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Duality for Dormant Opers

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Abstract. In the present paper, we prove that on a fixed, pointed stable curve over a field of characteristic $p > 0$, there exists a *canonical* duality between dormant $\mathfrak{sl}_n$-opers ($1 < n < p-1$) and dormant $\mathfrak{sl}_{(p-n)}$-opers, and that there exists a unique (up to isomorphism) dormant $\mathfrak{sl}_{(p-1)}$-oper.

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Introduction

The purpose of the present paper is to establish a *canonical* duality for dormant opers on a fixed algebraic curve of characteristic $p > 0$:

$$\boxed{\textit{dormant } \mathfrak{sl}_n\textit{-opers}} \quad \rightleftharpoons \quad \boxed{\textit{dormant } \mathfrak{sl}_{(p-n)}\textit{-opers}}$$

where n is an integer with $1 < n < p-1$ and $\mathfrak{sl}_n$ (resp., $\mathfrak{sl}_{(p-n)}$) denotes the special linear Lie algebra of rank n (resp., $p-n$).

0.1. Recall that a *dormant* $\mathfrak{sl}_n$*-oper* is, roughly speaking, a principal homogenous space over an algebraic curve equipped with a connection satisfying certain conditions, including the condition that its p-curvature vanishes

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identically. Various properties of (dormant) $\mathfrak{sl}_n$-opers in characteristic $p > 0$ and $n = 2$ were first discussed by S. Mochizuki in [14]. If n is general (but the underlying curve is assumed to be unpointed and smooth over an algebraically closed field), then the study of these objects has been carried out by K. Joshi, S. Ramanan, E. Z. Xia, J. K. Yu, C. Pauly, T. H. Chen, X. Zhu et al. (cf. [7], [8], [9], [3]). Also, formulations and background knowledge of dormant $\mathfrak{sl}_n$-opers (or, more generally, a dormant $(\mathfrak{g}, \hbar)$-opers for a semisimple Lie algebra $\mathfrak{g}$ and $\hbar \in k$) in the present paper were discussed in the author's papers (cf. [20], [21]). As we explained briefly in [21], § 0.2, dormant $\mathfrak{sl}_n$-opers and their moduli, which are our principal objects, contain diverse aspects and occur naturally in mathematics. At any rate, a detailed understanding of them in generalized setting will be of use in various areas relevant to the theory of opers in positive characteristic.

0.2. We shall describe the main theorem of the present paper. Let p a prime number, n a positive integer with $n < p$, (g, r) a pair of nonnegative integers satisfying the inequality $2g - 2 + r > 0$, k a perfect field of characteristic p, S a k-scheme, and $\mathfrak{X}_{/S} := (f : X \to S, \{\sigma_i : S \to X\}_{i=1}^r)$ a pointed stable curve over S of type (g, r) (cf. § 1.3). Denote by $\mathfrak{c}_n$ the GIT quotient of $\mathfrak{sl}_n$ by the adjoint action of PGL_n (= the projective linear group over k of rank n). In § 6.1, we define a certain subset $\mathfrak{c}_n(\mathbb{F}_p)^{\circledast}$ (cf. (158) for the precise definition of $\mathfrak{c}_n(\mathbb{F}_p)^{\circledast}$) of the set of $\mathbb{F}_p$-rational points $\mathfrak{c}_n(\mathbb{F}_p)$ of $\mathbb{F}_p$. Let $\vec{\rho}$ be an element of $(\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}$ (i.e., the product of r copies of $\mathfrak{c}_n(\mathbb{F}_p)^{\circledast}$). According to the discussion in § 6.1, to each such $\vec{\rho}$, one may associate another element $\vec{\rho}^{\bigstar}$ of $(\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}$. (Here, if $r = 0$, then we take $\vec{\rho} = \vec{\rho}^{\bigstar} = \emptyset$.) We shall write

$$(1) \qquad \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}} \quad \left(\text{resp., } \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n, \vec{\rho}, \mathfrak{X}_{/S}}\right)$$

for the moduli stack classifying dormant $\mathfrak{sl}_n$-opers (resp., dormant $\mathfrak{sl}_n$-opers of radii $\vec{\rho}$) on $\mathfrak{X}_{/S}$. Then, the main theorem of the present paper is the following assertion (cf. Theorem 6.2.1).

THEOREM A. *Suppose that $n < p - 1$.*

(i) *There exists a canonical isomorphism*

$$(2) \qquad \Theta^{\bigstar}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}} : \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-n)}, \mathfrak{X}_{/S}}$$

over S *satisfying that* $\Theta^{\bigstar}_{\mathfrak{sl}_{(p-n)},\mathfrak{X}_{/S}} \circ \Theta^{\bigstar}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}} = \mathrm{id}$.

(ii) *By restricting* $\Theta^{\bigstar}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}}$, *we obtain a canonical isomorphism*

$$\Theta^{\bigstar}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}} : \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-n)},\vec{\rho}^{\bigstar},\mathfrak{X}_{/S}} \tag{3}$$

over S *satisfying that* $\Theta^{\bigstar}_{\mathfrak{sl}_{(p-n)},\vec{\rho},\mathfrak{X}_{/S}} \circ \Theta^{\bigstar}_{\mathfrak{sl}_n,\vec{\rho}^{\bigstar},\mathfrak{X}_{/S}} = \mathrm{id}$.

Also, we obtain the following assertion (cf. Theorem 6.2.2), which is a generalization (to the case of pointed stable curves) of [6], Theorem A (ii). The unique dormant $\mathfrak{sl}_{(p-1)}$-oper asserted in *loc. cit.* was studied explicitly by means of (the Frobenius pull-back of) the sheaf of locally exact 1-forms.

Theorem B. *The structure morphism* $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-1)},\mathfrak{X}_{/S}} \rightarrow S$ *of* $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-1)},\mathfrak{X}_{/S}}$ *is an isomorphism. That is to say, there exists a unique (up to isomorphism) dormant* $\mathfrak{sl}_{(p-1)}$*-oper on* $\mathfrak{X}_{/S}$.

One may apply Theorems A and B to the study toward explicit computations of the number of dormant $\mathfrak{sl}_n$-opers. Indeed, let $\overline{\mathfrak{M}}_{g,r}$ denote the moduli stack classifying pointed stable curves over k of type (g,r) and $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},g,r}$ denote the moduli stack classifying pointed stable curves over k of type (g,r) equipped with a dormant $\mathfrak{sl}_n$-oper of radii $\vec{\rho}$ on it. Then, Theorem A allows us to generalize the result of [20], Theorem H, which give an explicit computation of the generic degree of $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},g,0}$ over $\overline{\mathfrak{M}}_{g,0}$ (cf. Corollary 6.3.1). Here, recall that since $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},g,0}$ is finite and generically étale over $\overline{\mathfrak{M}}_{g,r}$ (cf. [20], Theorem G), this generic degree coincides with the number of dormant $\mathfrak{sl}_n$-opers on a sufficiently general curve.

Moreover, by combining results in the present paper with results in p-adic Teichmüller theory, we will have (in § 6.4) a rather explicit understanding of the case where $n = p-2$. In particular, we will discuss (cf. Corollary 6.4.2) the structure of the fusion ring $\mathfrak{F}^{\mathrm{Zzz...}}_{p,\mathfrak{sl}_{(p-2)}}$ (cf. (178)) associated with the function $N^{\mathrm{Zzz...}}_{p,\mathfrak{sl}_{(p-2)},0}$ (cf. (175)) assigning, to each data of radii $\vec{\rho} \in (\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}$, the generic degree of $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-2)},\vec{\rho},0,r}/\overline{\mathfrak{M}}_{0,r}$.

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1. Preliminaries

Throughout the present paper, let us fix a prime p, a perfect field k of characteristic p (hence $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z} \subseteq k$), and a pair of nonnegative integers (g, r) satisfying that $2g - 2 + r > 0$.

1.1. For a field k' over $\mathbb{F}_p$, we shall denote by

$$2^{k'} \ \left(\text{resp., } 2^{k'}_{\sharp(-)=n}\right) \tag{4}$$

the set of subsets of k' (resp., the set of subsets of k' with cardinality n). Also, denote by

$$\mathbb{N}^{k'} \ \left(\text{resp., } \mathbb{N}^{k'}_{\sharp(-)=n}\right) \tag{5}$$

the set of multisets over k' (resp., the set of multisets over k' with cardinality n). (For the definition and various notations concerning a *multiset*, we refer to [18].) In particuler, $2^{k'} \subseteq \mathbb{N}^{k'}$ and $2^{k'}_{\sharp(-)=n} \subseteq \mathbb{N}^{k'}_{\sharp(-)=n}$.

The symmetric group $\mathfrak{S}_n$ of n letters acts, by permutation, on the product $k'^{\times n}$ of n copies of k'. The quotient set $\mathfrak{S}_n \backslash k'^{\times n}$ may be identified with $\mathbb{N}^{k'}_{\sharp(-)=n}$ and, in particular, we have a natural surjection

$$k'^{\times n} \twoheadrightarrow \left(\mathfrak{S}_n \backslash k'^{\times n} \cong\right) \ \mathbb{N}^{k'}_{\sharp(-)=n}. \tag{6}$$

Let $\tau_0 := [\tau_{0,1}, \cdots, \tau_{0,n}]$ be an element of $\mathbb{N}^{k'}_{\sharp(-)=n}$ and $a \in k'$. Then, we shall write

$$\begin{aligned} \tau_0^{+a} &:= [\tau_{0,1} + a, \cdots, \tau_{0,n} + a] \\ \big(\text{resp., } \tau_0^{-a} &:= [\tau_{0,1} - a, \cdots, \tau_{0,n} - a] \ \big) \in \mathbb{N}^{k'}_{\sharp(-)=n}. \end{aligned} \tag{7}$$

If, moreover, τ_0 is a subset of $\mathbb{F}_p$ (i.e., an element of $2^{\mathbb{F}_p}_{\sharp(-)=n} \subseteq \mathbb{N}^{k'}_{\sharp(-)=n}$), then we shall write

$$\tau_0^{\triangleright} := \mathbb{F}_p \setminus \tau_0 \ \ (\subseteq \mathbb{F}_p)\,, \tag{8}$$

and

$$\tau_0^{\triangledown} := [-\tau_{0,1}, -\tau_{0,2}, \cdots, -\tau_{0,n}] \ (\subseteq \mathbb{F}_p). \tag{9}$$

Next, let r be a positive integer, and $\vec{\tau} := (\tau_i)_{i=1}^r$ an r-tuple of multisets over k', i.e., an element of the product $(\mathbb{N}^{k'})^{\times r}$ of r-copies of $\mathbb{N}^{k'}$. Then, for each $\vec{a} := (a_i)_{i=1}^r \in k'^{\times r}$, we shall write

$$\vec{\tau}^{+\vec{a}} := (\tau_i^{+a_i})_{i=1}^r \quad \left(\text{resp., } \vec{\tau}^{-\vec{a}} := (\tau_i^{-a_i})_{i=1}^r\right). \tag{10}$$

If, moreover, $\vec{\tau}$ lies in $(2^{\mathbb{F}_p}_{\sharp(-)=n})^{\times r}$, then we shall write

$$\vec{\tau}^{\triangleright} := (\tau_i^{\triangleright})_{i=1}^r, \quad \vec{\tau}^{\triangledown} := (\tau_i^{\triangledown})_{i=1}^r, \quad \text{and} \quad \vec{\tau}^{\bigstar} := ((\tau_i^{\triangleright})^{\triangledown})_{i=1}^r, \tag{11}$$

all of which are elements of $(2^{\mathbb{F}_p}_{\sharp(-)=n})^{\times r}$. One verifies immediately the equalities

$$(\vec{\tau}^{\bigstar})^{\bigstar} = \vec{\tau} \tag{12}$$

and

$$(\vec{\tau}^{+\vec{a}})^{\bigstar} = (\vec{\tau}^{\bigstar})^{-\vec{a}}. \tag{13}$$

1.2. Let T be a scheme over k ($\supseteq \mathbb{F}_p$) and $f : Y \to T$ a scheme over T. Denote by $F_T : T \to T$ (resp., $F_Y : Y \to Y$) the absolute Frobenius morphism of T (resp., Y). The *Frobenius twist of* Y *over* T is, by definition, the base-change $Y_T^{(1)}$ ($:= Y \times_{T,F_T} T$) of $f : Y \to T$ via $F_T : T \to T$. Denote by $f^{(1)} : Y_T^{(1)} \to T$ the structure morphism of the Frobenius twist of Y over T. The *relative Frobenius morphism of* Y *over* T is the unique morphism $F_{Y/T} : Y \to Y_T^{(1)}$ over T that fits into a commutative diagram of the form

$$\begin{array}{ccccc} Y & \xrightarrow{F_{Y/T}} & Y_T^{(1)} & \xrightarrow{\mathrm{id}_Y \times F_T} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f^{(1)}}\downarrow & & {\scriptstyle f}\downarrow \\ T & \xrightarrow{\mathrm{id}_T} & T & \xrightarrow{F_T} & T. \end{array} \tag{14}$$

Here, the upper composite in this diagram coincides with F_Y and the right-hand square is, by the definition of $Y_T^{(1)}$, cartesian.

1.3. Denote by $\overline{\mathfrak{M}}_{g,r}$ the moduli stack of r-pointed stable curves (cf. [13], Definition 1.1) over k of genus g (i.e., of type (g,r)), and by $f_{\mathfrak{tau}} : \mathfrak{C}_{g,r} \to \overline{\mathfrak{M}}_{g,r}$ the tautological curve, with its r marked points $\mathfrak{s}_1, \cdots, \mathfrak{s}_r : \overline{\mathfrak{M}}_{g,r} \to \mathfrak{C}_{g,r}$. Recall (cf. [13], Corollary 2.6 and Theorem 2.7; [5], § 5) that $\overline{\mathfrak{M}}_{g,r}$ may be represented by a geometrically connected, proper, and smooth Deligne-Mumford stack over k of dimension $3g-3+r$. Also, recall (cf. [10], Theorem 4.5) that $\overline{\mathfrak{M}}_{g,r}$ has a natural log structure given by the divisor at infinity, where we shall denote the resulting log stack by $\overline{\mathfrak{M}}^{\mathrm{log}}_{g,r}$. Also, by taking the divisor which is the union of the $\mathfrak{s}_i$'s and the pull-back of the divisor at infinity of $\overline{\mathfrak{M}}_{g,r}$, we obtain a log structure on $\mathfrak{C}_{g,r}$; we denote the resulting log stack by $\mathfrak{C}^{\mathrm{log}}_{g,r}$. $f_{\mathfrak{tau}} : \mathfrak{C}_{g,r} \to \overline{\mathfrak{M}}_{g,r}$ extends naturally to a morphism $f^{\mathrm{log}}_{\mathfrak{tau}} : \mathfrak{C}^{\mathrm{log}}_{g,r} \to \overline{\mathfrak{M}}^{\mathrm{log}}_{g,r}$ of log stacks.

Next, let S be a scheme, or more generally, a stack over k and

$$\mathfrak{X}_{/S} := (f : X \to S, \{\sigma_i : S \to X\}_{i=1}^r) \tag{15}$$

a pointed stable curve over S of type (g,r), consisting of a (proper) semi-stable curve $f : X \to S$ over S of genus g and r marked points $\sigma_i : S \to X$ $(i = 1, \cdots, r)$. $\mathfrak{X}_{/S}$ determines its classifying morphism $s : S \to \overline{\mathfrak{M}}_{g,r}$ and an isomorphism $X \xrightarrow{\sim} S \times_{s,\overline{\mathfrak{M}}_{g,r}} \mathfrak{C}_{g,r}$ over S. By pulling-back the log structures of $\overline{\mathfrak{M}}^{\mathrm{log}}_{g,r}$ and $\mathfrak{C}^{\mathrm{log}}_{g,r}$, we obtain log structures on S and X respectively; we denote the resulting log stacks by S^{log} and X^{log}. The structure morphism $f : X \to S$ extends to a morphism $f^{\mathrm{log}} : X^{\mathrm{log}} \to S^{\mathrm{log}}$ of log stacks, which is log smooth (cf. [11], § 3; [10], Theorem 2.6). Write $\mathcal{T}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ for the sheaf of logarithmic derivations of X^{log} over S^{log} and $\Omega_{X^{\mathrm{log}}/S^{\mathrm{log}}} := \mathcal{T}^{\vee}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ for its dual, i.e., the sheaf of logarithmic differentials of X^{log} over S^{log}. Both $\mathcal{T}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ and $\Omega_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ are line bundles on X. For each $i = 1, \cdots, r$, one may obtain the *residue isomorphism*, which is, by definition, an isomorphism

$$\sigma_i^*(\Omega_{X^{\mathrm{log}}/S^{\mathrm{log}}}) \xrightarrow{\sim} \mathcal{O}_S \tag{16}$$

given by assigning $1 \in \mathcal{O}_S$ to any local section of the form $\sigma_i^*(d\mathrm{log}(x)) \in \sigma_i^*(\Omega_{X^{\mathrm{log}}/S^{\mathrm{log}}})$ (for a local function x defining the closed subscheme $\sigma_i : S \to X$ of X).

If $t : T \to S$ is an S-scheme, then we shall use the notation "$\mathfrak{X}_{/T}$" for indicating the base-change of $\mathfrak{X}_{/S}$ via t, i.e., the pointed stable curve

$$\mathfrak{X}_{/T} := (X \times_{S,t} T/T, \{\sigma_i \times_S \mathrm{id}_T : (T =)\ S \times_{S,t} T \to X \times_{S,t} T\}_{i=1}^r) \tag{17}$$

over T.

1.4. Let S, $\mathfrak{X}_{/S}$ be as above and $\mathcal{V}$ a vector bundle (i.e., a locally free coherent $\mathcal{O}_X$-module) on X. By an *S-connection* (resp., *S-log connection*) on $\mathcal{V}$, we mean (cf. [20], § 4.1) an $f^{-1}(\mathcal{O}_S)$-linear morphism

$$\nabla : \mathcal{V} \to \Omega_{X/S} \otimes \mathcal{V} \ \left(\text{resp., } \nabla : \mathcal{V} \to \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{V}\right) \tag{18}$$

satisfying the condition that

$$\nabla(a \cdot m) = d(a) \otimes m + a \cdot \nabla(m) \tag{19}$$

for local sections $a \in \mathcal{O}_X$ and $m \in \mathcal{V}$, where d denotes the universal derivation $\mathcal{O}_X \to \Omega_{X/S}$ ($\subseteq \Omega_{X^{\log}/S^{\log}}$).

If ∇ is an S-log connection on $\mathcal{V}$, then we shall write $\det(\nabla)$ (resp., $\nabla^\vee$) for the S-log connection on the determinant $\det(\mathcal{V})$ (resp., the dual $\mathcal{V}^\vee$) of $\mathcal{V}$ induced naturally by ∇. Also, for $m \geq 1$, we shall write $\nabla^{\otimes m}$ for the S-log connection on the m-fold tensor product $\mathcal{V}^{\otimes m}$ of $\mathcal{V}$ induced by ∇. If, moreover, we are given a vector bundle $\mathcal{V}'$ on X and an S-log connection ∇' on $\mathcal{V}'$, then we shall write $\nabla \otimes \nabla'$ for the S-log connection on the tensor product $\mathcal{V} \otimes \mathcal{V}'$ induced by ∇ and ∇'.

A *log integrable vector bundle* on $\mathfrak{X}_{/S}$ (of rank $m \geq 1$) is a pair $\mathfrak{F} := (\mathcal{F}, \nabla_\mathcal{F})$ consisting of a vector bundle $\mathcal{F}$ on X (of rank m) and an S-log connection $\nabla_\mathcal{F}$ on $\mathcal{F}$. If, moreover, $\mathcal{F}$ is of rank 1, then we shall refer to such an $\mathfrak{F}$ as a *log integrable line bundle* on $\mathfrak{X}_{/S}$.

Let $\mathfrak{F} := (\mathcal{F}, \nabla_\mathcal{F})$ and $\mathfrak{G} := (\mathcal{G}, \nabla_\mathcal{G})$ be log integrable vector bundles on $\mathfrak{X}_{/S}$. An *isomorphism of log integrable vector bundles* from $\mathfrak{F}$ to $\mathfrak{G}$ is an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{G}$ of $\mathcal{O}_X$-modules that is compatible with the respective S-log connections $\nabla_\mathcal{F}$ and $\nabla_\mathcal{G}$.

1.5. We recall the definition of the p-curvature of a logarithmic connection (cf., e.g., [19], § 3). Let $\mathfrak{F} := (\mathcal{F}, \nabla_\mathcal{F})$ be a log integrable vector bundle on $\mathfrak{X}_{/S}$. If ∂ is a logarithmic derivation corresponding to a local section of $\mathcal{T}_{X^{\log}/S^{\log}}$, then we shall denote by $\partial^{(p)}$ the p-th symbolic power of ∂ (i.e., "$\partial \mapsto \partial^{(p)}$" asserted in [16], Proposition 1.2.1), which is also a logarithmic derivation corresponding to a local section of $\mathcal{T}_{X^{\log}/S^{\log}}$. Then there exists uniquely an *$\mathcal{O}_X$-linear* morphism

$$\psi^{\nabla_\mathcal{F}} : \mathcal{T}^{\otimes p}_{X^{\log}/S^{\log}} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \tag{20}$$

determined by assigning

$$\partial^{\otimes p} \mapsto \nabla_{\mathcal{F}}(\partial)^{\circ p} - \nabla_{\mathcal{F}}(\partial^{(p)}) \tag{21}$$

for any local section $\partial \in \mathcal{T}_{X^{\log}/S^{\log}}$, where $\nabla_{\mathcal{F}}(\partial)^{\circ p}$ denotes the p-th iterate of the $f^{-1}(\mathcal{O}_S)$-linear endomorphism $\nabla_{\mathcal{F}}(\partial)$ of $\mathcal{F}$. We shall refer to $\psi^{\nabla_{\mathcal{F}}}$ as the *p-curvature* of $\nabla_{\mathcal{F}}$.

1.6. Next, we recall the monodromy of a logarithmic connection. Let $(\mathcal{F}, \nabla_{\mathcal{F}})$ be as above, and suppose that $r > 0$. For each $i = 1, \cdots, r$, consider the composite

$$\mathcal{F} \stackrel{\nabla_{\mathcal{F}}}{\to} \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F} \to \sigma_{i*}(\sigma_i^*(\Omega_{X^{\log}/S^{\log}}) \otimes \sigma_i^*(\mathcal{F})) \stackrel{\sim}{\to} \sigma_{i*}(\sigma_i^*(\mathcal{F})), \tag{22}$$

where the second arrow arises from the adjunction relation "$\sigma_i^*(-) \dashv \sigma_{i*}(-)$" (i.e., "the functor $\sigma_i^*(-)$ is left adjoint to the functor $\sigma_{i*}(-)$"), and the third arrow arises from the residue isomorphism (16). This composite corresponds (via the adjunction relation '$\sigma_i^*(-) \dashv \sigma_{i*}(-)$") to an $\mathcal{O}_S$-linear endomorphism $\sigma_i^*(\mathcal{F}) \to \sigma_i^*(\mathcal{F})$, equivalently, a global section

$$\mu_i^{\nabla_{\mathcal{F}}} \in \Gamma(S, \mathcal{E}nd_{\mathcal{O}_S}(\sigma_i^*(\mathcal{F}))). \tag{23}$$

DEFINITION 1.6.1. We shall refer to $\mu_i^{\nabla_{\mathcal{F}}}$ as the *monodromy* of $\nabla_{\mathcal{F}}$ at σ_i.

REMARK 1.6.2. Let $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$ be a log integrable *line* bundle on $\mathfrak{X}_{/S}$. Then, $\mathcal{E}nd_{\mathcal{O}_S}(\sigma_i^*(\mathcal{L})) \cong \mathcal{O}_S$, and hence, $\mu_i^{\nabla_{\mathcal{L}}}$ $(i = 1, \cdots, r)$ may be thought of as an element of $\Gamma(S, \mathcal{O}_S)$. In particular, it makes sense to ask whether $\mu_i^{\nabla_{\mathcal{L}}}$ lies in k $(\subseteq \Gamma(S, \mathcal{O}_S))$ or not.

REMARK 1.6.3. If $\mathcal{G}$ is a vector bundle on $X_S^{(1)}$, then one may define (cf. [20], § 3.3) canonically an S-log connection

$$\nabla_{\mathcal{G}}^{\mathrm{can}} : F_{X/S}^*(\mathcal{G}) \to \Omega_{X^{\log}/S^{\log}} \otimes F_{X/S}^*(\mathcal{G}) \tag{24}$$

on the pull-back $F_{X/S}^*(\mathcal{G})$ of $\mathcal{G}$, which is uniquely determined by the condition that the sections of the subsheaf $F_{X/S}^{-1}(\mathcal{G})$ $(\subseteq F_{X/S}^*(\mathcal{G}))$ are contained in $\mathrm{Ker}(\nabla_{\mathcal{G}}^{\mathrm{can}})$. We shall refer to $\nabla_{\mathcal{G}}^{\mathrm{can}}$ as the *canonical S-log connection* on $F_{X/S}^*(\mathcal{G})$. One verifies immediately that

$$\mathrm{Im}(\nabla_{\mathcal{G}}^{\mathrm{can}}) \subseteq \Omega_{X/S} \otimes F_{X/S}^*(\mathcal{G}) \ \left(\subseteq \Omega_{X^{\mathrm{log}}/S^{\mathrm{log}}} \otimes F_{X/S}^*(\mathcal{G})\right) \tag{25}$$

(i.e., $\nabla_{\mathcal{G}}^{\mathrm{can}}$ arises from a *non-logarithmic* connection on $F_{X/S}^*(\mathcal{G})$) and

$$\psi^{\nabla_{\mathcal{G}}^{\mathrm{can}}} = \mu_i^{\nabla_{\mathcal{G}}^{\mathrm{can}}} = 0 \tag{26}$$

for any i.

1.7. Let us write $m := \mathrm{rk}(\mathcal{F})$. For each $i = 1, \cdots, r$, denote by $\phi_i^{\nabla_{\mathcal{F}}}(t) \in \Gamma(S, \mathcal{O}_S)[t]$ the characteristic polynomial of $\mu_i^{\nabla_{\mathcal{F}}}$, i.e.,

$$\phi_i^{\nabla_{\mathcal{F}}}(t) := \det(t \cdot \mathrm{id}_{\sigma_i^*(\mathcal{F})} - \mu_i^{\nabla_{\mathcal{F}}}) = \sum_{j=0}^{m} a_{i,j}^{\nabla_{\mathcal{F}}} \cdot t^j, \tag{27}$$

where $a_{i,j}^{\nabla_{\mathcal{F}}} \in \Gamma(S, \mathcal{O}_S)$ (satisfying that $a_{i,m}^{\nabla_{\mathcal{F}}} = 1$).

Proposition 1.7.1. *Suppose further that $\psi^{\nabla_{\mathcal{F}}} = 0$. Then, for any $j = 0, \cdots, m$, the element $a_{i,j}^{\nabla_{\mathcal{F}}}$ lies in $\mathbb{F}_p$.*

Proof. The condition "$\psi^{\nabla_{\mathcal{F}}} = 0$" implies the equality

$$(\mu_i^{\nabla_{\mathcal{F}}})^p - \mu_i^{\nabla_{\mathcal{F}}} = 0. \tag{28}$$

Hence, (since S is of characteristic p) the following sequence of equalities holds:

$$\begin{aligned} \sum_{j=0}^{m} (a_{i,j}^{\nabla_{\mathcal{F}}})^p \cdot t^{jp} &= (\sum_{j=0}^{m} a_{i,j}^{\nabla_{\mathcal{F}}} \cdot t^j)^p \\ &= \det((t \cdot \mathrm{id}_{\sigma_i^*(\mathcal{F})} - \mu_i^{\nabla_{\mathcal{F}}})^p) \\ &= \det(t^p \cdot \mathrm{id}_{\sigma_i^*(\mathcal{F})} - (\mu_i^{\nabla_{\mathcal{F}}})^p) \\ &= \det(t^p \cdot \mathrm{id}_{\sigma_i^*(\mathcal{F})} - \mu_i^{\nabla_{\mathcal{F}}}) \\ &= \sum_{j=0}^{m} a_{i,j}^{\nabla_{\mathcal{F}}} \cdot t^{jp}. \end{aligned} \tag{29}$$

This yields the equality

$$(a_{i,j}^{\nabla_{\mathcal{F}}})^p = a_{i,j}^{\nabla_{\mathcal{F}}} \tag{30}$$

$(j = 0, \cdots, m)$, i.e., $a_{i,j}^{\nabla_{\mathcal{F}}} \in \mathbb{F}_p$. This completes the proof of Proposition 1.7.1. □

DEFINITION 1.7.2. Let $\mathfrak{F} := (\mathcal{F}, \nabla_{\mathcal{F}})$ be as above.

(i) Let $\tau_i := [\tau_{i,1}, \cdots, \tau_{i,m}]$ be a multiset over $\Gamma(S, \mathcal{O}_S)$ with cardinality m. We shall say that $\nabla_{\mathcal{F}}$ *is of exponent* τ_i at σ_i if $\phi_i^{\nabla_{\mathcal{F}}}(t)$ may be described as

$$\phi_i^{\nabla_{\mathcal{F}}}(t) = \prod_{j=1}^{m}(t - \tau_{i,j}). \tag{31}$$

(ii) Suppose that $r > 0$ and we are given an r-tuple $\vec{\tau} := (\tau_i)_{i=1}^r$ of multisets over k with cardinality n (i.e., an element of $(\mathbb{N}^k_{\sharp(-)=n})^{\times r}$). Then, we shall say that $\mathfrak{F}$ is *of exponent* $\vec{\tau}$ if $\nabla_{\mathcal{F}}$ is of exponent τ_i at σ_i for any $i \in \{1, \cdots, r\}$.

(iii) Suppose that $r = 0$. Then, we shall say, for convenience, that any log integrable vector bundle on $\mathfrak{X}_{/S}$ is *of exponent* $\emptyset$.

REMARK 1.7.3. Let $i \in \{1, \cdots, r\}$, $\tau_i \in \Gamma(S, \mathcal{O}_S)$, and let $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$ be a log integrable line bundle on $\mathfrak{X}_{/S}$. Then, $\nabla_{\mathcal{L}}$ is of exponent τ_i at σ_i if and only if $\mu_i^{\nabla_{\mathcal{L}}} = \tau_i$.

The following two propositions follow immediately from the various definitions involved.

PROPOSITION 1.7.4. *Let $\mathfrak{F} = (\mathcal{F}, \nabla_{\mathcal{F}})$ be a log integrable vector bundle on $\mathfrak{X}_{/S}$, and suppose that $\nabla_{\mathcal{F}}$ is of exponent $\tau_i := [\tau_{i,1}, \cdots, \tau_{i,m}]$ (where $\tau_{i,j} \in \Gamma(S, \mathcal{O}_S)$) at σ_i. Then, the S-log connection* $\det(\nabla_{\mathcal{F}})$ *on the line bundle* $\det(\mathcal{F})$ *is of exponent $\sum_{j=1}^m \tau_{i,j}$ at σ_i.*

Proposition 1.7.5. *Let $\mathfrak{F} := (\mathcal{F}, \nabla_{\mathcal{F}})$ be a log integrable vector bundle on $\mathfrak{X}_{/S}$ of exponent $\vec{\tau} := (\tau_i)_{i=1}^r \in (\mathbb{N}^k_{\sharp(-)=n})^{\times r}$ and $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$ a log integrable line bundle on $\mathfrak{X}_{/S}$ of exponent $\vec{a} := (a_i)_{i=1}^r \in k^{\times r}$ $(= (\mathbb{N}^k_{\sharp(-)=1})^{\times r})$. Then, the log integrable vector bundle*

$$\mathfrak{F} \otimes \mathfrak{L} := (\mathcal{F} \otimes \mathcal{L}, \nabla_{\mathcal{F}} \otimes \nabla_{\mathcal{L}}) \tag{32}$$

is of exponent $\vec{\tau}^{\,+\vec{a}}$ (cf. (10)).

2. Determinant Data

In this section, we shall recall the notion of a determinant data (cf. Definition 2.1.1, (i)), which was introduced in the author's paper (cf. [20], Definition 4.9.1 (i)). As we proved in [20] (cf. [20], Theorem D, or (165) displayed later), one may realize, after fixing an n-determinant data $\mathbb{U}$, each $\mathfrak{sl}_n$-oper as a certain integrable vector bundle, i.e., a $(\mathrm{GL}_n, \mathbb{U})$-oper. Since we have assumed that the ground field k is of positive characteristic, there exists (cf. Proposition 2.1.5) necessarily an n-determinant data with prescribed monodromy.

2.1. Let S and $\mathfrak{X}_{/S}$ be as before, and n a positive integer with $n \leq p$.

Definition 2.1.1 (cf. [20], Definition 4.9.1).

(i) An *n-determinant data* for $\mathfrak{X}_{/S}$ is a pair

$$\mathbb{U} := (\mathcal{B}, \nabla_0) \tag{33}$$

consisting of a line bundle $\mathcal{B}$ on X and an S-log connection ∇_0 on the line bundle $\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n}$.

(ii) Let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ and $\mathbb{U}' := (\mathcal{B}', \nabla_0')$ be n-determinant data for $\mathfrak{X}_{/S}$. An *isomorphism of n-determinant data* from $\mathbb{U}$ to $\mathbb{U}'$ is an isomorphism $\mathcal{B} \xrightarrow{\sim} \mathcal{B}'$ of $\mathcal{O}_X$-modules such that the induced isomorphism

$$\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n} \xrightarrow{\sim} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}'^{\otimes n} \tag{34}$$

is compatible with the respective S-log connections ∇_0 and ∇'_0.

REMARK 2.1.2. Let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be an n-determinant data for $\mathfrak{X}_{/S}$ and $s' : S' \to S$ a morphism of k-schemes. Then the base-change

$$s'^{*}(\mathbb{U}) := ((\mathrm{id}_X \times s')^*(\mathcal{B}), (\mathrm{id}_X \times s')^*(\nabla_0)) \tag{35}$$

via s' forms an n-determinant for $\mathfrak{X}_{/S'}$ (cf. (17)).

DEFINITION 2.1.3. We shall say that an n-determinant data $\mathbb{U} := (\mathcal{B}, \nabla_0)$ for $\mathfrak{X}_{/S}$ is *dormant* if $\psi^{\nabla_0} = 0$.

DEFINITION 2.1.4. Let $\vec{a}$ be an element of $k^{\times r}$. We shall say that an n-determinant data $\mathbb{U} := (\mathcal{B}, \nabla_0)$ for $\mathfrak{X}_{/S}$ is *of exponent* $\vec{a}$ if the log integrable line bundle $(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n}, \nabla_0)$ is of exponent $\vec{a}$ (cf. Definition 1.7.2 (ii) and (iii)).

PROPOSITION 2.1.5. *Suppose that* $r > 0$, *and let* $\vec{a} := (a_i)_{i=1}^r \in \mathbb{F}_p^{\times r}$. *Then, there exists a dormant n-determinant data* $\mathbb{U} := (\mathcal{B}, \nabla_0)$ *for* $\mathfrak{X}_{/S}$ *of exponent* $\vec{a}$.

PROOF. Since $n \leq p$, one may choose a pair of nonnegative integers (s,t) satisfying that $p \cdot s = n \cdot t + \frac{n(n-1)}{2}$. Let us take $\mathcal{B}' := \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes t}$. Then,

$$\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}'^{\otimes n} \xrightarrow{\sim} (\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes s})^{\otimes p} \xrightarrow{\sim} F_{X/S}^*((\mathrm{id}_X \times F_S)^*(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes s})). \tag{36}$$

Denote by ∇'_0 the S-log connection on $\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}'^{\otimes n}$ corresponding, via this composite isomorphism, to the canonical S-log connection on $F_{X/S}^*((\mathrm{id}_X \times F_S)^*(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes s}))$ (cf. Remark 1.6.3). Then, the pair $(\mathcal{B}', \nabla'_0)$ forms a dormant n-determinant data for $\mathfrak{X}_{/S}$ of exponent $(0, 0, \cdots, 0)$ (cf. (26)).

Now, for each $i = 1, \cdots, r$, we shall choose a nonnegative integer m_i such that $\overline{m_i \cdot n} = a_i$ in $\mathbb{F}_p$, where $\overline{m_i \cdot n}$ denotes the image of $m_i \cdot n \in \mathbb{Z}$ via the quotient $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$. In particular, $\sum_{i=1}^r (m_i \cdot n)\sigma_i$ is an effective relative divisor on X relative to S. By passing to the isomorphism

$$(37)\qquad (\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}'^{\otimes n})(-\sum_{i=1}^{r}(m_i \cdot n)\sigma_i) \xrightarrow{\sim} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}'(-\sum_{i=1}^{r} m_i\sigma_i)^{\otimes n},$$

we obtain an S-log connection ∇_0 on $\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}'(-\sum_{i=1}^{r} m_i\sigma_i)^{\otimes n}$ (with vanishing p-curvature) corresponding to the restriction of ∇'_0 to $(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}'^{\otimes n})(-\sum_{i=1}^{r}(m_i \cdot n)\sigma_i)$. This S-log connection is, by construction, of exponent $\overline{m_i \cdot n} = a_i$ at σ_i. Thus, we obtain a dormant n-determinant data

$$(38)\qquad (\mathcal{B} := \mathcal{B}'(-\sum_{i=1}^{r} m_i\sigma_i), \nabla_0)$$

which satisfies the required conditions, as desired. □

2.2. Let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be an n-determinant data for $\mathfrak{X}_{/S}$ and $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$ be a log integrable line bundle on $\mathfrak{X}_{/S}$. We shall consider the pair

$$(39)\qquad \mathbb{U} \otimes \mathfrak{L} := (\mathcal{B} \otimes \mathcal{L}, \nabla_0 \otimes \nabla_{\mathcal{L}}^{\otimes n}),$$

where we regard $\nabla_0 \otimes \nabla_{\mathcal{L}}^{\otimes n}$ (cf. § 1.4) as an S-log connection on

$$(40)\qquad \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes (\mathcal{B} \otimes \mathcal{L})^{\otimes n} \left(\cong (\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n}) \otimes \mathcal{L}^{\otimes n}\right).$$

One verifies that $\mathbb{U} \otimes \mathfrak{L}$ forms an n-determinant data for $\mathfrak{X}_{/S}$. If, moreover, $\mathbb{U}$ is dormant and $\psi^{\nabla_{\mathcal{L}}} = 0$, then $\mathbb{U} \otimes \mathfrak{L}$ turns out to be dormant.

2.3. For each line bundle $\mathcal{B}$ on X, one may construct, in a canonical manner, a dormant p-determinant data whose underlying line bundle coincides with $\mathcal{B}$.

Indeed, let us consider the natural composite isomorphism

$$(41)\qquad F_{X/S}^{*}((\mathrm{id}_X \times F_S)^{*}(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{p-1}{2}} \otimes \mathcal{B})) \xrightarrow{\sim} (\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{p-1}{2}} \otimes \mathcal{B})^{\otimes p} \xrightarrow{\sim} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{p(p-1)}{2}} \otimes \mathcal{B}^{\otimes p}.$$

By passing to this composite, we obtain an S-log connection $\nabla_{0,\mathcal{B}}^{\mathrm{can}}$ on $\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{p(p-1)}{2}} \otimes \mathcal{B}^{\otimes p}$ corresponding to the canonical S-log connection on $F_{X/S}^*((\mathrm{id}_X \times F_S)^*(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{p-1}{2}} \otimes \mathcal{B}))$ (cf. Remark 1.6.3). Thus, the pair

$$\mathbb{U}_{\mathcal{B}}^{\mathrm{can}} := (\mathcal{B}, \nabla_{0,\mathcal{B}}^{\mathrm{can}}) \tag{42}$$

forms a dormant p-determinant data for $\mathfrak{X}_{/S}$ satisfying that $\mu_i^{\nabla_{0,\mathcal{B}}^{\mathrm{can}}} = 0$ for any i (cf. (26)).

2.4. Let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be an n-determinant data for $\mathfrak{X}_{/S}$, and write

$$\mathcal{B}^{\nabla,n} := \Omega_{X^{\log}/S^{\log}}^{\otimes(n-1)} \otimes \mathcal{B}^{\vee}. \tag{43}$$

Then, we have a natural composite isomorphism

$$\begin{aligned}\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{n(n-1)}{2}} \otimes (\mathcal{B}^{\nabla,n})^{\otimes n} &\xrightarrow{\sim} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{n(n-1)}{2}} \otimes (\Omega_{X^{\log}/S^{\log}}^{\otimes(n-1)} \otimes \mathcal{B}^{\vee})^{\otimes n} \\ &\xrightarrow{\sim} (\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n})^{\vee}.\end{aligned} \tag{44}$$

The S-log connection $\nabla_0^{\vee}$ on $(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n})^{\vee}$ carries, by means of this composite isomorphism, an S-log connection on $\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{n(n-1)}{2}} \otimes (\mathcal{B}^{\nabla,n})^{\otimes n}$; we shall denote this connection by $\nabla_0^{\triangledown}$. Thus, we obtain an n-determinant data

$$\mathbb{U}^{\triangledown} := (\mathcal{B}^{\nabla,n}, \nabla_0^{\triangledown}) \tag{45}$$

for $\mathfrak{X}_{/S}$, which is referred to as the *dual n-determinant data* of $\mathbb{U}$. One verifies that there exists a natural isomorphism

$$(\mathbb{U}^{\triangledown})^{\triangledown} \xrightarrow{\sim} \mathbb{U} \tag{46}$$

of n-determinant data.

2.5. Let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be as above, and write

$$\mathcal{B}^{\triangleright,n} := \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes n} \otimes \mathcal{B}. \tag{47}$$

Consider the canonical composite isomorphism

$$(48)\qquad \begin{aligned}&(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{p(p-1)}{2}} \otimes \mathcal{B}^{\otimes p}) \otimes (\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n})^{\vee} \\ &\xrightarrow{\sim} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{1}{2}\cdot(p^2-p-n^2+n)} \otimes \mathcal{B}^{\otimes (p-n)} \\ &\xrightarrow{\sim} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{(p-n)(p-n-1)}{2}} \otimes (\mathcal{B}^{\triangleright, n})^{\otimes (p-n)}.\end{aligned}$$

The product $\nabla_{0,\mathcal{B}}^{\mathrm{can}} \otimes \nabla_0^{\vee}$ of the S-log connections carries, by means of this composite isomorphism, an S-log connection $\nabla_0^{\triangleright}$ on $\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{(p-n)(p-n-1)}{2}} \otimes (\mathcal{B}^{\triangleright, n})^{\otimes (p-n)}$. Thus, we obtain a $(p-n)$-determinant data

$$(49)\qquad \mathbb{U}^{\triangleright} := (\mathcal{B}^{\triangleright, n}, \nabla_0^{\triangleright})$$

for $\mathfrak{X}_{/S}$.

Moreover, we shall write

$$(50)\qquad \mathbb{U}^{\bigstar} := (\mathbb{U}^{\triangleright})^{\triangledown}$$

and refer to it as the *$\bigstar$-dual $(p-n)$-determinant data* of $\mathbb{U}$. If $\mathcal{B}^{\bigstar}$ denotes the underlying line bundle of $\mathbb{U}^{\bigstar}$, i.e.,

$$(51)\qquad \mathcal{B}^{\bigstar} := (\mathcal{B}^{\triangleright, n})^{\triangledown, p-n},$$

then there exist natural isomorphisms

$$(52)\qquad \mathcal{B}^{\bigstar} \xrightarrow{\sim} \Omega_{X^{\log}/S^{\log}}^{\otimes (p-1)} \otimes \mathcal{B}^{\vee} \ \text{ and } \ \mathcal{B}^{\bigstar} \xrightarrow{\sim} \mathcal{B}^{\triangledown, p}.$$

The following three propositions follow immediately from the various definitions involved.

Proposition 2.5.1. *There exists a canonical isomorphism*

$$(53)\qquad (\mathbb{U}^{\bigstar})^{\bigstar} \xrightarrow{\sim} \mathbb{U}$$

of n-determinant data.

Proposition 2.5.2. *Let $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$ be a log integrable line bundle on $\mathfrak{X}_{/S}$, and write $\mathfrak{L}^{\vee} := (\mathcal{L}^{\vee}, \nabla_{\mathcal{L}}^{\vee})$. Then, there exists a canonical isomorphism*

$$(54)\qquad (\mathbb{U} \otimes \mathfrak{L})^{\bigstar} \xrightarrow{\sim} \mathbb{U}^{\bigstar} \otimes \mathfrak{L}^{\vee}$$

of $(p-n)$*-determinant data.*

Proposition 2.5.3. *Let* $\mathcal{B}$ *be a line bundle on* X*. Then, there exists a canonical isomorphism*

$$\left(\mathbb{U}^{\mathrm{can}}_{\mathcal{B}^{\triangledown},p} \xrightarrow{\sim}\right) \ \mathbb{U}^{\mathrm{can}}_{\mathcal{B}^{\bigstar}} \xrightarrow{\sim} (\mathbb{U}^{\mathrm{can}}_{\mathcal{B}})^{\triangledown} \tag{55}$$

of p*-determinant data for* $\mathfrak{X}_{/S}$*.*

3. Opers on Pointed Stable Curves

In this section, we recall the definition of a (dormant) GL_n-oper (where GL_n denotes the general linear group over k of rank n) and consider the canonical construction of a dormant GL_p-oper by means of a line bundle (cf. Proposition 3.7.1).

Let S, $\mathfrak{X}_{/S}$, and n be as before.

3.1. First, recall the definition of a GL_n-oper, as follows.

Definition 3.1.1 (cf. [20], Definition 4.2.1).

(i) A GL_n*-oper* on $\mathfrak{X}_{/S}$ is a collection of data

$$\mathfrak{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n), \tag{56}$$

where

- $\mathcal{F}$ is a vector bundle on X of rank n;
- $\nabla_{\mathcal{F}}$ is an S-log connection $\mathcal{F} \to \Omega_{X^{\mathrm{log}}/S^{\mathrm{log}}} \otimes \mathcal{F}$ on $\mathcal{F}$;
- $\{\mathcal{F}^j\}_{j=0}^n$ is a decreasing filtration

$$0 = \mathcal{F}^n \subseteq \mathcal{F}^{n-1} \subseteq \cdots \subseteq \mathcal{F}^0 = \mathcal{F} \tag{57}$$

  on $\mathcal{F}$ by vector bundles on X,

satisfying the following three conditions:

(1) The subquotients $\mathcal{F}^j/\mathcal{F}^{j+1}$ $(0 \leq j \leq n-1)$ are line bundles;

(2) $\nabla_{\mathcal{F}}(\mathcal{F}^j) \subseteq \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}^{j-1}$ $(1 \leq j \leq n-1)$;

(3) The $\mathcal{O}_X$-*linear* morphism

$$\mathfrak{ks}^j_{\mathfrak{F}^\heartsuit} : \mathcal{F}^j/\mathcal{F}^{j+1} \to \Omega_{X^{\log}/S^{\log}} \otimes (\mathcal{F}^{j-1}/\mathcal{F}^j) \tag{58}$$

defined by assigning $\overline{a} \mapsto \overline{\nabla_{\mathcal{F}}(a)}$ for any local section $a \in \mathcal{F}^j$ (where $\overline{(-)}$'s denote the images in the respective quotients), which is well-defined by virtue of the condition (2), is an isomorphism.

(ii) Let $\mathfrak{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$, $\mathfrak{G}^\heartsuit := (\mathcal{G}, \nabla_{\mathcal{G}}, \{\mathcal{G}^j\}_{j=0}^n)$ be GL_n-opers on $\mathfrak{X}_{/S}$. An *isomorphism of* GL_n*-opers* from $\mathfrak{F}^\heartsuit$ to $\mathfrak{G}^\heartsuit$ is an isomorphism $(\mathcal{F}, \nabla_{\mathcal{F}}) \xrightarrow{\sim} (\mathcal{G}, \nabla_{\mathcal{G}})$ of log integrable vector bundles (cf. § 1.4) that is compatible with the respective filtrations $\{\mathcal{F}^j\}_{j=0}^n$ and $\{\mathcal{G}^j\}_{j=0}^n$.

Remark 3.1.2. Let $\mathfrak{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ be a GL_n-oper on $\mathfrak{X}_{/S}$ and fix $j \in \{0, \cdots, n-1\}$. By composing the isomorphisms

$$(\mathcal{F}^l/\mathcal{F}^{l+1}) \otimes \mathcal{T}^{\otimes(l-j)}_{X^{\log}/S^{\log}} \xrightarrow{\sim} (\mathcal{F}^{l+1}/\mathcal{F}^{l+2}) \otimes \mathcal{T}^{\otimes(l-j+1)}_{X^{\log}/S^{\log}} \tag{59}$$

arising from the isomorphisms $\mathfrak{ks}^l_{\mathfrak{F}^\heartsuit}$ $(j+1 \leq l \leq n-1)$ (cf. (58)), we obtain a canonical composite isomorphism

$$\mathcal{F}^j/\mathcal{F}^{j+1} \xrightarrow{\sim} (\mathcal{F}^{j+1}/\mathcal{F}^{j+2}) \otimes \mathcal{T}_{X^{\log}/S^{\log}} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{F}^{n-1} \otimes \mathcal{T}^{\otimes n-1-j}_{X^{\log}/S^{\log}} \tag{60}$$

between line bundles on X. Moreover, by using these isomorphisms for all j, we obtain a composite isomorphism

$$\begin{aligned} \mathfrak{det}_{\mathfrak{F}^\heartsuit} : \det(\mathcal{F}) &\xrightarrow{\sim} \bigotimes_{j=0}^{n-1} (\mathcal{F}^j/\mathcal{F}^{j+1}) \\ &\xrightarrow{\sim} \Big(\bigotimes_{j=0}^{n-1} \mathcal{T}^{\otimes(n-1-j)}_{X^{\log}/S^{\log}}\Big) \otimes (\mathcal{F}^{n-1})^{\otimes n} \\ &\xrightarrow{\sim} \mathcal{T}^{\otimes \frac{n(n-1)}{2}}_{X^{\log}/S^{\log}} \otimes (\mathcal{F}^{n-1})^{\otimes n}. \end{aligned} \tag{61}$$

In particular, the following equalities hold:

$$\deg(\mathcal{F}) = \deg(\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes (\mathcal{F}^{n-1})^{\otimes n}) \\ = n(n-1)(1-g) + n \cdot \deg(\mathcal{F}^{n-1}). \tag{62}$$

The following proposition will be used in the proof of Proposition 4.2.1.

PROPOSITION 3.1.3. *Let $\mathfrak{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ be a GL_n-oper on $\mathfrak{X}_{/S}$ and $(\mathcal{V}, \nabla_{\mathcal{V}})$ a log integrable vector bundle on $\mathfrak{X}_{/S}$. Suppose that we are given two morphisms ϕ_1, $\phi_2 : \mathcal{F} \to \mathcal{V}$ both of which are compatible with the respective S-log connections $\nabla_{\mathcal{F}}$, $\nabla_{\mathcal{V}}$, and satisfying that $\phi_1|_{\mathcal{F}^{n-1}} = \phi_2|_{\mathcal{F}^{n-1}}$. Then, the equality $\phi_1 = \phi_2$ holds.*

PROOF. Suppose that $\phi_1|_{\mathcal{F}^j} = \phi_2|_{\mathcal{F}^j}$ for some $j \in \{1, \cdots, n-1\}$. By the definition of a GL_n-oper, $\mathcal{F}^{j-1}$ may be generated by $\mathcal{F}^j$ and $\nabla_{\mathcal{F}}(\mathcal{F}^j)$. Hence, since both ϕ_1 and ϕ_2 are compatible with the S-log connections $\nabla_{\mathcal{F}}$ and $\nabla_{\mathcal{V}}$, the equality $\phi_1|_{\mathcal{F}^j} = \phi_2|_{\mathcal{F}^j}$ implies the equality $\phi_1|_{\mathcal{F}^{j-1}} = \phi_2|_{\mathcal{F}^{j-1}}$. Thus, the assertion follows from descending induction on j. □

DEFINITION 3.1.4. We shall say that a GL_n-oper $\mathfrak{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ on $\mathfrak{X}_{/S}$ is *dormant* if $\psi^{\nabla_{\mathcal{F}}} = 0$.

3.2. Let $\mathfrak{F} := (\mathcal{F}, \nabla_{\mathcal{F}})$ be a log integrable vector bundle on $\mathfrak{X}_{/S}$ of rank n. We shall write

$$\mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})} := F_{X/S}^*(F_{X/S*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))). \tag{63}$$

Note that although $\mathrm{Ker}(\nabla_{\mathcal{F}})$ is not an $\mathcal{O}_X$-module, one may equip, in a natural manner, its direct image $F_{X/S*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))$ via $F_{X/S}$ with a structure of $\mathcal{O}_{X_S^{(1)}}$-module. The $\mathcal{O}_{X_S^{(1)}}$-linear inclusion $F_{X/S*}(\mathrm{Ker}(\nabla_{\mathcal{F}})) \hookrightarrow F_{X/S*}(\mathcal{F})$ corresponds, via the adjunction relation "$F_{X/S}^*(-) \dashv F_{X/S*}(-)$", to an $\mathcal{O}_X$-linear morphism

$$\nu^{\nabla_{\mathcal{F}}} : \mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})} \to \mathcal{F}. \tag{64}$$

If we consider the canonical S-log connection $\nabla^{\mathrm{can}}_{F_{X/S*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))}$ on $\mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})}$ (cf. Remark 1.6.3), then the morphism $\nu^{\nabla_{\mathcal{F}}}$ is compatible with the respective connections $\nabla^{\mathrm{can}}_{F_{X/S*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))}$ and $\nabla_{\mathcal{F}}$. The morphism $\nu^{\nabla_{\mathcal{F}}}$ fits into the short exact sequence

$$0 \to \mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})} \overset{\nu^{\nabla_{\mathcal{F}}}}{\to} \mathcal{F} \to \Lambda_{\mathrm{sing}} \oplus \bigoplus_{i=1}^{r} \Lambda_i \to 0 \tag{65}$$

of $\mathcal{O}_X$-modules (cf. [20], the short exact sequence (872)), where Λ_{sing} and Λ_i $(i = 1, \cdots, r)$ are $\mathcal{O}_X$-modules supported on the nonsmooth locus of the semistable curve X (over S) and the locus $\mathrm{Im}(\sigma_i) \subseteq X$ respectively.

Let us recall the following proposition.

Proposition 3.2.1. *In the above notation, suppose further that* $\psi^{\nabla_{\mathcal{F}}} = 0$ *and* $S = \mathrm{Spec}(k')$ *for some algebraically closed field* k' *over* k*. Then, for each* i $(= 1, \cdots, r)$*, there exists uniquely a multiset* $[\tau_{i,1}, \cdots, \tau_{i,n}]$ *over* $\mathbb{Z}$ *with cardinality* n *satisfying the following three conditions:*

(1) $0 \leq \tau_{i,j} < p$ *for any* $j = 1, \cdots, n$*;*

(2) Λ_i *decomposes into the direct sum*

$$\Lambda_i \overset{\sim}{\to} \bigoplus_{j=1}^{n} \mathcal{O}_X(\tau_{i,j}\sigma_i)/\mathcal{O}_X; \tag{66}$$

(3) $\nabla_{\mathcal{F}}$ *is of exponent* $[\overline{\tau}_{i,1}, \cdots, \overline{\tau}_{i,n}]$ *at* σ_i *(cf. Definition 1.7.2 (i)), where each* $\overline{\tau}_{i,j}$ *denotes the image of* $\tau_{i,j}$ *via the quotient* $\mathbb{Z} \to \mathbb{F}_p$ $(\subseteq k)$*.*

Proof. The assertion follows from [17], Corollary 2.10, or the discussion in [20] following Lemma 8.3.2. □

3.3. Next, we consider GL_n-opers with prescribed determinant. Let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be an n-determinant data (cf. Definition 2.1.1, (i)) for $\mathfrak{X}_{/S}$.

Definition 3.3.1 (cf. [20], Definition 4.9.4).

(i) A $(\mathrm{GL}_n, \mathbb{U})$-*oper* on $\mathfrak{X}_{/S}$ is a collection of data

$$\mathfrak{F}^{\diamondsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^{\diamondsuit}}), \tag{67}$$

where

- the collection of data

$$\mathfrak{F}^{\diamondsuit\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n) \tag{68}$$

  is a GL_n-oper on $\mathfrak{X}_{/S}$ (which is referred to as the *underlying* GL_n*-oper of* $\mathfrak{F}^{\diamondsuit}$);
- $\eta_{\mathfrak{F}^{\diamondsuit}}$ is an isomorphism $\mathcal{B} \xrightarrow{\sim} \mathcal{F}^{n-1}$ of $\mathcal{O}_X$-modules such that the composite isomorphism

$$\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes \mathcal{B}^{\otimes n} \xrightarrow{\mathrm{id}\otimes \eta_{\mathfrak{F}^{\diamondsuit}}^{\otimes n}} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes \frac{n(n-1)}{2}} \otimes (\mathcal{F}^{n-1})^{\otimes n} \xrightarrow{\mathfrak{det}_{\mathfrak{F}^{\diamondsuit\heartsuit}}^{-1}} \det(\mathcal{F}) \tag{69}$$

  (cf. (61) for the definition of the isomorphism $\mathfrak{det}_{(-)}$) is compatible with ∇_0 and $\det(\nabla_{\mathcal{F}})$ (cf. § 1.4).

(ii) Let $\mathfrak{F}^{\diamondsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^{\diamondsuit}})$ and $\mathfrak{G}^{\diamondsuit} := (\mathcal{G}, \nabla_{\mathcal{G}}, \{\mathcal{G}^j\}_{j=0}^n, \eta_{\mathfrak{G}^{\diamondsuit}})$ be $(\mathrm{GL}_n, \mathbb{U})$-opers on $\mathfrak{X}_{/S}$. An *isomorphism of* $(\mathrm{GL}_n, \mathbb{U})$*-opers* from $\mathfrak{F}^{\diamondsuit}$ to $\mathfrak{G}^{\diamondsuit}$ is an isomorphism $\alpha : \mathfrak{F}^{\diamondsuit\heartsuit} \xrightarrow{\sim} \mathfrak{G}^{\diamondsuit\heartsuit}$ between the respective underlying GL_n-opers (cf. Definition 3.1.1, (ii)) whose restriction $\alpha|_{\mathcal{F}^{n-1}} : \mathcal{F}^{n-1} \xrightarrow{\sim} \mathcal{G}^{n-1}$ satisfies the equality $\alpha|_{\mathcal{F}^{n-1}} \circ \eta_{\mathfrak{F}^{\diamondsuit}} = \eta_{\mathfrak{G}^{\diamondsuit}}$.

Definition 3.3.2. Let $\vec{\tau} := (\tau_i)_{i=1}^r$ be an element of $(2^k_{\sharp(-)=n})^{\times r}$ (where we take $\vec{\tau} := \emptyset$ if $r = 0$). Then, we shall say that a $(\mathrm{GL}_n, \mathbb{U})$-oper $\mathfrak{F}^{\diamondsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^{\diamondsuit}})$ on $\mathfrak{X}_{/S}$ is *of exponent* $\vec{\tau}$ if the underlying log integrable vector bundle $(\mathcal{F}, \nabla_{\mathcal{F}})$ is of exponent $\vec{\tau}$ (cf. Definition 1.7.2 (ii) and (iii)).

Remark 3.3.3. Notice that the definition of a $(\mathrm{GL}_n, \mathbb{U})$-oper in Definition 3.3.1 differs slightly from the definition of a $(\mathrm{GL}_n, 1, \mathbb{U})$-oper proposed in [20], Definition 4.9.4.

REMARK 3.3.4. Let $s' : S' \to S$ be a morphism of k-schemes and $\mathfrak{F}^\heartsuit := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^\heartsuit})$ a $(\mathrm{GL}_n, \mathbb{U})$-oper on $\mathfrak{X}_{/S}$. Then, the collection of data

$$s'^*(\mathfrak{F}^\heartsuit) \tag{70}$$

obtained from pulling-back the collection of data $\mathfrak{F}^\heartsuit$ via $\mathrm{id}_X \times s' : X \times_S S' \to X$ forms a $(\mathrm{GL}_n, s'^*(\mathbb{U}))$-oper (cf. (35)) on $\mathfrak{X}_{/S'}$ (cf. (17)).

DEFINITION 3.3.5. We shall say that a $(\mathrm{GL}_n, \mathbb{U})$-oper $\mathfrak{F}^\heartsuit := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^\heartsuit})$ is *dormant* if $\psi^{\nabla_\mathcal{F}} = 0$.

REMARK 3.3.6. If there exists a dormant $(\mathrm{GL}_n, \mathbb{U})$-oper $\mathfrak{F}^\heartsuit := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^\heartsuit})$ on $\mathfrak{X}_{/S}$, then $\mathbb{U}$ is necessarily dormant (by [20], Proposition 3.2.2). Indeed, if Tr denotes the trace map $\mathrm{Tr} : \mathcal{E}nd(\mathcal{F}) \to \mathcal{O}_X$, then we have $\psi^{\nabla_0} = \psi^{\det(\nabla_\mathcal{F})} = \mathrm{Tr} \circ \psi^{\nabla_\mathcal{F}} = 0$ (cf. [8], Proposition 2.1.2 (iii)).

REMARK 3.3.7. Suppose that $n = 1$. Let us consider the 1-step filtration $\{\mathcal{B}^j\}_{j=0}^1$ on $\mathcal{B}$ given by $\mathcal{B}^0 := \mathcal{B}$ and $\mathcal{B}^1 := 0$. Then, the collection of data

$$\mathfrak{B}^\heartsuit := (\mathcal{B}, \nabla_0, \{\mathcal{B}^j\}_{j=0}^1, \mathrm{id}_\mathcal{B}) \tag{71}$$

forms a unique (up to isomorphism) $(\mathrm{GL}_1, \mathbb{U})$-oper on $\mathfrak{X}_{/S}$. If $\mathbb{U}$ is dormant, then $\mathfrak{B}^\heartsuit$ is tautologically dormant.

PROPOSITION 3.3.8. *Suppose that $r > 0$. Let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be an n-determinant data for $\mathfrak{X}_{/S}$ and $\mathfrak{F}^\heartsuit := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^\heartsuit})$ a dormant $(\mathrm{GL}_n, \mathbb{U})$-oper on $\mathfrak{X}_{/S}$ (hence $\mathbb{U}$ is dormant, as we explained in Remark 3.3.6). Then, there exists an element $\vec{\tau} := (\tau_i)_{i=1}^r$ of $(2^{\mathbb{F}_p}_{\sharp(-)=n})^{\times r}$ such that $\mathfrak{F}^\heartsuit$ is of exponent $\vec{\tau}$.*

PROOF. It follows from Proposition 1.7.1 that for each $i = 1, \cdots r$, the characteristic polynomial $\phi_i^{\nabla_\mathcal{F}}(t)$ of $\mu_i^{\nabla_\mathcal{F}}$ lies in $\mathbb{F}_p[t]$. Thus, we may assume, after possibly restricting X to a geometric fiber of $f : X \to S$, that $S = k'$ for some algebraically closed field k' over k. Let $[\tau_{i,1}, \cdots, \tau_{i,n}]$ $(i = 1, \cdots, r)$ be the multiset asserted in Proposition 3.2.1 of the case where the log integrable vector bundle under consideration is taken to be $(\mathcal{F}, \nabla_\mathcal{F})$.

(In particular, $\nabla_{\mathcal{F}}$ is of exponent $\tau_i := [\overline{\tau}_{i,1}, \cdots, \overline{\tau}_{i,n}]$ at σ_i.) But, according to [20], Proposition 8.3.3, the integers $\tau_{i,1}, \cdots, \tau_{i,n}$ are mutually distinct. This implies that τ_i is consequently a subset of $\mathbb{F}_p$, and hence, completes the proof of Proposition 3.3.8. □

3.4. Let $\mathfrak{F}^{\diamondsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^{\diamondsuit}})$ be a $(\mathrm{GL}_n, \mathbb{U})$-oper on $\mathfrak{X}_{/S}$ of exponent $\vec{\tau} \in (\mathbb{N}^k_{\sharp(-)=n})^{\times r}$, $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$ a log integrable line bundle on $\mathfrak{X}_{/S}$ of exponent $\vec{a} := (a_i)_{i=1}^r \in k^{\times r}$. Then, one verifies from Proposition 1.7.5 that the collection of data

$$\mathfrak{F}^{\diamondsuit}_{\otimes \mathfrak{L}} := (\mathcal{F} \otimes \mathcal{L}, \nabla_{\mathcal{F}} \otimes \nabla_{\mathcal{L}}, \{\mathcal{F}^j \otimes \mathcal{L}\}_{j=0}^n, \eta_{\mathfrak{F}^{\diamondsuit}} \otimes \mathrm{id}_{\mathcal{L}}) \tag{72}$$

forms a $(\mathrm{GL}_n, \mathbb{U} \otimes \mathfrak{L})$-oper (cf. (39)) on $\mathfrak{X}_{/S}$ of exponent $\vec{\tau}^{+\vec{a}}$ (cf. (10)).

3.5. Let $\mathfrak{F}^{\diamondsuit}$ be as above. If we write $(\mathcal{F}^\vee)^j$ for the kernel of the dual $\mathcal{F}^\vee \twoheadrightarrow (\mathcal{F}^{n-j})^\vee$ of the inclusion $\mathcal{F}^{n-j} \hookrightarrow \mathcal{F}$, then the collection of data

$$(\mathcal{F}^\vee, \nabla^\vee_{\mathcal{F}}, \{(\mathcal{F}^\vee)^j\}_{j=0}^n) \tag{73}$$

(cf. § 1.4 for the definition of $\nabla^\vee_{\mathcal{F}}$) forms a GL_n-oper on $\mathfrak{X}_{/S}$.

Moreover, consider the composite isomorphism

$$\begin{aligned} \eta^{\triangledown}_{\mathfrak{F}^{\diamondsuit}} : \mathcal{B}^{\triangledown, n} &\xrightarrow{\sim} (\mathcal{T}^{\otimes(n-1)}_{X^{\log}/S^{\log}} \otimes \mathcal{B})^\vee \xrightarrow{\sim} (\mathcal{T}^{\otimes(n-1)}_{X^{\log}/S^{\log}} \otimes \mathcal{F}^{n-1})^\vee \\ &\xrightarrow{\sim} (\mathcal{F}^0/\mathcal{F}^1)^\vee \xrightarrow{\sim} (\mathcal{F}^\vee)^{n-1} \end{aligned} \tag{74}$$

(cf. (43) for the definition of $\mathcal{B}^{\triangledown, n}$), where the second isomorphism arise from $\eta_{\mathcal{F}^{\diamondsuit}} : \mathcal{B} \xrightarrow{\sim} \mathcal{F}^{n-1}$ and the third isomorphism denotes the dual of the isomorphism (60) for the case where $j = 0$. One verifies that the collection of data

$$\mathfrak{F}^{\diamondsuit\triangledown} := (\mathcal{F}^\vee, \nabla^\vee_{\mathcal{F}}, \{(\mathcal{F}^\vee)^j\}_{j=0}^n, \eta^{\triangledown}_{\mathfrak{F}^{\diamondsuit}}) \tag{75}$$

forms a $(\mathrm{GL}_n, \mathbb{U}^\triangledown)$-oper on $\mathfrak{X}_{/S}$ (cf. (45) for the definition of $\mathbb{U}^\triangledown$). We shall refer to $\mathfrak{F}^{\diamondsuit\triangledown}$ as the *dual* $(\mathrm{GL}_n, \mathbb{U}^\triangledown)$*-oper of* $\mathfrak{F}^{\diamondsuit}$. If, moreover, $\mathfrak{F}^{\diamondsuit}$ is dormant, then $\mathfrak{F}^{\diamondsuit\triangledown}$ is immediately verified to be dormant.

Finally, there exists a natural isomorphism

$$(\mathfrak{F}^{\diamondsuit\triangledown})^\triangledown \xrightarrow{\sim} \mathfrak{F}^{\diamondsuit} \tag{76}$$

of $(\mathrm{GL}_n, \mathbb{U})$-opers.

3.6. Recall (cf. [20], § 4.4) that the sheaf of *logarithmic crystalline differential operators* (or *lcdo*'s for short) on X^{log} over S^{log} is the Zariski sheaf

$$\mathcal{D}^{<\infty}_{X^{\mathrm{log}}/S^{\mathrm{log}}} \tag{77}$$

on X generated, as a sheaf of rings, by $\mathcal{O}_X$ and $\mathcal{T}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ subject to the following relations:

- $f_1 * f_2 = f_1 \cdot f_2$;
- $f_1 * \xi_1 = f_1 \cdot \xi_1$;
- $\xi_1 * \xi_2 - \xi_2 * \xi_1 = [\xi_1, \xi_2]$;
- $f_1 * \xi_1 - \xi_1 * f_1 = \xi_1(f_1)$,

for local sections $f_1,\ f_2 \in \mathcal{O}_X$ and $\xi_1,\ \xi_2 \in \mathcal{T}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$, where $*$ denotes the multiplication in $\mathcal{D}^{<\infty}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$. In a usual sense, the *order* (≥ 0) of a given *lcdo* (i.e., a local section of $\mathcal{D}^{<\infty}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$) is well-defined. Hence, $\mathcal{D}^{<\infty}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ admits, for each $j \geq 0$, the subsheaf

$$\mathcal{D}^{<j}_{X^{\mathrm{log}}/S^{\mathrm{log}}} \ \left(\subseteq \mathcal{D}^{<\infty}_{X^{\mathrm{log}}/S^{\mathrm{log}}}\right) \tag{78}$$

consisting of *lcdo*'s of order $< j$. $\mathcal{D}^{<j}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ $(j = 0, 1, 2, \cdots, \infty)$ admits two different structures of $\mathcal{O}_X$-module — one as given by left multiplication (where we denote this $\mathcal{O}_X$-module by ${}^{l}\mathcal{D}^{<j}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$), and the other given by right multiplication (where we denote this $\mathcal{O}_X$-module by ${}^{r}\mathcal{D}^{<j}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$) —. In particular, we have

$$\mathcal{D}^{<0}_{X^{\mathrm{log}}/S^{\mathrm{log}}} = 0 \ \text{ and } \ {}^{l}\mathcal{D}^{<1}_{X^{\mathrm{log}}/S^{\mathrm{log}}} = {}^{r}\mathcal{D}^{<1}_{X^{\mathrm{log}}/S^{\mathrm{log}}} = \mathcal{O}_X. \tag{79}$$

The set $\{\mathcal{D}^{<j}_{X^{\mathrm{log}}/S^{\mathrm{log}}}\}_{j\geq 0}$ forms an increasing filtration on $\mathcal{D}^{<\infty}_{X^{\mathrm{log}}/S^{\mathrm{log}}}$ satisfying that

$$\bigcup_{j\geq 0} \mathcal{D}^{<j}_{X^{\mathrm{log}}/S^{\mathrm{log}}} = \mathcal{D}^{<\infty}_{X^{\mathrm{log}}/S^{\mathrm{log}}}, \ \text{ and } \ \mathcal{D}^{<j+1}_{X^{\mathrm{log}}/S^{\mathrm{log}}}/\mathcal{D}^{<j}_{X^{\mathrm{log}}/S^{\mathrm{log}}} \xrightarrow{\sim} \mathcal{T}^{\otimes j}_{X^{\mathrm{log}}/S^{\mathrm{log}}} \tag{80}$$

for any $j \geq 0$.

Let $\mathcal{F}$ be a vector bundle on X, and consider the tensor product $\mathcal{D}^{<j}_{X^{\log}/S^{\log}} \otimes \mathcal{F}$ of $\mathcal{F}$ with the $\mathcal{O}_X$-module ${}^r\mathcal{D}^{<j}_{X^{\log}/S^{\log}}$. In the following, we shall regard the $\mathcal{D}^{<j}_{X^{\log}/S^{\log}} \otimes \mathcal{F}$ as being equipped with a structure of $\mathcal{O}_X$-module arising from the structure of $\mathcal{O}_X$-module ${}^l\mathcal{D}^{<j}_{X^{\log}/S^{\log}}$ on $\mathcal{D}^{<j}_{X^{\log}/S^{\log}}$.

Next, $\nabla_{\mathcal{F}}$ be an S-log connection on $\mathcal{F}$. One may associate $\nabla_{\mathcal{F}}$ with a structure of left $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$-module

$$\nabla^{\mathcal{D}} : \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{F} \to \mathcal{F}, \tag{81}$$

(which is $\mathcal{O}_X$-linear) on $\mathcal{F}$ determined uniquely by the condition that $\nabla^{\mathcal{D}}(\partial \otimes v) = \langle \nabla(v), \partial \rangle$ for any local sections $v \in \mathcal{F}$ and $\partial \in \mathcal{T}_{X^{\log}/S^{\log}}$, where $\langle -, - \rangle$ denotes the pairing $(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}) \times \mathcal{T}_{X^{\log}/S^{\log}} \to \mathcal{F}$ induced by the natural paring $\mathcal{T}_{X^{\log}/S^{\log}} \times \Omega_{X^{\log}/S^{\log}} \to \mathcal{O}_X$. This assignment $\nabla \mapsto \nabla^{\mathcal{D}}$ determines a bijective correspondence between the set of S-log connections on $\mathcal{F}$ and the set of structures of left $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$-module $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{F} \to \mathcal{F}$ on $\mathcal{F}$.

3.7. Let $\mathcal{B}$ be an arbitrary line bundle on X, and recall the p-determinant data $\mathbb{U}^{\mathrm{can}}_{\mathcal{B}} := (\mathcal{B}, \nabla^{\mathrm{can}}_{0,\mathcal{B}})$ constructed in § 2.3. Then, one may construct a canonical $(\mathrm{GL}_p, \mathbb{U}^{\mathrm{can}}_{\mathcal{B}})$-oper as follows.

First, observe that there exists an $\mathcal{O}_X$-linear morphism

$$\Psi : \mathcal{T}^{\otimes p}_{X^{\log}/S^{\log}} \otimes \mathcal{B} \to \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B} \tag{82}$$

determined uniquely by assigning $\partial^{\otimes p} \otimes b \mapsto (\partial^p - \partial^{(p)}) \otimes b$ for any local sections $\partial \in \mathcal{T}_{X^{\log}/S^{\log}}$, $b \in \mathcal{B}$. We shall write

$$\mathcal{D}^{\Psi}_{\mathcal{B}} \tag{83}$$

for the quotient of the left $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$-module $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B}$ by the $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$-submodule generated by the image of Ψ.

The structure of left $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$-module on $\mathcal{D}^{\Psi}_{\mathcal{B}}$ corresponds (cf. § 3.6) to an S-log connection

$$\nabla_{\mathcal{D}^{\Psi}_{\mathcal{B}}} : \mathcal{D}^{\Psi}_{\mathcal{B}} \to \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{D}^{\Psi}_{\mathcal{B}}. \tag{84}$$

Next, let us write $\mathcal{D}_{\mathcal{B}}^{\Psi,j}$ for the image of $\mathcal{D}_{X^{\log}/S^{\log}}^{<p-j} \otimes \mathcal{B}$ via the natural surjection $\mathcal{D}_{X^{\log}/S^{\log}}^{<\infty} \otimes \mathcal{B} \twoheadrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi}$. $\{\mathcal{D}_{\mathcal{B}}^{\Psi,j}\}_{j=0}^{p}$ forms a p-step decreasing filtration

$$0 = \mathcal{D}_{\mathcal{B}}^{\Psi,p} \subseteq \mathcal{D}_{\mathcal{B}}^{\Psi,p-1} \subseteq \cdots \subseteq \mathcal{D}_{\mathcal{B}}^{\Psi,0} = \mathcal{D}_{\mathcal{B}}^{\Psi} \tag{85}$$

on $\mathcal{D}_{\mathcal{B}}^{\Psi}$ by vector bundles on X.

Finally, denote by

$$\eta_{\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}} : \mathcal{B} \xrightarrow{\sim} \mathcal{D}_{\mathcal{B}}^{\Psi,p-1} \tag{86}$$

the isomorphism obtained by restricting the surjection $\mathcal{D}_{X^{\log}/S^{\log}}^{<\infty} \otimes \mathcal{B} \twoheadrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi}$ to $\mathcal{D}_{X^{\log}/S^{\log}}^{<1} \otimes \mathcal{B}$ $(= \mathcal{B})$ (cf. (79)).

Then, the following proposition holds.

Proposition 3.7.1. *The collection of data*

$$\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit} := (\mathcal{D}_{\mathcal{B}}^{\Psi}, \nabla_{\mathcal{D}_{\mathcal{B}}^{\Psi}}, \{\mathcal{D}_{\mathcal{B}}^{\Psi,j}\}_{j=0}^{p}, \eta_{\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}}) \tag{87}$$

forms a dormant $(\mathrm{GL}_p, \mathbb{U}_{\mathcal{B}}^{\mathrm{can}})$*-oper on* $\mathfrak{X}_{/S}$ *of exponent* $(\mathbb{F}_p, \mathbb{F}_p, \cdots, \mathbb{F}_p)$.

Proof. One may verifies, by the various definitions involved, that $\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}$ is a dormant $(\mathrm{GL}_p, \mathbb{U}_{\mathcal{B}}^{\mathrm{can}})$-oper on $\mathfrak{X}_{/S}$. The remaining portion (i.e., $\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}$ is of exponent $(\mathbb{F}_p, \mathbb{F}_p, \cdots, \mathbb{F}_p)$) follows from Proposition 3.3.8 (since a subset of $\mathbb{F}_p$ with cardinality p coincides with $\mathbb{F}_p$ itself). □

Proposition 3.7.2. *Let us identify the p-determinant data* $\mathbb{U}_{\mathcal{B}^\star}^{\mathrm{can}}$ *with* $(\mathbb{U}_{\mathcal{B}}^{\mathrm{can}})^{\triangledown}$ *via the isomorphism asserted in Proposition 2.5.3. Then, there exists a canonical isomorphism*

$$\mathfrak{D}_{\mathcal{B}^\star}^{\Psi\diamondsuit} \xrightarrow{\sim} (\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit})^{\triangledown} \tag{88}$$

of $(\mathrm{GL}_p, (\mathbb{U}_{\mathcal{B}}^{\mathrm{can}})^{\triangledown})$*-opers, where*

$$(\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit})^{\triangledown} := (\mathcal{D}_{\mathcal{B}}^{\Psi\vee}, \nabla_{\mathcal{D}_{\mathcal{B}}^{\Psi}}^{\vee}, \{(\mathcal{D}_{\mathcal{B}}^{\Psi\vee})^{j}\}_{j=0}^{p}, \eta_{\mathfrak{D}_{\mathcal{B}}^{\Psi}}^{\triangledown}) \tag{89}$$

denotes the dual $(\mathrm{GL}_p, (\mathbb{U}_{\mathcal{B}}^{\mathrm{can}})^{\triangledown})$*-oper of* $\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}$ *(cf. § 3.5).*

Proof. Consider the $\mathcal{O}_X$-linear composite

$$\begin{aligned}\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B}^{\bigstar} &\xrightarrow{\sim} \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes (\mathcal{D}^{\Psi\vee}_{\mathcal{B}})^{p-1} \\ &\hookrightarrow \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{D}^{\Psi\vee}_{\mathcal{B}} \to \mathcal{D}^{\Psi\vee}_{\mathcal{B}},\end{aligned} \tag{90}$$

where

- the first arrow denotes the tensor product of the identity map of $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$ and the composite isomorphism

$$\mathcal{B}^{\bigstar} \xrightarrow{\sim} (\mathcal{T}^{\otimes(p-1)}_{X^{\log}/S^{\log}} \otimes \mathcal{B})^{\vee} \xrightarrow{\sim} (\mathcal{D}^{\Psi,0}_{\mathcal{B}}/\mathcal{D}^{\Psi,1}_{\mathcal{B}})^{\vee} \xrightarrow{\sim} (\mathcal{D}^{\Psi\vee}_{\mathcal{B}})^{p-1} \tag{91}$$

  arising from the various definitions involved;

- the second arrow denotes the natural inclusion;

- the third arrow denotes the morphism defining the structure of left $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$-module on $\mathcal{D}^{\Psi\vee}_{\mathcal{B}}$ corresponding to $\nabla^{\vee}_{\mathcal{D}^{\Psi}_{\mathcal{B}}}$ (cf. § 3.6).

Since $(\mathcal{D}^{\Psi\vee}_{\mathcal{B}}, \nabla^{\vee}_{\mathcal{D}^{\Psi}_{\mathcal{B}}})$ has vanishing p-curvature, this composite turns out to factor through the quotient $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B}^{\bigstar} \twoheadrightarrow \mathcal{D}^{\Psi}_{\mathcal{B}^{\bigstar}}$. The resulting morphism

$$\alpha_{\mathcal{B}} : \mathcal{D}^{\Psi}_{\mathcal{B}^{\bigstar}} \to \mathcal{D}^{\Psi\vee}_{\mathcal{B}}, \tag{92}$$

determines an isomorphism of $(\mathrm{GL}_p, (\mathbb{U}^{\mathrm{can}}_{\mathcal{B}})^{\triangledown})$-opers from $\mathfrak{D}^{\Psi\diamondsuit}_{\mathcal{B}^{\bigstar}}$ to $(\mathfrak{D}^{\Psi}_{\mathcal{B}})^{\triangledown}$, as desired. □

4. Duality for Dormant GL_n-Opers

In this section , we discuss duality between dormant GL_n-opers and dormant GL_{p-n}-opers. Let S, $\mathfrak{X}_{/S}$, n, $\mathbb{U}$ be as before. Suppose further that $n < p$ and $\mathbb{U}$ is dormant. (It follows from Proposition 2.1.5 that such a $\mathbb{U}$ necessarily exists.)

4.1. Let us consider a procedure for constructing a dormant $\mathrm{GL}_{(p-n)}$-oper by means of a dormant GL_n-oper.

Let $\vec{\tau} := (\tau_i)_{i=1}^r$ be an element of $2^{\mathbb{F}_p}_{\sharp(-)=n}$ (where $\vec{\tau} := \emptyset$ if $r = 0$) and $\mathfrak{F}^\diamondsuit := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^\diamondsuit})$ a dormant $(\mathrm{GL}_n, \mathbb{U})$-oper on $\mathfrak{X}_{/S}$ of exponent $\vec{\tau}$. Consider the composite

$$\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B} \overset{\mathrm{id}\otimes\eta_{\mathfrak{F}^\diamondsuit}}{\to} \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{F}^{n-1} \hookrightarrow \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{F} \to \mathcal{F}, \tag{93}$$

where the second arrow denotes the natural inclusion and the third arrow denotes the morphism defining the structure of left $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}}$-module on $\mathcal{F}$ corresponding to $\nabla_\mathcal{F}$. Since $(\mathcal{F}, \nabla_\mathcal{F})$ has vanishing p-curvature, this composite turns out to factor through the natural surjection $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B} \twoheadrightarrow \mathcal{D}^\Psi_\mathcal{B}$. We denote the resulting morphism by

$$\widetilde{\eta}_{\mathfrak{F}^\diamondsuit} : \mathcal{D}^\Psi_\mathcal{B} \to \mathcal{F}, \tag{94}$$

which is surjective and compatible with the respective S-log connections $\nabla_{\mathcal{D}^\Psi_\mathcal{B}}$ and $\nabla_\mathcal{F}$. In particular, by restricting $\nabla_{\mathcal{D}^\Psi_\mathcal{B}}$, one may construct an S-log connection

$$\nabla_{\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})} : \mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}) \to \Omega_{X^{\log}/S^{\log}} \otimes \mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}) \tag{95}$$

on $\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})$. Moreover, for each $j = 0, \cdots, p$, we shall write

$$\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^j := \mathcal{D}^{\Psi,j}_\mathcal{B} \cap \mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}). \tag{96}$$

The inclusions $\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^j \hookrightarrow \mathcal{D}^{\Psi,j}_\mathcal{B}$ of the cases where $j = p-n-1$ and $p-n$ give rise to a composite

$$\begin{aligned} \eta^\triangleright_{\mathfrak{F}^\diamondsuit} : \mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^{(p-n)-1} &\twoheadrightarrow \mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^{(p-n)-1}/\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^{p-n} \\ &\hookrightarrow \mathcal{D}^{\Psi,(p-n)-1}_\mathcal{B}/\mathcal{D}^{\Psi,p-n}_\mathcal{B} \\ &\overset{\sim}{\to} \mathcal{D}^{\Psi,p-1}_\mathcal{B} \otimes \mathcal{T}^{\otimes n}_{X^{\log}/S^{\log}} \\ &= \mathcal{B}^{\triangleright,n} \end{aligned} \tag{97}$$

(cf. (47) for the definition of $\mathcal{B}^{\triangleright,n}$), where the third arrow denotes the isomorphism (60) of the case where the GL_n-oper is taken to be the underlying GL_n-oper $\mathfrak{D}^{\Psi\diamondsuit\heartsuit}_\mathcal{B}$ of $\mathfrak{D}^{\Psi\diamondsuit}_\mathcal{B}$. Thus, we obtain a collection of data

$$\mathfrak{F}^{\diamondsuit\triangleright} := (\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}), \nabla_{\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})}, \{\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^j\}_{j=0}^{p-n}, \eta^\triangleright_{\mathfrak{F}^\diamondsuit}). \tag{98}$$

PROPOSITION 4.1.1. *$\mathfrak{F}^{\diamondsuit\triangleright}$ is a dormant $(\mathrm{GL}_{(p-n)}, \mathbb{U}^{\triangleright})$-oper on $\mathfrak{X}_{/S}$ of exponent $\vec{\tau}^{\triangleright}$ (cf. (49) for the definition of $\mathbb{U}^{\triangleright}$ and (11) for the definition of $\vec{\tau}^{\triangleright}$).*

PROOF. First, let us prove the claim that *the natural composite*

$$\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}}) \hookrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi} \twoheadrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi}/\mathcal{D}_{\mathcal{B}}^{\Psi,p-n} \tag{99}$$

is an isomorphism. To this end, (since both $\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})$ and $\mathcal{D}_{\mathcal{B}}^{\Psi}/\mathcal{D}_{\mathcal{B}}^{\Psi,p-n}$ are flat over S) we may assume, by considering various fibers over S, that $S = \mathrm{Spec}(k')$ for an algebraically closed field k' over k. Write

$$\mathrm{gr}^j := \mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})^j/\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})^{j+1} \tag{100}$$

$(j = 0, \cdots, p-1)$. The inclusion $\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}}) \hookrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi}$ yields an inclusion

$$\mathrm{gr}^j \hookrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi,j}/\mathcal{D}_{\mathcal{B}}^{\Psi,j+1} \tag{101}$$

into the line bundle $\mathcal{D}_{\mathcal{B}}^{\Psi,j}/\mathcal{D}_{\mathcal{B}}^{\Psi,j+1}$. If $x_1, \cdots, x_L$ denote the generic points of irreducible components of X, then the stalk $\mathrm{gr}^j_{x_l}$ of gr^j at x_l $(l = 1, \cdots, L)$ is either trivial or free of rank one. In particular, since the stalk of $\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})$ at x_l is free of rank n, the cardinality of the set $I_l := \{j \mid \mathrm{gr}^j_{x_l} \neq 0\}$ is exactly n. Here, recall that the inclusion $\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}}) \hookrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi}$ is tautologically compatible with the respective k'-log connections $\nabla_{\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})}$ and $\nabla_{\mathcal{D}_{\mathcal{B}}^{\Psi}}$. Thus, it follows from Proposition 3.7.1 that $\mathrm{gr}^{j+1}_{x_l} \neq 0$ implies $\mathrm{gr}^j_{x_l} \neq 0$. But this implies that $I_l = \{0, 1, \cdots, n-1\}$, and hence, the composite (99) is an isomorphism at every generic point in X. Let us observe two sequences of equalities

$$\begin{aligned} &\deg(\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}}) \\ &= \deg(\mathcal{D}_{\mathcal{B}}^{\Psi}) - \deg(\mathcal{F}) \\ &= p(p-1)(1-g) + p \cdot \deg(\mathcal{B}) - n(n-1)(1-g) - n \cdot \deg(\mathcal{B}) \\ &= (p-n)(p+n-1+\deg(\mathcal{B})) \end{aligned} \tag{102}$$

(where the second equality follows from (62)) and

$$
\begin{aligned}
(103)\qquad & \deg(\mathcal{D}_{\mathcal{B}}^{\Psi}/\mathcal{D}_{\mathcal{B}}^{\Psi,p-n}) \\
&= \deg(\mathcal{D}_{\mathcal{B}}^{\Psi}) - \sum_{j=0}^{n-1} \deg(\mathcal{D}_{\mathcal{B}}^{\Psi,p-j-1}/\mathcal{D}_{\mathcal{B}}^{\Psi,p-j}) \\
&= \deg(\mathcal{D}_{\mathcal{B}}^{\Psi}) - \sum_{j=0}^{n-1} (\mathcal{T}_{X^{\log}/S^{\log}}^{\otimes j} \otimes \mathcal{B}) \\
&= p(p-1)(1-g) + p\cdot\deg(\mathcal{B}) - \sum_{j=0}^{n-1}(j(2g-2)+\deg(\mathcal{B})) \\
&= p(p-1)(1-g) + p\cdot\deg(\mathcal{B}) - n(n-1)(g-1) - n\cdot\deg(\mathcal{B}) \\
&= (p-n)(p+n-1+\deg(\mathcal{B})).
\end{aligned}
$$

By comparing $\deg(\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}}))$ with $\deg(\mathcal{D}_{\mathcal{B}}^{\Psi}/\mathcal{D}_{\mathcal{B}}^{\Psi,p-n})$, we conclude that the composite (99) is an isomorphism of $\mathcal{O}_X$-modules. This completes the proof of the claim.

By virtue of the claim, the morphism (101) for any j and the composite $\eta_{\mathcal{F}}^{\triangleright}$ (cf. (97)) turn out to be isomorphisms. Hence, (since $\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit\heartsuit}$ is a GL_p-oper) the morphism $\mathrm{gr}^{j+1} \to \Omega_{X^{\log}/S^{\log}} \otimes \mathrm{gr}^j$ induced naturally by $\nabla_{\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})}$ (obtained in the same manner as (58)) is an isomorphism. Moreover, by the definition of $\nabla_0^{\triangleright}$ (cf. § 2.5), the composite isomorphism

$$
\begin{aligned}
(104)\qquad \det(\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})) &\xrightarrow{\sim} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{(p-n)(p-n-1)}{2}} \otimes (\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}}))^{(p-n)-1})^{\otimes(p-n)} \\
&\xrightarrow{\mathrm{id}\otimes(\eta_{\mathfrak{F}^{\diamondsuit}}^{\triangleright})^{\otimes n}} \mathcal{T}_{X^{\log}/S^{\log}}^{\otimes\frac{(p-n)(p-n-1)}{2}} \otimes (\mathcal{B}^{\triangleright})^{\otimes(p-n)}
\end{aligned}
$$

obtained in the same manner as (69) is compatible with the respective S-log connections $\det(\nabla_{\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})})$ and $\nabla_0^{\triangleright}$. Finally, observe that since $\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}$ is dormant (cf. Proposition 3.7.1), $\nabla_{\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^{\diamondsuit}})}$ has vanishing p-curvature.

Consequently, the collection of data $\mathfrak{F}^{\diamondsuit\triangleright}$ forms a dormant $(\mathrm{GL}_{(p-n)}, \mathbb{U}^{\triangleright})$-oper, and we complete the proof of Proposition 4.1.1. □

4.2. Consequently, by taking account of the discussions in § 3.5 and § 4.1, we obtain a dormant $(\mathrm{GL}_{(p-n)}, \mathbb{U}^{\bigstar})$-oper

$$
(105)\qquad \mathfrak{F}^{\diamondsuit\bigstar} := (\mathfrak{F}^{\diamondsuit\triangleright})^{\triangledown}
$$

on $\mathfrak{X}_{/S}$ of exponent $\vec{\tau}^\star$. The assignment $(-)^\star$ (i.e., $\mathfrak{F}^\diamondsuit \mapsto \mathfrak{F}^{\diamondsuit\star}$ for each dormant $(\mathrm{GL}_n, \mathbb{U})$-oper $\mathfrak{F}^\diamondsuit$) is compatible with both base-changing $s'^*(-)$ (cf. (70)) via any morphism $s' : S' \to S$ of k-schemes and tensoring $(-)_{\otimes \mathfrak{L}}$ (cf. § 3.4) with any log integrable line bundle $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$. More precisely, there exist a natural isomorphism

$$s'^*(\mathfrak{F}^{\diamondsuit\star}) \xrightarrow{\sim} (s'^*(\mathfrak{F}^\diamondsuit))^\star \tag{106}$$

of $(\mathrm{GL}_n, s'^*(\mathbb{U}))$-opers and a natural isomorphism

$$(\mathfrak{F}^{\diamondsuit\star})_{\otimes \mathfrak{L}^\vee} \xrightarrow{\sim} (\mathfrak{F}^\diamondsuit_{\otimes \mathfrak{L}})^\star \tag{107}$$

of $(\mathrm{GL}_n, \mathbb{U}^\star \otimes \mathfrak{L}^\vee)$-opers, where we identify $\mathbb{U}^\star \otimes \mathfrak{L}^\vee$ with $(\mathbb{U} \otimes \mathfrak{L})^\star$ via the isomorphism asserted in Proposition 2.5.2.

PROPOSITION 4.2.1. *Let us identify the n-determinant data $(\mathbb{U}^\star)^\star$ with $\mathbb{U}$ via the isomorphism asserted in Proposition 2.5.1. Then, there exists a natural isomorphism*

$$(\mathfrak{F}^{\diamondsuit\star})^\star \xrightarrow{\sim} \mathfrak{F}^\diamondsuit \tag{108}$$

of $(\mathrm{GL}_n, \mathbb{U})$-opers.

PROOF. For simplicity, we shall write

$$\mathfrak{F}^{\diamondsuit\triangleright} =: (\mathcal{E}, \nabla_{\mathcal{E}}, \{\mathcal{E}^j\}_{j=0}^n, \eta_{\mathfrak{E}^\diamondsuit}), \tag{109}$$

which is a dormant $(\mathrm{GL}_{(p-n)}, \mathbb{U}^\triangleright)$-oper on $\mathfrak{X}_{/S}$. The natural inclusion $\mathfrak{inc} : \mathcal{E} \hookrightarrow \mathcal{D}^\Psi_{\mathcal{B}}$ induces, via the respective quotients, an isomorphism $\overline{\mathfrak{inc}} : \mathcal{E}/\mathcal{E}^1 \xrightarrow{\sim} \mathcal{D}^\Psi_{\mathcal{B}}/\mathcal{D}^{\Psi,1}_{\mathcal{B}}$ (cf. the discussion in the proof of Proposition 4.1.1, which assert that the morphism (101) is an isomorphism). One verifies that the composite

$$\mathcal{B}^\star \xrightarrow{\sim} \Omega^{\otimes p}_{X^{\log}/S^{\log}} \otimes \mathcal{B}^\vee \xrightarrow{\sim} (\mathcal{D}^\Psi_{\mathcal{B}}/\mathcal{D}^{\Psi,1}_{\mathcal{B}})^\vee \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}^1)^\vee = (\mathcal{E}^\vee)^{(p-n)-1} \tag{110}$$

coincides with $\eta^\triangledown_{\mathfrak{E}^\diamondsuit}$, where

- the first arrow denotes the first isomorphism in (52);
- the second arrow denotes the dual of the isomorphism (60) of the case where the triple "$(\mathfrak{F}^\heartsuit, n, j)$" is taken to be "$(\mathfrak{D}^{\Psi\diamondsuit\heartsuit}_{\mathcal{B}}, p, 0)$";

- the third arrow denotes the dual of $\overline{\mathfrak{inc}}$.

On the other hand, this composite may be naturally considered as the restriction to $\mathcal{D}^{\Psi,p-1}_{\mathcal{B}^\bigstar}$ of the composite $\mathfrak{inc}^\vee \circ \alpha_{\mathcal{B}}$ of the isomorphism $\alpha_{\mathcal{B}} : \mathcal{D}^{\Psi}_{\mathcal{B}^\bigstar} \xrightarrow{\sim} \mathcal{D}^{\Psi\vee}_{\mathcal{B}}$ asserted in Proposition 3.7.2 (cf. (92)) and the dual $\mathfrak{inc}^\vee : \mathcal{D}^{\Psi\vee}_{\mathcal{B}} \twoheadrightarrow \mathcal{E}^\vee$ of $\mathfrak{inc}$. Thus, it follows from Proposition 3.1.3 that $\widetilde{\eta}^{\triangledown}_{\mathfrak{E}^\diamondsuit} = \mathfrak{inc}^\vee \circ \alpha_{\mathcal{B}}$. In particular, by taking the kernels of both $\widetilde{\eta}^{\triangledown}_{\mathfrak{E}^\diamondsuit}$ and $\mathfrak{inc}^\vee$, we have a canonical isomorphism

$$((\mathfrak{F}^{\diamondsuit\triangleright})^{\triangledown})^{\triangleright} \xrightarrow{\sim} (\mathfrak{F}^{\diamondsuit})^{\triangledown} \tag{111}$$

of GL_n-opers. This isomorphism induces a sequence of isomorphism

$$(\mathfrak{F}^{\diamondsuit\bigstar})^{\bigstar} = (((\mathfrak{F}^{\diamondsuit\triangleright})^{\triangledown})^{\triangleright})^{\triangledown} \xrightarrow{\sim} ((\mathfrak{F}^{\diamondsuit})^{\triangledown})^{\triangledown} \xrightarrow{\sim} \mathfrak{F}^{\diamondsuit}, \tag{112}$$

where the last isomorphism follows from (76). The completes the proof of Proposition 4.2.1. $\square$

4.3. We shall write $\mathfrak{Set}$ for the category of (small) sets. Also, for a scheme (or, more generally, a stack) S over k, write $\mathfrak{Sch}_{/S}$ for the category of relative S-schemes. Let us consider the $\mathfrak{Set}$-valued functor

$$\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\mathfrak{X}_{/S}} \left(\text{resp.}, \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\tau,\mathfrak{X}_{/S}}\right) : \mathfrak{Sch}_{/S} \to \mathfrak{Set} \tag{113}$$

on $\mathfrak{Sch}_{/S}$ which, to any S-schemes $t : T \to S$, assigns the set of isomorphism classes of dormant $(\mathrm{GL}_n, t^*(\mathbb{U}))$-opers on $\mathfrak{X}_{/T}$ (resp., dormant $(\mathrm{GL}_n, t^*(\mathbb{U}))$-opers on $\mathfrak{X}_T$ of exponent $\vec{\tau}$) (cf. Remark 2.1.2). As explained later (cf. § 6.2), both $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\mathfrak{X}_{/S}}$ and $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\tau,\mathfrak{X}_{/S}}$ may be represented by relative finite S-schemes. By virtue of Proposition 3.3.8, the relative S-scheme $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\mathfrak{X}_{/S}}$ decomposes into the disjoint union

$$\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\mathfrak{X}_{/S}} = \coprod_{\vec{\tau} \in (2^{\mathbb{F}_p}_{\sharp(-)=n})^{\times r}} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}}. \tag{114}$$

Theorem 4.3.1. *The assignment $\mathfrak{F}^{\diamondsuit} \mapsto \mathfrak{F}^{\diamondsuit\bigstar}$ constructed in § 4.2 defines a canonical isomorphism*

$$\Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}} : \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_{(p-n)},\mathbb{U}^\bigstar,\vec{\tau}^\bigstar,\mathfrak{X}_{/S}} \tag{115}$$

over S satisfying the equality

$$\Theta^{\bigstar}_{\mathrm{GL}_{(p-n)},\mathbb{U}^{\bigstar},\vec{\tau}^{\bigstar},\mathfrak{X}_{/S}} \circ \Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}} = \mathrm{id} \tag{116}$$

of automorphisms of $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathbb{U},\mathfrak{X}_{/S}}$.

PROOF. The assertion follows from the functoriality of the assignment $\mathfrak{F}^{\diamondsuit} \mapsto \mathfrak{F}^{\diamondsuit\bigstar}$ (cf. (106)) and Proposition 4.2.1. □

Also, the isomorphism $\Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}}$ just obtained satisfies the following property.

PROPOSITION 4.3.2. *Let $\vec{a}$ be an element of $\mathbb{F}_p^{\times r}$ (where we take $\vec{a} := \emptyset$ if $r = 0$) and $\mathfrak{L} := (\mathcal{L}, \nabla_{\mathcal{L}})$ be a log integrable line bundle on $\mathfrak{X}_{/S}$ of exponent $\vec{a}$. Then, the assignment $\mathfrak{F}^{\diamondsuit} \mapsto \mathfrak{F}^{\diamondsuit}_{\otimes\mathfrak{L}}$ (cf. (72)) determines an isomorphism*

$$\Xi_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S},\mathfrak{L}} : \mathfrak{Op}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}_{\mathrm{GL}_n,\mathbb{U}\otimes\mathfrak{L},\vec{\tau}^{+\vec{a}},\mathfrak{X}_{/S}} \tag{117}$$

over S. Moreover, the square diagram

$$\begin{array}{ccc} \mathfrak{Op}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}} & \xrightarrow{\Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S}}} & \mathfrak{Op}_{\mathrm{GL}_{(p-n)},\mathbb{U}^{\bigstar},\vec{\tau}^{\bigstar},\mathfrak{X}_{/S}} \\ {\scriptstyle \Xi_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S},\mathfrak{L}}}\downarrow & & \downarrow{\scriptstyle \Xi_{\mathrm{GL}_n,\mathbb{U},\vec{\tau},\mathfrak{X}_{/S},\mathfrak{L}}} \\ \mathfrak{Op}_{\mathrm{GL}_n,\mathbb{U}\otimes\mathfrak{L},\vec{\tau}^{+\vec{a}},\mathfrak{X}_{/S}} & \xrightarrow{\alpha\circ\Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U},\vec{\tau}^{+\vec{a}},\mathfrak{X}_{/S}}} & \mathfrak{Op}_{\mathrm{GL}_{(p-n)},\mathbb{U}^{\bigstar}\otimes\mathfrak{L}^{\vee},\vec{\tau}^{\bigstar},\mathfrak{X}_{/S}} \end{array} \tag{118}$$

is commutative, where the lower horizontal arrow denotes the composite of $\Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U}\otimes\mathfrak{L},\vec{\tau}^{+\vec{a}},\mathfrak{X}_{/S}}$ *and the isomorphism*

$$\alpha : \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_{(p-n)},(\mathbb{U}\otimes\mathfrak{L})^{\bigstar},(\vec{\tau}^{+\vec{a}})^{\bigstar},\mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_{(p-n)},\mathbb{U}^{\bigstar}\otimes\mathfrak{L}^{\vee},(\vec{\tau}^{\bigstar})^{-\vec{a}},\mathfrak{X}_{/S}} \tag{119}$$

arising, in an evident fashion, from both the equality $(\vec{\tau}^{+\vec{a}})^{\bigstar} = (\vec{\tau}^{\bigstar})^{-\vec{a}}$ (cf. (13)) and the isomorphism $(\mathbb{U} \otimes \mathfrak{L})^{\bigstar} \xrightarrow{\sim} \mathbb{U}^{\bigstar} \otimes \mathfrak{L}^{\vee}$ asserted in Proposition 2.5.2.

PROOF. The former assertion follows from the discussion in § 3.4. The latter assertion follows from the isomorphism (107). □

Moreover, the following corollary of Theorem 4.3.1 holds.

COROLLARY 4.3.3. *The structure morphism* $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_{(p-1)},\mathbb{U},\mathfrak{X}_{/S}} \to S$ *of* $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_{(p-1)},\mathbb{U},\mathfrak{X}_{/S}}$ *is an isomorphism. That is to say, there exists a unique (up to isomorphism) dormant* $\mathrm{GL}_{(p-1)}$*-oper on* $\mathfrak{X}_{/S}$.

PROOF. The assertion follows from Theorem 4.3.1 of the case where $n = 1$ and the discussion in Remark 3.3.7. □

5. Duality for Dormant GL_n-Opers on a Proper Smooth Curve

In this section, we shall consider the duality $(-)^\bigstar$ (cf. (105)) for dormant GL_n-opers of the case where the underlying curve is smooth and there are no marked points. In this case, one may describe the assignment $\mathfrak{F}^\diamondsuit \mapsto \mathfrak{F}^{\diamondsuit\bigstar}$ in a manner different from that provided in the previous section.

Suppose that $r = 0$ and X/S is smooth. In particular, the log structures on both X and S defined in § 1.3 are trivial (hence, $\Omega_{X^{\log}/S^{\log}} = \Omega_{X/S}$).

5.1. To begin with, we shall describe the dormant $(\mathrm{GL}_p, \mathbb{U}^{\mathrm{can}}_{\mathcal{B}})$-oper $\mathfrak{D}^{\Psi\diamondsuit}_{\mathcal{B}}$ (cf. (87)) in a manner different from that provided in § 3.7. Let $\mathcal{B}$ be a line bundle on X. We shall write

$$\mathcal{A}_{\mathcal{B}^\vee} := F^*_{X/S}(F_{X/S*}(\mathcal{B}^\vee)). \tag{120}$$

Since the relative Frobenius morphism $F_{X/S} : X \to X^{(1)}_S$ is finite and faithfully flat of degree p, one verifies that $\mathcal{A}_{\mathcal{B}^\vee}$ is a vector bundle on X of rank p. Denote by

$$\pi_{\mathcal{B}^\vee} : \mathcal{A}_{\mathcal{B}^\vee} \ \left(= F^*_{X/S}(F_{X/S*}(\mathcal{B}^\vee))\right) \twoheadrightarrow \mathcal{B}^\vee \tag{121}$$

the surjection determined by the adjunction relation "$F^*_{X/S}(-) \dashv F_{X/S*}(-)$". Then, $\pi_{\mathcal{B}^\vee}$ and the canonical S-connection $\nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^\vee)}$ (cf. Remark 1.6.3) on $\mathcal{A}_{\mathcal{B}^\vee}$ give rise to a p-step decreasing filtration $\{\mathcal{A}^j_{\mathcal{B}^\vee}\}^p_{j=0}$ as follows.

$$\begin{aligned}(122)\qquad &\mathcal{A}^0_{\mathcal{B}^\vee} := \mathcal{A}_{\mathcal{B}^\vee};\\ &\mathcal{A}^1_{\mathcal{B}^\vee} := \mathrm{Ker}(\mathcal{A}_{\mathcal{B}^\vee} \overset{\pi_{\mathcal{B}^\vee}}{\twoheadrightarrow} \mathcal{B}^\vee);\\ &\mathcal{A}^j_{\mathcal{B}^\vee} := \mathrm{Ker}(\mathcal{A}^{j-1}_{\mathcal{B}^\vee} \xrightarrow{\nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^\vee)}|_{\mathcal{A}^{j-1}_{\mathcal{B}^\vee}}} \Omega_{X/S}\otimes\mathcal{A}_{\mathcal{B}^\vee}\\ &\qquad\qquad \twoheadrightarrow \Omega_{X/S}\otimes(\mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^{j-1}_{\mathcal{B}^\vee}))\end{aligned}$$

$(j = 2, \cdots, p)$. In particular, $\pi_{\mathcal{B}^\vee}$ induces an isomorphism

$$(123)\qquad \overline{\pi}_{\mathcal{B}^\vee} : \mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^1_{\mathcal{B}^\vee} \xrightarrow{\sim} \mathcal{B}^\vee.$$

It follows from [20], Proposition 9.3.1 (or, [8], Theorem 3.1.6, for the case where $S = \mathrm{Spec}(\overline{k})$ for an algebraically closed field $\overline{k}$), that the collection of data

$$(124)\qquad \mathfrak{A}^\heartsuit_{\mathcal{B}^\vee} := (\mathcal{A}_{\mathcal{B}^\vee}, \nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^\vee)}, \{\mathcal{A}^j_{\mathcal{B}^\vee}\}^p_{j=0})$$

forms a dormant GL_p-oper on $\mathfrak{X}_{/S}$. We obtain an isomorphism

$$(125)\qquad \Omega^{\otimes(p-1)}_{X/S}\otimes(\mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^1_{\mathcal{B}^\vee}) \xrightarrow{\sim} \mathcal{A}^{p-1}_{\mathcal{B}^\vee}$$

determined by the composite isomorphism (60) of the case where the triple $(\mathfrak{F}^\heartsuit, n, j)$ is taken to be $(\mathfrak{A}^\heartsuit_{\mathcal{B}^\vee}, p, 0)$. Let us define $\eta_{\mathfrak{A}^\heartsuit_{\mathcal{B}^\vee}}$ to be the composite isomorphism

(126)

$$\eta_{\mathfrak{A}^\heartsuit_{\mathcal{B}^\vee}} : \left(\mathcal{B}^{\nabla,p} =\right)\ \Omega^{\otimes(p-1)}_{X/S}\otimes\mathcal{B}^\vee \xrightarrow{\mathrm{id}\otimes\overline{\pi}^{-1}_{\mathcal{B}^\vee}} \Omega^{\otimes(p-1)}_{X/S}\otimes\mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^1_{\mathcal{B}^\vee} \xrightarrow{(125)} \mathcal{A}^{p-1}_{\mathcal{B}^\vee}.$$

Then, the following proposition holds.

PROPOSITION 5.1.1. *The collection of data*

$$(127)\qquad \mathfrak{A}^\diamondsuit_{\mathcal{B}^\vee} := (\mathcal{A}_{\mathcal{B}^\vee}, \nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^\vee)}, \{\mathcal{A}^j_{\mathcal{B}^\vee}\}^p_{j=0}, \eta_{\mathfrak{A}^\heartsuit_{\mathcal{B}^\vee}})$$

forms a dormant $(\mathrm{GL}_p, \mathbb{U}^{\mathrm{can}\nabla}_{\mathcal{B}})$*-oper on* $\mathfrak{X}_{/S}$*. Moreover, there exists a canonical isomorphism*

$$(128)\qquad \gamma_{\mathcal{B}} : \mathfrak{D}^{\Psi\diamondsuit}_{\mathcal{B}} \xrightarrow{\sim} \mathfrak{A}^{\diamondsuit\nabla}_{\mathcal{B}^\vee}$$

of $(\mathrm{GL}_p, \mathbb{U}_{\mathcal{B}}^{\mathrm{can}})$*-opers.*

Proof. The former assertion follows immediately from the definition of $\eta_{\mathfrak{A}^{\heartsuit}_{\mathcal{B}^\vee}}$. We shall prove the latter assertion. Consider the composite

$$(129)\qquad \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B} \overset{\mathrm{id}\otimes \overline{\pi}^{\vee}_{\mathcal{B}^\vee}}{\to} \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes (\mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^1_{\mathcal{B}^\vee})^\vee \hookrightarrow \mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{A}^\vee_{\mathcal{B}^\vee} \overset{\nabla^{\mathrm{can}\vee\mathcal{D}}_{F_{X/S*}(\mathcal{B}^\vee)}}{\to} \mathcal{A}^\vee_{\mathcal{B}^\vee},$$

where the second arrow arises from the natural inclusion $(\mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^1_{\mathcal{B}^\vee})^\vee \hookrightarrow \mathcal{A}^\vee_{\mathcal{B}^\vee}$. Since $\nabla^{\mathrm{can}\vee}_{F_{X/S*}(\mathcal{B}^\vee)}$ has vanishing p-curvature, this composite factors through the quotient $\mathcal{D}^{<\infty}_{X^{\log}/S^{\log}} \otimes \mathcal{B} \twoheadrightarrow \mathcal{D}^{\Psi}_{\mathcal{B}}$. The resulting morphism

$$(130)\qquad \gamma_{\mathcal{B}} : \mathcal{D}^{\Psi}_{\mathcal{B}} \to \mathcal{A}^\vee_{\mathcal{B}^\vee}$$

is surjective and compatible with the respective connections $\nabla_{\mathcal{D}^{\Psi}_{\mathcal{B}}}$ and $\nabla^{\mathrm{can}\vee}_{F_{X/S*}(\mathcal{B}^\vee)}$ (cf. the discussion at the beginning of § 4.1). Since the vector bundles $\mathcal{D}^{\Psi}_{\mathcal{B}}$ and $\mathcal{A}^\vee_{\mathcal{B}^\vee}$ have the same rank p, $\gamma_{\mathcal{B}}$ turns out to be an isomorphism. One verifies that $\gamma_{\mathcal{B}}$ is compatible with the respective filtrations $\{\mathcal{D}^{\Psi,j}_{\mathcal{B}}\}_{j=0}^p$ and $\{(\mathcal{A}^\vee_{\mathcal{B}^\vee})^j\}_{j=0}^p$. Thus, by the definition of $\eta_{\mathfrak{A}^{\heartsuit}_{\mathcal{B}^\vee}}$, $\gamma_{\mathcal{B}}$ forms an isomorphism $\mathfrak{D}^{\Psi\diamondsuit}_{\mathcal{B}} \overset{\sim}{\to} \mathfrak{A}^{\diamondsuit\triangledown}_{\mathcal{B}^\vee}$ of $(\mathrm{GL}_p, \mathbb{U}^{\mathrm{can}}_{\mathcal{B}})$-opers. □

5.2. Now, let us fix an n-determinant data $\mathbb{U} := (\mathcal{B}, \nabla_0)$ (where $\mathcal{B}$ is as above) for $\mathfrak{X}_{/S}$ and a dormant $(\mathrm{GL}_n, \mathbb{U})$-oper $\mathfrak{F}^\diamondsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n, \eta_{\mathfrak{F}^\diamondsuit})$ on $\mathfrak{X}_{/S}$. Since $\nabla^\vee_{\mathcal{F}}$ has vanishing p-curvature, the morphism

$$(131)\qquad F^*_{X/S}(F_{X/S*}(\mathrm{Ker}(\nabla^\vee_{\mathcal{F}}))) \to \mathcal{F}^\vee$$

corresponding, via the adjunction relation "$F^*_{X/S}(-) \dashv F_{X/S*}(-)$", to the natural inclusion $F_{X/S*}(\mathrm{Ker}(\nabla^\vee_{\mathcal{F}})) \hookrightarrow F_{X/S*}(\mathcal{F}^\vee)$ is an isomorphism (cf. [12], § 5, p. 190, Theorem 5.1). Thus, we obtain a composite

$$(132)\qquad \lambda_{\mathfrak{F}^\diamondsuit} : \mathcal{F}^\vee \overset{\sim}{\to} F^*_{X/S}(F_{X/S*}(\mathrm{Ker}(\nabla^\vee_{\mathcal{F}}))) \to F^*_{X/S}(F_{X/S*}(\mathcal{B}^\vee))\ (= \mathcal{A}_{\mathcal{B}^\vee}),$$

where the first arrow denotes the inverse of (131) and the second arrow denotes the pull-back via $F_{X/S}$ of the composite

$$F_{X/S*}(\mathrm{Ker}(\nabla_{\mathcal{F}}^{\vee})) \hookrightarrow F_{X/S*}(\mathcal{F}^{\vee}) \twoheadrightarrow F_{X/S*}(\mathcal{F}^{n-1\vee}) \xrightarrow{F_{X/S*}(\eta^{\vee}_{\mathfrak{F}^{\diamondsuit}})} F_{X/S*}(\mathcal{B}^{\vee}). \tag{133}$$

Next, let us write

$$\mathcal{H} := \mathcal{A}_{\mathcal{B}^{\vee}}/\lambda_{\mathfrak{F}^{\diamondsuit}}(\mathcal{F}^{\vee}). \tag{134}$$

If $\pi_{\mathcal{H}} : \mathcal{A}_{\mathcal{B}^{\vee}} \twoheadrightarrow \mathcal{H}$ denotes the natural surjection, then $\mathcal{H}$ may be equipped with a $(p-n)$-step decreasing filtration $\{\mathcal{H}^j\}_{j=0}^{p-n}$ given by

$$\mathcal{H}^j := \pi_{\mathcal{H}}(\mathcal{A}_{\mathcal{B}^{\vee}}^{n+j}) \quad (j = 0, \cdots, p-n)\,. \tag{135}$$

For each $j \in \{0, \cdots, p-n\}$, $\pi_{\mathcal{H}}$ restricts to a morphism

$$\pi_{\mathcal{H}}^j : \mathcal{A}_{\mathcal{B}^{\vee}}^{n+j} \rightarrow \mathcal{H}^j. \tag{136}$$

On the other hand, since $\lambda_{\mathfrak{F}^{\diamondsuit}} : \mathcal{F}^{\vee} \rightarrow \mathcal{A}_{\mathcal{B}^{\vee}}$ is compatible with the respective S-connections $\nabla_{\mathcal{F}}^{\vee}$ and $\nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^{\vee})}$, $\nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^{\vee})}$ induces, via $\pi_{\mathcal{H}}$, an S-connection

$$\nabla_{\mathcal{H}} : \mathcal{H} \rightarrow \Omega_{X/S} \otimes \mathcal{H} \tag{137}$$

on $\mathcal{H}$. Then, the following lemma holds.

LEMMA 5.2.1. *For each $j \in \{0, \cdots, p-n\}$, the morphism $\pi_{\mathcal{H}}^j$ is an isomorphism. Moreover, the collection of data*

$$\mathfrak{H}^{\heartsuit} := (\mathcal{H}, \nabla_{\mathcal{H}}, \{\mathcal{H}^j\}_{j=0}^{p-n}) \tag{138}$$

forms a dormant $\mathrm{GL}_{(p-n)}$-oper on $\mathfrak{X}_{/S}$.

PROOF. Let us observe the following two facts (a), (b):

(a) The square diagram

$$
\begin{array}{ccc}
\mathcal{F}^\vee & \xrightarrow{\lambda_{\mathfrak{F}^\diamondsuit}} & \mathcal{A}_{\mathcal{B}^\vee} \\
\downarrow & & \downarrow \pi_{\mathcal{B}^\vee} \\
\mathcal{F}^{n-1\vee} & \xrightarrow[\eta^\vee_{\mathfrak{F}^\diamondsuit}]{} & \mathcal{B}^\vee
\end{array} \tag{139}
$$

is commutative, where the left-hand vertical arrow denotes the dual of the inclusion $\mathcal{F}^{n-1} \hookrightarrow \mathcal{F}$. Moreover, the kernels of the left-hand and right-hand vertical arrows coincide with $(\mathcal{F}^\vee)^1$ and $\mathcal{A}^1_{\mathcal{B}^\vee}$ respectively. In particular, (since $\eta^\vee_{\mathfrak{F}^\diamondsuit}$ is an isomorphism) the diagram (139) induces an isomorphism

$$
\mathcal{F}^\vee/(\mathcal{F}^\vee)^1 \xrightarrow{\sim} \mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^1_{\mathcal{B}^\vee}. \tag{140}
$$

(b) For any $j = 2, \cdots, n$, (by the definition of a GL_n-oper) the following equalities holds:

$$
\begin{aligned}
(\mathcal{F}^\vee)^j &= \mathrm{Ker}((\mathcal{F}^\vee)^{j-1} \overset{\nabla^\vee_{\mathcal{F}}|_{(\mathcal{F}^\vee)^{j-1}}}{\rightarrow} \Omega_{X/S} \otimes \mathcal{F}^\vee \\
&\qquad \twoheadrightarrow \Omega_{X/S} \otimes (\mathcal{F}^\vee/(\mathcal{F}^\vee)^{j-1})), \\
\mathcal{A}^j_{\mathcal{B}^\vee} &= \mathrm{Ker}(\mathcal{A}^{j-1}_{\mathcal{B}^\vee} \overset{\nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^\vee)}|_{\mathcal{A}^{j-1}_{\mathcal{B}^\vee}}}{\rightarrow} \Omega_{X/S} \otimes \mathcal{A}_{\mathcal{B}^\vee} \\
&\qquad \twoheadrightarrow \Omega_{X/S} \otimes (\mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^{j-1}_{\mathcal{B}^\vee})).
\end{aligned} \tag{141}
$$

These facts imply, by induction on j, that the morphism $\lambda_{\mathfrak{F}^\diamondsuit}$ restricts to morphisms

$$
(\mathcal{F}^\vee)^j \to \mathcal{A}^j_{\mathcal{B}^\vee} \quad (j = 0, \cdots, n). \tag{142}
$$

The resulting morphisms

$$
\overline{\lambda}^j_{\mathfrak{F}^\diamondsuit} : (\mathcal{F}^\vee)^j/(\mathcal{F}^\vee)^{j+1} \hookrightarrow \mathcal{A}^j_{\mathcal{B}^\vee}/\mathcal{A}^{j+1}_{\mathcal{B}^\vee} \quad (j = 0, \cdots, n-1) \tag{143}
$$

between the respective subquotients of $\mathcal{F}^\vee$ and $\mathcal{A}_{\mathcal{B}^\vee}$ make the square diagrams

$$
\begin{array}{ccc}
(\mathcal{F}^\vee)^j/(\mathcal{F}^\vee)^{j+1} & \xrightarrow{\overline{\lambda}^j_{\mathfrak{F}^\diamondsuit}} & \mathcal{A}^j_{\mathcal{B}^\vee}/\mathcal{A}^{j+1}_{\mathcal{B}^\vee} \\
\mathfrak{ks}^j_{\mathfrak{F}^\heartsuit} \downarrow \wr & & \wr \downarrow \mathfrak{ks}^j_{\mathfrak{A}^\heartsuit_{\mathcal{B}^\vee}} \\
\Omega_{X/S} \otimes ((\mathcal{F}^\vee)^{j-1}/(\mathcal{F}^\vee)^j) & \xrightarrow{\mathrm{id} \otimes \overline{\lambda}^{j-1}_{\mathfrak{F}^\diamondsuit}} & \Omega_{X/S} \otimes (\mathcal{A}^{j-1}_{\mathcal{B}^\vee}/\mathcal{A}^j_{\mathcal{B}^\vee})
\end{array} \tag{144}
$$

$(j = 0, \cdots, n)$ commute. By induction on j, the commutativity of (144) and the isomorphism (140) imply that all the $\overline{\lambda}^j_{\mathfrak{F}^\diamondsuit}$'s are isomorphisms. Hence, the composite $\mathcal{F}^\vee \overset{\lambda_{\mathfrak{F}^\diamondsuit}}{\to} \mathcal{A}_{\mathcal{B}^\vee} \twoheadrightarrow \mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^n_{\mathcal{B}^\vee}$ is an isomorphism, equivalently, $\pi^0_{\mathcal{H}}$ is an isomorphism. It follows that all the $\pi^j_{\mathcal{H}}$'s are isomorphisms. This completes the proof of the former assertion.

Next, let us consider the latter assertion. The morphisms $\pi^j_{\mathcal{H}}$ induce commutative square diagrams

(145)
$$\begin{array}{ccc} \mathcal{A}^{n+j}_{\mathcal{B}^\vee}/\mathcal{A}^{n+j+1}_{\mathcal{B}^\vee} & \longrightarrow & \mathcal{H}^j/\mathcal{H}^{j+1} \\ {\scriptstyle \mathfrak{ks}^j_{\mathfrak{A}^\heartsuit_{\mathcal{B}^\vee}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{ks}^j_{\mathfrak{H}^\heartsuit}} \\ \Omega_{X/S} \otimes (\mathcal{A}^{n+j-1}_{\mathcal{B}^\vee}/\mathcal{A}^{n+j}_{\mathcal{B}^\vee}) & \longrightarrow & \Omega_{X/S} \otimes (\mathcal{H}^{j-1}/\mathcal{H}^j) \end{array}$$

$(j = 0, \cdots, n-p)$, where the upper and lower horizontal arrows are isomorphisms arising from $\pi^j_{\mathcal{H}}$ and $\pi^{j-1}_{\mathcal{H}}$ respectively. Thus, the latter assertion follows from the commutativity of (145) and the fact (cf. Proposition 5.1.1) that $(\mathcal{A}_{\mathcal{B}^\vee}, \nabla^{\mathrm{can}}_{F_{X/S*}(\mathcal{B}^\vee)}, \{\mathcal{A}^j_{\mathcal{B}^\vee}\}^p_{j=0})$ is a GL_p-oper (which implies that the $\mathfrak{ks}^j_{\mathfrak{A}^\heartsuit_{\mathcal{B}^\vee}}$'s are isomorphisms). □

Finally, let us define $\eta_{\mathfrak{H}^\diamondsuit}$ to be the composite isomorphism

(146)
$$\eta_{\mathfrak{H}^\diamondsuit} : \mathcal{B}^\bigstar \overset{(52)}{\to} \Omega^{\otimes(p-1)}_{X/S} \otimes \mathcal{B}^\vee \overset{\mathrm{id}\otimes\overline{\pi}^{-1}_{\mathcal{B}^\vee}}{\to} \Omega^{\otimes(p-1)}_{X/S} \otimes (\mathcal{A}_{\mathcal{B}^\vee}/\mathcal{A}^1_{\mathcal{B}^\vee}) \overset{(125)}{\to} \mathcal{A}^{p-1}_{\mathcal{B}^\vee} \overset{\pi^{n-p-1}_{\mathcal{H}}}{\to} \mathcal{H}^{n-p-1}.$$

Then, by the following proposition, the assignment $\mathfrak{F}^\diamondsuit \mapsto \mathfrak{F}^{\diamondsuit\bigstar}$ may be identified with the assignment $\mathfrak{F}^\diamondsuit \mapsto \mathfrak{H}^\diamondsuit$ (cf. (147)).

PROPOSITION 5.2.2. *The collection of data*

(147)
$$\mathfrak{H}^\diamondsuit := (\mathcal{H}, \nabla_{\mathcal{H}}, \{\mathcal{H}^j\}^{p-n}_{j=0}, \eta_{\mathfrak{H}^\diamondsuit})$$

forms a dormant $(\mathrm{GL}_{p-n}, \mathbb{U}^\bigstar)$*-oper on* $\mathfrak{X}_{/S}$ *and is isomorphic to* $\mathfrak{F}^{\diamondsuit\bigstar}$*.*

PROOF. Denote by $\iota_{\mathfrak{D}^{\Psi\diamondsuit}_{\mathcal{B}}} : \mathcal{B} \hookrightarrow \mathcal{D}^{\Psi}_{\mathcal{B}}$ (resp., $\iota_{\mathfrak{F}^\diamondsuit} : \mathcal{B} \hookrightarrow \mathcal{F}$) the injection defined to be the composite of $\eta_{\mathcal{D}^{\Psi\diamondsuit}_{\mathcal{B}}} : \mathcal{B} \overset{\sim}{\to} \mathcal{D}^{\Psi,p-1}_{\mathcal{B}}$ (resp., $\eta_{\mathfrak{F}^\diamondsuit} : \mathcal{B} \overset{\sim}{\to}$

$\mathcal{F}^{n-1}$) and the inclusion $\mathcal{D}_{\mathcal{B}}^{\Psi, p-1} \hookrightarrow \mathcal{D}_{\mathcal{B}}^{\Psi}$ (resp., $\mathcal{F}^{n-1} \hookrightarrow \mathcal{F}$). The following equalities hold:

$$\widetilde{\eta}_{\mathfrak{F}^\diamondsuit} \circ \iota_{\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}} = \iota_{\mathfrak{F}^\diamondsuit} = \lambda^\vee_{\mathfrak{F}^\diamondsuit} \circ \pi^\vee_{\mathcal{B}^\vee} = \lambda^\vee_{\mathfrak{F}^\diamondsuit} \circ \gamma_{\mathcal{B}} \circ \iota_{\mathfrak{D}_{\mathcal{B}}^{\Psi\diamondsuit}}. \tag{148}$$

Hence, it follows from Proposition 3.1.3 that (since both $\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}$ and $\lambda^\vee_{\mathfrak{F}^\diamondsuit} \circ \gamma$ are compatible with the respective connections $\nabla_{\mathcal{D}_{\mathcal{B}}^{\Psi}}$ and $\nabla_{\mathcal{F}}$) the diagram

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{B}}^{\Psi} & \xrightarrow[\sim]{\gamma_{\mathcal{B}}} & \mathcal{A}^\vee_{\mathcal{B}^\vee} \\ & {\scriptstyle \widetilde{\eta}_{\mathfrak{F}^\diamondsuit}} \searrow \quad \swarrow {\scriptstyle \lambda^\vee_{\mathfrak{F}^\diamondsuit}} & \\ & \mathcal{F} & \end{array} \tag{149}$$

is commutative. This commutative diagram induces an isomorphism

$$(\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}), \nabla_{\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}}) \xrightarrow{\sim} (\mathcal{H}^\vee, \nabla^\vee_{\mathcal{H}}) \tag{150}$$

of (log) integrable vector bundles. It follows from the various definitions involved that the dual isomorphism

$$\delta : (\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^\vee, \nabla^\vee_{\widetilde{\eta}_{\mathfrak{F}^\diamondsuit}}) \xrightarrow{\sim} (\mathcal{H}, \nabla_{\mathcal{H}}) \tag{151}$$

of (150) is compatible with the respective filtrations $\{(\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^\vee)^j\}_{j=0}^{p-n}$ and $\{\mathcal{H}^j\}_{j=0}^{n-p}$, and moreover, satisfies the equality $\delta|_{(\mathrm{Ker}(\widetilde{\eta}_{\mathfrak{F}^\diamondsuit})^\vee)^{p-n-1}} \circ \eta^{\triangleright\nabla}_{\mathfrak{F}^\diamondsuit} = \eta_{\mathfrak{H}^\diamondsuit}$. This completes the proof of Proposition 5.2.2. □

Remark 5.2.3. Once the proposition just above is proved, one may describe the assignment $\mathfrak{F}^\diamondsuit \mapsto \mathfrak{H}^\diamondsuit$ ($\cong \mathfrak{F}^{\diamondsuit\bigstar}$) more briefly, as follows. For simplicity, suppose further that $S = \mathrm{Spec}(\overline{k})$ for an algebraically closed field $\overline{k}$ over k. Let $\mathbb{U}$ and $\mathfrak{F}^\diamondsuit$ be as above. Denote by $\alpha : F_{X/\overline{k}*}(\mathrm{Ker}(\nabla^\vee_{\mathcal{F}})) \to F_{X/\overline{k}*}(\mathcal{B}^\vee)$ the composite (133). Then, one verifies that $\mathfrak{H}^\diamondsuit$ is isomorphic to

$$(F^*_{X/\overline{k}}(\mathrm{Coker}(\alpha)), \nabla^{\mathrm{can}}_{\mathrm{Coker}(\alpha)}, \{F^*_{X/\overline{k}}(\mathrm{Coker}(\alpha))^{[j]}\}_{j=0}^{p-n}), \tag{152}$$

where $\{F^*_{X/\overline{k}}(\mathrm{Coker}(\alpha))^{[j]}\}_{j=0}^{p-n}$ denotes the Harder-Narasimhan filtration on the vector bundle $F^*_{X/\overline{k}}(\mathrm{Coker}(\alpha))$.

6. Duality for Dormant $\mathfrak{sl}_n$-Opers

In [20], we discussed the definition of a(n) (dormant) $\mathfrak{sl}_n$-oper (or, more generally, a $\mathfrak{g}$-oper for a semisimple Lie algebra $\mathfrak{g}$) on a pointed stable curve and various properties concerning their moduli. In this last section, we consider duality for dormant $\mathfrak{sl}_n$-opers induced by Theorem 4.3.1 and some applications (by means of results of p-adic Teichmüller theory) to understanding the moduli stack of dormant $\mathfrak{sl}_n$-opers of the case where $n = p-2$.

6.1. Suppose that $p-1 > n > 1$. We shall identify the Lie algebra $\mathfrak{pgl}_n$ of PGL_n with the spectrum of the symmetric algebra $\mathbb{S}_k(\mathfrak{pgl}_n^\vee)$ on $\mathfrak{pgl}_n$ over k. Since $n < p$, one may identify $\mathfrak{sl}_n$ with $\mathfrak{pgl}_n$ via the natural composite $\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n \twoheadrightarrow \mathfrak{pgl}_n$ and moreover, identify PGL_n with the adjoint group of $\mathfrak{sl}_n$. Write $\mathfrak{c}_n$ for the GIT quotient $\mathfrak{pgl}_n /\!\!/ \mathrm{PGL}_n$ of $\mathfrak{pgl}_n$ by the adjoint action of PGL_n, i.e., the spectrum $\mathrm{Spec}(\mathbb{S}_k(\mathfrak{pgl}_n^\vee)^{\mathrm{PGL}_n})$ of the ring of polynomial invariants on $\mathfrak{pgl}_n$. Denote by

$$\chi_n : \mathfrak{pgl}_n \to \mathfrak{c}_n \tag{153}$$

the so-called *Chevalley map*, i.e., the morphism over k corresponding to the inclusion $\mathbb{S}_k(\mathfrak{pgl}_n^\vee)^{\mathrm{PGL}_n} \hookrightarrow \mathbb{S}_k(\mathfrak{pgl}_n^\vee)$ of k-algebras. Also, denote by $\mathfrak{t}_n$ the Lie subalgebra of $\mathfrak{pgl}_n$ consisting of the image (via $\mathfrak{gl}_n \twoheadrightarrow \mathfrak{pgl}_n$) of diagonal matrices, and by $\kappa : \mathfrak{t}_n \hookrightarrow \mathfrak{pgl}_n$ the natural inclusion.

Note that the various Lie algebras and morphisms between them mentioned above may be defined over $\mathbb{F}_p$. Hence, it makes sense to speak of the sets of the $\mathbb{F}_p$-rational points $\mathfrak{pgl}_n(\mathbb{F}_p)$, $\mathfrak{c}_n(\mathbb{F}_p)$, $\mathfrak{t}_n(\mathbb{F}_p)$ of $\mathfrak{pgl}_n$, $\mathfrak{c}_n$, $\mathfrak{t}_n$ respectively, and of the maps between the respective sets of $\mathbb{F}_p$-rational points $\chi_n(\mathbb{F}_p) : \mathfrak{pgl}_n(\mathbb{F}_p) \to \mathfrak{c}_n(\mathbb{F}_p)$, $\kappa(\mathbb{F}_p) : \mathfrak{t}_n(\mathbb{F}_p) \to \mathfrak{pgl}_n(\mathbb{F}_p)$. The set $\mathfrak{t}_n(\mathbb{F}_p)$ may be identified with the quotient $\mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p)$, where Δ denotes the diagonal embedding $\mathbb{F}_p \hookrightarrow \mathbb{F}_p^{\times n}$. The symmetric group $\mathfrak{S}_n$ of n letters acts, by permutation, on $\mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p)$ in such a way that the surjection $\mathbb{F}_p^{\times n} \twoheadrightarrow \mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p)$ is compatible with the respective $\mathfrak{S}_n$-actions (cf. (6)). Hence, we have the double quotient $\mathfrak{S}_n \backslash \mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p)$. The natural surjection $\mathbb{F}_p^{\times n} \twoheadrightarrow \mathfrak{S}_n \backslash \mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p)$ factors through the surjection $\mathbb{F}_p^{\times n} \twoheadrightarrow \mathbb{N}^{\mathbb{F}_p}_{\sharp(-)=n}$ (cf. (6)), and we have the resulting surjection

$$\pi_n^{\mathbb{F}_p} : \mathbb{N}^{\mathbb{F}_p}_{\sharp(-)=n} \twoheadrightarrow \mathfrak{S}_n \backslash \mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p). \tag{154}$$

Next, observe that the composite

$$\mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p) \ (= \mathfrak{t}_n(\mathbb{F}_p)) \overset{\kappa(\mathbb{F}_p)}{\hookrightarrow} \mathfrak{pgl}_n(\mathbb{F}_p) \overset{\chi_n(\mathbb{F}_p)}{\to} \mathfrak{c}_n(\mathbb{F}_p) \tag{155}$$

factors through the surjection $\mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p) \twoheadrightarrow \mathfrak{S}_n \backslash \mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p)$. The resulting map of sets

$$\chi_n^{\mathbb{F}_p} : \mathfrak{S}_n \backslash \mathbb{F}_p^{\times n}/\Delta(\mathbb{F}_p) \to \mathfrak{c}_n(\mathbb{F}_p) \tag{156}$$

is injective.

For each element $a \in \mathbb{F}_p$ and $\rho_0 \in \mathrm{Im}(\chi_n^{\mathbb{F}_p})$ $(\subseteq \mathfrak{c}_n(\mathbb{F}_p))$, there exists a *unique* multiset

$$\rho_0^{\circledast a} := [\rho_{0,1}^{\circledast a}, \cdots, \rho_{0,n}^{\circledast a}] \tag{157}$$

(where $\rho_{0,j}^{\circledast a} \in \mathbb{F}_p$ for each $j \in \{1, \cdots, n\}$) over $\mathbb{F}_p$ with cardinality n (i.e., an element of $\mathbb{N}_{\sharp(-)=n}^{\mathbb{F}_p}$) satisfying that $\chi_n^{\mathbb{F}_p} \circ \pi_n^{\mathbb{F}_p}(\rho_0^{\circledast a}) = \rho$ and $\sum_{j=1}^n \rho_{0,j}^{\circledast a} = a$.

Let us define a subset $\mathfrak{c}_n(\mathbb{F}_p)^{\circledast}$ of $\mathfrak{c}_n(\mathbb{F}_p)$ to be

$$\mathfrak{c}_n(\mathbb{F}_p)^{\circledast} := \chi_n^{\mathbb{F}_p} \circ \pi_n^{\mathbb{F}_p}(2_{\sharp(-)=n}^{\mathbb{F}_p}). \tag{158}$$

If $\rho_0 \in \mathfrak{c}_n(\mathbb{F}_p)^{\circledast}$ and $a \in \mathbb{F}_p$, then (since $\rho_0^{\circledast a}$ is a subset of $\mathbb{F}_p$) the element

$$\rho_0^{\bigstar} := \chi_{(p-n)}^{\mathbb{F}_p} \circ \pi_{(p-n)}^{\mathbb{F}_p}((\rho_0^{\circledast a})^{\bigstar}) \tag{159}$$

is well-defined and lies in $\mathfrak{c}_{(p-n)}(\mathbb{F}_p)^{\circledast}$. Note that the element $\rho^{\bigstar}$ does not depend on the choice of a. Thus, the assignment $\rho \mapsto \rho^{\bigstar}$ defines a well-defined bijection of sets

$$\theta_n^{\bigstar} : \mathfrak{c}_n(\mathbb{F}_p)^{\circledast} \to \mathfrak{c}_{(p-n)}(\mathbb{F}_p)^{\circledast}, \tag{160}$$

which is verified to satisfy the equality $\theta_{(p-n)}^{\bigstar} \circ \theta_n^{\bigstar} = \mathrm{id}$.

For each $\vec{\rho} := (\rho_i)_{i=1}^r \in \mathfrak{c}_n(\mathbb{F}_p)^{\times r}$ and $\vec{a} := (a_i)_{i=1}^r \in \mathbb{F}_p^{\times r}$, we shall write

$$\vec{\rho}^{\,\circledast \vec{a}} := (\rho_i^{\circledast a_i})_{i=1}^r \in (\mathbb{N}_{\sharp(-)=n}^{\mathbb{F}_p})^{\times r}. \tag{161}$$

If, moreover, $\vec{\rho}$ lies in $(\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}$, then one obtains

$$\vec{\rho}^{\,\bigstar} := (\rho_i^{\bigstar})_{i=1}^r \in (\mathfrak{c}_{(p-n)}(\mathbb{F}_p)^{\circledast})^{\times r}. \tag{162}$$

In particular, $(\vec{\rho}^{\bigstar})^{\bigstar} = \vec{\rho}$.

6.2. Let $\vec{\rho} := (\rho_i)_{i=1}^r \in \mathfrak{c}_n(\mathbb{F}_p)^{\times r}$ (where $\vec{\rho} := \emptyset$ if $r = 0$). We shall write

$$\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}} \left(\text{resp., } \overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \vec{\rho}, \mathfrak{X}_{/S}}\right) : \mathfrak{Sch}_{/S} \to \mathfrak{Set} \tag{163}$$

for the moduli functor, as introduced in [20] § 3.6, classifying (the isomorphism classes of) dormant $\mathfrak{sl}_n$-opers on $\mathfrak{X}_{/S}$ (resp., dormant $\mathfrak{sl}_n$-opers on $\mathfrak{X}_{/S}$ of radii $\vec{\rho}$). (For convenience, we shall say that any dormant $\mathfrak{sl}_n$-oper is *of radii* $\emptyset$.) Both $\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}}$ and $\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \vec{\rho}, \mathfrak{X}_{/S}}$ may be represented by (possibly empty) relative finite S-schemes (cf. [20], Theorem C). Moreover, if $r > 0$, then $\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}}$ decomposes into the disjoint union

$$\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}} = \coprod_{\vec{\rho} \in \mathfrak{c}_n(\mathbb{F}_p)^{\times r}} \overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \vec{\rho}, \mathfrak{X}_{/S}}. \tag{164}$$

(cf. [20], Theorem C (i)).

Now, let $\vec{a} := (a_i)_{i=1}^r \in \mathbb{F}_p^{\times r}$, $\vec{\rho} := (\rho_i)_{i=1}^r \in \mathfrak{c}_n(\mathbb{F}_p)^{\times r}$ (where we take $\vec{a} := \emptyset$ and $\vec{\rho} := \emptyset$ if $r = 0$), and let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be a dormant n-determinant data for $\mathfrak{X}_{/S}$ of exponent $\vec{a}$. Also, let $\mathfrak{F}^{\diamondsuit}$ be a dormant $(\mathrm{GL}_n, \mathbb{U})$-oper on $\mathfrak{X}_{/S}$ (resp., a dormant $(\mathrm{GL}_n, \mathbb{U})$-oper on $\mathfrak{X}_{/S}$ of exponent $\vec{\rho}^{\circledast \vec{a}}$, where $\vec{\rho} \in \mathfrak{c}_n(\mathbb{F}_p)^{\times r}$). It induces, via the change of structure group $\mathrm{GL}_n \twoheadrightarrow \mathrm{PGL}_n$, a dormant $\mathfrak{sl}_n$-oper on $\mathfrak{X}_{/S}$ (resp., a dormant $\mathfrak{sl}_n$-oper on $\mathfrak{X}_{/S}$ of radii $\vec{\rho}$), which we denote by $\mathcal{F}^{\diamondsuit\spadesuit}$. The assignment $\mathcal{F}^{\diamondsuit} \mapsto \mathcal{F}^{\diamondsuit\spadesuit}$ defines a morphism

$$\begin{gathered} \Lambda_{\mathrm{GL}_n, \mathbb{U}, \mathfrak{X}_{/S}} : \overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathrm{GL}_n, \mathbb{U}, \mathfrak{X}_{/S}} \to \overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \mathfrak{X}_{/S}} \\ \left(\text{resp., } \Lambda_{\mathrm{GL}_n, \mathbb{U}, \vec{\rho}, \mathfrak{X}_{/S}} : \overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathrm{GL}_n, \mathbb{U}, \vec{\rho}^{\circledast \vec{a}}, \mathfrak{X}_{/S}} \to \overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathfrak{sl}_n, \vec{\rho}, \mathfrak{X}_{/S}}\right) \end{gathered} \tag{165}$$

over S. By [20], Corollary 4.14.3 (i), $\Lambda_{\mathrm{GL}_n, \mathbb{U}, \mathfrak{X}_{/S}}$ is an isomorphism. (In particular, $\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathrm{GL}_n, \mathbb{U}, \mathfrak{X}_{/S}}$, as well as $\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathrm{GL}_n, \mathbb{U}, \vec{\rho}^{\circledast \vec{a}}, \mathfrak{X}_{/S}}$, may be represented by a relative finite S-scheme.) On the other hand, it follows from Proposition 3.3.8 that if $r > 0$, then $\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathrm{GL}_n, \mathbb{U}, \mathfrak{X}_{/S}}$ decomposes into the disjoint union

$$\overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathrm{GL}_n, \mathbb{U}, \mathfrak{X}_{/S}} = \coprod_{\vec{\rho} \in (\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}} \overset{\text{Zzz...}}{\mathfrak{Op}}_{\mathrm{GL}_n, \mathbb{U}, \vec{\rho}^{\circledast \vec{a}}, \mathfrak{X}_{/S}}. \tag{166}$$

Hence, the decomposition (164) may be described as

$$\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}} = \coprod_{\vec{\rho}\in(\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}}, \tag{167}$$

and, by restricting $\Lambda_{\mathrm{GL}_n,\mathbb{U},\mathfrak{X}_{/S}}$, we obtain an isomorphism

$$\Lambda_{\mathrm{GL}_n,\mathbb{U},\vec{\rho},\mathfrak{X}_{/S}} : \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathrm{GL}_n,\mathbb{U},\vec{\rho}^{\,\divideontimes\vec{a}},\mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}} \tag{168}$$

over S. Consider the composite isomorphisms

$$\begin{aligned} \Theta^{\bigstar}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}} &:= \Lambda_{\mathrm{GL}_{(p-n)},\mathbb{U},\mathfrak{X}_{/S}} \circ \Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U},\mathfrak{X}_{/S}} \circ \Lambda^{-1}_{\mathrm{GL}_n,\mathbb{U},\mathfrak{X}_{/S}}, \\ \Theta^{\bigstar}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}} &:= \Lambda_{\mathrm{GL}_{(p-n)},\mathbb{U},(\vec{\rho}^{\,\divideontimes\vec{a}})^{\bigstar},\mathfrak{X}_{/S}} \circ \Theta^{\bigstar}_{\mathrm{GL}_n,\mathbb{U},\vec{\rho}^{\,\divideontimes\vec{a}},\mathfrak{X}_{/S}} \circ \Lambda^{-1}_{\mathrm{GL}_n,\mathbb{U},\vec{\rho},\mathfrak{X}_{/S}}. \end{aligned} \tag{169}$$

Proposition 4.3.2 implies that these morphisms do not depend on the choice of $\mathbb{U}$. By Theorem 4.3.1, the following theorem holds.

THEOREM 6.2.1 (= Theorem A).

(i) *For each positive integer n with $1 < n < p-1$, there exists a canonical isomorphism*

$$\Theta^{\bigstar}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}} : \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-n)},\mathfrak{X}_{/S}} \tag{170}$$

(i.e., the first isomorphism in (169)) over S satisfying that $\Theta^{\bigstar}_{\mathfrak{sl}_{(p-n)},\mathfrak{X}_{/S}} \circ \Theta^{\bigstar}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}} = \mathrm{id}$.

(ii) *If, moreover, $r > 0$ and we are given an element $\vec{\rho} \in (\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}$, then we obtain, by restricting $\Theta^{\bigstar}_{\mathfrak{sl}_n,\mathfrak{X}_{/S}}$, a canonical isomorphism*

$$\Theta^{\bigstar}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}} : \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}} \xrightarrow{\sim} \mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-n)},\vec{\rho}^{\bigstar},\mathfrak{X}_{/S}} \tag{171}$$

(i.e., the second isomorphism in (169)) over S satisfying that $\Theta^{\bigstar}_{\mathfrak{sl}_{(p-n)},\vec{\rho},\mathfrak{X}_{/S}} \circ \Theta^{\bigstar}_{\mathfrak{sl}_n,\vec{\rho}^{\bigstar},\mathfrak{X}_{/S}} = \mathrm{id}$.

Also, by Corollary 4.3.3, the following assertion holds.

THEOREM 6.2.2 (= Theorem B). *The structure morphism* $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-1)},\mathfrak{X}_{/S}} \to S$ *of* $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_{(p-1)},\mathfrak{X}_{/S}}$ *is an isomorphism. That is to say, there exists a unique (up to isomorphism) dormant* $\mathfrak{sl}_{(p-1)}$*-oper on* $\mathfrak{X}_{/S}$.

6.3. In order to achieve a detailed understanding of the moduli stack of $\mathfrak{sl}_{(p-n)}$-opers (of the case where n is, in a certain sense, sufficiently small relative to p), we will use, in this subsection, Theorem 6.2.1 and some results concerning the moduli stack of dormant $\mathfrak{sl}_n$-opers obtained in [20].

First, we shall recall the theory of dormant operatic fusion rings $\mathfrak{F}^{\mathrm{Zzz...}}_{p,\mathfrak{sl}_n}$ discussed in [20], § 7. For each integer n with $1 < n < p-1$ and $\vec{\rho} := (\rho_i)_{i=1}^r \in \mathfrak{c}_n(\mathbb{F}_p)^{\times r}$ (where $\vec{\rho} := \emptyset$ if $r = 0$), write

$$\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},g,r} \tag{172}$$

for $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},\mathfrak{X}_{/S}}$ of the case where the pointed stable curve $\mathfrak{X}_{/S}$ is taken to be the tautological pointed stable curve

$$(f_{\mathfrak{tau}} : \mathfrak{C}_{g,r} \to \overline{\mathfrak{M}}_{g,r}, \{\mathfrak{s}_i, : \overline{\mathfrak{M}}_{g,r} \to \mathfrak{C}_{g,r}\}_{i=1}^r) \tag{173}$$

(cf. § 1.3) over $\overline{\mathfrak{M}}_{g,r}$. That is, $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},g,r}$ is defined to be the stack in groupoids over $\mathfrak{Sch}_{/\mathrm{Spec}(k)}$ whose category of sections over a k-scheme S is the groupoid of the pairs $(\mathfrak{X}_{/S}, \mathcal{E}^{\spadesuit})$ consisting of a pointed stable curve $\mathfrak{X}_{/S}$ over S of type (g,r) and a dormant $\mathfrak{sl}_n$-opers on $\mathfrak{X}_{/S}$ of radii $\vec{\rho}$. (Indeed, it follows from [20], Proposition 2.2.5, that any (dormant) $\mathfrak{sl}_n$-oper does not have nontrivial automorphisms.)

According to [20], Theorem C and Theorem G, $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},g,r}$ is finite and generically étale over $\overline{\mathfrak{M}}_{g,r}$. Let

$$N^{\mathrm{Zzz...}}_{p,\mathfrak{sl}_n,\vec{\rho},g,r} \tag{174}$$

denote the generic degree of $\mathfrak{Op}^{\mathrm{Zzz...}}_{\mathfrak{sl}_n,\vec{\rho},g,r}$ over $\overline{\mathfrak{M}}_{g,r}$. For each finite set I, $\mathbb{N}^I$ denotes the free commutative monoid generated by I, and moreover, for each integer l, $\mathbb{N}^I_{\geq l}$ denotes the submonoid of $\mathbb{N}^I$ consisting of elements $x = \sum_{i=1}^m a_i\lambda_i$ (where $\lambda_i \in I$ and $a_i \in \mathbb{N}$ for each $i = 1, \cdots, m$) with $\sum_{i=1}^m a_i \geq l$. Here, recall from [20], § 5.8, that there exists an involution

$\lambda \mapsto \lambda^{\veebar}$ on $\mathfrak{c}_n(\mathbb{F}_p)$ that comes, via $\mathfrak{sl}_n \xrightarrow{\sim} \mathfrak{pgl}_n \xrightarrow{\chi_n} \mathfrak{c}_n$, from the involution on $\mathfrak{sl}_n$ given by assigning $(a_{i,j})_{i,j} \mapsto (-a_{n+1-j,n+1-i})_{i,j}$. This involution extends by linearity to an involution $x \mapsto x^{\veebar}$ on $\mathbb{N}^{I}_{\geq l}$ (for each $l \in \mathbb{Z}$). Then, the function

$$N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,g} : \begin{array}{rcl} \mathbb{N}^{\mathfrak{c}_n(\mathbb{F}_p)}_{\geq 3-2g} & \rightarrow & \mathbb{Z} \\ \sum_{i=1}^{r} \rho_i & \mapsto & N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,\vec{\rho},g,r} \end{array} \tag{175}$$

is verified to be well-defined, i.e., the value $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,\vec{\rho},g,r}$ does not depend on the ordering of $\rho_1, \cdots, \rho_r$ (cf. [20], the discussion following Proposition 7.5.2). It follows from [20], Theorem 7.10.4 (i), that if g_1, g_2 are nonnegative integers and $x \in \mathbb{N}^{\mathfrak{c}_n(\mathbb{F}_p)}_{\geq 3-2g_1}$, $y \in \mathbb{N}^{\mathfrak{c}_n(\mathbb{F}_p)}_{\geq 3-2g_2}$, then the collection of functions $\{N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,g}\}_{g\geq 0}$ satisfies the following rule:

$$N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,g_1+g_2}(x+y) = \sum_{\lambda \in \mathfrak{c}_n(\mathbb{F}_p)} N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,g_1}(x+\lambda) \cdot N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,g_2}(y+\lambda^{\veebar}). \tag{176}$$

Also, it follows from [20], Theorem E, that for any nonnegative integer g and any $x \in \mathbb{N}^{\mathfrak{c}_n(\mathbb{F}_p)}_{\geq 3-2g}$, the following equality holds:

$$N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,g}(x^{\veebar}) = N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,g}(x). \tag{177}$$

In particular, the functions $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,0}$ forms a pseudo-fusion rule (cf. [20], Definition 7.6.1) on the finite set $\mathfrak{c}_n(\mathbb{F}_p)$ (with involution $x \mapsto x^{\veebar}$), and hence, one obtains the fusion ring

$$\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n} \tag{178}$$

associated with $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,0}$ (cf. [19], (834)).

The decomposition (167) implies that $\mathfrak{c}(\mathbb{F}_p) \setminus \mathfrak{c}_n(\mathbb{F}_p)^{\circledast}$ is contained in the kernel (cf. [20], Remark 7.7.1) of $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,0}$, i.e., $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,\vec{\rho},0,r} \neq 0$ only if $\vec{\rho} \in (\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}$. Hence, the restriction

$$N'_n := N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,0}|_{\mathbb{N}^{\mathfrak{c}_n(\mathbb{F}_p)^{\circledast}}_{\geq 3}} \tag{179}$$

of $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,0}$ to $\mathbb{N}^{\mathfrak{c}_n(\mathbb{F}_p)^{\circledast}}_{\geq 3} \subseteq \mathbb{N}^{\mathfrak{c}_n(\mathbb{F}_p)}_{\geq 3}$ forms a pseudo-fusion rule on $\mathfrak{c}_n(\mathbb{F}_p)^{\circledast}$, and the natural inclusion $\mathfrak{F}_{N'_n} \hookrightarrow \mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n}$ gives rise to an isomorphism

$$(\mathfrak{F}_{N'_n})_{\text{red}} \xrightarrow{\sim} (\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n})_{\text{red}} \tag{180}$$

(cf. [20], Remark 7.7.1) between the reduced rings associated with $\mathfrak{F}_{N'_n}$ and $\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n}$ respectively. Let

$$\overline{\mathcal{C}as}_{p,\mathfrak{sl}_n} \tag{181}$$

be the element of $(\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n})_{\text{red}}$ defined to be the image of $\sum_{\lambda \in \mathfrak{c}_n(\mathbb{F}_p)} \lambda^{\veebar} \cdot \lambda \in \mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n}$ via the quotient $\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n} \twoheadrightarrow (\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n})_{\text{red}}$. Then, by means of Theorem 6.2.1, we have the following corollary. In particular, one may extend (cf. Corollary 6.3.1 (ii)) a result in the paper [20] (cf. [20], Theorem H) concerning an explicit computation of the value $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,\emptyset,g,0}$.

COROLLARY 6.3.1.

(i) *There exists a canonical isomorphism*

$$(\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n})_{\text{red}} \xrightarrow{\sim} (\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-n)}})_{\text{red}} \ \ (=: \mathfrak{F}) \tag{182}$$

of rings that sends $\overline{\mathcal{C}as}_{p,\mathfrak{sl}_n}$ *to* $\overline{\mathcal{C}as}_{p,\mathfrak{sl}_{(p-n)}}$ $(=: \overline{\mathcal{C}as})$*. In particular, for each* $\vec{\rho} = (\rho_i)_{i=1}^r \in (\mathfrak{c}_n(\mathbb{F}_p)^{\circledast})^{\times r}$ *(where* $\vec{\rho} := \emptyset$ *if* $r = 0$*), the following equalities hold:*

$$N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,\vec{\rho},g,r} = N^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-n)},\vec{\rho}^{\star},g,r} = \sum_{\chi \in \text{Hom}(\mathfrak{F},\mathbb{C})} \chi(\overline{\mathcal{C}as})^{g-1} \cdot \prod_{i=1}^{r} \chi(\rho_i), \tag{183}$$

where $\text{Hom}(\mathfrak{F}, \mathbb{C})$ *denotes the set of ring homomorphisms* $\mathfrak{F} \to \mathbb{C}$.

(ii) *Suppose that* $p > n \cdot \max\{g-1, 2\}$*. Then, the generic degrees* $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,\emptyset,g,0}$ *and* $N^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-n)},\emptyset,g,0}$ *are given by the following formula:*

$$(184)\qquad N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,\emptyset,g,0} = N^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-n)},\emptyset,g,0} = \frac{p^{(n-1)(g-1)-1}}{n!} \cdot \sum_{\substack{(\zeta_1,\cdots,\zeta_n)\in\mathbb{C}^{\times n} \\ \zeta_i^p=1,\ \zeta_i\neq\zeta_j(i\neq j)}} \frac{(\prod_{i=1}^n \zeta_i)^{(n-1)(g-1)}}{\prod_{i\neq j}(\zeta_i-\zeta_j)^{g-1}}.$$

Proof. Assertion (i) follows from [20] Theorem F, Theorem 6.2.1 (ii), the definitions of $N^{\text{Zzz...}}_{p,\mathfrak{sl}_n,0}$ and $\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_n}$, and the isomorphism (180) (as well as the isomorphism (180) of the case where the integer "n" is taken to be "$p-n$"). Assertion (ii) follows from [20], Theorem H, and Theorem 6.2.1 (i). □

6.4. In the following, we focus on the case where $n = p-2$. By means of Theorem 6.2.1 and results in the p-adic Teichmüller developed by S. Mochizuki (cf. [15]), one may prove Corollaries 6.4.1 and 6.4.2 described below.

Corollary 6.4.1. *Let $\vec{\rho} \in \mathfrak{c}_{(p-2)}(\mathbb{F}_p)^{\times r}$. Then, the stack $\mathfrak{Op}^{\text{Zzz...}}_{\mathfrak{sl}_{(p-2)},\vec{\rho},g,r}$ is a (possibly empty) geometrically connected, proper, and smooth Deligne-Mumford stack over k of dimension $3g-3+r$. Moreover, the natural morphism $\mathfrak{Op}^{\text{Zzz...}}_{\mathfrak{sl}_{(p-2)},\vec{\rho},g,r} \to \overline{\mathfrak{M}}_{g,r}$ is finite, faithfully flat, and generically étale.*

Proof. The assertion follows from Theorem 6.2.1 (ii) of the case $n=2$ and [15], Chap. II, § 2.8, Theorem 2.8, which asserts that $\mathfrak{Op}^{\text{Zzz...}}_{\mathfrak{sl}_2,\vec{\rho}^\star,g,r}$ satisfies the same properties as the properties desired for $\mathfrak{Op}^{\text{Zzz...}}_{\mathfrak{sl}_{(p-2)},\vec{\rho},g,r}$ in the statement. □

Next, we consider the structure of the reduced ring $(\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)}})_{\mathrm{red}}$ associated with $\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)}}$. (In order to perform any computation that we will need in the ring $\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)}}$, it suffices to understand the structure of $(\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)}})_{\mathrm{red}}$ (cf. Corollary 6.3.1 (i))).

Let us write

$$(185)\qquad \mathbb{F} := \{a \in \mathbb{Z} \mid 0 \leq a \leq \frac{p-3}{2}\}.$$

The composite

$$\mathbb{F} \to 2^{\mathbb{F}_p}_{\sharp(-)=(p-2)} \to \mathfrak{c}_{(p-2)}(\mathbb{F}_p) \tag{186}$$

is injective, where the first arrow denotes the map

$$\begin{aligned} \mathbb{F} &\to 2^{\mathbb{F}_p}_{\sharp(-)=(p-2)} \\ a &\mapsto \mathbb{F}_p \setminus \{0, \overline{-2a-1}\}. \end{aligned} \tag{187}$$

We shall regard $\mathbb{F}$ as a subset of $\mathfrak{c}_{(p-2)}(\mathbb{F}_p)$ via this composite injection.

Corollary 6.4.2. *The complement of $\mathbb{F}$ in $\mathfrak{c}_{(p-2)}(\mathbb{F}_p)$ is contained in the kernel (cf. [20], Remark 7.7.1) of $N^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)},0}$. Moreover, the structure constants $N^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)},0}(\alpha+\beta+\gamma)$ (where α, β, $\gamma \in \mathfrak{c}_{(p-2)}(\mathbb{F}_p)$) of the fusion ring $\mathfrak{F}^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)}}$ are given as follows:*

$$N^{\text{Zzz...}}_{p,\mathfrak{sl}_{(p-2)},0}(\alpha+\beta+\gamma) = \begin{cases} 1 & \textit{if } (\alpha,\beta,\gamma) \in W, \\ 0 & \textit{if } (\alpha,\beta,\gamma) \notin W, \end{cases} \tag{188}$$

where W denotes a subset of $\mathbb{F}^{\times 3}$ ($\subseteq \mathfrak{c}_{(p-2)}(\mathbb{F}_p)^{\times 3}$) defined to be

$$W := \left\{ (s,t,u) \in \mathbb{F}^{\times 3} \;\middle|\; s+t+u \leq p-2, \textit{ and } \begin{array}{l} 0 \leq s \leq t+u, \\ 0 \leq t \leq u+s, \\ 0 \leq u \leq s+t. \end{array} \right\}. \tag{189}$$

Proof. Consider the composite

$$\mathbb{F} \to \mathfrak{sl}_2(\mathbb{F}_p) \overset{\chi_2(\mathbb{F}_p)}{\to} \mathfrak{c}_2(\mathbb{F}_p), \tag{190}$$

where the first arrow denotes the map given by assigning $a \mapsto \begin{pmatrix} \overline{a} & 0 \\ 0 & -\overline{a} \end{pmatrix}$. This composite factors through the inclusion $\mathfrak{c}_2(\mathbb{F}_p)^{\circledast} \to \mathfrak{c}_2(\mathbb{F}_p)$. By passing to the resulting injection $w : \mathbb{F} \hookrightarrow \mathfrak{c}_2(\mathbb{F}_p)^{\circledast}$, we regard $\mathbb{F}$ as a subset of $\mathfrak{c}_2(\mathbb{F}_p)^{\circledast}$. According to the discussion in [20], § 7.11 (or, [15], Introduction, § 1.2, Theorem 1.3), the structure constants $N^{\text{Zzz...}}_{p,\mathfrak{sl}_2,0}(\alpha+\beta+\gamma)$ (where α, β, $\gamma \in \mathfrak{c}_2(\mathbb{F}_p)^{\circledast}$) of $\mathfrak{F}_{N_2'}$ (cf. (179)) are given as follows:

$$N^{\text{Zzz...}}_{p,\mathfrak{sl}_2,0}(\alpha+\beta+\gamma) = \begin{cases} 1 & \text{if } (\alpha,\beta,\gamma) \in W, \\ 0 & \text{if } (\alpha,\beta,\gamma) \notin W. \end{cases} \tag{191}$$

On the other hand, the composite

$$\mathbb{F} \overset{w}{\hookrightarrow} \mathfrak{c}_2(\mathbb{F}_p)^{\circledast} \overset{\theta_2^{\star}}{\rightarrow} \mathfrak{c}_{(p-2)}(\mathbb{F}_p)^{\circledast} \hookrightarrow \mathfrak{c}_{(p-2)}(\mathbb{F}_p) \tag{192}$$

coincides with the composite (186). Thus, the assertion follows from Corollary 6.3.1 (i) of the case $n = 2$. □

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