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Properties of Minimal Charts and Their Applications IV: Loops

By Teruo Nagase and Akiko Shima

Abstract. We investigate minimal charts with loops, a simple closed curve consisting of edges of label m containing exactly one white vertex. We shall show that there does not exist any loop in a minimal chart with exactly seven white vertices in this paper.

1. Introduction

Charts are oriented labeled graphs in a disk which represent surface braids (see [1],[5], and see Section 2 for the precise definition of charts, see [5, Chapter 14] for the definition of surface braids). In a chart there are three kinds of vertices; white vertices, crossings and black vertices. A C-move is a local modification of charts in a disk. The closures of surface braids are embedded oriented closed surfaces in 4-space $\mathbb{R}^4$ (see [5, Chapter 23] for the definition of the closures of surface braids). A C-move between two charts induces an ambient isotopy between the closures of the corresponding two surface braids.

We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat. In [11], we showed that there is no minimal chart with exactly five vertices (see Section 2 for the precise definition of minimal charts). Hasegawa proved that there exists a minimal chart with exactly six white vertices [2]. This chart represents the surface braid whose closure is ambient isotopic to a 2-twist spun trefoil. In [3] and [10], we investigated minimal charts with exactly four white vertices.

A *loop* of label m in a chart is a simple closed curve consisting of edges of label m which contains exactly one white vertex.

In this paper, we investigate properties of minimal charts and need to prove that there is no minimal chart with exactly seven white vertices. In

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particular we investigate a minimal chart containing a loop which bounds a disk containing three white vertices in its interior (see Lemma 3.2). We shall show the following theorem:

THEOREM 1.1. *There does not exist any loop in a minimal chart with exactly seven white vertices.*

Let Γ be a chart. If an object consists of some edges of Γ, arcs in edges of Γ and arcs around white vertices, then the object is called *a pseudo chart*.

The paper is organized as follows. In Section 2, we define charts and minimal charts. In Section 3, we give pseudo charts in a disk bounded by a loop in a minimal chart (Lemma 3.1 and Lemma 3.2). And we review a k-angled disk, a disk whose boundary consists of edges of label m and contains exactly k white vertices. In Section 4, we give useful lemmata and introduce notations. In Section 5, we give a proof of Lemma 3.2. In Section 6, we discuss about a solar eclipse, two loops with exactly one white vertex. In Section 7, we discuss about two kinds of subgraphs containing two loops of label m, one called a pair of eyeglasses and the other called a pair of skew eyeglasses. In Section 8, we prove Triangle Lemma(Lemma 8.3) for a 3-angled disk. In Section 9, we prove Theorem 1.1.

2. Preliminaries

Let n be a positive integer. An *n-chart* is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

(i) Every vertex has degree 1, 4, or 6.

(ii) The labels of edges are in $\{1, 2, \ldots, n-1\}$.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and the label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

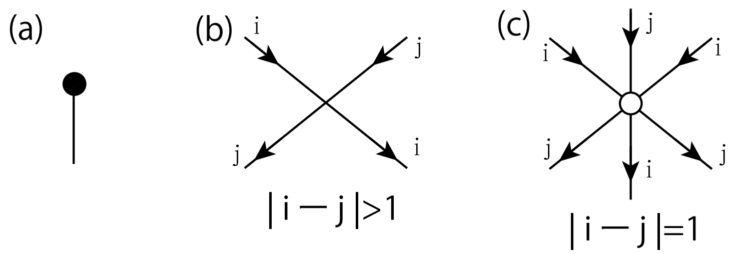

Fig. 1. (a) A black vertex. (b) A crossing. (c) A white vertex.

We call a vertex of degree 1 a *black vertex,* a vertex of degree 4 a *crossing*, and a vertex of degree 6 a *white vertex* respectively (see Fig. 1). Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward or outward is called a *middle arc* at the white vertex (see Fig. 1(c)). There are two middle arcs in a small neighborhood of each white vertex.

Now *C-moves* are local modifications of charts in a disk as shown in Fig. 2 (see [1], [5], [12] for the precise definition). These C-moves as shown in Fig. 2 are examples of C-moves.

Let Γ be a chart. For each label m, we denote by Γ_m the 'subgraph' of Γ consisting of all the edges of label m and their vertices. An edge of Γ is the closure of a connected component of the set obtained by taking out all white vertices and crossings from Γ. On the other hand, we assume that

> an *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m.

Thus any vertex of Γ_m is a black vertex or a white vertex. Hence any crossing of Γ is not considered as a vertex of Γ_m.

Let Γ be a chart, m a label of Γ, and e an edge of Γ or Γ_m. The edge e is called a *free edge* if it has two black vertices. The edge e is called a *terminal edge* if it has a white vertex and a black vertex. The edge e is a *loop* if it is a closed edge with only one white vertex. Note that free edges, terminal edges, and loops may contain crossings of Γ.

Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

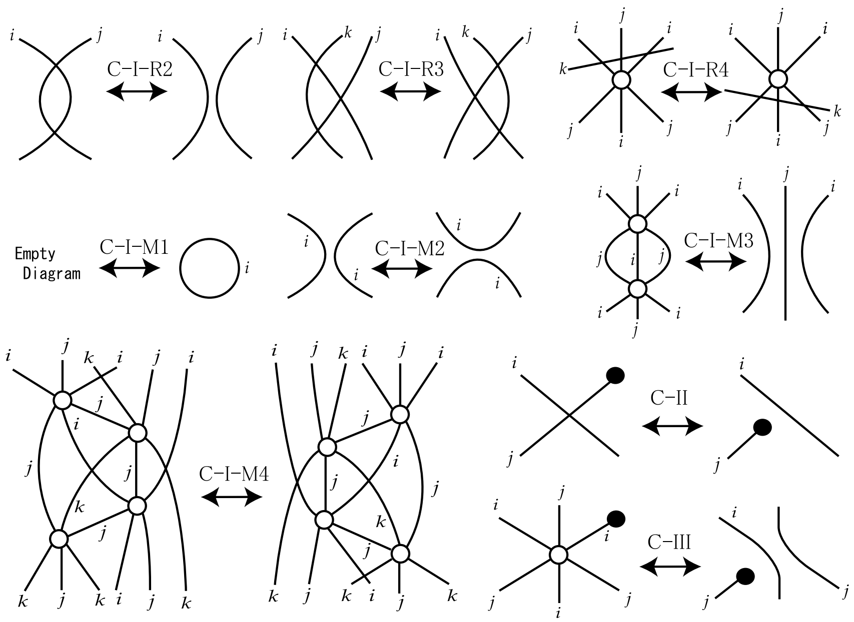

Fig. 2. For the C-III move, the edge containing the black vertex does not contain a middle arc in the left figure.

For each chart Γ, let $w(\Gamma)$ and $f(\Gamma)$ be the number of white vertices, and the number of free edges respectively. The pair $(w(\Gamma), -f(\Gamma))$ is called a *complexity* of the chart (see [4]). A chart Γ is called a *minimal chart* if its complexity is minimal among the charts C-move equivalent to the chart Γ with respect to the lexicographic order of pairs of integers.

We showed the difference of a chart in a disk and in a 2-sphere (see [6, Lemma 2.1]). This lemma follows from that there exists a natural one-to-one correspondence between $\{$charts in $S^2\}$/C-moves and $\{$charts in $D^2\}$/C-moves, conjugations ([5, Chapter 23 and Chapter 25]). To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk.

Assumption 1. *In this paper, all charts are contained in the* 2*-sphere* S^2.

We have the special point in the 2-sphere S^2, called *the point at infinity*, denoted by ∞. In this paper, all charts are contained in a disk such that the disk does not contain the point at infinity ∞.

Let Γ be a chart, and m a label of Γ. A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). A *ring* is a closed edge of Γ_m containing a crossing but not containing any white vertices. A hoop is said to be *simple* if one of the two complementary domains of the hoop does not contain any white vertices.

We can assume that all minimal charts Γ satisfy the following five conditions (see [6],[7],[8]):

Assumption 2. *No terminal edge of* Γ_m *contains a crossing. Hence any terminal edge of* Γ_m *contains a middle arc.*

Assumption 3. *No free edge of* Γ_m *contains a crossing. Hence any free edge of* Γ_m *is a free edge of* Γ*.*

Assumption 4. *All free edges and simple hoops in* Γ *are moved into a small neighborhood* U_∞ *of the point at infinity* ∞*. Hence we assume that* Γ *does not contain free edges nor simple hoops, otherwise mentioned.*

Assumption 5. *Each complementary domain of any ring and hoop must contain at least one white vertex.*

Assumption 6. *The point at infinity* ∞ *is moved in any complementary domain of* Γ*.*

In this paper for a set X in a space we denote the interior of X, the boundary of X and the closure of X by $\mathrm{Int}X$, ∂X and $Cl(X)$ respectively.

3. k-Angled Disks

Let ℓ be a loop of label m in a chart Γ. Let e be the edge of Γ_m containing the white vertex in ℓ with $e \neq \ell$. Then the loop ℓ bounds two disks on the 2-sphere. One of the two disks does not contain the edge e. The disk is called *the associated disk of the loop* ℓ (see Fig. 3).

Let Γ be a chart, m a label of Γ, D a disk, and k a positive integer. If ∂D consists of k edges of Γ_m, then D is called *a* k*-angled disk of* Γ_m.

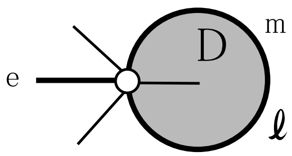

Fig. 3. The gray region is the associated disk of a loop ℓ.

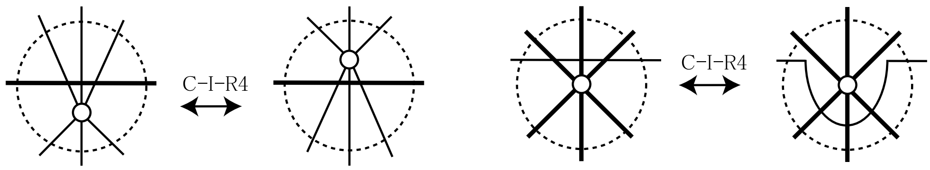

Fig. 4. C-moves keeping thicken figures fixed.

Note that the boundary ∂D may contain crossings, and each of two disks bounded by a loop of label m is a 1-angled disk of Γ_m.

Let Γ and Γ' be C-move equivalent charts. Suppose that a pseudo chart X of Γ is also a pseudo chart of Γ'. Then we say that Γ is modified to Γ' by *C-moves keeping X fixed.* In Fig. 4, we give examples of C-moves keeping pseudo charts fixed.

Let Γ be a chart, and m a label of Γ. Let D be a k-angled disk of Γ_m, and G a pseudo chart in D with $\partial D \subset G$. Let $r : D \to D$ be a reflection of D, and G^* the pseudo chart obtained from G by changing the orientations of all of the edges. Then the set $\{G, G^*, r(G), r(G^*)\}$ is called the *RO-family of the pseudo chart G.*

Warning. To draw a pseudo chart in a RO-family, we draw a part of $\Gamma \cap N$ here N is a regular neighborhood of a k-angled disk D. For example, we draw the pseudo chart in Fig. 5(a) for a pseudo chart in a RO-family in Fig. 5(b).

Let X be a set in a chart Γ. Let

$$w(X) = \text{the number of white vertices in } X,$$

$$c(X) = \text{the number of crossings in } X.$$

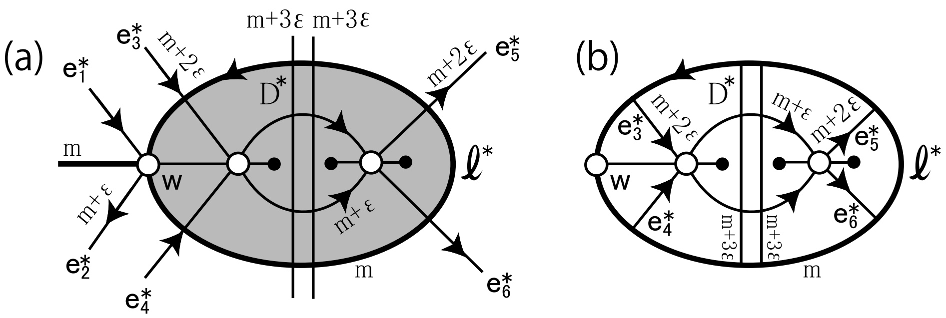

Fig. 5. The gray region is a disk D^*, the thick lines are edges of label m, and $\varepsilon \in \{+1, -1\}$.

LEMMA 3.1 ([7, Lemma 4.2 and Lemma 8.1]). *Let Γ be a minimal chart with a loop ℓ^* of label m with the associated disk D^*. Let ε be the integer in $\{+1, -1\}$ such that the white vertex in ℓ^* is in $\Gamma_{m+\varepsilon}$. Then we have the following:*

(1) *$w(\Gamma \cap \mathrm{Int}D^*) \geq 2$ and $w(\Gamma \cap (S^2 - D^*)) \geq 2$.*

(2) *If $w(\Gamma \cap \mathrm{Int}D^*) = 2$, then D^* contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 5*(a) *by C-moves in D^* keeping ∂D^* fixed.*

Let Γ be a minimal chart, and m a label of Γ. Let D be a k-angled disk of Γ_m. The pair of integers $(w(\Gamma \cap \mathrm{Int}D), c(\Gamma \cap \partial D))$ is called the *local complexity with respect to D*. Let $\mathbb{S}$ be the set of all minimal charts obtained from Γ by C-moves in a regular neighborhood of D keeping ∂D fixed. The chart Γ is said to be *locally minimal with respect to D* if its local complexity with respect to D is minimal among the charts in $\mathbb{S}$ with respect to the lexicographic order of pairs of integers.

The following lemma will be proved in Section 5.

LEMMA 3.2. *Let Γ be a minimal chart with a loop ℓ of label k. Let μ be the integer in $\{+1, -1\}$ such that the white vertex in ℓ is in $\Gamma_{k+\mu}$. If Γ is locally minimal with respect to the associated disk D of ℓ and if*

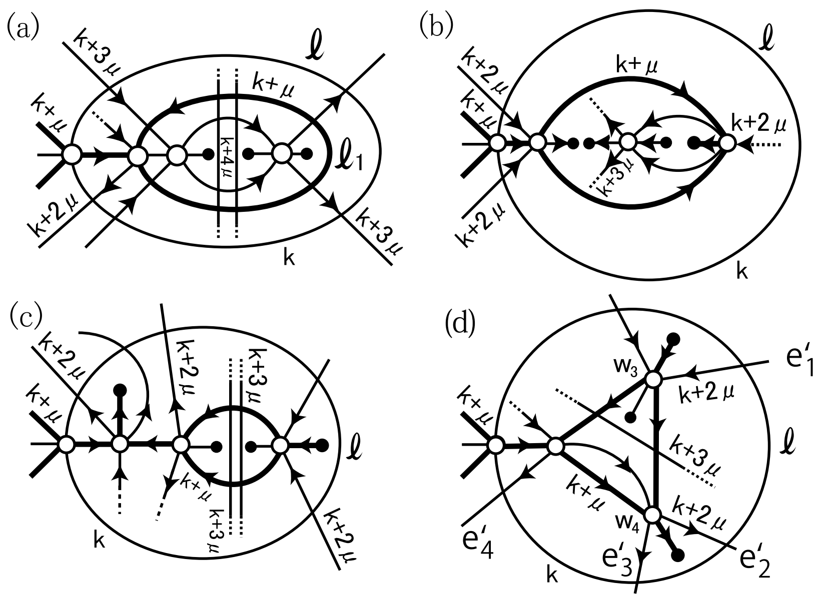

Fig. 6. The thick lines are edges of label $k+\mu$ where $\mu \in \{+1, -1\}$.

$w(\Gamma \cap \text{Int}D) \leq 3$, *then D contains a pseudo chart of the RO-families of the five pseudo charts as shown in Fig.* 5(a) *and Fig. 6.*

Let Γ be a chart, and m a label of Γ. An edge of Γ_m is called a *feeler* of a k-angled disk D of Γ_m if the edge intersects $N - \partial D$ where N is a regular neighborhood of ∂D in D.

The following lemma will be used in the proofs of Theorem 1.1 and Lemma 3.2.

Lemma 3.3 ([8, Theorem 1.1]). *Let* Γ *be a minimal chart. Let D be a* 2*-angled disk of* Γ_m *with at most one feeler such that* Γ *is locally minimal with respect to D. If* $w(\Gamma \cap \text{Int}D) \leq 1$, *then D contains a pseudo chart of the RO-families of the five pseudo charts as shown in Fig. 7.*

Let Γ be a chart, and D a k-angled disk of Γ_m. If any feeler of D is a terminal edge, then we say that D is *special*.

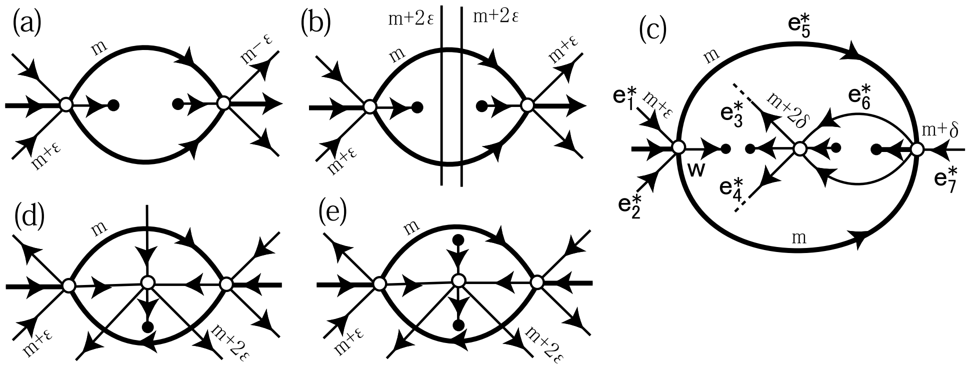

Fig. 7. (a), (b), (d), (e) 2-angled disks without feelers. (c) A 2-angled disk with one feeler. The thick lines are edges of label m, and $\varepsilon, \delta \in \{+1, -1\}$.

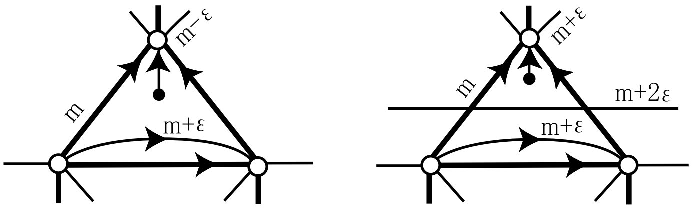

Fig. 8. The thick lines are edges of label m, and $\varepsilon \in \{+1, -1\}$.

The following lemma will be used in Case (e) and Case (f) of the proof of Lemma 3.2.

Lemma 3.4 ([8, Theorem 1.2]). *Let* Γ *be a minimal chart. Let* D *be a special* 3*-angled disk of* Γ_m *such that* Γ *is locally minimal with respect to* D*. If* $w(\Gamma \cap \mathrm{Int} D) = 0$*, then* D *contains a pseudo chart of the RO-families of the two pseudo charts as shown in Fig. 8.*

4. Useful Lemmata

Let Γ be a chart. Let D be a disk such that

(1) ∂D consists of an edge e_1 of Γ_m and an edge e_2 of Γ_{m+1}, and

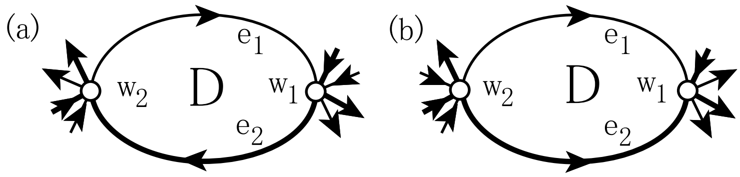

Fig. 9.

(2) any edge containing a white vertex in e_1 does not intersect the open disk Int$D$.

Note that ∂D may contain crossings. Let w_1 and w_2 be the white vertices in e_1. If the disk D satisfies one of the following conditions, then D is called *a lens of type* $(m, m+1)$ (see Fig. 9):

(i) Neither e_1 nor e_2 contains a middle arc.

(ii) One of the two edges e_1 and e_2 contains middle arcs at both white vertices w_1 and w_2 simultaneously.

The following lemma will be used in Case (a), Case (d) and Case (e) of the proof of Lemma 3.2.

LEMMA 4.1 ([6, Theorem 1.1]). *Let* Γ *be a minimal chart. Then there exist at least three white vertices in the interior of any lens.*

LEMMA 4.2 ([7, Lemma 6.1]). *Let* Γ *be a minimal chart. If a connected component* G *of* Γ_m *contains a white vertex, then* G *contains at least two white vertices.*

Let α be an arc, and p, q points in α. We denote by $\alpha[p, q]$ the subarc of α whose end points are p and q.

Let Γ be a chart, and m a label of Γ. Let α be an arc in an edge of Γ_m, and w a white vertex with $w \notin \alpha$. Suppose that there exists an arc β in Γ such that its end points are the white vertex w and an interior point p of the arc α. Then we say that *the white vertex* w *connects with the point* p *of* α *by the arc* β.

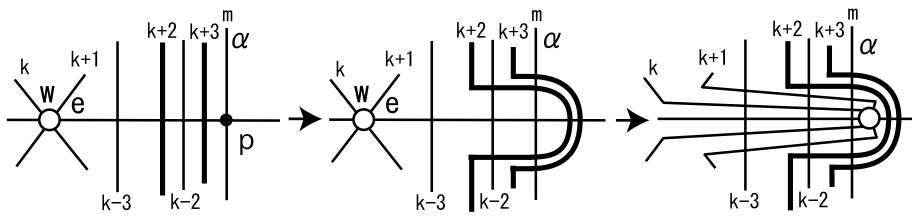

Fig. 10. Case (1): $k > m$ and $\varepsilon = +1$.

The following lemma will be used in Case (c) of the proof of Lemma 3.2.

LEMMA 4.3 ([6, Lemma 4.2]) (Shifting Lemma). *Let* Γ *be a chart and* α *an arc in an edge of* Γ_m*. Let* w *be a white vertex of* $\Gamma_k \cap \Gamma_h$ *where* $h = k + \varepsilon, \varepsilon \in \{+1, -1\}$*. Suppose that the white vertex* w *connects with a point* p *of the arc* α *by an arc in an edge* e *of* Γ_k*. Suppose that one of the following two conditions is satisfied:*

(1) $h > k > m$.

(2) $h < k < m$.

Then for any neighborhood V *of the arc* $e[w,p]$ *we can shift the white vertex* w *to* $e - e[w,p]$ *along the edge* e *by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves in* V *keeping* $\bigcup_{i<0} \Gamma_{k+i\varepsilon}$ *fixed* (*see Fig. 10*).

The following lemma will be used in Case (d) in the proof of Lemma 3.2 and Step 3, Step 6 and Step 8 in the proof of Theorem 1.1.

LEMMA 4.4 ([6, Lemma 5.4]). *If a minimal chart* Γ *contains the pseudo chart as shown in Fig. 11, then the interior of the disk* D^* *contains at least one white vertex, where* D^* *is the disk with the boundary* $e_3^* \cup e_4^* \cup e^*$.

We use the following notation:

In our argument, we often need a name for an unnamed edge by using a given edge and a given white vertex. For the convenience, we use the

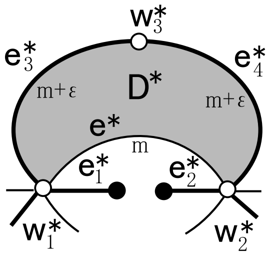

Fig. 11. The gray region is the disk D^*. The label of the edge e^* is m, and $\varepsilon \in \{+1, -1\}$.

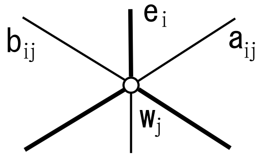

Fig. 12.

following naming: Let e', e_i, e'' be three consecutive edges containing a white vertex w_j. Here, the two edges e' and e'' are unnamed edges. There are six arcs in a neighborhood U of the white vertex w_j. If the three arcs $e' \cap U$, $e_i \cap U$, $e'' \cap U$ lie anticlockwise around the white vertex w_j in this order, then e' and e'' are denoted by a_{ij} and b_{ij} respectively (see Fig. 12). There is a possibility $a_{ij} = b_{ij}$ if they are contained in a loop.

Let Γ be a chart, and v a vertex. Let α be a short arc of Γ in a small neighborhood of v with $v \in \partial\alpha$. If the arc α is oriented to v, then α is called *an inward arc*, and otherwise α is called *an outward arc*.

Let Γ be an n-chart. Let F be a closed domain with $\partial F \subset \Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}$ for some label k of Γ, where $\Gamma_0 = \emptyset$ and $\Gamma_n = \emptyset$. By Condition (iii) for charts, in a small neighborhood of each white vertex, there are three inward arcs and three outward arcs. Also in a small neighborhood of each black vertex, there exists only one inward arc or one outward arc. We often use the following fact, when we fix (inward or outward) arcs near white vertices and black vertices:

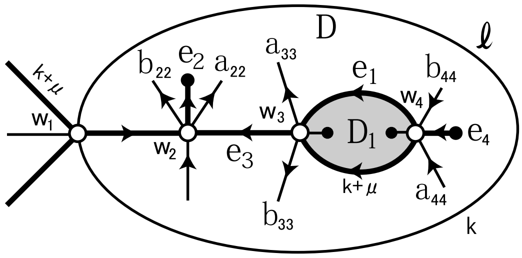

Fig. 13. The thick lines are edges of label $k+\mu$ where $\mu \in \{+1,-1\}$. The gray region is a 2-angled disk D_1.

> *The number of inward arcs contained in $F \cap \Gamma_k$ is equal to the number of outward arcs in $F \cap \Gamma_k$.*

When we use this fact, we say that we use *IO-Calculation with respect to Γ_k in F*. For example, in a minimal chart Γ, consider the pseudo chart as shown in Fig. 13 where $\mu \in \{+1,-1\}$ and

(1) D is the associated disk of a loop ℓ of label k with $w(\Gamma \cap \text{Int}D) = 3$,

(2) w_2, w_3, w_4 are white vertices in Int$D$ with $w_2, w_3, w_4 \in \Gamma_{k+\mu}$,

(3) none of the four edges $a_{33}, b_{33}, a_{44}, b_{44}$ contains a middle arc at w_3 nor w_4 (by Assumption 2 none of them is a terminal edge).

By (2), we have $w_i \in \Gamma_k$ or $w_i \in \Gamma_{k+2\mu}$ for each $i = 2, 3, 4$. Let $F = Cl(D - D_1)$. Then we can show that

if $w_2 \in \Gamma_{k+2\mu}$, then $w_3, w_4 \in \Gamma_k$ or $w_3, w_4 \in \Gamma_{k+2\mu}$.

For if not, then one of w_3, w_4 is in Γ_k and the other is in $\Gamma_{k+2\mu}$. If $w_3 \in \Gamma_k$ and $w_4 \in \Gamma_{k+2\mu}$, then considering $\partial D = \ell$ contains one inward arc and one outward arc, by (3) the number of inward arcs in $F \cap \Gamma_k$ is one, but the number of outward arcs in $F \cap \Gamma_k$ is three. This is a contradiction. Similarly if $w_3 \in \Gamma_{k+2\mu}$ and $w_4 \in \Gamma_k$, then we have the same contradiction. Thus if $w_2 \in \Gamma_{k+2\mu}$, then $w_3, w_4 \in \Gamma_k$ or $w_3, w_4 \in \Gamma_{k+2\mu}$. Instead of the above argument, we just say that

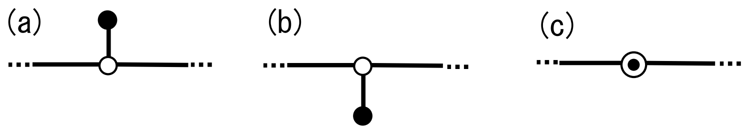

Fig. 14.

if $w_2 \in \Gamma_{k+2\mu}$*, then we have* $w_3, w_4 \in \Gamma_k$ *or* $w_3, w_4 \in \Gamma_{k+2\mu}$ *by IO-Calculation with respect to* Γ_k *in* F*.*

In our argument we often construct a chart Γ. On the construction of a chart Γ, for a white vertex w, among the three edges of Γ_m containing w, if one of the three edges is a terminal edge (see Fig. 14(a) and (b)), then we remove the terminal edge and put a black dot at the center of the white vertex as shown in Fig. 14(c). Namely Fig. 14(c) means Fig. 14(a) or Fig. 14(b).

5. Proof of Lemma 3.2

We start to prove Lemma 3.2. Let Γ be a minimal chart with a loop ℓ of label k such that Γ is locally minimal with respect to the associated disk D of ℓ. Suppose $w(\Gamma\cap\text{Int}D) \leq 3$. By Lemma 3.1, we only need to examine the case $w(\Gamma\cap\text{Int}D) = 3$. In this section we assume that

(1) w_1 is the white vertex in the loop ℓ of label k, and

(2) μ is the integer in $\{+1, -1\}$ with $w_1 \in \Gamma_{k+\mu}$.

By Lemma 3.1(1), the condition $w(\Gamma\cap\text{Int}D) = 3$ implies that the disk D contains at most one loop of label $k + \mu$. Thus we can easily prove that the disk D contains a pseudo chart of the RO-families of the six pseudo charts as shown in Fig. 15. We use the notations as shown in Fig. 15 throughout this section. There are six cases (see Fig. 15).

Case (a). Suppose that the disk D contains the pseudo chart as shown in Fig. 15(a), where

(a-1) the loop ℓ_1 is of label $k + \mu$, and bounds a disk D_1.

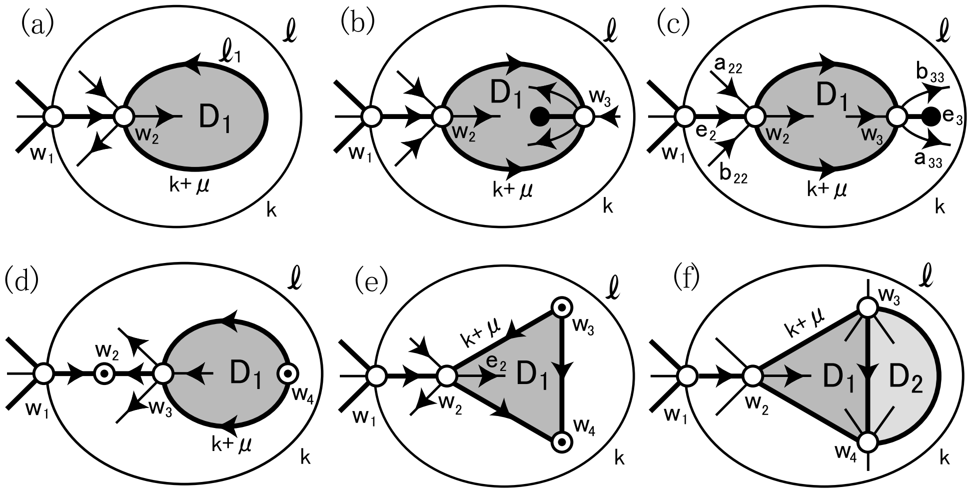

Fig. 15. The thick lines are edges of label $k+\mu$ where $\mu \in \{+1, -1\}$.

Since $w(\Gamma \cap \mathrm{Int} D) = 3$, by Lemma 3.1(1) the disk D_1 contains exactly two white vertices in its interior, say w_3, w_4. Further by Lemma 3.1(2) the disk D_1 contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 5(a). Comparing Fig. 5(a) and Fig. 15(a), we have $D^* = D_1, m = k+\mu, w = w_2$ and

(a-2) $e_1^*, e_2^* \subset \Gamma_{k+\mu+\varepsilon}$, $e_3^*, e_4^*, e_5^*, e_6^* \subset \Gamma_{k+\mu+2\varepsilon}$ where $\varepsilon \in \{+1, -1\}$ and

(a-3) none of $e_3^*, e_4^*, e_5^*, e_6^*$ contains a middle arc at w_3 nor w_4 (hence none of $e_3^*, e_4^*, e_5^*, e_6^*$ is a terminal edge by Assumption 2).

We claim that all of $e_3^*, e_4^*, e_5^*, e_6^*$ intersect $\ell = \partial D$. If not, then there exists a lens in D not containing any white vertices in its interior, because $e_3^* \neq e_6^*$ and $e_4^* \neq e_5^*$. This contradicts Lemma 4.1. Hence all of the four edges must intersect ∂D. Thus $2 \leq |(k+\mu+2\varepsilon) - k| = |\mu + 2\varepsilon|$. Hence $\varepsilon = \mu$ because $\varepsilon, \mu \in \{+1, -1\}$. Thus by (a-2), we have $e_1^*, e_2^* \subset \Gamma_{k+2\mu}$, and $e_3^*, e_4^*, e_5^*, e_6^* \subset \Gamma_{k+3\mu}$. Therefore we have the pseudo chart as shown in Fig. 6(a).

Case (b). Suppose that the disk D contains the pseudo chart as shown in Fig. 15(b), where (b-1) D_1 is a 2-angled disk of $\Gamma_{k+\mu}$ with *one feeler*.

Since $w(\Gamma \cap \text{Int}D) = 3$, we have $w(\Gamma \cap \text{Int}D_1) \leq 1$. Thus by Lemma 3.3, the disk D_1 contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 7(c) because the only pseudo chart contains a feeler. Comparing Fig. 7(c) and Fig. 15(b), we have $m = k + \mu, w = w_2$ and

(b-2) $e_1^*, e_2^* \subset \Gamma_{k+\mu+\varepsilon}$ where $\varepsilon \in \{+1, -1\}$,

(b-3) $e_3^*, e_4^* \subset \Gamma_{k+\mu+2\delta}$, $e_5^* \subset \Gamma_{k+\mu}$, $e_6^*, e_7^* \subset \Gamma_{k+\mu+\delta}$ where $\delta \in \{+1, -1\}$.

We shall show $\mu = \varepsilon = \delta$. If $\varepsilon = -\mu$, then we have $e_1^*, e_2^* \subset \Gamma_k$. Thus $e_1^*, e_2^* \subset Cl(D - D_1)$. Hence whether $e_7^* \subset \Gamma_k$ or not, we have a contradiction by IO-Calculation with respect to Γ_k in $Cl(D - D_1)$. Thus $\varepsilon = \mu$. If $\delta = -\mu$, then we have $e_3^*, e_4^* \subset \Gamma_{k-\mu}$. Thus $e_3^*, e_4^* \subset D$. Since $e_1^*, e_2^* \subset \Gamma_{k+2\mu}$ and since $e_7^* \subset \Gamma_k$, we have a contradiction by IO-Calculation with respect to $\Gamma_{k-\mu}$ in D. Thus we have $\delta = \mu$. Hence by (b-2) and (b-3), we have

(b-4) $e_1^*, e_2^* \subset \Gamma_{k+2\mu}$, $e_3^*, e_4^* \subset \Gamma_{k+3\mu}$, and $e_6^*, e_7^* \subset \Gamma_{k+2\mu}$.

Therefore we have the pseudo chart as shown in Fig. 6(b).

Case (c). Suppose that the disk D contains the pseudo chart as shown in Fig. 15(c) where

(c-1) D_1 is a 2-angled disk of $\Gamma_{k+\mu}$ without feelers,

(c-2) the edges in ∂D_1 are oriented from w_2 to w_3, and

(c-3) each of edges $a_{22}, b_{22}, a_{33}, b_{33}$ is of label k or $k + 2\mu$.

Let w_4 be the white vertex in $\text{Int}D$ different from w_2 and w_3. We show that we can push w_4 out D by C-moves. Since $w(\Gamma \cap \text{Int}D) = 3$, we have $w(\Gamma \cap \text{Int}D_1) \leq 1$. Thus considering the orientations of edges in ∂D_1, by (c-1), (c-2) and Lemma 3.3, the disk D_1 contains a pseudo chart of the RO-families of the two pseudo charts as shown in Fig. 7(a) and (b). Hence $w_4 \in D - D_1$. Let s be the label with $w_4 \in \Gamma_s \cap \Gamma_{s+1}$. There are three cases:

(i) for each $j = s, s + 1$ none of edges of Γ_j containing w_4 intersect ∂D,

(ii) for each $j = s, s + 1$ there exists an edge of Γ_j containing w_4 and intersecting ∂D,

(iii) otherwise.

Case (i). By Lemma 4.2, for each $j = s, s+1$ the connected component of Γ_j containing w_4 contains another white vertex w_2 or w_3. Hence there exists an edge e'_j of Γ_j with $e'_j \ni w_4$ containing w_2 or w_3. Here one of e'_s and e'_{s+1} is a_{22} or b_{22} and the other is a_{33} or b_{33}. Thus by (c-3), the difference of labels of e'_s and e'_{s+1} is 0 or ± 2. This is a contradiction, because e'_s is of label s and e'_{s+1} is of lable $s+1$. Hence Case (i) does not occur.

Case (ii). We have $|s-k| \geq 2$ and $|(s+1)-k| \geq 2$, because $\ell = \partial D$ is of label k. Hence we can show $s < s+1 < k$ or $k < s < s+1$. By Shifting Lemma(Lemma 4.3) we can push the white vertex w_4 from Int$D$ to the exterior of D by C-moves. However this contradicts the fact that Γ is locally minimal with respect to D. Hence Case (ii) does not occur.

Case (iii). Let e be the edge Γ_s or Γ_{s+1} with $e \ni w_4$ and $e \cap \partial D \neq \emptyset$, and t the label of e. Then $t \in \{s, s+1\}$. Let $j \in \{s, s+1\}$ be the label different from t. Then $t = j + \delta$ for some $\delta \in \{+1, -1\}$ and none of edges of Γ_j containing w_4 intersects ∂D. In a similar way to Case (i), we can show that there exists an edge e'_j of Γ_j with $e'_j \ni w_4$ containing w_2 or w_3. Here e'_j is a_{22}, b_{22}, a_{33} or b_{33}. By (c-3), we have $j = k$ or $j = k + 2\mu$. Thus we have $t = k + \delta$ or $t = k + 2\mu + \delta$. Since the edge e of label t intersects the loop $\ell = \partial D$ of label k, we have $|t - k| \geq 2$. Hence $\mu = \delta$ and $t = k + 3\mu$. Thus

(c-4) $w_4 \in \Gamma_{k+2\mu} \cap \Gamma_{k+3\mu}$.

There exists an arc β in a regular neighborhood of $e'_j \cup D_1 \cup e_2 \cup e_3$ in D connecting the vertex w_4 and a point x in ∂D such that Intβ intersects Γ transversely and

(c-5) $\beta \cap \Gamma_{k+\mu} = \emptyset$.

Let p be a point in Intβ with $\Gamma \cap \beta[w_4, p] = w_4$. By C-I-R2 moves, in a regular neighborhood N of $\beta[p, x]$ we can push all the arcs of $N \cap (\cup_{i \geq 2} \Gamma_{k+i\mu})$ from D along β so that $(\beta - w_4) \cap \Gamma_{k+i\mu} = \emptyset$ $(i \geq 2)$, and by (c-4) and (c-5) we can push the white vertex w_4 along β from Int$D$ to the exterior of D by C-I-R2 moves and C-I-R4 moves (cf. [6, Corollary 4.5]). However again this contradicts the fact Γ is locally minimal with respect to D. Hence the disk D does not contain the pseudo chart as shown in Fig. 15(c).

Case (d). Suppose that the disk D contains the pseudo chart as shown in Fig. 15(d). Since $w(\Gamma \cap \text{Int} D_1) = 0$, considering the orientation of edges on ∂D_1, Lemma 3.3 assures that the disk D_1 contains a pseudo chart of the RO-families of the two pseudo charts as shown in Fig. 7(a) and (b). So we have the pseudo chart as shown in Fig. 13 used for the explanation of IO-Calculation, here

(d-1) none of edges $a_{22}, b_{22}, a_{33}, b_{33}, a_{44}, b_{44}$ contains a middle arc at w_2, w_3 nor w_4 (by Assumption 2 none of them is a terminal edge).

Since $w_2 \in \Gamma_{k+\mu}$, we have also $w_2 \in \Gamma_k$ or $w_2 \in \Gamma_{k+2\mu}$. If $w_2 \in \Gamma_k$, then $w_4 \in \Gamma_k$ by IO-Calculation with respect to Γ_k in $Cl(D - D_1)$. Hence $w_3 \in \Gamma_{k+2\mu}$ by IO-Calculation with respect to Γ_k in $Cl(D - D_1)$. Thus by (d-1), we have $a_{22} = b_{44}$ and $b_{22} = a_{44}$. Since w_2, w_3, w_4 are all the white vertices in $\text{Int} D$, the disk D_2 bounded by $a_{22} \cup e_1 \cup e_3$ does not contain any white vertices in its interior. This contradicts Lemma 4.4 by setting $D^* = D_2, e_1^* = e_4, e_2^* = e_2, e^* = a_{22}, e_3^* = e_1, e_4^* = e_3$ in Fig. 11. Hence $w_2 \in \Gamma_{k+2\mu}$. Thus as we have done in the explanation of IO-Calculation, we have $w_3, w_4 \in \Gamma_k$ or $w_3, w_4 \in \Gamma_{k+2\mu}$ by IO-Calculation with respect to Γ_k in $Cl(D - D_1)$. If $w_3, w_4 \in \Gamma_k$, then there exist two lenses of type $(k, k+\mu)$ in D not containing any white vertices in their interiors. This contradicts Lemma 4.1. Thus $w_3, w_4 \in \Gamma_{k+2\mu}$. Looking at Fig. 13, we have the pseudo chart as shown in Fig. 6(c).

Case (e). Suppose that the disk D contains the pseudo chart as shown in Fig. 15(e) where

(e-1) w_3, w_4 are contained in terminal edges of label $k + \mu$, say e_3, e_4,

(e-2) the edge e_2 does not contain a middle arc at w_2 (hence e_2 is not a terminal edge by Assumption 2),

(e-3) the edge e_2 contains an outward arc at w_2.

By (e-1), any feeler of D_1 is a terminal edge. Thus the disk D_1 is a special 3-angled disk of $\Gamma_{k+\mu}$. Hence $w(\Gamma \cap \text{Int} D_1) = 0$ and Lemma 3.4 imply that the disk D_1 contains a pseudo chart of the RO-families of the two pseudo charts as shown in Fig. 8 each of which has no feeler. Hence neither e_3 nor e_4 is a feeler. That is (e-4) $e_3 \cup e_4 \subset Cl(D - D_1)$.

Comparing Fig. 8 and Fig. 15(e), we have $m = k + \mu$, and by (e-2) the edge e_2 is an edge of label $m + \varepsilon$ with two white vertices where $\varepsilon \in \{+1, -1\}$. Hence

(e-5) the edge e_2 is an edge of $\Gamma_{k+\mu+\varepsilon}$ with $\partial e_2 = \{w_2, w_3\}$ or $\partial e_2 = \{w_2, w_4\}$.

If $\partial e_2 = \{w_2, w_3\}$, then by (e-3) the edge e_2 is oriented from w_2 to w_3. This contradicts Condition (iii) of the definition of a chart. Hence $\partial e_2 = \{w_2, w_4\}$ (see Fig. 16). Thus

(e-6) $w_2, w_4 \in \Gamma_{k+\mu+\varepsilon}$.

On the other hand, by (e-1), we have

(e-7) $w_3 \in \Gamma_{k+\mu} \cap \Gamma_k$ or $w_3 \in \Gamma_{k+\mu} \cap \Gamma_{k+2\mu}$.

Suppose $\mu = -\varepsilon$. Then $w_2, w_4 \in \Gamma_k$ by (e-6). If $w_3 \in \Gamma_k$, then we have $a_{33} = b_{44}$ in Fig. 16 because the loop ℓ is of label k. Hence there exists a lens of type $(k, k + \mu)$ in $Cl(D - D_1)$ containing w_3, w_4 but not containing any white vertices in its interior. This contradicts Lemma 4.1. Hence $w_3 \in \Gamma_{k+2\mu}$ by (e-7). But we have a contradiction by IO-Calculation with respect to Γ_k in $Cl(D - D_1)$. Thus $\mu = \varepsilon$.

By (e-6), we have $w_2, w_4 \in \Gamma_{k+2\mu}$. If $w_3 \in \Gamma_k$, then we have a contradiction by IO-Calculation with respect to Γ_k in $Cl(D-D_1)$. Hence $w_3 \in \Gamma_{k+2\mu}$ by (e-7). Therefore we have the pseudo chart as shown in Fig. 6(d).

Case (f). Suppose that the disk D contains the pseudo chart as shown in Fig. 15(f). Since $w(\Gamma \cap \mathrm{Int} D_2) = 0$, by Lemma 3.3 the 2-angled disk D_2 contains a pseudo chart of the RO-families of the two pseudo charts as shown in Fig. 7(a) and (b). Hence both the two edges in ∂D_2 are oriented from w_3 to w_4. Thus

(f-1) the boundary of the 3-angled disk D_1 is oriented clockwise.

On the other hand, the disk D_1 does not have any feelers. Thus the disk D_1 is special. Since $w(\Gamma \cap \mathrm{Int} D_1) = 0$, by Lemma 3.4 the disk D_1 contains a pseudo chart of the RO-families of the two pseudo charts as shown in Fig. 8. This contradicts Condition (f-1). Hence D does not contain the pseudo chart as shown in Fig. 15(f).

Therefore we complete the proof of Lemma 3.2. □

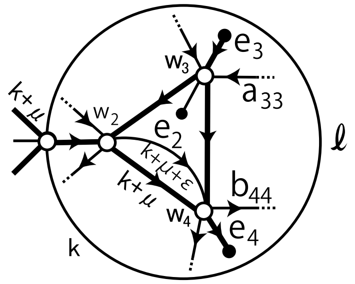

Fig. 16. The thick lines are edges of label $k+\mu$ where $\mu, \varepsilon \in \{+1, -1\}$.

6. Solar Eclipses

LEMMA 6.1 ([7, Corollary 1.3]). *Let Γ be a minimal chart with at most seven white vertices. Then there is no lens of Γ.*

A subgraph of a chart is called a *solar eclipse* if it consists of two loops and contains only one white vertex.

LEMMA 6.2 ([7, Lemma 8.4]). *Let Γ be a minimal chart with at most seven white vertices. If there exists a solar eclipse, then the associated disk of each loop of the solar eclipse contains at least three white vertices in its interior.*

LEMMA 6.3. *Let D be the associated disk of a loop ℓ of label m in a minimal chart Γ with $w(\Gamma) = 7$ such that Γ is locally minimal with respect to D. Let ε be the integer in $\{+1, -1\}$ such that the white vertex in ℓ is in $\Gamma_{m+\varepsilon}$. Then we have the following:*

(1) *If $w(\Gamma \cap \text{Int}D) = 2$, then $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) \geq 2$.*

(2) *If $w(\Gamma \cap \text{Int}D) = 3$, then $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) \geq 1$.*

(3) *If $w(\Gamma \cap \text{Int}D) = 3$ and if the disk D contains one of the two pseudo charts as shown in Fig. 6*(a) *and* (b) *here $m = k$ and $\varepsilon = \mu$, then $w(\Gamma_{m+3\varepsilon} \cap (S^2 - D)) \geq 1$.*

Proof. We show Statement (1). By Lemma 3.1(2), the disk D contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 5(a) here $D = D^*$. Since none of the four edges $e_3^*, e_4^*, e_5^*, e_6^*$ of $\Gamma_{m+2\varepsilon}$ in Fig. 5(a) contain a middle arc at a white vertex in Int$D$, all of four edges $e_3^*, e_4^*, e_5^*, e_6^*$ are not terminal edges by Assumption 2.

If $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) = 0$, then $e_3^* = e_5^*$ and $e_4^* = e_6^*$. Thus there exist two lenses of type $(m+\varepsilon, m+2\varepsilon)$. This contradicts Lemma 6.1.

Now suppose that $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) = 1$. Let D_1 be the 2-angled disk of $\Gamma_{m+2\varepsilon}$ in D. Since $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) = 1$ and $w(\Gamma_{m+2\varepsilon} \cap (D - D_1)) = 0$, there exists only one white vertex of $\Gamma_{m+2\varepsilon}$ in $S^2 - D_1$. Thus we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in $Cl(S^2 - D_1)$. Hence $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) \geq 2$.

We show Statement (3). Suppose that the disk D contains the pseudo chart as shown in Fig. 6(a) here $k = m$ and $\mu = \varepsilon$. Then the loop ℓ_1 of label $m + \varepsilon$ in D bounds the associated disk D_1 with $w(\Gamma \cap \text{Int} D_1) = 2$. By Lemma 6.3(1), we have $w(\Gamma_{(m+\varepsilon)+2\varepsilon} \cap (S^2 - D_1)) \geq 2$. Since $w(\Gamma_{m+3\varepsilon} \cap (D - D_1)) = 0$, we have $w(\Gamma_{m+3\varepsilon} \cap (S^2 - D)) \geq 2$. Thus clearly $w(\Gamma_{m+3\varepsilon} \cap (S^2 - D)) \geq 1$.

Suppose that the disk D contains the pseudo chart as shown in Fig. 6(b) here $k = m$ and $\mu = \varepsilon$. By Lemma 4.2, the condition $w(\Gamma_{m+3\varepsilon} \cap D) = 1$ implies $w(\Gamma_{m+3\varepsilon} \cap (S^2 - D)) \geq 1$.

We show Statement (2). Now suppose that

(i) $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) = 0$.

Since $w(\Gamma \cap \text{Int} D) = 3$, by Lemma 3.2 the disk D contains a pseudo chart of the RO-families of the four pseudo charts as shown in Fig. 6 where $k = m$ and $\mu = \varepsilon$.

Suppose that the disk D contains the pseudo chart as shown in Fig. 6(a). By (i), we have a loop ℓ_2 of label $m + 2\varepsilon$ so that $\ell_1 \cup \ell_2$ is a solar eclipse. But the associated disk of ℓ_1 contains only two white vertices in its interior. This contradicts Lemma 6.2.

Suppose that the disk D contains the pseudo chart as shown in Fig. 6(b). Let D_1 be the 2-angled disk of $\Gamma_{m+\varepsilon}$ in D. Since $w(\Gamma_{m+2\varepsilon} \cap (D - D_1)) = 0$, the condition (i) implies that $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D_1)) = 0$. Hence we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in $Cl(S^2 - D_1)$.

Suppose that the disk D contains the pseudo chart as shown in Fig. 6(c). Let D_1 be the 2-angled disk of $\Gamma_{m+\varepsilon}$ in D. Since there exists only one white vertex of $\Gamma_{m+2\varepsilon}$ in $D - D_1$, the condition (i) implies that $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D_1)) = 1$. Hence we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in $Cl(S^2 - D_1)$.

Suppose the disk D contains the pseudo chart as shown in Fig. 6(d). Let D_1 be the 3-angled disk of $\Gamma_{m+\varepsilon}$ in D. Since $w(\Gamma_{m+2\varepsilon} \cap (D - D_1)) = 0$, the condition (i) implies that $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D_1)) = 0$. Hence for the edge e'_1 in Fig. 6(d) we have $e'_1 = e'_2$, $e'_1 = e'_3$ or $e'_1 = e'_4$. If $e'_1 = e'_2$, then there exists a lens of type $(m+\varepsilon, m+2\varepsilon)$ containing w_3, w_4. This contradicts Lemma 6.1. If $e'_1 = e'_3$ or $e'_1 = e'_4$, then the edge e'_1 separates the disk $Cl(S^2 - D_1)$ into two disks. Let D' be the disk of the two disks containing the edge e'_2. Then we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in D'.

Therefore we have a contradiction for any cases. Hence $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) \geq 1$. □

PROPOSITION 6.4. *There is no solar eclipse in a minimal chart with exactly seven white vertices.*

PROOF. Suppose that there exists a solar eclipse G in a minimal chart Γ with $w(\Gamma) = 7$. Then G consists of two loops ℓ_1 and ℓ_2 with $\ell_1 \subset \Gamma_m$ and $\ell_2 \subset \Gamma_{m+1}$ for some label m. We can assume that Γ is locally minimal with respect to the associated disks D_1 and D_2 of the loops ℓ_1 and ℓ_2 respectively. By Lemma 6.2, we have $w(\Gamma \cap \mathrm{Int} D_1) \geq 3$ and $w(\Gamma \cap \mathrm{Int} D_2) \geq 3$. Since $w(\Gamma) = 7$ and $w(\ell_1 \cup \ell_2) = 1$, we have $w(\Gamma \cap \mathrm{Int} D_1) = 3$ and $w(\Gamma \cap \mathrm{Int} D_2) = 3$. By Lemma 3.2, the two disks D_1 and D_2 contain pseudo charts of the RO-families of the four pseudo charts as shown in Fig. 6. For $\ell_2 \subset \Gamma_{m+1}$ where $\ell = \ell_2, k = m+1, \mu = -1$ in Fig. 6 we have

(1) $w((\Gamma_m \cup \Gamma_{m-1}) \cap \mathrm{Int} D_2) = 3$.

Since $S^2 - D_1 \supset \mathrm{Int} D_2$, by (1) we have $w((\Gamma_m \cup \Gamma_{m-1}) \cap (S^2 - D_1)) \geq w((\Gamma_m \cup \Gamma_{m-1}) \cap \mathrm{Int} D_2) = 3$. Setting $\ell = \ell_1 \subset \Gamma_m$ and $\varepsilon = 1$ (because the white vertex in ℓ_1 is in Γ_{m+1}), applying Lemma 6.3(2) we have $w(\Gamma_{m+2} \cap (S^2 - D_1)) \geq 1$. Hence we have

$$\begin{aligned} w(\Gamma \cap (S^2 - D_1)) &\geq w((\Gamma_m \cup \Gamma_{m-1}) \cap (S^2 - D_1)) + w(\Gamma_{m+2} \cap (S^2 - D_1)) \\ &\geq 3 + 1 = 4. \end{aligned}$$

Thus

$$w(\Gamma) = w(\ell_1) + w(\Gamma \cap \mathrm{Int}D_1) + w(\Gamma \cap (S^2 - D_1)) \geq 1 + 3 + 4 = 8.$$

This contradicts the fact $w(\Gamma) = 7$. Therefore there is no solar eclipse in Γ. □

LEMMA 6.5. *Let ℓ be a loop of label m in a minimal chart Γ with $w(\Gamma) = 7$, and D the associated disk of ℓ. Then $w(\Gamma_m \cap (S^2 - D)) \leq 2$.*

PROOF. Suppose that $w(\Gamma_m \cap (S^2 - D)) \geq 3$. We shall show $w(\Gamma) \geq 8$. It suffices to prove the case that Γ is locally minimal with respect to D. Let ε be the integer in $\{+1, -1\}$ such that the white vertex in ℓ is in $\Gamma_{m+\varepsilon}$. By Lemma 3.1(1) we have $w(\Gamma \cap \mathrm{Int}D) \geq 2$.

If $w(\Gamma \cap \mathrm{Int}D) = 2$, then by Lemma 6.3(1) we have $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) \geq 2$. Since $w(\Gamma_m \cap (S^2 - D)) \geq 3$, we have $w(\Gamma \cap (S^2 - D)) \geq 3 + 2$. Thus

$$w(\Gamma) = w(\ell) + w(\Gamma \cap \mathrm{Int}D) + w(\Gamma \cap (S^2 - D)) \geq 1 + 2 + (3 + 2) = 8.$$

If $w(\Gamma \cap \mathrm{Int}D) = 3$, then by Lemma 6.3(2) we have $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D)) \geq 1$. Similarly we can show $w(\Gamma) \geq 1 + 3 + (3 + 1) = 8$. If $w(\Gamma \cap \mathrm{Int}D) \geq 4$, then we can show $w(\Gamma) \geq 1 + 4 + (3 + 0) = 8$. Hence we have $w(\Gamma) \geq 8$. This contradicts the fact $w(\Gamma) = 7$. Therefore $w(\Gamma_m \cap (S^2 - D)) \leq 2$. □

7. Pairs of Eyeglasses and Pairs of Skew Eyeglasses

LEMMA 7.1 (cf. [7, Lemma 6.2]). *Let Γ be a minimal chart, and m a label of Γ. Let G be a connected component of Γ_m containing a white vertex and a loop. If G contains at most three white vertices, then G is one of the three subgraphs as shown in Fig. 17.*

We call the subgraph as shown in Fig. 17(a) *a pair of eyeglasses.* We call the subgraphs as shown in Fig. 17(b) and (c) *pairs of skew eyeglasses of type* 1 *and* 2 respectively.

LEMMA 7.2. *Let G be a pair of eyeglasses in a minimal chart Γ with $w(\Gamma) = 7$. Let D_1, D_2 be the associated disks of loops in G. If Γ is locally minimal with respect to the disks D_1 and D_2, then $w(\Gamma \cap \mathrm{Int}D_1) = 2$ and $w(\Gamma \cap \mathrm{Int}D_2) = 2$.*

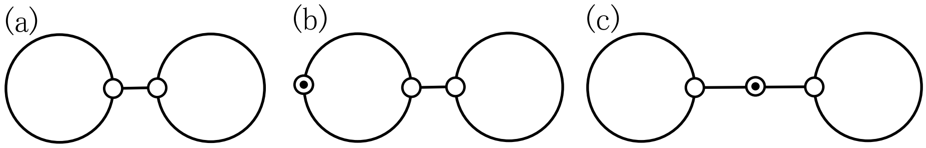

Fig. 17.

PROOF. Since $w(\Gamma\cap\text{Int}D_1)+w(\Gamma\cap\text{Int}D_2)+w(G) \leq w(\Gamma) = 7$, we have $w(\Gamma\cap\text{Int}D_1) + w(\Gamma\cap\text{Int}D_2) \leq 5$. Thus by Lemma 3.1(1) we have

$$(w(\Gamma \cap \text{Int}D_1), w(\Gamma \cap \text{Int}D_2)) = (2, 2), (2, 3) \text{ or } (3, 2).$$

Suppose that $(w(\Gamma\cap\text{Int}D_1), w(\Gamma\cap\text{Int}D_2)) = (2, 3)$. Then by $w(\Gamma) = 7$, we have

(1) $w(\Gamma \cap (S^2 - (D_1 \cup D_2))) = 0$.

Further by Lemma 3.1(2) the disk D_1 contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 5(a) (see Fig. 18(a)). Without loss of generality we can assume that the edge e_1 of Γ_m is oriented from w_2 to w_1. We use the notations as shown in Fig. 18(a). Hence

(2) $w(\Gamma_{m+\varepsilon} \cap D_1) = 3$.

One of the two edges a_{11}, b_{11} of $\Gamma_{m+\varepsilon}$ in Fig. 18(a) does not contain a middle arc at w_1, one of the two edges is not a terminal edge by Assumption 2. Thus by (1), there are three cases: $a_{11} = b_{11}, a_{11} \ni w_2$ or $b_{11} \ni w_2$. But if $a_{11} = b_{11}$, then $a_{11} \cup \ell_1$ is a solar eclipse. This contradicts Proposition 6.4. Hence we have

(3) $w_2 \in a_{11} \subset \Gamma_{m+\varepsilon}$ or $w_2 \in b_{11} \subset \Gamma_{m+\varepsilon}$.

Since $w(\Gamma\cap\text{Int}D_2) = 3$, by Lemma 3.2 the disk D_2 contains a pseudo chart of the RO-families of the four pseudo charts as shown in Fig. 6. By (3), for the label k and $\mu \in \{+1, -1\}$ in Fig. 6 we have $k = m$ and $\mu = \varepsilon$.

Suppose that D_2 contains one of the two pseudo charts as shown in Fig. 6(a) and (b). Then we have $w(\Gamma_{m+3\varepsilon}\cap(S^2-D_2)) \geq 1$ by Lemma 6.3(3).

Further $S^2 - D_2 \supset D_1$ and (2) imply that $w((\Gamma_{m+\varepsilon} \cup \Gamma_{m+3\varepsilon}) \cap (S^2 - D_2)) \geq 3 + 1 = 4$. Thus we have $w(\Gamma \cap (S^2 - D_2)) \geq 4$. Hence

$$w(\Gamma) = w(\ell_2) + w(\Gamma \cap \mathrm{Int}D_2) + w(\Gamma \cap (S^2 - D_2)) \geq 1 + 3 + 4 = 8.$$

This contradicts the fact $w(\Gamma) = 7$.

Suppose that D_2 contains the pseudo chart as shown in Fig. 6(c) (see Fig. 18(b)). Now let E_i be the 2-angled disk of $\Gamma_{m+\varepsilon}$ in D_i for $i = 1, 2$. Since $(D_1 \cup D_2) - (E_1 \cup E_2)$ contains only one white vertex of $\Gamma_{m+2\varepsilon}$, the condition (1) implies $w(\Gamma_{m+2\varepsilon} \cap (S^2 - (E_1 \cup E_2))) = 1$. Hence we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in $Cl(S^2 - (E_1 \cup E_2))$.

Suppose that D_2 contains the pseudo chart as shown in Fig. 6(d) (see Fig. 18(c)) where

(d-1) none of e_1^*, e_2^*, e_3^* in Fig. 18(c) contains a middle arc at w_4^* or w_5^* (by Assumption 2, none of e_1^*, e_2^*, e_3^* is a terminal edge),

(d-2) by (3) all seven white vertices are contained in the same connected component of $\Gamma_{m+\varepsilon}$.

Claim. All of e_1^*, e_2^*, e_3^* contain white vertices in $\mathrm{Int}D_1$.

For, by (1) and (d-1), for the edge e_1^*, there are four cases: $e_1^* = e_4^*$, $e_1^* = e_5^*$, the edge e_1^* is a loop or the edge e_1^* contains a white vertex in $\mathrm{Int}D_1$.

If $e_1^* = e_4^*$ or $e_1^* = e_5^*$, the edge e_1^* separates the disk $Cl(S^2 - D_3)$ into two disks, here D_3 is the 3-angled disk of $\Gamma_{m+\varepsilon}$ in D_2 (see Fig. 18(c)). Let D' be one of the two disks contains the edge e_2^*. Then we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in D'. If e_1^* is a loop, then the associated disk D^* of e_1^* contains w_2. Further by (d-2), the disk D^* must contain the three white vertices in D_1. Hence $S^2 - (D_3 \cup D^*)$ does not contain any white vertices. Thus $e_3^* = e_4^*$ and $e_2^* = e_5^*$. Hence there exists a lens of type $(m + \varepsilon, m + 2\varepsilon)$. This contradicts Lemma 6.1. Therefore e_1^* contains a white vertex in $\mathrm{Int}D_1$. Similarly we can show that each of the two edges e_2^*, e_3^* contains a white vertex in $\mathrm{Int}D_1$. Thus Claim holds.

But the three edges e_1^*, e_2^*, e_3^* of $\Gamma_{m+2\varepsilon}$ are oriented outward at the same white vertex in $\mathrm{Int}D_1$. This is a contradiction.

Thus we have a contradiction for any cases. Hence $(w(\Gamma \cap \mathrm{Int}D_1), w(\Gamma \cap \mathrm{Int}D_2)) \neq (2, 3)$. In a similar way as above, we can show

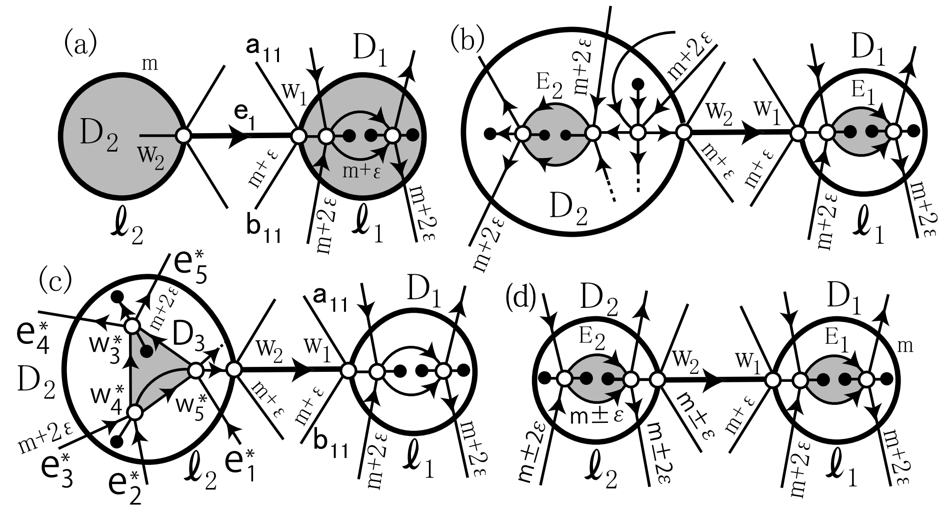

Fig. 18. (a) The gray regions are disks D_1, D_2. (b), (d) The gray regions are disks E_1, E_2. (c) The gray region is a 3-angled disk D_3.

that $(w(\Gamma\cap\text{Int}D_1), w(\Gamma\cap\text{Int}D_2)) \neq (3,2)$. Therefore $(w(\Gamma\cap\text{Int}D_1), w(\Gamma\cap\text{Int}D_2)) = (2,2)$. This completes the proof of Lemma 7.2. □

PROPOSITION 7.3. *There is no pair of eyeglasses in a minimal chart with exactly seven white vertices.*

PROOF. Suppose that there exists a pair of eyeglasses G in a minimal chart Γ with $w(\Gamma) = 7$. Without loss of generality we can assume that Γ is locally minimal with respect to the associated disks D_1 and D_2 of loops in G. By Lemma 7.2, we have $(w(\Gamma\cap\text{Int}D_1), w(\Gamma\cap\text{Int}D_2)) = (2,2)$. By Lemma 3.1(2) each of the disks D_1 and D_2 contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 5(a) (see Fig. 18(d)). Moreover there exists exactly one white vertex in $S^2 - (D_1 \cup D_2)$, say w'.

If $w' \in \Gamma_m$, then this contradicts Lemma 4.2. Thus $w' \notin \Gamma_m$.

If $w' \in \Gamma_{m+2\varepsilon}$, then there exists only one white vertex w' of $\Gamma_{m+2\varepsilon}$ in $S^2 - (E_1 \cup E_2)$ where E_i is the 2-angled disk E_i in D_i for $i = 1, 2$. Hence if $w_2 \in \Gamma_{m+\varepsilon}$ (resp. $w_2 \notin \Gamma_{m+\varepsilon}$, i.e. $w_2 \in \Gamma_{m-\varepsilon}$) then we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in $Cl(S^2-(E_1\cup E_2))$ (resp. $Cl(S^2-E_1)$). Thus $w'\notin\Gamma_{m+2\varepsilon}$.

Similarly we can show $w'\notin\Gamma_{m-2\varepsilon}$. Now $w'\notin\Gamma_m\cup\Gamma_{m\pm 2\varepsilon}$ implies $w'\notin\Gamma_{m\pm\varepsilon}$, because a white vertex in Γ_k is also in Γ_{k-1} or Γ_{k+1}. Hence the vertex w' is contained in Γ_k for some label k with $k\neq m, m\pm\varepsilon, m\pm 2\varepsilon$. However Γ_k contains exactly one white vertex w'. This contradicts Lemma 4.2. Therefore there is no pair of eyeglasses in Γ. $\square$

Proposition 7.4. *There is no pair of skew eyeglasses of type* 2 *in a minimal chart with exactly seven white vertices.*

Proof. Suppose that there exists a pair of skew eyeglasses G of type 2 in a minimal chart Γ with $w(\Gamma)=7$. We only show that the edge e_3 of Γ_m is oriented inward at w_3. We use the notations as shown in Fig. 19(a). By Lemma 3.1(1), the condition $w(\Gamma)=7$ implies $(w(\Gamma\cap\mathrm{Int}D_1), w(\Gamma\cap\mathrm{Int}D_2))=(2,2)$. Thus

(i) $w(\Gamma\cap(S^2-(G\cup D_1\cup D_2)))=0$.

For the edge a_{33} of $\Gamma_{m+\varepsilon}$ in Fig. 19(a), there are five cases: (1) $a_{33}=a_{11}$, (2) $a_{33}=b_{11}$, (3) $a_{33}=a_{22}$, (4) $a_{33}=b_{22}$, (5) a_{33} is a loop.

Case (1). The union $a_{33}\cup e_1$ bounds a lens. This contradicts Lemma 6.1.

Case (3). We have $b_{33}=b_{22}$. Thus $b_{33}\cup e_2$ bounds a lens. This contradicts Lemma 6.1.

Case (4). The union $a_{33}\cup e_2$ bounds a disk D' containing the edge b_{33}. By (i), we have $w(\Gamma\cap\mathrm{Int}D')=0$. Thus we have a contradiction by IO-Calculation with respect to $\Gamma_{m+\varepsilon}$ in D'.

Case (5). We have $b_{33}\ni w_2$ because the edge b_{33} does not contain a middle arc at w_3 (the edge b_{33} is not a terminal edge by Assumption 2). Thus $w_2\in\Gamma_{m+\varepsilon}$. Hence whether $w_1\in\Gamma_{m-\varepsilon}$ or $w_1\in\Gamma_{m+\varepsilon}$, we have $a_{11}=b_{11}$, because one of a_{11} and b_{11} does not contain a middle arc. Thus $a_{11}\cup\ell_1$ is a solar eclipse. This contradicts Proposition 6.4.

Hence Case (2) occurs. Similarly we can show $b_{33}=a_{22}$. Thus the two edges a_{11}, b_{22} are terminal edges. By Lemma 3.1(2), both of D_1 and D_2

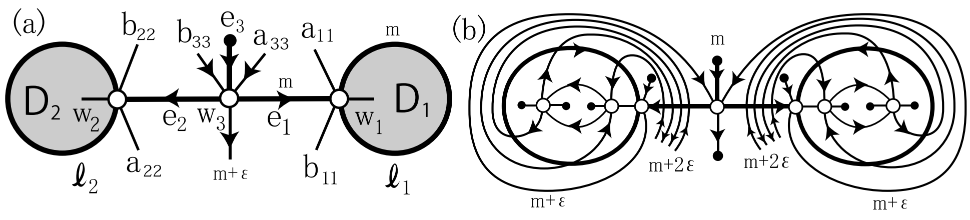

Fig. 19. The gray regions are disks D_1, D_2, m is a label, and $\varepsilon = \pm 1$.

contain pseudo charts of the RO-family of the pseudo chart as shown in Fig. 5(a) (see Fig. 19(b)). There are four edges of $\Gamma_{m+2\varepsilon}$ intersecting ∂D_1, say e'_1, e'_2, e'_3, e'_4. By Lemma 6.1, we have $e'_i \neq e'_j$ for $i \neq j$. Thus the four edges must contain white vertices in IntD_2. However it is impossible that Int$(e'_i)\cap$Int$(e'_j) = \emptyset$ for each pair i, j with $1 \leq i < j \leq 4$. Therefore there is no pair of skew eyeglasses of type 2 in Γ. □

8. Triangle Lemma

Let Γ be a chart, and D a disk. Let α be a simple arc in ∂D. We call a simple arc γ in an edge of Γ_k a (D, α)-*arc* of label k provided that $\partial\gamma \subset$Intα and Int$\gamma \subset$Int$D$. If there is no (D, α)-arc in Γ, then the chart Γ is said to be (D, α)-*arc free*.

Let Γ be a chart and D a disk. Let α be a simple arc in ∂D. For each $k = 1, 2, \cdots$, let Σ_k be the pseudo chart which consists of all arcs in $D \cap \Gamma_k$ intersecting the set $Cl(\partial D - \alpha)$. Let $\Sigma_\alpha = \cup_k \Sigma_k$.

The following two lemmata will be used in the proof of Lemma 8.3.

LEMMA 8.1 ([6, Lemma 3.2]) (Disk Lemma). *Let* Γ *be a minimal chart and* D *a disk. Let* α *be a simple arc in* ∂D. *Suppose that the interior of* α *contains neither white vertices, isolated points of* $D \cap \Gamma$, *nor arcs of* $D \cap \Gamma$. *If* Int$D$ *does not contain white vertices of* Γ, *then for any neighborhood* V *of* α, *there exists a* (D, α)*-arc free minimal chart* Γ' *obtained from the chart* Γ *by C-moves in* $V \cup D$ *keeping* Σ_α *fixed* (*see Fig. 20*).

LEMMA 8.2. *Let* Γ *be a chart,* e *an edge of* Γ_m, *and* w_1, w_2 *the white vertices in* e. *Suppose* $w_1 \in \Gamma_{m-1}$ *and* $w_2 \in \Gamma_{m+1}$. *Then for any neighbor-*

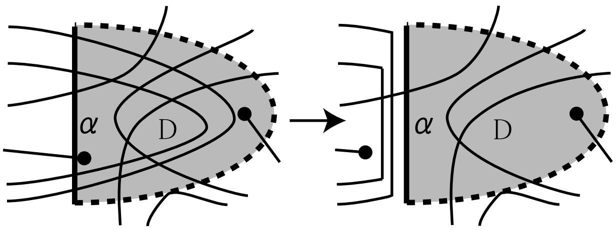

Fig. 20.

hood V of the edge e, there exists a chart Γ' obtained from the chart Γ by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves in V keeping $\Gamma_{m-1}\cup\Gamma_m\cup\Gamma_{m+1}$ such that the edge e does not contain any crossings.

PROOF. Let b be a point in $\text{Int}(e)$. Since $\text{Int}(e)\cap(\Gamma_{m-1}\cup\Gamma_{m+1})=\emptyset$, all the crossings of Γ_i $(i<m-1)$ in the edge e can be moved into the arc $e[b,w_2]$ by C-I-R2 moves and C-I-R3 moves. Similarly all the crossings of Γ_i $(i>m+1)$ in the edge e can be moved into the arc $e[w_1,b]$ by C-moves.

Since the white vertex w_1 is in $\Gamma_m\cap\Gamma_{m-1}$, we can push each crossing in the arc $e[w_1,b]$ to the side of w_1 along the edge e by C-I-R2 moves and C-I-R4 moves. Similarly we can push each crossing in the arc $e[b,w_2]$ to the side of w_2 along the edge e by C-moves. Hence the edge e does not contain any crossings. □

The following lemma will be used in Step 1 and Step 8 of the proof of Theorem 1.1.

LEMMA 8.3 (Triangle Lemma).

(1) *For a chart Γ, if there exists a* 3*-angled disk D_1 of Γ_m without feelers in a disk D as shown in Fig. 21*(a) *and if $w(\Gamma\cap\text{Int}D_1)=0$, then there exists a chart obtained from Γ by C-moves in D which contains the pseudo chart in D as shown in Fig. 21*(b).

(2) *For a minimal chart Γ, if there exists a* 3*-angled disk D_1 of Γ_m without feelers in a disk D as shown in Fig. 21*(c)*, then $w(\Gamma\cap\text{Int}D_1)\geq 1$.*

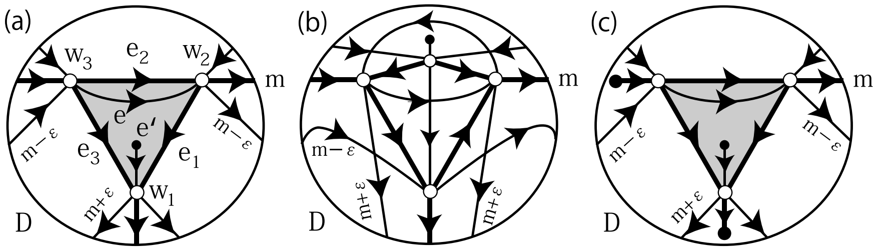

Fig. 21. The gray regions are 3-angled disks D_1. The thick lines are edges of label m, and $\varepsilon \in \{+1, -1\}$.

PROOF. We show Statement (1). We use the notations as shown in Fig. 21(a).

CLAIM 1. $D_1 \cap (\Gamma_{m-\varepsilon} \cup \Gamma_m \cup \Gamma_{m+\varepsilon}) = e \cup e' \cup \partial D_1$.

For, if D_1 contains a ring or a hoop of label $m-\varepsilon, m, m+\varepsilon$, say C, then C bounds a disk in D_1 not containing any white vertices. This contradicts Assumption 5. Hence the disk D_1 contains neither hoop nor ring of label $m-\varepsilon, m, m+\varepsilon$. Since the disk D_1 does not contain any free edges by Assumption 4, Claim 1 holds.

CLAIM 2. We can assume that neither e_1 nor e contains a crossing by C-moves.

For, $w_1 \in \Gamma_{m+\varepsilon}$ and $w_2 \in \Gamma_{m-\varepsilon}$. Thus by Lemma 8.2 we can assume that the edge e_1 of Γ_m does not contain any crossings. Moreover we can push the crossings in e to the side of w_3 by C-moves as follows. Since the label of the edge e is $m-\varepsilon$, we have $\mathrm{Int}(e) \cap (\Gamma_{m-2\varepsilon} \cup \Gamma_m) = \emptyset$. Hence Claim 1 implies that we have $\mathrm{Int}(e) \cap (\Gamma_{m-2\varepsilon} \cup \Gamma_m \cup \Gamma_{m+\varepsilon}) = \emptyset$. Since the white vertex w_3 is in $\Gamma_{m-\varepsilon} \cap \Gamma_m$, we can push each crossing in the edge e to the side of w_3 along e by C-I-R2 moves and C-I-R4 moves. Hence Claim 2 holds.

CLAIM 3. We can assume that $D_1 \cap \Gamma = e \cup e' \cup \partial D_1$ by C-moves.

For, Claim 2 assures that by applying Disk Lemma(Lemma 8.1) twice, we can assume that Γ is (D_1, e_2)-arc free and (D_1, e_3)-arc free. Since D_1 does

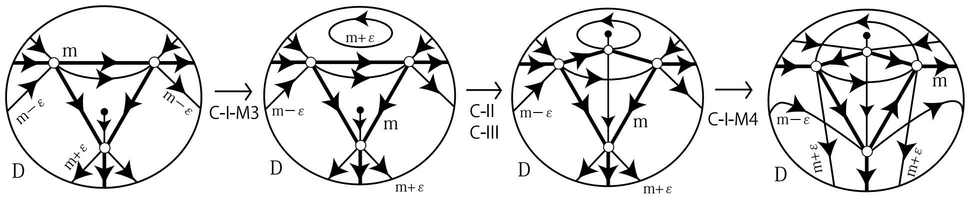

Fig. 22. The thick lines are edges of label m, and $\varepsilon \in \{+1, -1\}$.

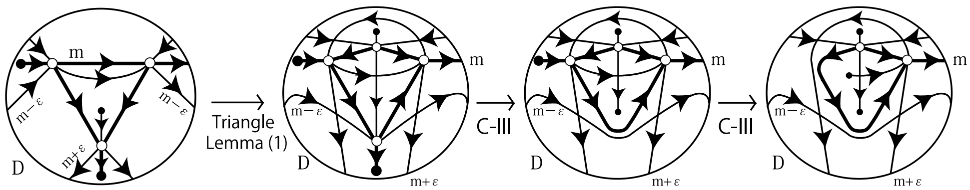

Fig. 23. The thick lines are edges of label m, and $\varepsilon \in \{+1, -1\}$.

not contain free edges, hoops nor rings by Assumption 4 and Assumption 5, we have $D_1 \cap \Gamma = e \cup e' \cup \partial D_1$. Hence Claim 3 holds.

As shown in Fig. 22, we can deform by C-moves in D from Γ to a chart containing the pseudo chart as shown in Fig. 21(b).

We show Statement (2). Suppose that a minimal chart Γ contains the pseudo chart as shown in Fig. 21(c). If $w(\Gamma \cap \text{Int} D_1) = 0$, then as shown in Fig. 23, by the help of Triangle Lemma(Lemma 8.3(1)), we can reduce the number of white vertices of Γ by C-moves. This contradicts the fact that Γ is a minimal chart. Hence $w(\Gamma \cap \text{Int} D_1) \geq 1$. □

9. Proof of Theorem 1.1

Now we start to prove Theorem 1.1. We show the theorem by contradiction. Suppose that there exists a minimal chart Γ with $w(\Gamma) = 7$ containing a loop ℓ of label m. Let D_1 be the associated disk of ℓ. Then $w(\Gamma_m \cap (S^2 - D_1)) \leq 2$ by Lemma 6.5. Thus the loop ℓ is contained in a connected component G of Γ_m with $w(G) \leq 3$. By Lemma 7.1, the graph G

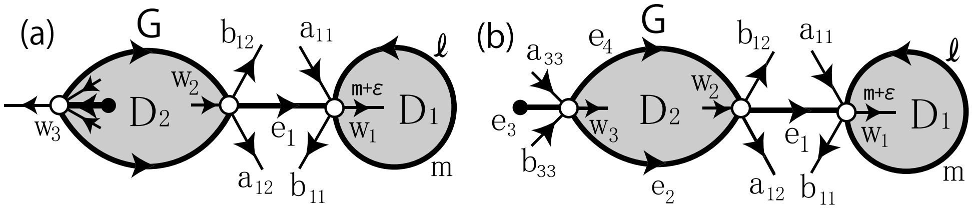

Fig. 24. The pair of skew eyeglasses G is the thick lines of label m, the gray regions are the disks D_1, D_2.

is a pair of eyeglasses or a pair of skew eyeglasses. By Proposition 7.3 and Proposition 7.4, the graph G is a pair of skew eyeglasses of type 1.

From now on, throughout this section, we assume that

(i) ℓ is the loop of label m in the pair of skew eyeglasses G of type 1,

(ii) w_1 is the white vertex in ℓ with $w_1 \in \Gamma_m \cap \Gamma_{m+\varepsilon}$ where $\varepsilon \in \{+1, -1\}$,

(iii) Γ is locally minimal with respect to D_1 and D_2 where D_1 is the associated disk of ℓ and D_2 is the 2-angled disk of Γ_m with $w_1 \notin D_2$ (see Fig. 24).

We shall prove Theorem 1.1 by eight steps: If necessary we change all orientation of edges, we can assume that the edge e_1 of label m is oriented from w_2 to w_1. We show only the case that ℓ is oriented anticlockwise. We use the notations as shown in Fig. 24.

Step 1. The disk D_2 is a 2-angled disk without feelers (i.e. Fig. 24(b)).

Step 2. There exists at least one white vertex of $\Gamma_{m-1} \cup \Gamma_{m+1}$ in $S^2 - (D_1 \cup D_2)$, say w_7.

Step 3. $w(\Gamma \cap \mathrm{Int} D_1) = 2$.

Step 4. There exists a white vertex in $S^2 - (D_1 \cup D_2)$ different from w_7, say w_6, and $w_6, w_7 \in \Gamma_{m+\varepsilon} \cap \Gamma_{m+2\varepsilon}$ and $w_2, w_3 \in \Gamma_{m+\varepsilon}$.

Step 5. The edge a_{12} contains one of the white vertices w_6, w_7.

Without loss of generality we can assume $w_6 \in a_{12}$.

Step 6. The edge b_{33} contains the white vertex w_7.

Step 7. The edge a_{33} contains the white vertex w_7.

Step 8. There does not exist a pair of skew eyeglasses of type 1.

PROOF OF STEP 1. Suppose that D_2 is a 2-angled disk with one feeler. We use the notations as shown in Fig. 24(a).

We shall show that Γ contains a pseudo chart as shown in Fig. 25. By IO-Calculation with respect to $\Gamma_{m\pm 1}$ in D_2 and $Cl(S^2 - (D_1 \cup D_2))$, there exists a white vertex of $\Gamma_{m-1} \cup \Gamma_{m+1}$ in $\text{Int}D_2$, say w_6, and there exists a white vertex of $\Gamma_{m-1} \cup \Gamma_{m+1}$ in $S^2 - (D_1 \cup D_2)$, say w_7. Now

$$(1) \quad \begin{aligned} 7 &= w(\Gamma) \\ &= w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) \\ &\quad + w(G) + w(\Gamma \cap (S^2 - (D_1 \cup D_2))) \\ &\geq w(\Gamma \cap \text{Int}D_1) + 1 + 3 + 1. \end{aligned}$$

Hence we have $w(\Gamma \cap \text{Int}D_1) \leq 2$. By Lemma 3.1(1), we have $w(\Gamma \cap \text{Int}D_1) = 2$. Thus by (1)

$$(2) \quad w(\Gamma \cap \text{Int}D_2) = 1 \text{ and } w(\Gamma \cap (S^2 - (D_1 \cup D_2))) = 1.$$

Since D_2 is a 2-angled disk with one feeler, by (2) and Lemma 3.3 the disk D_2 contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 7(c) (see D_2 in Fig. 25).

For the edge b_{11} in Fig. 24(a), there are two cases: $b_{11} = a_{11}$ or $b_{11} \ni w_7$. But if $b_{11} = a_{11}$, then we have a solar eclipse $b_{11} \cup \ell$. This contradicts Proposition 6.4. Hence $b_{11} \ni w_7$.

For the edge a_{12} in Fig. 24(a), there are two cases: $a_{12} = a_{11}$ or $a_{12} \ni w_7$. If $a_{12} = a_{11}$, then the edges e_1 and a_{12} separate the annulus $Cl(S^2 - (D_1 \cup D_2))$ into two regions. Let F be one of the two regions with $w_7 \notin F$. By (2), we have $w(\Gamma \cap \text{Int}F) = 0$. Thus we have a contradiction by IO-Calculation with respect to $\Gamma_{m+\varepsilon}$ in F. Hence $a_{12} \ni w_7$.

Since a_{12} and b_{11} are oriented inward at w_7, we have $b_{12} \not\ni w_7$. Thus we have $b_{12} = a_{11}$, because b_{12} does not contain a middle arc at w_2. Let e_7 be

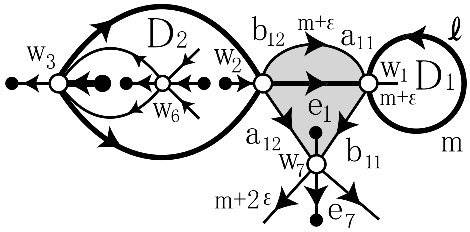

Fig. 25. The gray region is a 3-angled disk D_3.

the edge of $\Gamma_{m+\varepsilon}$ containing w_7 different from a_{12} and b_{11}. Then (2) implies that e_7 is a terminal edge.

We shall show $w_7 \in \Gamma_{m+2\varepsilon}$. Since $w_7 \in b_{11} \subset \Gamma_{m+\varepsilon}$, we have $w_7 \in \Gamma_m$ or $w_7 \in \Gamma_{m+2\varepsilon}$. If $w_7 \in \Gamma_m$, then there exists a connected component of Γ_m containing exactly one white vertex w_7. This contradicts Lemma 4.2. Thus $w_7 \in \Gamma_{m+2\varepsilon}$.

Let D_3 be the 3-angled disk bounded by $a_{11} \cup b_{11} \cup a_{12}$ with $e_1 \subset D_3$, If $e_7 \subset D_3$, then we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in D_3. Thus we have $e_7 \not\subset D_3$ (see Fig. 25). Since $w_7 \in S^2-(D_1 \cup D_2)$, (2) implies that

(3) $w(\Gamma \cap \text{Int}D_3) = 0$.

Thus the edge of $\Gamma_{m+2\varepsilon}$ containing w_7 in D_3 is a terminal edge. Therefore Γ contains a pseudo chart as shown in Fig. 25.

However in a neighborhood of D_3, the chart Γ contains the pseudo chart as shown in Fig. 21(c). Now (3) contradicts Triangle Lemma(Lemma 8.3(2)). Hence D_2 is a 2-angled disk without feelers (see Fig. 24(b)). This completes the proof of Step 1. □

PROOF OF STEP 2. By Step 1, the chart Γ contains the pseudo chart as shown in Fig. 24(b). We use the notations as shown in Fig. 24(b). Suppose that there do not exist any white vertices of $\Gamma_{m-1} \cup \Gamma_{m+1}$ in $S^2-(D_1 \cup D_2)$. By Assumption 2, the edge b_{11} is not a terminal edge. Thus there are three cases: (1) $b_{11} = a_{11}$, (2) $b_{11} = a_{33}$, (3) $b_{11} = b_{33}$.

Case (1). We have a solar eclipse $b_{11} \cup \ell$. This contradicts Proposition 6.4.

Case (2). We have $b_{33} = a_{12}$. Hence the union $b_{33} \cup e_2$ bounds a lens. This contradicts Lemma 6.1.

Case (3). We have $w_1, w_3 \in \Gamma_{m+\varepsilon}$, because $\partial b_{11} = \{w_1, w_3\}$. By Assumption 2, the edge a_{12} is not a terminal edge. Hence $a_{12} = a_{11}$ or $a_{12} = a_{33}$. Thus a_{12} intersects the edge b_{11}. Since $w_1 \in a_{12} \cap b_{11}$ or $w_3 \in a_{12} \cap b_{11}$, the two edges a_{12} and b_{11} are of label $m+\varepsilon$. This contradicts Condition (iv) of the definition of a chart.

Hence we have a contradiction for any cases. Thus there exists a white vertex of $\Gamma_{m-1} \cup \Gamma_{m+1}$ in $S^2 - (D_1 \cup D_2)$. This completes the proof of Step 2. □

Proof of Step 3. By Step 2

$$(*) \quad \begin{aligned} 7 &= w(\Gamma) \\ &\geq w(\Gamma \cap \mathrm{Int} D_1) + w(\Gamma \cap \mathrm{Int} D_2) + w(G) \\ &\quad + w(\Gamma \cap (S^2 - (D_1 \cup D_2)) \\ &\geq w(\Gamma \cap \mathrm{Int} D_1) + 0 + 3 + 1. \end{aligned}$$

Thus $w(\Gamma \cap \mathrm{Int} D_1) \leq 3$. By Lemma 3.1(1), we have $w(\Gamma \cap \mathrm{Int} D_1) \geq 2$. Hence it suffices to prove $w(\Gamma \cap \mathrm{Int} D_1) \neq 3$.

Suppose $w(\Gamma \cap \mathrm{Int} D_1) = 3$. By (*), Step 2 and $w(\Gamma) = 7$, we have

$$(**) \quad w(\Gamma \cap (S^2 - (D_1 \cup D_2)) = 1 \text{ and } w(\Gamma \cap \mathrm{Int} D_2) = 0.$$

Let w_7 be the white vertex in $S^2 - (D_1 \cup D_2)$. For the edge b_{33} in Fig. 24(b), there are four cases: (1) $b_{33} = a_{12}$, (2) $b_{33} = b_{12}$, (3) $b_{33} = b_{11}$, (4) $b_{33} \ni w_7$.

Case (1). The union $b_{33} \cup e_2$ bounds a lens. This contradicts Lemma 6.1.

Case (2). The edge b_{33} separates the annulus $Cl(S^2 - (D_1 \cup D_2))$ into a disk and an annulus. Let F be one of the disk and the annulus with $w_7 \notin F$. Since $w(\Gamma \cap (S^2 - (D_1 \cup D_2)) = 1$ by (**), we have $w(\Gamma \cap \mathrm{Int} F) = 0$. Thus we have a contradiction by IO-Calculation with respect to $\Gamma_{m\pm 1}$ in F.

Case (3). We have $w_1, w_3 \in \Gamma_{m+\varepsilon}$. For the edge a_{12} in Fig. 24(b), the edge is of label $m - \varepsilon$ or $m + \varepsilon$. If a_{12} is of label $m + \varepsilon$, then we have

$a_{12} \ni w_7$ and $a_{33} = b_{12}$, because neither a_{12} nor a_{33} is a terminal edge by Assumption 2. Thus the union $a_{33} \cup e_4$ bounds a lens. This contradicts Lemma 6.1. Hence a_{12} is of label $m - \varepsilon$. Since a_{12} is not a terminal edge, we have $w_7 \in a_{12} \subset \Gamma_{m-\varepsilon}$. Hence $w_7 \in \Gamma_{m-2\varepsilon}$ or $w_7 \in \Gamma_m$. If $w_7 \in \Gamma_m$, then there exists a connected component of Γ_m containing exactly one white vertex w_7. This contradicts Lemma 4.2. Hence $w_7 \in \Gamma_{m-2\varepsilon}$.

We shall show that $\Gamma_{m-2\varepsilon}$ contains exactly one white vertex w_7. Since $w(\Gamma \cap \text{Int}D_1) = 3$, $w_1 \in \Gamma_{m+\varepsilon}$, and since Γ is locally minimal with respect to D_1 by (iii), all the white vertices in $\text{Int}D_1$ are contained in $\Gamma_{m+\varepsilon} \cup \Gamma_{m+2\varepsilon}$ by Lemma 3.2. Thus none of the white vertices in $\text{Int}D_1$ is contained in $\Gamma_{m-2\varepsilon}$. Since $w_1, w_2, w_3 \in \Gamma_m$, none of w_1, w_2, w_3 is contained in $\Gamma_{m-2\varepsilon}$. Hence $\Gamma_{m-2\varepsilon}$ contains exactly one white vertex w_7. This contradicts Lemma 4.2.

Case (4). For the edge b_{11} in Fig. 24(b), there are three cases: $b_{11} = a_{11}$, $b_{11} = a_{33}$ or $b_{11} \ni w_7$. If $b_{11} = a_{11}$, then we have a solar eclipse $b_{11} \cup \ell$. This contradicts Proposition 6.4. If $b_{11} = a_{33}$ or $b_{11} \ni w_7$, then we have $a_{12} \ni w_7$ (see Fig. 26(a)). Since $w(\Gamma \cap \text{Int}D_2) = 0$ by (**), the disk D_2 contains a pseudo chart of the RO-family of the pseudo chart as shown in Fig. 7(b) by Lemma 3.3. Since $w(\Gamma \cap (S^2 - (D_1 \cup D_2))) = 1$ by (**), we have $w(\Gamma \cap \text{Int}D_3) = 0$, here D_3 is the disk with $\partial D_3 = e_2 \cup a_{12} \cup b_{33}$ and $w_1 \notin D_3$. This contradicts Lemma 4.4.

Thus we have a contradiction for any cases. Hence $w(\Gamma \cap \text{Int}D_1) \neq 3$. Therefore $w(\Gamma \cap \text{Int}D_1) = 2$. This completes the proof of Step 3. □

PROOF OF STEP 4. By Step 3, we have $w(\Gamma \cap \text{Int}D_1) = 2$. Since $w(\Gamma) = 7$, we have

(1) $w(\Gamma \cap (S^2 - D_1)) = 4$.

Since $w_2, w_3 \in S^2 - D_1$ and $w_7 \in S^2 - D_1$ by Step 2, there exists a white vertex in $S^2 - D_1$ different from w_2, w_3, w_7, say w_6. By Lemma 6.3(1) we have $w(\Gamma_{m+2\varepsilon} \cap (S^2 - D_1)) \geq 2$. Since $w_2, w_3 \in \Gamma_m$ (see Fig. 24(b)) we have $w_2, w_3 \notin \Gamma_{m+2\varepsilon}$. Hence we have

(2) $w_6, w_7 \in \Gamma_{m+2\varepsilon}$.

Since $w_7 \in \Gamma_{m-1} \cup \Gamma_{m+1}$ by Step 2, we have $w_7 \in \Gamma_{m+\varepsilon} \cap \Gamma_{m+2\varepsilon}$.

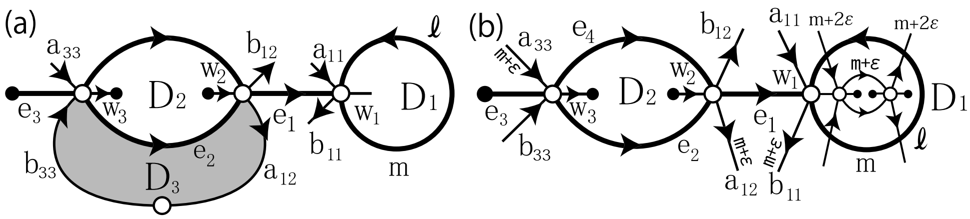

Fig. 26. The gray region is the disk D_3.

By Step 2 and (1), we have

$$\begin{aligned} 4 &= w(\Gamma \cap (S^2 - D_1)) \\ &= w(\Gamma \cap \mathrm{Int} D_2) + w(\Gamma \cap \partial D_2) + w(\Gamma \cap (S^2 - (D_1 \cup D_2))) \\ &\geq w(\Gamma \cap \mathrm{Int} D_2) + 2 + 1. \end{aligned}$$

Hence we have $w(\Gamma \cap \mathrm{Int} D_2) \leq 1$. Since the edges in ∂D_2 are oriented from w_3 to w_2, we have $w(\Gamma \cap \mathrm{Int} D_2) = 0$ by Lemma 3.3 and Step 1. Thus $w_6 \in S^2 - (D_1 \cup D_2)$.

By (2), we have $w_6 \in \Gamma_{m+2\varepsilon}$. Thus $w_6 \in \Gamma_{m+\varepsilon}$ or $w_6 \in \Gamma_{m+3\varepsilon}$. Suppose $w_6 \in \Gamma_{m+3\varepsilon}$. Since $w(\Gamma \cap \mathrm{Int} D_1) = 2$, by Lemma 3.1(2) the white vertices in $\mathrm{Int} D_1$ are in $\Gamma_{m+\varepsilon} \cap \Gamma_{m+2\varepsilon}$. Thus $\Gamma_{m+3\varepsilon}$ contains exactly one white vertex w_6. This contradicts Lemma 4.2. Hence $w_6 \in \Gamma_{m+\varepsilon}$. Thus $w_6 \in \Gamma_{m+\varepsilon} \cap \Gamma_{m+2\varepsilon}$.

We can show $w_2, w_3 \in \Gamma_{m-\varepsilon}$ or $w_2, w_3 \in \Gamma_{m+\varepsilon}$ by IO-Calculation with respect to $\Gamma_{m-\varepsilon}$ in $Cl(S^2 - (D_1 \cup D_2))$. If $w_2, w_3 \in \Gamma_{m-\varepsilon}$, then there exist two lenses of type $(m, m-\varepsilon)$. This contradicts Lemma 6.1. Thus $w_2, w_3 \in \Gamma_{m+\varepsilon}$. This completes the proof of Step 4. □

By Step 3, Step 4, Lemma 3.1(2) and Lemma 3.3, we have the pseudo chart as shown in Fig. 26(b). From now on, we use the notations as shown in Fig. 26(b).

Proof of Step 5. For the edge a_{12} in Fig. 26(b), there are four cases: (1) $a_{12} = a_{11}$, (2) $a_{12} = a_{33}$, (3) $a_{12} = b_{33}$, (4) a_{12} contains one of w_6, w_7.

Case (1). The edges a_{12}, e_1 separate the annulus $Cl(S^2 - (D_1 \cup D_2))$ into two regions. Each region must contain a white vertex in its interior.

In one of the regions, there exists a loop of label $m+\varepsilon$ whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.1(1).

Case (2). The edges a_{12} separates the annulus $Cl(S^2-(D_1\cup D_2))$ into two regions. In a similar way to Case (1), we can show that there exists a loop of label $m+\varepsilon$ whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.1(1).

Case (3). The union $a_{12}\cup e_2$ bounds a lens. This contradicts Lemma 6.1.

Thus Case (4) occurs. This completes the proof of Step 5. □

PROOF OF STEP 6. For the edge b_{33} in Fig. 26(b), there are four cases: (1) $b_{33}=b_{11}$, (2) $b_{33}=b_{12}$, (3) $b_{33}\ni w_6$, (4) $b_{33}\ni w_7$.

Case (1). The edges b_{11}, e_1 separate the annulus $Cl(S^2-(D_1\cup D_2))$ into two regions. Let F be one of the two regions containing $w_6\in a_{12}$. If $w_7\in F$, then $a_{33}=b_{12}$. Hence the union $a_{33}\cup e_4$ bounds a lens. This contradicts Lemma 6.1. If $w_7\notin F$, then there exists a loop of label $m+\varepsilon$ in F whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.1(1).

Case (2). The edges b_{11} separate the annulus $Cl(S^2-(D_1\cup D_2))$ into two regions. Let F be one of the two regions containing a_{33}. Then F must contain w_7. Thus there exists a loop of label $m+\varepsilon$ in F whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.1(1).

Case (3). By Lemma 4.4 the disk D_3 bounded by $a_{12}\cup b_{33}\cup e_2$ contains at least one white vertex in its interior as shown in Fig. 26(a). Thus $w_7\in D_3$. For the white vertex w_7, there are two edges of $\Gamma_{m+\varepsilon}$ not containing middle arcs. One contains w_6, and the other is a loop. Hence there exists a loop of label $m+\varepsilon$ in D_3 whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.1(1).

Hence Case (4) occurs. This completes the proof of Step 6. □

Proof of Step 7. For the edge a_{33} in Fig. 26(b), there are four cases: (1) $a_{33} = b_{12}$, (2) $a_{33} = b_{11}$, (3) $a_{33} \ni w_6$, and (4) $a_{33} \ni w_7$.

Case (1). The union $a_{33} \cup e_4$ bounds a lens. This contradicts Lemma 6.1.

Case (2). We have $b_{12} = a_{11}$. Let e_7 be an edge of $\Gamma_{m+\varepsilon}$ with $w_7 \in e_7$ different from b_{33} not containing a middle arc at w_7. By Assumption 2, the edge e_7 is not a terminal edge. Thus e_7 is a loop or $w_6 \in e_7$. If e_7 is a loop, then the associated disk of e_7 does not contain any white vertices in its interior. This contradicts Lemma 3.1(1). Hence $w_6 \in e_7$.

There are four edges of $\Gamma_{m+2\varepsilon}$ intersecting ∂D_1, say e'_1, e'_2, e'_3, e'_4. Since there is no lens by Lemma 6.1, each of the four edges must contain one of w_6, w_7. Thus there are two terminal edges of $\Gamma_{m+\varepsilon}$ in the disk D_3 containing w_6, w_7 respectively (see Fig. 27(a)). However it is impossible that for each pair $1 \leq i < j \leq 4$ with $\text{Int}(e'_i) \cap \text{Int}(e'_j) = \emptyset$.

Case (3). There exists an edge e_7 of $\Gamma_{m+\varepsilon}$ containing w_7 different from b_{33} not containing a middle arc at w_7. In a similar way to Case (2), we can show $w_6 \in e_7$. Let e'_7 be the edge of $\Gamma_{m+\varepsilon}$ containing w_7 different from b_{33} and e_7. Then e'_7 is a terminal edge. However we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in the disk D_3 bounded by $a_{33} \cup b_{33} \cup e_7$ containing e_3 (see Fig. 27(b) for the case $D_3 \supset e'_7$).

Hence Case (4) occurs. This completes the proof of Step 7. □

Proof of Step 8. So far, we have a pseudo chart as shown in Fig. 28(a). For the edge b_{11} in Fig. 28(a), there are three cases: (1) $b_{11} = a_{11}$, (2) $b_{11} \ni w_7$, (3) $b_{11} \ni w_6$.

Case (1). We have a solar eclipse. This contradicts Proposition 6.4.

Case (2). There exists a loop of label $m + \varepsilon$ containing w_6 whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.1(1).

Case (3). There are two cases: $b_{12} \ni w_7$ or $b_{12} = a_{11}$. But if $b_{12} \ni w_7$, then the union $b_{12} \cup a_{33} \cup e_4$ bounds a disk D' with $w(\Gamma \cap \text{Int}D') = 0$. This contradicts Lemma 4.4. Hence $b_{12} = a_{11}$.

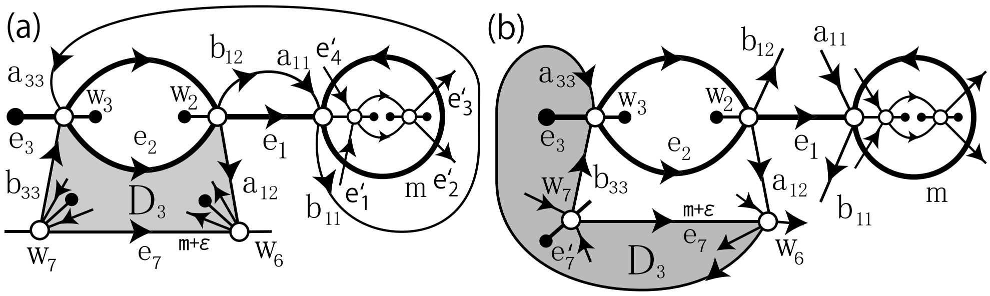

Fig. 27. The gray region is the disk D_3.

Let e_6 be the edge of $\Gamma_{m+\varepsilon}$ containing w_6 different from a_{12}, b_{11}. Let D_3 be the disk bounded by $a_{12} \cup b_{12} \cup b_{11}$ with $e_1 \subset D_3$. Then

(*) $w(\Gamma \cap \mathrm{Int} D_3) = 0$.

If $e_6 \subset D_3$, then we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in D_3. Thus we have $e_6 \not\subset D_3$ and there exists a terminal edge of $\Gamma_{m+2\varepsilon}$ containing w_6 in D_3.

Now e_6 is a terminal edge or $e_6 \ni w_7$. If e_6 is a terminal edge, then in a neighborhood of D_3, the chart Γ contains the pseudo chart as shown in Fig. 21(c). By Triangle Lemma(Lemma 8.3(2)), we have $w(\Gamma \cap \mathrm{Int} D_3) \geq 1$. This contradicts (*). Hence e_6 is not a terminal edge. Thus $e_6 \ni w_7$ (see Fig. 28(b)).

Let e_1', e_2', e_3' be the edges of $\Gamma_{m+2\varepsilon}$ incident with the 2-angled disk in D_1 in Fig. 28(b). For the edge e_1', there are four cases: (3-1) $e_1' = e_2'$, (3-2) $e_1' = e_3'$, (3-3) $e_1' = a_{66}$, (3-4) $e_1' = b_{66}$.

Case (3-1). There exists a lens of type $(m+\varepsilon, m+2\varepsilon)$. This contradicts Lemma 6.1.

Let e' be the edge of $\Gamma_{m+\varepsilon}$ in D_1 containing w_1, and D_4 the 2-angled disk of $\Gamma_{m+\varepsilon}$ in D_1.

Case (3-2). The edge e_1' separates the annulus $Cl(S^2 - D_4)$ into two regions. Let F be the one of the two regions containing the edge e_2'. Then $w(\Gamma \cap \mathrm{Int} F) = 0$. Hence we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in F.

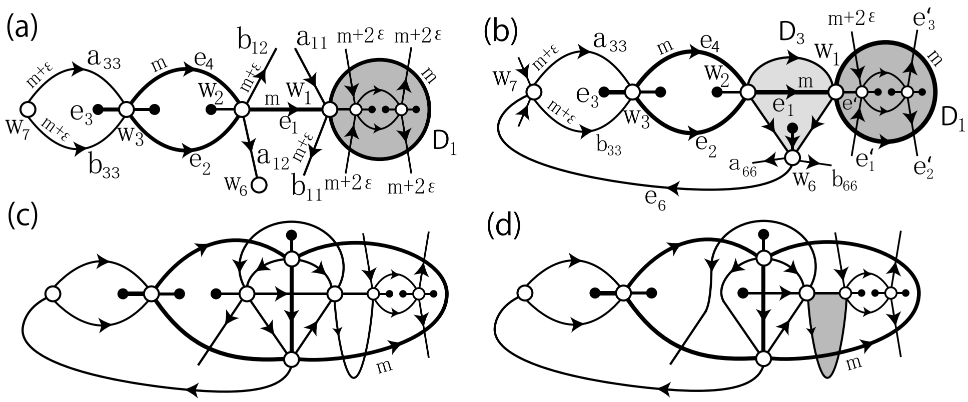

Fig. 28. The thick lines are edges of label m. (a) The gray region is the disk D_1. (b) The gray regions are the disks D_1, D_3. (d) The gray region is a lens.

Case (3-3). The simple arc $e'_1 \cup e'$ separates the annulus $Cl(S^2 - D_3)$ into two regions. Let F be the one of the two regions containing the edge b_{66}. Then Int$F$ does not contain any white vertices except w_3 and w_7. Hence we have a contradiction by IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in F.

Thus Case (3-4) occurs. In a neighborhood of D_3, the chart Γ contains the pseudo chart as shown in Fig. 21(a). By Triangle Lemma(Lemma 8.3(1)), we obtain a chart containing the pseudo chart as shown in Fig. 21(b) (see Fig. 28(c)). Apply a C-III move to the chart as shown in Fig. 28(c), we obtain a minimal chart as shown in Fig. 28(d). However there exists a lens of type $(m+\varepsilon, m+2\varepsilon)$. This contradicts Lemma 6.1. Therefore there does not exist a pair of skew eyeglasses of type 1. This completes the proof of Step 8, and the proof of Theorem 1.1. □

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