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Doubly Uniform Complete Law of Large Numbers for Independent Point Processes

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Abstract. We prove a law of large numbers in terms of uniform complete convergence of independent random variables taking values in functions of 2 parameters which share similar monotonicity properties as the increments of monotone functions in the initial and the final time parameters. The assumptions for the main result are the Hölder continuity on the expectations as well as moment conditions, while the sample functions may contain jumps.

1. Introduction

Let $T > 0$, which we fix throughout this paper. Put $\Delta = \{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leqq t_1 \leqq t_2 \leqq T\}$, and denote by $\mathcal{D}$ the set of functions $y: \Delta \to [0, \infty)$ which, for $(t_1, t_2) \in \Delta$, is non-increasing in t_1 and non-decreasing in t_2, right continuous in each variable, and satisfies $y(t, t) = 0$ for $t \in [0, T]$. An example is the increment $y(s, t) = z(t) - z(s)$ of an increasing right continuous function $z: [0, T] \to \mathbb{R}$.

We consider $\mathcal{D}$ valued random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We call a function $Z: \Omega \to \mathcal{D}$ a $\mathcal{D}$ valued random variable, if for each pair of rationals $(t_1, t_2) \in \Delta \cap \mathbb{Q}^2$, the function $Z(t_1, t_2): \Omega \to \mathbb{R}$ is a real valued random variable in the standard sense, i.e., Borel measurable function with finite expectation $\mathrm{E}[\, Z(t_1, t_2)\,]$. Monotonicity and right continuity imply that $Z(t_1, t_2)$ are random variables with finite expectations also on irrational points, and that the supremum with respect to $(t_1, t_2) \in \Delta$ is equal to the supremum on countable points $(t_1, t_2) \in \Delta \cap \mathbb{Q}^2$, hence is measurable. Since a $\mathcal{D}$ valued random variable is determined by a countable number of random variables, a sequence of independent $\mathcal{D}$ valued random variables makes sense.

In this paper we prove the following.

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THEOREM 1. *Let $r > 0$ and $q > 2$. For each $N \in \mathbb{N}$, let $Z_i^{(N)}$, $i = 1, 2, \dots, N$, be a sequence of independent, $\mathcal{D}$ valued random variables, and let $M^{(N)}$ be a positive real, and $w_i^{(N)}$, $i = 1, 2, \dots, N$, a sequence of nonnegative reals. Assume the following for each $i = 1, 2, \dots, N$ and $N \in \mathbb{N}$:*

$$
\begin{aligned}
&\text{(i)}\ \ \mathrm{E}[\, Z_i^{(N)}(0,T)^q \,]^{1/q} \leqq M^{(N)}, \\
&\text{(ii)}\ \ |\mathrm{E}[\, Z_i^{(N)}(t_1,t_2) - Z_i^{(N)}(s_1,s_2) \,]| \\
&\qquad \leqq M^{(N)} w_i^{(N)} \, (|t_1 - s_1|^r + |t_2 - s_2|^r), \\
&\qquad (s_1,s_2),\ (t_1,t_2) \in \Delta.
\end{aligned} \tag{1.1}
$$

Then the arithmetic average $Y^{(N)} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} Z_i^{(N)}$ satisfies

$$
\begin{aligned}
&\mathrm{E}[\, \sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_1,t_2) - \mathrm{E}[\, Y^{(N)}(t_1,t_2) \,]|^q \,] \\
&\leqq \frac{M^{(N)q} 2^{q-1}}{N^{q^2 r/(2qr+2r+2)}} \, (C_q^q \, (2T\, \overline{w}^{(N)1/r} + 1) + 2^{2q}), \\
&N = N_0, N_0 + 1, \dots ,
\end{aligned} \tag{1.2}
$$

where we put

$$
\overline{w}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} w_i^{(N)} , \tag{1.3}
$$

and N_0 is the smallest integer satisfying $N_0^{qr/(2qr+2r+2)} \geqq 2$, and C_q is the positive constant in Proposition 6, a constant depending only on q.

If in addition, $\{M^{(N)}, \overline{w}^{(N)}\}$ is bounded, and

$$
(q^2 - 2q - 2)r > 2 \tag{1.4}
$$

holds, then

$$
\begin{aligned}
&\lim_{N_0\to\infty} \sum_{N \geqq N_0} \mathrm{P}[\, \sup_{(t_1,t_2)\in\Delta} \left| \frac{1}{N} \sum_{i=1}^{N} (Z_i^{(N)}(t_1,t_2) - \mathrm{E}[\, Z_i^{(N)}(t_1,t_2) \,]) \right| \\
&\qquad > \epsilon \,] = 0
\end{aligned} \tag{1.5}
$$

for all $\epsilon > 0$.

The result (1.5) states the complete convergence (to 0) of Hsu and Robbins [6, 2, 3], of the sequence

$$\sup_{(t_1,t_2)\in\Delta} \left| \frac{1}{N} \sum_{i=1}^{N} (Z_i^{(N)}(t_1,t_2) - \mathrm{E}[\, Z_i^{(N)}(t_1,t_2)\,]) \right|, \ N = 1, 2, \dots ,$$

which in particular implies that this sequence converges almost surely. We shall in this paper refer to (1.5) the complete law of large numbers uniform in 2 parameters t_1 and t_2.

Note also that we assume global Hölder continuity properties (1.1)(ii) only on the expectation. In particular, the sample paths may contain jumps as in point processes. The condition on the expectation, on the other hand, is in contrast to the case of 1 variable (see §A). Comparing Lemma 4 and Lemma 10, the difference may be explained that a line is finitely ramified while a square is infinitely ramified; we can cut an interval into small pieces by a finite number of points, while this is impossible for a square.

Consideration of supremum in 2 variables appears when we consider law of large numbers for dependent random variables as a perturbation of that for independent random variables. Stronger bounds on independent variables imply clearer proofs when dependence is introduced as a perturbation. See [5] for an application of our result to the hydrodynamic limit of stochastic ranking process with position dependent intensities.

As a simpler example, We can apply Theorem 1 to the increment

$$Z_i^{(N)}(t_1,t_2) \mapsto Z_i^{(N)}(t_2) - Z_i^{(N)}(t_1)$$

of increasing processes $Z_i^{(N)}$, to obtain the following.

Corollary 2. *Let* $D_\uparrow = D_\uparrow([0,T],\mathbb{R})$ *be the set of non-decreasing, right continuous functions on a closed interval* $[0,T]$. *Let* $r > 0$ *and* $q > 2$. *For each* $N \in \mathbb{N}$, *let* $Z_i^{(N)} : \ \Omega \to D_\uparrow$, $i = 1, 2, \dots , N$, *be a sequence of independent,* $D_\uparrow$ *valued random variables, and let* $M_i^{(N)}$ *be a positive real, and* $w_i^{(N)}$, $i = 1, 2, \dots , N$, *a sequence of nonnegative reals. Assume the* *following for each* $i = 1, 2, \dots, N$ *and* $N \in \mathbb{N}$*:*

$$\text{(1.6)}\qquad \begin{aligned} &\textit{(i)}\ \ \mathrm{E}[\,|Z_i^{(N)}(T) - Z_i^{(N)}(0)|^q\,]^{1/q} \leqq M^{(N)}, \\ &\textit{(ii)}\ \ |\mathrm{E}[\,Z_i^{(N)}(t)\,] - \mathrm{E}[\,Z_i^{(N)}(s)\,]| \\ &\qquad \leqq M^{(N)} w_i^{(N)} |t-s|^r,\ \ s,t \in [0,T], \end{aligned}$$

where $\overline{w}^{(N)}$ *is as in (1.3).*

Then the arithmetic average $Y^{(N)}(t) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} Z_i^{(N)}(t)$ *satisfies*

$$\text{(1.7)}\qquad \begin{aligned} &\mathrm{E}[\sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_2) - Y^{(N)}(t_1) - \mathrm{E}[\,Y^{(N)}(t_2) - Y^{(N)}(t_1)\,]|^q\,] \\ &\leqq \frac{M^{(N)q} 2^{q-1}}{N^{q^2 r/(2qr+2r+2)}} (C_q^q\,(2T\,\overline{w}^{(N)1/r} + 1) + 2^{2q}), \\ &N = N_0, N_0+1, \dots, \end{aligned}$$

where N_0 *is the smallest integer satisfying* $N_0^{qr/(2qr+2r+2)} \geqq 2$*, and* C_q *is a positive constant depending only on* q*.*

If in addition, $\{M^{(N)}, \overline{w}^{(N)}\}$ *is bounded, and (1.4) holds, then a doubly uniform complete law of large numbers*

$$\text{(1.8)}\qquad \begin{aligned} \lim_{N_0\to\infty} \sum_{N \geqq N_0} \mathrm{P}[\sup_{(t_1,t_2)\in\Delta} \Bigg| \frac{1}{N}\sum_{i=1}^{N} &(Z_i^{(N)}(t_2) - Z_i^{(N)}(t_1)) \\ &- \mathrm{E}[\,Z_i^{(N)}(t_2) - Z_i^{(N)}(t_1)\,])\Bigg| > \epsilon\,] = 0 \end{aligned}$$

for all $\epsilon > 0$*.*

Unfortunately, (1.8) is also a direct consequence of a uniform law for single variable (such as Proposition 9 in §A). (Theorem 1 of course works for a wider class.)

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2. Uniform Finite Dimensional Approximation for Functions in $\mathcal{D}$

In this section we prove the following estimate for functions in $\mathcal{D}$, the set of functions on $[0,T]\times[0,T]$, non-increasing in the first variable and non-decreasing in the second variable and taking 0 at diagonal points. While this is the crucial estimate for the proof in §3 of the main theorem, the results in this section hold without reference to probability spaces.

Proposition 3. *Assume that $m\in\mathcal{D}$ satisfies $m(0,T)\leqq 1$ and is globally Hölder continuous, namely, that there exist positive constants r and C such that*

$$|m(t_1,t_2)-m(s_1,s_2)|\leqq C|t_1-s_1|^r+C|t_2-s_2|^r,\quad (s_1,s_2),\ (t_1,t_2)\in\Delta. \tag{2.1}$$

Then for any $n\in\mathbb{N}$ there exists a finite set $\Delta^*=\{(t_{k,1},t_{k,2})\in\Delta\mid k=1,2,\dots,K\}$, satisfying

$$K\leqq 2(n-1)T\,(C\,n)^{1/r}+1 \tag{2.2}$$

such that

$$\sup_{(t_1,t_2)\in\Delta}|y(t_1,t_2)-m(t_1,t_2)|\leqq\bigvee_{k=1}^{K}|y(t_{k,1},t_{k,2})-m(t_{k,1},t_{k,2})|+\frac{2}{n} \tag{2.3}$$

holds for any $y\in\mathcal{D}$, where $\bigvee_{k=1}^{K}c_k$ denotes the largest value in $c_1,\dots,c_K$.

The following lemma is the technical core of this section.

Lemma 4. *Assume that $m\in\mathcal{D}$ satisfies $m(0,T)\leqq 1$ and that there exists a positive constant r and C such that (2.1) holds. Then for any $n\in\mathbb{N}$ there exists a finite set $\Delta^*=\{(t_{i,1},t_{i,2})\in\Delta\mid i=1,2,\dots,K\}$, of size K satisfying a bound (2.2), such that for each $(t_1,t_2)\in\Delta$,*

$$\begin{aligned}&\textit{either}\ \ m(t_1,t_2)\leqq\frac{1}{n}\ \textit{or there exists}\ (u_1,u_2)\in\Delta_n^*\ \textit{such that}\\ &u_1\geqq t_1\,,\ u_2\leqq t_2\,,\quad\textit{and}\ \ m(t_1,t_2)\leqq m(u_1,u_2)+\frac{2}{n}\,,\end{aligned} \tag{2.4}$$

and

$$\text{(2.5)}\quad \begin{aligned}&\textit{there exists } (s_1,s_2)\in\Delta_n^* \textit{ such that}\\ &s_1 \leqq t_1\,,\ \ s_2 \geqq t_2\,,\quad \textit{and}\ \ m(t_1,t_2)\geqq m(s_1,s_2)-\frac{2}{n}\,.\end{aligned}$$

PROOF. Note that monotonicity in the definition of $\mathcal{D}$ implies that the maximum value of m is $m(0,T)$ and the minimum is $m(t,t)=0$, $t\in[0,T]$. We omit the trivial case of m being identically 0, and consider the case $m(0,T)>0$.

Fix n and put

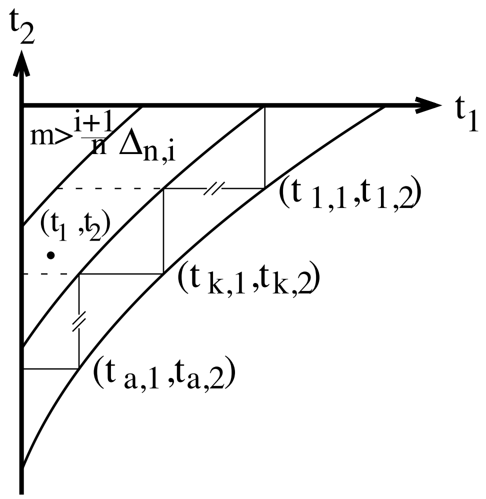
$$\text{(2.6)}\quad \Delta_{n,i}=\{(t_1,t_2)\in\Delta\mid \frac{i}{n}<m(t_1,t_2)\leqq\frac{i+1}{n}\},\quad i=0,1,2,\dots,n-1.$$

Denote by i_{max} the index such that $(0,T)\in\Delta_{n,i_{max}}$. Then $i_{max}\leqq n-1$, because m takes values in $[0,1]$, and there is a partition

$$\text{(2.7)}\qquad \Delta=\{(t_1,t_2)\in\Delta\mid m(t_1,t_2)=0\}\cup\bigcup_{i=0}^{i_{max}}\Delta_{n,i}\,.$$

If $m(t_1,t_2)=0$ or $(t_1,t_2)\in\Delta_{n,0}$ then $m(t_1,t_2)\leqq\dfrac{1}{n}$ and the first alternative in (2.4) holds. Fix $i\in\{1,2,\dots,i_{max}\}$, and define a positive integer a and reals $t_{0,2}$, $t_{k,1}$ and $t_{k,2}$ for $k=1,2,\dots,a$, inductively in k, as follows:

$$\text{(2.8)}\quad \begin{aligned}&t_{0,2}=T.\\ &\text{Assume that } t_{k,2} \text{ is defined for some } k.\text{ If}\\ &t_{k,2}\leqq\sup\{s\in[0,T]\mid m(0,s)<\frac{i}{n}\}\text{ then } a=k,\text{ otherwise,}\\ &\text{define } t_{k+1,1} \text{ and } t_{k+1,2} \text{ by}\\ &t_{k+1,1}=\inf\{s\in[0,T]\mid (s,t_{k,2})\in\Delta_{n,i-1}\},\\ &t_{k+1,2}=\inf\{s\in[0,T]\mid (t_{k+1,1},s)\in\Delta_{n,i-1}\}.\end{aligned}$$



Monotonicity of m implies that $t_{k,1}$ and $t_{k,2}$ are non-increasing in k, and the continuity of m implies

$$(2.9) \qquad m(t_{k,1}, t_{k-1,2}) = \frac{i}{n}, \quad m(t_{k,1}, t_{k,2}) = \frac{i-1}{n}, \quad k = 1, 2, \dots, a.$$

This and the assumption of Hölder continuity (2.1) imply

$$t_{k-1,2} - t_{k,2} \geqq (C\,n)^{-1/r} \quad \text{and} \quad t_{k,1} - t_{k+1,1} \geqq (C\,n)^{-1/r}, \ k = 1, 2, \dots,$$

so that $a \leqq T\,(C\,n)^{1/r}$. The definition of a also implies $\frac{i-1}{n} \leqq m(0, t_{a,2}) \leqq \frac{i}{n}$, hence, for $(t_1, t_2) \in \Delta_{n,i}$ (2.6) implies that there exists $k \in \{1, 2, \dots, a\}$ such that

$$(2.10) \qquad t_{k,2} < t_2 \leqq t_{k-1,2}\,,$$

which, with monotonicity and (2.9), further implies

$$m(t_{k,1}, t_2) \leqq m(t_{k,1}, t_{k-1,2}) = \frac{i}{n} < m(t_1, t_2),$$

which in turn implies

$$(2.11) \qquad t_1 < t_{k,1}$$

and

$$m(t_1,t_2) \leqq \frac{i}{n} + \frac{1}{n} = m(t_{k,1},t_{k,2}) + \frac{2}{n}\,. \tag{2.12}$$

Denote by $\Delta_+^* = \{(t_{\ell,1},t_{\ell,2}) \in \Delta \mid \ell = 1,2,\dots,a'\}$, the union of so obtained $\{(t_{k,1},t_{k,2}) \mid k = 1,\dots,a\}$ for all $i = 1,\dots,i_{max}$. In particular, since $a \leqq T\,(C\,n)^{1/r}$ for each i, and $i_{max} \leqq n-1$, $a' \leqq (n-1)\,T\,(C\,n)^{1/r}$. Combining (2.7), (2.10), (2.11), and (2.12), we have (2.4).

A proof of (2.5) is similar. Fix n, and put

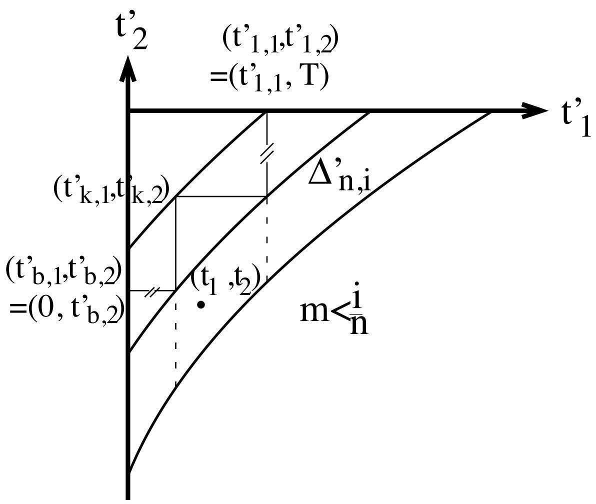
$$\Delta'_{n,i} = \{(t_1,t_2) \in \Delta \mid \frac{i}{n} \leqq m(t_1,t_2) < \frac{i+1}{n}\}, \quad i = 0,1,\dots,n. \tag{2.13}$$

Denote by $i'_{max} \leqq n$ the index such that $(0,T) \in \Delta'_{n,i'_{max}}$. Then there is a partition

$$\Delta = \bigcup_{i=0}^{i'_{max}} \Delta'_{n,i}\,. \tag{2.14}$$

If $(t_1,t_2) \in \Delta'_{n,i'_{max}} \cup \Delta'_{n,i'_{max}-1}$, then $t_1 \geqq 0$, $t_2 \leqq T$, and $m(t_1,t_2) \geqq m(0,T) - \dfrac{2}{n}$. Fix $i \in \{0,1,2,\dots,i'_{max}-2\}$, and define a positive integer b and reals $t'_{k,1}$ and $t'_{k,2}$ for $k = 1,2,\dots,b$, inductively in k, as follows:

$$\begin{aligned}
&t'_{1,2} = T,\\
&\text{Assume that } t'_{k,2} \text{ is defined for some } k. \text{ If}\\
&t'_{k,2} \leqq \sup\{s \in [0,T] \mid m(0,s) < \frac{i}{n}\} \text{ then } b = k \text{ and } t'_{b,1} = 0,\\
&\text{otherwise define } t'_{k,1} \text{ and } t'_{k+1,2} \text{ by}\\
&t'_{k,1} = \inf\{s \in [0,T] \mid (s,t'_{k,2}) \in \Delta'_{n,i+1}\},\\
&t'_{k+1,2} = \inf\{s \in [0,T] \mid (t'_{k,1},s) \in \Delta'_{n,i+1}\}.
\end{aligned} \tag{2.15}$$



Monotonicity of m implies that $t'_{k,1}$ and $t'_{k,2}$ are non-increasing in k, and the continuity of m implies

$$m(t'_{k,1}, t'_{k,2}) = \frac{i+2}{n}, \quad m(t'_{k,1}, t'_{k+1,2}) = \frac{i+1}{n}, \quad k = 1, 2, \dots, b. \tag{2.16}$$

This and the assumption of Hölder continuity (2.1) imply, as before, that $b \leqq T\,(C\,n)^{1/r}$ and $\dfrac{i+1}{n} \leqq m(0, t'_{b,2}) \leqq \dfrac{i+2}{n}$. Then for $(t_1, t_2) \in \Delta'_{n,i}$ either $t_1 \geqq t'_{1,1}$ or there exists $k \in \{2, 3, \dots, b\}$ such that

$$t'_{k,1} \leqq t_1 < t'_{k-1,1}\,, \tag{2.17}$$

which, with monotonicity and (2.16), further implies

$$m(t_1, t_2) < \frac{i+1}{n} = m(t'_{k-1,1}, t'_{k,2}) \leqq m(t_1, t'_{k,2}),$$

which in turn implies

$$t_2 < t'_{k,2} \tag{2.18}$$

and

$$m(t'_{k,1}, t'_{k,2}) = \frac{i+2}{n} \leqq m(t_1, t_2) + \frac{2}{n}\,. \tag{2.19}$$

Denote by $\Delta_-^* = \{(t'_{\ell,1}, t'_{\ell,2}) \in \Delta \mid \ell = 1, 2, \dots, b'\}$, the union of $\{(0, T)\}$ and so obtained $\{(t'_{k,1}, t'_{k,2}) \mid k = 1, \dots, b\}$ for all $i = 0, 1, \dots, i'_{max} - 2$. In particular, since $b \leqq T\,(C\,n)^{1/r}$ for each i, and $i'_{max} - 2 \leqq n - 2$, we have $b' \leqq (n-1)\,T\,(C\,n)^{1/r} + 1$. Combining (2.14), (2.17), (2.18), and (2.19), we have (2.5).

Finally, we put $\Delta_n^* = \Delta_+^* \cup \Delta_-^*$. Then

$$K = \sharp \Delta_n^* \leqq a' + b' \leqq 2(n-1) T\,(C\,n)^{1/r} + 1,$$

which proves (2.2). □

Proof of Proposition 3. Let $(t_1, t_2) \in \Delta$.

If $y(t_1, t_2) \geqq m(t_1, t_2)$, then, Lemma 4 and monotonicity of y imply that, there exists $(s_1, s_2) \in \Delta_n^*$ such that,

$$\begin{aligned}
&|y(t_1,t_2) - m(t_1,t_2)| = y(t_1,t_2) - m(t_1,t_2) \\
&\leqq (y(s_1,s_2) - m(s_1,s_2)) + (m(s_1,s_2) - m(t_1,t_2)) \\
&\leqq |y(s_1,s_2) - m(s_1,s_2)| + \frac{2}{n}\,.
\end{aligned}$$

Assume in the following that $y(t_1, t_2) \leqq m(t_1, t_2)$. Note that the assumptions on y implies that y is nonnegative, hence, if $m(t_1, t_2) \leqq \frac{1}{n}$, then

$$|y(t_1,t_2) - m(t_1,t_2)| = m(t_1,t_2) - y(t_1,t_2) \leqq \frac{1}{n}\,.$$

If $m(t_1, t_2) > \frac{1}{n}$, then Lemma 4 and monotonicity of y imply that, there exists (u_1, u_2) in Δ_n^* such that,

$$\begin{aligned}
&|y(t_1,t_2) - m(t_1,t_2)| = m(t_1,t_2) - y(t_1,t_2) \\
&\leqq (m(u_1,u_2) - y(u_1,u_2)) + (m(t_1,t_2) - m(u_1,u_2)) \\
&\leqq |y(u_1,u_2) - m(u_1,u_2)| + \frac{2}{n}\,.
\end{aligned}$$

Therefore (2.3) holds.

Lemma 4 also implies the claimed bound on K, the size of the set Δ_n^*. □

3. Proof of Main Theorem

PROPOSITION 5. *Let Y be a random variable taking values in $\mathcal{D}$. Assume that*

$$\begin{aligned}&|\mathrm{E}[\,Y(t_1,t_2)-Y(s_1,s_2)\,]| \\ &\leqq C(\mathrm{E}[\,Y(0,T)\,]\vee 1)\,(|t_1-s_1|^r+|t_2-s_2|^r),\quad (s_1,s_2),\ (t_1,t_2)\in\Delta,\end{aligned} \tag{3.1}$$

holds for some $r>0$ and $C>0$. Then for any $n\in\mathbb{N}$ and for any $q\geqq 1$,

$$\begin{aligned}&\mathrm{E}[\sup_{(t_1,t_2)\in\Delta}|Y(t_1,t_2)-\mathrm{E}[\,Y(t_1,t_2)\,]|^q\,] \\ &\leqq 2^{q-1}K_n\sup_{(t_1,t_2)\in\Delta}\mathrm{E}[\,|Y(t_1,t_2)-\mathrm{E}[\,Y(t_1,t_2)\,]|^q\,] \\ &\quad+2^{q-1}\left(\frac{2}{n}\right)^q(\mathrm{E}[\,Y(0,T)\,]^q\vee 1)\end{aligned} \tag{3.2}$$

holds, where K_n is K in (2.2).

PROOF. For each sample $\omega\in\Omega$, Proposition 3 with

$$y(t_1,t_2)=\frac{Y(\omega)(t_1,t_2)}{\mathrm{E}[\,Y(0,T)\,]\vee 1},\ m(t_1,t_2)=\frac{\mathrm{E}[\,Y(t_1,t_2)\,]}{\mathrm{E}[\,Y(0,T)\,]\vee 1}$$

implies that for any $n\in\mathbb{N}$ there exists $K=K_n\geqq 1$, satisfying (2.2), and a finite set

$$\Delta^*=\{(t_{k,1},t_{k,2})\in\Delta\mid k=1,2,\ldots,K\},$$

independent of sample ω, such that

$$\begin{aligned}&\sup_{(t_1,t_2)\in\Delta}|Y(\omega)(t_1,t_2)-\mathrm{E}[\,Y(t_1,t_2)\,]| \\ &\leqq\bigvee_{k=1}^{K}|Y(\omega)(t_{k,1},t_{k,2})-\mathrm{E}[\,Y(t_{k,1},t_{k,2})\,]|+\frac{2}{n}\,(\mathrm{E}[\,Y(0,T)\,]\vee 1).\end{aligned}$$

This, with an elementary inequality

$$a^p+b^p\leqq(a+b)^p\leqq 2^{p-1}(a^p+b^p), \tag{3.3}$$

which holds for all positive a and b with $p \geqq 1$, implies

$$\begin{aligned}
&\mathrm{E}[\sup_{(t_1,t_2)\in\Delta} |Y(t_1,t_2) - \mathrm{E}[\,Y(t_1,t_2)\,]|^q\,] \\
&\leqq 2^{q-1}\mathrm{E}[\,\left(\bigvee_{k=1}^{K} |Y(t_{k,1},t_{k,2}) - \mathrm{E}[\,Y(t_{k,1},t_{k,2})\,]|\right)^q\,] \\
&\quad + 2^{q-1}\left(\frac{2}{n}\right)^q (\mathrm{E}[\,Y(0,T)\,]^q \vee 1) \\
&= 2^{q-1}\mathrm{E}[\,\bigvee_{k=1}^{K} |Y(t_{k,1},t_{k,2}) - \mathrm{E}[\,Y(t_{k,1},t_{k,2})\,]|^q\,] \\
&\quad + 2^{q-1}\left(\frac{2}{n}\right)^q (\mathrm{E}[\,Y(0,T)\,]^q \vee 1) \\
&\leqq 2^{q-1}\sum_{k=1}^{K} \mathrm{E}[\,|Y(t_{k,1},t_{k,2}) - \mathrm{E}[\,Y(t_{k,1},t_{k,2})\,]|^q\,] \\
&\quad + 2^{q-1}\left(\frac{2}{n}\right)^q (\mathrm{E}[\,Y(0,T)\,]^q \vee 1),
\end{aligned}$$

which implies (3.2). □

We will see that Proposition 5 reduces the claim of Theorem 1 to the complete law of large numbers for real valued random variables. It is known [6, 2, 3] that a necessary and sufficient condition for the complete law of large numbers for identically distributed real valued random variables with finite expectation is the existence of variance, hence existence of the second order moment suffices [1, §10.4, Example 1]. A generalized result for the case of varying distribution is also known [7]. We will use the results in the following form. See the references for a proof.

PROPOSITION 6. *For each* $N \in \mathbb{N}$, *let* $\tilde{Z}_i^{(N)} : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, N$, *be a finite sequence of independent, real valued random variables, and put* $\tilde{Y}^{(N)} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \tilde{Z}_i^{(N)}$. *Assume that there exists* $q > 2$ *such that*

$$M^{(N)} := \max_{i\in\{1,\ldots,N\}} \mathrm{E}[\,|\tilde{Z}_i^{(N)}|^q\,]^{1/q} < \infty.$$

Then there exists a positive constant C_q depending only on q, (in particular, independent of N and $M^{(N)}$,) such that

$$\mathrm{E}[\,|\tilde{Y}^{(N)} - \mathrm{E}[\,\tilde{Y}^{(N)}\,]|^q\,]^{1/q} \leqq \frac{C_q M^{(N)}}{\sqrt{N}}, \tag{3.4}$$

hold.

We can for example put

$$C_q = \left(\frac{1}{2}(4k)^q + \frac{2k}{2k-q}(8k)^q\right)^{1/q},$$

in (3.4), where k is the smallest integer greater than $q/2$.

Proof of Theorem 1. Note first that the assumption (1.1)(i) implies

$$\sup_{\substack{i=1,2,\dots,N,\\ (t_1,t_2)\in\Delta}} \mathrm{E}[\,|Z_i^{(N)}(t_1,t_2)|^q\,]^{1/q} \leqq M^{(N)}, \tag{3.5}$$

because of monotonicity.

The assumption (1.1)(ii) imply that

$$Y := \frac{1}{M^{(N)}} Y^{(N)} = \frac{1}{N\,M^{(N)}} \sum_{i=1}^{N} Z_i^{(N)}$$

satisfies all the assumptions in Proposition 5, with $C = \overline{w}^{(N)}$ in (3.1). Proposition 5 and Proposition 6 therefore imply,

$$\begin{aligned} &\mathrm{E}[\sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_1,t_2) - \mathrm{E}[\,Y^{(N)}(t_1,t_2)\,]|^q\,] \\ &\leqq 2^{q-1} K_n \frac{(C_q\,M^{(N)})^q}{N^{q/2}} + 2^{q-1}\left(\frac{2M^{(N)}}{n}\right)^q \end{aligned}$$

for positive integers n and N, where K_n is K of (2.2) with $C = \overline{w}^{(N)}$. Now for each N, fix $n = n_N$ to be the largest integer not greater than $N^{rq/(2qr+2r+2)}$. If $N_0^{rq/(2qr+2r+2)} \geqq 2$, then for $N \geqq N_0$, $\frac{1}{2}N^{rq/(2qr+2r+2)} < n_N \leqq N^{rq/(2qr+2r+2)}$, and we have (1.2).

The assumption (1.4) and just proved result (1.2) imply

$$\mathrm{E}[\sum_{N=N_0}^{\infty} \sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_1,t_2) - \mathrm{E}[\, Y^{(N)}(t_1,t_2)\,]|^q\,] < \infty,$$

which implies, with Chebyshev's inequality,

$$\sum_{N=N_0}^{\infty} \mathrm{P}[\sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_1,t_2) - \mathrm{E}[\, Y^{(N)}(t_1,t_2)\,]| > \epsilon\,] < \infty$$

for all $\epsilon > 0$. Hence (1.5) holds. □

If the moment condition (1.1)(i) holds for arbitrarily large exponent q, the doubly uniform complete law of large numbers holds with 'order of fluctuation' arbitrary close to 1/2, as expected.

THEOREM 7. *For each $N \in \mathbb{N}$, let $Z_i^{(N)} : \Omega \to \mathcal{D}$, $i = 1,2,\dots,N$, be a sequence of independent, $\mathcal{D}$ valued random variables. Let $r > 0$, and for $N \in \mathbb{N}$, let $M^{(N)}$ be a positive real and $w_i^{(N)}$, $i = 1,2,\dots,N$, a sequence of nonnegative reals. Assume the following for each $i = 1,2,\dots,N$ and $N \in \mathbb{N}$:*

$$\begin{aligned} &\text{(i)}\ \ \mathrm{E}[\,|Z_i^{(N)}(0,T)|^q\,]^{1/q} \leqq M^{(N)},\ q \in \mathbb{N},\\ &\text{(ii)}\ \ |\mathrm{E}[\,Z_i^{(N)}(t_1,t_2)\,] - \mathrm{E}[\,Z_i^{(N)}(s_1,s_2)\,]|\\ &\qquad \leqq M^{(N)} w_i^{(N)}\,(|t_1-s_1|^r + |t_2-s_2|^r),\\ &\qquad (s_1,s_2),\ (t_1,t_2) \in \Delta. \end{aligned} \tag{3.6}$$

Then for any $\gamma \in (0,\frac{1}{2})$, and $p > 0$ the average $Y^{(N)} = \frac{1}{N}\sum_{i=1}^{N} Z_i^{(N)}$ satisfies

$$\begin{aligned} &\mathrm{E}[\sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_1,t_2) - \mathrm{E}[\,Y^{(N)}(t_1,t_2)\,]|^p\,]^{1/p}\\ &\leqq \frac{M^{(N)}}{N^{\gamma}} 2^{1-1/q} (C_q^q\,(2T(\overline{w}^{(N)})^{1/r}+1) + 2^{2q})^{1/q},\\ &N = N_0, N_0+1, \dots, \end{aligned} \tag{3.7}$$

where $\overline{w}^{(N)}$ is as in (1.3), C_q as in Proposition 6, $N_0 = N_0(r,q)$ is the smallest integer satisfying $N_0^{rq/(2rq+2r+2)} \geqq 2$, and $q = q(p,\gamma) = 3 \vee$

$\dfrac{r+1}{r}\,\dfrac{2\gamma}{1-2\gamma} \vee p$. *(In particular, q and N_0 are independent of N, $M^{(N)}$, and $\overline{w}^{(N)}$.)*

If in addition, $\{M^{(N)}, \overline{w}^{(N)}\}$ is bounded, then (1.5) holds.

Proof. As in the proof of Theorem 1, we have (1.2) for all N and q. Let $\gamma \in (0, \frac{1}{2})$ and $p > 0$, and choose $q = 3 \vee 2\dfrac{r+1}{r}\,\dfrac{\gamma}{1-2\gamma} \vee p$. Then $q \geqq 2\dfrac{r+1}{r}\,\dfrac{\gamma}{1-2\gamma}$ implies $\dfrac{rq}{2rq+2r+2} \geqq \gamma$, hence, the monotonicity of L_p norms and (1.2) imply

$$\begin{aligned}
&\mathrm{E}[\sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_1,t_2) - \mathrm{E}[\,Y^{(N)}(t_1,t_2)\,]|^p\,]^{1/p} \\
&\leqq \mathrm{E}[\sup_{(t_1,t_2)\in\Delta} |Y^{(N)}(t_1,t_2) - \mathrm{E}[\,Y^{(N)}(t_1,t_2)\,]|^q\,]^{1/q} \\
&\leqq \frac{M^{(N)}}{N^{\gamma}} 2^{1-1/q} (C_q^q\,(2T(\overline{w}^{(N)})^{1/r} + 1) + 2^{2q})^{1/q},
\end{aligned}$$

which proves (3.7). The argument for the proof of (1.5) in the proof of Theorem 1 also proves (1.5) in Theorem 7. □

4. Example of Point Processes with Dependent Increments

Application of the main result such as Theorem 7 in [5] requires a considerable preparation. Here we give a simpler example, similar to Corollary 2, which shares basic estimates with [5].

Consider a new large office building with a large number, say N, of lighting equipments. Each light bulb has a random lifetime, which may depend on the location ($i = 1, 2, \ldots, N$) in the building. The distribution of each lifetime also could depend in a mathematically cumbersome way on the latest time the light bulb burnt out, because the light bulb products in the market are updated according to e.g., advances in technology or regulation on materials. We would be interested in estimating the number of bulbs to be replaced in the time period $[t_1, t_2]$, which is the random number $N\,(Y^{(N)}(t_2) - Y^{(N)}(t_1))$ in terms of the notations in Corollary 2. (Note also that in these practical applications where N is finite and fixed, a complete law of large numbers as we consider should be natural than a strong law of large numbers which assumes relation between random variables with different N.)

As an example of point process with dependent increments we consider the point process $Z_i^{(N)}(t)$ with last-arrival-time dependent intensity [4, §3]. For each $N \in \mathbb{N}$ and $i = 1, 2, \dots, N$, the sequence $\tau_{i,0}^{(N)} = 0 < \tau_{i,1}^{(N)} < \cdots$ of arrival times are random times defined inductively by

$$\begin{aligned} &\mathrm{P}[\, t < \tau_{i,k}^{(N)} \mid \mathcal{F}_{\tau_{i,k-1}^{(N)}} \,] \\ &= \exp(-\int_{\tau_{i,k-1}^{(N)}}^{t} w_i^{(N)}(\tau_{i,k-1}^{(N)}, u)\, du) \quad \text{on} \quad t \geqq \tau_{i,k-1}^{(N)}\,, \end{aligned} \tag{4.1}$$

where, $w_i^{(N)}$ is a nonnegative continuously differentiable function defined on

$$(t_0, t) \in \Delta = \{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leqq t_1 \leqq t_2 \leqq T\},$$

to which we refer as the intensity function of the counting process

$$Z_i^{(N)}(t) = \max\{k \in \mathbb{Z}_+ \mid \tau_{i,k}^{(N)} \leqq t\}. \tag{4.2}$$

If $w_i^{(N)}$ is independent of the first variable, the definition of $Z_i^{(N)}$ reduces to that of the Poisson process with intensity function $w_i^{(N)}$, but in general, unlike the Poisson processes, $Z_i^{(N)}$ is not of independent increment.

THEOREM 8. *For $N \in \mathbb{N}$ and $i = 1, \dots, N$, let $w_i^{(N)} : \Delta \to [0, \infty)$ be a nonnegative continuously differentiable function, and $Z_i^{(N)}$ a process determined by (4.2). If $Z_i^{(N)}$, $i = 1, \dots, N$, are independent for each $N \in \mathbb{N}$, and*

$$C := \sup_{N \in \mathbb{N},\ i \in \{1,\dots,N} \ \sup_{(t_1,t_2) \in \Delta} w_i^{(N)}(t_1, t_2) < \infty,$$

then a doubly uniform complete law of large numbers

$$\begin{aligned} &\lim_{N_0 \to \infty} \sum_{N \geqq N_0} \mathrm{P}[\, \sup_{(t_1,t_2) \in \Delta} \left| \frac{1}{N} \sum_{i=1}^{N} (Z_i^{(N)}(t_1, t_2) - \mathrm{E}[\, Z_i^{(N)}(t_1, t_2)\,]) \right| \\ &> \epsilon \,] = 0, \quad \epsilon > 0, \end{aligned} \tag{4.3}$$

holds for the number of arrival times in the intervals

$$Z_i^{(N)}(t_1,t_2) := Z_i^{(N)}(t_2) - Z_i^{(N)}(t_1), \quad 0 \leqq t_1 \leqq t_2 \leqq T. \tag{4.4}$$

PROOF. Note that $Z_i^{(N)}(0)=0$. It is known [4, §3, Thm. 5] that

$$\begin{aligned}&\mathrm{E}[\, Z_i^{(N)}(t)\,(Z_i^{(N)}(t)-1)\cdots(Z_i^{(N)}(t)-q+1)\,]\\ &\leqq \Bigl(\int_0^t \max_{s\in[0,u]} w_i^{(N)}(s,u)\,du\Bigr)^q \leqq (CT)^q,\end{aligned}$$

holds for all positive integer q. For each positive integers q, N, and i, $Z \geqq 2q$ implies $Z-1 > Z-2 > \cdots > Z-q+1 > \frac{1}{2}Z > 0$, so that

$$\begin{aligned}&\mathrm{E}[\, |Z_i^{(N)}(T)|^q\,]\\ &= \mathrm{E}[\, Z_i^{(N)}(T)^q;\ Z_i^{(N)}(T) \geqq 2q\,] + \mathrm{E}[\, Z_i^{(N)}(T)^q;\ Z_i^{(N)}(T) < 2q\,]\\ &\leqq 2^q \mathrm{E}[\, Z_i^{(N)}(T)\cdots(Z_i^{(N)}(T)-q+1)\,] + (2q)^q\\ &\leqq (2CT)^q + (2q)^q.\end{aligned} \tag{4.5}$$

Let $0 \leqq s \leqq t \leqq T$ and put $\Omega_i^{(N)}(s,t) = \int_s^t w_i^{(N)}(s,u)\,du$. According to [4, §3], we have explicit formulas

$$\begin{aligned}&\mathrm{E}[\, Z_i^{(N)}(t) - Z_i^{(N)}(s)\,]\\ &= \sum_{k=1}^{\infty} \int_s^t \Bigl(\int_{0\leqq u_1 \leqq \cdots \leqq u_k} w_i^{(N)}(u_{k-1},u_k) e^{-\Omega_i^{(N)}(u_{k-1},u_k)}\\ &\qquad \times \prod_{\alpha=1}^{k-1} w_i^{(N)}(u_{\alpha-1},u_\alpha) e^{-\Omega_i^{(N)}(u_{\alpha-1},u_\alpha)}\Big|_{u_0=0}\, du_\alpha\Bigr)\, du_k\end{aligned}$$

and

$$\begin{aligned}&e^{-\Omega_i^{(N)}(0,t)}\\ &+ \sum_{k=1}^{\infty} \int_{0\leqq u_1 \leqq u_2 \leqq \cdots \leqq u_k \leqq t} e^{-\Omega_i^{(N)}(u_k,t)}\end{aligned}$$

$$\times \prod_{\alpha=1}^{k} w_i^{(N)}(u_{\alpha-1},u_\alpha)\, e^{-\Omega_i^{(N)}(u_{\alpha-1},u_\alpha)}\Big|_{u_0=0}\, du_\alpha$$
$$= 1.$$

Using nonnegativity of each terms and an assumption $w_i^{(N)}(u_{k-1},u_k) \leqq C$

$$\text{(4.6)} \qquad \mathrm{E}[\, Z_i^{(N)}(t) - Z_i^{(N)}(s)\,] \leqq C \int_s^t 1\, du \leqq C\,(t-s).$$

The estimates (4.5) and (4.6), and Theorem 7 in §3 with $M^{(N)} = 1$, $w_i^{(N)} = C$, and $r = 1$ imply (4.3). □

Appendix A. Complete Law of Large Numbers for Independent Monotone Function Valued Random Variables

Here we prove the following, 1 variable analog of Corollary 2.

PROPOSITION 9. *For each $N \in \mathbb{N}$, let $Z_i^{(N)} : \Omega \to D_\uparrow$, $i = 1, 2, \ldots, N$, be a sequence of independent, $D_\uparrow$ valued random variables, and assume that* $\mathrm{E}[\, Z_i^{(N)}(t)\,]$ *is continuous in $t \in [0,T]$ for all N and i, and that there exist positive constants q and M satisfying $q > 1+\sqrt{3}$ and*

$$\text{(A.1)} \qquad \begin{aligned} &\mathrm{E}[\, |Z_i^{(N)}(T)|^q\,]^{1/q} \vee \mathrm{E}[\, |Z_i^{(N)}(0)|^q\,]^{1/q} \leqq M, \\ &i = 1, 2, \ldots, N,\ \ N \in \mathbb{N}. \end{aligned}$$

Then a doubly uniform complete law of large numbers

$$\text{(A.2)} \qquad \begin{aligned} \lim_{N_0\to\infty} \sum_{N \geqq N_0} \mathrm{P}[\, \sup_{t\in[0,T]} \Big| \frac{1}{N} \sum_{i=1}^{N} (Z_i^{(N)}(t) - \mathrm{E}[\, Z_i^{(N)}(t)\,]) \Big| \\ > \epsilon\,] = 0, \quad \epsilon > 0, \end{aligned}$$

holds.

Basic idea for a proof is similar to that for the main theorem; to reduce the problem of function valued random variables to that of real valued random variables, and use a standard result (Proposition 6) of complete law of large numbers for real valued random variables.

We consider the functions on a finite interval $[0,T]$, and put $t_0 = 0$ and $t_{K+1} = T$ in the following Lemma 10.

LEMMA 10. *Let $m:\ [0,T] \to \mathbb{R}$ be a non-decreasing, continuous function on a closed interval $[0,T]$. Then for any $\delta > 0$ there exists a finite increasing sequence $\{t_k \mid k = 1,\dots,K\} \subset [0,T]$ of length K satisfying*

$$0 \leqq K < \frac{1}{\delta}(m(T) - m(0)), \tag{A.3}$$

such that for any non-decreasing function $y:\ [0,T] \to \mathbb{R}$,

$$\sup_{t\in[0,T]} |y(t) - m(t)| \leqq \bigvee_{k=0}^{K+1} |y(t_k) - m(t_k)| + \delta, \tag{A.4}$$

holds, where $\bigvee_k c_k = \max_k c_k$ denotes the largest number in $\{c_k\}$.

REMARK. Note that K is independent of y.

PROOF. If $m(T) - m(0) \leqq \delta$, put $K = 0$. Otherwise, let K be the largest integer satisfying (A.3) and define a non-decreasing sequence $0 \leqq t_1 \leqq \cdots \leqq t_K \leqq T$ by

$$t_k = \inf\{t \in [0,T] \mid m(t) - m(0) \geqq k\delta\}, \quad k = 1,\dots,K.$$

Then monotonicity and continuity and the choice of K imply

$$m(t_{k+1}) - m(t_k) \leqq \delta,\ k = 0,1,\dots,K. \tag{A.5}$$

For $t \in [0,T)$ choose $k \in \{0,1,\dots,K\}$ such that $t_k \leqq t < t_{k+1}$. Non-decreasing properties of y and m, with (A.5) imply, that if $y(t) \geqq m(t)$, then

$$\begin{aligned}
&|y(t) - m(t)| = y(t) - m(t)\\
&\leqq y(t_{k+1}) - m(t_k)\\
&\leqq |y(t_{k+1}) - m(t_{k+1})| + m(t_{k+1}) - m(t_k)\\
&\leqq |y(t_{k+1}) - m(t_{k+1})| + \delta,
\end{aligned}$$

while if $y(t) < m(t)$, then

$$\begin{aligned}|y(t) - m(t)| &= m(t) - y(t) \\ &\leqq (m(t_{k+1}) - m(t_k)) + (m(t_k) - y(t_k)) \\ &\leqq |y(t_k) - m(t_k)| + \delta,\end{aligned}$$

which proves (A.4). □

LEMMA 11. *Let $Y : \Omega \to D_\uparrow$ be a random variable taking values in $D_\uparrow$. Assume that $\mathrm{E}[\, Y(t)\,]$ is continuous in $t \in [0, T]$ and that there exists $q \geqq 1$ such that*

$$M := \mathrm{E}[\, |Y(0)|^q \,]^{1/q} \vee \mathrm{E}[\, |Y(T)|^q \,]^{1/q} < \infty. \tag{A.6}$$

Then for any $\delta > 0$,

$$\begin{aligned}&\mathrm{E}[\sup_{t\in[0,T]} |Y(t) - \mathrm{E}[\, Y(t)\,]|^q\,] \\ &\leqq \frac{2^{q-1}}{\delta}(\mathrm{E}[\, Y(T)\,] - \mathrm{E}[\, Y(0)\,] + 2\delta) \\ &\quad \times \sup_{t\in[0,T]} \mathrm{E}[\, |Y(t) - \mathrm{E}[\, Y(t)\,]|^q\,] + 2^{q-1}\delta^q,\end{aligned} \tag{A.7}$$

holds.

PROOF. Since Y is non-decreasing in t, it holds that $Y(t) \leqq |Y(0)| \vee |Y(T)|$, $t \in [0, T]$, and (A.6) implies $\mathrm{E}[\, |Y(t)|^q\,] \leqq M^q$, and the monotonicity of L_p norms with $q \geqq 1$ further implies $\mathrm{E}[\, |Y(t)|\,] \leqq \mathrm{E}[\, |Y(t)|^q\,]^{1/q} \leqq M$.

Lemma 10 with $m = \mathrm{E}[\, Y\,]$ implies that there exist $K \geqq 0$ and a sequence $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_K \leqq t_{K+1} = T$, such that

$$K < \frac{1}{\delta}(\mathrm{E}[\, Y(T)\,] - \mathrm{E}[\, Y(0)\,]), \tag{A.8}$$

and with $y = Y(\omega)$ for each sample $\omega \in \Omega$, that

$$\sup_{t\in[0,T]} |Y(\omega)(t) - \mathrm{E}[\, Y(t)\,]| \leqq \bigvee_{k=0}^{K+1} |Y(\omega)(t_k) - \mathrm{E}[\, Y(t_k)\,]| + \delta.$$

This, with (3.3) implies, as in the proof of Proposition 5,

$$\begin{aligned}&\mathrm{E}[\sup_{t\in[0,T]}|Y(t)-\mathrm{E}[\,Y(t)\,]|^q\,]\\&\leqq 2^{q-1}(K+2)\sup_{t\in[0,T]}\mathrm{E}[\,|Y(t)-\mathrm{E}[\,Y(t)\,]|^q\,]+2^{q-1}\delta^q.\end{aligned}$$

With (A.8) we have (A.7). □

We are ready to prove Proposition 9.

Proof of Proposition 9. For $N\in\mathbb{N}$ let $Y^{(N)}$ denote the arithmetic average $Y^{(N)}=\frac{1}{N}\sum_{i=1}^{N}Z_i^{(N)}$. Note that since $Z_i^{(N)}(t)$ is monotone in t, the assumption (A.1) implies $\mathrm{E}[\,|Z_i^{(N)}(t)|^q\,]^{1/q}\leqq M$, for all i, N, and t.

Fix $N\in\mathbb{N}$. Applying the monotonicity of L^p norms in p to the arithmetic average $\frac{1}{N}\sum_{i=1}^{N}\cdot_i$, we have

$$\begin{aligned}\mathrm{E}[\,|Y^{(N)}(t)|^q\,]^{1/q}&\leqq\left(\frac{1}{N}\sum_{i=1}^{N}\mathrm{E}[\,|Z_i^{(N)}(t)|^q\,]\right)^{1/q}\\&\leqq\max_{i\in\{1,2,\dots,N\}}\mathrm{E}[\,|Z_i^{(N)}(t)|^q\,]^{1/q}=:M^{(N)}.\end{aligned}$$

Hence Lemma 11 with $Y=Y^{(N)}$ and $C_q=M^{(N)}$ implies

$$\begin{aligned}&\mathrm{E}[\sup_{t\in[0,T]}|Y^{(N)}(t)-\mathrm{E}[\,Y^{(N)}(t)\,]|^q\,]\\&\leqq\frac{2^q}{\delta_N}(2M^{(N)}+\frac{3}{2}\delta_N)\sup_{t\in[0,T]}\mathrm{E}[\,|Y^{(N)}(t)-\mathrm{E}[\,Y(t)\,]|^q\,]+2^{q-1}\delta_N^q,\end{aligned}\tag{A.9}$$

for any $\delta_N>0$. Also, Proposition 6 implies, for $q>2$,

$$\mathrm{E}[\,(Y^{(N)}(t)-\mathrm{E}[\,Y^{(N)}(t)\,])^q\,]^{1/q}\leqq\frac{C_qM^{(N)}}{\sqrt{N}},\ N\in\mathbb{N}.\tag{A.10}$$

Substituting (A.10) in (A.9), and choosing $\delta_N=M^{(N)}\,N^{-q/(2q+2)}$, we have

$$\mathrm{E}[\sup_{t\in[0,T]}|Y^{(N)}(t)-\mathrm{E}[\,Y^{(N)}(t)\,]|^q\,]\leqq\frac{\tilde{C}_q^q{M^{(N)}}^q}{N^{q^2/(2q+2)}},\ N\in\mathbb{N},\tag{A.11}$$

for $\tilde{C}_q = 2\,(\frac{7}{2}C_q^q + \frac{1}{2})^{1/q}$, where C_q is as in Proposition 6.

Since by assumption $q > 1 + \sqrt{3}$ and $M^{(N)} \leqq M$, $N \in \mathbb{N}$, we have $\dfrac{q^2}{2q+2} > 1$, so that

$$\mathrm{E}[\,\sum_{N=1}^{\infty} \sup_{t\in[0,T]} |Y^{(N)}(t) - \mathrm{E}[\,Y^{(N)}(t)\,]|^q\,] < \infty,$$

which, as in the proof of Theorem 1, implies (A.2). □

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