J. Math. Sci. Univ. Tokyo
23 (2016), 791–866.

Clifford Quartic Forms and Local Functional Equations of Non-Prehomogeneous Type

By Takeyoshi Kogiso* and Fumihiro Sato*

Abstract. It is known that one can associate local zeta functions satisfying a functional equation to the irreducible relative invariant of an irreducible regular prehomogeneous vector space. We construct polynomials of degree 4 (called *Clifford quartic forms*) that cannot be obtained from prehomogeneous vector spaces, but for which one can associate local zeta functions satisfying functional equations. The Clifford quartic form is defined for each finite dimensional representation of the tensor product of the Clifford algebras of two positive definite real quadratic forms and cannot be a relative invariant of any prehomogeneous vector space except for a few low dimensional cases. We also classify the exceptional cases of small dimension, namely, we determine all the prehomogeneous vector spaces with Clifford quartic forms as a relative invariant.

Introduction

Let $(\mathbf{G}, \rho, \mathbf{V})$ be an irreducible regular prehomogeneous vector space defined over $\mathbb{R}$ and let $(\mathbf{G}, \rho^*, \mathbf{V}^*)$ be the dual of $(\mathbf{G}, \rho, \mathbf{V})$. Then, there exists an irreducible homogeneous polynomial $P(v)$ (resp. $P^*(v^*)$) on $\mathbf{V}$ (resp. $\mathbf{V}^*$) such that the complement $\mathbf{\Omega}$ (resp. $\mathbf{\Omega}^*$) in $\mathbf{V}$ (resp. $\mathbf{V}^*$) of the hypersurface defined by $P(v)$ (resp. $P^*(v^*)$) is a single $\mathbf{G}$-orbit. Let $\Omega_1, \dots, \Omega_\nu$ (resp. $\Omega_1^*, \dots, \Omega_\nu^*$) be the connected components of $\mathbf{\Omega} \cap \mathbf{V}(\mathbb{R})$ (resp. $\mathbf{\Omega}^* \cap \mathbf{V}^*(\mathbb{R})$). For a rapidly decreasing function Φ (resp. Φ^*) on $\mathbf{V}(\mathbb{R})$ (resp. $\mathbf{V}^*(\mathbb{R})$) and $i = 1, \dots, \nu$, the local zeta functions are defined by

$$\zeta_i(s, \Phi) = \int_{\Omega_i} |P(v)|^s \, \Phi(v) \, dv, \quad \zeta_i^*(s, \Phi^*) = \int_{\Omega_i^*} |P^*(v^*)|^s \, \Phi^*(v^*) \, dv^*.$$

*The first and second authors are partially supported by the grant in aid of scientific research of JSPS No.24540029, 25247001, 15K04800 and 24540049.
2010 *Mathematics Subject Classification.* Primary 11S40, 11E45, 11E88; Secondary 15A63, 15A66.
Key words: Clifford quartic forms, local functional equations, prehomogeneous vector spaces.

These integrals are absolutely convergent for $\Re(s) > 0$ and can be continued to meromorphic functions of s in $\mathbb{C}$. The fundamental theorem in the theory of prehomogeneous vector spaces ([30], [23], [17]) states that the local zeta functions satisfy a functional equation

$$\zeta_i^*(s, \hat{\Phi}) = \sum_{j=1}^{\nu} \gamma_{ij}(s)\zeta_i\left(-\frac{n}{d} - s, \Phi\right), \tag{0.1}$$

where $\hat{\Phi}$ is the Fourier transform of Φ, $n = \dim \mathbf{V}$, $d = \deg P$, and the gamma-factors $\gamma_{ij}(s)$ are meromorphic functions of s independent of Φ.

Since the local zeta functions can be defined for an arbitrary polynomial P, it is natural to ask whether there are any polynomials other than the ones obtained from the theory of prehomogeneous vector spaces satisfying a functional equation of the form (0.1). Such polynomials, if exist, should be analytically and arithmetically interesting objects.

In [8, Chapter XVI], Faraut and Koranyi constructed polynomials having a functional equation of the form (0.1) from representations of Euclidean Jordan algebras and observed that the polynomials cannot be obtained from prehomogeneous vector spaces for the simple Euclidean Jordan algebras of rank 2 (apart from some low-dimensional exceptions). This result gives the first example of non-prehomogeneous polynomials satisfying local functional equations. However it seems still unclear when the polynomials are actually non-prehomogeneous.

In [25], the second author considered the pullback of local functional equation by non-degenerate dual quadratic mappings. Let P (resp. P^*) be a homogeneous polynomial of degree d on a $\mathbb{C}$-vector space $\mathbf{V}$ (resp. $\mathbf{V}^*$) with $\mathbb{R}$-structure V (resp. V^*). Here $\mathbf{V}^*$ is the vector space dual to $\mathbf{V}$. Let $Q : \mathbf{W} \to \mathbf{V}$ and $Q^* : \mathbf{W}^* \to \mathbf{V}^*$ be non-degenerate dual quadratic mappings (in the sense explained in §1). If the local zeta functions attached to P and P^* satisfy a local functional equation of the form (0.1), then the local zeta functions attached to the pullbacks $\tilde{P} = P \circ Q$ and $\tilde{P}^* = P^* \circ Q^*$ also satisfy a similar functional equation (see Theorem 1.2). This generalizes the earlier result of Faraut and Koranyi (see [25, §2.2]).

In this paper, we examine the case where $\dim \mathbf{V} = p + q \geq 3$ and $P(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$ and classify the self-dual non-degenerate quadratic mappings Q to the quadratic space $(\mathbf{V}, P)$. As a result, we can obtain many examples of polynomials having local functional equations, but they are not obtained from prehomogeneous vector spaces. Our main results are summarized as follows:

- The self-dual quadratic mappings to $(\mathbf{V}, P)$ defined over $\mathbb{R}$ are in one to one correspondence to (not necessarily irreducible) representations of $C_p \otimes C_q$, where C_p and C_q are the real Clifford algebras of the positive definite quadratic forms $v_1^2 + \cdots + v_p^2$ and $v_{p+1}^2 + \cdots + v_{p+q}^2$ (Theorem 2.2). For a representation ρ of $C_p \otimes C_q$ on a real vector space $W = \mathbf{W}(\mathbb{R}) = \mathbb{R}^m$, we may assume that the images $S_i = \rho(e_i)$ of the standard basis $e_1, \ldots, e_{p+q}$ are symmetric matrices. Then the quadratic mapping $Q : W \to V$ given by $Q(w) = ({}^t w S_1 w, \ldots, {}^t w S_{p+q} w)$ is self-dual, and we call the polynomial

$$\tilde{P}(w) = P(Q(w)) = \sum_{i=1}^{p} ({}^t w S_i w)^2 - \sum_{j=p+1}^{p+q} ({}^t w S_j w)^2,$$

  the *Clifford quartic form* associated with ρ. In the special case where $(p, q) = (1, q)$ and S_1 is the identity matrix, $\tilde{P}$ coincides with the polynomial that Faraut-Koranyi [8] obtained from a representation of the simple Euclidean Jordan algebra of rank 2.

- The quadratic mapping Q given above is non-degenerate, if and only if the associated Clifford quartic form $\tilde{P}$ does not vanish identically, and then the local zeta functions for $\tilde{P}$ satisfy functional equations of the form (0.1) (Theorem 2.13). For simplicity we give here an explicit formula for the functional equation for $p \geq q \geq 2$. For a rapidly decreasing function Ψ on W, the local zeta functions are defined by

$$\begin{aligned} \tilde{\zeta}_+(s, \Psi) &= \int_{\tilde{P}(w)>0} \left|\tilde{P}(w)\right|^s \Psi(w)\, dw, \\ \tilde{\zeta}_-(s, \Psi) &= \int_{\tilde{P}(w)<0} \left|\tilde{P}(w)\right|^s \Psi(w)\, dw, \end{aligned}$$

  and they satisfy the functional equation

$$\begin{aligned} \begin{pmatrix} \tilde{\zeta}_+\left(s, \hat{\Psi}\right) \\ \tilde{\zeta}_-\left(s, \hat{\Psi}\right) \end{pmatrix} &= 2^{4s+m/2} \pi^{-4s-2-m/2} \Gamma(s+1) \Gamma\left(s + \frac{n}{2}\right) \\ &\quad \times \Gamma\left(s + 1 + \frac{m-2n}{4}\right) \Gamma\left(s + \frac{m}{4}\right) \end{aligned}$$

$$\times \sin \pi s \begin{pmatrix} \sin \pi \left(s + \frac{q-p}{2}\right) & -2 \sin \frac{\pi p}{2} \cos \frac{\pi q}{2} \\ -2 \sin \frac{\pi q}{2} \cos \frac{\pi p}{2} & \sin \pi \left(s + \frac{p-q}{2}\right) \end{pmatrix}$$
$$\times \begin{pmatrix} \tilde{\zeta}_+ \left(-\frac{m}{4} - s, \Psi\right) \\ \tilde{\zeta}_- \left(-\frac{m}{4} - s, \Psi\right) \end{pmatrix},$$

where $m = \dim W$ and $\hat{\Psi}$ is the Fourier transform of Ψ. The degenerate cases, in which $\tilde{P}(w) \equiv 0$, appear only for

$$(p, q, m) = (2,1,2), (3,1,4), (5,1,8), (9,1,16), (2,2,4), (3,3,8), (5,5,16)$$

(Theorem 3.1).

- We completely determine when the Clifford quartic form $\tilde{P}(w)$ becomes a relative invariant of some prehomogeneous vector space (Theorem 3.2). In particular, if $p + q \geq 12$, then, there exist no prehomogeneous vector spaces having $\tilde{P}$ as a relative invariant. If $p + q \leq 4$, then, $\tilde{P}$ is always a relative invariant of some prehomogeneous vector space. If $5 \leq p + q \leq 11$ and $p + q \neq 6$, then, $\tilde{P}$ is a relative invariant of a prehomogeneous vector space only for very low-dimensional cases. Thus most of the Clifford quartic forms are non-prehomogeneous.

- To classify the cases related to prehomogeneous vector spaces, we need solid knowledge on the group of symmetries of the Clifford quartic form $\tilde{P}$. The group contains $Spin(p, q) \times H_{p,q}(\rho)$ where $H_{p,q}(\rho)$ is the intersection of the orthogonal groups $O(S_1), \ldots, O(S_{p+q})$. Except for a few low-dimensional cases, the Lie algebra of the group of symmetries of $\tilde{P}$ coincides with $\mathfrak{so}(p, q) \times \mathrm{Lie}(H_{p,q}(\rho))$ (Theorem 3.3), and the structure of $\mathrm{Lie}(H_{p,q}(\rho))$ can be determined explicitly (Theorem 3.4).

Thus our results show that the class of homogeneous polynomials that satisfy local functional equations of the form (0.1) is broader than the class of relative invariants of regular prehomogeneous vector spaces. The characterization of such polynomials is an interesting open problem. In relation to this characterization problem (in a more general form), Etingof, Kazhdan, and Polishchuk ([7]) considered the following condition for a homogeneous rational function f on a finite-dimensional vector space $\mathbf{V}$:

$$v \mapsto \operatorname{grad} f(v) \text{ defines a birational mapping of } \mathbb{P}(\mathbf{V}) \longrightarrow \mathbb{P}(\mathbf{V}^*).$$

They called this condition *the projective semiclassical condition* (PSC). A function satisfying PSC is often called *homaloidal*. It is observed that the condition PSC is closely related to the existence of local functional equation. For example, regular prehomogeneous vector spaces have homaloidal relative invariant polynomials, and it is difficult to construct non-prehomogeneous homaloidal polynomials. The classification of homaloidal polynomials is a difficult problem and of considerable interest in algebraic geometry ([3], [5], [6]).

For a homaloidal homogeneous rational function f, there exists a rational function f^* satisfying the identity $f^*(\mathrm{grad}\log f(v)) = 1/f(v)$, which is called the multiplicative Legendre transform of f. In [7], the authors raised the following question and answered it affirmatively for cubic forms:

> "Is it true that any homaloidal polynomial whose multiplicative Legendre transform is also a polynomial is a relative invariant of a prehomogeneous vector space?"

However, it can be easily proved (Theorem 2.14) that every Clifford quartic form is homaloidal and its multiplicative Legendre transform coincides with the original Clifford quartic form (up to a constant multiple). Thus, the answer to the question above is negative, since Clifford quartic forms are nonprehomogeneous in general.

The organization of this paper is as follows: In §1, we will recall the pullback theorem of local functional equations in [25]. In §2, we introduce Clifford quartic forms, and calculate the functional equations satisfied by the local zeta functions of the Clifford quartic forms. In §3, we formulate the main theorems (Theorems 3.1, 3.2) and describe an outline of the proofs. The proofs of the main theorems will be given in §4, §5, §6 and §7. In §4, we classify the degenerate cases. In §5 and §6, we make a precise investigation on the group of symmetries of Clifford quartic forms. Classification of prehomogeneous cases will be done in §7.

As in the case of relative invariants of prehomogeneous vector spaces, the Clifford quartic forms are expected to enjoy rich arithmetic properties. In fact, with the Clifford quartic forms, we can associate global zeta functions satisfying a functional equation, which are analogues of genus zeta functions of quadratic forms. For the polynomials constructed by Faraut-Koranyi, Achab defined global zeta functions and proved their functional equations ([1], [2]). In her argument, it is crucial that $Q^{-1}(v)_{\mathbb{R}}$ $(P(v) \neq 0)$ is compact.

Her method does not apply to our general setting. Our method is based on the theory of automorphic pairs of distributions on prehomogeneous vector spaces ([32], [33]). We discuss the global zeta functions in a separate paper ([26]).

Acknowledgment. We thank H. Ochiai for helpful discussions and his suggestions for improvements in exposition. We also thank H. Ishi and T. Yoshino for informing us their works [11] and [18], respectively, and for interesting discussions.

Notation. We denote by $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, respectively, the field of real numbers, the field of complex numbers and the Hamilton quaternion algebra. For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we write

$$\begin{array}{rcl} M(m;\mathbb{K}) & & \text{for the matrix algebra of size } m \text{ over } \mathbb{K}, \\ M(m,n;\mathbb{K}) & & \text{for the set of } m \text{ by } n \text{ matrices with entries in } \mathbb{K}, \\ \mathrm{Sym}(m;\mathbb{K}) & = & \left\{ X \in M(m;\mathbb{K}) \;\middle|\; {}^tX = X \right\}, \\ \mathrm{Alt}(m;\mathbb{K}) & = & \left\{ X \in M(m;\mathbb{K}) \;\middle|\; {}^tX = -X \right\}. \end{array}$$

For a $w \in \mathbb{R}^m$ and an $S \in \mathrm{Sym}(m, \mathbb{R})$, we put $S[w] := {}^twSw$. We say that the signature of S (or of the quadratic form $S[w]$) is (p, q), if S is non-degenerate and has exactly p positive and q negative eigenvalues. For square matrices $A \in M(m;\mathbb{K})$ and $B \in M(n;\mathbb{K})$, we put $A \perp B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(m+n;\mathbb{K})$. The identity matrix and the zero matrix of size m are denoted by 1_m and 0_m, respectively. We write $1_{p,q}$ for $1_p \perp -1_q$. We put $\mathbf{e}[z] := \exp(2\pi\sqrt{-1}z)$. For a real vector space V, we denote the space of rapidly decreasing functions on V by $\mathcal{S}(V)$. We use the same symbols as those in [9, Chapter X, §2.1] to denote real classical Lie algebras. When we consider a complex classical group as a linear algebraic group, we use the corresponding bold face letters; e.g., $\mathbf{GL}(m) = GL(m, \mathbb{C})$.

1. Pullback of Local Functional Equations by Quadratic Mappings

In this section, we recall the main result of [25].

Let $\mathbf{V}$ (resp. $\mathbf{W}$) be a complex vector space of dimension n (resp. m) with real-structure V (resp. W) and $\mathbf{V}^*$ (resp. $\mathbf{W}^*$) the vector space dual to $\mathbf{V}$ (resp. $\mathbf{W}$). The dual vector space V^* (resp. W^*) of the real vector space V (resp. W) can be regarded as a real-structure of $\mathbf{V}^*$(resp. $\mathbf{W}$). Let P (resp. P^*) be an irreducible homogeneous polynomial of degree d on $\mathbf{V}$ (resp. $\mathbf{V}^*$) defined over $\mathbb{R}$. We put

$$\mathbf{\Omega} = \{ v \in \mathbf{V} \mid P(v) \neq 0 \}, \quad \Omega = \mathbf{\Omega} \cap V,$$
$$\mathbf{\Omega}^* = \{ v^* \in \mathbf{V}^* \mid P^*(v^*) \neq 0 \}, \quad \Omega^* = \mathbf{\Omega}^* \cap V^*.$$

We assume that

(A.1) there exists a biregular rational mapping $\phi : \mathbf{\Omega} \to \mathbf{\Omega}^*$ defined over $\mathbb{R}$.

Let

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_\nu, \quad \Omega^* = \Omega_1^* \cup \cdots \cup \Omega_\nu^*$$

be the decomposition of Ω and Ω^* into connected components. Note that (A.1) implies that the numbers of connected components of Ω and Ω^* are the same and we may assume that

$$\Omega_j^* = \phi(\Omega_j) \quad (j = 1, \ldots, \nu).$$

Suppose that we are given quadratic mappings $Q : \mathbf{W} \to \mathbf{V}$ and $Q^* : \mathbf{W}^* \to \mathbf{V}^*$ defined over $\mathbb{R}$. The mappings $B_Q : \mathbf{W} \times \mathbf{W} \to \mathbf{V}$ and $B_{Q^*} : \mathbf{W}^* \times \mathbf{W}^* \to \mathbf{V}^*$ defined by

$$B_Q(w_1, w_2) := Q(w_1 + w_2) - Q(w_1) - Q(w_2),$$
$$B_{Q^*}(w_1^*, w_2^*) := Q^*(w_1^* + w_2^*) - Q^*(w_1^*) - Q^*(w_2^*)$$

are bilinear. For given $v \in \mathbf{V}$ and $v^* \in \mathbf{V}^*$, the mappings $Q_{v^*} : \mathbf{W} \to \mathbb{C}$ and $Q_v^* : \mathbf{W}^* \to \mathbb{C}$ defined by

$$Q_{v^*}(w) = \langle Q(w), v^* \rangle, \quad Q_v^*(w^*) = \langle v, Q^*(w^*) \rangle$$

are quadratic forms on $\mathbf{W}$ and $\mathbf{W}^*$, respectively.

Let $q(w)$ and $q^*(w^*)$ be non-degenerate quadratic forms on W and W^*, respectively. Fix a basis of W and the basis of W^* dual to it and denote by S and S^* the matrices of q and q^*, respectively. We say that q^* is dual to

$q(w)$, if $S^* = S^{-1}$. Duality for quadratic forms can be extended to quadratic mappings.

DEFINITION 1.1. (1) The quadratic mapping Q (resp. Q^*) is said to be *non-degenerate*, if the open set $\tilde{\mathbf{\Omega}} := Q^{-1}(\mathbf{\Omega})$ (resp. $\tilde{\mathbf{\Omega}}^* = Q^{*-1}(\mathbf{\Omega}^*)$) is not empty and the rank of the differential of Q (resp. Q^*) at every $w \in \tilde{\mathbf{\Omega}}$ (resp. $w^* \in \tilde{\mathbf{\Omega}}^*$) is equal to n. (It is obvious that $m \geq n$, if Q is non-degenerate.)

(2) The quadratic mapping Q^* is said to be *dual* to Q with respect to the biregular mapping ϕ in (A.1), if the quadratic form $Q^*_v(w^*)$ on W^* is dual to the quadratic form $Q_{\phi(v)}(w)$ on W for any $v \in \mathbf{\Omega}$.

We assume that

(A.2) Q and Q^* are non-degenerate and dual to each other with respect to the biregular mapping ϕ in (A.1).

REMARK 1.1. (1) By the assumption, there exist non-zero constants α, β satisfying

$$\det(Q^*_v) = \alpha P(v)^{m/d}, \quad \det\left(\frac{\partial \phi(v)_i}{\partial v_j}\right) = \beta P(v)^{-2n/d}.$$

(2) In [25], the assumptions (A.1) and (A.3) (= (A.2) in the present paper) are erroneously formulated only by referring to real structure. Moreover Ω and Ω^* in [25, p.167, Lines 21 and 22] should be $\mathbf{\Omega}$ and $\mathbf{\Omega}^*$.

The main result in [25] is that, in the above setting, if $P(v)$ and $P^*(v^*)$ satisfy a local functional equation, then the pull backs $\tilde{P}(w) := P(Q(w))$ and $\tilde{P}^*(w^*) := P^*(Q^*(w^*))$ also satisfy a local functional equation. Let us give a precise formulation.

For an $s \in \mathbb{C}$ with $\Re(s) > 0$, we define a continuous function $|P(v)|^s_j$ on V by

$$|P(v)|^s_j = \begin{cases} |P(v)|^s, & v \in \Omega_j, \\ 0, & v \notin \Omega_j. \end{cases}$$

The function $|P(v)|^s_j$ can be extended to a tempered distribution depending on s in $\mathbb{C}$ meromorphically. Similarly we define $|P^*(v^*)|^s_j$ ($s \in \mathbb{C}$).

We denote the spaces of rapidly decreasing functions on the real vector spaces V and V^* by $\mathcal{S}(V)$ and $\mathcal{S}(V^*)$, respectively. For $\Phi \in \mathcal{S}(V)$ and $\Phi^* \in \mathcal{S}(V^*)$, we define the local zeta functions by setting

$$\zeta_i(s, \Phi) = \int_V |P(v)|_i^s \, \Phi(v) \, dv,$$
$$\zeta_i^*(s, \Phi^*) = \int_{V^*} |P^*(v^*)|_i^s \, \Phi^*(v^*) \, dv^* \quad (i = 1, \ldots, \nu).$$

It is well-known that the local zeta functions $\zeta_i(s, \Phi)$, $\zeta_i^*(s, \Phi^*)$ are absolutely convergent for $\Re(s) > 0$, and have analytic continuations to meromorphic functions of s in $\mathbb{C}$. We assume the following:

(A.3) A local functional equation of the form

$$\zeta_i^*(s, \hat{\Phi}) = \sum_{j=1}^{\nu} \Gamma_{ij}(s) \zeta_j(-\frac{n}{d} - s, \Phi) \quad (i = 1, \ldots, \nu) \tag{1.1}$$

holds for every $\Phi \in \mathcal{S}(V)$, where $\Gamma_{ij}(s)$ are meromorphic functions on $\mathbb{C}$ not depending on Φ with $\det(\Gamma_{ij}(s)) \neq 0$ and

$$\hat{\Phi}(v^*) = \int_V \Phi(v) \exp(-2\pi\sqrt{-1}\langle v, v^* \rangle) \, dv,$$

the Fourier transform of Φ.

We put

$$\tilde{P}(w) = P(Q(w)), \quad \tilde{P}^*(w^*) = P^*(Q^*(w^*))$$
$$\tilde{\Omega}_i = Q^{-1}(\Omega_i), \quad \tilde{\Omega}_i^* = Q^{*-1}(\Omega_i^*) \quad (i = 1, \ldots, \nu).$$

Some of $\tilde{\Omega}_i$'s and $\tilde{\Omega}_i^*$'s may be empty. We define $\left|\tilde{P}(w)\right|_i^s$ and $\left|\tilde{P}^*(w^*)\right|_i^s$ in the same manner as above. For $\Psi \in \mathcal{S}(W)$ and $\Psi^* \in \mathcal{S}(W^*)$, we define the zeta functions associated with $\tilde{P}$ and $\tilde{P}^*$ by

$$\tilde{\zeta}_i(s, \Psi) = \int_W \left|\tilde{P}(w)\right|_i^s \Psi(w) \, dw, \quad \tilde{\zeta}_i^*(s, \Psi^*) = \int_{W^*} \left|\tilde{P}^*(w^*)\right|_i^s \Psi^*(w^*) \, dw^*.$$

We denote by $\hat{\Psi}$ the Fourier transform of Ψ:

$$\hat{\Psi}(w^*) = \int_W \Psi(w) \exp(2\pi\sqrt{-1}\langle w, w^* \rangle) \, dw.$$

THEOREM 1.2 ([25], Theorem 4). *Under the assumptions* (A.1), (A.2), (A.3), *the zeta functions* $\tilde{\zeta}_i(s, \Psi)$ *and* $\tilde{\zeta}_i^*(s, \Psi^*)$ *satisfy the local functional equation*

$$\tilde{\zeta}_i^*\left(s, \hat{\Psi}\right) = \sum_{j=1}^{\nu} \tilde{\Gamma}_{ij}(s) \tilde{\zeta}_j\left(-\frac{m}{2d} - s, \Psi\right),$$

where the gamma factors $\tilde{\Gamma}_{ij}(s)$ *are given by*

$$\tilde{\Gamma}_{ij}(s) = 2^{2ds+m/2} |\alpha|^{1/2} |\beta|^{-1} \sum_{k=1}^{\nu} \mathbf{e}\left[\frac{p_k - q_k}{8}\right] \Gamma_{ik}(s) \Gamma_{kj}\left(s + \frac{m-2n}{2d}\right),$$

where α, β *are the constants defined in Remark* 1.1 (1) *and* (p_k, q_k) *is the signature of the quadratic form* Q_v^* *for* $v \in \Omega_k$.

REMARK 1.2. (1) The signature (p_k, q_k) of $Q_v^*(w^*)$ does not depend on the choice of v, since Ω_k is connected.

(2) In [25], the theorem is formulated for multi-variable zeta functions. Here we restrict ourselves to single variable zeta functions for simplicity.

The theory of prehomogeneous vector spaces (see [28], [30], [23], [17]) provides a lot of examples of P and P^* satisfying (A.1) and (A.3). Therefore, if one can construct dual non-degenerate quadratic mappings to a prehomogeneous vector space, then by Theorem 1.2, one obtains a new local functional equation. In [8, Chapter XVI], Faraut and Koranyi proved that, starting from a representation of a Euclidean Jordan algebra, one can construct polynomials satisfying local functional equations (see also Clerc [4]). Theorem 1.2 generalizes their result (see [25, §2.2]).

The Faraut-Koranyi construction is especially interesting in the case of the simple Euclidean Jordan algebras of rank 2, since the polynomials $\tilde{P}$ obtained in this case are *not* relative invariants of prehomogeneous vector spaces except for some low-dimensional cases, as is noticed in [4] (without specifying the low-dimensional exceptions explicitly). Let us explain this non-prehomogeneous example without referring to Jordan algebra. Let V be the $q+1$-dimensional real quadratic space of signature $(1, q)$. We fix a basis $\{e_0, e_1, \ldots, e_q\}$ of V, for which the quadratic form is given by

$$P(v) = v_0^2 - v_1^2 - \cdots - v_q^2.$$

Denote by C_q the Clifford algebra of the positive definite quadratic form $v_1^2+\cdots+v_q^2$ and consider a representation $S : C_q \to M(m;\mathbb{R})$ of C_q on an m-dimensional $\mathbb{R}$-vector space. We may assume that $S_i := S(e_i)$ $(i = 1, \ldots , q)$ are symmetric matrices. We denote by $W = \mathbb{R}^m$ the representation space of S, and define a quadratic mapping $Q : W \to V$ by

$$Q(w) = ({}^t ww)e_0 + \sum_{i=1}^{q} S_i[w]e_i.$$

Then, if $\tilde{P}(w) = P(Q(w)) = ({}^t ww)^2 - \sum_{i=1}^{q}(S_i[w])^2$ does not vanish identically, Q is a self-dual non-degenerate quadratic mapping and, by Theorem 1.2, $\tilde{P}$ satisfies a local functional equation. In the next section, we generalize this construction by classifying the self-dual non-degenerate quadratic mappings to real non-degenerate quadratic spaces of arbitrary signature.

REMARK 1.3. In [11], Ishi proved that, if V is the underlying vector space of a semisimple (not necessarily Euclidean) Jordan algebra, P is the determinant of the Jordan algebra and $Q : W \to V$ is a self-dual non-degenerate quadratic mapping, then the mapping $v \mapsto Q_v^*$ induces a representation of the Jordan algebra V.

2. Local Functional Equations of Clifford Quartic Forms

2.1. Self-dual quadratic mappings and representations of Clifford algebras

Let p, q be non-negative integers, V a real $p+q$-dimensional vector space and consider a quadratic form $P(v)$ of signature (p, q) on V. We often write $n = \dim V = p + q$. We assume that $n = p + q \geq 3$. Then the quadratic form $P(v)$ is absolutely irreducible. Fix a basis $\{e_1, \ldots , e_{p+q}\}$ of V, which is called the *standard* basis, satisfying

$$P\left(\sum_{i=1}^{p+q} v_i e_i\right) = \sum_{i=1}^{p} v_i^2 - \sum_{j=1}^{q} v_{p+j}^2.$$

We identify V with $\mathbb{R}^{p+q}$ with the standard basis, and also with its dual vector space via the standard inner product $(v, v^*) = v_1 v_1^* + \cdots + v_{p+q} v_{p+q}^*$. Put $\Omega = V \setminus \{P = 0\}$. We determine the quadratic mappings $Q : W \to V$

that are self-dual with respect to the biregular mapping $\phi : \Omega \longrightarrow \Omega$ defined by

$$\phi(v) := \frac{1}{2}\mathrm{grad}\log P(v) = \frac{1}{P(v)}(v_1, \dots, v_p, -v_{p+1}, \dots, -v_{p+q}).$$

By a quadratic mapping Q of $W = \mathbb{R}^m$ to $V = \mathbb{R}^{p+q}$, we mean a mapping defined by

$$Q(w) = (S_1[w], \dots, S_{p+q}[w])$$

for some real symmetric matrices $S_1, \dots, S_{p+q}$ of size m. For $v = (v_1, \dots, v_{p+q}) \in \mathbb{R}^{p+q}$, we put

$$S(v) = \sum_{i=1}^{p+q} v_i S_i.$$

Then, by definition, the mapping Q is self-dual with respect to ϕ if and only if

$$S(v)S(\phi(v)) = 1_m \quad (v \in \Omega).$$

If we put

$$\varepsilon_i = \begin{cases} 1 & (1 \leq i \leq p), \\ -1 & (p+1 \leq i \leq p+q), \end{cases} \tag{2.1}$$

this condition is equivalent to the polynomial identity

$$\sum_{i=1}^{p} v_i^2 S_i^2 - \sum_{j=1}^{q} v_{p+j}^2 S_{p+j}^2 + \sum_{1\leq i<j\leq p+q} v_i v_j \left(\varepsilon_j S_i S_j + \varepsilon_i S_j S_i\right) = P(v)1_m.$$

This identity holds if and only if

$$S_i^2 = 1_m \quad (1 \leq i \leq p+q), \tag{2.2}$$

$$S_i S_j = \begin{cases} S_j S_i & (1 \leq i \leq p < j \leq p+q \text{ or } 1 \leq j \leq p < i \leq p+q) \\ -S_j S_i & (1 \leq i, j \leq p \text{ or } p+1 \leq i, j \leq p+q). \end{cases} \tag{2.3}$$

This means that the linear map $S : V \to \mathrm{Sym}(m; \mathbb{R})$ can be extended to a representation of the tensor product of the Clifford algebra C_p of $v_1^2 + \cdots + v_p^2$

and the Clifford algebra C_q of $v_{p+1}^2 + \cdots + v_{p+q}^2$. We denote the tensor product $C_p \otimes C_q$ by $R_{p,q}$.

Conversely, if we are given a representation $\rho : R_{p,q} \to M(m;\mathbb{R})$, then we can obtain a self-dual quadratic mapping $Q : W = \mathbb{R}^m \to V$. Indeed, the images of the standard basis $S_1 = \rho(e_1), \ldots, S_{p+q} = \rho(e_{p+q})$ satisfy the relations above. (We always identify $e_i \otimes 1$ (resp. $1 \otimes e_i$) with e_i for $1 \leq i \leq p$ (resp. $p+1 \leq i \leq p+q$).) Moreover, since $R_{p,q}$ is semisimple, ρ is a direct sum of irreducible representations. Any irreducible representation of $R_{p,q}$ is a tensor product of an irreducible representation of C_p and an irreducible representation of C_q. Hence, S_i is of the form $(\rho_1 \otimes \rho_1')(e_i) \perp \cdots \perp (\rho_r \otimes \rho_r')(e_i)$ for some representations $\rho_1, \ldots, \rho_r$ of C_p and some representations $\rho_1', \ldots, \rho_r'$ of C_q. Therefore, by the following lemma (applied to the positive definite case), we may take symmetric matrices as $S_1, \ldots, S_{p+q}$ (by taking conjugate, if necessary), and then the mapping $Q(w) = (S_1[w], \ldots, S_{p+q}[w])$ is self-dual.

LEMMA 2.1. *Let P be a quadratic form on $V = \mathbb{R}^{p+q}$ of signature (p,q) and let $e_1, \ldots, e_{p+q}$ be the standard basis of V such that $P(\sum_{i=1}^{p+q} v_i e_i) = \sum_{i=1}^{p} v_i^2 - \sum_{j=1}^{q} v_{p+j}^2$. Denote by $C_{p,q}$ the Clifford algebra of the quadratic form P and let $\rho : C_{p,q} \to M(m;\mathbb{R})$ be a representation of $C_{p,q}$. Then, in the equivalence class of ρ, there exists a representation with the property that $\rho(e_i)$ is a symmetric matrix for $1 \leq i \leq p$ and a skew-symmetric matrix for $p+1 \leq i \leq p+q$.*

PROOF. By the definition of the Clifford algebra $C_{p,q}$, we have

$$e_i^2 = 1 \quad (1 \leq i \leq p), \quad e_i^2 = -1 \quad (p+1 \leq i \leq p+q),$$
$$e_i e_j = -e_i e_j \quad (i \neq j).$$

Hence, the multiplicative group G generated by $\{-1, e_1, \ldots, e_{p+q}\}$ is a finite group and ρ gives a group-representation of G on $\mathbb{R}^m$. Therefore, if we replace ρ by an equivalent representation if necessary, we may assume that every element in G is represented by an orthogonal matrix. Then $\rho(e_i) = \rho(e_i)^{-1} = {}^t\rho(e_i)$ for $1 \leq i \leq p$, and $\rho(e_i) = -\rho(e_i)^{-1} = -{}^t\rho(e_i)$ for $p+1 \leq i \leq p+q$. $\square$

Thus we have proved the following theorem on the correspondence between self-dual quadratic mappings and representations of $R_{p,q}$.

THEOREM 2.2. *Self-dual quadratic mappings Q of $W = \mathbb{R}^m$ to the quadratic space (V, P) correspond to representations ρ of $R_{p,q} = C_p \otimes C_q$ such that $\rho(V)$ is contained in* $\mathrm{Sym}(m; \mathbb{R})$.

We call the symmetric matrices $S_1 = \rho(e_1), \dots, S_{p+q} = \rho(e_{p+q})$ the *basis matrices* of ρ.

REMARK 2.1. The construction above is a generalization of the quadratic mappings obtained from representations of simple Euclidean Jordan algebra of rank 2 in the theory of Faraut-Koranyi [8]. In the case $(p, q) = (1, q)$, we have $C_1 \cong \mathbb{R} \oplus \mathbb{R}$ and $R_{1,q} = C_1 \otimes C_q \cong C_q \oplus C_q$. Hence representations of $R_{1,q}$ can be identified with the direct sum of two C_q-modules W_+ and W_-. On W_+ (resp. W_-), e_1 acts as multiplication by $+1$ (resp. -1). The quadratic mappings given by the Faraut-Koranyi construction correspond to representations of $R_{1,q}$ for which $W_- = \{0\}$.

Later we need the following lemma on canonical forms of basis matrices.

LEMMA 2.3. *Assume that $p \geq 2$ and let ρ be a representation of $R_{p,q}$. Then the dimension m of ρ is even. Put $d = m/2$. Let $S_1 = \rho(e_1), \dots, S_{p+q} = \rho(e_{p+q})$ be the basis matrices of ρ. Then the signature of S_i $(1 \leq i \leq p)$ is (d, d) and (by replacing ρ by an equivalent representation, if necessary,) we can take the basis matrices of the form*

$$S_1 = \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix}, \quad S_i = \begin{pmatrix} 0 & B_i \\ {}^tB_i & 0 \end{pmatrix} \ (2 \leq i \leq p),$$
$$S_{p+j} = \begin{pmatrix} A_j & 0 \\ 0 & A_j \end{pmatrix} \ (1 \leq j \leq q).$$

Here $A_1, \dots, A_q$ are the basis matrices of some d-dimensional representation of $R_{0,q} = C_q$, $B_2 = 1_d$, and $B_3, \dots, B_p$ are orthogonal and skew symmetric matrices. Moreover they satisfy the commutation relations

$$A_iA_j = -A_jA_i \ (1 \leq i < j \leq q), \quad B_iB_j = -B_jB_i \ (3 \leq i < j \leq p),$$
$$A_iB_j = B_jA_i \quad (1 \leq i \leq q, \ 3 \leq j \leq p).$$

PROOF. Since $S_1^2 = 1_m$, we may assume that $S_1 = 1_{r,m-r}$. Then, by the anti-commutativity $S_iS_j = -S_jS_i$, the symmetric matrices S_i $(2 \leq i \leq p)$ are of the form

$$S_i = \begin{pmatrix} 0_r & B_i \\ {}^tB_i & 0_{m-r} \end{pmatrix}, \quad B_i \in M(r, m-r; \mathbb{R}).$$

Since $S_i^2 = 1_m$, we have $r = m - r$ and B_i is orthogonal. This shows that m is even and the signatures of $S_1, \ldots, S_p$ are equal to (d, d) $(d = m/2)$. Put $\tilde{B}_2 = {}^tB_2 \perp 1_d$. By replacing all the S_i by $\tilde{B}_2 S_i \tilde{B}_2^{-1}$, we may assume that $S_1 = 1_{d,d}$ and $B_2 = 1_d$. The commutation relations $S_2S_i = -S_iS_2$ $(3 \leq i \leq p)$ imply that B_i $(3 \leq i \leq p)$ are skew-symmetric. The remaining part of the lemma is also a straightforward consequence of the commutation relations (2.2) and (2.3). □

2.2. Structure of $R_{p,q}$

It is well-known that the structure of C_p depends on $p \bmod 8$ as the following lemma shows (see [21], [22], [34]):

LEMMA 2.4. *The structure of C_p is given by the following table:*

p	C_p
$p \equiv 0 \pmod 8$	$M(2^{p/2}; \mathbb{R})$
$p \equiv 1 \pmod 8$	$M(2^{(p-1)/2}; \mathbb{R}) \oplus M(2^{(p-1)/2}; \mathbb{R})$
$p \equiv 2 \pmod 8$	$M(2^{p/2}; \mathbb{R})$
$p \equiv 3 \pmod 8$	$M(2^{(p-1)/2}; \mathbb{C})$
$p \equiv 4 \pmod 8$	$M(2^{(p-2)/2}; \mathbb{H})$
$p \equiv 5 \pmod 8$	$M(2^{(p-3)/2}; \mathbb{H}) \oplus M(2^{(p-3)/2}; \mathbb{H})$
$p \equiv 6 \pmod 8$	$M(2^{(p-2)/2}; \mathbb{H})$
$p \equiv 7 \pmod 8$	$M(2^{(p-1)/2}; \mathbb{C})$

We denote by $R_{p,q}^+$ the subalgebra of $R_{p,q}$ consisting of all the even elements, namely, the subalgebra generated by e_ie_j $(1 \leq i < j \leq p+q)$. The structure of $R_{p,q}$ and $R_{p,q}^+$ is easily seen from Lemma 2.4.

Lemma 2.5. *Put $n = p + q$. Then the structure of $R_{p,q}$ and $R^+_{p,q}$ is given by the following table:*

Type	$(R_{p,q}, R^+_{p,q})$	ℓ	r	$(\mathbb{K}, \mathbb{K}')$	$\{p \bmod 8, q \bmod 8\}$
I	(T, T')	$2^{n/2}$	$2^{n/2-1}$	$(\mathbb{R}, \mathbb{C})$	$\{0, 2\}, \{4, 6\}$
		$2^{(n-1)/2}$	$2^{(n-1)/2}$	$(\mathbb{C}, \mathbb{R})$	$\{0, 7\}, \{2, 3\}, \{3, 4\}, \{6, 7\}$
		$2^{(n-1)/2}$	$2^{(n-1)/2-1}$	$(\mathbb{C}, \mathbb{H})$	$\{0, 3\}, \{2, 7\}, \{3, 6\}, \{4, 7\}$
		$2^{n/2-1}$	$2^{n/2-1}$	$(\mathbb{H}, \mathbb{C})$	$\{0, 6\}, \{2, 4\}$
II	$(T, 2T')$	$2^{n/2}$	$2^{n/2-1}$	$(\mathbb{R}, \mathbb{R})$	$\{0, 0\}, \{2, 2\}, \{4, 4\}, \{6, 6\}$
		$2^{n/2-1}$	$2^{n/2-2}$	$(\mathbb{H}, \mathbb{H})$	$\{0, 4\}, \{2, 6\}$
III	$(2T, T')$	$2^{(n-1)/2}$	$2^{(n-1)/2}$	$(\mathbb{R}, \mathbb{R})$	$\{0, 1\}, \{1, 2\}, \{4, 5\}, \{5, 6\}$
		$2^{n/2-1}$	$2^{n/2-1}$	$(\mathbb{C}, \mathbb{C})$	$\{1, 3\}, \{1, 7\}, \{3, 5\}, \{5, 7\}$
		$2^{(n-3)/2}$	$2^{(n-3)/2}$	$(\mathbb{H}, \mathbb{H})$	$\{0, 5\}, \{1, 4\}, \{1, 6\}, \{2, 5\}$
IV	$(2T, 2T')$	$2^{n/2-1}$	$2^{n/2-1}$	$(\mathbb{C}, \mathbb{R})$	$\{3, 3\}, \{7, 7\}$
		$2^{n/2-1}$	$2^{n/2-2}$	$(\mathbb{C}, \mathbb{H})$	$\{3, 7\}$
V	$(4T, 2T')$	$2^{n/2-1}$	$2^{n/2-1}$	$(\mathbb{R}, \mathbb{R})$	$\{1, 1\}, \{5, 5\}$
		$2^{n/2-2}$	$2^{n/2-2}$	$(\mathbb{H}, \mathbb{H})$	$\{1, 5\}$

where T (resp. T') denotes the matrix algebra $M(\ell; \mathbb{K})$ (resp. $M(r; \mathbb{K}')$), and $2T$ (resp. $2T'$, $4T$) denotes $T \oplus T$ (resp. $T' \oplus T'$, $T \oplus T \oplus T \oplus T$).

The number of inequivalent irreducible representations of $R_{p,q}$ is equal to the number of simple components, namely, 1 for type I and II , 2 for type III and IV, and 4 for type V. The dimension over $\mathbb{R}$ of the irreducible representations of $R_{p,q}$ is given by $\ell \dim_{\mathbb{R}} \mathbb{K}$, which is a power of 2. As in Lemma 2.1, we denote by $C_{p,q}$ the Clifford algebra of the quadratic form P and by $C^+_{p,q}$ the subalgebra of $C_{p,q}$ of even elements. Then the algebra $R^+_{p,q}$ is isomorphic to $C^+_{p,q}$, while $R_{p,q}$ is not necessarily isomorphic to $C_{p,q}$. The isomorphism of $R^+_{p,q}$ to $C^+_{p,q}$ is given by $e_i e_j \mapsto \varepsilon_j \tilde{e}_i \tilde{e}_j$, where $\tilde{e}_i$ denotes the element $e_i \in V$ viewed as an element in $C_{p,q}$ and ε_i's are the same as in (2.1).

Let $\mathfrak{k}_{p,q}$ be the real vector space spanned by $e_i e_j$ $(1 \leq i < j \leq p + q)$. Then $\mathfrak{k}_{p,q}$ is a Lie subalgebra of $R_{p,q}$ with bracket product $[X, Y] := XY - YX$.

LEMMA 2.6. *The Lie algebra $\mathfrak{k}_{p,q}$ is isomorphic to*

$$\mathfrak{so}(p,q) = \left\{ X \in M(p+q;\mathbb{R}) \;\middle|\; {}^tX1_{p,q} + 1_{p,q}X = 0 \right\}$$

and a representation ρ of $R_{p,q}$ on a vector space W induces a Lie algebra representation of $\mathfrak{k}_{p,q}$, which is a direct sum of real (half-) spin representations of $\mathfrak{so}(p,q)$. The self-dual quadratic map $Q : W \to V$ corresponding to ρ is $\mathrm{Spin}(p,q)$*-equivariant. Here,* $\mathrm{Spin}(p,q)$ *denotes the real spin group, which is a double covering group of the identity component of the orthogonal group* $\mathrm{SO}(p,q)$.

PROOF. For $Y \in \mathfrak{k}_{p,q}$, we have

$$\frac{d}{dt} S_k[\exp(t\rho(Y))w]\bigg|_{t=0} = ({}^t\rho(Y)S_k + S_k\rho(Y))[w] \quad (w \in W).$$

In case $Y = e_ie_j$ $(i \neq j)$, the relations (2.2) and (2.3) imply that

$${}^t\rho(Y)S_k + S_k\rho(Y) = {}^t(S_iS_j)S_k + S_k(S_iS_j) = \begin{cases} 0 & (k \neq i,j), \\ 2S_j & (k = i), \\ -2\varepsilon_i\varepsilon_j S_i & (k = j). \end{cases}$$

This shows that a Lie algebra isomorphism f of $\mathfrak{k}_{p,q}$ onto $\mathfrak{so}(p,q)$ is defined by

$$f(e_ie_j) = X_{ij} := 2E_{ij} - 2\varepsilon_i\varepsilon_j E_{ji} \quad (1 \leq i < j \leq p+q)$$

(E_{ij} is the matrix unit) and Q satisfies the identity

$$\frac{d}{dt} Q(\exp(t\rho(Y))w)\bigg|_{t=0} = \frac{d}{dt}\exp(tf(Y))Q(w)\bigg|_{t=0}.$$

This shows that Q is $\mathrm{Spin}(p,q)$-equivariant, since $\mathrm{Spin}(p,q)$ is connected for $p+q \geq 3$ (see [34, Theorem 5.4.7]). The representation of $\mathfrak{k}_{p,q}$ on W generates the representation ρ of the algebra $R^+_{p,q}$ which is isomorphic to the even Clifford algebra $C^+_{p,q}$. Hence the representation of $\mathfrak{k}_{p,q}$ on W is equivalent to a direct sum of real (half-) spin representations. □

The following distinction between representations will play an important role later.

DEFINITION 2.7. For a representation ρ of $R_{p,q}$, denote by $\rho_{\mathbb{C}}$ the complexification of ρ, which is a representation of $R_{p,q} \otimes \mathbb{C}$. ρ is called *pure*, if the restriction of $\rho_{\mathbb{C}}$ to $R_{p,q}^{+} \otimes \mathbb{C}$ is isotypic, namely, it is a direct sum of several copies of a single irreducible representation of $R_{p,q}^{+} \otimes \mathbb{C}$. If $\rho_{\mathbb{C}}$ contains inequivalent irreducible representations of $R_{p,q}^{+} \otimes \mathbb{C}$, then ρ is called *mixed*.

2.3. Clifford quartic forms and local zeta functions

Let p, q be non-negative integers satisfying $n = p + q \geq 3$. Since $R_{p,q}$ is isomorphic to $R_{q,p}$ and the results are symmetric with respect to p and q, it is sufficient to describe our results only for the case where $p \geq q$.

Let $S_1, \ldots, S_{p+q}$ be the basis matrices of an m-dimensional representation ρ of $R_{p,q}$, and define the quadratic mapping $Q : W = \mathbb{R}^m \to V = \mathbb{R}^{p+q}$ by

$$Q(w) = (S_1[w], \ldots, S_{p+q}[w]) \quad (w \in W).$$

We put

$$P(v) = \sum_{i=1}^{p} v_i^2 - \sum_{j=1}^{q} v_{p+j}^2, \quad \tilde{P}(w) = P(Q(w)) = \sum_{i=1}^{p} S_i[w]^2 - \sum_{j=1}^{q} S_{p+j}[w]^2.$$

We call the polynomial $\tilde{P}(w)$ of degree 4 the *Clifford quartic form* associated with ρ.

LEMMA 2.8. *The quadratic mapping Q is non-degenerate if and only if the Clifford quartic form $\tilde{P}(w)$ does not vanish identically.*

PROOF. The only if part is obvious. Let us prove the if part. By Lemma 2.6, $GL(1) \times \mathrm{Spin}(p, q)$ acts on V (resp. W) as the vector representation (resp. (a direct sum of several copies of) the spin representation) and Q is $GL(1) \times \mathrm{Spin}(p, q)$-equivariant. Moreover the action of $GL(1) \times \mathrm{Spin}(p, q)$ on V gives a prehomogeneous vector space and the open orbit Ω is given by $\{v \in V | P(v) \neq 0\}$. Hence, the if part follows from [25, Lemma 6]. □

By Theorem 1.2 and Lemma 2.8, if the Clifford quartic form $\tilde{P}$ does not vanish identically, then the local zeta functions of $\tilde{P}$ satisfy a local functional equation with an explicit gamma factor. We see later in Theorem 3.1 that the quadratic mapping Q corresponding to an m-dimensional representation of $R_{p,q}$ is non-degenerate and $\tilde{P}$ does not vanish identically, if and only if (p,q,m) is different from any one of

$$(2,1,2),(3,1,4),(5,1,8),(9,1,16),(2,2,4),(3,3,8),(5,5,16).$$

Now we describe the explicit formula for the local functional equations.

LEMMA 2.9. *For $p \geq q \geq 0$ with $p+q \geq 3$, we put*

$$\Omega = \left\{ v \in \mathbb{R}^{p+q} \;\middle|\; P(v) \neq 0 \right\}, \quad P(v) = \sum_{i=1}^{p} v_i^2 - \sum_{j=1}^{q} v_{p+j}^2.$$

Then, the decomposition of Ω into connected components is given as follows:

(1) *If $(p,q) = (p,0)$ with $p \geq 3$, then Ω is connected and we have $\Omega = \Omega_+ := \{ v \in \mathbb{R}^p \mid P(v) > 0 \}$.*

(2) *If $(p,q) = (p,1)$ with $p \geq 2$, then we have $\Omega = \Omega_+ \cup \Omega_{-,+} \cup \Omega_{-,-}$, where*

$$\begin{aligned} \Omega_+ &= \left\{ v \in \mathbb{R}^{p+1} \;\middle|\; P(v) > 0 \right\}, \\ \Omega_{-,+} &= \left\{ v \in \mathbb{R}^{p+1} \;\middle|\; P(v) < 0, v_{p+1} > 0 \right\}, \\ \Omega_{-,-} &= \left\{ v \in \mathbb{R}^{p+1} \;\middle|\; P(v) < 0, v_{p+1} < 0 \right\}. \end{aligned}$$

(3) *If $p, q \geq 2$, then we have $\Omega = \Omega_+ \cup \Omega_-$, where*

$$\Omega_+ = \left\{ v \in \mathbb{R}^{p+q} \;\middle|\; P(v) > 0 \right\}, \quad \Omega_- = \left\{ v \in \mathbb{R}^{p+q} \;\middle|\; P(v) < 0 \right\}.$$

Put $\tilde{\Omega}_\pm = Q^{-1}(\Omega_\pm)$ and $\tilde{\Omega}_{-,\pm} = Q^{-1}(\Omega_{-,\pm})$. Then, for $\Psi \in \mathcal{S}(W)$, the local zeta functions of the Clifford quartic form $\tilde{P}$ is defined by

$$\begin{cases} \tilde{\zeta}_\pm(s,\Psi) = \displaystyle\int_{\tilde{\Omega}_\pm} \left| \tilde{P}(w) \right|^s \Psi(w)\, dw, \\ \tilde{\zeta}_{-,\pm}(s,\Psi) = \displaystyle\int_{\tilde{\Omega}_{-,\pm}} \left| \tilde{P}(w) \right|^s \Psi(w)\, dw. \end{cases} \tag{2.4}$$

By the general theory in §1, the local zeta functions $\tilde{\zeta}_{\pm}(s;\Psi)$, $\tilde{\zeta}_{-,\pm}(s;\Psi)$ can be continued to meromorphic functions of s in $\mathbb{C}$, and satisfy the functional equations in Theorem 1.2. Let us calculate several constants appearing in the gamma factors $\tilde{\Gamma}_{ij}(s)$ in Theorem 1.2.

For $v \in \mathbb{R}^{p+q}$ with $P(v) \neq 0$, the signature of the symmetric matrix $S(v) = \sum_{i=1}^{p+q} v_i S_i$ depends only on the connected component to which v belongs. For each connected component Ω_η, $\Omega_{-,\eta}$ $(\eta = \pm 1)$ we define $\gamma = \gamma_\eta, \gamma_{-,\eta}$ by

$$\gamma = \mathbf{e}\left[\frac{\sigma_+ - \sigma_-}{8}\right],$$

where σ_+ and σ_-, respectively, are the numbers of positive and negative eigenvalues of $S(v)$ for a point $v \in \Omega_\eta$ or $\Omega_{-,\eta}$. An explicit formula for the constants γ is given by the following lemma.

LEMMA 2.10. *Assume that $p \geq q \geq 0$.*

(1) *If $p \geq 3, q = 0$, then $\gamma = 1$.*

(2) *If $p \geq 2, q = 1$, then*

$$\gamma_+ = 1, \quad \gamma_{-,\eta} = \begin{cases} (\sqrt{-1})^{\eta(k_+ - k_-)} & (p = 2) \\ (-1)^{k_+ - k_-} & (p = 3) \\ 1 & (p \geq 4) \end{cases} \qquad (\eta = \pm),$$

where k_+ (resp. k_-) is the multiplicity in ρ of the irreducible representations of $R_{p,1}$ for which e_{p+1} acts as multiplication by $+1$ (resp. -1).

(3) *If $p \geq q \geq 2$, then $\gamma_+ = \gamma_- = 1$.*

PROOF. The constant γ does not depend on the choice of a representative v of $\Omega_{\eta\tau}$. Hence we may take $v = {}^t(\pm 1, 0, \ldots, 0)$ or $v = {}^t(0, \ldots, 0, \pm 1)$. Then $S(v) = \pm S_1$ or $\pm S_{p+q}$.

(1) If $p \geq 2$, by Lemma 2.3, the signature of S_1 is $(\frac{m}{2}, \frac{m}{2})$. Hence we have $\gamma = 1$.

(2) The proof of $\gamma_+ = 1$ is quite the same as that for (1). Let d be the dimension of irreducible representations of C_p. (Note that all the irreducible

representations are of the same dimension.) Then $S_{p+1} = I_{dk_+,dk_-}$ and we have $\sigma_+ - \sigma_- = d(k_+ - k_-)$. Since $d = 2$ for $p = 2$, $d = 4$ for $p = 3$, and $8|d$ for $p \geq 3$, this implies the fourth assertion.

(5) The last assertion is obvious from Lemma 2.3. □

LEMMA 2.11. *The constants α, β defined in Remark 1.1 (1) are given by*

$$\alpha = \pm 1, \quad \beta = (-1)^{q+1}.$$

PROOF. Put $n := p + q = \dim V$. For $v \in V$, we put $\varepsilon v = (\varepsilon_1 v_1, \dots, \varepsilon_n v_n)$, where ε_i's are given by (2.1). Then, by (2.2) and (2.3), we have $S(v)S(\varepsilon v) = P(v)1_m$ and $\det S(v) \det S(\varepsilon v) = P(v)^m$. Since $\det S(v) = \alpha P(v)^{m/2}$, and $\det S(\varepsilon v) = \alpha P(\varepsilon v)^{m/2} = \alpha P(v)^{m/2}$, this proves that $\alpha = \pm 1$. The Jacobian

$$\det\left(\frac{\partial \phi(v)_i}{\partial v_j}\right)$$
$$= \det\left\{ \frac{1}{P(v)} \begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{pmatrix} \right.$$
$$\left. - \frac{2}{P(v)^2} \begin{pmatrix} (\varepsilon_1 v_1)^2 & (\varepsilon_1 v_1)(\varepsilon_2 v_2) & \cdots & (\varepsilon_1 v_1)(\varepsilon_n v_n) \\ (\varepsilon_2 v_2)(\varepsilon_1 v_1) & (\varepsilon_2 v_2)^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & (\varepsilon_{n-1} v_{n-1})(\varepsilon_n v_n) \\ (\varepsilon_n v_n)(\varepsilon_1 v_1) & \cdots & (\varepsilon_n v_n)(\varepsilon_{n-1} v_{n-1}) & (\varepsilon_n v_n)^2 \end{pmatrix} \right\}$$

is an $SO(p,q)$-invariant homogeneous rational function of degree $-2n$ and is equal to $\beta P(v)^{-n}$. We can obtain β easily by taking $v = {}^t(1, 0, \dots, 0)$. □

We define the local zeta functions for the quadratic form $P(v)$ by (the analytic continuations of) the integrals

$$\zeta_\pm(s, \Phi) = \int_{\Omega_\pm} |P(v)|^s \, \Phi(v) \, dv,$$

$$\zeta_{-,\pm}(s, \Phi) = \int_{\Omega_{-,\pm}} |P(v)|^s \, \Phi(v) \, dv \quad (\Phi \in \mathcal{S}(\mathbb{R}^n)).$$

Lemma 2.12. *For any $\Phi \in \mathcal{S}(\mathbb{R}^n)$ $(n = p+q)$, the following functional equations hold:*
(1) *Assume that $(p,q) = (n,0)$. Then*

$$\zeta_+\left(s, \hat{\Phi}\right) = -\pi^{-(2s+n/2+1)}\Gamma(s+1)\Gamma\left(s+\frac{n}{2}\right)\sin(s\pi)\,\zeta_+\left(-s-\frac{n}{2}, \Phi\right).$$

(2) *Assume that $(p,q) = (n-1,1)$. Then*

$$\begin{pmatrix} \zeta_+\left(s,\hat{\Phi}\right) \\ \zeta_{-,+}\left(s,\hat{\Phi}\right) \\ \zeta_{-,-}\left(s,\hat{\Phi}\right) \end{pmatrix} = \pi^{-(2s+n/2+1)}\Gamma(s+1)\Gamma\left(s+\frac{n}{2}\right) \times \begin{pmatrix} -\cos(s\pi) & -\cos\left(\frac{n\pi}{2}\right) & -\cos\left(\frac{n\pi}{2}\right) \\ \frac{1}{2} & \frac{1}{2}\mathbf{e}\left[-\frac{2s+n}{4}\right] & \frac{1}{2}\mathbf{e}\left[\frac{2s+n}{4}\right] \\ \frac{1}{2} & \frac{1}{2}\mathbf{e}\left[\frac{2s+n}{4}\right] & \frac{1}{2}\mathbf{e}\left[-\frac{2s+n}{4}\right] \end{pmatrix} \times \begin{pmatrix} \zeta_+\left(-s-\frac{n}{2},\Phi\right) \\ \zeta_{-,+}\left(-s-\frac{n}{2},\Phi\right) \\ \zeta_{-,-}\left(-s-\frac{n}{2},\Phi\right) \end{pmatrix}.$$

(3) *Assume that $p, q \geq 2$ and put $n = p+q$. Then*

$$\begin{pmatrix} \zeta_+\left(s,\hat{\Phi}\right) \\ \zeta_-\left(s,\hat{\Phi}\right) \end{pmatrix} = \pi^{-(2s+n/2+1)}\Gamma(s+1)\Gamma\left(s+\frac{n}{2}\right) \times \begin{pmatrix} -\sin\left(\frac{(2s+q)\pi}{2}\right) & \sin\left(\frac{p\pi}{2}\right) \\ \sin\left(\frac{q\pi}{2}\right) & -\sin\left(\frac{(2s+p)\pi}{2}\right) \end{pmatrix} \times \begin{pmatrix} \zeta_+\left(-s-\frac{n}{2},\Phi\right) \\ \zeta_-\left(-s-\frac{n}{2},\Phi\right) \end{pmatrix}.$$

Proof. The first and the third functional equations are well-known (see, e.g., [17, §4.2]). The second functional equation can be derived from the two-variable functional equations given in [20, §, Theorem 2] or [24, Theorem 3.6] by specialization of a variable. (It is also contained in [31].) □

Now we have all the necessary data for the description of the local functional equations satisfied by the Clifford quartic forms.

For simplicity, we give explicit formulas for the local functional equations under the assumption that

(2.5) "the constants γ are equal to 1 and $m = \dim W \geq 8$."

THEOREM 2.13. *If Q is non-degenerate and the assumption* (2.5) *is satisfied, then the local zeta functions $\tilde{\zeta}_{\pm}(s,\Psi)$, $\tilde{\zeta}_{-,\pm}(s,\Psi)$ $(\Psi \in \mathcal{S}(W))$ satisfy the following local functional equations:*
(1) *Assume that $(p,q) = (n,0)$. Then*

$$\begin{aligned}\tilde{\zeta}_{+}\left(s,\hat{\Psi}\right) &= 2^{4s+m/2}\pi^{-4s-2-m/2}\Gamma(s+1)\Gamma\left(s+\frac{n}{2}\right) \\ &\quad \times \Gamma\left(s+1+\frac{m-2n}{4}\right)\Gamma\left(s+\frac{m}{4}\right) \\ &\quad \times \sin(\pi s)\sin\pi\left(s-\frac{n}{2}\right)\tilde{\zeta}_{+}\left(-\frac{m}{4}-s,\Psi\right).\end{aligned}$$

(2) *Assume that $(p,q) = (n-1,1)$. Then*

$$\begin{aligned}&\begin{pmatrix}\tilde{\zeta}_{+}\left(s,\hat{\Psi}\right) \\ \tilde{\zeta}_{-,+}\left(s,\hat{\Psi}\right) \\ \tilde{\zeta}_{-,-}\left(s,\hat{\Psi}\right)\end{pmatrix} \\ &= 2^{4s+m/2}\pi^{-4s-2-m/2}\Gamma(s+1)\Gamma\left(s+\frac{n}{2}\right)\Gamma\left(s+1+\frac{m-2n}{4}\right)\Gamma\left(s+\frac{m}{4}\right) \\ &\quad \times \sin\pi s\begin{pmatrix}-\sin\pi\left(s-\frac{n}{2}\right) & 0 & 0 \\ -\sin\left(\frac{n\pi}{2}\right) & -\sin\pi\left(s+\frac{n}{2}\right) & 0 \\ -\sin\left(\frac{n\pi}{2}\right) & 0 & -\sin\pi\left(s+\frac{n}{2}\right)\end{pmatrix} \\ &\quad \times \begin{pmatrix}\tilde{\zeta}_{+}\left(-\frac{m}{4}-s,\Psi\right) \\ \tilde{\zeta}_{-,+}\left(-\frac{m}{4}-s,\Psi\right) \\ \tilde{\zeta}_{-,-}\left(-\frac{m}{4}-s,\Psi\right)\end{pmatrix}.\end{aligned}$$

(3) *Assume that $p, q \geq 2$ and put $n = p + q$. Then*

$$\begin{pmatrix} \tilde{\zeta}_+\left(s, \hat{\Psi}\right) \\ \tilde{\zeta}_-\left(s, \hat{\Psi}\right) \end{pmatrix}$$

$$= 2^{4s+m/2}\pi^{-4s-2-m/2}\Gamma(s+1)\Gamma\left(s+\frac{n}{2}\right)\Gamma\left(s+1+\frac{m-2n}{4}\right)\Gamma\left(s+\frac{m}{4}\right)$$

$$\times \sin \pi s \begin{pmatrix} \sin \pi\left(s+\frac{q-p}{2}\right) & -2\sin\frac{\pi p}{2}\cos\frac{\pi q}{2} \\ -2\sin\frac{\pi q}{2}\cos\frac{\pi p}{2} & \sin\pi\left(s+\frac{p-q}{2}\right) \end{pmatrix} \begin{pmatrix} \tilde{\zeta}_+\left(-\frac{m}{4}-s, \Psi\right) \\ \tilde{\zeta}_-\left(-\frac{m}{4}-s, \Psi\right) \end{pmatrix}.$$

REMARK 2.2. The non-degenerate cases that are excluded by the assumption (2.5) are

$$(p, q) = (2, 1), (3, 1) \quad \text{and} \quad (p, q, m) = (3, 0, 4).$$

In these cases, the Clifford quartic forms are relative invariants of prehomogeneous vector spaces of rather simple structure (see Theorems 3.1 and 3.2) and the local functional equations are well-known. As we shall see in Theorem 3.2, the Clifford quartic form is not a relative invariant of any prehomogeneous vector space except for some low-dimensional cases, for which the local functional equations are new.

2.4. The Clifford quartic forms are homaloidal

A homogeneous rational function f on a finite-dimensional vector space $\mathbf{V}$ is called *homaloidal*, if

> the mapping $\mathbb{P}(\mathbf{V}) \longrightarrow \mathbb{P}(\mathbf{V}^*)$ defined by $v \mapsto \operatorname{grad} f(v)$ is birational, equivalently,
> the mapping $\mathbf{V} \longrightarrow \mathbf{V}^*$ defined by $v \mapsto \operatorname{grad}\log f(v)$ is birational.

For a homaloidal homogeneous rational function f, there exists a rational function f^* satisfying the identity $f^*(\operatorname{grad}\log f(v)) = 1/f(v)$, which is called the *multiplicative Legendre transform* of f. Following [6], we call a polynomial f a *homaloidal EKP-polynomial* if f is homaloidal and its multiplicative Legendre transform f^* is also a polynomial.

By definition, a regular prehomogeneous vector space has homaloidal relatively invariant polynomials. As we mentioned in the introduction, it is rather difficult to construct homaloidal polynomials that are not relative invariants of prehomogeneous vector spaces and the classification of homaloidal polynomials has been done only for some special cases:

- Cubic homaloidal EKP-polynomials are classified by Etingof-Kazhdan-Polishchuk ([7]).
- Homaloidal polynomials in 3 variables without multiple factors are classified by Dolgachev ([6]).
- In [3]. Bruno determined when a product of linear forms is homaloidal.

All the homaloidal polynomials classified in these works are relative invariants of prehomogeneous vector spaces, and Etingof, Kazhdan and Polishchuk ([7, §3.4, Question 1]) asked whether homaloidal EKP-polynomials are relative invariants of regular prehomogeneous vector spaces.

The following theorem shows that the Clifford quartic forms are counter examples of degree 4 to the question raised by Etingof, Kazhdan and Polishchuk, since most of Clifford quartic forms are non-prehomogeneous as will be shown in Theorem 3.2.

THEOREM 2.14. *Let $S_1, \dots, S_{p+q}$ be the basis matrices of a representation $R_{p,q}$. Then the Clifford quartic form*

$$\tilde{P}(w) = \sum_{i=1}^{p} S_i[w]^2 - \sum_{j=1}^{q} S_{p+j}[w]^2$$

is a homaloidal EKP-polynomials, unless it vanishes identically. The multiplicative Legendre transform of $\tilde{P}$ coincides with $\tilde{P}$ itself up to a constant factor.

PROOF. Since $\operatorname{grad} S_i[w] = 2S_i w$, we have

$$\operatorname{grad} \tilde{P}(w) = {}^t\left(\frac{\partial \tilde{P}}{\partial w_1}(w), \dots, \frac{\partial P}{\partial w_m}(w)\right) = 2^2 \sum_{i=1}^{p+q} \varepsilon_i S_i[w] S_i w,$$

where ε_i $(1 \le i \le p+q)$ are defined by (2.1). Let us calculate

$$\tilde{P}(\operatorname{grad} \tilde{P}(w)) = \sum_{i=1}^{p+q} \varepsilon_i S_i[\operatorname{grad} \tilde{P}(w)]^2$$

For any i, we have

$$\begin{aligned} 2^{-4}S_i[\operatorname{grad}\tilde{P}(w)] &= S_i\left[\sum_{j=1}^{p+q}\varepsilon_j S_j[w]S_j w\right] \\ &= \sum_{j=1}^{p+q} S_j[w]^2(S_jS_iS_j)\,[w] \\ &\quad + \sum_{1\le j<k\le p+q}\varepsilon_j\varepsilon_k S_j[w]S_k[w](S_jS_iS_k+S_kS_iS_j)\,[w]\,. \end{aligned}$$

From the commutation relations (2.2) and (2.3), it follows that

$$\begin{aligned} S_jS_iS_j &= \begin{cases} S_i & (i=j), \\ -\varepsilon_i\varepsilon_j S_i & (i\neq j), \end{cases} \\ S_jS_iS_k+S_kS_iS_j &= \begin{cases} 0 & (j\neq k,\ j\neq i, k\neq i), \\ 2S_j & (k=i), \\ 2S_k & (j=i). \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} 2^{-4}S_i[\operatorname{grad}\tilde{P}(w)] &= S_i[w]^3-\varepsilon_i S_i[w]\sum_{j\neq i}\varepsilon_j S_j[w]^2+2\varepsilon_i S_i[w]\sum_{j\neq i}\varepsilon_j S_j[w]^2 \\ &= S_i[w]^3+\varepsilon_i S_i[w]\sum_{j\neq i}\varepsilon_j S_j[w]^2 \\ &= \varepsilon_i S_i[w]\tilde{P}(w). \end{aligned}$$

Thus we obtain

$$\tilde{P}(\operatorname{grad}\tilde{P}(w)) = 2^8\tilde{P}(w)^3.$$

In other words,

$$\tilde{P}(\operatorname{grad}(\log\tilde{P})(w)) = \tilde{P}(\tilde{P}(w)^{-1}\operatorname{grad}\tilde{P}(w)) = 2^8\tilde{P}(w)^{-1}.$$

This shows that the multiplicative Legendre transform of $\tilde{P}$ coincides with $\tilde{P}$ itself (up to a constant factor). Consequently, by [7, Proposition 3.6], $\tilde{P}$ is a homaloidal EKP-polynomial. □

Remark 2.3. Some examples of non-prehomogeneous homaloidal polynomials are constructed by Ciliberto, Russo, and Simis ([5, §3]). Other examples of non-prehomogeneous (reducible) homaloidal rational functions are given by Letac and Massam ([19]).

3. Clifford Quartic Forms and Prehomogeneous Vector Spaces

3.1. Main results

Let ρ be a representation of $R_{p,q}$ on an m-dimensional real vector space W and let $\tilde{P}(w)$ be the associated Clifford quartic forms. Then, $\tilde{P}(w)$ is not a relative invariant of any prehomogeneous vector space except for some low-dimensional cases, and the local functional equation satisfied by $\tilde{P}(w)$ in Theorem 2.13 is not covered by the theory of prehomogeneous vector spaces. In this section we determine when $\tilde{P}(w)$ is a relative invariants of a prehomogeneous vector space.

The main results are the following:

Theorem 3.1. *The quadratic mapping Q associated to ρ is non-degenerate (equivalently, the Clifford quartic form does not vanish identically), if and only if*

$$(p,q,m) \neq (2,1,2),(3,1,4),(5,1,8),(9,1,16),(2,2,4),(3,3,8),(5,5,16).$$

Theorem 3.2. (1) *A Clifford quartic form is not a relative invariant of any prehomogeneous vector space if and only if*

$$\begin{cases} p+q=5,\ m>8; \\ p+q=6,\ m>16 \text{ and } \rho \text{ is mixed}; \\ p+q=7,8,9,\ m>16; \\ p+q=10,\ m>32,\ \text{ or } m=32 \text{ and } \rho \text{ is mixed}; \\ p+q=11,\ m>32; \\ p+q\geq 12. \end{cases}$$

(For the definition of mixed representation, see Definition 2.7.)

(2) *The prehomogeneous vector spaces having Clifford quartic forms as a relative invariant are the spaces listed in Table 1 in p.819. In Table 1, we denote by Λ_1 the standard representation of $\mathfrak{gl}(k,\mathbb{K})$ on $\mathbb{K}^k$ for $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$.*

We also denote by Λ_1 *the representations of classical groups obtained from the following natural inclusions*

$$SO(k_1,k_2)\hookrightarrow GL(k_1+k_2,\mathbb{R}),\quad SO(k,\mathbb{C})\hookrightarrow GL(k,\mathbb{C}),$$
$$SO^*(2k)\hookrightarrow GL(k,\mathbb{H}),$$
$$Sp(k_1,k_2)\hookrightarrow GL(k_1+k_2,\mathbb{H}),\quad Sp(k,\mathbb{R})\hookrightarrow GL(2k,\mathbb{R}),$$
$$Sp(k,\mathbb{C})\hookrightarrow GL(2k,\mathbb{C}),$$
$$U(k_1,k_2)\hookrightarrow GL(k_1+k_2,\mathbb{C}),$$

and by Λ_e *(resp.* Λ_o*) the even (resp. odd) spin representation of* $Spin(p,q)$*. Moreover the action of* $GL(1,\mathbb{R})$ *is the scalar multiplication.*

(3) *The Clifford quartic forms are absolutely irreducible, if and only if*

$$(p+q,m)\neq(3,2),(3,4),(4,4),(4,8),(5,8),(6,8),(8,16),(9,16),(10,16).$$

3.2. Structure of the groups of symmetries of Clifford quartic forms and strategy of the proofs of Main theorems

For the proofs of Theorems 3.1 and 3.2, it is necessary to determine the structure of the group

$$G_{p,q}(\rho):=\left\{g\in GL(W)\;\middle|\;\tilde{P}(gw)=\tilde{P}(w)\right\},$$

the group of symmetries of $\tilde{P}$. If $(GL(1,\mathbb{R})\times G_{p,q}(\rho),W)$ is not (a real form of) a prehomogeneous vector space, then there exist no prehomogeneous vector spaces with $\tilde{P}$ as a relative invariant.

Let $\mathfrak{g}_{p,q}(\rho)$ be the Lie algebra of $G_{p,q}(\rho)$. Differentiating the identity $\tilde{P}(\exp(tX)w)=\tilde{P}(w)$ $(X\in\mathfrak{gl}(W))$, we have

$$\mathfrak{g}_{p,q}(\rho)=\left\{X\in\mathfrak{gl}(W)\;\middle|\;\sum_{i=1}^{p+q}\varepsilon_iS_i[w]({}^tXS_i+S_iX)[w]=0\right\},$$

where ε_i's are as in (2.1).

First note that the group $\mathrm{Spin}(p,q)$ is contained in $G_{p,q}(\rho)$, since, by Lemma 2.6, we have

$$\tilde{P}(gw)=P(Q(gw))=P(gQ(w))=P(w)\quad(g\in\mathrm{Spin}(p,q)).$$

Table 1. Prehomogeneous vector spaces having Clifford quartic forms as a relative invariant.

(p,q)	prehomogeneous vector space
$(3,0)$	$(GL(1,\mathbb{H}) \times SO^*(2k), \Lambda_1 \otimes \Lambda_1)$
$(2,1)$	$(GL(2,\mathbb{R}) \times SO(k_1,k_2), \Lambda_1 \otimes \Lambda_1)$
$(4,0)$	$(GL(1,\mathbb{H}) \times GL(1,\mathbb{H}) \times GL(k,\mathbb{H}), (\Lambda_1 \otimes 1 \otimes \Lambda_1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1^*))$
$(3,1)$	$(GL(2,\mathbb{C}) \times SU(k_1,k_2), \Lambda_1 \otimes \Lambda_1)$
$(2,2)$	$(GL(2,\mathbb{R}) \times GL(2,\mathbb{R}) \times SL(k,\mathbb{R}), (\Lambda_1 \otimes 1 \otimes \Lambda_1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1^*))$
$(5,0)$	$(GL(1,\mathbb{R}) \times SO(8), \Lambda_1)$
$(4,1)$	$(GL(1,\mathbb{R}) \times SO(4,4), \Lambda_1)$
$(3,2)$	$(GL(1,\mathbb{R}) \times SO(4,4), \Lambda_1)$
$(6,0)$	$(GL(2,\mathbb{C}) \times SU(4), \Lambda_1 \otimes \Lambda_1)$
$(5,1)$	$(GL(2,\mathbb{H}) \times Sp(k_1,k_2), \Lambda_1 \otimes \Lambda_1)$ $(k_1+k_2 \geq 2)$
	$(GL(1,\mathbb{R}) \times SL(2,\mathbb{H}) \times SU(2) \times SU(2), (\Lambda_1 \otimes \Lambda_1 \otimes 1) \oplus (\Lambda_1^* \otimes 1 \otimes \Lambda_1))$
$(4,2)$	$(GL(2,\mathbb{C}) \times SU(2,2), \Lambda_1 \otimes \Lambda_1)$
$(3,3)$	$(GL(4,\mathbb{R}) \times Sp(k,\mathbb{R}), \Lambda_1 \otimes \Lambda_1)$ $(k \geq 2)$
	$(GL(1,\mathbb{R}) \times SL(4,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), (\Lambda_1 \otimes \Lambda_1 \otimes 1) \oplus (\Lambda_1^* \otimes 1 \otimes \Lambda_1))$
$(7,0)$	$(GL(2,\mathbb{R}) \times SO(8), \Lambda_1 \otimes \Lambda_1)$
$(6,1)$	$(GL(1,\mathbb{H}) \times SO^*(8), \Lambda_1 \otimes \Lambda_1)$
$(5,2)$	$(GL(1,\mathbb{H}) \times SO^*(8), \Lambda_1 \otimes \Lambda_1)$
$(4,3)$	$(GL(2,\mathbb{R}) \times SO(4,4), \Lambda_1 \otimes \Lambda_1)$
$(8,0)$	$(GL(1,\mathbb{R}) \times SO(8), \Lambda_1 \otimes 1) \oplus (GL(1,\mathbb{R}) \times SO(8), \Lambda_1 \otimes 1)$
$(7,1)$	$(GL(1,\mathbb{C}) \times SO(8,\mathbb{C}), \Lambda_1)$
$(5,3)$	$(GL(1,\mathbb{C}) \times SO(8,\mathbb{C}), \Lambda_1)$
$(4,4)$	$(GL(1,\mathbb{R}) \times SO(4,4), \Lambda_1 \otimes 1) \oplus (GL(1,\mathbb{R}) \times SO(4,4), \Lambda_1 \otimes 1)$
$(9,0)$	$(GL(1,\mathbb{R}) \times SO(16), \Lambda_1)$
$(8,1)$	$(GL(1,\mathbb{R}) \times SO(8,8), \Lambda_1)$
$(5,4)$	$(GL(1,\mathbb{R}) \times SO(8,8), \Lambda_1)$
$(9,1)$	$(GL(2,\mathbb{R}) \times Spin(9,1), \Lambda_1 \otimes \Lambda_\sharp)$ $(\sharp = e, o)$
$(7,3)$	$(GL(1,\mathbb{H}) \times Spin(7,3), \Lambda_1 \otimes \Lambda_\sharp)$ $(\sharp = e, o)$
$(5,5)$	$(GL(2,\mathbb{R}) \times Spin(5,5), \Lambda_1 \otimes \Lambda_\sharp)$ $(\sharp = e, o)$
$(10,1)$	$(GL(1,\mathbb{R}) \times Spin(10,2), \Lambda_\sharp)$ $(\sharp = e, o)$
$(9,2)$	$(GL(1,\mathbb{R}) \times Spin(10,2), \Lambda_\sharp)$ $(\sharp = e, o)$
$(6,5)$	$(GL(1,\mathbb{R}) \times Spin(6,6), \Lambda_\sharp)$ $(\sharp = e, o)$

Hence the Lie algebra $\mathfrak{k}_{p,q}$ ($\cong \mathfrak{so}(p,q)$) of $\mathrm{Spin}(p,q)$ is contained in $\mathfrak{g}_{p,q}(\rho)$. Here we identify $\mathfrak{k}_{p,q}$ with the image $\rho(\mathfrak{k}_{p,q}) \subset \mathfrak{gl}(W)$, which is the Lie subalgebra of $\mathfrak{gl}(W)$ spanned by $\{ S_iS_j \mid 1 \le i < j \le p+q\}$.

There exists another group contained in $G_{p,q}(\rho)$. Put

$$H_{p,q}(\rho) := \{g \in GL(W) \mid S_i[gw] = S_i[w] \ (1 \le i \le p+q)\} = \bigcap_{i=1}^{p+q} O(S_i).$$

Then, it is obvious that $H_{p,q}(\rho)$ is also contained in $G_{p,q}(\rho)$. Since $g \mapsto S_i\,{}^tg^{-1}S_i$ $(1 \le i \le p+q)$ are involutions commuting with each other, $H_{p,q}(\rho)$ is a reductive Lie group. The Lie algebra $\mathfrak{h}_{p,q}(\rho)$ of $H_{p,q}(\rho)$ is given by

$$\mathfrak{h}_{p,q}(\rho) = \left\{ X \in \mathfrak{gl}(W) \;\middle|\; {}^tXS_i + S_iX = 0 \ (i = 1, \ldots, p+q)\right\}.$$

Since $S_iS_jX = XS_iS_j$ for any $X \in \mathfrak{h}_{p,q}(\rho)$, we have $[\mathfrak{k}_{p,q}, \mathfrak{h}_{p,q}(\rho)] = \{0\}$. Moreover, from Lemma 2.6 it follows that $\mathfrak{k}_{p,q} \cap \mathfrak{h}_{p,q}(\rho) = \{0\}$. We define the subalgebra $\mathfrak{g}'_{p,q}(\rho)$ of $\mathfrak{g}_{p,q}(\rho)$ by

$$\mathfrak{g}'_{p,q}(\rho) = \mathfrak{k}_{p,q} + \mathfrak{h}_{p,q}(\rho) \cong \mathfrak{so}(p,q) \oplus \mathfrak{h}_{p,q}(\rho).$$

The following two theorems are the key to the proof of our main results.

Theorem 3.3. *Let ρ be an m-dimensional representation of $R_{p,q}$. Then, we have*

$$\mathfrak{g}_{p,q}(\rho) = \mathfrak{g}'_{p,q}(\rho)$$

except for the following low dimensional cases.

$p+q$	3	4	5	6	7	8	9	10	11
m	$2,4$	$4,8$	8	$8,16$	16	16	16	$16,32$	32

Theorem 3.4. *The Lie algebra $\mathfrak{h}_{p,q}(\rho)$ is isomorphic to the reductive*

Lie algebra given in the following table:

$(R_{p,q}, R^+_{p,q})$	$(\mathbb{K}, \mathbb{K}')$	$\{p \bmod 8, q \bmod 8\}$	$\mathfrak{h}_{p,q}(\rho)$
(T, T')	$(\mathbb{R}, \mathbb{C})$	$\{0,2\},\{4,6\}$	$\mathfrak{so}(k,\mathbb{C})$
	$(\mathbb{C}, \mathbb{R})$	$\{0,7\},\{2,3\},\{3,4\},\{6,7\}$	$\mathfrak{sp}(k,\mathbb{R})$
	$(\mathbb{C}, \mathbb{H})$	$\{0,3\},\{2,7\},\{3,6\},\{4,7\}$	$\mathfrak{so}^*(2k)$
	$(\mathbb{H}, \mathbb{C})$	$\{0,6\},\{2,4\}$	$\mathfrak{sp}(k,\mathbb{C})$
$(T, 2T')$	$(\mathbb{R}, \mathbb{R})$	$\{0,0\},\{2,2\},\{4,4\},\{6,6\}$	$\mathfrak{gl}(k,\mathbb{R})$
	$(\mathbb{H}, \mathbb{H})$	$\{0,4\},\{2,6\}$	$\mathfrak{gl}(k,\mathbb{H})$
$(2T, T')$	$(\mathbb{R}, \mathbb{R})$	$\{0,1\},\{1,2\},\{4,5\},\{5,6\}$	$\mathfrak{so}(k_1,k_2)$
	$(\mathbb{C}, \mathbb{C})$	$\{1,3\},\{1,7\},\{3,5\},\{5,7\}$	$\mathfrak{u}(k_1,k_2)$
	$(\mathbb{H}, \mathbb{H})$	$\{0,5\},\{1,4\},\{1,6\},\{2,5\}$	$\mathfrak{sp}(k_1,k_2)$
$(2T, 2T')$	$(\mathbb{C}, \mathbb{R})$	$\{3,3\},\{7,7\}$	$\mathfrak{sp}(k_1,\mathbb{R}) \oplus \mathfrak{sp}(k_2,\mathbb{R})$
	$(\mathbb{C}, \mathbb{H})$	$\{3,7\}$	$\mathfrak{so}^*(2k_1) \oplus \mathfrak{so}^*(2k_2)$
$(4T, 2T')$	$(\mathbb{R}, \mathbb{R})$	$\{1,1\},\{5,5\}$	$\mathfrak{so}(k_1,k_2) \oplus \mathfrak{so}(k_3,k_4)$
	$(\mathbb{H}, \mathbb{H})$	$\{1,5\}$	$\mathfrak{sp}(k_1,k_2) \oplus \mathfrak{sp}(k_3,k_4)$

Here k, k_1, k_2, k_3, k_4 are the multiplicities of irreducible representations in ρ. More precisely, if $R_{p,q} = T$, then $R_{p,q}$ has only one irreducible representation ρ_1 and k is the multiplicity of ρ_1 in ρ; if $R_{p,q} = T \oplus T$, then $R_{p,q}$ has two irreducible representations ρ_1, ρ_2 and k_1, k_2 are the multiplicities of ρ_1, ρ_2 in ρ; if $R_{p,q} = T \oplus T \oplus T \oplus T$, then $R_{p,q}$ has four irreducible representations $\rho_1, \rho_2, \rho_3, \rho_4$ and $R^+_{p,q}$ has two irreducible representations (the even and the odd half-spin representations) Λ_e, Λ_o. We may assume that $\rho_1|_{R^+_{p,q}} = \rho_2|_{R^+_{p,q}} = \Lambda_e$ and $\rho_3|_{R^+_{p,q}} = \rho_4|_{R^+_{p,q}} = \Lambda_o$. Then, k_1, k_2, k_3, k_4 are the multiplicities of $\rho_1, \rho_2, \rho_3, \rho_4$ in ρ.

For the proofs of Theorems 3.1 and 3.2, we have to know how the Lie algebra $\mathfrak{so}(p,q) \oplus \mathfrak{h}_{p,q}(\rho)$ acts on W. To describe the result, we need some notational preliminaries. We denote by Λ_1 the standard representation of $\mathfrak{gl}(k,\mathbb{K})$ on $\mathbb{K}^k$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We also denote by Λ_1 the representations of the classical Lie algebras obtained from the following natural inclusions

$$\mathfrak{so}(k_1,k_2) \hookrightarrow \mathfrak{gl}(k_1+k_2,\mathbb{R}), \quad \mathfrak{so}(k,\mathbb{C}) \hookrightarrow \mathfrak{gl}(k,\mathbb{C}), \quad \mathfrak{so}^*(2k) \hookrightarrow \mathfrak{gl}(k,\mathbb{H}),$$
$$\mathfrak{sp}(k_1,k_2) \hookrightarrow \mathfrak{gl}(k_1+k_2,\mathbb{H}), \quad \mathfrak{sp}(k,\mathbb{R}) \hookrightarrow \mathfrak{gl}(2k,\mathbb{R}), \quad \mathfrak{sp}(k,\mathbb{C}) \hookrightarrow \mathfrak{gl}(2k,\mathbb{C}),$$
$$\mathfrak{u}(k_1,k_2) \hookrightarrow \mathfrak{gl}(k_1+k_2,\mathbb{C}).$$

In case $R^+_{p,q}$ is a simple algebra, then we denote by Λ the representation of $\mathfrak{k}_{p,q} \cong \mathfrak{so}(p,q)$ induced by the unique irreducible representation of $R^+_{p,q}$. Namely Λ is the spin representation. In case $R^+_{p,q}$ is the direct sum of two simple algebras, then we denote by Λ_e and Λ_o the irreducible representations of $\mathfrak{k}_{p,q} \cong \mathfrak{so}(p,q)$ induced by the two irreducible representations of $R^+_{p,q}$. Namely Λ_e and Λ_o are the even and odd half-spin representations. Moreover we denote by 1 the trivial representation of a Lie algebra.

THEOREM 3.5. *Put* $\mathfrak{g}'_{p,q}(\rho) = \mathfrak{so}(p,q) \oplus \mathfrak{h}_{p,q}(\rho)$. *Then* $\mathfrak{g}'_{p,q}(\rho)$ *acts on the representation space* W *of* ρ *as follows.*

$(R_{p,q}, R^+_{p,q})$	$(\mathbb{K}, \mathbb{K}')$	$\{p \bmod 8, q \bmod 8\}$	Representation of $\mathfrak{g}'_{p,q}(\rho)$
(T, T')	$(\mathbb{R}, \mathbb{C})$	$\{0,2\}, \{4,6\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{so}(k,\mathbb{C}), \Lambda \otimes \Lambda_1)$
	$(\mathbb{C}, \mathbb{R})$	$\{0,7\}, \{2,3\}$ $\{3,4\}, \{6,7\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{sp}(k,\mathbb{R}), \Lambda \otimes \Lambda_1)$
	$(\mathbb{C}, \mathbb{H})$	$\{0,3\}, \{2,7\}$ $\{3,6\}, \{4,7\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{so}^*(2k), \Lambda \otimes \Lambda_1)$
	$(\mathbb{H}, \mathbb{C})$	$\{0,6\}, \{2,4\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{sp}(k,\mathbb{C}), \Lambda \otimes \Lambda_1)$
$(T, 2T')$	$(\mathbb{R}, \mathbb{R})$	$\{0,0\}, \{2,2\}$ $\{4,4\}, \{6,6\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{gl}(k,\mathbb{R}), \Lambda_e \otimes \Lambda_1 + \Lambda_o \otimes \Lambda_1^*)$
	$(\mathbb{H}, \mathbb{H})$	$\{0,4\}, \{2,6\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{gl}(k,\mathbb{H}), \Lambda_e \otimes \Lambda_1 + \Lambda_o \otimes \Lambda_1^*)$
$(2T, T')$	$(\mathbb{R}, \mathbb{R})$	$\{0,1\}, \{1,2\}$ $\{4,5\}, \{5,6\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{so}(k_1,k_2), \Lambda \otimes \Lambda_1)$
	$(\mathbb{C}, \mathbb{C})$	$\{1,3\}, \{1,7\}$ $\{3,5\}, \{5,7\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{u}(k_1,k_2), \Lambda \otimes \Lambda_1)$
	$(\mathbb{H}, \mathbb{H})$	$\{0,5\}, \{1,4\}$ $\{1,6\}, \{2,5\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{sp}(k_1,k_2), \Lambda \otimes \Lambda_1)$
$(2T, 2T')$	$(\mathbb{C}, \mathbb{R})$	$\{3,3\}, \{7,7\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{sp}(k_1,\mathbb{R}) \oplus \mathfrak{sp}(k_2,\mathbb{R}),$ $\Lambda_e \otimes \Lambda_1 \otimes 1 + \Lambda_o \otimes 1 \otimes \Lambda_1)$
	$(\mathbb{C}, \mathbb{H})$	$\{3,7\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{so}^*(2k_1) \oplus \mathfrak{so}^*(2k_2),$ $\Lambda_e \otimes \Lambda_1 \otimes 1 + \Lambda_o \otimes 1 \otimes \Lambda_1)$
$(4T, 2T')$	$(\mathbb{R}, \mathbb{R})$	$\{1,1\}, \{5,5\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{so}(k_1,k_2) \oplus \mathfrak{so}(k_3,k_4),$ $\Lambda_e \otimes \Lambda_1 \otimes 1 + \Lambda_o \otimes 1 \otimes \Lambda_1)$
	$(\mathbb{H}, \mathbb{H})$	$\{1,5\}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{sp}(k_1,k_2) \oplus \mathfrak{sp}(k_3,k_4),$ $\Lambda_e \otimes \Lambda_1 \otimes 1 + \Lambda_o \otimes 1 \otimes \Lambda_1)$

The remaining sections §4 – 7 are devoted to the proofs of Theorems 3.1 – 3.5. The outline of the proofs is as follows. First we prove the "if"-part of Theorem 3.1 in §4. Theorem 3.3 is proved in §5. We shall prove Theorem 3.4 in §6. From the proof of Theorem 3.4, we can easily read how the Lie algebra $\mathfrak{so}(p,q) \oplus \mathfrak{h}_{p,q}(\rho)$ acts on W. So Theorems 3.4 and 3.5 are proved at the same time.

The proofs of the "only if"-part of Theorems 3.1 and Theorem 3.2 are given in §7. In the proof we distinguish the following three cases:

(I) $(p,q,m) = (2,1,2), (3,1,4), (5,1,8), (9,1,16), (2,2,4), (3,3,8), (5,5,16)$.

(II) $(p+q,m) = (3,4), (4,8), (5,8), (6,16), (7,16), (8,16), (9,16), (10,32), (11,32)$.

(III) The remaining cases.

For Case (I), we prove in §7.1 that the $\mathfrak{gl}(1,\mathbb{R}) \oplus \mathfrak{g}'_{p,q}(\rho)$-module W gives a prehomogeneous vector space with no non-trivial relative invariants. Since the Clifford quartic form is a relative invariant of the prehomogeneous vector space, it vanishes identically. This proves the "only if"-part of Theorem 3.1.

For Case (II), we prove in §7.2 that $\mathfrak{g}_{p,q}(\rho)$ is strictly larger than $\mathfrak{g}'_{p,q}(\rho)$ and the $\mathfrak{gl}(1,\mathbb{R}) \oplus \mathfrak{g}_{p,q}(\rho)$-module W gives a prehomogeneous vector space except when $p+q = 10, m = 32$ and ρ is mixed.

For Case (III), we have $\mathfrak{g}_{p,q}(\rho) = \mathfrak{g}'_{p,q}(\rho)$, since the union of Case (I) and Case (II) is precisely the exceptional parameters (p,q,m) in Theorem 3.3. So, by the results of the classification of (not necessarily irreducible) prehomogeneous vector spaces ([29], [12], [13], [14], [15]), we can characterize the cases where the $\mathfrak{gl}(1,\mathbb{R}) \oplus \mathfrak{g}_{p,q}(\rho)$-module W gives a prehomogeneous vector space. This is done in §7.3.

Finally, in §7.4, we prove the irreducibility result (Theorem 3.2 (3)). The proof is based on the classification in Theorem 3.2 (2) and the calculation of the multiplicative Legendre transform in §2.4. This completes the proof of Theorem 3.2.

3.3. Non-prehomogeneous example: $(p,q) = (3,2)$

By Theorem 3.2, non-prehomogeneous Clifford quartic forms appear first for $p+q = 5$. We consider here this non-prehomogeneous case for $(p,q) = (3,2)$ in some detail. The algebra $R_{3,2}$ has a unique irreducible representation ρ_0 of degree 8. We may choose the basis matrices $S_i = \rho_0(e_i)$

$(1 \le i \le 5)$ as follows:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 1_2 \\ 0 & 0 & -1_2 & 0 \\ 0 & -1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & J & 0 \\ 0 & 0 & 0 & -J \\ -J & 0 & 0 & 0 \\ 0 & J & 0 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & J \\ 0 & 0 & J & 0 \\ 0 & -J & 0 & 0 \\ -J & 0 & 0 & 0 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 & 0 & H \\ 0 & 0 & -H & 0 \\ 0 & -H & 0 & 0 \\ H & 0 & 0 & 0 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} 0 & 0 & 0 & K \\ 0 & 0 & -K & 0 \\ 0 & -K & 0 & 0 \\ K & 0 & 0 & 0 \end{pmatrix},$$

where we put $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we have

$$\rho_0(\mathfrak{k}_{3,2}) = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \;\middle|\; X \in \mathfrak{sp}(2,\mathbb{R}) \right\},$$
$$\mathfrak{sp}(2,\mathbb{R}) = \left\{ X \in M(4;\mathbb{R}) \;\middle|\; {}^tXJ_2 + J_2X = 0 \right\},$$

where $\mathfrak{k}_{3,2}$ is the Lie algebra spanned by e_ie_j $(1 \le i < j \le 5)$ and $J_2 = J \perp J$. We identify the representation space W_0 of ρ_0 with $M(4,2;\mathbb{R})$ by $w = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto (uv)$. Then the action of $\mathfrak{k}_{3,2}$ is given by the left multiplication of $\mathfrak{sp}(2,\mathbb{R})$. Consider a reducible representation $\rho = \rho_0^{\oplus k}$. If $k \ge 2$, then by Theorems 3.1 and 3.2, the Clifford quartic form

$$\tilde{P}(w) = \sum_{i=1}^{3} S_i^{(k)}[w]^2 - \sum_{i=4}^{5} S_i^{(k)}[w]^2, \quad S_i^{(k)} = \overbrace{S_i \perp \cdots \perp S_i}^{k}$$

is non-prehomogeneous and satisfies the functional equation (Theorem 2.13)

$$\begin{pmatrix} \tilde{\zeta}_+\left(s, \hat{\Psi}\right) \\ \tilde{\zeta}_-\left(s, \hat{\Psi}\right) \end{pmatrix} = 2^{4s+4k-1}\pi^{-4s-4k-2}\Gamma(s+1)\Gamma\left(s+\frac{5}{2}\right)$$
$$\times \Gamma\left(s+2k-\frac{3}{2}\right)\Gamma\left(s+2k\right)$$

$$\times \begin{pmatrix} -\sin(2\pi s) & -4\sin(\pi s) \\ 0 & \sin(2\pi s) \end{pmatrix} \begin{pmatrix} \tilde{\zeta}_+ (-2k - s, \Psi) \\ \tilde{\zeta}_- (-2k - s, \Psi) \end{pmatrix}.$$

In this case, the Clifford quartic form has an expression in terms of more popular invariants. By Theorems 3.3, 3.4 and 3.5, $\tilde{P}$ is invariant under the action of the group $G_{3,2}(\rho)$ with Lie algebra

$$\mathfrak{g}_{3,2}(\rho) = \mathfrak{sp}(2, \mathbb{R}) \oplus \mathfrak{sp}(k, \mathbb{R}),$$
$$\mathfrak{sp}(k, \mathbb{R}) = \left\{ X \in M(2k; \mathbb{R}) \;\middle|\; {}^tXJ_k + J_kX = 0 \right\},$$

where $J_k = \overbrace{J \perp \cdots \perp J}^{k}$. We identify the representation space of ρ with $M(4, 2k; \mathbb{R})$. Then the action on $M(4, 2k; \mathbb{R})$ is given by $w \mapsto Xw + w\,{}^tY$ ($x \in \mathfrak{sp}(2, \mathbb{R})$, $Y \in \mathfrak{sp}(k, \mathbb{R})$) (see §6.2).

In the following lemma, we use the symbols $\mathbf{M}(4, 2k)$ and $\mathbf{Alt}(4)$, respectively, to denote $M(4, 2k; \mathbb{C})$ and $\mathrm{Alt}(4; \mathbb{C})$ viewed as the affine spaces of dimension $8k$ and 6.

Lemma 3.6. *We define the polynomials P_1, P_2 on $\mathbf{M}(4, 2k)$ by*

$$P_1(w) = \text{the Pfaffian of } wJ_k\,{}^tw, \quad P_2(w) = \mathrm{tr}(J_2wJ_k\,{}^tw).$$

Then the ring $\mathbb{C}[\mathbf{M}(4, 2k)]^{\mathbf{Sp}(2)\times\mathbf{Sp}(k)}$ of $\mathbf{Sp}(2) \times \mathbf{Sp}(k)$-invariants is generated by P_1, P_2 and is isomorphic to the polynomial ring of 2 variables.

Proof. The $\mathbf{Sp}(2)$-equivariant mapping $\phi : \mathbf{M}(4, 2k) \to \mathbf{Alt}(4)$ ($\phi(w) = wJ_k\,{}^tw$) induces an isomorphism $\mathbf{M}(4, 2k)//\mathbf{Sp}(k) \cong \mathbf{Alt}(4)$, namely, $\phi^* : \mathbb{C}[\mathbf{Alt}(4)] \to \mathbb{C}[\mathbf{M}(4, 2k)]^{\mathbf{Sp}(k)}$ is a $\mathbb{C}$-algebra isomorphism. Hence, $\phi^* : \mathbb{C}[\mathbf{Alt}(4)]^{\mathbf{Sp}(2)} \xrightarrow{\cong} \mathbb{C}[\mathbf{M}(4, 2k)]^{\mathbf{Sp}(2)\times\mathbf{Sp}(k)}$. Put $V_1 = \mathbb{C}J_2$ and $V_2 = \{Y \in \mathbf{Alt}(4) \mid \mathrm{tr}(J_2Y) = 0\}$. Then V_1 and V_2 are simple $\mathbf{Sp}(2)$-modules and $\mathbf{Alt}(4) = V_1 \oplus V_2$. It is known that $(\mathbf{GL}(1) \times \mathbf{GL}(1) \times \mathbf{Sp}(2), \mathbf{Alt}(4))$ is a regular prehomogeneous vector space and the fundamental relative invariants are given by the Pfaffian $Pf(Y)$ and $\mathrm{tr}(J_2Y)$ (see [12]). Here the first (resp. second) factor of $\mathbf{GL}(1) \times \mathbf{GL}(1)$ acts of V_1 (resp. V_2) as scalar multiplication. This shows that $\mathbb{C}[\mathbf{M}(4, 2k)]^{\mathbf{Sp}(2)} = \mathbb{C}[Pf(Y), \mathrm{tr}(J_2Y)]$ and hence $\mathbb{C}[\mathbf{M}(4, 2k)]^{\mathbf{Sp}(2)\times\mathbf{Sp}(k)} = \mathbb{C}[P_1, P_2]$, since $P_1 = \phi^*(Pf(Y))$ and $P_2 = \phi^*(\mathrm{tr}(J_2Y))$. The fundamental relative invariants are algebraically independent ([29, §4, Lemma 4], [17, Lemma 2.8]) and this implies that $\mathbb{C}[\mathbf{M}(4,2k)]^{\mathbf{Sp}(2)\times\mathbf{Sp}(k)}$ is isomorphic to the polynomial ring of 2 variables. □

The polynomial P_1 is of degree 4 and the polynomial P_2 is a quadratic form of signature $(4k, 4k)$. By Lemma 3.6, the Clifford quartic form is written as a linear combination of P_1 and P_2^2. Indeed we have

$$\tilde{P}(w) = -16P_1(w) + P_2(w)^2.$$

Note that $P_1(w)$ is the irreducible relative invariant of the prehomogeneous vector space $(\mathbf{GL}(4)\times\mathbf{Sp}(k), \Lambda_1\otimes\Lambda_1, \mathbf{M}(4,2k))$ and is viewed as the Clifford quartic form for $(p,q) = (3,3)$. Hence P_1 also satisfies a local functional equation. The polynomial P_2^2 also satisfy a local functional equation, since P_2 is a quadratic form. We define the local zeta functions $\tilde{\zeta}_{1,\pm}(f;s,\Psi)$ (resp. $\tilde{\zeta}_{2,+}(f;s,\Psi)$) for P_1 (resp. P_2^2) as in §2. Then we obtain the following functional equations from Theorem 2.13 for P_1 and Theorem 2.12 for P_2^2:

$$\begin{aligned}
\begin{pmatrix} \tilde{\zeta}_{1,+}\left(s,\hat{\Psi}\right) \\ \tilde{\zeta}_{1,-}\left(s,\hat{\Psi}\right) \end{pmatrix} &= 2^{4s+4k}\pi^{-4s-4k-2}\Gamma(s+1)\Gamma\left(s+3\right) \\
&\quad \times \Gamma\left(s+2k-2\right)\Gamma\left(s+2k\right) \\
&\quad \times \begin{pmatrix} \sin(\pi s)^2 & 0 \\ 0 & \sin(\pi s)^2 \end{pmatrix} \begin{pmatrix} \tilde{\zeta}_{1,+}\left(-2k-s,\Psi\right) \\ \tilde{\zeta}_{1,-}\left(-2k-s,\Psi\right) \end{pmatrix}; \\
\tilde{\zeta}_{2,+}\left(s,\hat{\Psi}\right) &= -2^{4s+4k-1}\pi^{-4s-4k-2}\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+1\right) \\
&\quad \times \Gamma\left(s+2k\right)\Gamma\left(s+2k+\frac{1}{2}\right) \\
&\quad \times \sin(2\pi s)\tilde{\zeta}_{2,+}\left(-2k-s,\Psi\right).
\end{aligned}$$

4. Proof of the "If"-Part of Theorem 3.1

The "if"-part of Theorem 3.1 is an immediate consequence of Lemma 2.8 and Lemma 4.1 below.

LEMMA 4.1. *If the Clifford quartic form $\tilde{P}(w)$ vanishes identically, then (p,q,m) is equal to one of*

$$(2,1,2), (3,1,4), (5,1,8), (9,1,16), (2,2,4), (3,3,8), (5,5,16).$$

PROOF. Assume that $\tilde{P}(w)$ vanishes identically. Then, by differentiating $\tilde{P}(w)$, we obtain

$$\frac{\partial \tilde{P}}{\partial w_j}(w) = 2\sum_{i=1}^{p} S_i[w]\frac{\partial}{\partial w_j}S_i[w] - 2\sum_{i=p+1}^{p+q} S_i[w]\frac{\partial}{\partial w_j}S_i[w] \equiv 0.$$

Since $S_i[w]$ are non-degenerate, $\frac{\partial}{\partial w_j}S_i[w]$ are non-zero linear forms. Hence it is sufficient to prove that, if there exist non-zero linear forms $f_i(w)$ ($i = 1, \dots, p+q$) satisfying the identity

$$\sum_{i=1}^{p+q} S_i[w]f_i(w) \equiv 0, \tag{4.1}$$

then (p, q, m) belongs to the list in the lemma.

We may assume that $p \geq q$ and $p \geq 2$. Then, by Lemma 2.3, m is even. Put $d = m/2$ and we may assume that $S_1, \dots, S_{p+q}$ are of the form given in Lemma 2.3. Then the identity (4.1) takes the form

$$\begin{aligned} 0 \equiv (u^2 - v^2)f_1(w) &+ 2\sum_{i=2}^{p}(u, B_i v)f_i(w) \\ &- \sum_{j=1}^{q}(A_j[u] + A_j[v])f_{p+j}(w), \end{aligned} \tag{4.2}$$

where we put $w = \begin{pmatrix} u \\ v \end{pmatrix}$ ($u, v \in \mathbb{R}^d$) and $(u, v) = {}^t uv$, $u^2 = (u, u)$, $v^2 = (v, v)$.

If $q = 0$, then the term that does not contain the variable v (resp. u) is given by $u^2 f_1(u, 0)$ (resp. $v^2 f_1(0, v)$). Hence, by (4.2), we have $f_1(u, 0) = f_1(0, v) = 0$ and $f_1(w) = f_1(u, 0) + f_1(0, v) = 0$. This contradicts the assumption that $f_1(w) \neq 0$. Therefore the relations of the form (4.1) implies that $q \geq 1$.

Consider the case where $q = 1$. Moreover we may assume that $(p, q, m) \neq (2, 1, 2)$. Then $m = 2d \geq 4$. Since $f_1(w) \neq 0$, we have $f_1(u, 0) \neq 0$ or $f_1(0, v) \neq 0$. Let us assume that $f_1(0, v) \neq 0$, since the case $f_1(u, 0) \neq 0$ can be treated quite similarly. Comparing the terms of degree 3 with respect to v on the left- and right-hand sides of the identity

$$0 \equiv (u^2 - v^2)f_1(w) + 2\sum_{i=2}^{p}(u, B_i v)f_i(w) - (A_1[u] + A_1[v])f_{p+1}(w), \tag{4.3}$$

we obtain

$$-v^2 f_1(0,v) - A_1[v] f_{p+1}(0,v) = 0.$$

Since $d \geq 2$ and v^2 is an irreducible polynomial over $\mathbb{R}$, v^2 divides $A_1[v]$. Moreover it follows from the relation $A_1^2 = 1_d$ that

$$A_1 = \pm 1_d, \quad f_{p+1}(0,v) = \mp f_1(0,v). \tag{4.4}$$

Now compare the terms of degree 2 and 1 with respect to u and v, respectively, in (4.3). Then we get

$$0 = u^2 f_1(0,v) + 2\sum_{i=2}^{p}(u, B_i v) f_i(u,0) - A_1[u] f_{p+1}(0,v).$$

From (4.4), it follows that

$$u^2 f_1(0,v) = -\sum_{i=2}^{p}(u, B_i v) f_i(u,0).$$

Since $f_1(0,v) \neq 0$, we can choose $v \in \mathbb{R}^d$ such that $f_1(0,v) > 0$. Then, for such a v, the quadratic form of u on the left-hand side is positive definite and of rank d. The number of positive eigenvalues of the quadratic form of u on the right-hand side is not greater than $p-1$. Thus we get

$$d \leq p-1 \text{ and } m = 2d \leq 2(p-1).$$

Note that $S_1, \ldots, S_p$ defines a representation of the Clifford algebra C_p, and the dimension d_0 of irreducible representations of C_p is given by the table,

p	2	3	4	5	6	7	8	9	$10 \leq p$
d_0	2	4	8	8	16	16	16	16	$2^{(p-1)/2} \leq d_0$
$2(p-1)$	2	4	6	8	10	12	14	16	$2(p-1)$

which we can get easily from Lemma 2.4. If $p \geq 10$, then $2^{(p-1)/2} > 2(p-1)$. Hence, the possibilities of $p \geq 2$ and m for $q = 1$ are

$$(2,1,2), \quad (3,1,4), \quad (5,1,8), \quad (9,1,16),$$

which are precisely the cases for $p \geq 2$ and $q = 1$ found on the list in Lemma 4.1.

Finally we consider the case for $p, q \geq 2$. Look at the terms in (4.2) including only u or only v to get

$$0 = u^2 f_1(u, 0) - \sum_{j=1}^{q} A_j[u] f_{p+j}(u, 0),$$
$$0 = -v^2 f_1(0, v) - \sum_{j=1}^{q} A_j[v] f_{p+j}(0, v).$$

One of these 2 identities should give a non-trivial relation for the d-dimensional representation of $R_{q,1}$ defined by $S_j = A_j$ $(j = 1, \ldots, q)$ and $S_{q+1} = 1_d$. By what we have proved for $q = 1$, this implies that

$$q \geq 2 \text{ and } d = 2, \quad q \geq 3 \text{ and } d = 4, \quad q \geq 5 \text{ and } d = 8, \quad q \geq 9 \text{ and } d = 16.$$

But, if $d = 2$, (resp. $4, 8, 16$), then $q \leq 2$ (resp. $4, 8, 16$). Hence we have $(q, d) = (2, 2), (3, 4), (5, 8), (9, 16)$. If we change the role of p and q, we also have $(p, d) = (2, 2), (3, 4), (5, 8), (9, 16)$. Hence the possibilities of (p, q, m) are

$$(2, 2, 4), \quad (3, 3, 8), \quad (5, 5, 16), \quad (9, 9, 32).$$

The case $(9, 9, 32)$ cannot occur, since the dimension of the irreducible representations of $R_{9,9}$ is equal to 2^8 (see Lemma 2.5), much bigger than 32. Thus we have obtained the list of (p, q, m) in Lemma 4.1. □

5. Proof of Theorem 3.3

The following lemma gives a sufficient condition for $\mathfrak{g}_{p,q}(\rho)$ to coincide with $\mathfrak{g}'_{p,q}(\rho) = \mathfrak{k}_{p,q} \oplus \mathfrak{h}_{p,q}(\rho)$.

Lemma 5.1. *Let ρ be a representation of $R_{p,q}$ such that the quadratic mapping associated to ρ is non-degenerate. If the basis matrices $S_1, S_2, \ldots,$ S_{p+q} of ρ satisfy the condition (♯) below, then $\mathfrak{g}_{p,q}(\rho) = \mathfrak{g}'_{p,q}(\rho)$:*

($\sharp$) *if* $\sum_{i=1}^{p+q} S_i[w]X_i[w] \equiv 0$ *for* $X_1, \dots, X_{p+q} \in \mathrm{Sym}(m;\mathbb{R})$ $(m = \deg \rho)$, *then we have* $X_i = \sum_{j=1}^{p+q} a_{ij}S_j$ $(i = 1, 2, \dots, p+q)$ *for some* a_{ij} *with* $a_{ij} = -a_{ji}$.

Proof. First note that, if ρ is non-degenerate, then $S_1, \dots, S_{p+q}$ are linearly independent. If not, the image of the associated quadratic mapping is contained in a hypersurface and the quadratic mapping can not be non-degenerate. For $X \in \mathfrak{g}_{p,q}(\rho)$, the assumption ($\sharp$) implies that

$$ {}^tXS_i + S_iX = \sum_{j=1}^{p+q} a_{ij}S_j \ (i = 1, \dots, p+q), $$

$$ a_{ij} = -\varepsilon_i\varepsilon_j a_{ji} \ (1 \le i < j \le p+q). $$

The coefficients a_{ij} defines an element (a_{ij}) in $\mathfrak{so}(p,q)$ and the mapping

$$ f : \mathfrak{g}_{p,q}(\rho) \longrightarrow \mathfrak{so}(p,q), \quad X \mapsto (a_{ij}) $$

gives a Lie algebra homomorphism. The mapping $\mathfrak{so}(p,q) \ni (a_{ij}) \mapsto \sum_{i<j} a_{ij}S_iS_j \in \rho(\mathfrak{k}_{p,q})$ is a section of f (see Lemma 2.6). It is obvious that

$$ \mathrm{Ker}(f) = \{X \in \mathfrak{g}_{p,q}(\rho) \mid {}^tXS_i + S_iX = 0 \ (1 \le i \le p+q)\} = \mathfrak{h}_{p,q}(\rho). $$

This proves that $\mathfrak{g}_{p,q} = \mathfrak{g}'_{p,q}$. □

Theorem 3.3 is an immediate consequence of Lemma 5.1 and the following lemma.

Lemma 5.2. *If* $p \ge q \ge 0$, $p+q \ge 3$ *and* $m > 4(p+q) - 8$, *then the condition* ($\sharp$) *in Lemma 5.1 holds for every* m*-dimensional representation of* $R_{p,q}$.

Indeed, denoting by m_0 the smallest dimension of representations of $R_{p,n-p}$ $(p = 0, 1, \dots, n)$, we have by Lemma 2.5 the following table.

$n = p+q$	3	4	5	6	7	8	9	10	11	12	$\cdots$
$4(p+q)-8$	4	8	12	16	20	24	28	32	36	40	$\cdots$
m_0	2	4	8	8	16	16	16	16	32	64	$\cdots$

Moreover $m_0 > 4(p+q)-8$ for $n = p+q \geq 12$. By Lemma 4.1 ("if"-part of Theorem 3.1), if $p+q \geq 3$ and $m > 4(p+q)-8$, then the quadratic mapping defined by an m-dimensional representation of $R_{p,q}$ is non-degenerate. Hence Theorem 3.3 follows immediately from Lemma 5.2.

We prove Lemma 5.2 by induction on $n = p+q$. Though we are interested in the case $p+q \geq 3$, we need the cases $p+q = 1, 2$ as the starting point of the inductive argument. Namely, we later use the fact that

> the condition ($\sharp$) in Lemma 5.1 holds for $p+q = 1, 2$, unless $p = q = 1$ and $S_1 = \pm S_2$.

For $p+q = 1$, this is obvious. For $p+q = 2$, this follows from that $S_1[x]$ and $S_2[x]$ are coprime in $\mathbb{R}[W]$ unless $p = q = 1$ and $S_1 = \pm S_2$.

Now we consider the case $p+q \geq 3$ with $p \geq q \geq 0$ and $m > 4(p+q)-8$. Since $p \geq 2$, we may choose the basis matrices $S_1, \ldots, S_{p+q}$ as given in Lemma 2.3. Then, the assumption $\sum_{i=1}^{p+q} S_i[w]X_i[w] = 0$ in ($\sharp$) can be written as follows:

$$\{(u,u)-(v,v)\}X_1[w] + 2\sum_{i=2}^{p}(u, B_i v)X_i[w] + \sum_{j=1}^{q}(A_j[u]+A_j[v])Y_j[w] = 0, \tag{5.1}$$

where we put $w = \begin{pmatrix} u \\ v \end{pmatrix}$ $(u, v \in \mathbb{R}^d)$ with $d = m/2$ and $Y_j = X_{p+j}$. We write the matrices $X_1, \ldots, X_p, Y_1, \ldots, Y_q$ in the form

$$X_i = \begin{pmatrix} X_i^{(1)} & X_i^{(2)} \\ {}^t X_i^{(2)} & X_i^{(3)} \end{pmatrix}, \quad X_i^{(1)}, X_i^{(3)} \in \mathrm{Sym}(d,\mathbb{R}),\ X_i^{(2)} \in M(d,\mathbb{R}),$$
$$Y_j = \begin{pmatrix} Y_j^{(1)} & Y_j^{(2)} \\ {}^t Y_j^{(2)} & Y_j^{(3)} \end{pmatrix}, \quad Y_j^{(1)}, Y_j^{(3)} \in \mathrm{Sym}(d,\mathbb{R}),\ Y_j^{(2)} \in M(d,\mathbb{R}).$$

Then the identity (5.1) is equivalent to the following 5 identities:

$$(u,u)X_1^{(1)}[u] + \sum_{j=1}^{q} A_j[u]Y_j^{(1)}[u] = 0, \tag{5.2}$$

$$-(v,v)X_1^{(3)}[v]+\sum_{j=1}^{q}A_j[v]Y_j^{(3)}[v]=0, \tag{5.3}$$

$$(u,u)X_1^{(3)}[v]-(v,v)X_1^{(1)}[u]+4\sum_{i=2}^{p}(u,B_iv)(u,X_i^{(2)}v)$$
$$+\sum_{j=1}^{q}(A_j[u]Y_j^{(3)}[v]+A_j[v]Y_j^{(1)}[u])=0,$$

$$(u,u)(u,X_1^{(2)}v)+(u,v)X_2^{(1)}[u]+\sum_{i=3}^{p}(u,B_iv)X_i^{(1)}[u] \tag{5.4}$$
$$+\sum_{j=1}^{q}A_j[u](u,Y_j^{(2)}v)=0,$$

$$-(v,v)(u,X_1^{(2)}v)+(u,v)X_2^{(3)}[v]+\sum_{i=3}^{p}(u,B_iv)X_i^{(3)}[v] \tag{5.5}$$
$$+\sum_{j=1}^{q}A_j[v](u,Y_j^{(2)}v)=0.$$

It is convenient to rewrite the third identity as follows:

$$-4\sum_{i=2}^{p}(u,B_iv)(u,X_i^{(2)}v) \tag{5.6}$$
$$=(u,u)X_1^{(3)}[v]-(v,v)X_1^{(1)}[u]+\sum_{j=1}^{q}(A_j[u]Y_j^{(3)}[v]+A_j[v]Y_j^{(1)}[u]).$$

The following lemma is a consequence of the identities (5.2), (5.3) and (5.6).

LEMMA 5.3. *There exist $a_{ij}\in\mathbb{R}$ satisfying $a_{ij}+a_{ji}=0$ and*

$$\begin{cases} X_1^{(1)}=\sum_{j=1}^{q}a_{1,p+j}A_j, \\ Y_j^{(1)}=a_{p+j,1}1_d+\sum_{k=1}^{q}a_{p+j,p+k}A_k \quad (1\le j\le q), \end{cases} \tag{5.7}$$

$$
\begin{cases} X_1^{(3)} = \displaystyle\sum_{j=1}^{q} a_{1,p+j} A_j, \\ Y_j^{(3)} = -a_{p+j,1} 1_d + \displaystyle\sum_{k=1}^{q} a_{p+j,p+k} A_k \quad (1 \leq j \leq q), \end{cases} \tag{5.8}
$$

$$
X_i^{(2)} = \sum_{j=2}^{p} a_{ij} B_j \quad (2 \leq i \leq p). \tag{5.9}
$$

Proof. First note that, from (5.7) and (5.8), we see that the right-hand side of (5.6) is identically zero and

$$
\sum_{i=2}^{p} (u, B_i v)(u, X_i^{(2)} v) = 0.
$$

This implies the identity (5.9), since, by the induction assumption, the condition (♯) holds for the $2d$-dimensional representation of C_{p-1} determined by $S_2, \ldots, S_p$. Hence it is sufficient to prove the identities (5.7), (5.8). In the subsequent discussion, we have to distinguish two cases, namely, the case where $q = 1$ and $A_1 = \pm 1_d$ and the case where $q \neq 1$ or $A_1 \neq \pm 1_d$. We refer to the first case as Case A and the second case as Case B.

Case A: We put $A_1 = \varepsilon 1_d$ ($\varepsilon = \pm 1$). Then the identities (5.2) and (5.3) become

$$
(u,u) X_1^{(1)}[u] + \varepsilon (u,u) Y_1^{(1)}[u] = 0, \quad -(v,v) X_1^{(3)}[v] + \varepsilon (v,v) Y_1^{(3)}[v] = 0.
$$

This implies that

$$
Y_1^{(1)} = -\varepsilon X_1^{(1)}, \quad Y_1^{(3)} = \varepsilon X_1^{(3)}. \tag{5.10}
$$

We substitute (5.10) to (5.6) to get

$$
-2 \sum_{i=2}^{p} (u, B_i v)(u, X_i^{(2)} v) = (u,u) X_1^{(3)}[v] - (v,v) X_1^{(1)}[u].
$$

Fix v and consider the left-hand side of this identity as a quadratic form of u. Then the rank of the quadratic form is not greater than $2(p-1)$.

Since $d > 2(p+1) - 4 = 2(p-1)$ by the assumption of Lemma 5.2, it is a degenerate quadratic form of u. Hence by the expression on the right-hand side we have $\det(X_1^{(3)}[v]1_d - (v,v)X_1^{(1)}) = 0$. Since v is arbitrary, this shows that $X_1^{(3)} = \lambda 1_d$ for some eigenvalue λ of $X_1^{(1)}$, namely, $X_1^{(3)}$ is a scalar matrix. Similarly we can prove that $X_1^{(1)}$ is also a scalar matrix and we get

$$X_1^{(1)} = a_{1,p+1}A_1, \quad X_1^{(3)} = b_{1,p+1}A_1$$

for some constants $a_{1,p+1}, b_{1,p+1}$. Then, since $(u,u)X_1^{(3)}[v] - (v,v)X_1^{(1)}[u]$ is degenerate, we have

$$a_{1,p+1} = b_{1,p+1}.$$

Furthermore (5.10) implies that

$$Y_1^{(1)} = -\varepsilon a_{1,p+1}A_1 = -a_{1,p+1}1_d, \quad Y_1^{(3)} = \varepsilon a_{1,p+1}A_1 = a_{1,p+1}1_d.$$

Hence putting $a_{p+1,1} = -a_{1,p+1}$, we obtain

$$Y_1^{(1)} = a_{p+1,1}1_d, \quad Y_1^{(3)} = -a_{p+1,1}1_d$$

and $a_{p+1,p+1} = b_{p+1,p+1} = 0$. This proves the identities (5.7) and (5.8).

Case B: In this case, since $q > 1$ or $A_1 \neq \pm 1_d$, by the induction hypothesis, we may assume that the d-dimensional representation of $R_{1,q}$ determined by $1_d, A_1, \ldots, A_q$ satisfies the condition (♯). Indeed, if $q = 1$ and $A_1 \neq \pm 1_d$, we have already seen it. If $q \geq 2$, then it is enough to note that the quadratic mapping defined by $1_d, A_1, \ldots, A_q$ is non-degenerate. This is a consequence of the inequality $d > 4q - 4 = 4(q+1) - 8$ and Lemma 4.1 ("if"-part of Theorem 3.1) applied to the case $(q, 1)$. Hence from the identities (5.2) and (5.3) we have

$$\text{(5.11)} \qquad \begin{cases} X_1^{(1)} = \displaystyle\sum_{j=1}^{q} a_{1,p+j}A_j, \\ Y_j^{(1)} = a_{p+j,1}1_d + \displaystyle\sum_{k=1}^{q} a_{p+j,p+k}A_k \quad (1 \leq j \leq q), \end{cases}$$

$$\text{(5.12)} \qquad \begin{cases} X_1^{(3)} = \displaystyle\sum_{j=1}^{q} b_{1,p+j}A_j, \\ Y_j^{(3)} = -b_{p+j,1}1_d + \displaystyle\sum_{k=1}^{q} b_{p+j,p+k}A_k \quad (1 \leq j \leq q), \end{cases}$$

where a_{ij}, b_{ij} satisfy that $a_{ij} + a_{ji} = 0$, $b_{ij} + b_{ji} = 0$. We rewrite the right-hand side of (5.6) by using (5.11) and (5.12) to get

$$\begin{aligned}(5.13)\qquad -4\sum_{i=2}^{p}(u, B_i v)(u, X_i^{(2)} v) &= -(u,u)\sum_{j=1}^{q}(a_{p+j,1} + b_{1,p+j})A_j[v] \\ &\quad + \sum_{j=1}^{q} A_j[u]\Bigg\{(a_{1,p+j} + b_{p+j,1})(v,v) \\ &\qquad - \sum_{k=1}^{q}(a_{p+j,p+k} + b_{p+k,p+j})A_k[v]\Bigg\}.\end{aligned}$$

As in Case A, the quadratic form of u (for an arbitrarily fixed v) defined by the left-hand side of (5.13) is degenerate. Indeed, if $q = 0$, then the right-hand side vanishes, and if $q > 0$, then the rank of the quadratic form is not greater than $2(p-1)$ ($\leq 2(p+q)-4 < d$). The linear combination of $(u,u), A_1[u], \ldots, A_q[u]$ on the right-hand side of (5.13) is degenerate, only when

$$\begin{aligned}(5.14)\qquad &\left(\sum_{j=1}^{q}(a_{p+j,1} + b_{1,p+j})A_j[v]\right)^2 \\ &\quad - \sum_{j=1}^{q}\Bigg\{(a_{1,p+j} + b_{p+j,1})(v,v) \\ &\qquad - \sum_{k=1}^{q}(a_{p+j,p+k} + b_{p+k,p+j})A_k[v]\Bigg\}^2 = 0.\end{aligned}$$

If $q = 1$ and $A_1 \neq \pm 1_d$, then this follows from that $c_1 A_1[u] - c_2(u,u)$ is degenerate only when $c_1 = \pm c_2$, since the eigenvalues of A_1 are ± 1. If $q \geq 2$, then, since the quadratic mapping defined by $1_d, A_1, \ldots, A_q$ is non-degenerate and self-dual, this follows from the identity

$$\det\left(c_0 1_d + \sum_{j=1}^{q} c_j A_j\right) = \pm\left(c_0^2 - \sum_{j=1}^{q} c_j^2\right)^{d/2},$$

which is obtained by Remark 1.1 (1) and the proof of Lemma 2.11. Furthermore in Case B, the quadratic forms $(u,u), A_1[u], \dots, A_q[u]$ are algebraically independent. Indeed, as we have already noted, if $q > 1$, then the quadratic mapping defined by $1_d, A_1, \dots, A_q$ is non-degenerate. Since the image of a non-degenerate quadratic mapping can not be contained in a low-dimensional algebraic set, this implies the algebraic independence. If $q = 1$, then $A_1^2 = 1_d$ and $A_1 \neq \pm 1_d$. Hence it is obvious that (u,u) and $A_1[u]$ is algebraically independent. Therefore, comparing the coefficients of $(v,v)^2$ on the both sides of (5.14), we obtain

$$\sum_{j=1}^{q} (a_{1,p+j} + b_{p+j,1})^2 = 0.$$

Since the constants $a_{1,p+j}$, $b_{p+j,1}$ are real numbers, we have

$$a_{1,p+j} + b_{p+j,1} = 0 \quad (1 \leq j \leq q).$$

and, by the skew-symmetry of a_{ij}, b_{ij},

$$a_{p+j,1} + b_{1,p+j} = 0 \quad (1 \leq j \leq q).$$

Hence, by (5.14), we get

$$\sum_{j=1}^{q} \left\{ \sum_{k=1}^{q} (a_{p+j,p+k} + b_{p+k,p+j}) A_k[v] \right\}^2 = 0.$$

This implies

$$\sum_{k=1}^{q} (a_{p+j,p+k} + b_{p+k,p+j}) A_k[v] = 0 \quad (1 \leq j \leq q).$$

Since the basis matrices $A_1, \dots, A_q$ are linearly independent, we have

$$a_{p+j,p+k} + b_{p+k,p+j} = 0 \quad (1 \leq j, k \leq q).$$

By using the skew-symmetry of a_{ij}, b_{ij}, we also have

$$a_{p+j,1} = b_{p+j,1}, \quad a_{p+j,p+k} = b_{p+j,p+k}. \tag{5.15}$$

Now the identities (5.7) and (5.8) follow from (5.11), (5.12) and (5.15). □

To proceed further, we need the following lemma.

Lemma 5.4. *We assume that the identity*

$$X[u](u,v)-(u,u)(u,Xv)=\sum_{i=1}^{r} Y_i[u](u,B_iv) \tag{5.16}$$

holds for $X, Y_1, \dots, Y_r \in \mathrm{Sym}(d,\mathbb{R})$ *and* $B_1, \dots, B_r \in \mathrm{Alt}(d,\mathbb{R})$. *Let* s *be the number of symmetric matrices* Y_i *that are not scalar matrices. If* $d > 2s$, *then* X *is a scalar matrix and the both sides of* (5.16) *are identically zero.*

Proof. We may assume that X is a diagonal matrix with diagonal entries $\lambda_1 \geq \cdots \geq \lambda_d$. Then, since

$$(X[u]1_d-(u,u)X)u=-\sum_{i=1}^{r} Y_i[u]B_iu,$$

by comparing the k-th entries of the left- and right-hand sides, we have

$$\left(\sum_{i\neq k}(\lambda_i-\lambda_k)u_i^2\right)u_k=-\sum_{i=1}^{r} Y_i[u]\psi_{ik}(u),$$

where $\psi_{ik}(u)$ is the linear form of u that gives the k-th entry of B_iu. The left-hand side of this identity is of degree 1 with respect to u_k. Since B_i is a skew-symmetric matrix, u_k does not appear in $\psi_{ik}(u)$. We consider $Y_i[u]$ as a quadratic polynomial of u_k and denote by $\phi_{ik}(u)$ the linear form of $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_d$ appearing as the coefficient of u_k. We may assume that Y_i are scalar matrices for $i \geq s+1$. Then $\phi_{ik}(u) = 0$ for $i \geq s+1$ and we have

$$\sum_{i\neq k}(\lambda_i-\lambda_k)u_i^2=-\sum_{i=1}^{s}\psi_{ik}(u)\phi_{ik}(u).$$

Note that the number of positive eigenvalues and that of negative eigenvalues of the quadratic form on the right-hand side are smaller than or equal to s. Consider the case where $k = 1$. Then the left-hand side of this identity is negative semidefinite. Hence the multiplicity of λ_1 is not smaller than $d-s$. Similarly, if we consider the case where $k = d$, the left-hand side of this identity is positive semidefinite and the multiplicity of λ_d is at least $d - s$. Therefore, if X has distinct eigenvalues, then $d \geq 2(d - s)$ namely $2s \geq d$. This contradicts the assumption $d > 2s$. Hence, X is a scalar matrix and the both sides of (5.16) are identically zero. □

Let us return to the proof of Lemma 5.2 and examine the identities (5.4) and (5.5). Substitute $v = u$ to (5.4) and (5.5). Then, since B_i $(i \geq 3)$ are skew-symmetric matrices and hence $(u, B_i u) = 0$, we obtain

$$(u,u)(X_1^{(2)} + X_2^{(1)})[u] + \sum_{j=1}^{q} A_j[u] Y_j^{(2)}[u] = 0, \tag{5.17}$$

$$(u,u)(X_2^{(3)} - X_1^{(2)})[u] + \sum_{j=1}^{q} A_j[u] Y_j^{(2)}[u] = 0. \tag{5.18}$$

Here we have to treat Case A and Case B separately as in the proof of Lemma 5.3.

Case A: In this case, $q = 1$, $A_1 = \varepsilon 1_d$ $(\varepsilon = \pm 1)$ and the identities (5.17) and (5.18) become

$$(u,u)(X_1^{(2)} + X_2^{(1)} + \varepsilon Y_1^{(2)})[u] = 0, \quad (u,u)(X_2^{(3)} - X_1^{(2)} + \varepsilon Y_1^{(2)})[u] = 0.$$

Hence, putting

$$\begin{cases} Z^{+} = X_1^{(2)} + \varepsilon Y_1^{(2)}, \quad Z^{-} = X_1^{(2)} - \varepsilon Y_1^{(2)}, \\ Z_s^{\pm} = \dfrac{1}{2}(Z^{\pm} + {}^t Z^{\pm}), \quad Z_a^{\pm} = \dfrac{1}{2}(Z^{\pm} - {}^t Z^{\pm}), \end{cases} \tag{5.19}$$

we have

$$Z_s^{+} = -X_2^{(1)}, \quad Z_s^{-} = X_2^{(3)}.$$

This yields that

$$X_1^{(2)} = Z_a^{+} - X_2^{(1)} - \varepsilon Y_1^{(2)} = Z_a^{-} + X_2^{(3)} + \varepsilon Y_1^{(2)}. \tag{5.20}$$

By substituting (5.20) to (5.4) and (5.5), we get

$$(u,u)(u,Z_a^+v)+(u,v)X_2^{(1)}[u]-(u,u)(u,X_2^{(1)}v)+\sum_{i=3}^{p}(u,B_iv)X_i^{(1)}[u]=0,$$

$$-(v,v)(u,Z_a^-v)+(u,v)X_2^{(3)}[v]-(v,v)(u,X_2^{(3)}v)+\sum_{i=3}^{p}(u,B_iv)X_i^{(3)}[v]=0.$$

We rewrite these identities as follows:

$$(u,v)X_2^{(1)}[u]-(u,u)(u,X_2^{(1)}v)=-(u,u)(u,Z_a^+v)-\sum_{i=3}^{p}(u,B_iv)X_i^{(1)}[u],$$

$$(u,v)X_2^{(3)}[v]-(v,v)(u,X_2^{(3)}v)=(v,v)(u,Z_a^-v)-\sum_{i=3}^{p}(u,B_iv)X_i^{(3)}[v].$$

Lemma 5.4 can apply to these identities, since s in Lemma 5.4 is not greater than $p-2$ and $d>2(p+q)-4>2s$. Thus we obtain for some α,β

$$X_2^{(1)}=\alpha 1_d,\quad X_2^{(3)}=\beta 1_d, \tag{5.21}$$

$$(u,u)(u,Z_a^+v)+\sum_{i=3}^{p}(u,B_iv)X_i^{(1)}[u]=0, \tag{5.22}$$

$$(v,v)(u,Z_a^-v)-\sum_{i=3}^{p}(u,B_iv)X_i^{(3)}[v]=0. \tag{5.23}$$

Summing (5.22) (resp. (5.23)) and the identity obtained by exchanging u and v in (5.22) (resp. (5.23)), we get

$$\{(u,u)-(v,v)\}(u,Z_a^+v)+\sum_{i=3}^{p}(u,B_iv)(X_i^{(1)}[u]-X_i^{(1)}[v])=0, \tag{5.24}$$

$$\{(u,u)-(v,v)\}(u,Z_a^-v)-\sum_{i=3}^{p}(u,B_iv)(X_i^{(3)}[u]-X_i^{(3)}[v])=0. \tag{5.25}$$

Since the condition (♯) holds for the $2d$-dimensional representation of $R_{p-1,0}$ determined by $S_1,S_3,\dots,S_p$ by induction hypothesis, there exist constants

$\gamma_{i,1}, \gamma_{1,i}, \delta_{i,1}, \delta_{1,i} \in \mathbb{R}$ $(3 \leq i \leq p)$ satisfying $\gamma_{i,1} = -\gamma_{1,i}$, $\delta_{i,1} = -\delta_{1,i}$ and

$$Z_a^+ = \sum_{i=3}^{p} \gamma_{1,i} B_i, \quad Z_a^- = \sum_{i=3}^{p} \delta_{1,i} B_i, \tag{5.26}$$

$$X_i^{(1)} = \gamma_{i,1} 1_d, \quad X_i^{(3)} = -\delta_{i,1} 1_d \quad (3 \leq i \leq p). \tag{5.27}$$

From (5.20), (5.21) and (5.26), we have

$$Z^+ = -\alpha 1_d + \sum_{i=3}^{p} \gamma_{1,i} B_i, \quad Z^- = \beta 1_d + \sum_{i=3}^{p} \delta_{1,i} B_i.$$

Then, since $X_1^{(2)} = \frac{1}{2}(Z^+ + Z^-)$, $Y_1^{(2)} = \frac{\varepsilon}{2}(Z^+ - Z^-)$, we obtain

$$X_1^{(2)} = \sum_{i=2}^{p} a_{1,i} B_i, \quad Y_1^{(2)} = \sum_{i=2}^{p} a_{p+1,i} B_i, \tag{5.28}$$

where we put

$$a_{1,2} = \frac{1}{2}(\beta - \alpha), \quad a_{p+1,2} = -\frac{\varepsilon}{2}(\alpha + \beta)$$
$$a_{1,i} = \frac{1}{2}(\gamma_{1,i} + \delta_{1,i}), \quad a_{p+1,i} = \frac{\varepsilon}{2}(\gamma_{1,i} - \delta_{1,i}) \quad (3 \leq i \leq p).$$

Now the identity (5.27) takes the form

$$\begin{cases} X_i^{(1)} = (a_{1,i} + \varepsilon a_{p+1,i}) 1_d, \\ X_i^{(3)} = (-a_{1,i} + \varepsilon a_{p+1,i}) 1_d \quad (3 \leq i \leq p). \end{cases} \tag{5.29}$$

Furthermore, by (5.21), we have

$$X_2^{(1)} = -a_{1,2} 1_d - a_{p+1,2} \varepsilon 1_d, \quad X_2^{(3)} = a_{1,2} 1_d - a_{p+1,2} \varepsilon 1_d.$$

Then, putting $a_{2,1} = -a_{1,2}$, $a_{2,p+1} = -a_{p+1,2}$, we obtain

$$X_2^{(1)} = a_{2,1} 1_d + a_{2,p+1} \varepsilon 1_d, \quad X_2^{(3)} = -a_{2,1} 1_d + a_{2,p+1} \varepsilon 1_d. \tag{5.30}$$

Lemma 5.2 for Case A follows from Lemma 5.3, (5.28), (5.29) and (5.30).

Case B: We consider (5.17) and (5.18) for Case B, namely, under the assumption $q \neq 1$ or $A_1 \neq \pm 1_d$. Put

$$X_{1,s}^{(2)} = \frac{1}{2}(X_1^{(2)} + {}^tX_1^{(2)}), \quad X_{1,a}^{(2)} = \frac{1}{2}(X_1^{(2)} - {}^tX_1^{(2)}),$$
$$Y_{i,s}^{(2)} = \frac{1}{2}(Y_i^{(2)} + {}^tY_i^{(2)}), \quad Y_{i,a}^{(2)} = \frac{1}{2}(Y_i^{(2)} - {}^tY_i^{(2)}).$$

Since, by the induction assumption, the condition (♯) holds for d-dimensional representations of $R_{1,q}$ for Case B, we have from (5.17) and (5.18)

$$(5.31) \qquad \begin{cases} X_{1,s}^{(2)} + X_2^{(1)} = X_2^{(3)} - X_{1,s}^{(2)} = \sum_{j=1}^{q} \alpha_j A_j, \\ Y_{i,s}^{(2)} = -\alpha_i 1_d + \sum_{j=1}^{q} \alpha_{ij} A_j \quad (\alpha_{ij} = -\alpha_{ji}). \end{cases}$$

By substituting (5.31) to (5.4) and (5.5), we obtain

$$(5.32) \qquad \begin{aligned} &(u,u)(u, X_{1,a}^{(2)} v) + (u,u)(u, X_{1,s}^{(2)} v) - (u,v) X_{1,s}^{(2)}[u] \\ &\quad + \sum_{i=3}^{p} (u, B_i v) X_i^{(1)}[u] + \sum_{i=1}^{q} A_i[u](u, Y_{i,a}^{(2)} v) \\ &\quad + \sum_{i,j=1}^{q} \alpha_{ij} A_i[u](u, A_j v) = 0, \end{aligned}$$

$$(5.33) \qquad \begin{aligned} &-(v,v)(u, X_{1,a}^{(2)} v) - (v,v)(u, X_{1,s}^{(2)} v) + (u,v) X_{1,s}^{(2)}[v] \\ &\quad + \sum_{i=3}^{p} (u, B_i v) X_i^{(3)}[v] + \sum_{i=1}^{q} A_i[v](u, Y_{i,a}^{(2)} v) \\ &\quad + \sum_{i,j=1}^{q} \alpha_{ij} A_i[v](u, A_j v) = 0. \end{aligned}$$

Let us prove that

$$\alpha_{ij} = 0 \quad (1 \leq i, j \leq q).$$

Denote by $f_1(u,v)$ (resp. $f_2(u,v)$) the left-hand side of (5.32) (resp. (5.33)) and consider $f_1(u,v)+f_1(v,u)+f_2(u,v)+f_2(v,u)$. Then we obtain

$$\begin{aligned}&2\big((u,u)-(v,v)\big)(u,X^{(2)}_{1,a}v)\\&\quad+\sum_{i=3}^{p}(u,B_iv)\{(X^{(1)}_i-X^{(3)}_i)[u]-(X^{(1)}_i-X^{(3)}_i)[v]\}\\&\quad+2\sum_{i,j=1}^{q}\alpha_{ij}(A_i[u]+A_i[v])(u,A_jv)=0\end{aligned}$$

This can be written as

$$S_1[w]T_1[w]+\sum_{i=3}^{p}S_i[w]T_i[w]+\sum_{j=1}^{q}S_{p+j}[w]T_{p+j}[w]=0,$$

$$T_1=\begin{pmatrix}0 & X^{(2)}_{1,a}\\ {}^tX^{(2)}_{1,a} & 0\end{pmatrix},$$

$$T_i=\begin{pmatrix}\frac{1}{2}\left(X^{(1)}_i-X^{(3)}_i\right) & 0\\ 0 & -\frac{1}{2}\left(X^{(1)}_i-X^{(3)}_i\right)\end{pmatrix}\ (3\le i\le p),$$

$$T_{p+j}=\sum_{k=1}^{q}\alpha_{jk}\begin{pmatrix}0 & A_k\\ A_k & 0\end{pmatrix}\ (1\le j\le q).$$

Since, by the induction assumption, the condition (♯) holds for $2d$-dimensional representations of $R_{p-1,q}$, every T_i $(i\neq 2)$ is a linear combination of $S_1,S_3,\dots,S_{p+q}$. Then, for j $(1\le j\le q)$, the matrix $\sum_{k=1}^{q}\alpha_{jk}A_k$, the off-diagonal block of T_{p+j}, is a linear combination of $B_3,\dots,B_p$, which is skew-symmetric. Since $\sum_{k=1}^{q}\alpha_{jk}A_k$ is symmetric, this equals 0. The linear independence of $A_1,\dots,A_q$ implies that $\alpha_{jk}=0$ $(1\le j,k\le q)$. Moreover, from (5.31), we get

$$Y^{(2)}_{i,s}=a_{p+i,2}1_d, \tag{5.34}$$

where we write $a_{p+i,2}$ for $-\alpha_i$. By $\alpha_{ij}=0$, the identities (5.32) and (5.33) become

$$\begin{aligned}&(u,v)X^{(2)}_{1,s}[u]-(u,u)(u,X^{(2)}_{1,s}v)\\&\quad=(u,u)(u,X^{(2)}_{1,a}v)+\sum_{i=3}^{p}(u,B_iv)X^{(1)}_i[u]+\sum_{i=1}^{q}A_i[u](u,Y^{(2)}_{i,a}v),\end{aligned}$$

$$- (u,v)X^{(2)}_{1,s}[v] + (v,v)(u, X^{(2)}_{1,s}v)$$
$$= -(v,v)(u, X^{(2)}_{1,a}v) + \sum_{i=3}^{p}(u, B_i v)X^{(3)}_i[v] + \sum_{i=1}^{q} A_i[v](u, Y^{(2)}_{i,a}v).$$

We can apply Lemma 5.4 to these identities, since s in Lemma 5.4 is not greater than $p+q-2$ and $d > 2(p+q-2) \geq 2s$. Therefore we have

$$X^{(2)}_{1,s} = a_{12}1_d \tag{5.35}$$

for some constant a_{12}, and

$$(u,u)(u, X^{(2)}_{1,a}v) + \sum_{i=3}^{p}(u, B_i v)X^{(1)}_i[u] + \sum_{i=1}^{q} A_i[u](u, Y^{(2)}_{i,a}v) = 0$$
$$-(v,v)(u, X^{(2)}_{1,a}v) + \sum_{i=3}^{p}(u, B_i v)X^{(3)}_i[v] + \sum_{i=1}^{q} A_i[v](u, Y^{(2)}_{i,a}v) = 0.$$

Summing these two identities, we obtain

$$\big((u,u) - (v,v)\big)(u, X^{(2)}_{1,a}v) + \sum_{i=3}^{p}(u, B_i v)(X^{(1)}_i[u] + X^{(3)}_i[v])$$
$$+ \sum_{i=1}^{q}(A_i[u] + A_i[v])(u, Y^{(2)}_{i,a}v) = 0.$$

We can rewrite this as follows:

$$S_1[w]Z_1[w] + \sum_{i=3}^{p+q} S_i[w]Z_i[w] = 0,$$
$$Z_1 = \begin{pmatrix} 0 & X^{(2)}_{1,a} \\ {}^tX^{(2)}_{1,a} & 0 \end{pmatrix}, \quad Z_i = \begin{pmatrix} X^{(1)}_i & 0 \\ 0 & X^{(3)}_i \end{pmatrix} \quad (3 \leq i \leq p),$$
$$Z_{p+j} = \begin{pmatrix} 0 & Y^{(2)}_{j,a} \\ {}^tY^{(2)}_{j,a} & 0 \end{pmatrix} \quad (1 \leq j \leq q).$$

Since any $2d$-dimensional representation of $R_{p-1,q}$ satisfies the condition ($\sharp$) by the induction assumption, we obtain

$$Z_i = \sum_{\substack{j=1 \\ j \neq 2}}^{p+q} a_{ij}S_j \quad (i = 1, 3, \ldots, p+q)$$

for some constants a_{ij} with $a_{ij} = -a_{ji}$. This identity together with Lemma 5.3, (5.31), (5.34) and (5.35) implies Lemma 5.2 for Case B.

6. Proof of Theorems 3.4 and 3.5

In this section we calculate the Lie algebra $\mathfrak{h}_{p,q}(\rho)$ and prove Theorems 3.4 and 3.5. Our calculation is based on the following Lemma.

Lemma 6.1. *Let $\rho : R_{p,q} \to M(m;\mathbb{R})$ be a representation of $R_{p,q}$ such that the basis matrices $S_i = \rho(e_i)$ $(1 \le i \le p+q)$ are symmetric matrices. Put*

$$\mathcal{A} = \mathcal{A}_{p,q}(\rho) := \left\{ X \in M(m;\mathbb{R}) \;\middle|\; \rho(Y)X = X\rho(Y) \ (\forall Y \in R^+_{p,q}) \right\}.$$

If $r \in (R_{p,q})^\times$ is an odd element, then we have

$$\mathfrak{h}_{p,q}(\rho) = \left\{ X \in \mathcal{A} \;\middle|\; {}^tX\rho(r) + \rho(r)X = 0 \right\}.$$

Proof. By the definition of $\mathfrak{h}_{p,q}(\rho)$,

$$\mathfrak{h}_{p,q}(\rho) = \left\{ X \in M(m;\mathbb{R}) \;\middle|\; {}^tXS_i + S_iX = 0 \ (1 \le i \le p+q) \right\}.$$

If $X \in \mathfrak{h}_{p,q}(\rho)$, then

$$S_iS_jX = -S_i{}^tXS_j = XS_iS_j.$$

Since S_iS_j $(1 \le i < j \le p)$ generate $\rho(R^+_{p,q})$, X is in $\mathcal{A}$. Thus $\mathfrak{h}_{p,q}(\rho) \subset \mathcal{A}$. Put $r' = e_ir$. Then r' is an element of $(R^+_{p,q})^\times$. Hence, for $X \in \mathcal{A}$, we have

$$\begin{aligned} {}^tX\rho(r) + \rho(r)X &= {}^tX\rho(e_i)\rho(r') + \rho(e_i)\rho(r')X \\ &= {}^tX\rho(e_i)\rho(r') + \rho(e_i)X\rho(r') \\ &= \left({}^tXS_i + S_iX\right)\rho(r'). \end{aligned}$$

Thus, if $X \in \mathcal{A}$, we have

$$\begin{aligned} {}^tX\rho(r) + \rho(r)X = 0 &\iff {}^tXS_i + S_iX = 0 \text{ for some } i \\ &\iff {}^tXS_i + S_iX = 0 \text{ for all } i \iff X \in \mathfrak{h}_{p,q}(\rho). \end{aligned}$$

This proves the lemma. □

Note that, we may take $r = e_i$. Then we have $\rho(r)^2 = 1$ and the map $X \mapsto -\rho(r)\,{}^tX\rho(r)^{-1} = -S_i\,{}^tXS_i$ is an involutive automorphism of $\mathcal{A}$. Hence $(\mathcal{A}, \mathfrak{h}_{p,q}(\rho))$ is a symmetric Lie algebra, if we regard $\mathcal{A}$ as a Lie algebra (see (6.1) below).

Let T' and $\mathbb{K}'$ be as in Theorem 3.4. It is often convenient to consider the representation space $W = \mathbb{R}^m$ as a $\mathbb{K}'$-vector space. Then the algebra $\mathcal{A}$ is of the form

$$\mathcal{A} = \begin{cases} M(k;\mathbb{K}') = \mathfrak{gl}(k,\mathbb{K}') & \text{if } R^+_{p,q} = T', \\ M(k_1;\mathbb{K}') \oplus M(k_2;\mathbb{K}') = \mathfrak{gl}(k_1,\mathbb{K}') \oplus \mathfrak{gl}(k_2,\mathbb{K}') & \\ & \text{if } R^+_{p,q} = T' \oplus T', \end{cases} \tag{6.1}$$

where k, k_1, k_2 are the multiplicities of irreducible representations corresponding to the simple components of $R^+_{p,q}$ in $\rho|_{R^+_{p,q}}$. In the sequel we denote the transpose of $X \in \mathcal{A}$ as a matrix in $M(m;\mathbb{R})$ by ${}^TX$ to distinguish it from the transpose as a matrix in $M(k;\mathbb{K}')$. Then ${}^TX$ corresponds to $X^* = {}^t\bar{X}$ in $M(k;\mathbb{K}')$, where $X \mapsto \bar{X}$ denotes the conjugate in $\mathbb{K}'$.

We put

$$\hat{e}_p = e_1 \cdots e_p \quad \hat{f}_q = e_{p+1} \cdots e_{p+q}.$$

Then it is easy to check the following identities

$$\begin{cases} \hat{e}_p^2 = \begin{cases} 1 & (p \equiv 0,1 \pmod 4), \\ -1 & (p \equiv 2,3 \pmod 4), \end{cases} \\ \hat{f}_q^2 = \begin{cases} 1 & (q \equiv 0,1 \pmod 4), \\ -1 & (q \equiv 2,3 \pmod 4), \end{cases} \end{cases} \tag{6.2}$$

$$\begin{cases} e_i\hat{e}_p = (-1)^{p-1}\hat{e}_p e_i \quad (i = 1, \dots, p), \\ e_{p+i}\hat{f}_q = (-1)^{q-1}\hat{f}_q e_{p+i} \quad (i = 1, \dots, q), \end{cases} \tag{6.3}$$

$$e_{p+i}\hat{e}_p = \hat{e}_p e_{p+i} \quad (i = 1, \dots, q), \quad e_i\hat{f}_q = \hat{f}_q e_i \quad (i = 1, \dots, p).$$

Now we are in a position to prove Theorems 3.4 by case by case examination. Theorem 3.5 on the action of $\mathfrak{g}'_{p,q}(\rho)$ is an immediate consequence of the description of $\mathfrak{h}_{p,q}(\rho)$ below.

6.1. The case: (T, T'), $(\mathbb{K}, \mathbb{K}') = (\mathbb{R}, \mathbb{C})$

In this case, $\{\bar{p}, \bar{q}\} = \{0, 2\}, \{4, 6\}$ and

$$R_{p,q} = M(2^{n/2}; \mathbb{R}) \supset R_{p,q}^{+} = M(2^{(n-2)/2}; \mathbb{C}) \quad (n = p + q).$$

We may assume that $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$. Then $\hat{e}_p^2 = 1, \hat{f}_q^2 = -1$. The center of $R_{p,q}^{+}$, which is isomorphic to $\mathbb{C}$, is given by $\mathbb{R} + \mathbb{R}\mathbf{i}$ ($\mathbf{i} := \hat{e}_p \hat{f}_q$), since $\mathbf{i}^2 = -1$ and $\mathbf{i}$ is a central element of $R_{p,q}^{+}$. Let W_0 be a simple $R_{p,q}$-module (unique up to isomorphism). Then W_0 is simple also as an $R_{p,q}^{+}$-module and naturally identified with $\mathbb{C}^{2^{(n-2)/2}}$. More generally, a not necessarily simple $R_{p,q}$-module $W = \overbrace{W_0 \oplus \cdots \oplus W_0}^{k}$ is identified with $M(2^{(n-2)/2}, k; \mathbb{C})$. Under the identification, $\mathcal{A} = M(k, \mathbb{C})$ and the action of $X \in \mathcal{A}$ on $w \in W$ is given by

$$X \cdot w = (w_1, \ldots, w_k)\, {}^t\bar{X}$$
$$(X \in \mathcal{A} = M(k, \mathbb{C}),\ w = (w_1, \ldots, w_k) \in W = M(2^{(n-2)/2}, k, \mathbb{C})).$$

Let us take e_1 as r in Lemma 6.1 and calculate

$$\mathfrak{h}_{p,q} = \left\{ X \in \mathcal{A} \;\middle|\; {}^T X \rho(r) + \rho(r) X = 0 \right\}.$$

For $\alpha = a + b\mathbf{i} \in \mathbb{C}$, by (6.3) , we have

$$e_1 \alpha = a e_1 + b e_1 \hat{e}_p \hat{f}_q = a e_1 - b \hat{e}_p \hat{f}_q e_1 = \bar{\alpha} e_1.$$

From this relation, we have

$$\begin{aligned} &\left({}^T X \rho(e_1) + \rho(e_1) X\right) \cdot (w_1, \ldots, w_k) \\ &\quad = (e_1 w_1, \ldots, e_1 w_k) X + e_1\left((w_1, \ldots, w_k)\, {}^t\bar{X}\right) \\ &\quad = (e_1 w_1, \ldots, e_1 w_k) X + (e_1 w_1, \ldots, e_1 w_k)\, {}^t X \\ &\quad = (e_1 w_1, \ldots, e_1 w_k)(X + {}^t X). \end{aligned}$$

Thus we have

$${}^T X \rho(e_1) + \rho(e_1) X = 0 \quad \Longleftrightarrow \quad {}^t X + X = 0 \quad (\text{in } M(k; \mathbb{C}))$$

and

$$\mathfrak{h}_{p,q} = \left\{ X \in M(k; \mathbb{C}) \;\middle|\; {}^t X + X = 0 \right\} = \mathfrak{so}(k, \mathbb{C}).$$

6.2. The case: (T, T'), $(\mathbb{K}, \mathbb{K}') = (\mathbb{C}, \mathbb{R})$

In this case, $\{\bar{p}, \bar{q}\} = \{0, 7\}, \{2, 3\}, \{3, 4\}, \{6, 7\}$ and

$$R_{p,q} = M(2^{(n-1)/2}; \mathbb{C}) \supset R^+_{p,q} = M(2^{(n-1)/2}; \mathbb{R}).$$

We may assume that $p \equiv 3 \pmod 4$, $q \equiv 0 \pmod 2$. By (6.2), $\hat{e}_p^2 = -1$. Since p is odd, $\mathbf{i} := \hat{e}_p$ is a central element of $R_{p,q}$ and the center of $R_{p,q}$ is given by $\mathbb{C}1_{2^{(n-1)/2}} = \mathbb{R} + \mathbb{R}\mathbf{i}$. Let W_0 be a simple $R_{p,q}$-module. Then $W_0 = \mathbb{C}^{2^{(n-1)/2}} = \mathbb{R}^{2^{(n-1)/2}} + \mathbf{i}\mathbb{R}^{2^{(n-1)/2}}$ is a direct sum of simple $R^+_{p,q}$-modules. In general a not necessarily simple $R_{p,q}$-module $W = \overbrace{W_0 \oplus \cdots \oplus W_0}^{k}$ is identified with $M(2^{(n-1)/2}, 2k; \mathbb{R})$. Under this identification, $\mathcal{A} = M(2k; \mathbb{R})$ and the action of $X \in \mathcal{A}$ on $w \in W$ is given

$$X \cdot w = (w_1, \ldots, w_{2k})\,{}^tX$$
$$(X \in \mathcal{A} = M(2k; \mathbb{R}), w = (w_1, \ldots, w_{2k}) \in W = M(2^{(n-1)/2}, 2k; \mathbb{R}))$$

Since p is odd, we may take $\mathbf{i}$ as r in Lemma 6.1 and

$$\mathfrak{h}_{p,q} = \left\{ X \in \mathcal{A} \,\middle|\, {}^TX\rho(\mathbf{i}) + \rho(\mathbf{i})X = 0 \right\}.$$

Since

$$\mathbf{i}(u + \mathbf{i}v) = -v + \mathbf{i}u \quad (u, v \in \mathbb{R}^{2^{(n-1)/2}}),$$

the action of $r = \mathbf{i}$ on W coincides with the action of

$$(6.4) \qquad J_k := \overbrace{J_1 \perp \cdots \perp J_1}^{k} \in M(2k; \mathbb{R}) = \mathcal{A}, \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on W. Thus we have

$${}^TX\rho(r) + \rho(r)X = 0 \iff {}^tXJ_k + J_kX = 0.$$

Hence,

$$\mathfrak{h}_{p,q} = \left\{ X \in M(2k; \mathbb{R}) \,\middle|\, {}^tXJ_k + J_kX = 0 \right\} = \mathfrak{sp}(k, \mathbb{R}).$$

6.3. The case: (T,T'), $(\mathbb{K},\mathbb{K}')=(\mathbb{C},\mathbb{H})$

In this case, $\{\bar{p},\bar{q}\}=\{0,3\},\{2,7\},\{3,6\},\{4,7\}$, and

$$R_{p,q}=M(2^{(n-1)/2};\mathbb{C})\supset R^+_{p,q}=M(2^{(n-3)/2};\mathbb{H}).$$

We may assume that $p\equiv 3 \pmod 4$, $q\equiv 0 \pmod 2$. As in the case of $(\mathbb{K},\mathbb{K}')=(\mathbb{C},\mathbb{R})$, the center of $R_{p,q}$ is given by $\mathbb{C}1_{2^{(n-1)/2}}=\mathbb{R}+\mathbb{R}\mathbf{i}$ with $\mathbf{i}:=\hat{e}_p$. We write $\mathbb{H}=\mathbb{C}+\mathbb{C}j$. Then $\alpha j=j\bar{\alpha}$ for $\alpha\in\mathbb{C}$.

Let W_0 be a simple $R_{p,q}$-module. Then $W_0=\mathbb{C}^{2^{(n-2)/2}}$ and W_0 is also simple as an $R^+_{p,q}$-module. We identify W_0 with $\mathbb{H}^{2^{(n-3)/2}}=\mathbb{C}^{2^{(n-3)/2}}+j\mathbb{C}^{2^{(n-3)/2}}$. Then a not necessarily simple $R_{p,q}$-module $W=\overbrace{W_0\oplus\cdots\oplus W_0}^{k}$ is identified with $M(2^{(n-3)/2},2k;\mathbb{C})$. Then, $\mathcal{A}=M(k;\mathbb{H})$ may be considered as a subalgebra of $M(2k;\mathbb{C})$. Since $\mathbb{H}$ is identified with

$$\left\{\begin{pmatrix}\alpha & \bar{\beta}\\ -\beta & \bar{\alpha}\end{pmatrix}\;\middle|\;\alpha,\beta\in\mathbb{C}\right\}=\left\{X\in M(2;\mathbb{C})\;\middle|\;J_1\bar{X}J_1^{-1}=X\right\},$$

we have

$$\mathcal{A}=M(2k;\mathbb{C})^\sigma=\{X\in M(2k;\mathbb{C})\mid\sigma(X)=X\},\quad\sigma(X)=J_k\bar{X}J_k^{-1},$$

where J_k is the skew symmetric matrix given by (6.4). The action of $X\in\mathcal{A}$ on $w\in W$ is then given by

$$X\cdot w=(w_1,\dots,w_k)\,{}^t\bar{X}.$$

Since p is odd, we may take $\mathbf{i}$ as r in Lemma 6.1 and

$$\mathfrak{h}_{p,q}=\left\{X\in\mathcal{A}\;\middle|\;{}^TX\rho(\mathbf{i})+\rho(\mathbf{i})X=0\right\}.$$

Since

$$\mathbf{i}(u+jv)=(u-jv)\sqrt{-1}\quad(u,v\in\mathbb{C}^{2^{(n-3)/2}}),$$

the action of $r=\mathbf{i}$ on W coincides with the action of $\sqrt{-1}H_k\in\mathcal{A}$, where

$$H_k:=\overbrace{\begin{pmatrix}-1&0\\0&1\end{pmatrix}\perp\cdots\perp\begin{pmatrix}-1&0\\0&1\end{pmatrix}}^{k}\in M(2k;\mathbb{C}).$$

Hence we have

$$ {}^{T}X\rho(r) + \rho(r)X = 0 \quad \Longleftrightarrow \quad {}^{t}\bar{X}H_k + H_kX = 0. $$

Therefore

$$ \begin{aligned} \mathfrak{h}_{p,q} &= \left\{X \in \mathcal{A} \,\middle|\, {}^{t}\bar{X}H_k + H_kX = 0\right\} \\ &= \left\{X \in \mathcal{A} \,\middle|\, {}^{t}XH_kJ_k + H_kJ_kX = 0\right\}. \end{aligned} $$

Put

$$ U_k = \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{pmatrix} \perp \cdots \perp \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{pmatrix}}^{k}. $$

Then ${}^{t}U_kU_k = H_kJ_k$. The mapping $X \mapsto Y = U_kXU_k^{-1}$ stabilizes $\mathcal{A} = M(k,\mathbb{H})$ in $M(2k;\mathbb{C})$ and gives an isomorphism

$$ \begin{aligned} \mathfrak{h}_{p,q} &\cong \left\{Y \in \mathcal{A} \,\middle|\, {}^{t}Y + Y = 0\right\} \\ &= \left\{Y \in M(2k;\mathbb{C}) \,\middle|\, {}^{t}Y + Y = 0,\ {}^{t}\bar{Y}J_k + J_kY = 0\right\} = \mathfrak{so}^*(2k). \end{aligned} $$

6.4. The case: (T,T')**,** $(\mathbb{K},\mathbb{K}') = (\mathbb{H},\mathbb{C})$

In this case, $\{\bar{p},\bar{q}\} = \{0,6\},\{2,4\}$ and

$$ R_{p,q} = M(2^{(n-2)/2};\mathbb{H}) \supset R^{+}_{p,q} = M(2^{(n-2)/2};\mathbb{C}). $$

We may assume that $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$. Then, $\hat{e}_p^2 = 1$, $\hat{f}_q^2 = -1$. Hence $\mathbf{i} := \hat{e}_p\hat{f}_q$ satisfies that $\mathbf{i}^2 = -1$ and the center of $R^{+}_{p,q}$ is given by $\mathbb{C}1_{2^{(n-1)/2}} = \mathbb{R} + \mathbb{R}\mathbf{i}$. Let W_0 be a simple $R_{p,q}$-module. Then $W_0 = \mathbb{H}^{2^{(n-2)/2}}$ decomposes to the direct sum of two isomorphic simple $R^{+}_{p,q}$-modules: $W_0 = \mathbb{H}^{2^{(n-2)/2}} = \mathbb{C}^{2^{(n-2)/2}} + \mathbb{C}^{2^{(n-2)/2}}j$. In general, a not necessarily simple $R_{p,q}$-module $W = \overbrace{W_0 \oplus \cdots \oplus W_0}^{k}$ is identified with $M(2^{(n-2)/2}, 2k;\mathbb{C})$. Under this identification, $\mathcal{A} = M(2k,\mathbb{C})$ and the action of $X \in \mathcal{A}$ on $w \in W$ is given by

$$ X \cdot w = (w_1, \ldots, w_k)\,{}^{t}\bar{X}. $$

Since $\mathbf{j} := j1_{2^{(n-2)/2}}$ anticommutes with $\mathbf{i} = \hat{e}_p\hat{f}_q$, $\mathbf{j}$ is an odd element of $R_{p,q}$. Hence we may take $\mathbf{j}$ as r in Lemma 6.1 and

$$\mathfrak{h}_{p,q} = \left\{ X \in \mathcal{A} \;\middle|\; {}^T X\rho(\mathbf{j}) + \rho(\mathbf{j})X = 0 \right\}.$$

Denote by $\mathbf{c}$ the complex conjugation on $W = M(2^{(n-2)/2}, 2k; \mathbb{C})$. Then, since

$$\mathbf{j}(u + vj) = -\bar{v} + \bar{u}j \quad (u, v \in \mathbb{C}^{2^{(n-2)/2}}),$$

the action of $r = \mathbf{j}$ on W coincides with the action of $J_k\mathbf{c}$. Hence we have

$$\begin{aligned} ({}^T X\rho(r) + \rho(r)X) \cdot w &= (\mathbf{j}w)X + \mathbf{j}(w\,{}^t\bar{X}) = \bar{w}\,{}^t J_k X + \bar{w}\,{}^t X\,{}^t J_k \\ &= \bar{w}({}^t J_k X + {}^t X\,{}^t J_k). \end{aligned}$$

Then we have

$${}^T X\rho(r) + \rho(r)X = 0 \quad \Longleftrightarrow \quad {}^t X J_k + J_k X = 0.$$

Therefore

$$\mathfrak{h}_{p,q} = \mathfrak{sp}(k, \mathbb{C}).$$

6.5. The case: $(T, T' \oplus T')$

In this case,

$$\{\bar{p}, \bar{q}\} = \begin{cases} \{0,0\}, \{2,2\}, \{4,4\}, \{6,6\} & (\mathbb{K} = \mathbb{K}' = \mathbb{R}) \\ \{0,4\}, \{2,6\} & (\mathbb{K} = \mathbb{K}' = \mathbb{H}) \end{cases}$$

and

$$R_{p,q} = M(\ell; \mathbb{K}) \supset R^+_{p,q} = M(r; \mathbb{K}) \oplus M(r; \mathbb{K}),$$

where ℓ, r are as in Lemma 2.5.

We have $p \equiv q \pmod 4$. Then

$$c^\pm = \frac{1}{2}\left(1 \pm \hat{e}_p\hat{f}_q\right)$$

are central orthogonal idempotents of $R^+_{p,q}$. For an $R_{p,q}$-module W, put $W^\pm = c^\pm W$. Then $W^\pm$ are the isotypic components of W as $R^+_{p,q}$-module and $W = W^+ \oplus W^-$. If W_0 is a simple $R_{p,q}$-module, the decomposition $W_0 = W_0^+ \oplus W_0^-$ gives two (non-isomorphic) simple $R^+_{p,q}$-modules. Then we have the following decomposition

$$W = \overbrace{W_0 \oplus \cdots \oplus W_0}^{k} = \overbrace{W_0^+ \oplus \cdots \oplus W_0^+}^{k} \oplus \overbrace{W_0^- \oplus \cdots \oplus W_0^-}^{k}$$

and the action of $\mathcal{A} = M(k;\mathbb{K})$ on W is given by

$$(X_1, X_2) \cdot (w_+, w_-) = (w_+ {}^t\bar{X}_1, w_- {}^t\bar{X}_2) \quad ((X_1, X_2) \in M(k;\mathbb{K}) \oplus M(k;\mathbb{K})).$$

Since p is even, we have

$$e_1 c^+ = c^- e_1, \quad e_1 c^- = c^+ e_1.$$

Hence $\rho(e_1)$ induces a linear isomorphism

$$\phi : W_0^+ \longrightarrow W_0^-, \quad \psi : W_0^- \longrightarrow W_0^+.$$

Since $e_1^2 = 1$, we have $\psi = \phi^{-1}$. Hence the action of e_1 on W is given by

$$\begin{aligned} &e_1 \cdot (w_1^+, \dots, w_k^+, w_1^-, \dots, w_k^-) \\ &\quad = (\phi^{-1}(w_1^-), \dots, \phi^{-1}(w_k^-), \phi(w_1^+), \dots, \phi(w_k^+)). \end{aligned}$$

Taking e_1 as r in Lemma 6.1, we have, for $X = (X_1, X_2) \in \mathcal{A}$,

$$\begin{aligned} &({}^T X \rho(r) + \rho(r) X)(w^+, w^-) \\ &\quad = (\phi^{-1}(w^-) X_1 + \phi^{-1}(w^- \, {}^t\bar{X}_2), \phi(w^+) X_2 + \phi(w^+ \, {}^t\bar{X}_1). \end{aligned}$$

Hence, we have

$${}^T X \rho(r) + \rho(r) X = 0 \quad \iff \quad X_2 = -\phi \circ {}^t\bar{X}_1 \circ \phi^{-1}.$$

Therefore

$$\mathfrak{h}_{p,q}(\rho) = \left\{ (X_1, -\phi \circ {}^t\bar{X}_1 \circ \phi^{-1}) \;\middle|\; X_1 \in M(k,\mathbb{K}) \right\} \cong \mathfrak{gl}(k,\mathbb{K}).$$

6.6. The case: $(T \oplus T, T')$

In this case,

$$\{\bar{p}, \bar{q}\} = \begin{cases} \{0,1\}, \{1,2\}, \{4,5\}, \{5,6\} & (\mathbb{K} = \mathbb{K}' = \mathbb{R}) \\ \{1,3\}, \{1,7\}, \{3,5\}, \{5,7\} & (\mathbb{K} = \mathbb{K}' = \mathbb{C}) \\ \{0,5\}, \{1,4\}, \{1,6\}, \{2,5\} & (\mathbb{K} = \mathbb{K}' = \mathbb{H}) \end{cases}$$

and

$$R_{p,q} = M(\ell; \mathbb{K}) \oplus M(\ell; \mathbb{K}) \supset R^+_{p,q} = M(r; \mathbb{K}),$$

where ℓ, r are as in Lemma 2.5. We may assume that $p \equiv 1 \pmod 4$. Then

$$c^{\pm} = \frac{1}{2}(1 \pm \hat{e}_p)$$

are central orthogonal idempotents of $R_{p,q}$. For an $R_{p,q}$-module W, put $W^{\pm} = c^{\pm} W$. Then $W^{\pm}$ are the isotypic components as $R_{p,q}$-module and $W = W^+ \oplus W^-$. Let W_0^+ (resp. W_0^-) be the simple $R_{p,q}$-module contained in W^+ (resp. W^-). Then we have

$$W^+ = \overbrace{W_0^+ \oplus \cdots \oplus W_0^+}^{k_1}, \quad W^- = \overbrace{W_0^- \oplus \cdots \oplus W_0^-}^{k_2}$$

for some k_1, k_2. Since W_0^+ is isomorphic to W_0^- as $R^+_{p,q}$-module, we have $\mathcal{A} = M(k_1 + k_2; \mathbb{K})$. The action of $\mathcal{A} = M(k_1 + k_2; \mathbb{K})$ on W is then given by

$$X \cdot w = w\, {}^t\bar{X}, \quad X \in M(k_1 + k_2; \mathbb{K}).$$

Since p is odd, we may take $\hat{e}_p$ as r in Lemma 6.1. Then we have

$$rc^{\pm} = \pm c^{\pm}$$

Hence, the action of r on W^+ (resp. W^-) is $+1$ (resp. -1) and the action of r on W coincides with the action of

$$I_{k_1,k_2} := \begin{pmatrix} 1_{k_1} & 0 \\ 0 & -1_{k_2} \end{pmatrix} \in \mathcal{A}.$$

Hence we have

$$^{T}X\rho(r)+\rho(r)X={}^{t}\bar{X}I_{k_1,k_2}+I_{k_1,k_2}X.$$

Therefore

$$\begin{aligned}\mathfrak{h}_{p,q}(\rho) &= \left\{X\in M(k_1+k_2;\mathbb{H})\;\middle|\;{}^{t}\bar{X}I_{k_1,k_2}+I_{k_1,k_2}X=0\right\}\\ &\cong \begin{cases}\mathfrak{so}(k_1,k_2) & (\mathbb{K}=\mathbb{R}),\\ \mathfrak{u}(k_1,k_2) & (\mathbb{K}=\mathbb{C}),\\ \mathfrak{sp}(k_1,k_2) & (\mathbb{K}=\mathbb{H}).\end{cases}\end{aligned}$$

6.7. The case: $(T\oplus T, T'\oplus T')$

In this case,

$$\{\bar{p},\bar{q}\}=\begin{cases}\{3,3\},\{7,7\} & (\mathbb{K},\mathbb{K}')=(\mathbb{C},\mathbb{R})\\ \{3,7\} & (\mathbb{K},\mathbb{K}')=(\mathbb{C},\mathbb{H})\end{cases}$$

and

$$R_{p,q}=M(\ell;\mathbb{K})\oplus M(\ell;\mathbb{K})\supset R^{+}_{p,q}=M(r;\mathbb{K}')\oplus M(r;\mathbb{K}'),$$

where ℓ, r are as in Lemma 2.5. Since $p\equiv q\equiv 3 \pmod 4$, we have $\hat{e}_p^2=\hat{f}_q^2=-1$ and

$$c^{\pm}=\frac{1}{2}\left(1\pm\hat{e}_p\hat{f}_q\right)$$

are central orthogonal idempotents of $R_{p,q}$ and of $R^{+}_{p,q}$. The algebra $R_{p,q}$ has two (non-isomorphic) simple modules W_0^{+}, W_0^{-} which satisfy

$$c^{+}W_0^{+}=W_0^{+},\quad c^{+}W_0^{-}=\{0\},\quad c^{-}W_0^{+}=\{0\},\quad c^{-}W_0^{-}=W_0^{-}.$$

Similarly $R^{+}_{p,q}$ has two(non-isomorphic) simple modules W_1^{+}, W_1^{-} which are not isomorphic which satisfy

$$c^{+}W_1^{+}=W_1^{+},\quad c^{+}W_1^{-}=\{0\},\quad c^{-}W_1^{+}=\{0\},\quad c^{-}W_1^{-}=W_1^{-}.$$

In the case where $\mathbb{K}'=\mathbb{R}$, the $R_{p,q}$-simple modules $W_0^{\pm}$ are the direct sum of two copies of the simple $R^{+}_{p,q}$-module $W_1^{\pm}$, and, in the case where $\mathbb{K}'=\mathbb{H}$, $W_0^{\pm}$ are simple as $R^{+}_{p,q}$-module:

$$W_0^{\pm}=\begin{cases}W_1^{\pm}\oplus W_1^{\pm} & (\mathbb{K}'=\mathbb{R}),\\ W_1^{\pm} & (\mathbb{K}'=\mathbb{H}).\end{cases}$$

A (not necessarily simple) $R_{p,q}$-module W is written as

$$W = \overbrace{W_0^+ \oplus \cdots \oplus W_0^+}^{k_1} \oplus \overbrace{W_0^- \oplus \cdots \oplus W_0^-}^{k_2}$$

and we have

$$\mathcal{A} = \begin{cases} M(2k_1;\mathbb{R}) \oplus M(2k_2;\mathbb{R}) & (\mathbb{K}' = \mathbb{R}), \\ M(k_1;\mathbb{H}) \oplus M(k_2;\mathbb{H}) & (\mathbb{K}' = \mathbb{H}). \end{cases}$$

Since p is odd, we may take $\hat{e}_p$ as r in Lemma 6.1. Then, by the calculation as in §6.2 or §6.3, the action of r on W coincides with the action of $(J_{k_1}, J_{k_2}) \in \mathcal{A}$ or $(H_{k_1}, H_{k_2}) \in \mathcal{A}$ according as $\mathbb{K}' = \mathbb{R}$ or $\mathbb{K}' = \mathbb{H}$. Therefore

$$\mathfrak{h}_{p,q}(\rho) = \begin{cases} \mathfrak{sp}(k_1, \mathbb{R}) \oplus \mathfrak{sp}(k_2, \mathbb{R}) & (\mathbb{K}' = \mathbb{R}), \\ \mathfrak{so}^*(2k_1) \oplus \mathfrak{so}^*(2k_2) & (\mathbb{K}' = \mathbb{H}). \end{cases}$$

6.8. The case: $(T \oplus T \oplus T \oplus T, T' \oplus T')$

In this case,

$$\{\bar{p}, \bar{q}\} = \begin{cases} \{1,1\}, \{5,5\} & (\mathbb{K} = \mathbb{K}' = \mathbb{R}) \\ \{1,5\} & (\mathbb{K} = \mathbb{K}' = \mathbb{H}) \end{cases}$$

and

$$\begin{aligned} R_{p,q} &= M(\ell;\mathbb{K}) \oplus M(\ell;\mathbb{K}) \oplus M(\ell;\mathbb{K}) \oplus M(\ell;\mathbb{K}) \\ &\supset R_{p,q}^+ = M(r;\mathbb{K}) \oplus M(r;\mathbb{K}), \end{aligned}$$

where ℓ, r are as in Lemma 2.5. Since $p \equiv q \equiv 1 \mod 4$, we have $\hat{e}_p^2 = \hat{f}_q^2 = 1$ and $\hat{e}_p, \hat{f}_q$ are central elements of $R_{p,q}$. Put

$$c_p^{\pm} = \frac{1}{2}\left(1 \pm \hat{e}_p\right), \quad c_q^{\pm} = \frac{1}{2}\left(1 \pm \hat{f}_q\right).$$

Then the elements

$$c^{++} = c_p^+ c_q^+, \quad c^{+-} = c_p^+ c_q^-, \quad c^{-+} = c_p^- c_q^+, \quad c^{--} = c_p^- c_q^-$$

give central orthogonal idempotents of $R_{p,q}$ and the elements

$$c^+ = c^{++} + c^{--} = \frac{1}{2}\left(1 + \hat{e}_p \hat{f}_q\right), \quad c^- = c^{+-} + c^{+-} = \frac{1}{2}\left(1 - \hat{e}_p \hat{f}_q\right)$$

give central orthogonal idempotents of $R^+_{p,q}$. The algebra $R_{p,q}$ has 4 (non-isomorphic) simple modules $W_0^{\pm\pm}$ corresponding to the idempotents $c^{\pm\pm}$. The simple $R_{p,q}$-modules W_0^{++} and W_0^{--} (resp. W_0^{+-} and W_0^{-+}) are isomorphic and simple as $R^+_{p,q}$-module. A not necessarily simple $R_{p,q}$-module W is written as

$$\begin{aligned} W \quad = \quad & \overbrace{W_0^{++} \oplus \cdots \oplus W_0^{++}}^{k_1} \oplus \overbrace{W_0^{--} \oplus \cdots \oplus W_0^{--}}^{k_2} \oplus \overbrace{W_0^{+-} \oplus \cdots \oplus W_0^{+-}}^{k_3} \\ & \oplus \overbrace{W_0^{-+} \oplus \cdots \oplus W_0^{-+}}^{k_4}. \end{aligned}$$

Since $W_0^{++} \cong W_0^{--}$ and $W_0^{+-} \cong W_0^{-+}$ as $R^+_{p,q}$-module, we have

$$\mathcal{A} = \begin{cases} M(k_1 + k_2; \mathbb{R}) \oplus M(k_3 + k_4; \mathbb{R}) & (\mathbb{K}' = \mathbb{R}), \\ M(k_1 + k_2; \mathbb{H}) \oplus M(k_3 + k_4; \mathbb{H}) & (\mathbb{K}' = \mathbb{H}). \end{cases}$$

Since p is odd, we may take $\hat{e}_p$ as r in Lemma 6.1. Then, by the calculation as in §6.5, the action of r on

$$W^+ := \overbrace{W_0^{++} \oplus \cdots \oplus W_0^{++}}^{k_1} \oplus \overbrace{W_0^{--} \oplus \cdots \oplus W_0^{--}}^{k_2},$$

$$W^- := \overbrace{W_0^{+-} \oplus \cdots \oplus W_0^{+-}}^{k_3} \oplus \overbrace{W_0^{-+} \oplus \cdots \oplus W_0^{-+}}^{k_4}$$

coincides with the action of I_{k_1,k_2}, I_{k_3,k_4}, respectively. Therefore

$$\mathfrak{h}_{p,q}(\rho) = \begin{cases} \mathfrak{so}(k_1, k_2) \oplus \mathfrak{so}(k_3, k_4) & (\mathbb{K}' = \mathbb{R}), \\ \mathfrak{sp}(k_1, k_2) \oplus \mathfrak{sp}(k_3, k_4) & (\mathbb{K}' = \mathbb{H}). \end{cases}$$

7. Proof of Main Theorems

A prehomogeneous vector space (over $\mathbb{C}$) is, by definition, a triple $(\mathbf{G}, \pi, \mathbf{W})$ of a connected linear algebraic group $\mathbf{G}$ defined over $\mathbb{C}$, a finite-dimensional complex vector space $\mathbf{W}$ and a rational representation π of $\mathbf{G}$ on $\mathbf{V}$ with a Zariski-open $\mathbf{G}$-orbit. For basic results on prehomogeneous vector spaces, refer to [17] and [29]. We say that a finite dimensional representation $(\mathfrak{g}, \pi, \mathbf{W})$ of a complex Lie algebra $\mathfrak{g}$ is a prehomogeneous vector space, if $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$ and π is the infinitesimal representation of a prehomogeneous vector space $(\mathbf{G}, \pi, \mathbf{W})$. Here, by abuse of notation, we use the same symbol π to denote the infinitesimal representation.

Denote by τ the representation of $\mathfrak{g}'_{p,q}(\rho)$ on W given in Theorem 3.5 and consider the triple $(\mathfrak{gl}(1,\mathbb{R}) \oplus \mathfrak{g}'_{p,q}(\rho), \Lambda_1 \otimes \tau, W)$, where $\mathfrak{gl}(1,\mathbb{R})$ acts on W as scalar multiplication.

7.1. Case (I)

Let us consider Case (I), namely, the cases

$$(p+q, m) = (3,2), (4,4), (6,8), (10,16).$$

In these cases, by Theorem 3.5 the triple $(\mathfrak{gl}(1,\mathbb{R}) \oplus \mathfrak{g}'_{p,q}(\rho), \Lambda_1 \otimes \tau, W)$ is a real form of the following prehomogeneous vector space:

$$\begin{aligned}
(p+q,m) = (3,2) &\qquad (\mathfrak{gl}(2,\mathbb{C}), \Lambda_1, \mathbb{C}^2);\\
(p+q,m) = (4,4) &\qquad (\mathfrak{gl}(2,\mathbb{C}), \Lambda_1, \mathbb{C}^2) \oplus (\mathfrak{gl}(2,\mathbb{C}), \Lambda_1, \mathbb{C}^2);\\
(p+q) = (6,8) &\qquad (\mathfrak{gl}(4,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}), \Lambda_1 \otimes \Lambda_1, M(4,2;\mathbb{C}));\\
(p+q,m) = (10,16) &\qquad (\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C}), \Lambda_1 \otimes \Lambda_\sharp, \mathbb{C}^{16}))\ (\sharp = e, o).
\end{aligned}$$

These prehomogeneous vector spaces have no relative invariants (see [29, §7]). Thus we have proved the “only if”-part of Theorem 3.1. Since the “if”-part of the theorem was proved in Lemma 4.1, this completes the proof of Theorem 3.1.

7.2. Case (II)

Let us consider Case (II), namely, the cases

$$\begin{aligned}
(p+q,m) \;=\; &(3,4),(4,8),(5,8),(6,16),(7,16),(8,16),\\
&(9,16),(10,32),(11,32).
\end{aligned}$$

By Theorem 3.5 the complexification of $(\mathfrak{gl}(1,\mathbb{R}) \oplus \mathfrak{g}'_{p,q}(\rho), \Lambda_1 \otimes \tau, W)$ is given by

$$\begin{aligned}
(p+q,m) = (3,4) &\qquad (\mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C}), \Lambda_1 \otimes \Lambda_1, M(2;\mathbb{C}));\\
(p+q,m) = (4,8) &\qquad (\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C});\\
&\qquad\quad \Lambda_1 \otimes ((\Lambda_1 \otimes 1 \otimes \Lambda_1) \oplus (1 \otimes \Lambda_1 \otimes \Lambda_1^*)),
\end{aligned}$$

$$M(2;\mathbb{C}) \oplus M(2;\mathbb{C}));$$

$$(p+q,m) = (5,8) \qquad (\mathfrak{sp}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C}), \Lambda_1 \otimes \Lambda_1, M(4,2;\mathbb{C}));$$

$$(p+q,m) = (6,16) \qquad (\mathfrak{gl}(4,\mathbb{C}) \oplus \mathfrak{sp}(2,\mathbb{C}), \Lambda_1 \otimes \Lambda_1, M(4;\mathbb{C})), \text{ or}$$

$$(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C});$$
$$\Lambda_1 \otimes ((\Lambda_1 \otimes 1 \otimes \Lambda_1) \oplus (\Lambda_1^* \otimes \Lambda_1 \otimes 1)),$$
$$M(4,2;\mathbb{C}) \oplus M(4,2;\mathbb{C}));$$

$$(p+q,m) = (7,16) \qquad (\mathfrak{so}(7,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C}), \Lambda \otimes \Lambda_1, M(8,2;\mathbb{C}));$$

$$(p+q,m) = (8,16) \qquad (\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(8,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C}),$$
$$\Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \oplus (\Lambda_1 \otimes \Lambda_1^*)), \mathbb{C}^8 \oplus \mathbb{C}^8);$$

$$(p+q,m) = (9,16) \qquad (\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(9,\mathbb{C}), \Lambda_1 \otimes \Lambda, \mathbb{C}^{16});$$

$$(p+q,m) = (10,32) \qquad (\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C}),$$
$$\Lambda_1 \otimes (\Lambda_e \oplus \Lambda_e), \mathbb{C}^{16} \oplus \mathbb{C}^{16}), \text{ or}$$
$$(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C}),$$
$$\Lambda_1 \otimes (\Lambda_o \oplus \Lambda_o), \mathbb{C}^{16} \oplus \mathbb{C}^{16}), \text{ or}$$
$$(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C}), \Lambda_1 \otimes (\Lambda_e \oplus \Lambda_o), \mathbb{C}^{16} \oplus \mathbb{C}^{16});$$

$$(p+q,m) = (11,32) \qquad (\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(11,\mathbb{C}), \Lambda_1 \otimes \Lambda_\sharp, \mathbb{C}^{32}) \quad (\sharp = e, o).$$

These representations give prehomogeneous vector spaces except $(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C}), \Lambda_1 \otimes (\Lambda_e \oplus \Lambda_o), \mathbb{C}^{16} \oplus \mathbb{C}^{16})$. This can be easily seen for $p+q = 3,4,5,6,8$ and we have:

$(p+q,m)$	CQF	$\mathfrak{g}_{p,q}(\rho) \otimes \mathbb{C}$
$(3,4)$	$(\det w)^2 \ (w \in M(2;\mathbb{C}))$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$
(4,8)	$\det w_1 \det w_2 \ ((w_1,w_2) \in M(2;\mathbb{C})$ $\oplus M(2;\mathbb{C}))$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ $\oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})$
$(5,8)$	$Pf(J_4[w])^2 \ (w \in M(4,2;\mathbb{C}))$	$\mathfrak{so}(8,\mathbb{C})$
$(6,16)$	$\det w \ (w \in M(4;\mathbb{C}))$	$\mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(4,\mathbb{C})$
(6,16)	$\det({}^t w_1 w_2) \ ((w_1,w_2) \in M(4,2;\mathbb{C})$ $\oplus M(4,2;\mathbb{C}))$	$\mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ $\oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})$
$(8,16)$	$q_1(w_1)q_2(w_2) \ ((w_1,w_2) \in \mathbb{C}^8 \oplus \mathbb{C}^8)$	$\mathfrak{so}(8,\mathbb{C}) \oplus \mathfrak{so}(8,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})$

Here "CQF" means "Clifford quartic form", Pf denotes the Pfaffian of an alternating matrix, $J_4 = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$, and q_1, q_2 are quadratic forms in 8 variables.

For $p+q = 7$ the triple is the prehomogeneous vector space (17) in [29, §7, I)] and the Clifford quartic form is given by its irreducible relative invariant, which is the same as the irreducible relative invariant of the space (15) in [29, §7, I)]. Hence we have $\mathfrak{g}_{p,q}(\rho) \otimes \mathbb{C} \cong \mathfrak{so}(8, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

For $p + q = 9$ the triple is the prehomogeneous vector space (19) in [29, §7, I)] and its fundamental relative invariant is a quadratic form. Hence the Clifford quartic form is the square of a quadratic form and we have $\mathfrak{g}_{p,q}(\rho) \otimes \mathbb{C} \cong \mathfrak{so}(16, \mathbb{C})$.

For $p + q = 10$, if the two direct summand is equivalent (namely, ρ is pure over $\mathbb{C}$), then the triple is the prehomogeneous vector space of type (17) on the list in [12, §3]. Hence the Clifford quartic form is the same as the irreducible relative invariant of the space (20) in [29, §7, I)] and we have $\mathfrak{g}_{p,q}(\rho) \otimes \mathbb{C} \cong \mathfrak{so}(10, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. If the two direct summand is inequivalent (namely, ρ is mixed), then we have $\mathfrak{g}_{p,q}(\rho) = \mathfrak{so}(10) \oplus \mathfrak{gl}(1)$ by direct calculation and hence $(\mathfrak{gl}(1) \oplus \mathfrak{g}_{p,q}(\rho), \Lambda_1 \otimes \tau, W)$ is not a prehomogeneous vector space (see [12, Proposition 2.24]).

For $p + q = 11$ the triple is the prehomogeneous vector space (22) in [29, §7, I)] and the Clifford quartic form is given by its irreducible relative invariant, which is the same as the irreducible relative invariant of the space (23) in [29, §7, I)]. Hence we have $\mathfrak{g}_{p,q}(\rho) \otimes \mathbb{C} \cong \mathfrak{so}(12, \mathbb{C})$.

Thus we have proved that, in Case (II), $\mathfrak{g}_{p,q}(\rho)$ is always strictly larger than $\mathfrak{g}'_{p,q}(\rho)$. Hence the converse of Theorem 3.3 also holds. Moreover the triples $(\mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{g}_{p,q}(\rho), \Lambda_1 \otimes \tau, W)$ are prehomogeneous vector spaces except the case where $(p + q, m) = (10, 32)$ and ρ is mixed.

7.3. Case (III)

For Case (III), by Theorem 3.3, we have $\mathfrak{g}_{p,q}(\rho) = \mathfrak{g}'_{p,q}(\rho)$. We determine when $(\mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{g}_{p,q}(\rho), \Lambda_1 \otimes \tau, W)$ is a (real form of a) prehomogeneous vector space.

First we consider the low-dimensional cases $p+q \leq 6$ and $p+q = 8$, where the (half-) spin representations of $\mathfrak{so}(p, q)$ are equivalent to the standard or rather simple tensor representations of classical Lie algebras (see [9, Chapter X, §6.4]).

The case $p + q = 3$

In this case, the representation τ is a real form of the representation

$$(\mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(k, \mathbb{C}), \Lambda_1 \otimes \Lambda \otimes \Lambda_1, M(2, k; \mathbb{C}))$$
$$\cong (\mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{so}(k, \mathbb{C}), \Lambda_1 \otimes \Lambda_1, M(2, k; \mathbb{C})) \quad (k \geq 3).$$

This representation gives a prehomogeneous vector space of type (15) in [29, §7 I)]. The fundamental relative invariant is of degree 4 and it is the Clifford quartic form.

The case $p+q=4$

In this case, the representation τ is a real form of the representation

$$\begin{aligned}(&\mathfrak{gl}(1,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{gl}(k,\mathbb{C}),\\ &\quad\Lambda_1\otimes((\Lambda_1\otimes 1\otimes\Lambda_1)\oplus(1\otimes\Lambda_1\otimes\Lambda_1^*)),\\ &\qquad\qquad M(2,k;\mathbb{C})\oplus M(2,k;\mathbb{C})),\quad (k\geq 3).\end{aligned}$$

This representation gives a prehomogeneous vector space and the fundamental relative invariant is $\det(w_1\,{}^tw_2)$ $(w=w_1,w_2)\in M(2,k;\mathbb{C})\oplus M(2,k;\mathbb{C}))$ and this is the Clifford quartic form.

The case $p+q=5$

In this case, the representation τ is a real form of the representation

$$\begin{aligned}&(\mathfrak{gl}(1,\mathbb{C})\oplus\mathfrak{so}(5,\mathbb{C})\oplus\mathfrak{sp}(k,\mathbb{C}),\Lambda_1\otimes\Lambda\otimes\Lambda_1,\mathbb{C}^{8k})\\ &\quad\cong(\mathfrak{gl}(1,\mathbb{C})\oplus\mathfrak{sp}(2,\mathbb{C})\oplus\mathfrak{sp}(k,\mathbb{C}),\\ &\qquad\quad\Lambda_1\otimes\Lambda_1\otimes\Lambda_1, M(4,2k;\mathbb{C})),\quad (k\geq 2).\end{aligned}$$

By the classification of irreducible prehomogeneous vector spaces ([29, §3, Proposition 21]), this does not give a prehomogeneous vector space.

The case $p+q=6$

In this case, the representation τ is a real form of the representation

$$\begin{aligned}(&\mathfrak{gl}(1,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sp}(k_1,\mathbb{C})\oplus\mathfrak{sp}(k_2,\mathbb{C}),\\ &\quad\Lambda_1\otimes((\Lambda_1\otimes\Lambda_1\otimes 1)\oplus(\Lambda_1^*\otimes 1\otimes\Lambda_1)), M(4,2k_1;\mathbb{C})\oplus M(4,2k_2)).\end{aligned}$$

Note that $k_1+k_2\geq 3$ for Case (III). We may assume that $k_1\geq k_2$. Then the representations above give prehomogeneous vector spaces only when $k_2=0$, equivalently, the representation ρ is pure over $\mathbb{C}$. In this case the prehomogeneous vector space is of type (13) in [29, §7 I)] and its fundamental relative invariant is the Clifford quartic form. By the classification of 3-simple prehomogeneous vector spaces in [16] (or direct calculation using [29, §3, Proposition 21]), one can check that the representation is nonprehomogeneous for $k_2 \geq 1$ (the case where the representation ρ is mixed).

The case $p+q=8$

In this case, the representation τ is a real form of the representation

$$(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{so}(8,\mathbb{C}) \oplus \mathfrak{gl}(k,\mathbb{R}),\\ \Lambda_1 \otimes ((\Lambda_e \otimes \Lambda_1) \oplus (\Lambda_o \otimes \Lambda_1^*)), M(8,k;\mathbb{C}) \oplus M(8,k;\mathbb{C})).$$

Note that $k \geq 2$ for Case (III). Then, by the classification of 2-simple prehomogeneous vector spaces ([13], [14], [15]), these are not prehomogeneous vector spaces.

The case $p+q=7$ **or** $p+q \geq 9$

By [29] (see also [13, Theorem 1.5]), the irreducible prehomogeneous vector spaces (defined over $\mathbb{C}$) containing the spin or half-spin representations of $\mathbf{Spin}(p+q)$ $(p+q=7, \text{ or } \geq 9)$ are given by

$$(7.1)\quad \begin{array}{ll} (\mathbf{Spin}(7) \times \mathbf{GL}(k), \Lambda \otimes \Lambda_1) & (k=1,2,3,5,6,7), \\ (\mathbf{Spin}(9) \times \mathbf{GL}(k), \Lambda \otimes \Lambda_1) & (k=1,15), \\ (\mathbf{Spin}(10) \times \mathbf{GL}(k), \Lambda_\sharp \otimes \Lambda_1) & (k=1,2,3,13,14,15,\ \sharp=e,o), \\ (\mathbf{Spin}(11) \times \mathbf{GL}(k), \Lambda \otimes \Lambda_1) & (k=1,31), \\ (\mathbf{Spin}(12) \times \mathbf{GL}(k), \Lambda_\sharp \otimes \Lambda_1) & (k=1,31,\ \sharp=e,o), \\ (\mathbf{Spin}(14) \times \mathbf{GL}(k), \Lambda_\sharp \otimes \Lambda_1) & (k=1,63,\ \sharp=e,o), \\ (\mathbf{Spin}(p+q) \times \mathbf{GL}(k), \Lambda_\sharp \otimes \Lambda_1) & (k \geq \deg \Lambda_\sharp,\ \sharp=e,o). \end{array}$$

By Theorem 3.5, the complexification of the representation τ of $\mathfrak{gl}(1) \oplus \mathfrak{g}_{p,q}(\rho) = \mathfrak{gl}(1) \oplus \mathfrak{g}'_{p,q}(\rho)$ on W is equivalent to one of the following:

(a) $(\mathfrak{gl}(1) \oplus \mathfrak{so}(p+q) \oplus \mathfrak{so}(k), \Lambda_1 \otimes \Lambda_e \otimes \Lambda_1)$ $(p+q \equiv 1,3 \pmod 8)$,

(b) $(\mathfrak{gl}(1) \oplus \mathfrak{so}(p+q) \oplus \mathfrak{sp}(k), \Lambda_1 \otimes \Lambda_e \otimes \Lambda_1)$ $(p+q \equiv 5,7 \pmod 8)$,

(c) $(\mathfrak{gl}(1) \oplus \mathfrak{so}(p+q) \oplus \mathfrak{gl}(k), \Lambda_1 \otimes (\Lambda_e \otimes \Lambda_1 \oplus \Lambda_o \otimes \Lambda_1^*))$ $(p+q \equiv 0,4 \pmod 8)$,

(d) $(\mathfrak{gl}(1) \oplus \mathfrak{so}(p+q) \oplus \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2), \Lambda_1 \otimes (\Lambda_e \otimes \Lambda_1 \otimes 1 \oplus \Lambda_o \otimes 1 \otimes \Lambda_1^*))$ $(p+q \equiv 2 \pmod 8)$,

(e) $(\mathfrak{gl}(1) \oplus \mathfrak{so}(p+q) \oplus \mathfrak{sp}(k_1) \oplus \mathfrak{sp}(k_2), \Lambda_1 \otimes (\Lambda_e \otimes \Lambda_1 \otimes 1 \oplus \Lambda_o \otimes 1 \otimes \Lambda_1^*))$ $(p+q \equiv 6 \pmod 8)$.

Assume that $(\mathfrak{gl}(1) \oplus \mathfrak{g}'_{p,q}(\rho), \Lambda_1 \otimes \tau, W)$ is a prehomogeneous vector space. Then, every direct summand is also a prehomogeneous vector space and it must coincide with one of the representations in (7.1). Hence, for Cases (a), (b), and (d), we have

$$\begin{cases} k = 1, p+q = 9, 11 & \text{(Case (a))}, \\ k = 1, p+q = 7 & \text{(Case (b))}, \\ k_1, k_2 = 0, 1, p+q = 10 & \text{(Case (d))}. \end{cases}$$

These cases belong to Case (I) or Case (II). Therefore the representation τ belonging to Case (III) does not give a prehomogeneous vector space in Cases (a), (b) and (d). For Case (e), the direct summand of τ does not coincide with any one of prehomogeneous vector spaces in (7.1). Hence no prehomogeneous vector spaces appear in Case (e). Finally let us consider Case (c). If $\deg \Lambda_e = \deg \Lambda_o \leq k$, by [14, Proposition 1.15], the representation does not give a prehomogeneous vector space. If $\deg \Lambda_e = \deg \Lambda_o > k$ and if a direct summand of the representation coincides with one of prehomogeneous vector spaces in (7.1), then we have $p+q = 12$ and $k = 1$. This does not give a prehomogeneous vector space by [12, Proposition 2.32].

Summing up the results in §7.1, §7.2, and §7.3, we obtain Theorem 3.2, (1) and (2).

7.4. Irreducibility of Clifford quartic forms

Let us prove Theorem 3.2, (3). It is sufficient for the proof to consider everything over $\mathbb{C}$. As we have seen in Theorem 3.1, the Clifford quartic form $\tilde{P}(w)$ vanishes identically for $(p+q, m) = (3,2), (4,4), (6,8), (10,16)$. The case by case examination in §7.2 shows that $\tilde{P}(w)$ is a product of two quadratic forms for $(p+q, m) = (3,4), (4,8), (5,8), (8,16), (9,16)$. By the classification of the cases where $\tilde{P}(w)$ is a relative invariant of a prehomogeneous vector space (see Table 1), $\tilde{P}(w)$ is absolutely irreducible for the other prehomogeneous cases.

Now we consider the cases where $\tilde{P}(w)$ is not a relative invariant of any prehomogeneous vector space, namely Case (III) treated in §7.3 and the case where $(p+q, m) = (10, 32)$ and ρ is mixed. We prove that $\tilde{P}(w)$ is absolutely irreducible in these cases. Since $\deg \tilde{P}(w) = 4$, it is sufficient to prove that

(a) $\tilde{P}(w)$ does not have a linear factor, and

(b) $\tilde{P}(w)$ is not a product of two irreducible quadratic forms.

(a) Assume that a linear form $\ell(w)$ divides $\tilde{P}(w)$. Then, $\ell(w)$ is also $\mathfrak{g}_{p,q}(\rho)$-invariant, and the kernel of $\ell(w)$ is an invariant subspace of codimension 1. Recall that $\mathfrak{g}_{p,q}(\rho) = \mathfrak{g}'_{p,q}(\rho)$ for Case (III), and $\mathfrak{g}_{p,q}(\rho) = \mathfrak{so}(p,q) \oplus \mathfrak{gl}(1) \supset \mathfrak{g}'_{p,q}(\rho) = \mathfrak{so}(p,q)$ for the unique non-prehomogeneous case $(p+q, m) = (10, 32)$. By Theorem 3.5, no $\mathfrak{g}_{p,q}(\rho)$-invariant subspaces of codimension 1 appear in these cases. This implies that $\tilde{P}(w)$ does not have a linear factor.

(b) Assume that $\tilde{P}(w) = q_1(w)q_2(w)$ for irreducible quadratic forms $q_1(w) = T_1[w], q_2(w) = T_2[w]$. Here T_1 and T_2 are symmetric matrices of size $m = \dim W$. If $q_1(w)$ and $q_2(w)$ are proportional, then $\tilde{P}(w)$ is a relative invariant of the prehomogeneous vector space $(\mathbf{GL}(1) \times \mathbf{SO}(T_1), \Lambda_1, \mathbf{W})$. Hence we may assume that $q_1(w)$ and $q_2(w)$ are coprime. It is obvious that $\mathfrak{g}_{p,q}(\rho) = \mathfrak{so}(T_1) \cap \mathfrak{so}(T_2)$. First we show that T_1 and T_2 are non-degenerate. If $r = \operatorname{rank} T_1 < m$, then there exists some $g \in GL(m, \mathbb{C})$ such that $({}^t g T_1 g, {}^t g T_2 g)$ is of the form

$$\left(\begin{pmatrix} T_1^{(1)} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} T_2^{(1)} & T_2^{(2)} \\ {}^t T_2^{(2)} & T_2^{(3)} \end{pmatrix} \right),$$

$$T_1^{(1)} \in \mathrm{Sym}(r; \mathbb{C}),\ T_1^{(1)} \neq 0,\ T_2^{(1)} \in \mathrm{Sym}(r; \mathbb{C}),$$

$$T_2^{(2)} \in M(r, m-r; \mathbb{C}),\ T_2^{(3)} \in \mathrm{Sym}(m-r; \mathbb{C}).$$

Since the Hessian of a homaloidal polynomial does not vanish identically, it follows from Theorem 2.14 that $\det T_2^{(3)} \neq 0$. Put $h = \begin{pmatrix} 1_r & -S_2^{(2)} {S_2^{(3)}}^{-1} \\ 0 & 1_{n-r} \end{pmatrix}$. Then $({}^t(gh)T_1(gh), {}^t(gh)T_2(gh))$ is of the form

$$\left(\begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} Y_2 & 0 \\ 0 & Y_3 \end{pmatrix} \right),\ \det Y_1 \neq 0,\ \det Y_3 \neq 0,$$

and the Lie algebra $\mathfrak{g}_{p,q}(\rho)$ is isomorphic to $(\mathfrak{so}(Y_1) \cap \mathfrak{so}(Y_2)) \oplus \mathfrak{so}(Y_3)$. Hence $\mathfrak{g}_{p,q}(\rho)$ has a subalgebra isomorphic to $\mathfrak{so}(Y_3)$ and the restriction to the subalgebra of the representation of $\mathfrak{g}_{p,q}(\rho)$ on W includes the vector representation of $\mathfrak{so}(Y_3)$. In the cases we are now considering, this can not happen. Indeed Theorem 3.5 implies that, if $p+q \geq 5$, this can happen only for the case $(p+q,m)=(8,16)$, which is a case where $\tilde{P}(w)$ is a relative invariant of a prehomogeneous vector space. Hence T_1 and T_2 are non-degenerate.

By differentiating the identity $\tilde{P}(w)=q_1(w)q_2(w)$, we have

$$\begin{aligned}\operatorname{grad}(\tilde{P}(w)) &= q_1(w)\operatorname{grad}(q_2(w))+q_2(w)\operatorname{grad}(q_1(w))\\ &= 2T_1[w]T_2w+2T_2[w]T_1w.\end{aligned}$$

Hence, by the relation $\tilde{P}(\operatorname{grad}(\tilde{P}(w)))=2^8\tilde{P}(w)^3$ proved in Theorem 2.14, we have

$$T_1\Big[T_1[w]T_2w+T_2[w]T_1w\Big]\cdot T_2\Big[T_1[w]T_2w+T_2[w]T_1w\Big]=2^4T_1[w]^3T_2[w]^3.$$

This implies that there exist an integer i $(0\le i\le 3)$ and a non-zero constant c satisfying

$$T_1\Big[T_1[w]T_2w+T_2[w]T_1w\Big]=cT_1[w]^iT_2[w]^{3-i}. \tag{7.2}$$

By expanding the left-hand side of this identity, we have

$$\begin{aligned}&T_1[w]^2(T_2T_1T_2)[w]+T_1[w]T_2[w](T_1^2T_2+T_2T_1^2)[w]+T_2[w]^2T_1^3[w]\\ &\quad=cT_1[w]^iT_2[w]^{3-i}.\end{aligned} \tag{7.3}$$

By exchanging T_1 and T_2, if necessary, we may assume that $i=0,1$. Then, the first term of the left hand side is divided by $T_2[w]$. Since $T_1[w]$ and $T_2[w]$ are coprime, $T_2T_1T_2[w]$ is a constant multiple of $T_2[w]$. Hence there exists a non-zero constant $c_1\neq 0$ such that $T_2T_1T_2=c_1T_2$. Since T_2 is non-degenerate, this implies that $T_1T_2=c_11_m$, and the relation (7.3) can be written as

$$3c_1T_1[w]^2T_2[w]+T_2[w]^2T_1^3[w]=cT_1[w]^iT_2[w]^{3-i}.$$

However, since $T_1[w]^2$ does not divide the right hand side, this is a contradiction. Thus we have proved that $\tilde{P}(w)$ cannot be a product of two irreducible quadratic forms. This completes the proof of Theorem 3.2.

References

[1] Achab, D., Représentations des algèbres de rang 2 et fonctions zêta associées, Ann. Inst. Fourier **45** (1995), 437–451.

[2] Achab, D., Zeta functions of Jordan algebras representations, Ann. Inst. Fourier **45** (1995), 1283–1303.
[3] Bruno, A., On homaloidal polynomials, Michigan Math. J. **55** (2007), 347–354.
[4] Clerc, J.-L., Zeta distributions associated to a representation of a Jordan algebra, Math. Z. **239** (2002), 263–276.
[5] Ciliberto, C., Russo, F. and A. Simis, Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian, Adv. in Math. **218** (2008), 1759–1805.
[6] Dolgachev, I., Polar Cremona transformations, Michigan Math. J. **48** (2000), 191–202.
[7] Etingof, P., Kazhdan, D. and A. Polishchuk, When is the Fourier transform of an elementary function elementary?, Selecta Math. **8** (2002), 27–66.
[8] Faraut, J. and A. Koranyi, *Analysis on symmetric cones*, Oxford University Press, 1994.
[9] Helgason, S., *Differential geometry, Lie groups and symmetric spaces*, Academic Press, 1978.
[10] Igusa, J., On functional equations of complex powers, Invent. Math. **85** (1986), 1–29.
[11] Ishi, H., Self-dual quadratic maps and representations of Jordan algebras, Preprint, 2012.
[12] Kimura, T., A classification of prehomogeneous vector spaces of simple algebraic groups with scalar multiplications, J. Algebra **83** (1983), 72–100.
[13] Kimura, T., Kasai, S., Inuzuka, M. and O. Yasukura, A classification of 2-simple prehomogeneous vector spaces of type I, J. Algebra **114** (1988), 369–400.
[14] Kimura, T., Kasai, S., Taguchi, M. and M. Inuzuka, Some P.V.-equivalence and a classification of 2-simple prehomogeneous vector spaces of type II, Trans. Amer. Math. Soc. **308** (1988), 433–494.
[15] Kimura, T., Kogiso, T. and K. Sugiyama, Relative invariants of 2-simple prehomogeneous vector spaces of type I, J. Algebra **308** (2007), 445–483.
[16] Kimura, T., Ueda, K. and T. Yoshigaki, A classification of 3-simple prehomogeneous vector spaces of nontrivial type, Japan. J. Math. **22** (1996), 159–198.
[17] Kimura, T., *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs **215**, 2003, American Mathematical Society.
[18] Kobayashi, T. and T. Yoshino, Compact Clifford-Klein forms of symmetric spaces – Revisited, Pure. Appl. Math. Quaterly **1** (2005), 591–663.
[19] Letac, G. and H. Massam, Wishart distributions for decomposable graphs, Ann of Statistics **35** (2007), 1278–1323.
[20] Muro, M., A note on the holonomic system of invariant hyperfunctions on a certain prehomogeneous vector space, *Algebraic Analysis* Vol.II, Academic Press, 1989, 493–503.

[21] Ono, T., *Variations on a theme of Euler: quadratic forms, elliptic curves and Hopf map*, 1994, Springer-Verlag.

[22] Porteous, I. R., *Clifford algebras and the classical groups*, Cambridge studies in advanced mathematics 50, 1995, Cambridge University Press.

[23] Sato, F., Zeta functions in several variables associated with prehomogeneous vector spaces I: Functional equations, Tôhoku Math. J. **34** (1982), 437–483.

[24] Sato, F., On functional equations of zeta distributions, Adv. Studies in pure Math. **15** (1989), 465–508.

[25] Sato, F., Quadratic maps and nonprehomogeneous local functional equations, Comment. Math. Univ. St. Pauli **56** (2007), 163–184.

[26] Sato, F., Zeta functions of quadratic mappings, in preparation.

[27] Sato, F. and T. Kogiso, Representations of Clifford algebras and local functional equations, RIMS Kôkyuroku Bessatsu **B36** (2012), 53–66.

[28] Sato, M., Theory of prehomogeneous vector spaces (Notes taken by T. Shintani in Japanese), Sugaku no Ayumi **15** (1970), 85–157.

[29] Sato, M. and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. **65** (1977), 1–155.

[30] Sato, M. and T. Shintani, On zeta functions associated with prehomogenous vector spaces, Ann. of Math. **100** (1974), 131–170.

[31] Suzuki, T., Fourier transforms of the complex powers of relative invariants of prehomogeneous vector spaces, Master Thesis (in Japanese), Nagoya University, 1975.

[32] Tamura, K., L functions associated with automorphic distributions on prehomogeneous vector spaces, Master Thesis (in Japanese), Rikkyo University, 2010.

[33] Tamura, K., Zeta functions attached to automorphic pairs of distributions on prehomogeneous vector spaces, Preprint, 2016.

[34] Varadarajan, V. S., *Supersymmetry for mathematicians: an introduction*, 2004, American Mathematical Society.

[35] Weil, A., Sur certaines groupes d'opérateurs unitaires, Acta. Math. **111** (1964), 143–211; *Collected Papers* III, 1–69.

(Received October 5, 2015)
(Revised June 23, 2016)

Takeyoshi Kogiso
Department of Mathematics
Josai University
1-1 Keyakidai, Sakado
Saitama 350-0295, Japan
E-mail: kogiso@math.josai.ac.jp

Fumihiro Sato
Institute for Mathematics and Computer Science
Tsuda College
2-1-1 Tsuda-machi, Kodaira-shi
Tokyo 187-8577, Japan
E-mail: fsato@tsuda.ac.jp