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Hermitian Tanno Connection and Bochner Type Curvature Tensors of Contact Riemannian Manifolds

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Abstract. On a contact Riemannian manifold, considering the curvature of hermitian Tanno connection, we introduce Bochner type curvature tensors. Some of them are pseudo-conformally invariant under gauge transformation and so are the others if and only if the associated almost complex structure is integrable.

Introduction

Let (M, θ) be a $(2n+1)$-dimensional contact manifold with a contact form θ. We have, hence, a unique vector field ξ satisfying $\theta(\xi) = 1$ and $\mathcal{L}_\xi \theta = 0$, where $\mathcal{L}_\xi$ is the Lie differentiation by ξ. Let us equip M with a Riemannian metric g and a $(1,1)$-tensor field J satisfying $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X, Y)$ and $J^2 X = -X + \theta(X)\xi$ for any vector fields X, Y. (In this paper we adopt such a notation as $d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])$.) To study the geometry of contact Riemannian manifold (M, θ, g, J), Tanno ([10]) introduced a linear connection ${}^*\nabla = {}^*\nabla^{(\theta)}$, called the Tanno connection in this paper (the TWT connection, or, the Tanaka-Webster-Tanno connection, in [9], etc.), defined by ${}^*\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X \xi + (\nabla^g_X \theta)(Y)\xi$ (∇^g is the Levi-Civita connection), which has really potential applications to the geometry of contact Riemannian structure. In general its action does not commute with that of the almost complex structure J, however. In fact, Tanno ([10, Proposition 3.1]) indicated

$$({}^*\nabla_X J)Y = \mathcal{Q}(Y, X) := (\nabla^g_X J)Y + (\nabla^g_X \theta)(JY)\,\xi + \theta(Y)J\nabla^g_X \xi.$$

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In [6] (and [5]), the first author adopted a modified connection $\nabla = \nabla^{(\theta)}$, called the hermitian Tanno connection, defined by

$$\nabla_X Y = {}^*\nabla_X Y - \frac{1}{2} J\mathcal{Q}(Y,X) = \begin{cases} {}^*\nabla_X(f\xi) & : Y = f\xi \quad (f \in C^\infty(M)), \\ \frac{1}{2}\left({}^*\nabla_X Y - J\,{}^*\nabla_X JY\right) & : Y \in \Gamma(H) \end{cases}$$

($H := \ker\theta$), so that we have $\nabla J = 0$, which simplified the study of Kohn-Rossi heat kernel on contact Riemannian manifolds. We expect the commutativity will make it plainer to investigate into the case of general J. In this paper, assuming $n \geq 2$, we intend to construct Bochner type tensors associated with the curvature $F(\nabla)$ (Theorems A and B and Corollary C in the introduction): Some of them are pseudo-conformally invariant under the gauge transformation $\theta \Rightarrow e^{2f}\theta$ together with (2.1) ($f \in C^\infty(M)$) (considered in [8], [12], etc.) and so are the others if and only if J is integrable, i.e., $[\Gamma(H_+), \Gamma(H_+)] \subset \Gamma(H_+)$, where we set $H_\pm = \{X \in H \otimes \mathbb{C} \mid JX = \pm iX\}$. In §4 (Theorems 4.2 and 4.4), as by-products, such tensors associated with $F({}^*\nabla)$ canonically deduced from those will be also presented.

Study on conformal invariance of curvature tensors (in the Riemannian case) will originate in the ones on the Weyl conformal curvature (e.g. [1, Chap. 1.G]) and the Bochner curvature ([3]). In the contact Riemmanian case, study on pseudo-conformal invariance has been developed as well. Our study is motivated directly by such works related to the curvature of canonical connection by Sakamoto-Takemura ([7]) and the curvature $F({}^*\nabla)$ by Tanno ([11], [12], [13]).

Unlike $F({}^*\nabla)(X,Y)$, the action of $F(\nabla)(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ on $\Gamma(H \otimes \mathbb{C})$ is decomposed orthogonally into the direct sum of $F(\nabla)(X,Y) : \Gamma(H_\pm) \to \Gamma(H_\pm)$ (cf. Proposition 1.2(2)) and, compared with those by Tanno, our Bochner type tensors are expressed rather clearly. Indeed we define the tensors $B(\nabla)^0 \in \Gamma(H_+ \otimes H_+^* \otimes H_-^* \otimes H_+^*)$ and $B(\nabla)^+ \in \Gamma(H_+ \otimes H_+^* \otimes H_+^* \otimes H_+^*)$ by

$$\begin{aligned} B(\nabla)^0(X,\overline{Y})Z \;=\; & F(\nabla)(X,\overline{Y})Z - \frac{1}{n+2}\Big\{\mathrm{Ric}^\nabla(Z,\overline{Y})X + \mathrm{Ric}^\nabla(X,\overline{Y})Z \\ & - g(Z,\overline{Y})\,\mathrm{ric}^\nabla(X) - g(X,\overline{Y})\,\mathrm{ric}^\nabla(Z)\Big\} \\ & + \frac{s^\nabla}{(n+1)(n+2)}\Big\{g(Z,\overline{Y})X + g(X,\overline{Y})Z\Big\}, \end{aligned}$$

$$B(\nabla)^+(X,Y)Z = F(\nabla)(X,Y)Z - \frac{1}{n-1}\Big\{\mathrm{Ric}(\nabla)(Z,Y)X - \mathrm{Ric}(\nabla)(Z,X)Y\Big\}$$

$(X, Y, Z \in \Gamma(H_+))$, where Ric^∇, etc., are the pseudohermitian Ricci curvature, etc., defined by

$$\mathrm{Ric}^\nabla(X,Y) = \mathrm{tr}_{H_+}\Big(Z \mapsto F(\nabla)(X,Y)Z\Big),$$
$$\mathrm{ric}^\nabla(Y) = \mathrm{ric}^\nabla_+(Y) \in H_+ \text{ with } g(\overline{X}, \mathrm{ric}^\nabla(Y)) = \mathrm{Ric}^\nabla(\overline{X}, Y),$$
$$\mathrm{Ric}(\nabla)(X,Y) = \mathrm{tr}_{TM}\Big(Z \mapsto F(\nabla)(Z,Y)X\Big)$$

and s^∇ is the pseudohermitian scalar curvature, i.e., $s^\nabla = \sum \mathrm{Ric}^\nabla(\xi_\alpha, \overline{\xi_\alpha})$ ($\{\xi_\alpha\}_{\alpha=1}^n$ is a unitary basis of H_+). It will be noteworthy that the tensor $B(\nabla)^0$ coincides with the Bochner curvature of Kähler manifolds of complex dimension n in appearance and vanishes in the ignored case $n = 1$ (cf. Proposition 1.2(3)).

Theorem A. *The tensor $B(\nabla)^0$ is pseudo-conformally invariant and so is the tensor $B(\nabla)^+$ in the case $n = 2$. In the case $n \geq 3$, $B(\nabla)^+$ is pseudo-conformally invariant if and only if J is integrable.*

Remark. By referring to (2.6) and (2.7), it is obvious that the tensor $B(\nabla)^0$ with $(\mathrm{Ric}^\nabla, \mathrm{ric}^\nabla, s^\nabla)$ replaced by $(\mathrm{Ric}(\nabla), -\mathrm{ric}(\nabla), s(\nabla)/2)$ is also pseudo-conformally invariant, where $\mathrm{ric}(\nabla)(Y) = \mathrm{ric}(\nabla)_+(Y)$ is defined similarly and $s(\nabla)$ is the ordinary scalar curvature.

Following the idea in [13, §3] (and [12, §3]), let us choose arbitrarily a nowhere vanishing $(2n+1)$-form ω and take a smooth function h defined by $dV_\theta = \pm e^h \omega$, where dV_θ is the volume element, i.e., $dV_\theta = \theta \wedge (d\theta)^n / n!$. Then we denote by $\Xi^\omega_\pm = \mathrm{grad}_\pm h$ the $H_\pm$-components of the gradient vector field $\Xi^\omega = \mathrm{grad}\, h$ and consider the tensor $U^+(\Xi^\omega :) \in \Gamma(H_+ \otimes H_+^* \otimes H_+^* \otimes H_+^*)$ defined by

$$U^+(\Xi^\omega : X, Y, Z) = \frac{1}{n-1}\Big\{g(\mathcal{Q}(Z,X) - 2\mathcal{Q}(X,Z), J\Xi^\omega_+)Y - g(\mathcal{Q}(Z,Y) - 2\mathcal{Q}(Y,Z), J\Xi^\omega_+)X\Big\} + g(\mathcal{Q}(X,Z), Y)J\Xi^\omega_+ + g(\mathcal{Q}(X,Y) - \mathcal{Q}(Y,X), J\Xi^\omega_+)Z,$$

which vanishes in the case $n=2$ (cf. Proposition 3.1).

THEOREM B. *In the case* $n \geq 3$, *the tensor* $B(\nabla)^+ - \frac{U^+(\Xi^\omega :)}{2(n+1)}$ *is pseudo-conformally invariant.*

REMARK. Referring to [12, §3] (and [13, §3]), instead of ω one may take a linear connection on TM to define similarly an associated pseudo-conformally invariant tensor.

By considering the identity $\overline{g(F(\nabla)(X,Y)Z,\overline{W})} = -g(\overline{Z}, F(\nabla)(\overline{X},\overline{Y})W)$, the above tensors obviously provide the other types of tensors, $B(\nabla)^-, U^-(\Xi^\omega :) \in \Gamma(H_+ \otimes H_-^* \otimes H_-^* \otimes H_+^*)$ defined by

$$\begin{aligned}
B(\nabla)^-(\overline{X},\overline{Y})Z &= F(\nabla)(\overline{X},\overline{Y})Z \\
&\quad + \frac{1}{n-1}\Big\{g(Z,\overline{X})\,\mathrm{ric}(\nabla)(\overline{Y}) - g(Z,\overline{Y})\,\mathrm{ric}(\nabla)(\overline{X})\Big\}, \\
U^-(\Xi^\omega : \overline{X},\overline{Y},Z) &= -\frac{1}{n-1}\Big\{g(Z,\overline{Y})\Big(\mathcal{Q}(J\Xi_-^\omega,\overline{X}) - 2\mathcal{Q}(\overline{X},J\Xi_-^\omega)\Big) \\
&\quad - g(Z,\overline{X})\Big(\mathcal{Q}(J\Xi_-^\omega,\overline{Y}) - 2\mathcal{Q}(\overline{Y},J\Xi_-^\omega)\Big)\Big\} \\
&\quad - g(\mathcal{Q}(\overline{X},J\Xi_-^\omega),\overline{Y})Z \\
&\quad - g(Z,J\Xi_-^\omega)\Big(\mathcal{Q}(\overline{X},\overline{Y}) - \mathcal{Q}(\overline{Y},\overline{X})\Big),
\end{aligned}$$

the latter of which vanishes in the case $n=2$.

COROLLARY C. (1) *In the case* $n=2$, *the tensor* $B(\nabla)^-$ *is pseudo-conformally invariant. In the case* $n \geq 3$, $B(\nabla)^-$ *is pseudo-conformally invariant if and only if* J *is integrable.* (2) *In the case* $n \geq 3$, *the tensor* $B(\nabla)^- - \frac{U^-(\Xi^\omega :)}{2(n+1)}$ *is pseudo-conformally invariant.*

After preliminaries in §1 and §2, we will prove Theorems A and B in §3. Here we will assemble some properties of the connections for quick reference. Refer to [10], [6], [5] for more detailed explanation. We have ${}^*\nabla\theta = \nabla\theta = 0$, ${}^*\nabla g = \nabla g = 0$, but the torsion tensors do not vanish: in fact, $T({}^*\nabla)(Z,W) = 0$, $T({}^*\nabla)(Z,\overline{W}) = ig(Z,\overline{W})\xi$, $T(\nabla)(Z,W) = [J,J](Z,W)/4 := (-[Z,W] + [JZ,JW] - J[JZ,W] - J[Z,JW])/4$, $T(\nabla)(Z,\overline{W}) = ig(Z,\overline{W})\xi$ $(Z,W \in \Gamma(H_+))$. If we set ${}^*\tau X = T({}^*\nabla)(\xi,X)$, etc., then ${}^*\tau = \tau$ and $\tau\circ J + J\circ\tau = 0$. We take and fix a local unitary frame

$\xi_\bullet = (\xi_0 = \xi, \xi_1, \ldots, \xi_n, \xi_{\bar{1}}, \ldots, \xi_{\bar{n}})$ of the bundle $TM \otimes \mathbb{C} = \mathbb{C}\xi \oplus H_+ \oplus H_-$ ($\xi_{\bar{\alpha}} := \overline{\xi_\alpha} \in H_-$, $g(\xi_\alpha, \xi_\beta) = 0$, $g(\xi_\alpha, \xi_{\bar{\beta}}) = \delta_{\alpha\beta}$, $1 \le \alpha, \beta \le n$) and denote its dual frame by $\theta^\bullet = (\theta^0 = \theta, \theta^1, \ldots, \theta^n, \theta^{\bar{1}}, \ldots, \theta^{\bar{n}})$. As usual, the Greek indices α, β, ... vary from 1 to n, the block Latin indices A, B, ... vary in $\{0, 1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ and the symbol $\sum$ may be omitted (in an unusual manner). Then we have

$$\begin{aligned}
\tau &= \xi_\alpha \otimes \theta^{\bar{\gamma}} \cdot \tau^\alpha_{\bar{\gamma}} + \xi_{\bar{\alpha}} \otimes \theta^\gamma \cdot \tau^{\bar{\alpha}}_\gamma \quad (\tau^{\bar{\alpha}}_\gamma = \tau^{\bar{\gamma}}_\alpha),\\
\mathcal{Q} &= \xi_\alpha \otimes \theta^{\bar{\beta}} \otimes \theta^{\bar{\gamma}} \cdot \mathcal{Q}^\alpha_{\bar{\beta}\bar{\gamma}}\\
&\qquad + \xi_{\bar{\alpha}} \otimes \theta^\beta \otimes \theta^\gamma \cdot \mathcal{Q}^{\bar{\alpha}}_{\beta\gamma} \quad (\mathcal{Q}^{\bar{\alpha}}_{\beta\gamma} = -\mathcal{Q}^{\bar{\beta}}_{\alpha\gamma} = -\mathcal{Q}^{\bar{\beta}}_{\gamma\alpha} - \mathcal{Q}^{\bar{\gamma}}_{\alpha\beta}).
\end{aligned}$$

If we set ${}^*\nabla \xi_B = \xi_A \cdot \omega({}^*\nabla)^A_B$, $\nabla \xi_B = \xi_A \cdot \omega(\nabla)^A_B$, then

$$\begin{aligned}
&\omega({}^*\nabla)^\alpha_\beta = \omega(\nabla)^\alpha_\beta, \quad \omega({}^*\nabla)^{\bar{\alpha}}_{\bar{\beta}} = \omega(\nabla)^{\bar{\alpha}}_{\bar{\beta}},\\
&\omega({}^*\nabla)^{\bar{\alpha}}_\beta(\xi_\gamma) = -\frac{i}{2}\mathcal{Q}^{\bar{\alpha}}_{\beta\gamma}, \quad \omega({}^*\nabla)^\alpha_{\bar{\beta}}(\xi_{\bar{\gamma}}) = \frac{i}{2}\mathcal{Q}^\alpha_{\bar{\beta}\bar{\gamma}}
\end{aligned}$$

and the others vanish. Recall that J is integrable if and only if the Tanno tensor $\mathcal{Q}$ vanishes ([10, Proposition 2.1]), and if $\mathcal{Q} = 0$, then obviously the connections ${}^*\nabla$ and ∇ coincide and, further, they coincide with the Tanaka-Webster connection (cf. [10, Proposition 3.1], [6, Lemma 1.1], [4, §1.2]).

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1. The Curvature Tensor of the Hermitian Tanno Connection

Since, by definition,

$$\begin{aligned}
g({}^*\nabla_X Y, Z) &= g(\nabla^g_X Y, Z) - g(\theta(Y)\,\tau X, Z) + g(g(\tau X, Y)\xi, Z)\\
&\qquad + \frac{1}{2}\Big\{ - g(\theta(X)JY, Z) - g(\theta(Y)JX, Z) - g(g(X, JY)\xi, Z)\Big\}
\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned}
\nabla^g_X Y &= {}^*\nabla_X Y - g((\tau + \frac{1}{2}J)X, Y)\xi\\
&\qquad + \theta(Y)(\tau + \frac{1}{2}J)X + \theta(X)(\tau + \frac{1}{2}J)Y - \theta(X)\,\tau Y.
\end{aligned}$$

In addition, obviously

$$
\begin{aligned}
[X,Y] &= {}^*\nabla_X Y - {}^*\nabla_Y X - T({}^*\nabla)(X,Y) \\
&= {}^*\nabla_X Y - {}^*\nabla_Y X + g(X,JY)\xi - \theta(X)\,\tau Y + \theta(Y)\,\tau X.
\end{aligned}
$$

Therefore we obtain:

PROPOSITION 1.1 (cf. Tanno [10, §6 and §8]).
(1) (*cf. Blair-Dragomir* [*2*, (36)]) *For any* $X, Y, Z \in \Gamma(TM)$, *we have*

$$
\begin{aligned}
F(\nabla^g)(X,Y)Z = {}& F({}^*\nabla)(X,Y)Z + (LX \wedge LY)Z \\
& - \frac{1}{2}g(X,JY)JZ + \theta(Z)S^\tau(X,Y) \\
& - g(S^\tau(X,Y),Z)\xi + \theta(Z)(\theta\wedge\mathcal{O})(X,Y) - g((\theta\wedge\mathcal{O})(X,Y),Z)\xi \\
& + \frac{1}{2}\Big\{\theta(Z)S^J(X,Y) - g(S^J(X,Y),Z)\xi \\
& \quad - \theta(X)({}^*\nabla_Y J)Z + \theta(Y)({}^*\nabla_X J)Z\Big\},
\end{aligned}
$$

where we set

$$
\begin{aligned}
& L = \tau + \frac{1}{2}J, \quad \mathcal{O} = \tau^2 + J\tau - \frac{1}{4}, \quad (X\wedge Y)Z = g(X,Z)Y - g(Y,Z)X, \\
& (\theta\wedge\mathcal{O})(X,Y) = \theta(X)\mathcal{O}(Y) - \theta(Y)\mathcal{O}(X), \\
& S^\tau(X,Y) = ({}^*\nabla_X\tau)Y - ({}^*\nabla_Y\tau)X, \quad S^J(X,Y) = ({}^*\nabla_X J)Y - ({}^*\nabla_Y J)X.
\end{aligned}
$$

(2) (*cf. Blair-Dragomir* [*2*, Theorem 3], *Seshadri* [*9*, Proposition 3.2]) *As for the curvature form* $F({}^*\nabla) = d\omega({}^*\nabla) + \omega({}^*\nabla)\wedge\omega({}^*\nabla)$: *It is reduced to* $F({}^*\nabla) = \sum_{A\neq 0, B\neq 0} \xi_A \otimes \theta^B \cdot F({}^*\nabla)^A_B$, $F({}^*\nabla)^A_B = -F({}^*\nabla)^{\bar B}_{\bar A}$, *and, setting* $F({}^*\nabla)^\alpha_{\beta\gamma\bar\lambda} = g(F({}^*\nabla)(\xi_\gamma, \xi_{\bar\lambda})\xi_\beta, \xi_{\bar\alpha})$, *etc., we have*

$$
\begin{aligned}
(1.1)\quad F({}^*\nabla)^\alpha_\beta = {}& F({}^*\nabla)^\alpha_{\beta\gamma\bar\lambda}\,\theta^\gamma\wedge\theta^{\bar\lambda} + i\tau^\alpha_{\bar\lambda}\,\theta^{\bar\beta}\wedge\theta^{\bar\lambda} - i\tau^{\bar\beta}_\gamma\,\theta^\gamma\wedge\theta^\alpha \\
& + \Big\{({}^*\nabla_{\xi_{\bar\alpha}}\tau)^{\bar\gamma}_\beta + \frac{i}{2}\tau^\nu_{\bar\alpha}\mathcal{Q}^{\bar\gamma}_{\nu\beta}\Big\}\theta^\gamma\wedge\theta \\
& - \Big\{({}^*\nabla_{\xi_\beta}\tau)^\gamma_{\bar\alpha} - \frac{i}{2}\tau^{\bar\nu}_\beta\mathcal{Q}^\gamma_{\bar\nu\bar\alpha}\Big\}\theta^{\bar\gamma}\wedge\theta \\
& + \frac{i}{4}({}^*\nabla_{\xi_{\bar\alpha}}\mathcal{Q})^{\bar\lambda}_{\gamma\beta}\,\theta^\gamma\wedge\theta^\lambda + \frac{i}{4}({}^*\nabla_{\xi_\beta}\mathcal{Q})^\lambda_{\bar\gamma\bar\alpha}\,\theta^{\bar\gamma}\wedge\theta^{\bar\lambda},
\end{aligned}
$$

$$
(1.2)\quad \left\{
\begin{aligned}
& F({}^*\nabla)^\alpha_{\beta\gamma\bar\lambda} = F({}^*\nabla)^\lambda_{\gamma\beta\bar\alpha}, \\
& F({}^*\nabla)^\alpha_{\beta\gamma\bar\lambda}\,\theta^\gamma\wedge\theta^{\bar\lambda} = F(\nabla^g)^\alpha_{\beta\gamma\bar\lambda}\,\theta^\gamma\wedge\theta^{\bar\lambda} \\
& \qquad\qquad - \tau^{\bar\beta}_\gamma\tau^\alpha_{\bar\lambda}\,\theta^\gamma\wedge\theta^{\bar\lambda} + \frac{1}{4}\,\theta^\alpha\wedge\theta^{\bar\beta} + \frac{1}{2}\delta_{\beta\alpha}\,\theta^\gamma\wedge\theta^{\bar\gamma}
\end{aligned}
\right.
$$

and

$$F({}^*\nabla)^{\bar\alpha}_{\bar\beta} = \Big\{({}^*\nabla_{\xi_\alpha}\tau)^{\bar\gamma}_{\beta} - ({}^*\nabla_{\xi_\beta}\tau)^{\bar\gamma}_{\alpha}\Big\}\theta^\gamma\wedge\theta + \frac{i}{2}\tau^{\gamma}_{\bar\nu}\Big\{\mathcal{Q}^{\bar\nu}_{\beta\alpha} - \mathcal{Q}^{\bar\nu}_{\alpha\beta}\Big\}\theta^{\bar\gamma}\wedge\theta$$
$$+ \frac{i}{2}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\alpha}_{\beta\gamma}\,\theta^\gamma\wedge\theta^{\bar\lambda} + \frac{i}{2}({}^*\nabla_{\xi_\lambda}\mathcal{Q})^{\bar\alpha}_{\beta\gamma}\,\theta^\gamma\wedge\theta^\lambda$$
$$+ \frac{i}{2}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\alpha}_{\beta\bar\gamma}\,\theta^{\bar\gamma}\wedge\theta^{\bar\lambda}.$$

(3) *We have*

$$\mathrm{Ric}^{{}^*\nabla}(\xi_\alpha,\xi_\beta) = \frac{i}{2}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\alpha\lambda},$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_\beta) = \frac{i}{2}\Big\{({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\lambda\alpha} + ({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\alpha}_{\lambda\beta}\Big\} + i(n-1)\tau^{\bar\beta}_{\alpha},$$
$$\mathrm{Ric}^{{}^*\nabla}(\xi_\alpha,\xi_{\bar\beta}) = F({}^*\nabla)^{\lambda}_{\lambda\alpha\bar\beta}, \quad \mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_{\bar\beta}) = \mathrm{Ric}^{{}^*\nabla}(\xi_\alpha,\xi_{\bar\beta}) - \frac{1}{2}\mathcal{Q}^{\bar\alpha}_{\lambda\mu}\mathcal{Q}^{\beta}_{\bar\lambda\bar\mu}.$$

Proof. As for (1) and (2): By lengthy calculation following [4, §1.4] and [2, §4] we obtain the formulas. (The terms in the last line of the above expression of $F(\nabla^g)(X,Y)Z$ are omitted from [2, (36)] and the last term in the above one of $F({}^*\nabla)^{\bar\alpha}_{\bar\beta}$ is omitted from the corresponding expression in [9, Proposition 3.2].) As for (3): By (2),

$$\mathrm{Ric}^{{}^*\nabla}(\xi_\alpha,\xi_\beta) = g(F({}^*\nabla)(\xi_\alpha,\xi_\beta)\xi_\lambda,\xi_{\bar\lambda}) = F({}^*\nabla)^{\lambda}_{\lambda\alpha\beta}$$
$$= \frac{i}{4}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\alpha\lambda} - \frac{i}{4}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\alpha}_{\beta\lambda} - i\tau^{\bar\lambda}_{\alpha}\delta_{\lambda\beta} + i\tau^{\bar\lambda}_{\beta}\delta_{\lambda\alpha} = \frac{i}{2}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\alpha\lambda},$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_\beta) = g(F({}^*\nabla)(\xi_\lambda,\xi_\beta)\xi_\alpha,\xi_{\bar\lambda}) + g(F({}^*\nabla)(\xi_{\bar\lambda},\xi_\beta)\xi_\alpha,\xi_\lambda)$$
$$= \frac{i}{2}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\lambda\alpha} + i(n-1)\tau^{\bar\beta}_{\alpha} + \frac{i}{2}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\alpha}_{\lambda\beta}$$

and

$$\begin{aligned}(1.3)\quad \mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_{\bar\beta}) &= g(F({}^*\nabla)(\xi_\lambda,\xi_{\bar\beta})\xi_\alpha,\xi_{\bar\lambda}) + g(F({}^*\nabla)(\xi_{\bar\lambda},\xi_{\bar\beta})\xi_\alpha,\xi_\lambda)\\ &= g(F({}^*\nabla)(\xi_\alpha,\xi_{\bar\beta})\xi_\lambda,\xi_{\bar\lambda}) + 2g(F({}^*\nabla)(\xi_{\bar\lambda},\xi_{\bar\beta})\xi_\alpha,\xi_\lambda)\\ &= \mathrm{Ric}^{{}^*\nabla}(\xi_\alpha,\xi_{\bar\beta}) - \frac{1}{2}\mathcal{Q}^{\bar\alpha}_{\lambda\mu}\mathcal{Q}^{\beta}_{\bar\lambda\bar\mu}.\end{aligned}$$

Note that $g(F({}^*\nabla)(\xi_{\bar\lambda},\xi_{\bar\beta})\xi_\alpha,\xi_\lambda)$ is equal to

$$\frac{i}{2}({}^*\nabla_{\xi_{\bar\beta}}\mathcal{Q})^{\bar\lambda}_{\alpha\bar\lambda} - \frac{i}{2}({}^*\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\lambda}_{\alpha\bar\beta} = \frac{1}{4}\mathcal{Q}^{\bar\lambda}_{\alpha\mu}\{\mathcal{Q}^{\mu}_{\bar\lambda\bar\beta} - \mathcal{Q}^{\mu}_{\bar\beta\bar\lambda}\} = \frac{1}{4}\mathcal{Q}^{\bar\lambda}_{\alpha\mu}\mathcal{Q}^{\beta}_{\bar\lambda\bar\mu}$$

$$= -\frac{1}{4}\mathcal{Q}^{\bar\alpha}_{\lambda\mu}\mathcal{Q}^{\beta}_{\bar\lambda\bar\mu}. \ \square$$

PROPOSITION 1.2.

(1) *For any $X, Y, Z \in \Gamma(TM)$, we have*

$$\begin{aligned}
F(\nabla^g)(X,Y)Z &= F(\nabla)(X,Y)Z + (LX\wedge LY)Z - \frac{1}{2}g(X,JY)JZ \\
&+ \theta(Z)S^{\nabla,\tau}(X,Y) - g(S^{\nabla,\tau}(X,Y),Z)\xi \\
&+ \theta(Z)(\theta\wedge\mathcal{O})(X,Y) - g((\theta\wedge\mathcal{O})(X,Y),Z)\xi \\
&+ \frac{1}{2}J\Big\{(\nabla_X\mathcal{Q})(Z,Y) - (\nabla_Y\mathcal{Q})(Z,X) + \mathcal{Q}(Z,T(\nabla)(X,Y))\Big\} \\
&+ \frac{1}{4}\Big\{\mathcal{Q}(\mathcal{Q}(Z,Y),X) - \mathcal{Q}(\mathcal{Q}(Z,X),Y)\Big\} - \frac{1}{2}\theta(X)\mathcal{Q}(Z,Y) \\
&+ \frac{1}{2}\theta(Y)\mathcal{Q}(Z,X) \\
&+ \frac{1}{2}\theta(Z)\Big\{\mathcal{Q}(Y,X) + J\mathcal{Q}(\tau Y,X) + J\tau\mathcal{Q}(Y,X) \\
&\qquad - \mathcal{Q}(X,Y) - J\mathcal{Q}(\tau X,Y) - J\tau\mathcal{Q}(X,Y)\Big\} \\
&- \frac{1}{2}\Big\{g(\mathcal{Q}(Y,X),Z) + g(J\mathcal{Q}(\tau Y,X),Z) + g(J\tau\mathcal{Q}(Y,X),Z) \\
&\qquad - g(\mathcal{Q}(X,Y),Z) - g(J\mathcal{Q}(\tau X,Y),Z) - g(J\tau\mathcal{Q}(X,Y),Z)\Big\}\xi,
\end{aligned}$$

where we set $S^{\nabla,\tau}(X,Y) = (\nabla_X\tau)Y - (\nabla_Y\tau)X$.

(2) *As for the curvature form $F(\nabla) = d\omega(\nabla) + \omega(\nabla)\wedge\omega(\nabla)$: It is reduced to $F(\nabla) = \xi_\alpha\otimes\theta^\beta\cdot F(\nabla)^\alpha_\beta + \xi_{\bar\alpha}\otimes\theta^{\bar\beta}\cdot F(\nabla)^{\bar\alpha}_{\bar\beta}$, $F(\nabla)^\alpha_\beta = -F(\nabla)^{\bar\beta}_{\bar\alpha}$, and we have*

$$\begin{aligned}
(1.4)\quad F(\nabla)^\alpha_\beta &= F(\nabla)^\alpha_{\beta\gamma\bar\lambda}\,\theta^\gamma\wedge\theta^{\bar\lambda} + i\tau^\alpha_{\bar\lambda}\,\theta^{\bar\beta}\wedge\theta^{\bar\lambda} - i\tau^{\bar\beta}_\gamma\,\theta^\gamma\wedge\theta^\alpha \\
&+ \Big\{(\nabla_{\xi_{\bar\alpha}}\tau)^{\bar\gamma}_\beta + \frac{i}{2}\tau^\nu_{\bar\alpha}\mathcal{Q}^{\bar\gamma}_{\nu\beta}\Big\}\theta^\gamma\wedge\theta \\
&- \Big\{(\nabla_{\xi_\beta}\tau)^\gamma_{\bar\alpha} - \frac{i}{2}\tau^{\bar\nu}_\beta\mathcal{Q}^\gamma_{\bar\nu\bar\alpha}\Big\}\theta^{\bar\gamma}\wedge\theta \\
&+ \frac{i}{4}(\nabla_{\xi_{\bar\alpha}}\mathcal{Q})^{\bar\lambda}_{\gamma\beta}\,\theta^\gamma\wedge\theta^\lambda + \frac{i}{4}(\nabla_{\xi_\beta}\mathcal{Q})^\lambda_{\bar\gamma\bar\alpha}\,\theta^{\bar\gamma}\wedge\theta^{\bar\lambda},
\end{aligned}$$

$$(1.5)\quad \begin{cases} F(\nabla)^{\alpha}_{\beta\gamma\bar{\lambda}} = F(\nabla)^{\lambda}_{\gamma\beta\bar{\alpha}} + \dfrac{1}{4}\Big\{\mathcal{Q}^{\alpha}_{\bar{\mu}\bar{\lambda}}\mathcal{Q}^{\bar{\mu}}_{\beta\gamma} - \mathcal{Q}^{\lambda}_{\bar{\mu}\bar{\alpha}}\mathcal{Q}^{\bar{\mu}}_{\gamma\beta}\Big\}, \\ F(\nabla)^{\alpha}_{\beta\gamma\bar{\lambda}}\,\theta^{\gamma}\wedge\theta^{\bar{\lambda}} = F(\nabla^{g})^{\alpha}_{\beta\gamma\bar{\lambda}}\,\theta^{\gamma}\wedge\theta^{\bar{\lambda}} \\ \qquad - \Big\{\dfrac{1}{4}\mathcal{Q}^{\bar{\beta}}_{\mu\gamma}\mathcal{Q}^{\alpha}_{\bar{\mu}\bar{\lambda}} + \tau^{\bar{\beta}}_{\gamma}\tau^{\alpha}_{\bar{\lambda}}\Big\}\theta^{\gamma}\wedge\theta^{\bar{\lambda}} + \dfrac{1}{4}\,\theta^{\alpha}\wedge\theta^{\bar{\beta}} + \dfrac{1}{2}\delta_{\beta\alpha}\,\theta^{\gamma}\wedge\theta^{\bar{\gamma}}. \end{cases}$$

(3) *We have*

$$\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_\beta) = \frac{i}{2}(\nabla_{\xi_{\bar{\lambda}}}\mathcal{Q})^{\bar{\beta}}_{\alpha\lambda}, \quad \mathrm{Ric}(\nabla)(\xi_\alpha,\xi_\beta) = \frac{i}{2}(\nabla_{\xi_{\bar{\lambda}}}\mathcal{Q})^{\bar{\beta}}_{\lambda\alpha} + i(n-1)\tau^{\bar{\alpha}}_{\beta},$$

$$\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_{\bar{\beta}}) = F(\nabla)^{\lambda}_{\lambda\alpha\bar{\beta}}, \quad \mathrm{Ric}(\nabla)(\xi_\alpha,\xi_{\bar{\beta}}) = \mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_{\bar{\beta}}) - \frac{1}{4}\mathcal{Q}^{\bar{\alpha}}_{\lambda\mu}\mathcal{Q}^{\beta}_{\bar{\mu}\bar{\lambda}}.$$

Proof. (1) follows from Proposition 1.1(1) and the formulas

$$(1.6)\quad \begin{aligned} F({}^*\nabla)(X,Y)Z &= F(\nabla)(X,Y)Z + \frac{1}{2}J(\nabla_X\mathcal{Q})(Z,Y) \\ &\quad - \frac{1}{2}J(\nabla_Y\mathcal{Q})(Z,X) + \frac{1}{2}J\mathcal{Q}(Z,T(\nabla)(X,Y)) \\ &\quad + \frac{1}{4}\mathcal{Q}(\mathcal{Q}(Z,Y),X) - \frac{1}{4}\mathcal{Q}(\mathcal{Q}(Z,X),Y), \\ ({}^*\nabla_X\mathcal{Q})(Z,Y) &= (\nabla_X\mathcal{Q})(Z,Y) \\ &\quad + \frac{1}{2}J\mathcal{Q}(Z,\mathcal{Q}(Y,X)) + \frac{1}{2}J\mathcal{Q}(\mathcal{Q}(Z,Y),X) \\ &\quad + \frac{1}{2}J\mathcal{Q}(\mathcal{Q}(Z,X),Y), \\ ({}^*\nabla_X\tau)Y &= (\nabla_X\tau)Y + \frac{1}{2}J\mathcal{Q}(\tau Y,X) + \frac{1}{2}J\tau\mathcal{Q}(Y,X). \end{aligned}$$

As for (2): Obviously we have

$$F(\nabla)^{\alpha}_{\beta} = F({}^*\nabla)^{\alpha}_{\beta} - \omega({}^*\nabla)^{\alpha}_{\bar{\mu}}\wedge\omega({}^*\nabla)^{\bar{\mu}}_{\beta} = F({}^*\nabla)^{\alpha}_{\beta} + \frac{1}{4}\mathcal{Q}^{\alpha}_{\bar{\mu}\bar{\lambda}}\mathcal{Q}^{\bar{\mu}}_{\beta\gamma}\,\theta^{\gamma}\wedge\theta^{\bar{\lambda}}.$$

Hence, (1.4) follows from (1.1) and the identities

$$(1.7)\quad \begin{aligned} &F({}^*\nabla)^{\alpha}_{\beta\gamma\bar{\lambda}} = F(\nabla)^{\alpha}_{\beta\gamma\bar{\lambda}} - \frac{1}{4}\mathcal{Q}^{\alpha}_{\bar{\mu}\bar{\lambda}}\mathcal{Q}^{\bar{\mu}}_{\beta\gamma}, \\ &({}^*\nabla_{\xi_{\bar{\alpha}}}\tau)^{\bar{\gamma}}_{\beta} = (\nabla_{\xi_{\bar{\alpha}}}\tau)^{\bar{\gamma}}_{\beta}, \quad ({}^*\nabla_{\xi_{\bar{\alpha}}}\mathcal{Q})^{\bar{\lambda}}_{\gamma\beta} = (\nabla_{\xi_{\bar{\alpha}}}\mathcal{Q})^{\bar{\lambda}}_{\gamma\beta} \end{aligned}$$

which the above formulas imply. (1.5) can be also easily deduced from (1.2). As for (3): We have

$$\begin{aligned}
&\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_\beta)=g(F(\nabla)(\xi_\alpha,\xi_\beta)\xi_\lambda,\xi_{\bar\lambda})=\frac{i}{2}(\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\lambda\alpha},\\
&\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_\beta)=g(F(\nabla)(\xi_\lambda,\xi_\beta)\xi_\alpha,\xi_{\bar\lambda})+g(F(\nabla)(\xi_{\bar\lambda},\xi_\beta)\xi_\alpha,\xi_\lambda)\\
&\qquad=g(F(\nabla)(\xi_\lambda,\xi_\beta)\xi_\alpha,\xi_{\bar\lambda})=\frac{i}{2}(\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\lambda\alpha}+i(n-1)\tau^{\bar\alpha}_{\beta},
\end{aligned}$$

and, by (1.7) and the argument at (1.3), we obtain

$$\begin{aligned}
&\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_{\bar\beta})=g(F(\nabla)(\xi_\lambda,\xi_{\bar\beta})\xi_\alpha,\xi_{\bar\lambda})=F(\nabla)^{\lambda}_{\alpha\lambda\bar\beta}\\
&=F({}^*\nabla)^{\lambda}_{\alpha\lambda\bar\beta}+\frac14\mathcal{Q}^{\lambda}_{\bar\mu\bar\beta}\mathcal{Q}^{\bar\mu}_{\alpha\lambda}=F({}^*\nabla)^{\lambda}_{\lambda\alpha\bar\beta}-\frac14\mathcal{Q}^{\bar\alpha}_{\lambda\mu}\mathcal{Q}^{\beta}_{\bar\lambda\bar\mu}+\frac14\mathcal{Q}^{\lambda}_{\bar\mu\bar\beta}\mathcal{Q}^{\bar\mu}_{\alpha\lambda}\\
&=F(\nabla)^{\lambda}_{\lambda\alpha\bar\beta}-\frac14\mathcal{Q}^{\lambda}_{\bar\mu\bar\beta}\mathcal{Q}^{\bar\mu}_{\lambda\alpha}-\frac14\mathcal{Q}^{\bar\alpha}_{\lambda\mu}\mathcal{Q}^{\beta}_{\bar\lambda\bar\mu}+\frac14\mathcal{Q}^{\lambda}_{\bar\mu\bar\beta}\mathcal{Q}^{\bar\mu}_{\alpha\lambda}\\
&=\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_{\bar\beta})-\frac14\mathcal{Q}^{\bar\alpha}_{\lambda\mu}\mathcal{Q}^{\beta}_{\bar\mu\bar\lambda}.\ \square
\end{aligned}$$

2. The Curvature Tensor of the Gauge Transform $\tilde\nabla=\nabla^{(e^{2f}\theta)}$

According to [12] (or [8]), we consider the gauge transformation $\xi_\bullet\Rightarrow\tilde\xi_\bullet$, $\theta^\bullet\Rightarrow\tilde\theta^\bullet$ given by

$$\begin{aligned}
&\tilde\xi_0=e^{-2f}\xi_0-e^{-2f}2i\xi_{\bar\mu}(f)\xi_\mu+e^{-2f}2i\xi_\mu(f)\xi_{\bar\mu},\\
(2.1)\quad&\tilde\xi_\alpha=e^{-f}\xi_\alpha,\quad \tilde\xi_{\bar\alpha}=e^{-f}\xi_{\bar\alpha},\\
&\tilde\theta^0=e^{2f}\theta^0,\quad \tilde\theta^\beta=e^f\theta^\beta+e^f2i\xi_{\bar\beta}(f)\theta^0,\quad \tilde\theta^{\bar\beta}=e^f\theta^{\bar\beta}-e^f2i\xi_\beta(f)\theta^0.
\end{aligned}$$

We set $\tilde\omega^A_B=\omega(\tilde\nabla)^A_B=\tilde g(\tilde\nabla\tilde\xi_B,\tilde\xi_{\bar A})$ and $({}^*\tilde\nabla_{\tilde\xi_\gamma}J)\tilde\xi_\beta=\tilde\xi_{\bar\alpha}\cdot\tilde{\mathcal{Q}}^{\bar\alpha}_{\beta\gamma}$, etc.

LEMMA 2.1. *We have*

$$\begin{aligned}
&e^f\tilde\omega^\alpha_\beta(\tilde\xi_\gamma)=\omega^\alpha_\beta(\xi_\gamma)+\delta_{\alpha\beta}\xi_\gamma(f)+\delta_{\alpha\gamma}2\xi_\beta(f),\\
&e^f\tilde\omega^\alpha_\beta(\tilde\xi_{\bar\gamma})=\omega^\alpha_\beta(\xi_{\bar\gamma})-\delta_{\alpha\beta}\xi_{\bar\gamma}(f)-\delta_{\beta\gamma}2\xi_{\bar\alpha}(f),\\
&\quad e^{2f}\tilde\tau^{\bar\alpha}_\beta=\tau^{\bar\alpha}_\beta+2i\Big(\xi_\beta\xi_\alpha-\nabla_{\xi_\beta}\xi_\alpha\Big)(f)-4i\xi_\alpha(f)\xi_\beta(f)+\xi_{\bar\mu}(f)\mathcal{Q}^{\bar\mu}_{\beta\alpha},\\
&\quad e^f\tilde{\mathcal{Q}}^{\bar\alpha}_{\beta\gamma}=\mathcal{Q}^{\bar\alpha}_{\beta\gamma}.
\end{aligned}$$

Proof. The lemma is shown in a way similar to [2, Lemma 10]. As for $e^{2f}\tilde{\tau}^{\bar\alpha}_{\beta}$: As in [2] we know

$$\begin{aligned} e^{2f}\tilde{\tau}^{\bar\alpha}_{\beta} &= \tau^{\bar\alpha}_{\beta} + i\xi_\beta\xi_\alpha(f) + i\xi_\alpha\xi_\beta(f) - 4i\xi_\beta(f)\xi_\alpha(f) \\ &\quad - i\omega^{\mu}_{\beta}(\xi_\alpha)\,\xi_\mu(f) - i\omega^{\mu}_{\alpha}(\xi_\beta)\,\xi_\mu(f) + i\omega^{\bar\mu}_{\beta}(\xi_\alpha)\,\xi_{\bar\mu}(f) + i\omega^{\bar\mu}_{\alpha}(\xi_\beta)\,\xi_{\bar\mu}(f) \\ &= \tau^{\bar\alpha}_{\beta} + i\Big(\xi_\alpha\xi_\beta - \nabla_{\xi_\alpha}\xi_\beta\Big)(f) + i\Big(\xi_\beta\xi_\alpha - \nabla_{\xi_\beta}\xi_\alpha\Big)(f) \\ &\quad - 4i\xi_\alpha(f)\xi_\beta(f) + \frac{1}{2}(\mathcal{Q}^{\bar\mu}_{\alpha\beta} + \mathcal{Q}^{\bar\mu}_{\beta\alpha})\xi_{\bar\mu}(f). \end{aligned}$$

In addition,

$$\begin{aligned} &\Big(\xi_\alpha\xi_\beta - \nabla_{\xi_\alpha}\xi_\beta\Big) - \Big(\xi_\beta\xi_\alpha - \nabla_{\xi_\beta}\xi_\alpha\Big) = T(\nabla)(\xi_\beta,\xi_\alpha) \\ &\quad = T(^*\nabla)(\xi_\beta,\xi_\alpha) - \omega(^*\nabla)^{\bar\mu}_{\alpha}(\xi_\beta)\xi_{\bar\mu} + \omega(^*\nabla)^{\bar\mu}_{\beta}(\xi_\alpha)\xi_{\bar\mu} = \frac{i}{2}(\mathcal{Q}^{\bar\mu}_{\alpha\beta} - \mathcal{Q}^{\bar\mu}_{\beta\alpha})\xi_{\bar\mu}. \end{aligned}$$

Thus we obtain the formula for $e^{2f}\tilde{\tau}^{\bar\alpha}_{\beta}$. □

Proposition 2.2. *We have*

$$\begin{aligned} (2.2)\quad & F(\tilde\nabla)(\xi_\gamma,\xi_\lambda)\xi_\beta - F(\nabla)(\xi_\gamma,\xi_\lambda)\xi_\beta \\ &= 2\Big(\xi_\gamma\xi_\beta - \nabla_{\xi_\gamma}\xi_\beta\Big)(f)\xi_\lambda - 4\xi_\beta(f)\xi_\gamma(f)\xi_\lambda - 2\Big(\xi_\lambda\xi_\beta - \nabla_{\xi_\lambda}\xi_\beta\Big)(f)\xi_\gamma \\ &\quad + 4\xi_\beta(f)\xi_\lambda(f)\xi_\gamma + \xi_{\bar\mu}(f)\, i(\mathcal{Q}^{\bar\mu}_{\gamma\lambda} - \mathcal{Q}^{\bar\mu}_{\lambda\gamma})\xi_\beta + \xi_{\bar\mu}(f)\, i\mathcal{Q}^{\bar\lambda}_{\gamma\beta}\xi_\mu, \\ (2.3)\quad & F(\tilde\nabla)(\xi_\gamma,\xi_{\bar\lambda})\xi_\beta - F(\nabla)(\xi_\gamma,\xi_{\bar\lambda})\xi_\beta \\ &= -2\Big(\xi_{\bar\lambda}\xi_\beta - \nabla_{\xi_{\bar\lambda}}\xi_\beta\Big)(f)\xi_\gamma - \delta_{\beta\lambda}4\xi_\nu(f)\xi_{\bar\nu}(f)\xi_\gamma \\ &\quad - 2\Big(\xi_{\bar\lambda}\xi_\gamma - \nabla_{\xi_{\bar\lambda}}\xi_\gamma\Big)(f)\xi_\beta - \delta_{\gamma\lambda}4\xi_\nu(f)\xi_{\bar\nu}(f)\xi_\beta \\ &\quad - \delta_{\beta\lambda}2\Big(\xi_\gamma\xi_{\bar\mu} - \nabla_{\xi_\gamma}\xi_{\bar\mu}\Big)(f)\xi_\mu - \delta_{\gamma\lambda}2\Big(\xi_\beta\xi_{\bar\mu} - \nabla_{\xi_\beta}\xi_{\bar\mu}\Big)(f)\xi_\mu. \end{aligned}$$

Proof. As for (2.2): Referring to Lemma 2.1, etc., we know

$$(2.4)\quad \begin{aligned} &\tilde\nabla_{\xi_\gamma}\xi_\beta - \nabla_{\xi_\gamma}\xi_\beta = 2\xi_\gamma(f)\xi_\beta + 2\xi_\beta(f)\xi_\gamma, \\ &\tilde\nabla_{\xi_{\bar\gamma}}\xi_\beta - \nabla_{\xi_{\bar\gamma}}\xi_\beta = -\delta_{\beta\gamma}2\xi_{\bar\mu}(f)\xi_\mu, \\ &[\xi_\gamma,\xi_\lambda] = {}^*\nabla_{\xi_\gamma}\xi_\lambda - {}^*\nabla_{\xi_\lambda}\xi_\gamma = \nabla_{\xi_\gamma}\xi_\lambda - \nabla_{\xi_\lambda}\xi_\gamma + \frac{i}{2}(\mathcal{Q}^{\bar\mu}_{\gamma\lambda} - \mathcal{Q}^{\bar\mu}_{\lambda\gamma})\xi_{\bar\mu}. \end{aligned}$$

Hence

$$\begin{aligned}\tilde{\nabla}_{\xi_\gamma}\tilde{\nabla}_{\xi_\lambda}\xi_\beta &= \nabla_{\xi_\gamma}\nabla_{\xi_\lambda}\xi_\beta + 2\xi_\gamma(f)\nabla_{\xi_\lambda}\xi_\beta + 2(\nabla_{\xi_\lambda}\xi_\beta)(f)\xi_\gamma \\ &\quad + 2\xi_\lambda(f)\nabla_{\xi_\gamma}\xi_\beta + 2\xi_\beta(f)\nabla_{\xi_\gamma}\xi_\lambda + 2\xi_\gamma\xi_\lambda(f)\xi_\beta + 2\xi_\gamma\xi_\beta(f)\xi_\lambda \\ &\quad + 4\xi_\lambda(f)\xi_\gamma(f)\xi_\beta + 4\xi_\beta(f)\xi_\gamma(f)\xi_\lambda + 8\xi_\beta(f)\xi_\lambda(f)\xi_\gamma, \\ \tilde{\nabla}_{[\xi_\gamma,\xi_\lambda]}\xi_\beta &= \nabla_{[\xi_\gamma,\xi_\lambda]}\xi_\beta + 2(\nabla_{\xi_\gamma}\xi_\lambda)(f)\xi_\beta + 2\xi_\beta(f)\nabla_{\xi_\gamma}\xi_\lambda \\ &\quad - 2(\nabla_{\xi_\lambda}\xi_\gamma)(f)\xi_\beta - 2\xi_\beta(f)\nabla_{\xi_\lambda}\xi_\gamma - \xi_{\bar\mu}(f)\, i(\mathcal{Q}^{\bar\beta}_{\gamma\lambda} - \mathcal{Q}^{\bar\beta}_{\lambda\gamma})\xi_\mu,\end{aligned}$$

which imply (2.2). As for (2.3): In addition to (2.4), we have

$$[\xi_\gamma, \xi_{\bar\lambda}] = \nabla_{\xi_\gamma}\xi_{\bar\lambda} - \nabla_{\xi_{\bar\lambda}}\xi_\gamma - i\delta_{\gamma\lambda}\xi.$$

Hence

$$\begin{aligned}\tilde{\nabla}_{\xi_\gamma}\tilde{\nabla}_{\xi_{\bar\lambda}}\xi_\beta &= \nabla_{\xi_\gamma}\nabla_{\xi_{\bar\lambda}}\xi_\beta + 2\xi_\gamma(f)\nabla_{\xi_{\bar\lambda}}\xi_\beta + 2(\nabla_{\xi_{\bar\lambda}}\xi_\beta)(f)\xi_\gamma \\ &\quad + \delta_{\beta\lambda}2(\nabla_{\xi_\gamma}\xi_{\bar\mu})(f)\xi_\mu - \delta_{\beta\lambda}4\xi_\gamma(f)\xi_{\bar\mu}(f)\xi_\mu \\ &\quad - \delta_{\beta\lambda}4\xi_\mu(f)\xi_{\bar\mu}(f)\xi_\gamma - \delta_{\beta\lambda}2\xi_\gamma\xi_{\bar\mu}(f)\xi_\mu, \\ \tilde{\nabla}_{[\xi_\gamma,\xi_{\bar\lambda}]}\xi_\beta &= \nabla_{[\xi_\gamma,\xi_{\bar\lambda}]}\xi_\beta - g(\xi_\beta, \nabla_{\xi_\gamma}\xi_{\bar\lambda})2\xi_{\bar\mu}(f)\xi_\mu \\ &\quad - 2(\nabla_{\xi_{\bar\lambda}}\xi_\gamma)(f)\xi_\beta - 2\xi_\beta(f)\nabla_{\xi_{\bar\lambda}}\xi_\gamma - i\delta_{\gamma\lambda}\Big(\tilde{\nabla}_\xi\xi_\beta - \nabla_\xi\xi_\beta\Big),\end{aligned}$$

which imply (2.3). □

PROPOSITION 2.3. *We have*

$$\mathrm{Ric}^{\tilde{\nabla}}(\xi_\alpha, \xi_\beta) - \mathrm{Ric}^{\nabla}(\xi_\alpha, \xi_\beta) = (n+2)\xi_{\bar\mu}(f)\, i\mathcal{Q}^{\bar\beta}_{\alpha\mu},$$

$$\begin{aligned}\text{(2.5)}\quad \mathrm{Ric}(\tilde{\nabla})(\xi_\alpha, \xi_\beta) - \mathrm{Ric}(\nabla)(\xi_\alpha, \xi_\beta) &= \xi_{\bar\mu}(f)\, i(\mathcal{Q}^{\bar\mu}_{\alpha\beta} - 2\mathcal{Q}^{\bar\mu}_{\beta\alpha}) \\ &\quad - 2(n-1)(\xi_\beta\xi_\alpha - \nabla_{\xi_\beta}\xi_\alpha)(f) + 4(n-1)\xi_\alpha(f)\xi_\beta(f),\end{aligned}$$

$$\begin{aligned}\text{(2.6)}\quad \mathrm{Ric}^{\tilde{\nabla}}(\xi_\alpha, \xi_{\bar\beta}) - \mathrm{Ric}^{\nabla}(\xi_\alpha, \xi_{\bar\beta}) &= \mathrm{Ric}(\tilde{\nabla})(\xi_\alpha, \xi_{\bar\beta}) - \mathrm{Ric}(\nabla)(\xi_\alpha, \xi_{\bar\beta}) \\ &= -2(n+2)(\xi_{\bar\beta}\xi_\alpha - \nabla_{\xi_{\bar\beta}}\xi_\alpha)(f) \\ &\quad + \delta_{\alpha\beta}\Big\{ - 2(\xi_{\bar\nu}\xi_\nu - \nabla_{\xi_{\bar\nu}}\xi_\nu)(f) \\ &\qquad - 4(n+1)\xi_\nu(f)\xi_{\bar\nu}(f) + 2(n+1)\, i\xi(f)\Big\}\end{aligned}$$

and

$$\begin{aligned}\text{(2.7)}\quad e^{2f}s^{\tilde{\nabla}} - s^{\nabla} &= \frac{1}{2}\{e^{2f}s(\tilde{\nabla}) - s(\nabla)\} \\ &= -4(n+1)(\xi_{\bar\nu}\xi_\nu - \nabla_{\xi_{\bar\nu}}\xi_\nu)(f) - 4n(n+1)\xi_\nu(f)\xi_{\bar\nu}(f) \\ &\quad + 2n(n+1)\, i\xi(f).\end{aligned}$$

Proof. Referring to Proposition 1.2(3) and Lemma 2.1, we have

$$\begin{aligned}
&\mathrm{Ric}^{\tilde\nabla}(\xi_\alpha,\xi_\beta)=e^{2f}\mathrm{Ric}^{\tilde\nabla}(\tilde\xi_\alpha,\tilde\xi_\beta)=\frac{i}{2}e^{2f}(\tilde\nabla_{\tilde\xi_{\bar\lambda}}\tilde{\mathcal{Q}})^{\bar\beta}_{\alpha\lambda}\\
&=\frac{i}{2}(\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\alpha\lambda}+i\xi_{\bar\mu}(f)\mathcal{Q}^{\bar\mu}_{\alpha\beta}+i\xi_{\bar\mu}(f)\mathcal{Q}^{\bar\beta}_{\mu\alpha}+i\xi_{\bar\mu}(f)\mathcal{Q}^{\bar\beta}_{\alpha\mu}+\delta_{\lambda\lambda}\,i\xi_{\bar\mu}(f)\mathcal{Q}^{\bar\beta}_{\alpha\mu}\\
&=\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_\beta)+(n+2)\xi_{\bar\mu}(f)\,i\mathcal{Q}^{\bar\beta}_{\alpha\mu}.
\end{aligned}$$

Similarly we know

$$\begin{aligned}
&\mathrm{Ric}(\tilde\nabla)(\xi_\alpha,\xi_\beta)=\frac{i}{2}e^{2f}(\tilde\nabla_{\tilde\xi_{\bar\lambda}}\tilde{\mathcal{Q}})^{\bar\beta}_{\lambda\alpha}+i(n-1)e^{2f}\tilde\tau^{\bar\alpha}_{\bar\beta}\\
&=\frac{i}{2}(\nabla_{\xi_{\bar\lambda}}\mathcal{Q})^{\bar\beta}_{\lambda\alpha}+i\xi_{\bar\lambda}(f)\mathcal{Q}^{\bar\beta}_{\lambda\alpha}+\delta_{\beta\lambda}\xi_{\bar\mu}(f)\,i\mathcal{Q}^{\bar\mu}_{\lambda\alpha}+\delta_{\lambda\lambda}\xi_{\bar\mu}(f)\,i\mathcal{Q}^{\bar\beta}_{\mu\alpha}\\
&\quad+\delta_{\alpha\lambda}\,\xi_{\bar\gamma}(f)\,i\mathcal{Q}^{\bar\beta}_{\lambda\gamma}+i(n-1)\tau^{\bar\alpha}_{\bar\beta}-2(n-1)\Big(\xi_\beta\xi_\alpha-\nabla_{\xi_\beta}\xi_\alpha\Big)(f)\\
&\quad+4(n-1)\xi_\alpha(f)\xi_\beta(f)+(n-1)\xi_{\bar\mu}(f)\,i\mathcal{Q}^{\bar\mu}_{\beta\alpha}\\
&=\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_\beta)+\xi_{\bar\mu}(f)\,i(\mathcal{Q}^{\bar\mu}_{\alpha\beta}-2\mathcal{Q}^{\bar\mu}_{\beta\alpha})\\
&\quad-2(n-1)\Big(\xi_\beta\xi_\alpha-\nabla_{\xi_\beta}\xi_\alpha\Big)(f)+4(n-1)\xi_\alpha(f)\xi_\beta(f).
\end{aligned}$$

Last, referring also to (2.3),

$$\begin{aligned}
&\mathrm{Ric}^{\tilde\nabla}(\xi_\alpha,\xi_{\bar\beta})=g(F(\tilde\nabla)(\xi_\alpha,\xi_{\bar\beta})\xi_\lambda,\xi_{\bar\lambda})\\
&=\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_{\bar\beta})-2(n+2)\Big(\xi_{\bar\beta}\xi_\alpha-\nabla_{\xi_{\bar\beta}}\xi_\alpha\Big)(f)\\
&\quad+\delta_{\alpha\beta}\Big\{-2\Big(\xi_{\bar\nu}\xi_\nu-\nabla_{\xi_{\bar\nu}}\xi_\nu\Big)(f)+i2(n+1)\xi(f)-4(n+1)\xi_\nu(f)\xi_{\bar\nu}(f)\Big\},
\end{aligned}$$

etc. □

3. The Proofs of Theorems A and B

As for the tensor $B(\nabla)^0$: (2.3) implies

$$\begin{aligned}
&g(F(\tilde\nabla)(\xi_\gamma,\xi_{\bar\lambda})\xi_\beta,\xi_{\bar\alpha})-g(F(\nabla)(\xi_\gamma,\xi_{\bar\lambda})\xi_\beta,\xi_{\bar\alpha})\\
&=-\delta_{\gamma\alpha}2\Big(\xi_{\bar\lambda}\xi_\beta-\nabla_{\xi_{\bar\lambda}}\xi_\beta\Big)(f)-\delta_{\beta\alpha}2\Big(\xi_{\bar\lambda}\xi_\gamma-\nabla_{\xi_{\bar\lambda}}\xi_\gamma\Big)(f)\\
&\quad-\delta_{\beta\lambda}2\Big(\xi_{\bar\alpha}\xi_\gamma-\nabla_{\xi_{\bar\alpha}}\xi_\gamma\Big)(f)-\delta_{\gamma\lambda}2\Big(\xi_{\bar\alpha}\xi_\beta-\nabla_{\xi_{\bar\alpha}}\xi_\beta\Big)(f)\\
&\quad+2(\delta_{\alpha\gamma}\delta_{\beta\lambda}+\delta_{\alpha\beta}\delta_{\gamma\lambda})\Big(i\xi(f)-2\xi_\nu(f)\xi_{\bar\nu}(f)\Big)
\end{aligned}$$

and, by (2.6) and (2.7), we have

$$i\xi(f) - 2\xi_\nu(f)\xi_{\bar\nu}(f) = \frac{e^{2f}s^{\tilde\nabla} - s^\nabla}{2n(n+1)} + \frac{2}{n}\Big(\xi_{\bar\nu}\xi_\nu - \nabla_{\xi_{\bar\nu}}\xi_\nu\Big)(f),$$

$$\Big(\xi_{\bar\beta}\xi_\alpha - \nabla_{\xi_{\bar\beta}}\xi_\alpha\Big)(f) = -\frac{1}{2(n+2)}\Big\{\mathrm{Ric}^{\tilde\nabla}(\xi_\alpha, \xi_{\bar\beta}) - \mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\beta})\Big\} + \delta_{\alpha\beta}\Big\{\frac{e^{2f}s^{\tilde\nabla} - s^\nabla}{2n(n+2)} + \frac{1}{n}\Big(\xi_{\bar\nu}\xi_\nu - \nabla_{\xi_{\bar\nu}}\xi_\nu\Big)(f)\Big\}.$$

Thus we obtain the equality

$$\begin{aligned} &g(F(\tilde\nabla)(\xi_\gamma, \xi_{\bar\lambda})\xi_\beta, \xi_{\bar\alpha}) - \frac{\delta_{\gamma\alpha}}{n+2}\mathrm{Ric}^{\tilde\nabla}(\xi_\beta, \xi_{\bar\lambda}) - \frac{\delta_{\beta\alpha}}{n+2}\mathrm{Ric}^{\tilde\nabla}(\xi_\gamma, \xi_{\bar\lambda}) \\ &\quad - \frac{\delta_{\beta\lambda}}{n+2}\mathrm{Ric}^{\tilde\nabla}(\xi_\gamma, \xi_{\bar\alpha}) - \frac{\delta_{\gamma\lambda}}{n+2}\mathrm{Ric}^{\tilde\nabla}(\xi_\beta, \xi_{\bar\alpha}) + \frac{\delta_{\beta\lambda}\delta_{\gamma\alpha} + \delta_{\gamma\lambda}\delta_{\beta\alpha}}{(n+1)(n+2)}\, e^{2f}s^{\tilde\nabla} \\ &= g(F(\nabla)(\xi_\gamma, \xi_{\bar\lambda})\xi_\beta, \xi_{\bar\alpha}) - \frac{\delta_{\gamma\alpha}}{n+2}\mathrm{Ric}^\nabla(\xi_\beta, \xi_{\bar\lambda}) - \frac{\delta_{\beta\alpha}}{n+2}\mathrm{Ric}^\nabla(\xi_\gamma, \xi_{\bar\lambda}) \\ &\quad - \frac{\delta_{\beta\lambda}}{n+2}\mathrm{Ric}^\nabla(\xi_\gamma, \xi_{\bar\alpha}) - \frac{\delta_{\gamma\lambda}}{n+2}\mathrm{Ric}^\nabla(\xi_\beta, \xi_{\bar\alpha}) + \frac{\delta_{\beta\lambda}\delta_{\gamma\alpha} + \delta_{\gamma\lambda}\delta_{\beta\alpha}}{(n+1)(n+2)}\, s^\nabla, \end{aligned}$$

which means that $B(\nabla)^0$ is a pseudo-conformal invariant.

As for the tensor $B(\nabla)^+$: Since (2.2) and (2.5) imply

$$\begin{aligned} &g(F(\tilde\nabla)(\xi_\gamma, \xi_\lambda)\xi_\beta, \xi_{\bar\alpha}) - g(F(\nabla)(\xi_\gamma, \xi_\lambda)\xi_\beta, \xi_{\bar\alpha}) \\ &= \delta_{\alpha\lambda}2\Big(\xi_\gamma\xi_\beta - \nabla_{\xi_\gamma}\xi_\beta\Big)(f) - \delta_{\alpha\lambda}4\xi_\beta(f)\xi_\gamma(f) - \delta_{\alpha\gamma}2\Big(\xi_\lambda\xi_\beta - \nabla_{\xi_\lambda}\xi_\beta\Big)(f) \\ &\quad + \delta_{\alpha\gamma}4\xi_\beta(f)\xi_\lambda(f) + \xi_{\bar\alpha}(f)\, i\mathcal{Q}^{\bar\lambda}_{\gamma\beta} + \delta_{\alpha\beta}\xi_{\bar\mu}(f)\, i(\mathcal{Q}^{\bar\mu}_{\gamma\lambda} - \mathcal{Q}^{\bar\mu}_{\lambda\gamma}) \\ &= \frac{1}{n-1}\Big\{\delta_{\alpha\gamma}\mathrm{Ric}(\tilde\nabla)(\xi_\beta, \xi_\lambda) - \delta_{\alpha\lambda}\mathrm{Ric}(\tilde\nabla)(\xi_\beta, \xi_\gamma)\Big\} \\ &\quad - \frac{1}{n-1}\Big\{\delta_{\alpha\gamma}\mathrm{Ric}(\nabla)(\xi_\beta, \xi_\lambda) - \delta_{\alpha\lambda}\mathrm{Ric}(\nabla)(\xi_\beta, \xi_\gamma)\Big\} \\ &\quad + g(U^+(\mathrm{grad}\, f : \xi_\gamma, \xi_\lambda, \xi_\beta), \xi_{\bar\alpha}), \end{aligned}$$

we have

$$B(\tilde\nabla)^+(\xi_\gamma, \xi_\lambda)\xi_\beta - B(\nabla)^+(\xi_\gamma, \xi_\lambda)\xi_\beta = U^+(\mathrm{grad}\, f : \xi_\gamma, \xi_\lambda, \xi_\beta). \tag{3.1}$$

In addition, we know

$$U^+(\mathrm{grad}\, f :) = \frac{\tilde U^+(\tilde\Xi^\omega :)}{2(n+1)} - \frac{U^+(\Xi^\omega :)}{2(n+1)}, \tag{3.2}$$

where the first term on the right hand side is the one associated with $\tilde{\theta}$. (Note that the form ω has been fixed.) Indeed we have $\tilde{U}^+(\tilde{\Xi}^\omega :) = U^+(e^{2f}\tilde{\Xi}^\omega :)$ and $e^{2f}\tilde{\Xi}^\omega_+ - \Xi^\omega_+ = e^{2f}\tilde{\xi}_{\bar{\mu}}(2(n+1)f + h)\,\tilde{\xi}_\mu - \xi_{\bar{\mu}}(h)\,\xi_\mu = 2(n+1)\,\mathrm{grad}_+ f$. Consequently Theorem B holds. Next, let us consider the tensor $\mathcal{U}_- \in \Gamma(H_- \otimes H_+^* \otimes H_+^* \otimes H_+^* \otimes H_-^*)$ defined by $g(U^+(\mathrm{grad}\, f : X, Y, Z), \overline{W}) = g(\mathcal{U}_-(X, Y, Z, \overline{W}), J\mathrm{grad}_+ f)$. For justifying the remaining assertions for $B(\nabla)^+$, then it will suffice to prove:

Proposition 3.1. *The tensor $U^+(\mathrm{grad}\, f\ :)$ vanishes for any $f \in C^\infty(M)$ if and only if the tensor $\mathcal{U}_-$ vanishes. In addition, we have:*

(1) *In the case $n = 2$, $\mathcal{U}_-$ vanishes.*

(2) *In the case $n \geq 3$, $\mathcal{U}_-$ vanishes if and only if the Tanno tensor $\mathcal{Q}$ vanishes.*

Proof. The first assertion is valid because, for a given point P and a given $\xi_{\bar{\mu}}$, there exists a smooth function f such that $\xi_{\bar{\mu}}(f)(P) \neq 0$ and $\xi_{\bar{\nu}}(f)(P) = 0$ $(\nu \neq \mu)$. Let us show (1) and (2). We have

$$\begin{aligned}
&\mathcal{U}_-(\xi_\gamma, \xi_\lambda, \xi_\beta, \xi_{\bar{\alpha}})\\
&= \frac{1}{n-1}\left\{\delta_{\alpha\lambda}(\mathcal{Q}^{\bar{\mu}}_{\beta\gamma} - 2\mathcal{Q}^{\bar{\mu}}_{\gamma\beta})\xi_{\bar{\mu}} - \delta_{\alpha\gamma}(\mathcal{Q}^{\bar{\mu}}_{\beta\lambda} - 2\mathcal{Q}^{\bar{\mu}}_{\lambda\beta})\xi_{\bar{\mu}}\right\}\\
&\quad + \mathcal{Q}^{\bar{\lambda}}_{\gamma\beta}\xi_{\bar{\alpha}} + \delta_{\alpha\beta}\mathcal{Q}^{\bar{\lambda}}_{\gamma\mu}\xi_{\bar{\mu}}.
\end{aligned}$$

As for (2): It is obvious that, if $\mathcal{Q} = 0$, then $\mathcal{U}_- = 0$. The converse is also true because, if $\alpha \notin \{\gamma, \lambda\}$, then $\mathcal{U}_-(\xi_\gamma, \xi_\lambda, \xi_\alpha, \xi_{\bar{\alpha}}) = 2\mathcal{Q}^{\bar{\lambda}}_{\gamma\alpha}\,\xi_{\bar{\alpha}} + \sum_{\mu\neq\alpha}\mathcal{Q}^{\bar{\lambda}}_{\gamma\mu}\,\xi_{\bar{\mu}}$. As for (1): Obviously we have $\mathcal{U}_-(\xi_\gamma, \xi_\lambda, \xi_\beta, \xi_{\bar{\alpha}}) = -\mathcal{U}_-(\xi_\lambda, \xi_\gamma, \xi_\beta, \xi_{\bar{\alpha}})$. By straightforward computation we know $\mathcal{U}_-(\xi_1, \xi_2, \xi_\beta, \xi_{\bar{1}}) = \mathcal{U}_-(\xi_1, \xi_2, \xi_\beta, \xi_{\bar{2}}) = 0$. Thus (1) is certainly true. □

4. The Tanno Connection and Bochner Type Curvature Tensors

In this section, we will present Bochner type tensors associated with $F(^*\nabla)$ deduced immediately from those associated with $F(\nabla)$. As stated in the introduction, Tanno ([11], [12], [13]) also proposed some tensors of those kinds, which are too involved to be reviewed quickly. We notice that, whatever they may be, the differences from ours merely consist of pseudo-conformally invariant terms and gap terms such as $\frac{U^+(\Xi^\omega :)}{2(n+1)}$ at (3.2).

From now on, we assume $X, Y, Z \in \Gamma(H)$ and decompose them into $X = X_+ + X_- \in \Gamma(H \otimes \mathbb{C}) = \Gamma(H_+) \oplus \Gamma(H_-)$, etc. We set

$$
\begin{aligned}
&b(\nabla)^{\mathbb{R}}(X,Y)Z = 2\,\mathrm{Re}\Big(b(\nabla)^{\mathbb{C}}(X,Y)Z\Big) \\
&= 2\,\mathrm{Re}\Big\{(B(\nabla)^0 - F(\nabla))(X_+,Y_-)Z_+ + (B(\nabla)^0 - F(\nabla))(X_-,Y_+)Z_+ \\
&\qquad + (B(\nabla)^+ - F(\nabla))(X_+,Y_+)Z_+ + (B(\nabla)^- - F(\nabla))(X_-,Y_-)Z_+\Big\},
\end{aligned}
$$

where we put $B(\nabla)^0(X_-,Y_+)Z_+ = -B(\nabla)^0(Y_+,X_-)Z_+$.

PROPOSITION 4.1. *We have*

$$
\begin{aligned}
(4.1)\qquad &\{F(^*\tilde{\nabla})(X,Y)Z + b(\tilde{\nabla})^{\mathbb{R}}(X,Y)Z\} \\
&\quad - \{F(^*\nabla)(X,Y)Z + b(\nabla)^{\mathbb{R}}(X,Y)Z\} \\
&= 2\,\mathrm{Re}\Big\{U^+(\mathrm{grad}\, f : X_+,Y_+,Z_+) + U^-(\mathrm{grad}\, f : X_-,Y_-,Z_+)\Big\} \\
&\quad + \frac{1}{2}J\Big\{(\tilde{\nabla}_X\tilde{\mathcal{Q}})(Z,Y) - (\nabla_X\mathcal{Q})(Z,Y)\Big\} \\
&\quad - \frac{1}{2}J\Big\{(\tilde{\nabla}_Y\tilde{\mathcal{Q}})(Z,X) - (\nabla_Y\mathcal{Q})(Z,X)\Big\} \\
&= 2\,\mathrm{Im}\Big\{iU^+(\mathrm{grad}\, f : X_+,Y_+,Z_+) + iU^-(\mathrm{grad}\, f : X_-,Y_-,Z_+) \\
&\qquad + g(X,\mathrm{grad}_+ f)\mathcal{Q}(Z,Y) - g(Y,\mathrm{grad}_+ f)\mathcal{Q}(Z,X) \\
&\qquad + g(Z,\mathrm{grad}_+ f)\Big(\mathcal{Q}(X,Y_-) - \mathcal{Q}(Y_-,X)\Big) \\
&\qquad + g(\mathcal{Q}(Z,Y),\mathrm{grad}_+ f)X_- - g(\mathcal{Q}(Z,X),\mathrm{grad}_+ f)Y_- \\
&\qquad - g(Z,Y_-)\mathcal{Q}(\mathrm{grad}_+ f,X) + g(Z,X_-)\mathcal{Q}(\mathrm{grad}_+ f,Y) \\
&\qquad + \Big(g(X_-,Y) - g(Y_-,X)\Big)\mathcal{Q}(Z,\mathrm{grad}_+ f) \\
&\qquad - \Big(g(\mathcal{Q}(Z,X),Y_-) - g(\mathcal{Q}(Z,Y),X_-)\Big)\mathrm{grad}_+ f\Big\} \\
&=: 2\,\mathrm{Im}\Big(U^{\mathbb{C}}(\mathrm{grad}\, f : X,Y,Z)\Big) =: U^{\mathbb{R}}(\mathrm{grad}\, f : X,Y,Z).
\end{aligned}
$$

PROOF. Referring to (1.6), (3.1), etc.,

$$
\begin{aligned}
&F(^*\tilde{\nabla})(X,Y)Z - F(^*\nabla)(X,Y)Z = \{F(\tilde{\nabla})(X,Y)Z - F(\nabla)(X,Y)Z\} \\
&\qquad + \{F(^*\tilde{\nabla})(X,Y)Z - F(\tilde{\nabla})(X,Y)Z\}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \{F({}^*\nabla)(X,Y)Z - F(\nabla)(X,Y)Z\} \\
&= -b(\tilde{\nabla})^{\mathbb{R}}(X,Y)Z + b(\nabla)^{\mathbb{R}}(X,Y)Z \\
&\quad + 2\,\mathrm{Re}\Big\{U^{+}(\mathrm{grad}\, f : X_+, Y_+, Z_+) + U^{-}(\mathrm{grad}\, f : X_-, Y_-, Z_+)\Big\} \\
&\quad + \frac{1}{2} J\{(\tilde{\nabla}_X \tilde{\mathcal{Q}})(Z,Y) - (\nabla_X \mathcal{Q})(Z,Y)\} \\
&\quad - \frac{1}{2} J\{(\tilde{\nabla}_Y \tilde{\mathcal{Q}})(Z,X) - (\nabla_Y \mathcal{Q})(Z,X)\} \\
&\quad + \frac{1}{2} J\tilde{\mathcal{Q}}(Z, T(\tilde{\nabla})(X,Y)) + \frac{1}{4}\tilde{\mathcal{Q}}(\tilde{\mathcal{Q}}(Z,Y),X) - \frac{1}{4}\tilde{\mathcal{Q}}(\tilde{\mathcal{Q}}(Z,X),Y) \\
&\quad - \frac{1}{2} J\mathcal{Q}(Z, T(\nabla)(X,Y)) - \frac{1}{4}\mathcal{Q}(\mathcal{Q}(Z,Y),X) + \frac{1}{4}\mathcal{Q}(\mathcal{Q}(Z,X),Y).
\end{aligned}
$$

The last two lines vanish because $\mathcal{Q}(Z, T(\nabla)(X,Y))$, $\mathcal{Q}(\mathcal{Q}(Z,Y),X)$, etc., are pseudo-conformally invariant. Hence we obtain the first equality at (4.1). The second one follows from Lemma 2.1. Indeed, we have

$$
\begin{aligned}
&(\tilde{\nabla}_{\xi_\gamma}\tilde{\mathcal{Q}})(\xi_\beta,\xi_\lambda) - (\nabla_{\xi_\gamma}\mathcal{Q})(\xi_\beta,\xi_\lambda) = e^{3f}(\tilde{\nabla}_{\tilde{\xi}_\gamma}\tilde{\mathcal{Q}})(\tilde{\xi}_\beta,\tilde{\xi}_\lambda) - (\nabla_{\xi_\gamma}\mathcal{Q})(\xi_\beta,\xi_\lambda) \\
&\quad = \Big\{ - 4\xi_\gamma(f)\mathcal{Q}^{\bar{\nu}}_{\beta\lambda} - 2\xi_\nu(f)\mathcal{Q}^{\bar{\gamma}}_{\beta\lambda} - 2\xi_\beta(f)\mathcal{Q}^{\bar{\nu}}_{\gamma\lambda} - 2\xi_\lambda(f)\mathcal{Q}^{\bar{\nu}}_{\beta\gamma}\Big\}\xi_{\bar{\nu}}, \\
&(\tilde{\nabla}_{\xi_{\bar{\gamma}}}\tilde{\mathcal{Q}})(\xi_\beta,\xi_\lambda) - (\nabla_{\xi_{\bar{\gamma}}}\mathcal{Q})(\xi_\beta,\xi_\lambda) = e^{3f}(\tilde{\nabla}_{\tilde{\xi}_{\bar{\gamma}}}\tilde{\mathcal{Q}})(\tilde{\xi}_\beta,\tilde{\xi}_\lambda) - (\nabla_{\xi_{\bar{\gamma}}}\mathcal{Q})(\xi_\beta,\xi_\lambda) \\
&\quad = \Big\{2\xi_{\bar{\gamma}}(f)\mathcal{Q}^{\bar{\nu}}_{\beta\lambda} + \delta_{\nu\gamma}\,2\xi_{\bar{\alpha}}(f)\mathcal{Q}^{\bar{\alpha}}_{\beta\lambda} + \delta_{\beta\gamma}\,2\xi_{\bar{\alpha}}(f)\mathcal{Q}^{\bar{\nu}}_{\alpha\lambda} + \delta_{\lambda\gamma}\,2\xi_{\bar{\alpha}}(f)\mathcal{Q}^{\bar{\nu}}_{\beta\alpha}\Big\}\xi_{\bar{\nu}}, \\
&(\tilde{\nabla}_{\xi_\gamma}\tilde{\mathcal{Q}})(\xi_\beta,\xi_{\bar{\lambda}}) - (\nabla_{\xi_\gamma}\mathcal{Q})(\xi_\beta,\xi_{\bar{\lambda}}) = (\tilde{\nabla}_{\xi_{\bar{\gamma}}}\tilde{\mathcal{Q}})(\xi_\beta,\xi_{\bar{\lambda}}) - (\nabla_{\xi_{\bar{\gamma}}}\mathcal{Q})(\xi_\beta,\xi_{\bar{\lambda}}) = 0. \ \square
\end{aligned}
$$

THEOREM 4.2. *The tensor* $F({}^*\nabla) + b(\nabla)^{\mathbb{R}} \in \Gamma(H \otimes H^* \otimes H^* \otimes H^*)$ *is pseudo-conformally invariant if and only if* J *is integrable. The tensor* $F({}^*\nabla) + b(\nabla)^{\mathbb{R}} - \frac{1}{2(n+1)}U^{\mathbb{R}}(\Xi^\omega :)$ *is pseudo-conformally invariant.*

PROOF. The second half is obvious. Indeed,

$$
\begin{aligned}
(4.2)\quad \{F({}^*\tilde{\nabla}) + b(\tilde{\nabla})^{\mathbb{R}}\} - \{F({}^*\nabla) + b(\nabla)^{\mathbb{R}}\} &= U^{\mathbb{R}}(\mathrm{grad}\, f :) \\
&= \frac{\tilde{U}^{\mathbb{R}}(\tilde{\Xi}^\omega :)}{2(n+1)} - \frac{U^{\mathbb{R}}(\Xi^\omega :)}{2(n+1)}.
\end{aligned}
$$

As for the first half: If $\mathcal{Q} = 0$, then obviously we have $U^{\mathbb{R}}(\mathrm{grad}\, f :) = 0$. We want to prove the converse. Let us complexify the domain of $U^{\mathbb{R}}(\mathrm{grad}\, f :)$

naturally. Then we have

$$\begin{aligned}
U^{\mathbb{R}}(\operatorname{grad} f : X_+, Y_+, Z_+) &= U^+(\operatorname{grad} f : X_+, Y_+, Z_+) \\
&\quad - g(X_+, J\operatorname{grad}_- f)\mathcal{Q}(Z_+, Y_+) + g(Y_+, J\operatorname{grad}_- f)\mathcal{Q}(Z_+, X_+) \\
&\quad - g(Z_+, J\operatorname{grad}_- f)\Big(\mathcal{Q}(X_+, Y_+) - \mathcal{Q}(Y_+, X_+)\Big) \\
&\quad + \Big(g(\mathcal{Q}(Z_+, X_+), Y_+) - g(\mathcal{Q}(Z_+, Y_+), X_+)\Big) J\operatorname{grad}_- f.
\end{aligned}$$

We define the tensors $\mathcal{U}^{\mathbb{R}}_{\mp} \in \Gamma(H_\mp \otimes H_+^* \otimes H_+^* \otimes H_+^* \otimes H_\mp^*)$ by $g(U^{\mathbb{R}}(\operatorname{grad} f : X_+, Y_+, Z_+), W_\mp) = g(\mathcal{U}^{\mathbb{R}}_{\mp}(X_+, Y_+, Z_+, W_\mp), J\operatorname{grad}_\pm f)$, which are expressed as

$$\begin{aligned}
\mathcal{U}^{\mathbb{R}}_{-}(X_+, Y_+, Z_+, W_-) &= \mathcal{U}_-(X_+, Y_+, Z_+, W_-), \\
\mathcal{U}^{\mathbb{R}}_{+}(X_+, Y_+, Z_+, W_+) &= g(\mathcal{Q}(W_+, Y_+), Z_+)X_+ + g(\mathcal{Q}(Z_+, X_+), W_+)Y_+ \\
&\quad + g(\mathcal{Q}(Y_+, W_+), X_+)Z_+ + g(\mathcal{Q}(X_+, Z_+), Y_+)W_+.
\end{aligned}$$

Now, in the case $n \geq 3$, if $U^{\mathbb{R}}(\operatorname{grad} f :) = 0$ for any f, then we have $\mathcal{U}_- = \mathcal{U}^{\mathbb{R}}_- = 0$, which yields $\mathcal{Q} = 0$ (cf. Proposition 3.1). In the case $n = 2$, $U^{\mathbb{R}}(\operatorname{grad} f :) = 0$ for any f implies that $\mathcal{U}^{\mathbb{R}}_+(\xi_1, \xi_2, \xi_1, \xi_2) = 2\mathcal{Q}^{\bar{1}}_{22}\xi_1 + 2\mathcal{Q}^{\bar{2}}_{11}\xi_2$ vanishes. Namely we have $\mathcal{Q} = 0$. □

Last, let us introduce another pair of such tensors, which are expressed only in components explicitly related to ${}^*\nabla$.

PROPOSITION 4.3. *We have*

$$\begin{aligned}
&\operatorname{Ric}^{\tilde{\nabla}}(\xi_\alpha, \xi_\beta) - \operatorname{Ric}^{\nabla}(\xi_\alpha, \xi_\beta) = \operatorname{Ric}^{{}^*\tilde{\nabla}}(\xi_\alpha, \xi_\beta) - \operatorname{Ric}^{{}^*\nabla}(\xi_\alpha, \xi_\beta), \\
&\operatorname{Ric}^{\tilde{\nabla}}(\xi_\alpha, \xi_{\bar\beta}) - \operatorname{Ric}^{\nabla}(\xi_\alpha, \xi_{\bar\beta}) = \operatorname{Ric}^{{}^*\tilde{\nabla}}(\xi_\alpha, \xi_{\bar\beta}) - \operatorname{Ric}^{{}^*\nabla}(\xi_\alpha, \xi_{\bar\beta}), \\
&\operatorname{Ric}(\tilde{\nabla})(\xi_\alpha, \xi_\beta) - \operatorname{Ric}(\nabla)(\xi_\alpha, \xi_\beta) = \frac{1}{2}\{\operatorname{Ric}({}^*\tilde{\nabla})(\xi_\alpha, \xi_\beta) + i(n-1)\,{}^*\tilde{\tau}_\alpha^{\bar\beta}\} \\
&\qquad - \frac{1}{2}\{\operatorname{Ric}({}^*\nabla)(\xi_\alpha, \xi_\beta) + i(n-1)\,{}^*\tau_\alpha^{\bar\beta}\} \\
&\qquad + \frac{n+2}{2}\, i(\mathcal{Q}_{\alpha\beta} - \mathcal{Q}_{\beta\alpha})(f), \\
&\operatorname{Ric}(\tilde{\nabla})(\xi_\alpha, \xi_{\bar\beta}) - \operatorname{Ric}(\nabla)(\xi_\alpha, \xi_{\bar\beta}) = \operatorname{Ric}({}^*\tilde{\nabla})(\xi_\alpha, \xi_{\bar\beta}) - \operatorname{Ric}({}^*\nabla)(\xi_\alpha, \xi_{\bar\beta}).
\end{aligned} \tag{4.3}$$

PROOF. By Propositions 1.1(3) and 1.2(3), etc.,

$$\mathrm{Ric}^{*\nabla}(\xi_\alpha,\xi_\beta)=\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_\beta),$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_\beta)+i(n-1)\,{}^*\tau_\alpha^{\bar\beta}=\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_\beta)+\mathrm{Ric}(\nabla)(\xi_\beta,\xi_\alpha),$$
$$\mathrm{Ric}^{*\nabla}(\xi_\alpha,\xi_{\bar\beta})=\mathrm{Ric}^{\nabla}(\xi_\alpha,\xi_{\bar\beta})-\frac{1}{4}\mathcal{Q}^{\bar\lambda}_{\mu\alpha}\mathcal{Q}^{\mu}_{\bar\lambda\bar\beta},$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_{\bar\beta})=\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_{\bar\beta})+\frac{1}{4}\mathcal{Q}^{\bar\lambda}_{\alpha\mu}\mathcal{Q}^{\beta}_{\bar\mu\bar\lambda},$$

which imply the proposition except (4.3). As for (4.3): The second formula says

$$\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_\beta)=\frac{1}{2}\{\mathrm{Ric}({}^*\nabla)(\xi_\alpha,\xi_\beta)+i(n-1)\,{}^*\tau_\alpha^{\bar\beta}\}+\frac{1}{2}\{\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_\beta)-\mathrm{Ric}(\nabla)(\xi_\beta,\xi_\alpha)\}$$

and, referring to (2.5), we have

$$\begin{aligned}&\{\mathrm{Ric}(\tilde\nabla)(\xi_\alpha,\xi_\beta)-\mathrm{Ric}(\tilde\nabla)(\xi_\beta,\xi_\alpha)\}-\{\mathrm{Ric}(\nabla)(\xi_\alpha,\xi_\beta)-\mathrm{Ric}(\nabla)(\xi_\beta,\xi_\alpha)\}\\&=3i(\mathcal{Q}_{\alpha\beta}-\mathcal{Q}_{\beta\alpha})(f)-2(n-1)T(\nabla)(\xi_\alpha,\xi_\beta)(f)\\&=(n+2)i(\mathcal{Q}_{\alpha\beta}-\mathcal{Q}_{\beta\alpha})(f).\end{aligned}$$

Hence we obtain (4.3). □

Accordingly, let us define $b({}^*\nabla)^{\mathbb{C}}(X,Y)Z$ to be $b(\nabla)^{\mathbb{C}}(X,Y)Z$ with $(\mathrm{Ric}^{\nabla},\mathrm{ric}^{\nabla},s^{\nabla})$ replaced by $(\mathrm{Ric}^{*\nabla},\mathrm{ric}^{*\nabla},s^{*\nabla})$ and with $\mathrm{Ric}(\nabla)(W_1,W_2)$, $\mathrm{ric}(\nabla)(W)$ replaced by

$$\frac{1}{2}\{\mathrm{Ric}({}^*\nabla)(W_1,W_2)+i(n-1)\,g({}^*\tau W_1,W_2)\},$$
$$\frac{1}{2}\{\mathrm{ric}({}^*\nabla)(W)-i(n-1)\,{}^*\tau W\},$$

and set $b({}^*\nabla)^{\mathbb{R}}(X,Y)Z=2\,\mathrm{Re}\Big(b({}^*\nabla)^{\mathbb{C}}(X,Y)Z\Big)$. Further, recalling $n\geq 2$, let us set

$$\begin{aligned}{}^*U^{\mathbb{C}}(\mathrm{grad}\,f:X,Y,Z)&=U^{\mathbb{C}}(\mathrm{grad}\,f:X,Y,Z)\\&\quad+\frac{n+2}{2(n-1)}\Big\{g(\mathcal{Q}(Z,X)-\mathcal{Q}(X,Z),\mathrm{grad}_+f)Y_+\\&\qquad-g(\mathcal{Q}(Z,Y)-\mathcal{Q}(Y,Z),\mathrm{grad}_+f)X_+\\&\qquad+g(Z_-,Y)\mathcal{Q}(X,\mathrm{grad}_+f)-g(Z_-,X)\mathcal{Q}(Y,\mathrm{grad}_+f)\Big\}\end{aligned}$$

and ${}^*U^{\mathbb{R}}(\operatorname{grad} f : X, Y, Z) = 2\operatorname{Im}\Big({}^*U^{\mathbb{C}}(\operatorname{grad} f : X, Y, Z)\Big)$. Then we have:

THEOREM 4.4. *The tensor* $F({}^*\nabla) + b({}^*\nabla)^{\mathbb{R}} \in \Gamma(H \otimes H^* \otimes H^* \otimes H^*)$ *is pseudo-conformally invariant. if and only if* J *is integrable. The tensor* $F({}^*\nabla) + b({}^*\nabla)^{\mathbb{R}} - \frac{1}{2(n+1)}{}^*U^{\mathbb{R}}(\Xi^\omega :)$ *is pseudo-conformally invariant.*

PROOF. (4.2) and Proposition 4.3 imply

$$\begin{aligned}\{F({}^*\tilde\nabla) + b({}^*\tilde\nabla)^{\mathbb{R}}\} - \{F({}^*\nabla) + b({}^*\nabla)^{\mathbb{R}}\} &= {}^*U^{\mathbb{R}}(\operatorname{grad} f :) \\ &= \frac{{}^*\tilde U^{\mathbb{R}}(\tilde\Xi^\omega :)}{2(n+1)} - \frac{{}^*U^{\mathbb{R}}(\Xi^\omega :)}{2(n+1)}.\end{aligned}$$

Hence the second half holds. As for the first half: The H_+-component of ${}^*U^{\mathbb{R}}(\operatorname{grad} f : X_+, Y_+, Z_+)$ is equal to

$$\begin{aligned}{}^*U^{\mathbb{R}}_+(\operatorname{grad} f : X_+, Y_+, Z_+) &= U^+(\operatorname{grad} f : X_+, Y_+, Z_+) \\ &\quad + \frac{n+2}{2(n-1)}\Big\{g(\mathcal{Q}(Z_+, Y_+) - \mathcal{Q}(Y_+, Z_+), J\operatorname{grad}_+ f)X_+ \\ &\quad - g(\mathcal{Q}(Z_+, X_+) - \mathcal{Q}(X_+, Z_+), J\operatorname{grad}_+ f)Y_+\Big\}.\end{aligned}$$

Let us define the tensor ${}^*\mathcal{U}^{\mathbb{R}}_- \in \Gamma(H_- \otimes H^*_+ \otimes H^*_+ \otimes H^*_+ \otimes H^*_-)$ by $g({}^*U^{\mathbb{R}}_+(\operatorname{grad} f : X_+, Y_+, Z_+), W_-) = g({}^*\mathcal{U}^{\mathbb{R}}_-(X_+, Y_+, Z_+, W_-), J\operatorname{grad}_+ f)$. Then

$$\begin{aligned}{}^*\mathcal{U}^{\mathbb{R}}_-(X_+, Y_+, Z_+, W_-) &= \mathcal{U}_-(X_+, Y_+, Z_+, W_-) \\ &\quad + \frac{n+2}{2(n-1)}\Big\{g(X_+, W_-)\Big(\mathcal{Q}(Z_+, Y_+) - \mathcal{Q}(Y_+, Z_+)\Big) \\ &\quad - g(Y_+, W_-)\Big(\mathcal{Q}(Z_+, X_+) - \mathcal{Q}(X_+, Z_+)\Big)\Big\}.\end{aligned}$$

We assume ${}^*U^{\mathbb{R}}(\operatorname{grad} f :) = 0$ for any f, so that ${}^*\mathcal{U}^{\mathbb{R}}_- = 0$. In the case $n \geq 3$, if $\alpha \notin \{\gamma, \lambda\}$, then $\mathcal{U}_-(\xi_\gamma, \xi_\lambda, \xi_\alpha, \xi_{\bar\alpha}) = {}^*\mathcal{U}^{\mathbb{R}}_-(\xi_\gamma, \xi_\lambda, \xi_\alpha, \xi_{\bar\alpha}) = 0$. Consequently we know $\mathcal{Q} = 0$ (cf. the proof of Proposition 3.1(2)). In the case $n = 2$, since ${}^*\mathcal{U}^{\mathbb{R}}_-(\xi_1, \xi_2, \xi_\beta, \xi_{\bar 1}) = 2\mathcal{Q}^{\bar 2}_{\beta\mu}\xi_{\bar\mu}$, ${}^*\mathcal{U}^{\mathbb{R}}_- = 0$ certainly implies $\mathcal{Q} = 0$. □

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