} > 0$ (< 0) and $W_{\xi\xi} > 0$ (< 0).*

Proof. The first assertion follows from (2.30), the second is a consequence of $W_{\xi\xi} = (F_{pp})^{-1}$. □

Let us now add some remarks on the global invertibility of Hölder's transformation. For this purpose we assume that $G \subset \hat{M}$ is a normal domain of type S, i.e., $G = \{(x, z, p) : (x, z) \in U, p \in B(x, z)\}$, where $B(x, z)$ is a domain in $\mathbb{R}^n$ which is star-shaped with respect to $p = 0$, and U is a domain in M. We also assume that

$$F(x, z, p) \neq 0 \quad \text{and} \quad \Phi(x, z, p) \neq 0 \quad \text{on } G. \tag{2.31}$$

Fix $(x, z) \in U$ and consider the mapping $\mathbf{y}(x, z, \cdot) : B(x, z) \to \mathbb{R}^n$ defined by

$$\mathbf{y}(x, z, p) := p/F(x, z, p).$$

Clearly we have $\mathbf{y}(x, z, 0) = 0$. Now fix some unit vector e in $\mathbb{R}^n$ and choose I as the largest interval in $\mathbb{R}^n$ such that $\lambda e \in B(x, z)$ for $\lambda \in I$. We consider the function $\varphi : I \to \mathbb{R}$ defined by

$$\varphi(\lambda) = \lambda/F(x, z, \lambda e).$$

Because of

$$\varphi'(\lambda) = -\Phi(x, z, \lambda e)/F^2(x, z, \lambda e)$$

we have $\varphi'(\lambda) \neq 0$ on I. Therefore $\mathbf{y}(x, z, \cdot)$ maps the segment $\Sigma = \{\lambda e : \lambda \in I\}$ bijectively onto the segment $\Sigma^* = \{\lambda^* e : \lambda^* \in \varphi(I)\}$ We then conclude that $\mathbf{y}(x, z, \cdot)$ maps $B(x, z)$ bijectively onto a domain $B_*(x, z)$ of $\mathbb{R}^n$ which is also star-shaped with respect to the origin, and so we obtain

Proposition 2.5. *If $F \in C^2(G)$ satisfies (2.31) on a normal domain G of type S, then $\mathcal{H}_F$ yields a C^2–diffeomorphism of G onto $G_* := \mathcal{H}_F(G)$, and G_* is also a normal domain of type S.*

3. The commuting diagram

Suppose that $F \in C^2(G)$ satisfies assumptions (A) and (B); i.e. $\mathcal{H}_F$ and $\mathcal{L}_F$ are diffeomorphisms of G onto G_* and G^* respectively. Then H, Ψ and W, M are well-defined and non-vanishing; thus we can define the mappings $\mathcal{H}_W$ and $\mathcal{L}_H$.

PROPOSITION 3.1. *If $F \in C^2(G)$ satisfies assumptions (A) and (B), and if $\mathcal{H}_F$ and $\mathcal{L}_H$ are diffeomorphisms, then we have*

$$\mathcal{L}_H \circ \mathcal{H}_F = \mathcal{H}_W \circ \mathcal{L}_F \tag{3.1}$$

and

$$\Psi \circ \mathcal{L}_H^{-1} = \frac{1}{W \circ \mathcal{H}_W^{-1}} \tag{3.2}$$

PROOF. (i) The mappings $\mathcal{L}_F$ and $\mathcal{H}_F$ are described by

$$p \mapsto \xi = F_p(x,z,p) \quad \text{and} \quad \xi \mapsto v = \frac{\xi}{W(x,z,\xi)}$$

respectively. Since $W(x,z,\xi) = \Phi(x,z,p)$, the composition $\mathcal{H}_W \circ \mathcal{L}_F$ is given by

$$p \mapsto v = \frac{F_p(x,z,p)}{p \cdot F_p(x,z,p) - F(x,z,p)} \tag{3.3}$$

(ii) On the other hand, $\mathcal{H}_F$ and $\mathcal{L}_H$ are given by $p \mapsto y = \dfrac{p}{F(x,z,p)}$ and $y \mapsto v = H_y(x,z,y)$ respectively.

By Proposition 2.2 we have

$$H_y(x,z,y) = \frac{F_p(x,z,p)}{\Phi(x,z,p)},$$

and therefore also $\mathcal{L}_H \circ \mathcal{H}_F$ is described by (3.3). This proves (3.1).

(iii) On account of (3.1) and of

$$\Psi = (1/\Phi) \circ \mathcal{H}_F^{-1} \quad , \quad W = \Phi \circ \mathcal{L}_F^{-1}$$

we infer that

$$\begin{aligned} \Psi \circ \mathcal{L}_H^{-1} &= (1/\Phi) \circ \mathcal{H}_F^{-1} \circ \mathcal{L}_F^{-1} = (1/\Phi) \circ (\mathcal{L}_H \circ \mathcal{H}_F)^{-1} \\ &= (1/\Phi) \circ (\mathcal{H}_W \circ \mathcal{L}_F)^{-1} = (1/\Phi) \circ \mathcal{L}_F^{-1} \circ \mathcal{H}_W^{-1} = (1/W) \circ \mathcal{H}_W^{-1}, \end{aligned}$$

and thus the Proposition is proved. □

This result has the following consequences.

1. Suppose that the assumptions of Proposition 3.1 are satisfied. Then we can define *Haar's transformation* $\mathcal{R}_F$ by

$$\mathcal{R}_F := \mathcal{L}_H \circ \mathcal{H}_F = \mathcal{H}_W \circ \mathcal{L}_F. \tag{3.4}$$

In coordinates the mapping $\mathcal{R}_F : (x, z, p) \mapsto (x, z, v)$ is given by (3.3); it describes a diffeomorphism of G onto $\mathcal{G} := \mathcal{R}_F(G)$.

Let us now introduce the *Haar transform* L of F by setting

$$L := \frac{1}{\Phi \circ \mathcal{R}_F^{-1}}. \tag{3.5}$$

In coordinates $F(x, z, p)$ and its Haar transform $L(x, z, v)$ are related by

$$L(x, z, v) = \frac{1}{p \cdot F_p(x, z, p) - F(x, z, p)}. \tag{3.6}$$

By (3.1) we have

$$(\mathcal{L}_H \circ \mathcal{H}_F)^{-1} = \mathcal{L}_F^{-1} \circ \mathcal{H}_W^{-1} = \mathcal{L}_W \circ \mathcal{H}_L,$$

that is,

$$\mathcal{R}_F^{-1} = \mathcal{R}_L. \tag{3.7}$$

In other words, Haar's transformation is in the same sense an involution as the transformations of Legendre and Hölder. Moreover we infer from (2.23) that

$$\mathcal{R}_F = \mathcal{A}_H = \mathcal{A}_W^{-1}. \tag{3.8}$$

All these results are described by the following commuting diagram:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{matrix} x,z,p;\ F \\ G \end{matrix}\ \boxed{\mathrm{I}} \arrow[r, "\mathcal{H}_F"] \arrow[d, "\mathcal{L}_F"'] \arrow[dr, "\mathcal{R}_F"'] & \boxed{\mathrm{III}}\ \begin{matrix} x,z,y;\ H \\ G_* \end{matrix} \arrow[d, "\mathcal{L}_H"] \\
\begin{matrix} x,z,\xi;\ W \\ G^* \end{matrix}\ \boxed{\mathrm{II}} \arrow[r, "\mathcal{H}_W"'] & \boxed{\mathrm{IV}}\ \begin{matrix} x,z,v;\ L \\ \mathcal{G} \end{matrix}
\end{tikzcd} \tag{3.9}
$$

One can freely move from one corner to the other, thereby obtaining the corresponding functions F, H, W, L as results of the marked transformations, each of which is on involution in the sense described above.

2. Suppose that F satisfies

$$F(x,z,p) \neq 0 \ , \ \Phi(x,z,p) \neq 0 \ , \ det\, F_{pp}(x,z,p) \neq 0. \tag{3.10}$$

Then $\mathcal{L}_F$ and $\mathcal{H}_F$ are local diffeomorphisms according to Proposition 2.1 and (2.15), and thus the Legendre transform $W = \Phi \circ \mathcal{L}_F^{-1}$ and the Hölder transform $H = (1/F) \circ \mathcal{H}_F^{-1}$ are locally well-defined. By (2.27) we have

$$W(x,z,\xi) \neq 0 \ , \ \mathrm{M}(x,z,\xi) \neq 0, \tag{3.11}$$

and (2.26) implies

$$\det H_{yy} = (F/\Phi)^{n+2} \det F_{pp} \tag{3.12}$$

whence

$$\det H_{yy} \neq 0. \tag{3.13}$$

Thus also $\mathcal{H}_W$ and $\mathcal{L}_H$ are local diffeomorphisms, and consequently the *commuting diagram* (3.9) *is locally valid provided that assumption* (3.10) *is satisfied.*

It remains the question to formulate conditions on F such that diagram (3.9) is globally valid. It will turn out that, essentially, the following assumption is sufficient for this purpose:

$$F(x,z,p) \neq 0 \ , \ \Phi(x,z,p) \neq 0 \ , \ F_{pp}(x,z,p) > 0 \quad (\text{or } < 0). \tag{3.14}$$

In fact, if G and G_* are normal domains of type C, then the definiteness of F_{pp} and H_{yy} respectively is sufficient for $\mathcal{L}_F$ and $\mathcal{L}_H$ to be globally invertible, and, according to Corollary 2.1, the definiteness of F_{pp} implies that of H_{yy}. Note that G_* is of type S (cf. Proposition 2.5); however, if G is of type C, then G_* is in general only of type S and not of type C.

If we assume that both G and $G_* = \mathcal{H}_F(G)$ are normal domains of type C and that F satisfies (3.14), then $\mathcal{L}_F, \mathcal{H}_F, \mathcal{L}_H$, and $\mathcal{R}_F = \mathcal{L}_H \circ \mathcal{H}_F$ are (globally) invertible. Since we can operate locally with the commuting diagram, we infer that also $\mathcal{H}_W = \mathcal{R}_H \circ \mathcal{L}_F^{-1}$ is invertible, and so we are in the situation of Proposition 3.1. Similarly we can argue if $G^* = \mathcal{L}_F(G)$ is of type C. Thus we obtain

THEOREM 3.1. *If $F \in C^2(G)$ satisfies* (3.14), *and if both G and $G_* = \mathcal{H}_F(G)$ (or $G^* = \mathcal{L}_F(G)$ respectively) are normal domains of type C, then $\mathcal{H}_F, \mathcal{L}_H, \mathcal{L}_F, \mathcal{H}_W$ are diffeomorphisms satisfying*

$$\mathcal{L}_H \circ \mathcal{H}_F = \mathcal{H}_W \circ \mathcal{L}_F \qquad \textit{and} \qquad \Psi \circ \mathcal{L}_H^{-1} = (1/W) \circ \mathcal{H}_W^{-1}.$$

The next result is an obvious but possibly useful modification of the preceding theorem.

THEOREM 3.2. *Let G be a normal domain of type C, and suppose that $F \in C^2(G)$. Moreover, let G'_* be a normal domain of type C contained in $G_* = \mathcal{H}_F(G)$ and $G' := \mathcal{H}_F^{-1}(G'_*)$. Then $F|_{G'}$ generates a commuting diagram as described in Proposition* 3.1.

Example. Consider the function

$$F(x,z,p) = -\ \frac{1}{\omega(x,z)}\sqrt{1+|p|^2}\ \ ,\ \omega(x,z) > 0$$

defined on $G = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. Its adjoint is given by

$$\Phi(x,z,p) = \frac{1}{\omega(x,z)}\ \frac{1}{\sqrt{1+|p|^2}}.$$

The transforms H, L, W are found to be

$$H(x,z,y) = -\sqrt{\omega^2(x,z) - |y|^2}\ ,\ \ W(x,z,\xi) = \sqrt{\omega^{-2}(x,z) - |\xi|^2},$$

$$L(x,z,v) = \omega(x,z)\sqrt{1+|v|^2}.$$

Moreover, we have $\mathcal{G} := \mathcal{R}_F(G) = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, while

$$\begin{aligned} G^* &:= \mathcal{L}_F(G) = \{(x,z,\xi) : (x,z) \in \mathbb{R}^n \times \mathbb{R},\ \ |\xi| < 1/\omega(x,z)\}, \\ G^* &:= \mathcal{H}_F(G) = \{(x,z,y) : (x,z) \in \mathbb{R}^n \times \mathbb{R},\ \ |y| < \omega(x,z)\}. \end{aligned}$$

Haar's transformation $\mathcal{R}_F$ is given by $v = -p$, while the other four transformations of the diagram are described by

$$y = \frac{-\sigma p}{\sqrt{1+|p|^2}}\ ,\ \ v = \frac{y}{\sqrt{\sigma^2 - |y|^2}}\ ,\ \ \xi = \frac{-\sigma p}{\sqrt{1+|p|^2}}\ ,\ \ v = \frac{\xi}{\sqrt{\sigma^2 - |\xi|^2}}.$$

Here we have $\sigma := 1/\omega(x,z)$.

4. Relations to contact transformations

It might be useful to remind the reader of the connection of the preceding discussion with Lie's theory of contact transformations. To simplify matters we drop the variables x, z and consider only $p = (p_1, \ldots, p_n)$. To obtain a higher point of view it is, however, useful to introduce $n+1$ further variables φ and $\pi = (\pi_1, \ldots, \pi_n)$ and to operate in the contact space $\mathcal{M}$ of points $e = (p, \varphi, \pi) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ equipped with the contact from

$$\omega = d\varphi - \pi \cdot dp, \tag{4.1}$$

that is,

$$\omega = d\varphi - \pi_i \ dp_i$$

(summation with the respect to i from 1 to n). According to Lie a smooth mapping $\mathcal{T} : G \to \mathcal{M}$ of a domain G in $\mathcal{M}$ is said to be a contact transformation if there is a nonvanishing function $\rho : G \to \mathbb{R}$ such that

$$\mathcal{T}^*\omega = \rho\omega. \tag{4.2}$$

If $\mathcal{T}$ is given by

$$\bar{p} = A(p, \varphi, \pi) \ , \ \ \bar{\varphi} = Z(p, \varphi, \pi) \ , \ \ \bar{\pi} = B(p, \varphi, \pi) \ ,$$

(4.2) means that

$$dZ - B_i \ dA_i = \rho[d\varphi - \pi_i dp_i] \ , \ \ \rho \neq 0.$$

It is well known that every contact transformation is an immersion. In the sequel we shall only consider contact transformations which are diffeomorphisms.

Consider now a smooth function $F(p), p \in \mathcal{P} \subset \mathbb{R}^n$. We interpret its graph as a hypersurface $\mathcal{S}_F$ in $\mathbb{R}^{n+1}$,

$$\mathcal{S}_F = \{(p, \varphi) : \varphi = F(p) \ , p \in \mathcal{P}\}. \tag{4.3}$$

With $\mathcal{S}_F$ we associate the n–strip $\mathcal{E}_F : \mathcal{P} \to \mathcal{M}$ of its surface elements $\mathcal{E}_F(p) := (p, F(p), F_p(p))$ satisfying $\mathcal{E}_F^*\omega = 0$. (In Lie's terminology immersions $\mathcal{E} : \mathcal{P} \to \mathcal{M}$ of n–dimensional domains $\mathcal{P}$ into $\mathcal{M}$ satisfying $\mathcal{E}^*\omega = 0$ are denoted as *n–dimensionale Elementvereine.*) Then, for any contact transformation $\mathcal{T} : \mathcal{G} \to \mathcal{M}$ defined on a domain G containing $\mathcal{E}_F(\mathcal{P})$, we can define a new n–strip $\mathcal{T} \circ \mathcal{E}_F$ which is given by

$$(\mathcal{T} \circ \mathcal{E}_F)(p) = (A(\mathcal{E}_F(p)), Z(\mathcal{E}(p)), B(\mathcal{E}_F(p))) \tag{4.4}$$

Set

$$\mathbf{f}(p) := A(\mathcal{E}_F(p)) \tag{4.5}$$

and suppose that the mapping $p \mapsto \bar{p} = \mathbf{f}(p)$ yields a diffeomorphism of $\mathcal{P}$ onto $\bar{\mathcal{P}} = \mathbf{f}(\mathcal{P}) \subset \mathbb{R}^n$. Then we can reparametrize $\mathcal{T} \circ \mathcal{E}_F$ by means of $\mathbf{f}$, thus obtaining a new n–strip $\bar{\mathcal{E}}_F$ defined as

$$\bar{\mathcal{E}}_F := \mathcal{T} \circ \mathcal{E}_F \circ \mathbf{f}^{-1} \tag{4.6}$$

The strip condition (implied by the fact that $\mathcal{T}$ is a contact transformation) means that

(4.7) $$\bar{\mathcal{E}}_F^* \ \omega = 0.$$

Let us introduce the *$\mathcal{T}$-adjoint of F*, $\Phi : \mathcal{P} \to \mathbb{R}^n$, by

(4.8) $$\Phi := Z \circ \mathcal{E}_F \ , \text{ i.e. } \Phi(p) = Z(p, F(p), F_p(p)),$$

and set

(4.9) $$\bar{F} := \Phi \circ \mathbf{f}^{-1}.$$

Then (4.5) and (4.7) imply

(4.10) $$\bar{\mathcal{E}}_F(\bar{p}) = (\bar{p}, \bar{F}(\bar{p}), \bar{F}_{\bar{p}}(\bar{p})) \ , \ \bar{p} \in \bar{\mathcal{P}},$$

i.e. we have found that

(4.11) $$\bar{\mathcal{E}}_F = \mathcal{E}_{\bar{F}}$$

To see the analogy between these formulas and those of Sections 2 and 3, we write $\mathcal{T}_F$ for $\mathbf{f}$, i.e.

(4.12) $$\mathcal{T}_F := A \circ \mathcal{E}_F$$

and we call $\mathcal{T}_F$ the *$\mathcal{T}$–transformation* generated by F, and $\bar{F} : \bar{\mathcal{P}} \to \mathbb{R}$ is said to be the *$\mathcal{T}$–transform* of F. If $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is an involution, i.e. $\mathcal{T} \circ \mathcal{T} = id$, then the mappings $\mathcal{T}_F : \mathcal{P} \to \bar{\mathcal{P}}$ are involutions in the sense that $\mathcal{T}_F^{-1} = \mathcal{T}_{\bar{F}}$, provided that $\mathcal{T}_F$ is invertible, and $\bar{\bar{F}} = F$, i.e. the $\mathcal{T}$-transform of $\bar{F}$ is F.

For example, Legendre's contact transformation $\mathcal{L}$ is the involutory contact transformation $\mathcal{L} : \mathcal{M} \to \mathcal{M}$ defined by

$$\bar{p} = \bar{\pi} \ , \ \bar{\varphi} = p \cdot \pi - \varphi \ , \ \bar{\pi} = p.$$

The directrix equation of $\mathcal{L}$ is $p \cdot \bar{p} - \varphi - \bar{\varphi} = 0$, which is the polar equation of a paraboloid $|p|^2 - 2\varphi = 0$. Given a function $F(p)$, $\mathcal{L}_F$ is the classical Legendre transformation considered in Section 2. Moreover, the $\mathcal{L}$–adjoint $\Phi(p)$ is just $\Phi = p \cdot F_p - F$, and the $\mathcal{L}$–transform of F is the Legendre transform W.

Secondly, consider the contact transformation $\mathcal{R}$ defined by

$$\bar{p} = \frac{\pi}{p \cdot \pi - \varphi} \ , \ \bar{\varphi} = \frac{1}{p \cdot \pi - \varphi} \ , \ \bar{p} = \frac{p}{\varphi} \ .$$

Except for an additional reflection $\bar{p} = p$, $\bar{\varphi} = -\varphi$, $\bar{\pi} = -\pi$, this transformation is derived from the directrix equation $p \cdot \bar{p} + \varphi\bar{\varphi} - 1$, the polar equation of the unit sphere $|p|^2 + \varphi^2 - 1 = 0$. Note that $\mathcal{R}$ is just the classical transformation by reciprocal polars, and $\mathcal{R}_F$ is Haar's transformation generated by F. As $\mathcal{R}$ is an involution, we obtain that $\mathcal{R}_F$ is an involution in the sense that $\mathcal{R}_F^{-1} = \mathcal{R}_L$ where $L = \bar{F}$ is the Haar transform of F, and $\bar{L} = F$.

Thirdly we consider the contact transformation $\mathcal{H}$ introduced by Carathéodory (see [1], p. 403), which is given by

$$\bar{p} = \frac{p}{\varphi}\ , \quad \bar{\varphi} = \frac{1}{\varphi}\ , \quad \bar{\pi} = \frac{\pi}{p \cdot \pi - \varphi}\ .$$

This transformation is also an involution, and $\mathcal{H}_F$ is just Hölder's transformation of Section 2, and $\bar{F}$ is the Hölder transform H of F. Moreover, one easily checks that

$$\mathcal{H} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{H}$$

and that

$$\mathcal{R} = \mathcal{L} \circ \mathcal{H}.$$

The geometric interpretation of the "derived transformations" $\mathcal{H}_F, \mathcal{L}_F, \mathcal{H}_F, \mathcal{L}_H$ yields now without computation the commuting diagram of Section 3,

$$\mathcal{R}_F = \mathcal{H}_W \circ \mathcal{L}_F = \mathcal{L}_H \circ \mathcal{H}_F.$$

Another useful commuting diagram is derived from the apsidal transformation $\mathcal{A}$ which commutes with the polar reciprocity $\mathcal{R}$ in the sense that $\mathcal{R} \circ \mathcal{A} = -\mathcal{A} \circ \mathcal{R}$. Following Herglotz [6], an elegant formulation of $\mathcal{A}$ is obtained if we introduce for $e = (p, \varphi, \pi)$ homogeneous coordinates $E = (Q, N)$ defined by

$$Q = (p, \varphi)\ ,\ \ N = \left(\frac{\pi}{p \cdot \pi - \varphi}\ ,\ -\frac{1}{p \cdot \pi - \varphi} \right).$$

Note that the points (Q, N) lie on the quadric $N \cdot Q = 1$. In the new coordinates E the polar reciprocity $\mathcal{R}$ is expressed by

$$\mathcal{R}(Q, N) = (N, Q).$$

The apsidal transformation $\mathcal{A}(Q,N) = (\bar{Q},\bar{N})$ is defined by

$$\sigma\bar{Q} = Q - |Q|^2 N \ , \ \sigma\bar{N} = |N|^2 Q - N \ , \ \sigma := \pm\sqrt{|Q|^2|N|^2 - 1}.$$

Since $\mathcal{A}$ provides a 1-to-2 correspondence in $\mathcal{M}$, we can sloppily write $\mathcal{R}\circ\mathcal{A} = \mathcal{A}\circ\mathcal{R}$. This fact can be used to show that $\mathcal{R}$ maps Fresnel surfaces into Fresnel surfaces (cf. [6]). Just as we have in [7] used the commuting diagram (3.9) to obtain a fourfold picture of the calculus of variations, one might use transformations $\mathcal{T}_F$ derived from any other contact transformation $\mathcal{T}$ to obtain further interesting pictures, but $\mathcal{A}$ might be particularly useful.

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