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Uniformization of Cyclic Quotients of Multiplicative A-singularities

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Abstract. This work is motivated by the canonical model of degenerations of Riemann surfaces. For a quotient space A_{d-1}/Γ of a 'multiplicative' A-singularity A_{d-1} in $\mathbb{C}^{n+1}$ under a certain cyclic group action Γ on A_{d-1}, we *explicitly* construct a small finite abelian subgroup G of $GL(n, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$. A resolution of $\mathbb{C}^n/G$ gives a decomposition of the monodromy (a *higher-dimensional fractional Dehn twist*) of a degeneration $A_{d-1}/\Gamma \to \mathbb{C}$ into subtwists along the exceptional set (it seems that T. Ashikaga's work on resolutions is related to this). Moreover: (1) We give a numerical criterion for a certain subgroup of $GL(n, \mathbb{C})$ to be small. (2) For a certain family of subgroups of $GL(n, \mathbb{C})$, we show that if one subgroup of this family is small, then all subgroups of this family are small (*equi-smallness theorem*).

1. Introduction

Let d be a positive integer and consider the following two complex varieties:

$$V = \{(x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1} \ : \ x_1^2 + x_2^2 + \cdots + x_n^2 = t^d\},$$
$$W = \{(z_1, z_2, \ldots, z_n, t) \in \mathbb{C}^{n+1} \ : \ z_1 z_2 \cdots z_n = t^d\}.$$

We say that V is an *additive A-singularity* and W is a *multiplicative A-singularity.* If $n = 2$, they are isomorphic via $(x_1, x_2) = (z_1 + \mathrm{i}z_2,\, z_1 - \mathrm{i}z_2)$. In contrast if $n \geq 3$, they are not isomorphic: The singular locus of V is isolated, while that of W is not isolated — the former is the origin, while the latter is the union of ${}_nC_2$ hyperplanes $H_{ij} = \{z_i = z_j = t = 0\}$, $1 \leq i < j \leq n$.

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Now let $f : V \to \mathbb{C}$ and $g : W \to \mathbb{C}$ be projections $f(x_1, x_2, \ldots, x_n, t) = t$, $g(z_1, z_2, \ldots, z_n, t) = t$. A smooth fiber $f^{-1}(s)$ (resp. $g^{-1}(s)$), as $s \to 0$, degenerates to the singular fiber $f^{-1}(0)$ (resp. $g^{-1}(0)$). When $n = 2$, the topological monodromy of $f : V \to \mathbb{C}$ (and $g : W \to \mathbb{C}$) is a $(-d)$-*Dehn twist* (Figure 1.1). When $n \geq 3$, the topological monodromy of $f : V \to \mathbb{C}$ is a *generalized Dehn twist*, and is described by using the *double covering method* (see [AGV], p.6). The topological monodromy of $g : W \to \mathbb{C}$ is another generalization of a Dehn twist. *In what follows, we exclusively consider W, and write it as A_{d-1}.*

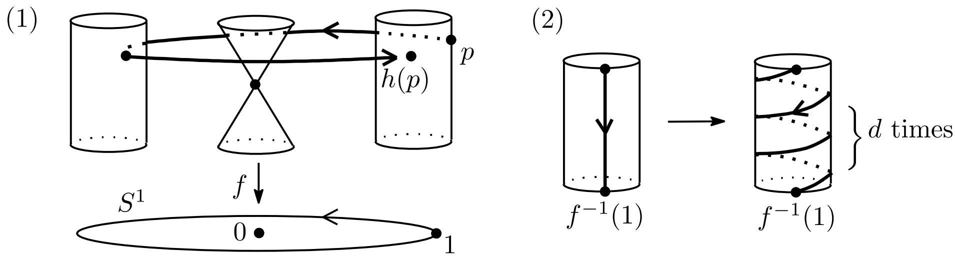


Fig. 1.1. (1) The topological monodromy of $f : V \to \mathbb{C}$. (2) It is a $(-d)$-Dehn twist.

We next introduce a *fractional* Dehn twist. Where a and m $(0 < a < m)$ and b and n $(0 < b < n)$ are two pairs of relatively prime integers, an $\left(\frac{a}{m}, \frac{b}{n}\right)$-*fractional Dehn twist* is a self-homeomorphism of an annulus $[0, 1] \times S^1$ illustrated in Figure 1.2. It is explicitly given by $(t, e^{\mathrm{i}\theta}) \mapsto (t, e^{2\pi\mathrm{i}\{(1-t)a/m - tb/n\}} e^{\mathrm{i}\theta})$.

More generally, where κ is an integer, an $\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-*fractional Dehn twist* is defined as the composite map of a $(+\kappa)$-Dehn twist and an $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist (Figure 1.3). If $\frac{a}{m} + \frac{b}{n} + \kappa > 0$, the $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist appears as the topological monodromy of a degeneration: Set $c := \gcd(m, n)$, $m' := m/c$, $n' := n/c$ and $d := n'a + m'b + m'n'c\kappa$, or $d = m'n'c\left(\frac{a}{m} + \frac{b}{n} + \kappa\right)$. Let Γ be the cyclic group acting on A_{d-1} generated by an automorphism $\gamma : (z, w, t) \in A_{d-1} \mapsto (e^{2\pi\mathrm{i}a/m} z, e^{2\pi\mathrm{i}b/n} w, e^{2\pi\mathrm{i}/m'n'c} t) \in A_{d-1}$. The induced map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$

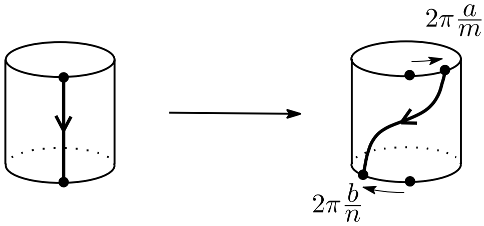


Fig. 1.2. An $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist.

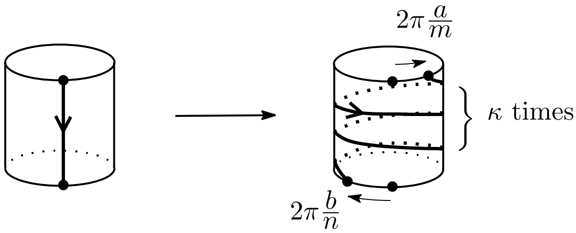


Fig. 1.3. An $\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

by a Γ-invariant map $\Phi : (z, w, t) \in A_{d-1} \mapsto t^{m'n'c} \in \mathbb{C}$ is a degeneration whose topological monodromy is the $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

We point out that $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ arises as a *local model* of a degeneration of Riemann surfaces; recall that a proper surjective holomorphic map $\pi : M \to \Delta$ from a smooth complex surface M to $\Delta := \{s \in \mathbb{C} : |s| < 1\}$ is a *degeneration of Riemann surfaces* (of genus g) if $\pi^{-1}(0)$ is singular and $\pi^{-1}(s)$ for $s \neq 0$ is a Riemann surface (of genus g). Figure 1.4 (1) illustrates an example of a singular fiber, which consists of cores, branches and a trunk. Contracting the branches and the trunk of this singular fiber yields the *canonical model* $\pi' : M' \to \Delta$ of $\pi : M \to \Delta$; the branches and the trunk become *cyclic quotient singularities* of M' (because the contraction of a chain of projective lines yields a cyclic quotient singularity). The singular fiber $(\pi')^{-1}(0)$ is thus as illustrated in Figure 1.4 (2). Let $p \in \pi^{-1}(0)$ be

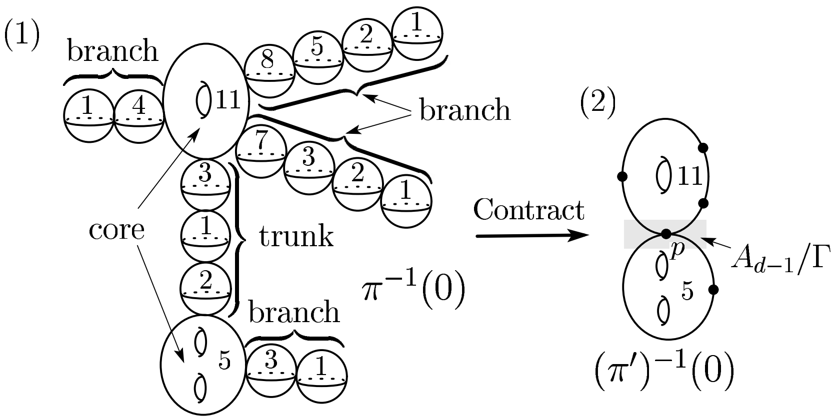


Fig. 1.4. Intersections of irreducible components are *transversal*. The positive integer on an irreducible component denotes the *multiplicity* of that component. The five bold points on $(\pi')^{-1}(0)$ denote the *cyclic quotient singularities* of M'.

the point resulting from the contraction of the trunk. A neighborhood of $p \in M'$ is then isomorphic to A_{d-1}/Γ (for $a/m = 4/11$, $b/n = 3/5$, $\kappa = 0$). Moreover the restriction $\pi'|_{A_{d-1}/\Gamma}$ coincides with $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$, and the topological monodromy of $\pi'|_{A_{d-1}/\Gamma}$ is a $-\left(\frac{4}{11}, \frac{3}{5}, 0\right)$-fractional Dehn twist.

More generally, for any trunk (see Figure 1.5), the same holds: *A neighborhood of its contraction is isomorphic to* A_{d-1}/Γ *(for some* a/m, b/n, κ*), and the local topological monodromy is a* $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$*-fractional Dehn twist, and* A_{d-1}/Γ *is a cyclic quotient singularity.*

In the above, the contraction of a trunk yields A_{d-1}/Γ, which is a cyclic quotient singularity. In fact, for *any* Γ (that is, for *any* a/m, b/n, κ), the quotient A_{d-1}/Γ is a cyclic quotient singularity, that is, $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$ for some cyclic group $G = \langle g \rangle$, where g is of the form $(u, v) \mapsto (e^{2\pi \mathrm{i}/l}u,\ e^{2\pi \mathrm{i}q/l}v)$ where l and q are some relatively prime positive integers. This is the starting point of our present work — we generalize it to the higher-dimensional case in order to apply it to degenerations of complex manifolds.

Let a_i and m_i $(i = 1, 2, \ldots, n)$ be relatively prime integers such that $0 < a_i < m_i$. Set $c := \gcd(m_1, m_2, \ldots, m_n)$ and $m_i' := m_i/c$. Take an

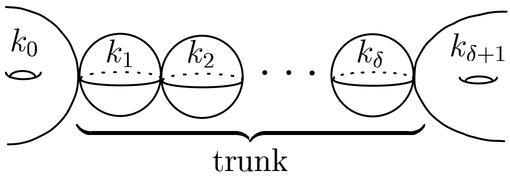


Fig. 1.5. A trunk is a chain of projective lines connecting cores. (k_0, $k_1, \dots, k_{\delta+1}$ are multiplicities.)

integer κ such that $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, and set

$$d := \left(\sum_{i=1}^{n} a_i m_1' \cdots \check{m}_i' \cdots m_n' \right) + m_1' m_2' \cdots m_n' c\kappa,$$

where $\check{m}_i'$ means the omission of m_i'. Or

$$d = m_1' m_2' \cdots m_n' c \Big(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa \Big). \tag{1.1}$$

Now let γ be an automorphism of $\mathbb{C}^{n+1}$ given by

$$\gamma : \ (x_1, \dots, x_n, t) \ \longmapsto \ (e^{2\pi \mathrm{i} a_1/m_1} x_1, \ \dots, e^{2\pi \mathrm{i} a_n/m_n} x_n, \ e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t).$$

Then (1.1) ensures that γ preserves $A_{d-1} := \big\{ (x_1, x_2, \dots, x_n, t) \in \mathbb{C}^{n+1} : x_1 x_2 \cdots x_n = t^d \big\}$. Let Γ be the cyclic group generated by γ. Let $\Phi : A_{d-1} \to \mathbb{C}$ be a Γ-invariant holomorphic map given by $\Phi(x_1, x_2, \cdots, x_n, t) = t^{m_1' m_2' \cdots m_n' c}$, and $\overline{\Phi}$ denote the holomorphic map on A_{d-1}/Γ induced by Φ. The topological monodromy of $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ is called a $-\Big(\frac{a_1}{m_1}, \frac{a_2}{m_2}, \cdots, \frac{a_n}{m_n}, \kappa \Big)$*-fractional Dehn twist*. This will be described in [SaTa].

The present paper shows that the cyclic quotient A_{d-1}/Γ is *uniformized* by a small abelian group. Here a finite subgroup of $GL(n, \mathbb{C})$ is *small* if it contains no pseudo-reflections. The following was originally proved by the second author:

(i) **Uniformization theorem for dimension 2** *There exists a small cyclic group $G \subset GL(2, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$* (Theorem 2.1). (This ensures that the minimal resolution of A_{d-1}/Γ is obtained by the *Hirzebruch-Jung resolution*.)

(ii) *Moreover under this isomorphism,* $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ *corresponds to the map* $\overline{\phi} : \mathbb{C}^2/G \to \mathbb{C}$ *induced by the* G*-invariant map* $\phi : \mathbb{C}^2 \to \mathbb{C}$, $\phi(u,v) = u^n v^m$ (Lemma 2.4).

This is generalized as follows (a diagonal matrix $\begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}$ is denoted by $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$):

MAIN THEOREM A. (i) *There exists a small finite abelian group* $G \subset GL(n, \mathbb{C})$ *such that* $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ (Theorem 6.3), *where* G *is cyclic only when* $n = 2$. *Next set* $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$ *where* $\check{m}'_i$ *means the omission of* m'_i. *Then* l_i *is a positive integer* (Remark 3.1) *and* G *is generated by the diagonal matrices* Q, R_1, R_2, $\ldots$, R_{n-1} *given by*
$Q = \mathrm{diag}\big(e^{2\pi i l_1 a_1/cd},\ e^{2\pi i l_2 a_2/cd}, \ldots, e^{2\pi i l_{n-1} a_{n-1}/cd},\ e^{2\pi i l_n (a_n + m_n \kappa)/cd}\big)$ *and*
$R_i = \mathrm{diag}\big(1, \ldots, 1,\ e^{2\pi i l_i m'_i/d},\ 1, \ldots, 1,\ e^{-2\pi i l_n m'_n/d}\big)$, *where* $e^{2\pi i l_i m'_i/d}$ *lies in the* i*th place* (Corollary 7.13).
(ii) *Under the isomorphism in* (i), $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ *corresponds to the map* $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$ *induced by the* G*-invariant map* $\phi : \mathbb{C}^n \to \mathbb{C}$, $\phi(v_1, v_2, \ldots v_n) = v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$ *where* $k_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c$ (Theorem 6.6 (2)).

REMARK. A resolution of $\mathbb{C}^n/G$ gives a decomposition of the monodromy (a *higher-dimensional fractional Dehn twist*) of $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ into subtwists along the exceptional set. It seems that T. Ashikaga's work on resolutions [Ash], [AsIs] is related to this.

The construction of G in Main Theorem A uses the following diagram of coverings:

(1.2)
$$\begin{array}{ccccc} & & & \widetilde{A}_{d-1} = \mathbb{C}^n & \\ & & \swarrow^{q} & & \searrow^{p} \\ & \mathbb{C}^n & & & A_{d-1}, \\ \swarrow^{r} & & & & \\ \mathbb{C}^n & & & & \end{array}$$

where p, q and r are covering maps given by

- $p(X_1, X_2, \ldots, X_n) = (X_1^d, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n)$ (note: $p : \widetilde{A}_{d-1} \to \mathbb{C}^n$ is the universal covering of A_{d-1}),

- $q(X_1, X_2, \ldots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \ldots, X_n^{m'_n})$,
- $r(u_1, u_2, \ldots, u_n) = (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$, where l_i is the positive integer appearing in Main Theorem A.

We *lift and descend* Γ with respect to the diagram (1.2): Lift Γ to a group $\widetilde{\Gamma}$ (acting on $\widetilde{A}_{d-1}$), and then descend $\widetilde{\Gamma}$ to a group H (acting on $\mathbb{C}^n$), and next descend H to a group G (acting on $\mathbb{C}^n$). Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong \mathbb{C}^n/G$ and $G \subset GL(n, \mathbb{C})$ is a small finite abelian group. We remark that in the case $n = 2$, H is *always* small, so the descent with respect to r is actually unnecessary. Even for $n \geq 3$, it may occur that H is small. Indeed:

Main Theorem B (Theorem 5.14 (2)). *The finite abelian group H is small if and only if* $\gcd(m'_i, m'_j) = 1$ *for any i, j such that $i \neq j$.*

Next let P be the pseudo-reflection subgroup of H, that is, P is generated by all pseudo-reflections of H. Regard κ as a 'parameter', and write $\widetilde{\Gamma}$, H, P as $\widetilde{\Gamma}_\kappa$, H_κ, P_κ. Then the following holds:

Main Theorem C (Lemma 6.7 and Theorem 6.8).

(1) *The pseudo-reflection subgroup P_κ of H_κ does not depend on κ: Let κ_0 denote the least integer among κ in the definition of d, then*

$$P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots .$$

(2) (**Equi-smallness**) *If H_{κ_0} is small, then H_κ is small for any κ, and if H_{κ_0} is not small, then H_κ is not small for any κ.*

2. Uniformization Theorem for Dimension 2

Let a and m $(0 < a < m)$ and b and n $(0 < b < n)$ be two pairs of relatively prime integers, and set $c := \gcd(m, n)$, $m' := \frac{m}{c}$, $n' := \frac{n}{c}$. (Note that m' and n' are integers.) Take an integer κ such that $\frac{a}{m} + \frac{b}{n} + \kappa > 0$, and set $d := an' + bm' + m'n'c\kappa$. Let γ be the automorphism of $\mathbb{C}^3$ given by $\gamma : (z, w, t) \mapsto (e^{2\pi i a/m} z, e^{2\pi i b/n} w, e^{2\pi i/m'n'c} t)$. Then γ preserves $A_{d-1} := \{zw = t^d\}$ in $\mathbb{C}^3$. Let Γ be the cyclic group generated by γ. Then:

THEOREM 2.1 (Uniformization theorem [Tak]). *There exists a small cyclic group* $G \subset GL(2,\mathbb{C})$ *such that* $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$. *Here* G *is explicitly given as follows*: *Let* a^* $(0 < a^* < m)$ *be the integer such that* $aa^* \equiv 1 \bmod m$, *and let* q $(0 < \mathsf{q} < cd)$ *be the integer such that* $\mathsf{q} \equiv \dfrac{a^*d - n'}{m'} \bmod cd$ (the right hand side is indeed an integer; see Remark 2.2 below). *Then* G *is generated by the automorphism* g *of* $\mathbb{C}^2$ *given by* $g : (u,v) \mapsto (e^{2\pi \mathrm{i}/cd}u,\ e^{2\pi \mathrm{i}\mathsf{q}/cd}v)$.

REMARK 2.2. Substituting $d := an' + bm' + m'n'c\kappa$ into $\dfrac{a^*d - n'}{m'}$ yields $\dfrac{aa^* - 1}{m'}n' + a^*b + a^*n'c\kappa$. Here since $aa^* \equiv 1 \bmod m$, we may write $aa^* - 1 = Km\,(= Km'c)$, where K is an integer. Then $\dfrac{a^*d - n'}{m'} = Kn'c + a^*b + a^*n'c\kappa$.

PROOF. Note first that the universal covering $p : \widetilde{A}_{d-1}\ (= \mathbb{C}^2) \to A_{d-1}$ of A_{d-1} is a d-fold covering given by $p(X,Y) = (X^d, Y^d, XY)$. Next let $q : \widetilde{A}_{d-1} \to \mathbb{C}^2$ be an $m'n'$-fold covering given by $q(X,Y) = (X^{m'}, Y^{n'})$, and consider the following diagram:

$$\begin{array}{ccc} & \widetilde{A}_{d-1} = \mathbb{C}^2 & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathbb{C}^2 & & A_{d-1}. \end{array} \tag{2.1}$$

Let $\widetilde{\Gamma}$ be the lift of Γ with respect to p, and G be the descent of $\widetilde{\Gamma}$ with respect to q. Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^2/G$.

We next show that G is generated by g. For $j = 1,2,\ldots,m'n'c$ and $k = 1,2,\ldots,d$, let $\widetilde{\gamma}_{j,k} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by $\widetilde{\gamma}_{j,k} : (X,Y) \mapsto (e^{2\pi \mathrm{i}(ja+km)/md}X,\ e^{2\pi \mathrm{i}\{j(b+n\kappa)-kn\}/nd}Y)$, and $g_{j,k} : \mathbb{C}^2 \to \mathbb{C}^2$ be the automorphism given by $g_{j,k} : (u,v) \mapsto (e^{2\pi \mathrm{i}(ja+km)/cd}u,\ e^{2\pi \mathrm{i}\{j(b+n\kappa)-kn\}/cd}v)$. Then for each $j = 1,2,\ldots,m'n'c$, the set of all lifts of $\gamma^j \in \Gamma$ with respect to p is $\{\widetilde{\gamma}_{j,k} : k = 1,2,\ldots,d\}$, and for any j, k, the descent of $\widetilde{\gamma}_{j,k}$ with respect to q is $g_{j,k}$. Hence $\widetilde{\Gamma}$ and G are explicitly given by

$$\widetilde{\Gamma} = \bigl\{\widetilde{\gamma}_{j,k} : j = 1,2,\ldots,m'n'c,\ k = 1,2,\ldots,d\bigr\},$$
$$G = \bigl\{g_{j,k} : j = 1,2,\ldots,m'n'c,\ k = 1,2,\ldots,d\bigr\}.$$

Therefore G is generated by the following two automorphisms α, β:

$$\alpha : \ (u,v) \longmapsto (e^{2\pi \mathrm{i} a/cd}u, e^{2\pi \mathrm{i}(b+n\kappa)/cd}v),$$
$$\beta : \ (u,v) \longmapsto (e^{2\pi \mathrm{i} m'/d}u, e^{-2\pi \mathrm{i} n'/d}v).$$

Let l $(0 < l < cd)$ be the integer such that $l \equiv \dfrac{1-aa^*}{m} \bmod cd$. Then by Corollary 7.17,

$$\alpha^{a^*}\beta^l = g, \qquad g^a = \alpha, \qquad g^m = \beta.$$

Hence $g \in G$ and G is generated by g.

We next show that G is small. Recall that G is generated by $g : (u,v) \mapsto (e^{2\pi \mathrm{i}/cd}u,\ e^{2\pi \mathrm{i}\mathsf{q}/cd}v)$. Here q and cd are relatively prime (Lemma 2.3 (2) below), so G is small. □

Explicit form of $A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$: Since $\widetilde{\Gamma}$ is the lift of Γ with respect to p, the map p induces an isomorphism $\overline{p} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to A_{d-1}/\Gamma$, and since G is the descent of $\widetilde{\Gamma}$ with respect to q, the map q induces an isomorphism $\overline{q} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to \mathbb{C}^2/G$. The isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$ in the uniformization theorem (Theorem 2.1) is then given by $\Psi := \overline{q} \circ \overline{p}^{\,-1} : \ A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$. We show that this map is explicitly given by

$$\Psi\big([x,y,t]\big) = \big[x^{m'/d},\ y^{n'/d}\big], \tag{2.2}$$

where $[x,y,t] \in A_{d-1}/\Gamma$ and $\big[x^{m'/d},\ y^{n'/d}\big] \in \mathbb{C}^2/G$ denote the images of $(x,y,t) \in A_{d-1}$ and $\big(x^{m'/d},\ y^{n'/d}\big) \in \mathbb{C}^2$ respectively. To see (2.2), first note that since $p(X,Y) = (X^d, Y^d, XY)$, we have $\overline{p}\big([X,Y]\big) = \big[X^d, Y^d, XY\big]$, so $\overline{p}^{\,-1}\big([x,\ y,\ t]\big) = \big[x^{1/d},\ y^{1/d}\big]$. Next since $q(X,\ Y) = (X^{m'},\ Y^{n'})$, we have $\overline{q}\big([x^{1/d},\ y^{1/d}]\big) = \big[x^{m'/d},\ y^{n'/d}\big]$. Hence $\overline{q} \circ \overline{p}^{\,-1}\big([x,y,t]\big) = \big[x^{m'/d},\ y^{n'/d}\big]$.

Supplement Let a^* $(0 < a^* < m)$ be the integer such that $aa^* \equiv 1 \bmod m$, and let q $(0 < \mathsf{q} < cd)$ be the integer such that $\mathsf{q} \equiv \dfrac{a^*d - n'}{m'} \bmod cd$, where the right hand side is indeed an integer (Remark 2.2). Similarly let b^* $(0 < b^* < n)$ be the integer such that $bb^* \equiv 1 \bmod n$, and let r $(0 < \mathsf{r} < cd)$ be the integer such that $\mathsf{r} \equiv \dfrac{b^*d - m'}{n'} \bmod cd$, where the right hand side is an integer as for q.

LEMMA 2.3.

(1) $\mathsf{q}\mathsf{r} \equiv 1 \bmod cd$, *that is,* $\mathsf{r} = \mathsf{q}^*$.

(2) q *and* cd *are relatively prime.*

PROOF. (1): It suffices to show that $\dfrac{a^*d - n'}{m'}\dfrac{b^*d - m'}{n'} \equiv 1 \bmod cd$. Here

$$\frac{a^*d - n'}{m'}\frac{b^*d - m'}{n'} = d\left(\frac{aa^* - 1}{m'}b^* + \frac{bb^* - 1}{n'}a^* + a^*b^*c\kappa\right) + 1.$$

Write $aa^* - 1 = Km\,(= Km'c)$ and $bb^* - 1 = Ln\,(= Ln'c)$. Then

$$\begin{aligned}\frac{a^*d - n'}{m'}\frac{b^*d - m'}{n'} &= cd\big(Kb^* + La^* + a^*b^*\kappa\big) + 1 \\ &\equiv 1 \bmod cd.\end{aligned}$$

(2): Since $\mathsf{q}\mathsf{r} \equiv 1 \bmod cd$, $\mathsf{q}\mathsf{r} = 1 + Mcd$ for some integer M. Then $\mathsf{q}\mathsf{r} - Mcd = 1$. Here $\gcd(\mathsf{q}, cd)$ divides the left hand side, so divides 1, thus $\gcd(\mathsf{q}, cd) = 1$. $\square$

Correspondence between functions Let $\Phi : A_{d-1} \to \mathbb{C}$ be a holomorphic map given by $\Phi(z, w, t) = t^{m'n'c}$. Then Φ is Γ-invariant, so induces a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$. As explained in § Introduction, the topological monodromy of $\overline{\Phi}$ is a $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$ in the uniformization theorem, the holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to a holomorphic map on $\mathbb{C}^2/G$. *This map is explicitly given.* First let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic map defined by $\phi(u, v) = u^n v^m$. Then ϕ is G-invariant. To see this, recall that by Theorem 2.1, the cyclic group G is generated by $g : (u, v) \mapsto (e^{2\pi \mathrm{i}/cd}u,\, e^{2\pi \mathrm{i}\mathsf{q}/cd}v)$, where q $(0 < \mathsf{q} < cd)$ is the integer such that $\mathsf{q} \equiv \dfrac{a^*d - n'}{m'} \bmod cd$. Then

$$\begin{aligned}\phi \circ g(u, v) &= \phi(e^{2\pi \mathrm{i}/cd}u,\, e^{2\pi \mathrm{i}\mathsf{q}/cd}v) = e^{2\pi \mathrm{i}c(n' + m'\mathsf{q})/cd}u^n v^m \\ &= e^{2\pi \mathrm{i}ca^*d/cd}u^n v^m \qquad \text{by } n' + m'\mathsf{q} \equiv a^*d \bmod cd \\ &= e^{2\pi \mathrm{i}a^*}u^n v^m = u^n v^m \\ &= \phi(u, v).\end{aligned}$$

Thus ϕ is G-invariant, so induces a holomorphic map $\overline{\phi} : \mathbb{C}^2/G \to \mathbb{C}$.

LEMMA 2.4 ([Tak]). *Under the isomorphism* $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$ *given by* (2.2), $\overline{\Phi}$ *corresponds to* $\overline{\phi}$, *that is,* $\overline{\Phi} = \overline{\phi} \circ \Psi$.

PROOF. Note first that

$$\begin{aligned}\overline{\phi} \circ \Psi\big([x, y, t]\big) &= \overline{\phi}\left([x^{m'/d},\, y^{n'/d}]\right) \\ &= x^{m'n/d} y^{n'm/d} = (xy)^{m'n'c/d}.\end{aligned}$$

Here $xy = t^d$ (because $(x, y, t) \in A_{d-1}$), so $\overline{\phi} \circ \Psi\big([x, y, t]\big) = t^{m'n'c}$. Thus $\overline{\phi} \circ \Psi\big([x, y, t]\big) = \overline{\Phi}\big([x, y, t]\big)$. $\square$

Where $\mathfrak{r} : R \to A_{d-1}/\Gamma$ is the minimal resolution of A_{d-1}/Γ, the composite map $\pi := \overline{\Phi} \circ \mathfrak{r} : R \to \mathbb{C}$ is a degeneration. As we see immediately, *thanks to* the uniformization theorem, the degeneration $\pi : R \to \mathbb{C}$ is isomorphic to a degeneration which is easy to describe.

Where $\mathfrak{r}' : R' \to \mathbb{C}^2/G$ is the minimal resolution of $\mathbb{C}^2/G$, the composite map $\pi' := \overline{\phi} \circ \mathfrak{r}' : R' \to \mathbb{C}$ is a degeneration. Since A_{d-1}/Γ and $\mathbb{C}^2/G$ are isomorphic (Theorem 2.1), two minimal resolutions $\mathfrak{r} : R \to A_{d-1}/\Gamma$ and $\mathfrak{r}' : R' \to \mathbb{C}^2/G$ are isomorphic, that is, there exists an isomorphism $\widetilde{\Psi} : R \to R'$ that makes the following diagram commute:

(2.3)
$$\begin{array}{ccc} R & \xrightarrow[\cong]{\widetilde{\Psi}} & R' \\ \mathfrak{r}\downarrow & & \downarrow\mathfrak{r}' \\ A_{d-1}/\Gamma & \xrightarrow[\cong]{\Psi} & \mathbb{C}^2/G. \end{array}$$

THEOREM 2.5. *The following diagram commutes*:

(2.4)
$$\begin{array}{ccc} R & \xrightarrow[\cong]{\widetilde{\Psi}} & R' \\ & {}_{\pi}\searrow \quad \swarrow_{\pi'} & \\ & \mathbb{C}. & \end{array}$$

Hence two degenerations $\pi := \overline{\Phi} \circ \mathfrak{r} : R \to \mathbb{C}$ *and* $\pi' := \overline{\phi} \circ \mathfrak{r}' : R' \to \mathbb{C}$ *are isomorphic.*

PROOF. By Lemma 2.4, the following diagram commutes:

$$
\begin{array}{ccc}
A_{d-1}/\Gamma & \xrightarrow[\cong]{\Psi} & \mathbb{C}^2/G \\
& \overline{\Phi} \searrow \quad \swarrow \overline{\phi} & \\
& \mathbb{C}. &
\end{array}
\tag{2.5}
$$

Combining the commutative diagrams (2.3) and (2.5) yields the commutative diagram (2.4). □

The degeneration $\pi' := \overline{\phi} \circ \mathfrak{r}' : R' \to \mathbb{C}$ may be described as follows: Since G is cyclic, $\mathbb{C}^2/G$ has a (unique) cyclic quotient singularity, which is resolved by a chain of projective lines (*Hirzebruch-Jung resolution*). Accordingly the singular fiber $(\pi')^{-1}(0)$ of $\pi' : R' \to \mathbb{C}$ is as illustrated in Figure 2.1 (see also Remark 2.6).

REMARK 2.6. The multiplicities of the singular fiber $(\pi')^{-1}(0)$ in Figure 2.1 is explicitly determined from m, n, a, b, κ. Let a^* and b^* ($0 < a^* < m$, $0 < b^* < n$) be the integers such that $aa^* \equiv 1 \bmod m$ and $bb^* \equiv 1 \bmod n$. Define then two sequences of integers $m_0 > m_1 > \cdots > m_\lambda = 1$ and

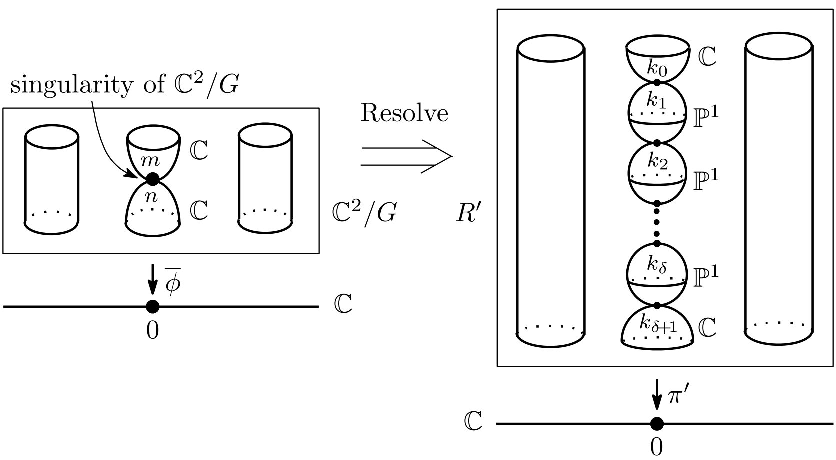
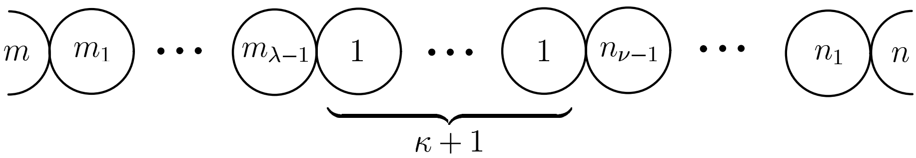


Fig. 2.1. The positive integers $k_0, k_1, \ldots, k_{\delta+1}$ are multiplicities. They are explicitly determined from Γ, more specifically, from m, n, a, b, κ (Remark 2.6).

(1) $\kappa \geq 0$



(2) $\kappa = -1$

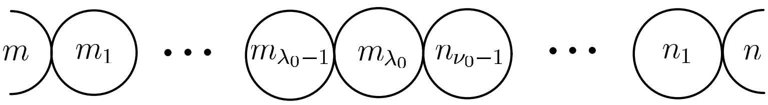


Fig. 2.2. The singular fibers for (1) $\kappa \geq 0$ and (2) $\kappa = -1$. A circle stands for $\mathbb{P}^1$ and a hemicircle for $\mathbb{C}$. (Each intersection is a node.)

$n_0 > n_1 > \cdots > n_\nu = 1$ inductively by the division algorithm with *negative* residues:

$$\begin{cases} m_0 := m, \quad m_1 := a^*, \\ m_{i-1} = s_i m_i - m_{i+1} \quad (0 < m_{i+1} < m_i), \qquad i = 1, 2, \ldots, \lambda - 1, \end{cases}$$

$$\begin{cases} n_0 := n, \quad n_1 := b^*, \\ n_{i-1} = t_i n_i - n_{i+1} \quad (0 < n_{i+1} < n_i), \qquad i = 1, 2, \ldots, \nu - 1. \end{cases}$$

Then:

(i) If $\kappa \geq 0$, then $(\pi')^{-1}(0)$ is as illustrated in (1) of Figure 2.2.

(ii) If $\kappa = -1$, then there exists a unique pair of integers λ_0 and ν_0 ($0 < \lambda_0 < \lambda$, $0 < \nu_0 < \nu$) such that $m_{\lambda_0+1} + n_{\nu_0+1} = m_{\lambda_0} = n_{\lambda_0}$, and $(\pi')^{-1}(0)$ is as illustrated in (2) of Figure 2.2.

3. Lifting and Descent

3.1. Diagram of covering maps

We generalize the uniformization theorem for dimension 2 (Theorem 2.1) to an arbitrary dimension. First let a_i and m_i ($i = 1, 2, \ldots, n$) be relatively prime integers such that $0 < a_i < m_i$. If κ is an integer satisfying $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, then

$$\kappa \geq -n + 1. \tag{3.1}$$

Indeed since $0 < a_i < m_i$, we have $0 < \frac{a_i}{m_i} < 1$ $(i = 1, 2, \ldots, n)$, so $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} < n$, thus $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa < n + \kappa$. Here the left hand side is positive by assumption, so $0 < n + \kappa$, that is, $-n + 1 \leq \kappa$.

Next set $c := \gcd(m_1, m_2, \ldots, m_n)$, $m'_i := m_i/c$ and

$$d := \left(\sum_{i=1}^{n} a_i m'_1 \cdots \check{m}'_i \cdots m'_n\right) + m'_1 m'_2 \cdots m'_n c\kappa, \tag{3.2}$$

where $\check{m}'_i$ means the omission of m'_i. Note that $d > 0$, indeed

$$d = m'_1 m'_2 \cdots m'_n c \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa\right) > 0. \tag{3.3}$$

Rewrite the equation on the left hand side as

$$\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} = \frac{d}{m'_1 m'_2 \cdots m'_n c} - \kappa.$$

Then $e^{2\pi \mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} = e^{2\pi \mathrm{i} d/m'_1 m'_2 \cdots m'_n c}$. Here $e^{-2\pi \mathrm{i}\kappa} = 1$, so

$$e^{2\pi \mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} = e^{2\pi \mathrm{i} d/m'_1 m'_2 \cdots m'_n c}. \tag{3.4}$$

Now let γ be an automorphism of $\mathbb{C}^{n+1}$ given by

$$\gamma : \ (x_1, \ldots, x_n, t) \ \mapsto \ (e^{2\pi \mathrm{i} a_1/m_1} x_1, \ \ldots, e^{2\pi \mathrm{i} a_n/m_n} x_n, \ e^{2\pi \mathrm{i}/m'_1 m'_2 \cdots m'_n c} t).$$

Then γ preserves $A_{d-1} := \{(x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1} \ : \ x_1 x_2 \cdots x_n = t^d\}$, that is, γ maps A_{d-1} to itself. Namely if $x_1 x_2 \cdots x_n = t^d$, then

$$(e^{2\pi \mathrm{i} a_1/m_1} x_1)(e^{2\pi \mathrm{i} a_2/m_2} x_2) \cdots (e^{2\pi \mathrm{i} a_n/m_n} x_n) = (e^{2\pi \mathrm{i}/m'_1 m'_2 \cdots m'_n c} t)^d,$$

that is, $e^{2\pi \mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} x_1 x_2 \cdots x_n = e^{2\pi \mathrm{i} d/m'_1 m'_2 \cdots m'_n c} t^d$. This indeed holds by (3.4). Now let Γ be the cyclic group generated by the automorphism γ of A_{d-1}.

The universal covering $p : \widetilde{A}_{d-1}$ $(= \mathbb{C}^n) \to A_{d-1}$ of A_{d-1} is a d^{n-1}-fold covering given by $p : (X_1, X_2, \ldots, X_n) \mapsto (X_1^d, \ X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n)$.

Consider the following diagram of coverings:

(3.5)
$$\mathbb{C}^n \xleftarrow{\;r\;} \mathbb{C}^n \xleftarrow{\;q\;} \widetilde{A}_{d-1} = \mathbb{C}^n \xrightarrow{\;p\;} A_{d-1},$$

where

- $q : (X_1, X_2, \ldots, X_n) \mapsto (X_1^{m'_1}, X_2^{m'_2}, \ldots, X_n^{m'_n})$ is an $m'_1 m'_2 \cdots m'_n$-fold covering,
- $r : (u_1, u_2, \ldots, u_n) \mapsto (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$ is an $l_1 l_2 \cdots l_n$-fold covering. Here
$$l_i := \frac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)} \qquad (i = 1, 2, \ldots, n),$$
where $\check{m}'_i$ means the omission of m'_i. Note that l_i is a positive integer (see Remark 3.1 below).

Remark 3.1. $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$ is a (positive) integer, because from the definition of lcm, the denominator $\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)$ divides the numerator $m'_1 \cdots \check{m}'_i \cdots m'_n$.

Now let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p, H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and G be the descent of H with respect to the covering r. We will show that G is a small finite abelian group such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ (the uniformization theorem). We begin with some preparation.

3.2. $\widetilde{\Gamma}$, H and G are finite groups

We first show that $\widetilde{\Gamma}$ is a group.

(i) $1 \in \widetilde{\Gamma}$: This is the trivial lift of $1 \in \Gamma$ (that is the identity map of $\widetilde{A}_{d-1}$).

(ii) $\xi \in \widetilde{\Gamma} \Rightarrow \xi^{-1} \in \widetilde{\Gamma}$: If ξ is a lift of $\gamma^j \in \Gamma$, then ξ^{-1} is a lift of $\gamma^{-j} \in \Gamma$.

(iii) $\xi_1, \xi_2 \in \widetilde{\Gamma} \Rightarrow \xi_1\xi_2 \in \widetilde{\Gamma}$: If ξ_1, ξ_2 are lifts of $\gamma^j, \gamma^k \in \Gamma$, then $\xi_1\xi_2$ is a lift of $\gamma^{j+k} \in \Gamma$.

We next show that H is a group as follows (similarly we can show that G is a group):

(i)' $1 \in H$: This is the descent of $1 \in \widetilde{\Gamma}$.

(ii)' $h \in H \Rightarrow h^{-1} \in H$: If h is the descent of $\xi \in \widetilde{\Gamma}$, then h^{-1} is the descent of $\xi^{-1} \in \widetilde{\Gamma}$.

(iii)' $h_1, h_2 \in H \Rightarrow h_1 h_2 \in H$: If h_1, h_2 are the descents of $\xi_1, \xi_2 \in \widetilde{\Gamma}$, then $h_1 h_2$ is the descent of $\xi_1 \xi_2 \in \widetilde{\Gamma}$.

The orders of $\widetilde{\Gamma}$, H and G are determined as follows (below, $|\widetilde{\Gamma}|$, $|H|$ and $|G|$ denote the orders):

Order of $\widetilde{\Gamma}$: Since $\widetilde{\Gamma}$ is the lift of Γ with respect to the d^{n-1}-fold covering p, we have $|\widetilde{\Gamma}| = d^{n-1}|\Gamma|$. Here $|\Gamma| = m'_1 m'_2 \cdots m'_n c$, so $|\widetilde{\Gamma}| = m'_1 m'_2 \cdots m'_n c d^{n-1}$.

Order of H: Since H is the descent of $\widetilde{\Gamma}$ (or $\widetilde{\Gamma}$ is the lift of H) with respect to the $m'_1 m'_2 \cdots m'_n$-fold covering q, we have $|\widetilde{\Gamma}| = m'_1 m'_2 \cdots m'_n |H|$. Here $|\widetilde{\Gamma}| = m'_1 m'_2 \cdots m'_n c d^{n-1}$ so $|H| = c d^{n-1}$.

Order of G: Since G is the descent of H (or H is the lift of G) with respect to the $l_1 l_2 \cdots l_n$-fold covering r, we have $|H| = l_1 l_2 \cdots l_n |G|$. Here $|H| = c d^{n-1}$, so $|G| = \dfrac{c d^{n-1}}{l_1 l_2 \cdots l_n}$. (This is indeed an integer. See Remark 3.3 below.)

The results obtained in this section are summarized as follows:

PROPOSITION 3.2. *Let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and let G be the descent of H with respect to the covering r. Then*:

(1) *The lift $\widetilde{\Gamma}$ of Γ is a finite group of order $m'_1 m'_2 \cdots m'_n c d^{n-1}$.* (In fact, $\widetilde{\Gamma}$ is abelian. See Lemma 4.7.)

(2) *The descent H of $\widetilde{\Gamma}$ is a finite group of order $c d^{n-1}$.* (In fact, H is abelian. See Lemma 4.8 (3).)

(3) *The descent G of H is a finite group of order $\dfrac{c d^{n-1}}{l_1 l_2 \cdots l_n}$.* (In fact, G is abelian. See Lemma 6.1 (C).)

Remark 3.3. The fact that $|G| = \dfrac{cd^{n-1}}{l_1 l_2 \cdots l_n}$ is an integer is reconfirmed as follows (we show this only for $n=3$): Using

$$\begin{cases} d = m_1' m_2' m_3' c \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa \right) \quad \text{(see (3.3))}, \\ l_1 := \dfrac{m_2' m_3'}{\mathrm{lcm}(m_2', m_3')}, \; l_2 := \dfrac{m_1' m_3'}{\mathrm{lcm}(m_1', m_3')}, \; l_3 := \dfrac{m_1' m_2'}{\mathrm{lcm}(m_1', m_2')}, \end{cases}$$

rewrite $|G| = \dfrac{cd^2}{l_1 l_2 l_3}$ as

$$\begin{aligned} |G| &= c \Big\{ \prod_{i \neq j} \mathrm{lcm}(m_i', m_j') \Big\} c^2 \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa \right)^2 \\ &= c \prod_{i \neq j} \mathrm{lcm}(m_i', m_j') \left\{ (c\kappa)^2 + \sum_{i=1}^{3} \left(\frac{2a_i c\kappa}{m_i'} + \frac{a_i^2}{(m_i')^2} \right) + \sum_{i \neq j} \frac{2a_i a_j}{m_i' m_j'} \right\}. \end{aligned}$$

Here $\prod_{i \neq j} \mathrm{lcm}(m_i', m_j') = \mathrm{lcm}(m_1', m_2')\mathrm{lcm}(m_1', m_3')\mathrm{lcm}(m_2', m_3')$ is divisible by m_i', $(m_i')^2$, $m_i' m_j'$, so the last expression is indeed an integer.

4. Determination of H

We keep the notation concerning the diagram (3.5). Moreover we adopt the following notation: For $j = 1, 2, \ldots, m_1' m_2' \cdots m_n' c$,

- $\mathrm{Lift}^{(j)}$: The set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering map $p : \widetilde{A}_{d-1} \to A_{d-1}$.
- $q_* \big(\mathrm{Lift}^{(j)} \big)$: The descent of $\mathrm{Lift}^{(j)}$ with respect to the covering map $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.
- $r_* \circ q_* \big(\mathrm{Lift}^{(j)} \big)$: The descent of $q_* \big(\mathrm{Lift}^{(j)} \big)$ with respect to the covering map $r : \mathbb{C}^n \to \mathbb{C}^n$.

Then

- $\widetilde{\Gamma} = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} \mathrm{Lift}^{(j)}$ is the lift of Γ with respect to the covering map $p : \widetilde{A}_{d-1} \to A_{d-1}$.

- $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\widetilde{\Gamma}$ with respect to the covering map $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.
- $G = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of H with respect to the covering map $r : \mathbb{C}^n \to \mathbb{C}^n$.

Actually, $\widetilde{\Gamma} = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$ is a disjoint union. Namely, if $j \neq k$, then $\mathrm{Lift}^{(j)} \cap \mathrm{Lift}^{(k)} = \emptyset$. On the other hand, $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$ and $G = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ are *not* disjoint unions. In fact, a descent of an element of $\mathrm{Lift}^{(j)}$ may coincide with that of an element of $\mathrm{Lift}^{(k)}$ $(j \neq k)$. In this case, $q_*\big(\mathrm{Lift}^{(j)}\big) \cap q_*\big(\mathrm{Lift}^{(k)}\big) \neq \emptyset$, and moreover, $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big) \cap r_* \circ q_*\big(\mathrm{Lift}^{(k)}\big) \neq \emptyset$.

In what follows, we write $\widetilde{\Gamma}$ as a disjoint union: $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$.

4.1. The lifts of each element of Γ

We next determine the set $\mathrm{Lift}^{(j)}$ of all lifts of $\gamma^j \in \Gamma$ with respect to the covering p. For $j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c$, we first define a set $\Lambda^{(j)}$ of n-tuples of integers as follows:

$$\Lambda^{(j)} := \Big\{(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \,:\, 0 \leq p_i < d,\ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}\Big\}. \tag{4.1}$$

LEMMA 4.1. *The number of elements of* $\Lambda^{(j)}$ *is* d^{n-1}.

PROOF. Setting $\Xi := \{(p_1, p_2, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1} \,:\, 0 \leq p_i < d\}$, consider a map $\varphi : \Lambda^{(j)} \to \Xi$ given by $(p_1, p_2, \ldots, p_{n-1}, p_n) \longmapsto (p_1, p_2, \ldots, p_{n-1})$. Here Ξ consists of d^{n-1} elements, thus it suffices to show that φ is bijective.

Surjectivity: We show that for any $(p_1, p_2, \ldots, p_{n-1}) \in \Xi$, the inverse image $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is not empty. Set $N := j\kappa - \sum_{i=1}^{n-1} p_i$ and let p_n $(0 \leq p_n < d)$ be the integer such that $p_n \equiv N \bmod d$. Then $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$. Moreover $\varphi(p_1, p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_{n-1})$, thus $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is not empty.

Injectivity: We show that for any $(p_1, p_2, \ldots, p_{n-1}) \in \Xi$, the inverse image $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is a single point. Note that $(p_1, p_2, \ldots, p_n)$ is contained in $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ precisely when p_n satisfies $\sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}$, that is, $p_n \equiv j\kappa - \sum_{i=1}^{n-1} p_i \bmod d$. Such an integer p_n $(0 \leq p_n < d)$ is unique, so $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is a single point. $\square$

Let $\Lambda^{(j)}$ be the set given by (4.1). For each $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ by

$$(X_1, X_2, \ldots, X_n) \longmapsto \left(e^{2\pi \mathrm{i}(ja_1 + m_1 p_1)/m_1 d} X_1,\ e^{2\pi \mathrm{i}(ja_2 + m_2 p_2)/m_2 d} X_2, \ldots, e^{2\pi \mathrm{i}(ja_n + m_n p_n)/m_n d} X_n\right).$$

Lemma 4.2. *For any* $(p_1, p_2, \ldots, p_n), (p'_1, p'_2, \ldots, p'_n) \in \Lambda^{(j)}$, *the following hold*:

(1) *For* $i = 1, 2, \ldots, n$,

$$e^{2\pi \mathrm{i}(ja_i + m_i p_i)/m_i d} = e^{2\pi \mathrm{i}(ja_i + m_i p'_i)/m_i d} e^{2\pi \mathrm{i}(p_i - p'_i)/d}.$$

(2) *If* $(p_1, p_2, \ldots, p_n) \neq (p'_1, p'_2, \ldots, p'_n)$, *say* $p_i \neq p'_i$ *for some* i, *then* $e^{2\pi \mathrm{i}(ja_i + m_i p_i)/m_i d} \neq e^{2\pi \mathrm{i}(ja_i + m_i p'_i)/m_i d}$.

(3) *If* $(p_1, p_2, \ldots, p_n) \neq (p'_1, p'_2, \ldots, p'_n)$, *then* $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \neq \widetilde{\gamma}^{(j)}_{p'_1, p'_2, \ldots, p'_n}$.

Proof. (1): From $\dfrac{ja_i + m_i p_i}{m_i d} - \dfrac{ja_i + m_i p'_i}{m_i d} = \dfrac{p_i - p'_i}{d}$, we have $\dfrac{ja_i + m_i p_i}{m_i d} = \dfrac{ja_i + m_i p'_i}{m_i d} + \dfrac{p_i - p'_i}{d}$, which yields the equation in assertion.

(2): Since $0 \leq p_i < d$ and $0 \leq p'_i < d$, $p_i \neq p'_i$ implies $p_i \not\equiv p'_i \bmod d$, accordingly $\dfrac{p_i - p'_i}{d} \not\equiv 0 \bmod \mathbb{Z}$. Hence $e^{2\pi \mathrm{i}(p_i - p'_i)/d} \neq 1$ in (1), implying that $e^{2\pi \mathrm{i}(ja_i + m_i p_i)/m_i d} \neq e^{2\pi \mathrm{i}(ja_i + m_i p'_i)/m_i d}$.

(3): This follows from (2). □

We next show the following:

COROLLARY 4.3. *The number of elements of* $\{\tilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\}$ *is* d^{n-1}.

PROOF. By (3) of Lemma 4.2, the number of elements in the set $\{\tilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\}$ coincides with that of $\Lambda^{(j)}$, and by Lemma 4.1, it is d^{n-1}. □

Recall that $d = m'_1 m'_2 \cdots m'_n c \left(\sum_{i=1}^{n} \dfrac{a_i}{m_i} + \kappa \right)$ (see (3.3)), so

$$\sum_{i=1}^{n} \frac{a_i}{m_i} = \frac{d}{m'_1 m'_2 \cdots m'_n c} - \kappa. \tag{4.2}$$

LEMMA 4.4. *For any* $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$,

$$\sum_{i=1}^{n} \frac{ja_i + m_i p_i}{m_i d} \equiv \frac{j}{m'_1 m'_2 \cdots m'_n c} \bmod \mathbb{Z}.$$

PROOF. Using (4.2), the left hand side is rewritten as

$$\sum_{i=1}^{n} \frac{ja_i + m_i p_i}{m_i d} = \frac{j}{m'_1 m'_2 \cdots m'_n c} - \frac{j\kappa}{d} + \sum_{i=1}^{n} \frac{p_i}{d}.$$

Here $\sum_{i=1}^{n} \dfrac{p_i}{d} \equiv \dfrac{j\kappa}{d} \bmod \mathbb{Z}$ (because $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$), so $\sum_{i=1}^{n} \dfrac{ja_i + m_i p_i}{m_i d} \equiv \dfrac{j}{m'_1 m'_2 \cdots m'_n c} \bmod \mathbb{Z}$. □

COROLLARY 4.5. *For each* j, *let* $\mathrm{Lift}^{(j)}$ *be the set of all lifts of* $\gamma^j \in \Gamma$ *with respect to the covering* $p : \widetilde{A}_{d-1} \to A_{d-1}$. *Then the following hold*:

(1) *The number of elements of* $\mathrm{Lift}^{(j)}$ *is* d^{n-1}.

(2) *For any* $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \in \mathrm{Lift}^{(j)}$.

(3) $\mathrm{Lift}^{(j)} = \bigl\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\bigr\}$.

Proof. (1): Since the covering p is d^{n-1}-fold, for each j, $\gamma^j \in \Gamma$ has d^{n-1} lifts, so $\mathrm{Lift}^{(j)}$ consists of d^{n-1} elements.

(2): It suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{A}_{d-1} & \xrightarrow{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}} & \widetilde{A}_{d-1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ A_{d-1} & \xrightarrow{\gamma^j} & A_{d-1}. \end{array}$$

For $(X_1, \ldots, X_n) \in \widetilde{A}_{d-1}$,

$$\begin{aligned} &p \circ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}(X_1, \ldots, X_n) \\ &= p\bigl(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/m_1d}X_1,\, \ldots,\, e^{2\pi \mathrm{i}(ja_n+m_np_n)/m_nd}X_n\bigr) \\ &= \bigl(e^{2\pi \mathrm{i}ja_1/m_1}X_1,\, \ldots,\, e^{2\pi \mathrm{i}ja_n/m_n}X_n,\, e^{2\pi \mathrm{i}\sum_{i=1}^{n}\{(ja_i+m_ip_i)/m_id\}}X_1X_2\cdots X_n\bigr). \end{aligned}$$

Here $e^{2\pi \mathrm{i}\sum_{i=1}^{n}\{(ja_i+m_ip_i)/m_id\}} = e^{2\pi \mathrm{i}j/m'_1m'_2\cdots m'_nc}$ by Lemma 4.4, thus

$$\begin{aligned} &p \circ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}(X_1, \ldots, X_n) \\ &\qquad = \bigl(e^{2\pi \mathrm{i}ja_1/m_1}X_1,\, \ldots,\, e^{2\pi \mathrm{i}ja_n/m_n}X_n,\, e^{2\pi \mathrm{i}j/m'_1m'_2\cdots m'_nc}X_1X_2\cdots X_n\bigr). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\gamma^j \circ p(X_1, \ldots, X_n) \\ &\qquad = \bigl(e^{2\pi \mathrm{i}ja_1/m_1}X_1,\, \ldots,\, e^{2\pi \mathrm{i}ja_n/m_n}X_n,\, e^{2\pi \mathrm{i}j/m'_1m'_2\cdots m'_nc}X_1X_2\cdots X_n\bigr). \end{aligned}$$

Hence $p \circ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} = \gamma^j \circ p$, confirming the assertion.

(3): From (2), $\bigl\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\bigr\} \subset \mathrm{Lift}^{(j)}$. Here "$\subset$" is "$=$", because the numbers of elements of both sets are equal, indeed they consist of d^{n-1} elements ((1) and Corollary 4.3). □

The following will be used in later discussion:

COROLLARY 4.6. *$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ descends to γ^j. Moreover if $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ is of the form $(X_1,\dots,X_i,\dots,X_n)\mapsto(X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n)$, then it descends to γ^j of the form*

$$(x_1,\dots,x_n,t)\longmapsto(x_1,\dots,e^{2\pi\mathrm{i}ja_i/m_i}x_i,\dots,x_n,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}t).$$

PROOF. The first statement follows from Corollary 4.5 (3). The second one is restated as $p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}=\gamma^j\circ p$, which is confirmed as follows:

$$\begin{aligned}
p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}&(X_1,\dots,X_i,\dots,X_n)\\
&=p(X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n)\\
&=(X_1^d,\dots,e^{2\pi\mathrm{i}ja_i/m_i}X_i^d,\dots,X_n^d,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_1X_2\cdots X_n)\\
&=\gamma^j\circ p(X_1,\dots,X_i,\dots,X_n).\ \square
\end{aligned}$$

By Corollary 4.5 (3), $\mathrm{Lift}^{(j)}=\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}\ :\ (p_1,p_2,\dots,p_n)\in\Lambda^{(j)}\right\}$. Since $\widetilde{\Gamma}=\coprod_{j=1}^{m'_1m'_2\cdots m'_nc}\mathrm{Lift}^{(j)}$ (disjoint union), we have

$$\widetilde{\Gamma}=\coprod_{j=1}^{m'_1m'_2\cdots m'_nc}\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}\ :\ (p_1,p_2,\dots,p_n)\in\Lambda^{(j)}\right\}.$$

Or

$$\widetilde{\Gamma}=\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}\ :\ (p_1,p_2,\dots,p_n)\in\Lambda^{(j)},\ j=1,2,\dots,m'_1m'_2\cdots m'_nc\right\}.$$

We thus obtain:

LEMMA 4.7. *The lift $\widetilde{\Gamma}$ of Γ consists of the automorphisms $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}:\widetilde{A}_{d-1}\to\widetilde{A}_{d-1}$ given by*

$$(X_1,\dots,X_n)\ \longmapsto\ \left(e^{2\pi\mathrm{i}(ja_1+m_1p_1)/m_1d}X_1,\dots,e^{2\pi\mathrm{i}(ja_n+m_np_n)/m_nd}X_n\right),$$

where $(p_1,p_2,\dots,p_n)\in\Lambda^{(j)}$ and $j=1,2,\dots m'_1m'_2\cdots m'_nc$. (In particular, any two elements of $\widetilde{\Gamma}$ commute, so $\widetilde{\Gamma}$ is abelian.)

4.2. Determination of H

Recall that $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$, where $\mathrm{Lift}^{(j)}$ denotes the set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$. Accordingly $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$, where $q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\mathrm{Lift}^{(j)}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$. We determine $q_*\big(\mathrm{Lift}^{(j)}\big)$. To that end, for $j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c$ and $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $h^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$(u_1, \ldots, u_n) \longmapsto \big(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/cd} u_1, \ldots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/cd} u_n\big).$$

LEMMA 4.8.

(1) *$h^{(j)}_{p_1,p_2,\ldots,p_n}$ is the descent of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.*

(2) $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$.

(3) $H = \Big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}, j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c\Big\}$.
(Thus any two elements of H commute, that is, H is abelian.)

PROOF. (1): Indeed since $\big(e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id}\big)^{m'_i} = e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}$, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{A}_{d-1} & \xrightarrow{\ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}\ } & \widetilde{A}_{d-1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{C}^n & \xrightarrow{\ h^{(j)}_{p_1,p_2,\ldots,p_n}\ } & \mathbb{C}^n. \end{array}$$

(2): By Corollary 4.5 (3), $\mathrm{Lift}^{(j)} = \big\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$, accordingly by (1), $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$.

(3): This follows from $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$ and (2). □

5. The Pseudo-Reflection Subgroup of H

5.1. Cyclic subgroups Γ_i of Γ and $\widetilde{\Gamma}_i$ of $\widetilde{\Gamma}$

Let $\gamma : A_{d-1} \to A_{d-1}$ be the automorphism given by

$$(5.1) \qquad \gamma : (x_1, \ldots, x_n, t) \longmapsto (e^{2\pi \mathrm{i} a_1/m_1} x_1, \ldots, e^{2\pi \mathrm{i} a_n/m_n} x_n, e^{2\pi \mathrm{i}/m'_1 m'_2 \cdots m'_n c} t).$$

(The order of γ is $m'_1 m'_2 \cdots m'_n c$.) Consider the cyclic group Γ generated by γ:

$$\Gamma = \{\gamma^j : j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c\}.$$

Let Γ_i $(i = 1, 2, \ldots, n)$ be the subgroup of Γ consisting of automorphisms of the form

$$(x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto (x_1, \ldots, e^{2\pi \mathrm{i} j a_i/m_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i} j/m'_1 m'_2 \cdots m'_n c} t),$$

that is,

$$(\sharp) \qquad e^{2\pi \mathrm{i} j a_k/m_k} = 1 \quad (k = 1, 2, \ldots, \check{i}, \ldots, n).$$

LEMMA 5.1. *For $j \in \mathbb{Z}$,*

$$\gamma^j \in \Gamma_i \iff j \textit{ is a multiple of } \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c.$$

PROOF. $\Longrightarrow$: If $\gamma^j \in \Gamma_i$, then from $(\sharp)$, $j a_k$ is divisible by m_k $(k = 1, 2, \ldots, \check{i}, \ldots, n)$. Here a_k and m_k are relatively prime, so j is divisible by m_k $(k = 1, 2, \ldots, \check{i}, \ldots, n)$. In particular, j is a multiple of $\mathrm{lcm}(m_1, \ldots, \check{m}_i, \ldots m_n) = \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots m'_n)c$.

$\Longleftarrow$: If j is a multiple of $\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots m'_n)c$, then j is divisible by m_k $(k = 1, 2, \ldots, \check{i}, \ldots, n)$, so $\dfrac{j a_k}{m_k}$ is an integer. Thus $e^{2\pi \mathrm{i} k a_k/m_k} = 1$ $(k = 1, 2, \ldots, \check{i}, \ldots, n)$, so $\gamma^j \in \Gamma_i$. $\square$

From Lemma 5.1, the following holds:

COROLLARY 5.2. *Γ_i is generated by $\gamma_i := \gamma^{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c}$.*

This element is explicitly given by

$$\gamma_i : (x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto \Big(x_1, \ldots, e^{2\pi \mathrm{i} a_i \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_i} x_i, \ldots, x_n, \\ e^{2\pi \mathrm{i} \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_1 m'_2 \cdots m'_n} t\Big).$$

Here $e^{2\pi \mathrm{i} \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_1 m'_2 \cdots m'_n} = e^{2\pi \mathrm{i}/m'_i l_i}$, because

$$\frac{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}{m'_1 m'_2 \cdots m'_n} = \frac{1}{m'_i} \frac{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}{m'_1 \cdots \check{m}'_i \cdots m'_n} = \frac{1}{m'_i l_i}.$$

Thus

$$\gamma_i : (x_1, \ldots, x_i, \ldots, x_n, t) \\ \longmapsto \Big(x_1, \ldots, e^{2\pi \mathrm{i} a_i \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i}/m'_i l_i} t\Big).$$

Set $L_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)$, then

$$\gamma_i : (x_1, \ldots, x_n, t) \longmapsto \big(x_1, \ldots, e^{2\pi \mathrm{i} a_i L_i/m'_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i}/m'_i l_i} t\big).$$

For $k \in \mathbb{Z}$,

$$\gamma_i^k : (x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto \big(x_1, \ldots, e^{2\pi \mathrm{i} a_i L_i k/m'_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i} k/m'_i l_i} t\big).$$

In particular,

$$\gamma_i^k = \mathrm{id} \text{ if and only if } e^{2\pi \mathrm{i} a_i L_i k/m'_i} = 1 \text{ and } e^{2\pi \mathrm{i} k/m'_i l_i} = 1.$$

Here:

(A) $e^{2\pi \mathrm{i} a_i L_i k/m'_i} = 1$ if and only if $\dfrac{L_i k}{m'_i}$ is an integer (because a_i and m'_i are relatively prime).

(B) $e^{2\pi \mathrm{i} k/m'_i l_i} = 1$ if and only if $\dfrac{k}{m'_i l_i}$ is an integer.

We restate (A). First write $\dfrac{L_i}{m'_i}$ as $\dfrac{L'_i}{m''_i}$ where L'_i and m''_i are relatively prime positive integers (or, $L'_i := \dfrac{L_i}{\gcd(L_i, m'_i)}$ and $m''_i := \dfrac{m'_i}{\gcd(L_i, m'_i)}$). Then $\dfrac{L_i k}{m'_i} \left(=\dfrac{L'_i k}{m''_i}\right)$ is an integer if and only if m''_i divides k. Thus (A) is restated as:

(A)' $e^{2\pi \mathrm{i} a_i L_i k/m_i'} = 1$ if and only if m_i'' divides k.

From (A)' and (B),

$$\gamma_i^k = \mathrm{id} \text{ if and only if } k \text{ is a common multiple of } m_i'' \text{ and } m_i' l_i.$$

Here m_i' is a multiple of m_i'' (because $m_i'' := \dfrac{m_i'}{\gcd(L_i, m_i')}$). Thus any common multiple of m_i'' and $m_i' l_i$ is necessarily a multiple of $m_i' l_i$. Therefore:

LEMMA 5.3. *$\gamma_i^k = \mathrm{id}$ if and only if k is a multiple of $m_i' l_i$. In particular, the order of γ_i is $m_i' l_i$.*

We summarize the above results (Corollary 5.2 and Lemma 5.3) as follows:

COROLLARY 5.4. *For each $i = 1, 2, \ldots, n$, let Γ_i be the subgroup of Γ consisting of automorphisms of the form*

$$(x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto (x_1, \ldots, e^{2\pi \mathrm{i} j a_i/m_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i} j/m_1' m_2' \cdots m_n' c} t).$$

Then Γ_i is a cyclic group of order $m_i' l_i$ generated by the automorphism

$$\gamma_i : (x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto \left(x_1, \ldots, e^{2\pi \mathrm{i} a_i L_i/m_i'} x_i, \ldots, x_n, e^{2\pi \mathrm{i}/m_i' l_i} t\right),$$

where $L_i := \mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')$ and $l_i := \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{L_i}$.

Let $p : \widetilde{A}_{d-1} (= \mathbb{C}^n) \to A_{d-1}$ be the covering of A_{d-1} given by

$$p(X_1, X_2, \ldots, X_n) = (X_1^d, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n),$$

and $\widetilde{\Gamma}$ be the lift of Γ with respect to p. Next let $\xi_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by

$$\text{(5.2)} \qquad \xi_i : (X_1, \ldots, X_i, \ldots, X_n) \longmapsto (X_1, \ldots, e^{2\pi \mathrm{i}/m_i' l_i} X_i, \ldots, X_n).$$

Then:

LEMMA 5.5.

(1) *The order of ξ_i is $m_i' l_i$.* (The order of γ_i is also $m_i' l_i$ by Lemma 5.3.)

(2) $\xi_i \in \widetilde{\Gamma}$. *In fact,* ξ_i *is a lift of* $\gamma_i \in \Gamma_i\,(\subset \Gamma)$*, that is, the following diagram commutes*:

$$\begin{array}{ccc} \widetilde{A}_{d-1} & \xrightarrow{\quad \xi_i \quad} & \widetilde{A}_{d-1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ A_{d-1} & \xrightarrow{\quad \gamma_i \quad} & A_{d-1}. \end{array}$$

Proof. (1) is clear. We show (2). It suffices to show that $p \circ \xi_i = \gamma_i \circ p$. Note first that

$$\begin{aligned} &p \circ \xi_i(X_1, \ldots, X_i, \ldots, X_n) \\ &\qquad = (X_1^d, \ldots, e^{2\pi \mathrm{i} d/m_i' l_i} X_i^d, \ldots, X_n^d, e^{2\pi \mathrm{i}/m_i' l_i} X_1 X_2 \cdots X_n). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\gamma_i \circ p(X_1, \ldots, X_i, \ldots, X_n) \\ &\qquad = (X_1^d, \ldots, e^{2\pi \mathrm{i} a_i L_i/m_i'} X_i^d, \ldots, X_n^d, e^{2\pi \mathrm{i}/m_i' l_i} X_1 X_2 \cdots X_n). \end{aligned}$$

Thus to show that $p \circ \xi_i = \gamma_i \circ p$, it suffices to show that $e^{2\pi \mathrm{i} d/m_i' l_i} = e^{2\pi \mathrm{i} a_i L_i/m_i'}$, that is,

$$\frac{d}{m_i' l_i} \equiv \frac{a_i L_i}{m_i'} \mod \mathbb{Z}. \tag{5.3}$$

Since $d = m_1' m_2' \cdots m_n' c\Big(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa\Big)$ and $l_i = \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{L_i}$, the left hand side of (5.3) is

$$\begin{aligned} \frac{d}{m_i' l_i} &= \frac{a_1 L_i}{m_1'} + \frac{a_2 L_i}{m_2'} + \cdots + \frac{a_n L_i}{m_n'} + c\kappa L_i \\ &\equiv \frac{a_1 L_i}{m_1'} + \frac{a_2 L_i}{m_2'} + \cdots + \frac{a_n L_i}{m_n'} \mod \mathbb{Z}. \end{aligned}$$

Here $L_i := \mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')$ is divisible by m_k' ($k = 1, 2, \ldots, \check{i}, \ldots, n$), so $\dfrac{a_k L_i}{m_k'} \in \mathbb{Z}$, that is, $\dfrac{a_k L_i}{m_k'} \equiv 0 \mod \mathbb{Z}$ ($k = 1, 2, \ldots, \check{i}, \ldots, n$), hence $\dfrac{d}{m_i' l_i} \equiv \dfrac{a_i L_i}{m_i'} \mod \mathbb{Z}$, confirming (5.3). □

As we saw in the paragraph above Lemma 4.7,

$$\widetilde{\Gamma} = \Big\{ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1,p_2,\ldots,p_n) \in \Lambda^{(j)},\ j = 1,2,\ldots,m'_1 m'_2 \cdots m'_n c \Big\},$$

where $\Lambda^{(j)} = \Big\{ (p_1,p_2,\ldots,p_n) \in \mathbb{Z}^n \,:\, 0 \le p_i < d,\ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z} \Big\}$ and $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is the automorphism given by

$$(X_1,\ldots,X_n) \longmapsto \big(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/m_1 d} X_1,\ldots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/m_n d} X_n \big).$$

Here Corollary 4.6 states that (i) $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ descends to γ^j and (ii) moreover if $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ is of the form $(X_1,\ldots,X_n) \mapsto (X_1,\ldots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_i d} X_i, \ldots, X_n)$, then it descends to γ^j of the form

$$(x_1,\ldots,x_n,t) \longmapsto (x_1,\ldots, e^{2\pi \mathrm{i} ja_i/m_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_i d} t).$$

Note the following:

LEMMA 5.6. *In the case of* (ii)*, there exists an integer s_i such that* $e^{2\pi \mathrm{i} ja_i/m_i} = e^{2\pi \mathrm{i} a_i L_i s_i/m'_i}$ *and* $e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_i d} = e^{2\pi \mathrm{i} s_i/m'_i l_i}$.

PROOF. Since the γ^j in (ii) is an element of Γ_i, and Γ_i is generated by γ_i (Corollary 5.4), there exists an integer s_i such that $\gamma^j = \gamma_i^{s_i}$. Here

$$\begin{cases} \gamma^j : \ (x_1,\ldots,x_n,t) \ \mapsto \ (x_1,\ldots, e^{2\pi \mathrm{i} ja_i/m_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_i d} t), \\ \gamma_i^{s_i} : \ (x_1,\ldots,x_n,t) \ \mapsto \ \big(x_1,\ldots, e^{2\pi \mathrm{i} a_i L_i s_i/m'_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i} s_i/m'_i l_i} t\big), \end{cases}$$

so $e^{2\pi \mathrm{i} ja_i/m_i} = e^{2\pi \mathrm{i} a_i L_i s_i/m'_i}$ and $e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_i d} = e^{2\pi \mathrm{i} s_i/m'_i l_i}$. □

Let $\xi_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by

$$\xi_i : \ (X_1,\ldots,X_i,\ldots,X_n) \longmapsto (X_1,\ldots, e^{2\pi \mathrm{i}/m'_i l_i} X_i, \ldots, X_n).$$

Then $\xi_i \in \widetilde{\Gamma}$ (Lemma 5.5 (2)). In fact, $\xi_i \in \widetilde{\Gamma} \cap \Xi_i$, where Ξ_i $(i = 1,2,\ldots,n)$ is the multiplicative group of automorphisms consisting of scalar multiplication of the ith coordinate of $\widetilde{A}_{d-1}$ $(= \mathbb{C}^n)$:

$$\Xi_i := \Big\{ (X_1,\ldots,X_i,\ldots,X_n) \longmapsto (X_1,\ldots,\lambda X_i,\ldots,X_n) \,:\, \lambda \in \mathbb{C}^\times \Big\}.$$

Setting $\widetilde{\Gamma}_i := \widetilde{\Gamma} \cap \Xi_i$, we claim that ξ_i in fact generates $\widetilde{\Gamma}_i$, that is, any element of $\widetilde{\Gamma}_i$ is a power of ξ_i. To see this, note that $\widetilde{\Gamma}_i$ consists of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ of the form

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (X_1, \dots, X_i, \dots, X_n) \\ \longmapsto (X_1, \dots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id} X_i, \dots, X_n).$$

Here for each $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \in \widetilde{\Gamma}_i$, there exists an integer s_i such that $e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id} = e^{2\pi \mathrm{i} s_i/m'_i l_i}$ (Lemma 5.6). Then

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (X_1, \dots, X_i, \dots, X_n) \longmapsto (X_1, \dots, e^{2\pi \mathrm{i} s_i/m'_i l_i} X_i, \dots, X_n),$$

so $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} = \xi_i^{s_i}$, confirming that ξ_i generates $\widetilde{\Gamma}_i$. Here the order of ξ_i is $m'_i l_i$ (Lemma 5.5 (1)), so the order of the cyclic group $\widetilde{\Gamma}_i$ is $m'_i l_i$.

We formalize the above result as follows:

Proposition 5.7. *For each $i = 1, 2, \dots, n$, let $\widetilde{\Gamma}_i$ be the subgroup of $\widetilde{\Gamma}$ consisting of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ of the form*

$$(X_1, \dots, X_i, \dots, X_n) \longmapsto (X_1, \dots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id} X_i, \dots, X_n).$$

Then $\widetilde{\Gamma}_i$ is a cyclic group of order $m'_i l_i$ generated by the automorphism

$$\xi_i : (X_1, \dots, X_i, \dots, X_n) \longmapsto (X_1, \dots, e^{2\pi \mathrm{i}/m'_i l_i} X_i, \dots, X_n),$$

where $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \dots, \check{m}'_i, \dots, m'_n)}$.

5.2. Cyclic subgroups H_i of H

We have described cyclic subgroups $\widetilde{\Gamma}_i$ ($i = 1, 2, \dots, n$) of $\widetilde{\Gamma}$. We next describe subgroups of H corresponding to them. Here H is the descent of $\widetilde{\Gamma}$ with respect to the covering map $q : \widetilde{A}_{d-1} \,(= \mathbb{C}^n) \to \mathbb{C}^n$ given by

$$q(X_1, X_2, \dots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n}).$$

Explicitly H is given by (Lemma 4.8 (3)):

$$H = \left\{ h^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \dots, m'_1 m'_2 \cdots m'_n c \right\},$$

where $h^{(j)}_{p_1,p_2,\dots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is the automorphism given by

$$(u_1, \dots, u_n) \longmapsto \bigl(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/cd}u_1, \dots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/cd}u_n\bigr).$$

Now let H_i $(i = 1, 2, \dots, n)$ be the subgroup of H consisting of $h^{(j)}_{p_1,p_2,\dots,p_n}$ of the form

$$(u_1, \dots, u_i, \dots, u_n) \longmapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}u_i, \dots, u_n). \tag{5.4}$$

Let $h_i : \mathbb{C}^n \to \mathbb{C}^n$ be the automorphism given by

$$h_i : (u_1, \dots, u_i, \dots, u_n) \longmapsto (u_1, \dots, e^{2\pi \mathrm{i}/l_i}u_i, \dots, u_n). \tag{5.5}$$

Then $h_i \in H$. In fact, h_i is the descent of $\xi_i \in \widetilde{\Gamma}_i\,(\subset \widetilde{\Gamma})$, that is, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{A}_{d-1} & \xrightarrow{\ \xi_i\ } & \widetilde{A}_{d-1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{C}^n & \xrightarrow{\ h_i\ } & \mathbb{C}^n. \end{array}$$

Since $\widetilde{\Gamma}_i$ is a cyclic group generated by ξ_i (Proposition 5.7) and h_i is the descent of ξ_i with respect to q, the descent of $\widetilde{\Gamma}_i$ is a cyclic group generated by h_i. As we show subsequently, this cyclic group coincides with H_i.

To show this, it suffices to show that for any $h^{(j)}_{p_1,p_2,\dots,p_n} \in H_i$, there exists an element of $\widetilde{\Gamma}_i$ that descends to $h^{(j)}_{p_1,p_2,\dots,p_n}$. Here

$$\begin{cases} h^{(j)}_{p_1,p_2,\dots,p_n} : \ (u_1, \dots, u_n) \longmapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}u_i, \dots, u_n), \\ q : \ (X_1, X_2, \dots, X_n) \longmapsto (X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n}). \end{cases}$$

Thus an automorphism $\zeta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ given by

$$\zeta : (X_1, \dots, X_n) \longmapsto (X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i+m_ip_i)/m_id\}}X_i, \dots, X_n) \tag{5.6}$$

descends to $h^{(j)}_{p_1,p_2,\dots,p_n}$. We show that in fact $\zeta \in \widetilde{\Gamma}$ (then from the form of ζ, $\zeta \in \widetilde{\Gamma}_i$, so ζ is a lift of $h^{(j)}_{p_1,p_2,\dots,p_n}$).

Step 1. Since $q(X_1, X_2, \dots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n})$, the set of all lifts of $h^{(j)}_{p_1,p_2,\dots,p_n} : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}u_i, \dots, u_n)$ with respect to the covering q consists of automorphisms

$$(X_1, \ldots, X_n) \mapsto (e^{2\pi \mathrm{i} k_1/m'_1} X_1, \ldots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d + k_i/m'_i\}} X_i, \ldots, e^{2\pi \mathrm{i} k_n/m'_n} X_n),$$

where $k_1, k_2, \ldots, k_n$ are integers.

Step 2. Since $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \in \widetilde{\Gamma}$ is a lift of $h^{(j)}_{p_1,p_2,\ldots,p_n}$ with respect to q (Lemma 4.8 (1)), $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ coincides with one of the automorphisms in Step 1. Namely for some integers $k_1, k_2, \ldots, k_n$,

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (X_1, \ldots, X_i, \ldots, X_n) \longmapsto (e^{2\pi \mathrm{i} k_1/m'_1} X_1, \ldots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d + k_i/m'_i\}} X_i, \ldots, e^{2\pi \mathrm{i} k_n/m'_n} X_n).$$

Next for each $k = 1, 2, \ldots, n$, take the automorphism

$$\xi_k : (X_1, \ldots, X_n) \mapsto (X_1, \ldots, e^{2\pi \mathrm{i}/m'_k l_k} X_k, \ldots, X_n).$$

The composite automorphism $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \xi_1^{-l_1 k_1} \xi_2^{-l_2 k_2} \cdots \xi_n^{-l_n k_n}$ is then given by

$$(X_1, \ldots, X_i, \ldots, X_n) \longmapsto (X_1, \ldots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d\}} X_i, \ldots, X_n).$$

This coincides with the automorphism ζ given by (5.6), thus

$$\zeta = \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \xi_1^{-l_1 k_1} \xi_2^{-l_2 k_2} \cdots \xi_n^{-l_n k_n}.$$

Step 3. Since $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \in \widetilde{\Gamma}$ and $\xi_k \in \widetilde{\Gamma}$ $(k = 1, 2, \ldots, n)$, we have $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \xi_1^{-l_1 k_1} \xi_2^{-l_2 k_2} \cdots \xi_n^{-l_n k_n} \in \widetilde{\Gamma}$. Hence $\zeta \in \widetilde{\Gamma}$, confirming the assertion.

We thus obtained the following:

Lemma 5.8. *For each* $h^{(j)}_{p_1,p_2,\ldots,p_n} \in H_i$, *there exists an element of* $\widetilde{\Gamma}_i$ *that descends to it* (*with respect to the covering* q). *In fact, the automorphism* $\zeta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ *given by* $\zeta : (X_1, \ldots, X_n) \mapsto (X_1, \ldots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d\}} X_i, \ldots, X_n)$ *is an element of* $\widetilde{\Gamma}_i$ *that descends to*

$$h^{(j)}_{p_1,p_2,\ldots,p_n} : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \ldots, u_n).$$

COROLLARY 5.9. *H_i is the descent of $\widetilde{\Gamma}_i$ with respect to the covering q.*

The descent of $\widetilde{\Gamma}_i$ with respect to the covering q is a cyclic group generated by h_i in (5.5). On the other hand, this descent coincides with H_i (Corollary 5.9). Thus:

LEMMA 5.10. *H_i is a cyclic group generated by the automorphism $h_i : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}/l_i} u_i, \dots, u_n)$. Thus the order of H_i is l_i.*

5.3. The pseudo-reflection subgroup of H

We retain the notation above. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$. Let H_i $(i = 1, 2, \dots, n)$ be the subgroup of H consisting of $h^{(j)}_{p_1, p_2, \dots, p_n}$ of the form

$$h^{(j)}_{p_1,p_2,\dots,p_n} : (u_1, \dots, u_i, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \dots, u_n). \tag{5.7}$$

In fact, H_i is a cyclic group of order l_i generated by h_i (Lemma 5.10). Note that if $i \neq j$, then $H_i \cap H_j = \{1\}$. In particular,

$$H_1 H_2 \cdots H_n = H_1 \times H_2 \times \cdots \times H_n. \tag{5.8}$$

Note also that the set of all pseudo-reflections in H is given by $\left(\bigcup_{i=1}^{n} H_i \right) \setminus \{1\}$.

Here a *pseudo-reflection* is a diagonalizable matrix such that one of its eigenvalues is a root of unity (distinct from 1) and all other eigenvalues are 1. Note that the identity matrix is *not* a pseudo-reflection.

Now let P be the *pseudo-reflection subgroup* of H that is the subgroup generated by all pseudo-reflections in H, that is, by $\left(\bigcup_{i=1}^{n} H_i \right) \setminus \{1\}$. Here H_i $(i = 1, 2, \dots, n)$ is a cyclic group generated by h_i, so P is generated by $h_1, h_2, \dots, h_n$, thus $P = H_1 H_2 \cdots H_n = H_1 \times H_2 \times \cdots \times H_n$ (see (5.8)). Since the order of H_i is l_i, the order of P is $l_1 l_2 \cdots l_n$. This confirms the following:

PROPOSITION 5.11. *Where H_i $(i = 1, 2, \dots, n)$ is a cyclic subgroup of H generated by the automorphism $h_i : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}/l_i} u_i,$*

$\ldots, u_n$), the pseudo-reflection subgroup P of H is the direct product $P = H_1 \times H_2 \times \cdots \times H_n$ and the order of P is $l_1 l_2 \cdots l_n$.

In particular, $P = \{1\}$ if and only if $l_1 = l_2 = \cdots = l_n = 1$. Thus:

COROLLARY 5.12. *H is small if and only if $l_1 = l_2 = \cdots = l_n = 1$.*

Now let G be the descent of H with respect to the $l_1 l_2 \cdots l_n$-fold covering $r : \mathbb{C}^n \to \mathbb{C}^n$ given by $r(u_1, u_2, \ldots, u_n) = (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$. Then $l_1 = l_2 = \cdots = l_n = 1$ *if and only if r is the identity map, or equivalently $H = G$.* This, combined with Corollary 5.12, gives the following:

LEMMA 5.13.

$$\begin{aligned} H \textit{ is small} &\iff l_1 = l_2 = \cdots = l_n = 1 \\ &\iff r \textit{ is the identity map} \\ &\iff H = G. \end{aligned}$$

The following arithmetic results are proved later (Corollary 5.19):

(1) If $n = 2$, then $l_1 = l_2 = 1$.

(2) If $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(m'_j, m'_k) = 1$ for any $j \neq k$.

This, combined with Lemma 5.13, yields the following:

THEOREM 5.14 (Numerical criterion of smallness).

(1) *If $n = 2$, then H is always small.*

(2) *If $n \geq 3$, then H is small if and only if* $\gcd(m'_i, m'_j) = 1$ *for any i, j such that $i \neq j$.*

Example 5.15. If $n = 3$, $a_1 = a_2 = a_3 = 1$, $m_1 = 2$, $m_2 = 4$, $m_3 = 6$ and $\kappa = 0$, then $c = \gcd(m_1, m_2, m_3) = 2$, $m'_1 = 1$, $m'_2 = 2$, $m'_3 = 3$ and $d = 2+3+6 = 11$. In this case, Γ is generated by the automorphism γ of A_{d-1} ($= A_{10}$) given by $\gamma(x_1, x_2, x_3, t) \mapsto (e^{2\pi i/2} x_1,\ e^{2\pi i/4} x_2,\ e^{2\pi i/6} x_3,\ e^{2\pi i/12} t)$. Let $\widetilde{\Gamma}$

be the lift of Γ with respect to the covering $p : \widetilde{A}_{10} \to A_{10}$, $p(X_1, X_2, X_3) = (X_1^{11}, X_2^{11}, X_3^{11}, X_1X_2X_3)$, and let H be the descent of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{10} \to \mathbb{C}^3$, $q(X_1, X_2, X_3) = (X_1, X_2^2, X_3^3)$. Then, since $\gcd(m_1', m_2') = 1$, $\gcd(m_1', m_3') = 1$ and $\gcd(m_2', m_3') = 1$, Theorem 5.14 ensures that H is small.

5.4. Supplement: Arithmetic result

This section is devoted to proving an arithmetic result used in §5.3.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be positive integers such that $\gcd(\lambda_1, \lambda_2, \dots, \lambda_n) = 1$, where $n \geq 2$. Set $l_i := \dfrac{\lambda_1 \cdots \check{\lambda}_i \cdots \lambda_n}{\operatorname{lcm}(\lambda_1, \dots, \check{\lambda}_i, \dots, \lambda_n)}$, where $\check{\lambda}_i$ means the omission of λ_i. Note that l_i is a positive integer (cf. Remark 3.1). We show that if $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$.

REMARK 5.16. If $n = 2$, this equivalence is vacuous, because $l_1 = l_2 = 1$ *always* holds (and $\gcd(\lambda_1, \lambda_2) = 1$ by assumption). In fact $l_1 = \dfrac{\lambda_1}{\gcd(\lambda_1)} = 1$ and $l_2 = \dfrac{\lambda_2}{\gcd(\lambda_2)} = 1$.

We begin with some preparation:

LEMMA 5.17. *For any i, j, k such that i, j and k are distinct, $l_i \geq \gcd(\lambda_j, \lambda_k)$.*

PROOF. We only show the assertion for $i = 1$, $j = 2$ and $k = 3$ (the assertion for other cases are similarly shown). Note first that $\lambda_2\lambda_3 = \gcd(\lambda_2, \lambda_3) \cdot \operatorname{lcm}(\lambda_2, \lambda_3)$. Multiplying $\lambda_4 \cdots \lambda_n$ to this yields:

$$\lambda_2\lambda_3\lambda_4 \cdots \lambda_n = \gcd(\lambda_2, \lambda_3) \cdot \operatorname{lcm}(\lambda_2, \lambda_3)\lambda_4 \cdots \lambda_n.$$

Here, since $\operatorname{lcm}(\lambda_2, \lambda_3)\lambda_4 \cdots \lambda_n \geq \operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_n)$,

$$\lambda_2\lambda_3\lambda_4 \cdots \lambda_n \geq \gcd(\lambda_2, \lambda_3) \cdot \operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_n).$$

Dividing this by $\operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4, \dots, \lambda_n)$,

$$\frac{\lambda_2\lambda_3\lambda_4 \cdots \lambda_n}{\operatorname{lcm}(\lambda_2, \lambda_3, \lambda_4 \cdots \lambda_n)} \geq \gcd(\lambda_2, \lambda_3).$$

Since the left hand side is l_1, we have $l_1 \geq \gcd(\lambda_2, \lambda_3)$. (Note: If $n = 3$, then the equality holds. In fact, $l_1 = \dfrac{\lambda_2\lambda_3}{\mathrm{lcm}(\lambda_2, \lambda_3)} = \gcd(\lambda_2, \lambda_3)$.) $\square$

We next show that:

LEMMA 5.18. *For each $i = 1, 2, \ldots, n$,*

$$l_i = 1 \iff \gcd(\lambda_j, \lambda_k) = 1 \text{ for any } j \neq k \text{ (distinct from } i\text{)}.$$

PROOF. $\Longrightarrow$: By Lemma 5.17, for any i, j, k such that i, j and k are distinct, $l_i \geq \gcd(\lambda_j, \lambda_k)$. In particular if $l_i = 1$, then $\gcd(\lambda_j, \lambda_k) = 1$.

$\Longleftarrow$: If $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$ such that j and k distinct from i, then $\mathrm{lcm}(\lambda_1, \ldots, \check{\lambda}_i, \ldots, \lambda_n) = \lambda_1 \cdots \check{\lambda}_i \cdots \lambda_n$, and thus $l_i = 1$. $\square$

From Lemma 5.18, $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$. (Actually if $n = 2$, then $l_1 = l_2 = 1$ always holds (Remark 5.16).)

Now let $m_1, m_2, \ldots, m_n$ be positive integers. Set $c := \gcd(m_1, m_2, \ldots, m_n)$ and $m_i' := \frac{m_i}{c}$ $(i = 1, 2, \ldots, n)$. Then $m_1', m_2', \ldots, m_n'$ are positive integers such that $\gcd(m_1', m_2', \ldots, m_n') = 1$. So we may apply the above to obtain the following:

COROLLARY 5.19. *Let $m_1, m_2, \ldots, m_n$ be positive integers. Set $c := \gcd(m_1, m_2, \ldots, m_n)$, $m_i' := \frac{m_i}{c}$ and $l_i := \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{\mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')}$, where $\check{m}_i'$ means the omission of m_i'.* (Note that l_i is a positive integer (cf. Remark 3.1).) *Then the following hold*:

(1) *If $n = 2$, then $l_1 = l_2 = 1$.*

(2) *If $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(m_j', m_k') = 1$ for any $j \neq k$.*

6. Uniformization Theorem for Arbitrary Dimension

6.1. Determination of G

Recall the diagram (3.5) for the covering maps p, q, r:

$$\begin{array}{ccccc} & & \widetilde{A}_{d-1} = \mathbb{C}^n & & \\ & \overset{q}{\swarrow} & & \overset{p}{\searrow} & \\ & \mathbb{C}^n & & & A_{d-1}. \\ \overset{r}{\swarrow} & & & & \\ \mathbb{C}^n & & & & \end{array} \tag{6.1}$$

Then

- $\widetilde{\Gamma} = \coprod_{j=1}^{m_1' m_2' \cdots m_n' c} \mathrm{Lift}^{(j)}$ (disjoint union) is the lift of Γ with respect to p, where $\mathrm{Lift}^{(j)}$ is the set of all lifts of $\gamma^j \in \Gamma$.
- $H = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\widetilde{\Gamma}$ with respect to q, where $q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\mathrm{Lift}^{(j)}$.
- $G = \bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of H with respect to r, where $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $q_*\big(\mathrm{Lift}^{(j)}\big)$.

Here $\mathrm{Lift}^{(j)} = \big\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$ (Corollary 4.5 (3)) and $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$ (Lemma 4.8 (2)). We next determine $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$. For $j = 1, 2, \ldots, m_1' m_2' \cdots m_n' c$ and $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $g^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$(v_1, \ldots, v_n) \longmapsto \big(e^{2\pi \mathrm{i} l_1 (j a_1 + m_1 p_1)/cd} v_1, \ldots, e^{2\pi \mathrm{i} l_n (j a_n + m_n p_n)/cd} v_n\big).$$

Then as for Lemma 4.8, we can show the following:

LEMMA 6.1.

(A) *$g^{(j)}_{p_1,p_2,\ldots,p_n}$ is the descent of $h^{(j)}_{p_1,p_2,\ldots,p_n}$ with respect to the covering $r : \mathbb{C}^n \to \mathbb{C}^n$.*

(B) $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{g^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$.

(C) $G = \Big\{ g^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1,p_2,\ldots,p_n) \in \Lambda^{(j)},\ j = 1,2,\ldots,m'_1 m'_2 \cdots m'_n c \Big\}$.
(In particular, any two elements of G commute, so G is abelian.)

6.2. Uniformization theorem

Let H be the descent of $\widetilde{\Gamma}$ with respect to the $m'_1 m'_2 \cdots m'_n$-fold covering $q : \mathbb{C}^n \to \mathbb{C}^n$ and P be the pseudo-reflection subgroup of H, that is, P is generated by all pseudo-reflections in H. The descent G of H with respect to the $l_1 l_2 \cdots l_n$-fold covering $r : \mathbb{C}^n \to \mathbb{C}^n$ is regarded as the quotient group H/P. Indeed the kernel of the surjective homomorphism $r_* : H \to G$ (given by $r_*(h) :=$ descent of h) is P, so $G \cong H/P$. Thus G is obtained from H by collapsing the pseudo-reflections in H, consequently:

Proposition 6.2. *G contains no pseudo-reflections, that is, is a small group.*

Now $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong \mathbb{C}^n/G$. Here G is a finite abelian group (Proposition 3.2 (3)) and small (Proposition 6.2). The following is thus established:

Theorem 6.3 (Uniformization theorem). *Let Γ be the cyclic group generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by*

$$\gamma : (x_1,\ldots,x_n,t) \longmapsto (e^{2\pi i a_1/m_1} x_1, \ldots, e^{2\pi i a_n/m_n} x_n,\ e^{2\pi i/m'_1 m'_2 \cdots m'_n c}\, t).$$

Then there exists a small finite abelian group $G \subset GL(n,\mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$.

We explicitly give the isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in the uniformization theorem. The covering maps p, q and r appearing in the diagram (6.1) induce isomorphisms $\overline{p} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to A_{d-1}/\Gamma$ and $\overline{q} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to \mathbb{C}^n/H$ and $\overline{r} : \mathbb{C}^n/H \to \mathbb{C}^n/G$. The isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in the uniformization theorem (Theorem 6.3) is then given by

$$\Psi := \overline{r} \circ \overline{q} \circ \overline{p}^{\,-1} :\ A_{d-1}/\Gamma \xrightarrow{\ \cong\ } \mathbb{C}^n/G. \tag{6.2}$$

Explicitly:

Lemma 6.4. $\Psi\big([x_1,\ldots,x_n,t]\big) = \big[x_1^{m'_1 l_1/d}, \ldots, x_n^{m'_n l_n/d}\big]$, *where* $[x_1, \dots, x_n, t] \in A_{d-1}/\Gamma$ *and* $\big[x_1^{m'_1 l_1/d}, \dots, x_n^{m'_n l_n/d}\big] \in \mathbb{C}^n/G$ *denote the images of* $(x_1, \dots, x_n, t) \in A_{d-1}$ *and* $\big(x_1^{m'_1 l_1/d}, \dots, x_n^{m'_n l_n/d}\big) \in \mathbb{C}^n$ *respectively.*

PROOF. Since $p(X_1, X_2, \dots, X_n) = (X_1^d, X_2^d, \dots, X_n^d, X_1 X_2 \cdots X_n)$, we have $\overline{p}\big([X_1, X_2, \dots, X_n]\big) = [X_1^d, X_2^d, \dots, X_n^d, X_1 X_2 \cdots X_n]$, so

$$\overline{p}^{\,-1}\big([x_1, x_2, \dots, x_n, t]\big) = \big[x_1^{1/d},\, x_2^{1/d},\, \dots, x_n^{1/d}\big].$$

Next since $q(X_1, X_2, \dots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n})$ and $r(u_1, u_2, \dots, u_n) = (u_1^{l_1}, u_2^{l_2}, \dots, u_n^{l_n})$, we have $\overline{q}\big([X_1, X_2, \dots, X_n]\big) = [X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n}]$ and $\overline{r}\big([u_1, u_2, \dots, u_n]\big) = [u_1^{l_1}, u_2^{l_2}, \dots, u_n^{l_n}]$, so

$$\begin{aligned}\overline{r} \circ \overline{q}\big([x_1^{1/d}, x_2^{1/d}, \dots, x_n^{1/d}]\big) &= \overline{r}\big([x_1^{m'_1/d}, x_2^{m'_2/d}, \dots, x_n^{m'_n/d}]\big)\\ &= \big[x_1^{m'_1 l_1/d}, x_2^{m'_2 l_2/d}, \dots, x_n^{m'_n l_n/d}\big].\end{aligned}$$

Hence $\Psi := \overline{r} \circ \overline{q} \circ \overline{p}^{\,-1}$ is explicitly given by

$$\Psi\big([x_1, x_2, \dots, x_n, t]\big) = \big[x_1^{m'_1 l_1/d},\, x_2^{m'_2 l_2/d}, \dots, x_n^{m'_n l_n/d}\big]. \ \square$$

6.3. Correspondence between functions

We use the notation in §6.2. Besides, let $\Phi : A_{d-1} \to \mathbb{C}$ be a holomorphic map given by $\Phi(x_1, x_2, \dots, x_n, t) = t^{m'_1 m'_2 \cdots m'_n c}$. Then Φ is Γ-invariant, so induces a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$. As we explained in § Introduction, the topological monodromy of $\overline{\Phi}$ is a $-\Big(\dfrac{a_1}{m_1}, \dfrac{a_2}{m_2}, \dots, \dfrac{a_n}{m_n}, \kappa\Big)$-*fractional Dehn twist*: If $n = 2$, then the topological monodromy of $\overline{\Phi}$ is the $-\Big(\dfrac{a_1}{m_1}, \dfrac{a_2}{m_2}, \kappa\Big)$-fractional Dehn twist.

Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ in (6.2), $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to a holomorphic map on $\mathbb{C}^n/G$. We describe this map. To that end, we need the following:

LEMMA 6.5. *For an element* $g \in G$ *given by*

$$(v_1, \dots, v_n) \longmapsto \big(e^{2\pi i l_1 (j a_1 + m_1 p_1)/cd} v_1,\, \dots, e^{2\pi i l_n (j a_n + m_n p_n)/cd} v_n\big),$$

write $\eta_i = e^{2\pi \mathrm{i} l_i (ja_i + m_i p_i)/cd}$ $(i = 1, 2, \ldots, n)$. *Next for* $i = 1, 2, \ldots, n$, *set* $k_i := \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c$, *where* $\check{m}'_i$ *means the omission of* m'_i. *Then* $\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} = 1$.

Proof. Since $l_i = \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$, we have $k_i l_i = m'_1 \cdots \check{m}'_i \cdots m'_n c$, so

$$\begin{aligned}\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} &= e^{2\pi \mathrm{i} k_1 l_1 (ja_1 + m_1 p_1)/cd} e^{2\pi \mathrm{i} k_2 l_2 (ja_2 + m_2 p_2)/cd} \cdots \\ &\qquad \cdots e^{2\pi \mathrm{i} k_n l_n (ja_n + m_n p_n)/cd} \\ &= e^{2\pi \mathrm{i} m'_1 m'_2 \cdots m'_n c \sum_{i=1}^{n} (ja_i/m_i + p_i)/d}.\end{aligned}$$

Here $\sum_{i=1}^{n} p_i/d = j\kappa/d$ (because $(p_1, p_2, \ldots p_n) \in \Lambda^{(j)}$), so

$$\begin{aligned}e^{2\pi \mathrm{i} m'_1 m'_2 \cdots m'_n c \sum_{i=1}^{n} (ja_i/m_i + p_i)/d} &= e^{2\pi \mathrm{i} j m'_1 m'_2 \cdots m'_n c \left(\sum_{i=1}^{n} a_i/m_i + \kappa \right)/d} \\ &= e^{2\pi \mathrm{i} j} \qquad \text{by (3.3).}\end{aligned}$$

Hence $\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} = e^{2\pi \mathrm{i} j} = 1$. □

We next show the following (this generalizes Lemma 2.4):

Theorem 6.6. *Let* $\phi : \mathbb{C}^n \to \mathbb{C}$ *be a holomorphic map given by* $\phi(v_1, v_2, \ldots v_n) = v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$, *where* $k_i := \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c$. *Then*:

(1) ϕ *is* G*-invariant. In particular, this induces a holomorphic map* $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$.

(2) *Under the isomorphism* $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ *in* (6.2), $\overline{\Phi}$ *corresponds to* $\overline{\phi}$, *that is,* $\overline{\Phi} = \overline{\phi} \circ \Psi$.

Proof. (1): For $(v_1, v_2, \ldots, v_n) \in \mathbb{C}^n$ and an element $g \in G$ given by $g : (v_1, v_2, \ldots, v_n) \mapsto (\eta_1 v_1, \eta_2 v_2, \ldots, \eta_n v_n)$,

$$\begin{aligned}\phi \circ g(v_1, v_2, \ldots, v_n) &= \phi(\eta_1 v_1,\ \eta_2 v_2, \ldots, \eta_n v_n) \\ &= (\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n}) v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} \\ &= \eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} \phi(v_1, v_2, \ldots, v_n) \\ &= \phi(v_1, v_2, \ldots, v_n) \qquad \text{by Lemma 6.5.}\end{aligned}$$

Thus $\phi \circ g = \phi$, confirming the assertion.

(2): Note first that

$$\begin{aligned}&\overline{\phi} \circ \Psi\big([x_1, x_2, \dots, x_n, t]\big)\\ &\quad = \overline{\phi}\big([x_1^{m'_1 l_1/d},\, x_2^{m'_2 l_2/d},\, \dots, x_n^{m'_n l_n/d}]\big) \quad \text{(Lemma 6.4)}\\ &\quad = x_1^{m'_1 l_1 k_1/d} x_2^{m'_2 l_2 k_2/d} \cdots x_n^{m'_n l_n k_n/d}.\end{aligned}$$

Here since $k_i l_i = m'_1 \cdots \check{m}'_i \cdots m'_n c$, we have $m'_i l_i k_i = m'_1 m'_2 \cdots m'_n c$. Thus the last expression is rewritten as

$$\begin{aligned}x_1^{m'_1 l_1 k_1/d} x_2^{m'_2 l_2 k_2/d} \cdots x_n^{m'_n l_n k_n/d} &= (x_1 x_2 \cdots x_n)^{m'_1 m'_2 \cdots m'_n c/d}\\ &= t^{m'_1 m'_2 \cdots m'_n c} \qquad \text{because } x_1 x_2 \cdots x_n = t^d.\end{aligned}$$

Hence $\overline{\phi} \circ \Psi\big([x_1, x_2, \dots, x_n, t]\big) = \overline{\Phi}\big([x_1, x_2, \dots, x_n, t]\big)$. $\square$

6.4. Equi-smallness theorem

Let Γ be the cyclic group generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by

$$\gamma:\ (x_1, \dots, x_n, t) \longmapsto (e^{2\pi \mathrm{i} a_1/m_1} x_1,\, \dots, e^{2\pi \mathrm{i} a_n/m_n} x_n,\, e^{2\pi \mathrm{i}/m'_1 m'_2 \cdots m'_n c} t),$$

where $d := \sum_{k=1}^{n} a_k m'_1 \cdots \check{m}'_k \cdots m'_n + m'_1 m'_2 \cdots m'_n c\kappa$. Here κ is an integer satisfying $(*)$ $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$. Then $\kappa \geq -n+1$ (see (3.1)).

Let $\widetilde{\Gamma}$ be the lift of Γ and H is the descent of $\widetilde{\Gamma}$. The pseudo-reflection subgroup P of H is generated by the automorphisms $h_i : \mathbb{C}^n \to \mathbb{C}^n$ ($i = 1, 2, \dots, n$) given by $h_i :\ (u_1, \dots, u_i, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}/l_i} u_i, \dots, u_n)$ (Proposition 5.11). Here $l_i = \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\operatorname{lcm}(m'_1, \dots, \check{m}'_i, \dots, m'_n)}$ does not depend on κ. Thus:

LEMMA 6.7. *The pseudo-reflection subgroup P of H does not depend on κ.*

In what follows, regarding κ as a 'parameter', write $\widetilde{\Gamma}$, H, P as $\widetilde{\Gamma}_\kappa$, H_κ, P_κ. These are subgroups of $GL(n, \mathbb{C})$. From Lemma 6.7,

$$P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots, \tag{6.3}$$

where κ_0 denotes the least integer in the set S of integers κ satisfying $(*)$. If H_{κ_0} is small, then $P_{\kappa_0} = \{1\}$ and by (6.3), $P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots = \{1\}$. Thus H_κ is small for any $\kappa \in S$. This confirms the following:

THEOREM 6.8 (Equi-smallness). *Let S be the set of integers κ satisfying $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, and let κ_0 denote the least integer in S. Then H_{κ_0} is small $\Longleftrightarrow$ H_κ is small for any $\kappa \in S$. (In other words, H_{κ_0} is not small $\Longleftrightarrow$ H_κ is not small for any $\kappa \in S$.)*

Example 6.9. (i): When $n = 3$, $a_1 = a_2 = a_3 = 1$, $m_1 = 2$, $m_2 = 4$ and $m_3 = 6$, $c = \gcd(m_1, m_2, m_3) = 2$, $m'_1 = 1$, $m'_2 = 2$ and $m'_3 = 3$. Then $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$, and thus $\frac{11}{12} + \kappa > 0$. Hence $\kappa_0 = 0$. Here by Example 5.15, H_{κ_0} is small. Thus by Theorem 6.8, H_κ is small for any integer κ such that $\kappa \geq 0$.

(ii): When $n = 3$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $m_1 = 2$, $m_2 = 3$ and $m_3 = 4$, $c = \gcd(m_1, m_2, m_3) = 1$, $m'_1 = 2$, $m'_2 = 3$ and $m'_3 = 4$. Then $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$, and thus $\frac{23}{12} + \kappa > 0$. Hence $\kappa_0 = -1$. Here since $\gcd(m'_1, m'_3) = 2$, Theorem 5.14 ensures that H_{κ_0} is not small. Thus by Theorem 6.8, H_κ is not small for any integer κ such that $\kappa \geq -1$.

7. Generators of $\widetilde{\Gamma}$, H and G

Let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and G be the descent of H with respect to the covering r. Then G is a small finite abelian group such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$. We explicitly give generators of $\widetilde{\Gamma}$, H, G.

7.1. Generators of $\widetilde{\Gamma}$

Recall that (see the paragraph above Lemma 4.7)

$$\widetilde{\Gamma} = \left\{ \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c \right\},$$

where $\Lambda^{(j)} = \left\{ (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \,:\, 0 \leq p_i < d,\ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z} \right\}$ and $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is an automorphism given by

$$(X_1, \ldots, X_n) \longmapsto \left(e^{2\pi\mathrm{i}(ja_1 + m_1 p_1)/m_1 d} X_1, \ldots, e^{2\pi\mathrm{i}(ja_n + m_n p_n)/m_n d} X_n \right).$$

Recall that Γ is generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by

$$\gamma : (x_1, \dots, x_n, t) \longmapsto (e^{2\pi \mathrm{i} a_1/m_1} x_1, \dots, e^{2\pi \mathrm{i} a_n/m_n} x_n, e^{2\pi \mathrm{i}/(m'_1 m'_2 \cdots m'_n c)} t).$$

The automorphism $\delta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ given by

$$\begin{aligned}&(X_1, X_2, \dots, X_n)\\ &\qquad \longmapsto (e^{2\pi \mathrm{i} a_1/m_1 d} X_1, e^{2\pi \mathrm{i} a_2/m_2 d} X_2, \dots, e^{2\pi \mathrm{i}(a_n + m_n \kappa)/m_n d} X_n)\end{aligned}$$

is a lift of $\gamma \in \Gamma$ with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$. Hence $\delta \in \widetilde{\Gamma}$. The automorphism $\eta_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ $(i = 1, 2, \dots, n-1)$ given by

$$(X_1, X_2, \dots, X_n) \longmapsto (X_1, \dots, X_{i-1}, e^{2\pi \mathrm{i}/d} X_i, X_{i+1}, \dots, e^{-2\pi \mathrm{i}/d} X_n)$$

is a lift of the identity $1 \in \Gamma$ with respect to the covering p. Hence $\eta_i \in \widetilde{\Gamma}$ $(i = 1, 2, \dots, n-1)$.

LEMMA 7.1. *Any element of $\widetilde{\Gamma}$ is expressed by δ, η_1, $\eta_2, \dots, \eta_{n-1} \in \widetilde{\Gamma}$. In fact, $\widetilde{\gamma}^{(j)}_{p_1, p_2, \dots, p_n} \in \widetilde{\Gamma}$ is expressed as $\widetilde{\gamma}^{(j)}_{p_1, p_2, \dots, p_n} = \delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}}$.*

PROOF. It suffices to show that $\widetilde{\gamma}^{(j)}_{p_1, p_2, \dots, p_n} \delta^{-j} \eta_1^{-p_1} \eta_2^{-p_2} \cdots \eta_{n-1}^{-p_{n-1}}$ is the identity. For brevity, express $\widetilde{\gamma}^{(j)}_{p_1, p_2, \dots, p_n}(\vec{x}) = A\vec{x}$, $\delta(\vec{x}) = B\vec{x}$ and $\eta_i(\vec{x}) = C_i \vec{x}$, where
$A = \mathrm{diag}\big(e^{2\pi \mathrm{i}(j a_1 + m_1 p_1)/m_1 d},\ e^{2\pi \mathrm{i}(j a_2 + m_2 p_2)/m_2 d}, \dots, e^{2\pi \mathrm{i}(j a_n + m_n p_n)/m_n d}\big)$,
$B = \mathrm{diag}\big(e^{2\pi \mathrm{i} a_1/m_1 d},\ e^{2\pi \mathrm{i} a_2/m_2 d}, \dots, e^{2\pi \mathrm{i} a_{n-1}/m_{n-1} d},\ e^{2\pi \mathrm{i}(a_n + m_n \kappa)/m_n d}\big)$,
$C_i = \mathrm{diag}\big(1, \dots, 1,\ e^{2\pi \mathrm{i}/d},\ 1, \dots, 1,\ e^{-2\pi \mathrm{i}/d}\big)$, where $e^{2\pi \mathrm{i}/d}$ lies in the ith place. Then $\widetilde{\gamma}^{(j)}_{p_1, p_2, \dots, p_n} \delta^{-j} \eta_1^{-p_1} \eta_2^{-p_2} \cdots \eta_{n-1}^{-p_{n-1}}(\vec{x}) = AB^{-j} C_1^{-p_1} C_2^{-p_2} \cdots C_{n-1}^{-p_{n-1}} \vec{x}$. It thus suffices to show that the matrix $D := AB^{-j} C_1^{-p_1} C_2^{-p_2} \cdots C_{n-1}^{-p_{n-1}}$ is the identity matrix. Since A, B, C_i are diagonal, D is also diagonal, so it suffices to show that any of its diagonal entries is 1. This is confirmed as follows:

- For $i = 1, 2, \dots, n-1$, the (i, i) entry of D is
$$e^{2\pi \mathrm{i}(j a_i + m_i p_i)/m_i d} \big(e^{2\pi \mathrm{i} a_i/m_i d}\big)^{-j} \big(e^{2\pi \mathrm{i}/d}\big)^{-p_i} = 1.$$
- The (n, n) entry of D is
$$\begin{aligned}&e^{2\pi \mathrm{i}(j a_n + m_n p_n)/m_n d} \big(e^{2\pi \mathrm{i}(a_n + m_n \kappa)/m_n d}\big)^{-j} \big(e^{2\pi \mathrm{i}/d}\big)^{p_1 + \cdots + p_{n-1}}\\ &\qquad = e^{2\pi \mathrm{i}(p_1 + \cdots + p_n - j\kappa)/d}.\end{aligned}$$

Here since $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, we have $\sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}$, and thus $e^{2\pi \mathrm{i}(p_1+\cdots+p_n-j\kappa)/d} = 1$. $\square$

Lemma 7.1 implies that:

Corollary 7.2. *$\widetilde{\Gamma}$ is generated by δ, $\eta_1, \eta_2, \ldots, \eta_{n-1}$, or as a subgroup of $GL(n, \mathbb{C})$, generated by the matrices B, C_1, $C_2, \ldots, C_{n-1}$ appearing in the proof of Lemma 7.1.*

7.2. Relations among generators of $\widetilde{\Gamma}$

Recall that $\widetilde{\Gamma}$ is a finite abelian group of order $m'_1 m'_2 \cdots m'_n c d^{n-1}$ (Proposition 3.2 (1)) and is generated by δ, η_1, η_2, $\ldots$, η_{n-1} (Corollary 7.2). These generators are generally *not* independent. In fact, the following holds (the proof is the same as that of Lemma 7.1):

Lemma 7.3. $\delta^{m'_1 m'_2 \cdots m'_n c} = \eta_1^{a_1 m'_2 m'_3 \cdots m'_n} \eta_2^{a_2 m'_1 m'_3 \cdots m'_n} \cdots \eta_{n-1}^{a_{n-1} m'_1 \cdots m'_{n-2} m'_n}$.

If the order of δ is $m'_1 m'_2 \cdots m'_n c$, then this relation is actually vacuous. To see this, we need the following:

Lemma 7.4.

(1) *Express $\delta(\vec{x}) = B\vec{x}$, where B is the matrix appearing in the proof of Lemma 7.1. Then $\det B = e^{2\pi \mathrm{i}/m'_1 m'_2 \cdots m'_n c}$.*

(2) *If $\delta^k = 1$, then k is a multiple of $m'_1 m'_2 \cdots m'_n c$. In particular, the order of δ is a multiple of $m'_1 m'_2 \cdots m'_n c$.*

(3) $\mathrm{lcm}(m'_1, m'_2, \ldots, m'_n) cd$ *is a multiple of $m'_1 m'_2 \cdots m'_n c$, and $\delta^{\mathrm{lcm}(m'_1, m'_2, \ldots, m'_n) cd} = 1$.*

(4) *Write $\mathrm{lcm}(m'_1, m'_2, \ldots, m'_n) cd = N m'_1 m'_2 \cdots m'_n c$ where N is a positive integer. Then the order of δ is $l m'_1 m'_2 \cdots m'_n c$ for some positive integer l $(1 \le l \le N)$.*

PROOF. We show the assertions only for $n = 3$ (the proof is the same for any n).

(1): Since $B = \begin{pmatrix} e^{2\pi\mathrm{i}a_1/m_1 d} & 0 & 0 \\ 0 & e^{2\pi\mathrm{i}a_2/m_2 d} & 0 \\ 0 & 0 & e^{2\pi\mathrm{i}(a_3+m_3\kappa)/m_3 d} \end{pmatrix}$, we have $\det B = e^{2\pi\mathrm{i}a_1/m_1 d} e^{2\pi\mathrm{i}a_2/m_2 d} e^{2\pi\mathrm{i}(a_3+m_3\kappa)/m_3 d} = e^{2\pi\mathrm{i}(a_1/m_1+a_2/m_2+a_3/m_3+\kappa)/d}$. Here $(a_1/m_1 + a_2/m_2 + a_3/m_3 + \kappa)/d = 1/m'_1 m'_2 m'_3 c$ (because $d := m'_1 m'_2 m'_3 c(a_1/m_1 + a_2/m_2 + a_3/m_3 + \kappa)$), confirming the assertion.

(2): If $\delta^k = 1$, then $B^k = I$ (the identity matrix), so $\det(B^k) = 1$. Then $e^{2\pi\mathrm{i}k/m'_1 m'_2 m'_3 c} = 1$ by (1). Thus k is a multiple of $m'_1 m'_2 m'_3 c$.

(3): We first show that $\mathrm{lcm}(m'_1, m'_2, m'_3)cd$ is a multiple of $m'_1 m'_2 m'_3 c$, for which it is sufficient to demonstrate that $\dfrac{\mathrm{lcm}(m'_1, m'_2, m'_3)cd}{m'_1 m'_2 m'_3 c}$ is an integer. Using $d := m'_1 m'_2 m'_3 c\left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa\right)$, we rewrite:

$$\begin{aligned}\frac{\mathrm{lcm}(m'_1, m'_2, m'_3)cd}{m'_1 m'_2 m'_3 c} &= \mathrm{lcm}(m'_1, m'_2, m'_3)c\left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa\right) \\ &= \frac{\mathrm{lcm}(m'_1, m'_2, m'_3)c}{m_1}a_1 + \frac{\mathrm{lcm}(m'_1, m'_2, m'_3)c}{m_2}a_2 + \frac{\mathrm{lcm}(m'_1, m'_2, m'_3)c}{m_3}a_3 \\ &\quad + \mathrm{lcm}(m'_1, m'_2, m'_3)c\kappa.\end{aligned}$$

Since $m_i = m'_i c$, the last expression is equal to

$$\frac{\mathrm{lcm}(m'_1, m'_2, m'_3)}{m'_1}a_1 + \frac{\mathrm{lcm}(m'_1, m'_2, m'_3)}{m'_2}a_2 + \frac{\mathrm{lcm}(m'_1, m'_2, m'_3)}{m'_3}a_3 + \mathrm{lcm}(m'_1, m'_2, m'_3)c\kappa.$$

This is an integer, because

$$\frac{\mathrm{lcm}(m'_1, m'_2, m'_3)}{m'_1}, \frac{\mathrm{lcm}(m'_1, m'_2, m'_3)}{m'_2}, \frac{\mathrm{lcm}(m'_1, m'_2, m'_3)}{m'_3} \text{ are integers.}$$

Thus $\dfrac{\mathrm{lcm}(m'_1, m'_2, m'_3)cd}{m'_1 m'_2 m'_3 c}$ is an integer, confirming the assertion.

We next show that $\delta^{\mathrm{lcm}(m'_1, m'_2, m'_3)cd} = 1$. For an integer k, the automorphism $\delta^k : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is given by

$$(X_1, X_2, X_3) \longmapsto (e^{2\pi\mathrm{i}a_1 k/m_1 d}X_1, e^{2\pi\mathrm{i}a_2 k/m_2 d}X_2, e^{2\pi\mathrm{i}(a_3+m_3\kappa)k/m_3 d}X_3).$$

Here if $k = \mathrm{lcm}(m_1', m_2', m_3')cd$, then

$$k/m_1 d = \mathrm{lcm}(m_1', m_2', m_3')/m_1', \qquad k/m_2 d = \mathrm{lcm}(m_1', m_2', m_3')/m_2',$$
$$k/m_3 d = \mathrm{lcm}(m_1', m_2', m_3')/m_3',$$

hence $k/m_1 d$, $k/m_2 d$, $k/m_3 d$ are integers, consequently $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} : (X_1, X_2, X_3) \longmapsto (X_1, X_2, X_3)$, so $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} = 1$.

(4): This follows from (2) and (3). □

Since $\eta_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ $(i = 1, 2, \ldots, n-1)$ is given by

$$(X_1, X_2, \ldots, X_n) \longmapsto (X_1, \ldots, X_{i-1}, e^{2\pi \mathrm{i}/d} X_i, X_{i+1}, \ldots, e^{-2\pi \mathrm{i}/d} X_n),$$

the order of η_i is d.

Lemma 7.5.

(1) *There is no nontrivial relation among $\eta_1, \eta_2, \ldots, \eta_{n-1}$: If $\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_{n-1}^{k_{n-1}} = 1$, then $\eta_1^{k_1} = \eta_2^{k_2} = \cdots = \eta_{n-1}^{k_{n-1}} = 1$.*

(2) *Let k be an integer such that $\delta^k \neq 1$. If δ^k is expressed by $\eta_1, \eta_2, \ldots, \eta_{n-1}$, that is, $\delta^k = \eta_1^{l_1} \eta_2^{l_2} \cdots \eta_{n-1}^{l_{n-1}}$ for some integers $l_1, l_2, \ldots, l_{n-1}$, then k is a multiple of $m_1' m_2' \cdots m_n' c$.*

(3) *If an integer k is not a multiple of $m_1' m_2' \cdots m_n' c$, then $\delta^k \neq 1$. Moreover δ^k cannot be expressed by $\eta_1, \eta_2, \ldots, \eta_{n-1}$.*

(4) *Let $\langle \delta \rangle$ and $\langle \eta_1, \eta_2, \ldots, \eta_{n-1} \rangle$ denote the subgroups of $GL(n, \mathbb{C})$ generated by δ and $\eta_1, \eta_2, \ldots, \eta_{n-1}$ respectively. If the order of δ is $m_1' m_2' \cdots m_n' c$, then $\langle \delta \rangle \cap \langle \eta_1, \eta_2, \ldots, \eta_{n-1} \rangle = \{1\}$.*

Proof. We show this for $n = 3$ (the proof is the same for any n).

(1): The automorphism $\eta_1^{k_1} \eta_2^{k_2} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is given by $(X_1, X_2, X_3) \mapsto (e^{2\pi \mathrm{i} k_1/d} X_1,\ e^{2\pi \mathrm{i} k_2/d} X_2,\ e^{-2\pi \mathrm{i}(k_1+k_2)/d} X_n)$. If $\eta_1^{k_1} \eta_2^{k_2} = 1$, then $e^{2\pi \mathrm{i} k_1/d} = 1$, $e^{2\pi \mathrm{i} k_2/d} = 1$, $e^{-2\pi \mathrm{i}(k_1+k_2)/d} = 1$. Accordingly $\eta_1^{k_1} = 1$ and $\eta_2^{k_2} = 1$ hold.

(2): Suppose that $\delta^k = \eta_1^{l_1} \eta_2^{l_2}$. Here since $\delta \in \widetilde{\Gamma}$ is a lift of $\gamma \in \Gamma$, $\delta^k \in \widetilde{\Gamma}$ is a lift of $\gamma^k \in \Gamma$ and since $\eta_1, \eta_2 \in \widetilde{\Gamma}$ are lifts of $1 \in \Gamma$, $\eta_1^{l_1} \eta_2^{l_2} \in \widetilde{\Gamma}$ is a lift of $1 \in \Gamma$. The relation $\delta^k = \eta_1^{l_1} \eta_2^{l_2}$ thus implies that δ^k is a lift of both γ^k

and 1, so $\gamma^k = 1$. Since the order of γ is $m_1'm_2'm_3'c$, this implies that k is a multiple of $m_1'm_2'm_3'c$.

(3): Since the order of δ is a multiple of $m_1'm_2'm_3'c$ (Lemma 7.4 (2)), if an integer k is not a multiple of $m_1'm_2'm_3'c$, then $\delta^k \neq 1$. The rest is a restatement of (2).

(4): This can be shown by contradiction. If $\langle\delta\rangle \cap \langle\eta_1, \eta_2\rangle \neq \{1\}$, then there exist elements $\delta^k \neq 1$ of $\langle\delta\rangle$ and $\eta_1^{l_1}\eta_2^{l_2} \neq 1$ of $\langle\eta_1, \eta_2\rangle$ such that $\delta^k = \eta_1^{l_1}\eta_2^{l_2}$. Then (2) implies that k is a multiple of $m_1'm_2'm_3'c$. But $\delta^{m_1'm_2'm_3'c} = 1$ by assumption, accordingly $\delta^k = 1$. This contradicts that $\delta^k \neq 1$. □

By (4) of Lemma 7.4, the order of δ is $lm_1'm_2'\cdots m_n'c$ for some positive integer l $(1 \leq l \leq N)$, where $N = \dfrac{\mathrm{lcm}(m_1', m_2', \ldots, m_n')cd}{m_1'm_2'\cdots m_n'c}$. The following holds:

Corollary 7.6.

(1) *If the order of δ is $m_1'm_2'\cdots m_n'c$, then the relation in Lemma 7.3 is vacuous, that is,* $\delta^{m_1'm_2'\cdots m_n'c} = \eta_1^{a_1m_2'm_3'\cdots m_n'} = \cdots = \eta_{n-1}^{a_{n-1}m_1'\cdots m_{n-2}'m_n'} = 1$.

(2) *If the order of δ is $m_1'm_2'\cdots m_n'c$, then $\widetilde{\Gamma}$ is isomorphic to the product of cyclic groups $\langle\delta\rangle \times \langle\eta_1\rangle \times \langle\eta_2\rangle \times \cdots \times \langle\eta_{n-1}\rangle$, where $\langle\delta\rangle$ and $\langle\eta_i\rangle$ denote the cyclic groups generated by δ and η_i respectively.*

Proof. We show this for $n = 3$ (the proof is the same for other cases).

(1): If the order of δ is $m_1'm_2'm_3'c$, then $\delta^{m_1'm_2'm_3'c} = 1$, so $\eta_1^{a_1m_2'm_3'}\eta_2^{a_2m_1'm_3'} = 1$ by Lemma 7.3. Consequently $\eta_1^{a_1m_2'm_3'} = \eta_2^{a_2m_1'm_3'} = 1$ by Lemma 7.5 (1), confirming the assertion.

(2): By Lemma 7.5 (4), if the order of δ is $m_1'm_2'm_3'c$, then $\langle\delta\rangle \cap \langle\eta_1, \eta_2\rangle = \{1\}$. Since $\widetilde{\Gamma}$ is generated by δ, η_1, η_2 (Corollary 7.2), we obtain $\widetilde{\Gamma} \cong \langle\delta\rangle \times \langle\eta_1, \eta_2\rangle$. Here $\langle\eta_1, \eta_2\rangle = \langle\eta_1\rangle \times \langle\eta_2\rangle$ because there is no nontrivial relation between η_1 and η_2 (Lemma 7.5 (1)). Hence $\widetilde{\Gamma} \cong \langle\delta\rangle \times \langle\eta_1\rangle \times \langle\eta_2\rangle$, confirming the assertion. □

Remark 7.7. If the order of δ is greater than $m_1'm_2'\cdots m_n'c$, then $\widetilde{\Gamma}$ is not isomorphic to $\langle\delta\rangle \times \langle\eta_1\rangle \times \langle\eta_2\rangle \times \cdots \times \langle\eta_{n-1}\rangle$, because there is a nontrivial relation among δ, η_1, $\eta_2, \ldots, \eta_{n-1}$ (Lemma 7.3).

7.3. Generators of H and relations among them

Recall that (see Lemma 4.8 (3))

$$H = \left\{ h^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \dots, m'_1 m'_2 \cdots m'_n c \right\},$$

where $h^{(j)}_{p_1,p_2,\dots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism given by

$$(u_1, \dots, u_n) \longmapsto \left(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/cd} u_1, \dots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/cd} u_n\right).$$

Recall that $\widetilde{\Gamma}$ is generated by δ, η_1, $\eta_2, \dots, \eta_{n-1}$ (Corollary 7.2), where

$$\begin{aligned} \delta : (X_1, X_2, \dots, X_n) & \\ \longmapsto (e^{2\pi \mathrm{i}a_1/m_1d} X_1, & e^{2\pi \mathrm{i}a_2/m_2d} X_2, \dots, e^{2\pi \mathrm{i}(a_n+m_n\kappa)/m_nd} X_n), \\ \eta_i : (X_1, X_2, \dots, X_n) & \longmapsto (X_1, \dots, X_{i-1}, e^{2\pi \mathrm{i}/d} X_i, X_{i+1}, \dots, e^{-2\pi \mathrm{i}/d} X_n). \end{aligned}$$

Let α, β_i $(i = 1, 2, \dots, n-1)$ be automorphisms of $\mathbb{C}^n$ given by

$$\begin{aligned} \alpha &: (u_1, u_2, \dots, u_n) \longmapsto (e^{2\pi \mathrm{i}a_1/cd} u_1, e^{2\pi \mathrm{i}a_2/cd} u_2, \dots, e^{2\pi \mathrm{i}(a_n+m_n\kappa)/cd} u_n), \\ \beta_i &: (u_1, u_2, \dots, u_n) \longmapsto (u_1, \dots, u_{i-1}, e^{2\pi \mathrm{i}m'_i/d} u_i, u_{i+1}, \dots, e^{-2\pi \mathrm{i}m'_n/d} u_n). \end{aligned}$$

They are respectively the descents of δ, $\eta_i \in \widetilde{\Gamma}$ (with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$), hence α, $\beta_i \in H$.

LEMMA 7.8. *Any element of H is expressed by α, β_1, $\beta_2, \dots, \beta_{n-1}$. In fact, $h^{(j)}_{p_1,p_2,\dots,p_n} \in H$ is expressed as $h^{(j)}_{p_1,p_2,\dots,p_n} = \alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}}$.*

PROOF. Since α, $\beta_i \in H$ are the descents of δ, $\eta_i \in \widetilde{\Gamma}$ respectively, $\alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}} \in H$ is the descent of $\delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}} \in \widetilde{\Gamma}$. On the other hand, $h^{(j)}_{p_1,p_2,\dots,p_n} \in H$ is the descent of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \in \widetilde{\Gamma}$ (Lemma 4.8 (1)). The relation $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} = \delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}}$ (in Lemma 7.1) then implies $h^{(j)}_{p_1,p_2,\dots,p_n} = \alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}}$. □

Lemma 7.8 implies that H is generated by α, β_1, $\beta_2, \dots, \beta_{n-1}$. Here α and β_i are expressed by the following diagonal matrices:
$S = \mathrm{diag}\left(e^{2\pi \mathrm{i}a_1/cd}, e^{2\pi \mathrm{i}a_2/cd}, \dots, e^{2\pi \mathrm{i}a_{n-1}/m_{n-1}d}, e^{2\pi \mathrm{i}(a_n+m_n\kappa)/cd}\right)$ and
$T_i = \mathrm{diag}\left(1, \dots, 1, e^{2\pi \mathrm{i}m'_i/d}, 1, \dots, 1, e^{-2\pi \mathrm{i}m'_n/d}\right)$, where $e^{2\pi \mathrm{i}m'_i/d}$ lies in the ith place. Thus:

COROLLARY 7.9. *H is generated by α, β_1, $\beta_2, \ldots, \beta_{n-1}$, or as a subgroup of $GL(n, \mathbb{C})$, generated by the matrices S, T_1, $T_2, \ldots, T_{n-1}$.*

Here $\alpha, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are actually *not* independent. In fact, there are relations among them:

LEMMA 7.10. *The generators α, β_1, $\beta_2, \ldots, \beta_{n-1}$ of H satisfy the following relations*:

(a) $\alpha^{m'_1 m'_2 \cdots m'_{n-1} c} = \beta_1^{a_1 m'_2 m'_3 \cdots m'_{n-1}} \beta_2^{a_2 m'_1 m'_3 \cdots m'_{n-1}} \cdots \beta_{n-1}^{a_{n-1} m'_1 m'_2 \cdots m'_{n-2}}$.

(b) *For $i = 1, 2, \ldots, n-1$,*

$$\alpha^{m'_1 \cdots \check{m}'_i \cdots m'_n c} = \beta_1^{a_1 m'_2 \cdots \check{m}'_i \cdots m'_n} \cdots \beta_i^{(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/m'_i} \cdots \cdots \beta_{n-1}^{a_{n-1} m'_1 \cdots \check{m}'_i \cdots m'_{n-2} m'_n},$$

where note that $(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/m'_i$ is an integer.

REMARK 7.11. The existence of nontrivial relations among α, β_1, β_2, $\ldots, \beta_{n-1}$ implies that $H = \langle \alpha, \beta_1, \beta_2, \ldots, \beta_{n-1} \rangle$ is *not* isomorphic to the product of cyclic groups $\langle \alpha \rangle \times \langle \beta_1 \rangle \times \langle \beta_2 \rangle \times \cdots \times \langle \beta_{n-1} \rangle$.

7.4. Generators of G and relations among them

Recall that (see Lemma 6.1 (C))

$$G = \left\{ g^{(j)}_{p_1, p_2, \ldots, p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c \right\},$$

where $g^{(j)}_{p_1, p_2, \ldots, p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism given by

$$(v_1, \ldots, v_n) \longmapsto \left(e^{2\pi i l_1 (j a_1 + m_1 p_1)/cd} v_1, \ldots, e^{2\pi i l_n (j a_n + m_n p_n)/cd} v_n \right).$$

Recall that H is generated by α, β_1, $\beta_2, \ldots, \beta_{n-1}$ (Corollary 7.9), where

$$\alpha : (u_1, u_2, \ldots, u_n) \longmapsto (e^{2\pi i a_1/cd} u_1, e^{2\pi i a_2/cd} u_2, \ldots, e^{2\pi i (a_n + m_n \kappa)/cd} u_n),$$
$$\beta_i : (u_1, u_2, \ldots, u_n) \longmapsto (u_1, \ldots, u_{i-1}, e^{2\pi i m'_i/d} u_i, u_{i+1}, \ldots, e^{-2\pi i m'_n/d} u_n).$$

Let f, g_i $(i = 1, 2, \ldots, n-1)$ be automorphisms of $\mathbb{C}^n$ given by

$$f : (v_1, v_2, \ldots, v_n) \longmapsto (e^{2\pi i l_1 a_1/cd} v_1, e^{2\pi i l_2 a_2/cd} v_2, \ldots, e^{2\pi i l_n (a_n + m_n \kappa)/cd} v_n),$$
$$g_i : (v_1, v_2, \ldots, v_n) \longmapsto (v_1, \ldots, v_{i-1}, e^{2\pi i l_i m'_i/d} v_i, v_{i+1}, \ldots, e^{-2\pi i l_n m'_n/d} v_n).$$

They are respectively the descents of α, $\beta_i \in H$ (with respect to the covering $r : \mathbb{C}^n \to \mathbb{C}^n$), hence $f, g_i \in G$. As for Lemma 7.8, we can show the following:

LEMMA 7.12. *Any element of G is expressed by f, g_1, $g_2, \dots, g_{n-1}$. In fact, $g^{(j)}_{p_1,p_2,\dots,p_n} \in G$ is expressed as $g^{(j)}_{p_1,p_2,\dots,p_n} = f^j g_1^{p_1} g_2^{p_2} \cdots g_{n-1}^{p_{n-1}}$.*

Lemma 7.12 implies that:

COROLLARY 7.13. *G is generated by $f, g_1, g_2, \dots, g_{n-1}$, where f and g_i are expressed by the diagonal matrices*
$Q = \operatorname{diag}\bigl(e^{2\pi \mathrm{i} l_1 a_1/cd},\ e^{2\pi \mathrm{i} l_2 a_2/cd}, \dots, e^{2\pi \mathrm{i} l_{n-1} a_{n-1}/cd},\ e^{2\pi \mathrm{i} l_n (a_n + m_n \kappa)/cd}\bigr)$ *and*
$R_i = \operatorname{diag}\bigl(1, \dots, 1,\ e^{2\pi \mathrm{i} l_i m_i'/d},\ 1, \dots, 1,\ e^{-2\pi \mathrm{i} l_n m_n'/d}\bigr)$, *where $e^{2\pi \mathrm{i} l_i m_i'/d}$ lies in the ith place.*

Here $f, g_1, g_2, \dots, g_{n-1}$ are actually *not* independent. In fact, there are relations among them:

LEMMA 7.14. *The generators f, g_1, $g_2, \dots, g_{n-1}$ of G satisfy the following ralations*:

(a) $$f^{\operatorname{lcm}(m_1', m_2', \dots, m_{n-1}')c} = g_1^{a_1 \operatorname{lcm}(m_1', m_2', \dots, m_{n-1}')/m_1'} \cdots g_{n-1}^{a_{n-1} \operatorname{lcm}(m_1', m_2', \dots, m_{n-1}')/m_{n-1}'},$$
where note that $a_k \operatorname{lcm}(m_1', m_2', \dots, m_{n-1}')/m_k'$ $(k = 1, 2, \dots, n-1)$ is an integer (because m_k' divides $\operatorname{lcm}(m_1', m_2', \dots, m_{n-1}')$*).*

(b) *For $i = 1, 2, \dots, n-1$,*
$$\begin{aligned} &f^{\operatorname{lcm}(m_1', \dots, \check{m}_i', \dots, m_n')c} \\ &\quad = g_1^{a_1 \operatorname{lcm}(m_1', \dots, \check{m}_i', \dots, m_n')/m_1'} \cdots g_i^{(a_i m_1' \cdots \check{m}_i' \cdots m_n' - d)/l_i m_i'} \cdots \\ &\qquad \cdots g_{n-1}^{a_{n-1} \operatorname{lcm}(m_1', \dots, \check{m}_i', \dots, m_n')/m_{n-1}'}, \end{aligned}$$
where note that $(a_i m_1' \cdots \check{m}_i' \cdots m_n' - d)/l_i m_i'$ is an integer and for $k = 1, 2, \dots, \check{i}, \dots, n$, $a_k \operatorname{lcm}(m_1', \dots, \check{m}_i', \dots, m_n')/m_k'$ is an integer (because m_k' divides $\operatorname{lcm}(m_1', \dots, \check{m}_i', \dots, m_n')$*).*

REMARK 7.15. The existence of nontrivial relations among f, g_1, g_2, $\dots, g_{n-1}$ implies that $G = \langle f, g_1, g_2, \dots, g_{n-1} \rangle$ is *not* isomorphic to the product of cyclic groups $\langle f \rangle \times \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_{n-1} \rangle$.

CASE $n = 2$. Let a_1^* $(0 < a_1^* < m_1)$ be the integer such that $a_1 a_1^* \equiv 1 \bmod m_1$. If $n = 2$, then G is a cyclic group generated by $g : (u_1, u_2) \mapsto (e^{2\pi \mathrm{i}/cd} u_1, e^{2\pi \mathrm{i}\mathsf{q}/cd} u_2)$, where q $(0 < \mathsf{q} < cd)$ is the integer such that $\mathsf{q} \equiv \dfrac{a_1^* d - m_2'}{m_1'} \bmod cd$ (Theorem 2.1). Note that $\dfrac{a_1^* d - m_2'}{m_1'}$ is an integer (cf. Lemma 2.3 (1)). Here the automorphism g is expressed by the matrix $P := \begin{pmatrix} e^{2\pi \mathrm{i}/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}\mathsf{q}/cd} \end{pmatrix}$, and as a subgroup of $GL(2, \mathbb{C})$, G is generated by P. On the other hand by Corollary 7.13, G is generated by two matrices $Q = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & \\ 0 & e^{2\pi \mathrm{i}(b+n\kappa)/cd} \end{pmatrix}$ and $R_1 = \begin{pmatrix} e^{2\pi \mathrm{i}m'/d} & \\ 0 & e^{-2\pi \mathrm{i}n'/d} \end{pmatrix}$. Note that $l_1 = l_2 = 1$, thus $G = H$, $f = \alpha$, $g_1 = \beta_1$. We describe the relations among P and Q, R_1.

For simplicity, write m_1, m_2, a_1, a_2, a_1^*, β_1, R_1 as m, n, a, b, a^*, β, R, and set $c := \gcd(m, n)$, $m' := \frac{m}{c}$, $n' := \frac{n}{c}$ and $d := an' + bm' + m'n'c\kappa$.

PROPOSITION 7.16. *The matrices P, Q, $R \in GL(2, \mathbb{C})$ expressing the automorphisms g, α, β are related as follows*:

(1) $P^a = Q$, $P^m = R$.

(2) *Noting that $\dfrac{1 - aa^*}{m}$ is an integer* (*because* $aa^* \equiv 1 \bmod m$), *let l* $(0 < l < cd)$ *be the integer such that $l \equiv \dfrac{1 - aa^*}{m} \bmod cd$. Then* $Q^{a^*} R^l = P$.

PROOF. (1): We first show $P^a = Q$. Since $a\mathsf{q} \equiv \dfrac{a(a^* d - n')}{m'} \equiv \dfrac{d - an'}{m'} \equiv b + n\kappa \bmod cd$,

$$P^a = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}a\mathsf{q}/cd} \end{pmatrix} = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}(b+n\kappa)/cd} \end{pmatrix} = Q.$$

We next show $P^m = R$. Since $m\mathsf{q} \equiv \dfrac{m(a^* d - n')}{m'} \equiv a^* cd - cn' \equiv -cn' \bmod cd$,

$$P^m = \begin{pmatrix} e^{2\pi \mathrm{i}m/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}m\mathsf{q}/cd} \end{pmatrix} = \begin{pmatrix} e^{2\pi \mathrm{i}m'/d} & 0 \\ 0 & e^{-2\pi \mathrm{i}n'/d} \end{pmatrix} = R.$$

(2): We first show $P^{aa^*+ml} = P$. Since $l \equiv \dfrac{1-aa^*}{m} \bmod cd$ and $aa^* + m\dfrac{1-aa^*}{m} = 1$, we have $aa^* + ml \equiv 1 \bmod cd$. Hence

$$e^{2\pi \mathrm{i}(aa^*+ml)/cd} = e^{2\pi \mathrm{i}/cd}, \qquad e^{2\pi \mathrm{i}(aa^*+ml)\mathsf{q}/cd} = e^{2\pi \mathrm{i}\mathsf{q}/cd}.$$

Accordingly, $P^{aa^*+ml} = P$. Then $(P^a)^{a^*}(P^m)^l = P$. Here since $P^a = Q$ and $P^m = R$ hold by (1), $Q^{a^*}R^l = P$. The assertion is thus confirmed. □

COROLLARY 7.17. *The automorphisms* g, α, $\beta : \mathbb{C}^2 \to \mathbb{C}^2$ *are related as follows*:

(1) $g^a = \alpha$, $g^m = \beta$.

(2) *Noting that* $\dfrac{1-aa^*}{m}$ *is an integer* (*because* $aa^* \equiv 1 \bmod m$), *let* l ($0 < l < cd$) *be the integer such that* $l \equiv \dfrac{1-aa^*}{m} \bmod cd$. *Then* $\alpha^{a^*}\beta^l = g$.

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