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Microlocal Resolvent Estimates, Revisited

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Abstract. Let H be a Schrödinger type operator with long-range perturbation. We study the wave front set of the distribution kernel of $(H-\lambda \mp i0)^{-1}$, where λ is in the absolutely continous spectrumof H. The result is a refinement of the microlocal resolvent estimate of Isozaki-Kitada [5, 6]. We prove the result for a class of pseudodifferential operators on manifolds so that they apply to discrete Schrödinger operators and higher order operators on the Euclidean space. The proof relies on propagation estimates, whereas the original proof of Isozaki-Kitada relies on a construction of parametrices.

1. Introduction

In this Introduction, we present our main results for Schrödinger operators for simplicity. The results under more general settings are explained in Section 2. Let

$$H = -\frac{1}{2}\triangle + V(x) \quad \text{on } L^2(\mathbb{R}^d),\ d \geq 1,$$

be a Schrödinger operator with a potential $V \in C^\infty(\mathbb{R}^d)$, real-valued. We suppose there is $\mu > 0$ such that for any multi-index $\alpha \in \mathbb{Z}_+^d$,

$$\left|\partial_x^\alpha V(x)\right| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^d,$$

with some $C_\alpha > 0$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. Then it is well-known that $\sigma_{\rm ess}(H) = [0,\infty)$; H has no positive eigenvalues; H is absolutely continuous on $(0,\infty)$, and

$$(H-\lambda \mp i0)^{-1} = \lim_{\varepsilon \to +0} (H-\lambda \mp i\varepsilon)^{-1}, \quad \lambda > 0,$$

exist as operators from $L^2(\mathbb{R}^d, \langle x\rangle^s dx)$ to $L^2(\mathbb{R}^d, \langle x\rangle^{-s} dx)$ with $s > 1/2$. We denote the Fourier transform by $\mathcal{F}$, and we write $\hat{H} = \mathcal{F} H \mathcal{F}^*$. Then

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the above claim implies $(\hat{H} - \lambda \mp i0)^{-1}$ exist as operators from $H^s(\mathbb{R}^d)$ to $H^{-s}(\mathbb{R}^d)$, and thus they have distribution kernel of order at most 1. We denote their distribution kernels by $K^{\pm}(\lambda) \in \mathcal{S}'(\mathbb{R}^{2d})$. We investigate the wave front set of $K^{\pm}(\lambda)$. We use somewhat nonstandard notation to represent a point in $T^*\mathbb{R}^{2d} \cong \mathbb{R}^{4d}$: We denote

$$(x, \xi, y, \eta) \in T^*\mathbb{R}^{2d}, \quad \text{where } (\xi, \eta) \in \mathbb{R}^{2d}, \text{ and } (x, y) \in T^*_{(\xi,\eta)}(\mathbb{R}^{2d}),$$

i.e., ξ, η denote points in $\mathbb{R}^d$ (the Fourier space), and x, y denote points in the cotangent spaces at ξ, η, respectively. We also use a special notation on the wave front set:

$$\mathrm{WF}'(K) = \bigl\{(x, \xi, -y, \eta) \in T^*\mathbb{R}^d \bigm| (x, \xi, y, \eta) \in \mathrm{WF}(K)\bigr\}$$

for a distribution $K \in \mathcal{S}'(\mathbb{R}^{2d})$, where $\mathrm{WF}(K)$ denotes the wave front set of K.

We denote

$$\begin{aligned}
\Sigma_0 &= \bigl\{(x, \xi, x, \xi) \bigm| (x, \xi) \in T^*\mathbb{R}^d\bigr\}, \\
\Sigma_\pm(\lambda) &= \bigl\{(x + t\xi, \xi, x, \xi) \bigm| (x, \xi) \in T^*\mathbb{R}^d, \tfrac{1}{2}|\xi|^2 = \lambda, \pm t \geq 0\bigr\}, \\
\Sigma'_\pm(\lambda) &= \bigl\{(t\xi, \xi) \bigm| \tfrac{1}{2}|\xi|^2 = \lambda, \pm t \geq 0\bigr\} \times \bigl\{(t\xi, \xi) \bigm| \tfrac{1}{2}|\xi|^2 = \lambda, \mp t \geq 0\bigr\}
\end{aligned}$$

for $\lambda > 0$.

Theorem 1.1. *For $\lambda > 0$,*

$$\mathrm{WF}'(K^{\pm}(\lambda)) \subset \Sigma_0 \cup \Sigma_\pm(\lambda) \cup \Sigma'_\pm(\lambda).$$

Remark 1.1. Σ_0 denotes the diagonal set, and $\mathrm{WF}((\text{kernel of } A)) \subset \Sigma_0$ if A is a pseudodifferential operator. $\Sigma_\pm(\lambda)$ represent the *free propagation* parts, and it is easy to show $\mathrm{WF}(K^{\pm}(\lambda)) = \Sigma_0 \cup \Sigma_\pm(\lambda)$ if $V = 0$. Thus only the third part $\Sigma'_\pm(\lambda)$ describes the singularities generated by the perturbation V.

Remark 1.2. A microlocal resolvent estimate of this form was proved in [15] for the short range case (in more general setting as in Section 2), and applied to the analysis of scattering matrices. The proof relies on a construction of parametrices.

The (two-sided) microlocal resolvent estimates of Isozaki-Kitada [5, 6] follow easily from Theorem 1.1.

Corollary 1.2. *Let $a_\pm \in S^0_{1,0}(\mathbb{R}^d)$, i.e., $a_\pm \in C^\infty(\mathbb{R}^{2d})$ and for any multi-indices α, β,*

$$\left|\partial_x^\alpha \partial_\xi^\beta a_\pm(x,\xi)\right| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}, \quad x, \xi \in \mathbb{R}^d$$

with some $C_{\alpha\beta} > 0$. Suppose there are $0 < c_1 < c_2$ and $-1 < \gamma_- < \gamma_+ < 1$ such that

$$\mathrm{supp}[a_\pm] \subset \left\{ (x,\xi) \;\middle|\; \pm\frac{x\cdot\xi}{|x|\,|\xi|} \ge \pm\gamma_\pm, c_1 \le |\xi| \le c_2, |x| \ge 1 \right\}.$$

Let $A_\pm = a_\pm(x, D_x)$. Then for any $N > 0$,

$$\langle x\rangle^N A_\mp (H - \lambda \mp i0)^{-1} A_\pm^* \langle x\rangle^N \in B(L^2(\mathbb{R}^d)), \quad \lambda > 0.$$

Proof. It suffices to show $\mathcal{F} A_\mp (H-\lambda \mp i0)^{-1} A_\pm^* \mathcal{F}^*$ are bounded from $H^{-N}(\mathbb{R}^d)$ to $H^N(\mathbb{R}^d)$, $\forall N > 0$, i.e., they are smoothing operators. We note the distribution kernels of $\mathcal{F} A_\mp (H-\lambda \mp i0)^{-1} A_\pm^* \mathcal{F}^*$ are given by $a_\pm(-D_\xi, \xi)\overline{a_\mp}(D_\eta, \eta) K^\pm(\lambda; \xi, \eta)$. We also note that if $x\cdot\xi \ge \gamma_+|x|\,|\xi|$ then

$$(x+t\xi)\cdot\xi \ge \gamma_+|x|\,|\xi| + t|\xi|^2 \ge \gamma_+|x+t\xi|\,|\xi|, \quad t \ge 0,$$

and thus we have $\dfrac{x\cdot(x+t\xi)}{|x|\,|x+t\xi|} \ge \gamma_+$. This implies that if $(x,\xi) \in \mathrm{supp}[a_+]$ then $(x+t\xi, \xi) \notin \mathrm{supp}[a_-]$, $t \ge 0$. Hence we learn that the essential support of $(a_-(x,\xi)a_+(y,\eta))$ does not intersect $\Sigma_+(\lambda)$. It is easy to show the essential support of $(a_-(x,\xi)a_+(y,\eta))$ does not intersect Σ_0 and $\Sigma_+'(\lambda)$. These imply $\mathcal{F} A_-(H-\lambda-i0)^{-1} A_+^* \mathcal{F}^*$ is smoothing. Similarly, we can show $\mathcal{F} A_+(H-\lambda+i0)^{-1} A_-^* \mathcal{F}^*$ is smoothing, and we complete the proof. □

Remark 1.3. Corollary 1.2 was proved by Isozaki and Kitada [5, 7], and it is analogous to two sided resolvent estimates of Mourre [12] (see also

Gérard [3]). The microlocal resolvent estimate of the above form is used to analyze long-range scattering and scattering matrices ([6, 8]).

In Section 2, we formulate our main results in more general settings. In Section 3, we prove our main theorem assuming a key lemma (Proposition 3.1), which is proved in Section 4. We discuss so-called one-sided microlocal resolvent estimates in Section 5.

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2. Model and Main Theorem

Here we formulate our model and state our main results that applies to higher order operators on $\mathbb{R}^d$ as well as various difference operators on $\mathbb{Z}^d$.

Let M be a d-dimensional C^∞ Riemannian manifold with a smooth density m, and let $p_0(\xi)$, $\xi \in M$, be a real-valued smooth function on M. In this paper we suppose M is either a compact manifold or a Euclidean space. If $M = \mathbb{R}^d$, we also assume $p_0 \in S^\ell(\mathbb{R}^d)$ with some ℓ, i.e.,

$$\left|\partial_\xi^\alpha p_0(\xi)\right| \leq C_\alpha \langle\xi\rangle^{\ell-|\alpha|}, \quad \xi \in \mathbb{R}^d$$

with any $\alpha \in \mathbb{Z}_+^d$, and it is elliptic, i.e.,

$$|p_0(\xi)| \geq c_0|\xi|^\ell - c_1, \quad \xi \in \mathbb{R}^d$$

with some $c_0, c_1 > 0$. The density $m(\xi)$ is also supposed to be bounded from above, and from below by a positive constant. These restrictions are not essential, and we assume these so that we can apply the functional calculus of pseudodifferential operators. We may consider more general case if necessary by generalizing the functional calculus theorem.

Let $\mu \in (0, 1]$ and let $\hat{V}$ be a pseudodifferential operator with the symbol $V(x, \xi) \in S_{1,0}^{-\mu}$ so that $\hat{V} = V(-D_\xi, \xi)$, and we write $V = \hat{V}$ where there is no confusion. We note we consider the pseudodifferential operators on manifolds in the sense of [16] Sect. 14.2.2, and the assumption implies $V \in C^\infty(T^*M)$ and for any multi-indices $\alpha, \beta \in \mathbb{Z}_+^d$ there is $C_{\alpha\beta K}$ in each local coordinate patch $K \Subset M$, such that

$$\left|\partial_x^\alpha \partial_\xi^\beta V(x,\xi)\right| \le C_{\alpha\beta K}\langle x\rangle^{-\mu-|\alpha|}, \quad \xi \in K, x \in T_\xi^* M,$$

where the length of x is defined by the Riemannian metric on $T_\xi^* M$. We also recall that the symbol is well-defined globally only modulo $S_{1,0}^{-\mu-1}$, and the full symbol is meaningful only in each local coordinate patch. If $M = \mathbb{R}^d$, we assume $V(x,\xi)(p_0(\xi)+i)^{-1} \in S_{1,0}^0(\mathbb{R}^d)$, globally.

We denote $\mathcal{H} = L^2(M, m)$, and

$$H_0\varphi(\xi) = p_0(\xi)\varphi(\xi) \quad \text{for } \varphi \in D(H_0) = \big\{\varphi \in \mathcal{H} \;\big|\; p_0\varphi \in \mathcal{H}\big\}.$$

It is easy to see H_0 is self-adjoint. We suppose $\hat{V}$ is symmetric, and hence

$$H = H_0 + V, \quad D(H) = D(H_0)$$

is self-adjoint on $\mathcal{H}$. We note V is bounded if M is compact, and H_0-bounded if $M = \mathbb{R}^d$ by the above assumptions.

Let $I \Subset \mathbb{R}$ be an interval, and we consider $(H - \lambda \mp i0)^{-1}$ for $\lambda \in I$. We define the velocity by

$$v(\xi) = dp_0(\xi) \in T_\xi^* M, \quad \xi \in M.$$

We suppose $p_0^{-1}(I) = \big\{\xi \in M \;\big|\; p_0(\xi) \in I\big\}$ is compact, and

$$v(\xi) \ne 0 \quad \text{for } \xi \in p_0^{-1}(I),$$

i.e., I does not contain critical values of p_0. Under this assumption, it is easy to see that the next claims follow from the standard Mourre theory (see, e.g., [11], [1], [15] Section 2): $\sigma_{\mathrm{p}}(H) \cap I$ is discrete, each eigenvalues are finite dimensional, and for $\lambda \in I \setminus \sigma_{\mathrm{p}}(H)$, $s > 1/2$, the limits

$$(H - \lambda \mp i0)^{-1} = \lim_{\varepsilon \to +0} (H - \lambda \mp i\varepsilon)^{-1} \in B(H^s, H^{-s})$$

exist. Let $K^\pm(\lambda)$ be the distribution kernels of $(H - \lambda \mp i0)^{-1}$, and we consider the microlocal singularities of $K^\pm(\lambda)$. As well as in the previous section, we represent a point in T^*M by

$$(x,\xi) \in T^*M, \text{ where } \xi \in M, x \in T_\xi^* M,$$

and also $(x,\xi,y,\eta)\in T^*(M\times M)$, where $(\xi,\eta)\in M\times M$, $x\in T^*_\xi M$ and $y\in T^*_\eta M$. We set Σ_0, $\Sigma_\pm(\lambda)$, $\Sigma'_\pm(\lambda)\subset T^*(M\times M)$ as

$$\begin{aligned}\Sigma_0 &= \big\{(x,\xi,x,\xi)\ \big|\ (x,\xi)\in T^*M\big\},\\ \Sigma_\pm(\lambda) &= \big\{(x+tv(\xi),\xi,x,\xi)\ \big|\ p_0(\xi)=\lambda, \pm t\ge 0\big\},\\ \Sigma'_\pm(\lambda) &= \big\{(tv(\xi),\xi)\ \big|\ p_0(\xi)=\lambda, \pm t\ge 0\big\}\times\big\{(tv(\xi),\xi)\ \big|\ p_0(\xi)=\lambda, \mp t\ge 0\big\}.\end{aligned}$$

Then our main result is stated as follows:

Theorem 2.1. *Let $\lambda\in I\setminus\sigma_{\mathrm{p}}(H)$. Then*

$$\mathrm{WF}'(K^\pm(\lambda))\subset\Sigma_0\cup\Sigma_\pm(\lambda)\cup\Sigma'_\pm(\lambda).$$

Microlocal resolvent estimates of Isozaki-Kitada type follows from this analogously to the previous section.

Corollary 2.2. *Let $\lambda\in I\setminus\sigma_{\mathrm{p}}(H)$, $a_\pm\in S^0_{1,0}(M)$, and suppose*

$$\mathrm{supp}[a_\pm]\subset\left\{(x,\xi)\in T^*M\ \middle|\ \pm\frac{x\cdot v(\xi)}{|x|\,|v(\xi)|}\ge\pm\gamma_\pm, p_0(\xi)\in K\right\},$$

where $-1<\gamma_-<\gamma_+<1$, $K\Subset M$. We set $A_\pm=a_\pm(-D_\xi,\xi)$. Then $A_\mp(H-\lambda\mp i0)^{-1}A^*_\pm$ are smoothing operators, bounded from $H^{-N}(M)$ to $H^N(M)$ with any N.

Examples. (1) A straightforward application is a differential operator on $\mathbb{R}^d$. Let H_0 be an m-th order symmetric elliptic partial differential operator with constant coefficients. We may write $H_0=p_0(D_x)$ with a real-valued polynomial of degree m. Suppose

$$V=\sum_{|\alpha|\le m-1}b_\alpha(x)D_x^\alpha$$

with $b_\xi\in C^\infty(\mathbb{R}^d)$ for each $\alpha\in\mathbb{Z}^d_+$, $|\alpha|\le m-1$. Suppose moreover that V is symmetric and $\big|\partial_\xi^\beta b_\alpha(x)\big|\le C_{\alpha\beta}\langle x\rangle^{-\mu-|\beta|}$ for each α and β. Let $I\Subset\mathbb{R}$ be an interval that does not contain critical points of $p_0(\xi)$. Then Theorem 2.1 applies for $\lambda\in I\setminus\sigma_{\mathrm{p}}(H)$.

(2) Another typical application is a difference operator on $\mathbb{Z}^d$. Let H_0 be a finite difference operator with constant coefficients, i.e.,

$$H_0u(n) = \sum_{m\in K} \gamma_m u(n-m), \quad n \in \mathbb{Z}^d,$$

where $K \subset \mathbb{Z}^d$ is a finite subset, and $\gamma_m \in \mathbb{C}$, $m \in K$. We suppose H_0 is symmetric. Then

$$p_0(\xi) = \sum_{m\in K} \gamma_m e^{i\xi\cdot m}$$

is a real-valued trigonometric polynomial on the torus $M = \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. Suppose $V(n)$ is the restriction of a smooth real-valued function $\tilde{V}(x)$ on $\mathbb{R}^d$ which satisfy $\left|\partial_\xi^\alpha V(x)\right| \le C_\alpha \langle x\rangle^{-\mu-|\alpha|}$ for each $\alpha \in \mathbb{Z}_+^d$. Then we can apply Theorem 2.1 to $H = H_0 + V$. We refer Nakamura [15] Section 7 for the detail of the construction.

3. Proof of Theorem 2.1

Here we prove our main theorem assuming a proposition, which is proved in the next section.

3.1. Notation

We use several classes of symbols. We denote the standard Kohn-Nirenberg symbol class of order m by S^m, i.e., $a \in S^m$ if $a \in C^\infty(T^*M)$ and for any multi-indices $\alpha, \beta \in \mathbb{Z}_+^d$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta}\langle x\rangle^{m-|\alpha|}, \quad \xi \in M, x \in T_\xi^* M$$

in each (relatively compact) local coordinate with some $C_{\alpha\beta} > 0$. We often use h-dependent symbols. We denote $a(h,x,\xi) \in S_h^m$ if $a(h,\cdot,\cdot) \in C^\infty(T^*M)$, $h \in (0,1]$, and for any $\alpha, \beta \in \mathbb{Z}_+^d$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(h,x,\xi)\right| \le C_{\alpha\beta} \min\left(\langle x\rangle^{m-|\alpha|}, h^{-m+|\alpha|}\right)$$

for $\xi \in M$, $x \in T_\xi^* M$, $h \in (0,1]$ with some $C_{\alpha\beta} > 0$. For example, $a(hx,\xi) \in S_h^0$ if $a(x,\xi) \in C_0^\infty(T^*M)$ is supported away from $\{x = 0\}$. Similarly, we use

(h,t)-dependent symbols, usually supported in the region: $|x| = O(h^{-1}+t)$. We denote $a \in S^m_{h,t}$ if $a(h,t,\cdot,\cdot) \in C^\infty(T^*M)$, and for any $\alpha, \beta \in \mathbb{Z}^d_+$,

$$\big|\partial_x^\alpha \partial_\xi^\beta a(h,t,x,\xi)\big| \leq C_{\alpha\beta} \min\big(\langle x\rangle^{m-|\alpha|}, (h^{-1}+t)^{m-|\alpha|}\big)$$

for $\xi \in M$, $x \in T^*_\xi M$, $h \in (0,1]$, $t \geq 0$ with some $C_{\alpha\beta} > 0$.

Our results are independent of the choice of quantizations, but it is convenient to use a quantization so that the quantization of a real symbol is symmetric. We hence use the Weyl quantization and we denote it by $\mathrm{Op}(a)$ with additional weights, e.g., as in (4.3). We also denote the quantization of $a(h,hx,\xi)$ by $\mathrm{Op}^h(a)$. We refer Hörmander [4] Vol. 3 or Zworski [16] for the pseudodifferential operator calculus.

3.2. Semiclassical reduction

We consider the "+" case only. The "−" case can be handled similarly. We suppose

$$(x_1, \xi_1, -x_2, \xi_2) \notin \Sigma_0 \cup \Sigma_+(\lambda) \cup \Sigma'_+(\lambda),$$

where $\lambda \in I \setminus \sigma_{\mathrm{p}}(H)$, $(x_1, x_2) \neq 0$, and we show $(x_1, \xi_1, x_2, \xi_2) \notin \mathrm{WF}'(K^+(\lambda))$. By the well-known semiclassical characterization of the wave front set (see, e.g., Martinez [10]), it suffice to show the existence of $a_0 \in C^\infty(T^*(M \times M))$ such that

$$a_0(x_1, \xi_1, -x_2, \xi_2) \neq 0$$

and

$$\big\|a_0(-hD_\xi, \xi, -hD_\eta, \eta) K^+(\lambda; \xi, \eta)\big\|_{L^2} \leq C_N h^N, \quad h \in (0,1],$$

with any N. We consider the case

$$a_0(x, \xi, -y, \eta) = a_1(x, \xi) a_2(y, \eta),$$

where $a_1, a_2 \in C_0^\infty(T^*M)$ are real-valued. We note, by the definition of the distribution kernel, the distribution kernel of $\mathrm{Op}^h(a_1)(H - \lambda - i0)^{-1}\mathrm{Op}^h(a_2)$ is given by

$$a_1(-hD_\xi, \xi) a_2(hD_\eta, \eta) K^+(\lambda, \xi, \eta) = a_0(-hD_\xi, \xi, -hD_\eta, \eta) K^+(\lambda, \xi, \eta).$$

We also note that for any $b \in C_0^\infty(\mathbb{R}^{2d})$,

$$\|\mathrm{Op}^h(b)\|_{HS} = (2\pi)^{-d/2}\|b\|_{L^2},$$

where $\|\cdot\|_{HS}$ and $\|\cdot\|_{B(L^2)}$ denote the Hilbert-Schmidt norm and the operator norm, respectively. Combining them, we learn

$$\begin{aligned}
&\big\|a_0(-hD_\xi, \xi, -hD_\eta, \eta)K^+(\lambda, \xi, \eta)\big\|_{L^2} \\
&\quad = \big\|\mathrm{Op}^h(a_1)(H-\lambda-i0)^{-1}\mathrm{Op}^h(a_2)\big\|_{HS} \\
&\quad \le Ch^{-d/2}\big\|\mathrm{Op}^h(a_1)(H-\lambda-i0)^{-1}\mathrm{Op}^h(a_2)\big\|_{B(L^2)}.
\end{aligned}$$

Thus it suffices to find $a_1, a_2 \in C_0^\infty(T^*M)$ such that $a_1(x_1,\xi_1) \neq 0$, $a_2(x_2,\xi_2) \neq 0$ and

$$\big\|\mathrm{Op}^h(a_1)(H-\lambda-i0)^{-1}\mathrm{Op}^h(a_2)\big\|_{B(L^2)} \le C_N h^N, \quad h \in (0,1], \tag{3.1}$$

for any N. In the following, we denote the operator norm of an operator A by $\|A\|$ without subscripts.

3.3. Case 1

At first we consider the easy case, i.e., either $p_0(\xi_1) \neq \lambda$ or $p_0(\xi_2) \neq \lambda$. For the moment we suppose $p_0(\xi_2) \neq \lambda$. Then we choose $a_2 \in C_0^\infty(T^*M)$ such that $a_2(x_2,\xi_2) = 1$ and

$$\mathrm{supp}[a_2] \subset \big\{(x,\xi) \;\big|\; |p_0(\xi)-\lambda| > 3\varepsilon\big\}$$

with some $\varepsilon > 0$. We then choose $f \in C_0^\infty(\mathbb{R})$ such that $f(z) = 1$ on $(\lambda-\varepsilon, \lambda+\varepsilon)$ and $\mathrm{supp}[f] \subset [\lambda-2\varepsilon, \lambda+2\varepsilon]$. By the functional calculus, $f(H)$ is a pseudodifferential operator with the symbol in S^0, and the symbol is supported in $p_0^{-1}([\lambda-2\varepsilon, \lambda+2\varepsilon])$ modulo the class of smoothing operators $S^{-\infty} = \bigcap_N S^{-N}$ (see, e.g., Dimassi-Sjöstrand [2], Zworski [16]). Hence, by the asymptotic expansion, we learn $f(H)\mathrm{Op}^h(a_2)$ has a symbol in $S_h^{-\infty} = \bigcap_N S_h^{-N}$. In particular, we have

$$\big\|\langle D_\xi\rangle f(H)\mathrm{Op}^h(a_2)\big\| \le C_N h^N, \quad h \in (0,1],$$

with any N. On the other hand, noting $(z-1)^{-1}(1-f(z)) \in S^0(\mathbb{R})$, we learn $(H-\lambda-i0)^{-1}(1-f(H))$ is a pseudodifferential operator with the symbol in S^0. We may suppose $\mathrm{supp}[a_1] \cap \mathrm{supp}[a_2] = \emptyset$, and hence

$$\big\|\mathrm{Op}^h(a_1)(H-\lambda-i0)^{-1}(1-f(H))\mathrm{Op}^h(a_2)\big\| \le C_N h^N \tag{3.2}$$

with any N. Combining these, we have

$$\begin{aligned}
&\|\mathrm{Op}^h(a_1)(H-\lambda-i0)^{-1}\mathrm{Op}^h(a_2)\| \\
&\leq \|\mathrm{Op}^h(a_1)\langle D_\xi\rangle\| \, \|\langle D_\xi\rangle^{-1}(H-\lambda-i0)^{-1}\langle D_\xi\rangle^{-1}\| \, \|\langle D_\xi\rangle f(H)\mathrm{Op}^h(a_2)\| \\
&\quad + \|\mathrm{Op}^h(a_1)(H-\lambda-i0)^{-1}(1-f(H))\mathrm{Op}^h(a_2)\| \\
&\leq C_N' h^{N-2}, \quad h\in(0,1],
\end{aligned}$$

since $\|\mathrm{Op}^h(a_j)\langle D_\xi\rangle\| = O(h^{-1})$ as $h\to+0$. This proves (3.1). The case $p_0(\xi_1)\neq\lambda$ is handled similarly.

3.4. Case 2

We now suppose $p_0(\xi_1)=p_0(\xi_2)=\lambda$. We choose $f\in C_0^\infty(\mathbb{R})$ so that $\mathrm{supp}[f]\Subset(I\setminus\sigma_{\mathrm{p}}(H))$ and $f=1$ on $[\lambda-\varepsilon,\lambda+\varepsilon]$ with some $\varepsilon>0$. Since $(H-\lambda-i0)^{-1}(1-f(H))$ is a pseudodifferential operator, (3.2) holds as well. Thus it suffices to consider $\mathrm{Op}^h(a_1)(H-\lambda-i0)^{-1}f(H)\mathrm{Op}^h(a_2)$. We recall

$$(H-\lambda-i0)^{-1} = i\lim_{\varepsilon\to+0}\int_0^\infty e^{-it(H-\lambda-i\varepsilon)}dt = i\int_0^\infty e^{it\lambda}e^{-itH}dt$$

in the weak sense. Thus it suffices to show

$$\int_0^\infty \|\mathrm{Op}^h(a_1)e^{-itH}f(H)\mathrm{Op}^h(a_2)\|dt \leq C_N h^N, \quad h\in(0,1], \tag{3.3}$$

for any N.

PROPOSITION 3.1. *Let $(x_1,\xi_1,-x_2,\xi_2)\notin\Sigma_0\cup\Sigma_+(\lambda)\cup\Sigma_+'(\lambda)$, and $p_0(\xi_1)=p_0(\xi_2)=\lambda$. If a_j are supported in sufficiently small neighborhoods of (x_j,ξ_j), $j=1,2$, then for any N there is C_N such that*

$$\|\mathrm{Op}^h(a_1)e^{-itH}f(H)\mathrm{Op}^h(a_2)\| \leq C_N h^N, \quad h\in(0,1], t\geq 0.$$

REMARK 3.1. Here we do not assume $\lambda\notin\sigma_{\mathrm{p}}(H)$. Thus the integrability in t does not necessarily hold. We also note that we assume $(x_1,x_2)\neq 0$, but one of $\{x_1,x_2\}$ may be 0.

We prove Proposition 3.1 in the next section, and we complete the proof of Theorem 2.1 assuming Proposition 3.1. By the multiple commutator estimate (Jensen-Mourre-Perry [9]), we have the following standard local decay estimate: for any $\nu > \kappa > 0$, there is C such that

$$\big\|\langle D_\xi\rangle^{-\nu} e^{-itH} f(H) \langle D_\xi\rangle^{-\nu}\big\| \le C\langle t\rangle^{-\kappa}, \quad t \in \mathbb{R}, \tag{3.4}$$

provided f is supported in $I \setminus \sigma_{\mathrm{p}}(H)$. We choose $\kappa = 2$, $\nu = 3$, and then we have

$$\begin{aligned}
&\big\|\mathrm{Op}^h(a_1) e^{-itH} f(H) \mathrm{Op}^h(a_2)\big\| \\
&\quad \le \big\|\mathrm{Op}^h(a_1)\langle D_\xi\rangle^3\big\| \, \big\|\langle D_\xi\rangle^{-3} e^{-itH} f(H)\langle D_\xi\rangle^{-3}\big\| \, \big\|\langle D_\xi\rangle^3 \mathrm{Op}^h(a_2)\big\| \\
&\quad \le Ch^{-6}\langle t\rangle^{-2}, \quad h \in (0,1], t \in \mathbb{R},
\end{aligned}$$

where we have used $\big\|\mathrm{Op}^h(a_j)\langle D_\xi\rangle^3\big\| = O(h^{-3})$. For an arbitrary $M > 0$, we set $N = 2M + 6$ in Proposition 3.1, and $T = h^{-M-6}$. Then we learn

$$\begin{aligned}
\int_0^\infty \big\|\mathrm{Op}^h(a_1) e^{-itH} f(H) \mathrm{Op}^h(a_2)\big\| dt &\le \int_0^T \cdots dt + \int_T^\infty \cdots dt \\
&\le C_N h^{2M+6} h^{-M-6} + Ch^{-6} h^{M+6} \\
&\le CM' h^M, \quad h \in (0,1].
\end{aligned}$$

This implies (3.3), and hence Theorem 2.1. □

4. Propagation Estimate : Proof of Proposition 3.1

We employ propagation estimate argument similar to that in Nakamura [13]. We note that the Egorov-type argument works for each t, but not uniformly in $t > 0$. Thus we cannot apply the Egorov-type argument directly here. We also note that the following symbol calculus is carried out mostly in a small local coordinate patch near ξ_2, and hence we can compute asymptotic expansions as in the Euclidean space case.

Let $(x_1, \xi_1, -x_2, \xi_2)$ be as in the proposition. Since $(x_1, \xi_1, -x_2, \xi_2) \notin \Sigma'_+(\lambda)$, either $x_1 + tv(\xi_1) \neq 0$ for $t \le 0$, or $x_2 + tv(\xi_2) \neq 0$ for $t \ge 0$. We first consider the latter case, i.e.,

$$x_2 + tv(\xi_2) \neq 0, \quad t \ge 0.$$

We remark that this assumption implies $x_2 \neq 0$, but the case $x_1 = 0$ is not excluded. Then there exist $\delta_1, \delta_2 > 0$ such that

$$\Omega(t) \cap \{(x_1, \xi_1)\} = \emptyset, \quad \Omega(t) \cap (\{0\} \times M) = \emptyset, \quad \text{for } t \ge 0, \tag{4.1}$$

where

$$\Omega(t) = \big\{(x,\xi) \bigm| |x-(x_2+tv(\xi_2))| \le 3\delta_1(1+t), |\xi-\xi_2| \le \delta_2\big\}.$$

We may also suppose δ_2 is so small that:

$$\text{if } |\xi-\xi_2| \le 2\delta_2 \text{ then } |v(\xi)-v(\xi_2)| < \delta_1/2. \tag{4.2}$$

We choose $\Phi \in C^\infty([0,\infty))$ such that $\Phi(s) = 1$ if $s \le 1/2$; $\Phi(s) = 0$ if $s \ge 1$; $\Phi(s) > 0$ if $s < 1$; and $\Phi'(s) \le 0$ for $s \le 1$. We also write $\Psi(s) = \Phi(s)^2$, $s \ge 0$. We now set

$$a_j(x,\xi) = \Phi\biggl(\frac{|x-x_j|}{\delta_1}\biggr)\Phi\biggl(\frac{|\xi-\xi_j|}{\delta_2}\biggr), \quad (x,\xi) \in T^*M, j = 1,2,$$

then $a_j \in S^0$, and $a_j(hx,\xi) \in S^0_h$. We also set

$$\phi_0(t,x,\xi) = \Phi\biggl(\frac{|x-y(t)|}{\delta_1(h^{-1}+t)}\biggr)\Phi\biggl(\frac{|\xi-\xi_2|}{\delta_2}\biggr), \quad (x,\xi) \in T^*M, t \ge 0,$$

where

$$y(t) = h^{-1}x_2 + tv(\xi_2) = h^{-1}(x_2 + htv(\xi_2)).$$

We note, by the condition (4.1),

$$|y(t)| \ge 3\delta_1 h^{-1}(1+ht), \quad t \ge 0.$$

On the other hand, by the support property of Φ, we have

$$|x-y(t)| \le \delta_1 h^{-1}(1+ht)$$

on the support of $\phi_0(t;\cdot,\cdot)$. Hence we learn

$$2\delta_1 h^{-1}(1+ht) \le |x| \le Ch^{-1}(1+ht)$$

with some C on the support of $\phi_0(t;\cdot,\cdot)$, and this implies $\phi_0 \in S^0_{h,t}$. We denote the support of $\phi_0(t,\cdot,\cdot)$ by

$$\Omega_0(t) = \big\{(x,\xi) \bigm| |x-y(t)| \le \delta_1(h^{-1}+t), |\xi-\xi_2| \le \delta_2\big\}.$$

We now quantize $\phi_0(t;x,\xi)$ by

$$\mathrm{Op}(\phi_0(t;\cdot,\cdot)) = \gamma^* m(\xi)^{-1/2}\chi(\xi)\phi_0^W(t;-D_\xi,\xi)\chi(\xi)m(\xi)^{1/2}(\gamma^{-1})^* \tag{4.3}$$

where γ is the identification map of the local coordinate patch; $\chi(\xi) \in C_0^\infty(\mathbb{R}^d)$ such that $\chi(\xi) = 1$ if $|\xi - \xi_2| \le \delta_2$ and $\chi(\xi) = 0$ if $|\xi - \xi_2| \ge 2\delta_2$; and $m(\xi)$ is the density. We use the same quantization in the following. We note $\mathrm{Op}(a)$ is symmetric if a is real-valued. We also note all the symbols here are supported in δ_2-neighborhood of ξ_2 modulo $S^{-\infty}_{h,t}$-terms.

We then define $\psi_0(t,x,\xi)$ be the symbol of $\mathrm{Op}(\phi_0(t;\cdot,\cdot))^*\mathrm{Op}(\phi_0(t;\cdot,\cdot))$ in the local coordinate patch. Clearly $\psi_0 \in S^0_{h,t}$ and the principal symbol is

$$\psi_0^0(t,x,\xi) = \Psi\biggl(\frac{|x-y(t)|}{\delta_1(h^{-1}+t)}\biggr)\Psi\biggl(\frac{|\xi-\xi_2|}{\delta_2}\biggr),$$

i.e., $\psi_0 - \psi_0^0 \in S^{-1}_{h,t}$. We note ψ_0 is supported in $\Omega_0(t)$ modulo $S^{-\infty}_{h,t}$.

Then we compute

$$\begin{aligned}
&\partial_t\psi_0^0(t,x,\xi) + v(\xi)\cdot\partial_x\psi_0^0(t,x,\xi)\\
&= \frac{1}{\delta_1(h^{-1}+t)}\biggl\{-\frac{|x-y(t)|}{h^{-1}+t} + \frac{x-y(t)}{|x-y(t)|}\cdot(v(\xi)-v(\xi_2))\biggr\}\times\\
&\quad\times\Psi'\biggl(\frac{|x-y(t)|}{\delta_1(h^{-1}+t)}\biggr)\Psi\biggl(\frac{|\xi-\xi_2|}{\delta_2}\biggr).
\end{aligned}$$

Since

$$\frac{\delta_1}{2}(h^{-1}+t) \le |x-y(t)|, \quad |v(\xi)-v(\xi_2)| \le \frac{\delta_1}{2}$$

on the support, we learn $\{\cdots\}$ in the RHS is nonpositive. Recalling $\Psi'(s) \le 0$, we learn

$$\partial_t\psi_0^0(t,x,\xi) + v(\xi)\cdot\partial_x\psi_0^0(t,x,\xi) \ge 0, \quad (x,t)\in T^*M, t\ge 0. \tag{4.4}$$

We also note $\partial_t\psi_0^0, \partial_x\psi_0^0 \in S^{-1}_{h,t}$. Then by the sharp Gårding inequality and asymptotic expansions, we learn

$$\partial_t\mathrm{Op}(\psi_0^0) + i[H_0, \mathrm{Op}(\psi_0^0)] \ge \mathrm{Op}(r_0^0)$$

with some $r_0^0 \in S^{-2}_{h,t}$. We then have, using the assumption on V,

$$\partial_t\mathrm{Op}(\psi_0) + i[H, \mathrm{Op}(\psi_0)] \ge \mathrm{Op}(r_0)$$

with some $r_0 \in S_{h,t}^{-1-\mu}$, supported in $\Omega(t)$ modulo $S_{h,t}^{-\infty}$.

Now we choose constants γ_j, $j = 1, 2, \dots$, so that $1 < \gamma_1 < \gamma_2 < \cdots < 2$. Let $C_j > 0$, $j = 1, 2, \dots$, be constants decided later. We then set

$$\psi_j(t,x,\xi) = C_j h^{(j-1)\mu}\big(h^\mu - (h^{-1}+t)^{-\mu}\big)\Psi\bigg(\frac{|x-y(t)|}{\gamma_j\delta_1(h^{-1}+t)}\bigg)\Psi\bigg(\frac{|\xi-\xi_2|}{\gamma_j\delta_2}\bigg)$$

for $(x,\xi) \in T^*M$, $t \geq 0$ and $j = 1, 2, \dots$. By direct computations, we see $\psi_j \in h^{j\mu}S_{h,t}^0$ and $\partial_t\psi_j \in h^{j\mu}S_{h,t}^{-1}$. Moreover, we have

$$\begin{aligned}&\partial_t\psi_j + v(\xi)\cdot\partial_x\psi_j \\ &\quad\geq \mu C_j h^{(j-1)\mu}(h^{-1}+t)^{-1-\mu}\Psi\bigg(\frac{|x-y(t)|}{\gamma_j\delta_1(h^{-1}+t)}\bigg)\Psi\bigg(\frac{|\xi-\xi_2|}{\gamma_j\delta_2}\bigg),\end{aligned}$$

which is proved similarly to (4.4). We set

$$\Omega_j(t) = \big\{(x,\xi) \bigm| |x-y(t)| \leq \gamma_j\delta_1(1+t), |\xi-\xi_2| \leq \gamma_j\delta_2\big\}.$$

Then $\psi_j(t,x,\xi)$ are supported in $\Omega_j(t)$, and

$$\partial_t\psi_j + v(\xi)\cdot\partial_x\psi_j(t,x,\xi) \geq \mu\kappa_j C_j h^{(j-1)\mu}(h^{-1}+t)^{-1-\mu} \quad \text{on } \Omega_{j-1}(t),$$

$j = 1, 2, \dots$, where $\kappa_j > 0$ are constants depending only on the choice of $\{\gamma_j\}$ and Ψ. Hence, if we choose C_1 sufficiently large, we have

$$\partial_t\psi_1 + v(\xi)\cdot\partial_x\psi_1 + r_0 \geq 0 \quad \text{on } T^*M \times ([0,\infty).$$

Then by using the sharp Gårding inequality again, we have

$$\partial_t \mathrm{Op}(\psi_0+\psi_1) + i[H, \mathrm{Op}(\psi_1+\psi_2)] \geq \mathrm{Op}(r_1)$$

with some $r_1 \in h^\mu S_{h,t}^{-1-\mu}$, supported in $\Omega_1(t)$ modulo $S_{h,t}^{-\infty}$. Repeating this procedure, we decide $C_2, C_3, \dots$, and we have

$$\partial_t\Big(\mathrm{Op}\Big(\textstyle\sum_{j=1}^m \psi_j\Big)\Big) + i\Big[H, \mathrm{Op}\Big(\textstyle\sum_{j=1}^m \psi_j\Big)\Big] \geq \mathrm{Op}(r_m),$$

where $r_m \in h^{m\mu}S_{h,t}^{-1-\mu}$, supported in $\Omega_m(t)$ modulo $S_{h,t}^{-\infty}$. In particular, we have

$$\int_0^\infty \big\|\mathrm{Op}(r_m)\big\|dt \leq Ch^{m\mu}\int_0^\infty (h^{-1}+t)^{-1-\mu}dt \leq C'h^{(m+1)\mu}.$$

We fix m large enough so that $(m+1)\mu > 2N$, where N is the exponent in Proposition 3.1.

Then we set

$$\psi(t,x,\xi) = \sum_{j=1}^{m} \psi_j(t,x,\xi) \in S^0_{h,t}.$$

We summarize the properties of ψ.

LEMMA 4.1. *ψ and $F(t) = \mathrm{Op}(\psi(t,\cdot,\cdot))$ satisfy the following properties:*

(1) *$\psi \in S^0_{h,t}$ and $F(0) = |\mathrm{Op}^h(a_2)|^2$.*

(2) *ψ is supported in*

$$\tilde{\Omega}(t) = \big\{(x,\xi) \;\big|\; |x-y(t)| \le 2\delta_1(h^{-1}+t), |\xi - \xi_2| \le 2\delta_2\big\}$$

modulo $S^{-\infty}_{h,t}$.

(3) *$F(t)$ satisfies the energy inequality:*

$$\partial_t F(t) + i[H, F(t)] \ge R(t),$$

where $\int_0^\infty \|R(t)\| dt \le Ch^{2N}$.

PROOF OF PROPOSITION 3.1. We recall the Heisenberg equation:

$$\frac{d}{dt}\big(e^{itH}F(t)e^{-itH}\big) = e^{itH}\big(\partial_t F(t) + i[H,F(t)]\big)e^{-itH},$$

and hence we have

$$\frac{d}{dt}\big(e^{itH}F(t)e^{-itH}\big) \ge e^{itH}R(t)e^{-itH}.$$

Integrating this inequality, we learn

$$e^{itH}F(t)e^{-itH} - F(0) \ge \int_0^t e^{itH}R(t)e^{-itH}dt \ge -Ch^{2N}$$

for all $t \ge 0$ by Lemma 4.1(3). Then, by using Lemma 4.1(1), we have

$$e^{-itH}|\mathrm{Op}^h(a_2)|^2 e^{itH} \le F(t) + Ch^{2N},$$

and hence

$$\left|\mathrm{Op}^h(a_1)e^{-itH}\mathrm{Op}^h(a_2)\right|^2 \le \mathrm{Op}^h(a_1)F(t)\mathrm{Op}^h(a_1) + Ch^{2N}.$$

We recall $\mathrm{supp}[a_1(h\cdot,\xi)] \cap \mathrm{supp}[\psi(t,\cdot,\cdot)] = \emptyset$; $\psi \in S^0_{h,t}$ and hence $\psi(t,\cdot,\cdot)$ is uniformly bounded in S^0_h. Then, by the asymptotic expansion, we learn $\|\mathrm{Op}^h(a_1)F(t)\| = O(h^{2N})$, $h \to +0$, uniformly in $t \ge 0$. These imply

$$\left\|\mathrm{Op}^h(a_1)e^{-itH}\mathrm{Op}^h(a_2)\right\|^2 \le Ch^{2N},$$

and we complete the proof of Proposition 3.1, provided $x_2 + tv(\xi_2) \ne 0$ for $t \ge 0$.

We now turn to the case $x_1 + tv(\xi_1) \ne 0$ for $t \le 0$. We consider

$$(\mathrm{Op}^h(a_1)e^{-itH}\mathrm{Op}^h(a_2))^* = \mathrm{Op}^h(a_2)e^{itH}\mathrm{Op}^h(a_1),$$

and replace t by $-t$. Then is is easy to check $(x_2, \xi_2, -x_1, \xi_1)$ satisfies the conditions in the other case. Thus the conclusion follows from the same argument as above. □

5. One-Sided Estimates

In Isozaki-Kitada [5, 7], another kind of estimates, called one-sided microlocal resolvent estimates, are proved. In this section, we formulate the one-sided estimates under our setting, and we show they are proved by the same method used to prove Theorem 2.1.

Theorem 5.1. *Let $\lambda \in I \setminus \sigma_{\mathrm{p}}(H)$ and suppose $a_\pm \in S^0(M)$ such that*

$$\mathrm{supp}[a_\pm] \subset \left\{ (x,\xi) \in T^*M \;\middle|\; \pm\frac{x \cdot v(\xi)}{|x|\,|v(\xi)|} > \pm(-1+\varepsilon), p_0(\xi) \in K \right\}$$

where $\varepsilon > 0$, $K \Subset M$. Let $\nu > 1$, $0 < s < \nu - 1$. Then $(H - \lambda \mp i0)^{-1}\mathrm{Op}(a_\pm)$ are bounded from $H^{-s}(M)$ to $H^{-\nu}(M)$.

We consider the "+" case only. The other case is proved similarly. Suppose $(x_2, \xi_2) \in \mathrm{supp}[a_+]$. Then $x_2 + tv(\xi_2) \ne 0$ for $t \ge 0$, and we can construct the symbols used in Section 4. We use the same notation as in Section 4, and we use the same time-dependent symbol $\psi(t, x, \xi)$ constructed from $a_2(x,\xi)$, which is supported in a small neighborhood of (x_2,ξ_2). Let $f \in C_0^\infty(\mathbb{R})$ also as in Section 4. We choose $\chi \in C_0^\infty(M)$ such that $\chi(\xi)f(H) = f(H)$ modulo $S^{-\infty}$. We then set

$$\zeta(t,x,\xi) = \Psi\biggl(\frac{2|x|}{\delta_1(h^{-1}+t)}\biggr)\chi(\xi), \quad (x,\xi) \in T^*M, t \geq 0,$$

and $\bar{\zeta}(t,x,\xi) = \chi(\xi) - \zeta(t,x,\xi) = (1-\Psi(\cdots))\chi(\xi)$. We note $\zeta, \bar{\zeta} \in S^0_{h,t}$. Then we observe $\|\mathrm{Op}(\zeta)\mathrm{Op}(\psi)\| = O(h^{2N})$ as $h \to +0$, uniformly in $t \geq 0$, again as in Section 4, since $\mathrm{supp}[\zeta] \cap \mathrm{supp}[\psi] = \emptyset$ modulo $S^{-\infty}_{h,t}$. Thus we arrive at the following estimate, analogously to Proposition 3.1:

Lemma 5.2. *For any N, there is $C_N > 0$ such that*

$$\bigl\|\mathrm{Op}(\zeta(t,\cdot,\cdot))e^{-itH}f(H)\mathrm{Op}^h(a_2)\bigr\| \leq C_N h^N, \quad h \in (0,1], t \geq 0.$$

We then have, using the decomposition $1 = \zeta + \bar{\zeta} + (1-\chi)$,

$$\begin{aligned}
&\bigl\|\langle D_\xi\rangle^{-\nu} e^{-itH} f(H)\mathrm{Op}^h(a_2)\bigr\| \\
&\leq \bigl\|\langle D_\xi\rangle^{-\nu}\mathrm{Op}(\bar{\zeta})\bigr\| \bigl\|e^{-itH}f(H)\mathrm{Op}^h(a_2)\bigr\| \\
&\quad + \bigl\|\langle D_\xi\rangle^{-\nu}\bigr\| \bigl\|\mathrm{Op}(\zeta)e^{-itH}f(H)\mathrm{Op}^h(a_2)\bigr\| \\
&\quad + \bigl\|\langle D_\xi\rangle^{-\nu}\bigr\| \bigl\|(1-\chi(\xi))f(H)\bigr\| \bigl\|e^{-itH}f(H)\mathrm{Op}^h(a_2)\bigr\| \\
&\leq C(h^{-1}+t)^{-\nu} + C_N h^N
\end{aligned}$$

for $h \in (0,1]$, $t \geq 0$. On the other hand, by the local decay estimate (3.4), we have

$$\bigl\|\langle D_\xi\rangle^{-\nu} e^{-itH} f(H)\mathrm{Op}^h(a_2)\bigr\| \leq Ch^{-\nu}\langle t\rangle^{-\kappa}, \quad t \in \mathbb{R},$$

with $1 < \kappa < \nu$. By setting $T = h^{-N/2}$ and choosing N large enough, we have

$$\begin{aligned}
\int_0^\infty \bigl\|\langle D_\xi\rangle^{-\nu} e^{-itH} f(H)\mathrm{Op}^h(a_2)\bigr\| dt &\leq \int_0^T \cdots dt + \int_T^\infty \cdots dt \\
&\leq C\int_0^\infty (h^{-1}+t)^{-\nu} dt + C_N h^{N/2} + Ch^{-\nu+(\kappa-1)N/2} \leq Ch^{\nu-1}.
\end{aligned}$$

Thus we obtain the following:

Lemma 5.3. *Let $\nu > 1$. Then*

$$\big\|\langle D_\xi\rangle^{-\nu}(H-\lambda-i0)^{-1}\mathrm{Op}^h(a_2)\big\| \le Ch^{\nu-1}, \quad h\in(0,1].$$

Now suppose $\tilde{a}\in S^0(M)$ such that its essential support in contained in a small conic neighborhood of (x_2,ξ_2). Then by the standard Littlewood-Paley decomposition argument, we learn

$$(H-\lambda-i0)^{-1}f(H)\mathrm{Op}(\tilde{a}) \quad \text{is bounded from } H^{-s}(M) \text{ to } H^{-\nu}(M),$$

where $0<s<\nu-1$. Since $(H-\lambda-i0)^{-1}(1-f(H))$ is a pseudodifferential operator, it is also bounded from $H^{-s}(M)$ to $H^{-s}(M)\subset H^{-\nu}(M)$. Thus $(H-\lambda-i0)^{-1}\mathrm{Op}(\tilde{a})$ is bounded from $H^{-s}(M)$ to $H^{-\nu}(M)$. Now Theorem 5.1 follows by the partition of unity argument. □

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