J. Math. Sci. Univ. Tokyo
30 (2023), 365–369.

A Note on Characterizing Pluriharmonic Functions via the Ohsawa–Takegoshi Extension Theorem

By Takahiro Inayama

Abstract. For a continuous function, we prove that the function is pluriharmonic if and only if the equality part of the optimal Ohsawa–Takegoshi L^2-extension theorem is satisfied with respect to the metric having the function as a weight. This partially resolves the conjecture proposed by the author.

1. Introduction

On a one-dimensional complex domain, as is well known, a subharmonic function is characterized by the mean value inequality, and when the equality of the inequality holds, the function becomes a harmonic function. Subharmonic functions play an important role in complex analysis and geometry, and are also used in important theorems such as Hörmander's L^2-estimate [5] and Ohsawa–Takegoshi's L^2-extension theorem [9].

On the other hand, recent research has revealed that the fact that the optimal Ohsawa–Takegoshi L^2-extension theorem holds itself guarantees the subharmonicity of the weight. This property is called the minimal extension property [4] or the optimal L^p-extension property [1, 2] in a general setting, and has been widely studied and applied by various experts (cf. [3, 4, 1, 2, 6, 7]). In other words, subharmonic functions can be characterized by the inequality part of the optimal Ohsawa–Takegoshi L^2-extension theorem. Based on the analogy with the above, we propose the following conjecture in [7, Appendix A]:

Conjecture 1.1. *Let φ be an upper semi-continuous function on a domain $\Omega \subset \mathbb{C}^n$. Then the following are equivalent:*

(1) *φ is pluriharmonic.*

2020 *Mathematics Subject Classification.* 32A10, 32A36.

Key words: Ohsawa–Takegoshi extension theorem, L^2-extension, plurisubharmonic function, pluriharmonic function, L^2-extension index.

(2) *$\varphi > -\infty$ and for any holomorphic cylinder $a + P_{r,s,A}$, where $(a, r, s, A) \in \Omega_{\widetilde{\delta}}$ (see the notation below), there exists a unique holomorphic function f on $a + P_{r,s,A}$ satisfying $f(a) = 1$ and*

$$\int_{a+P_{r,s,A}} |f|^2 e^{-\varphi} \leq |P_{r,s,A}| e^{-\varphi(a)},$$

where $|P_{r,s,A}|$ is the volume of $P_{r,s,A}$.

We proved this conjecture in the case that φ was smooth by using the theory of the L^2-extension index in [7]. Our aim of this paper is to give an affirmative answer to the above conjecture for a *continuous* function φ. To clarify the claim of the theorem, several symbols will be prepared. Let $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$, $\mathbb{B}_s^m = \{z \in \mathbb{C}^m \mid |z| < s\}$ and $P_{r,s,A} = A(\Delta_r \times \mathbb{B}_s^{n-1})$ for $r, s > 0$ and $A \in \mathbf{U}(n)$, where $\mathbf{U}(n)$ is the set of all n-dimensional unitary groups. Here $P_{r,s,A}$ is a holomorphic cylinder. We let $\Omega_{\widetilde{\delta}} = \{(a, r, s, A) \in \Omega \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbf{U}(n) \mid a + P_{r,s,A} \subset \Omega\}$. Then the main theorem can be stated as follows.

THEOREM 1.2. *Let φ be a* ***continuous*** *function on Ω. Then φ is pluriharmonic if and only if for any $(a, r, s, A) \in \Omega_{\widetilde{\delta}}$, there exists a unique holomorphic function f on $a + P_{r,s,A}$ satisfying $f(a) = 1$ and*

$$\int_{a+P_{r,s,A}} |f|^2 e^{-\varphi} \leq |P_{r,s,A}| e^{-\varphi(a)}.$$

Note that the above f satisfies the equality

$$\int_{a+P_{r,s,A}} |f|^2 e^{-\varphi} = |P_{r,s,A}| e^{-\varphi(a)}.$$

For the proof, we use the terms of the L^2-extension index and the characterization of log-plurisubharmonicity.

Acknowledgment. The author is supported by Japan Society for the Promotion of Science, Grant-in-Aid for Research Activity Start-up (Grant No. 21K20336) and Grant-in-Aid for Early-Career Scientists (Grant No. 23K12978). He also expresses his gratitude to Dr. Wang Xu for fruitful discussions, sharing various ideas and his generous response.

2. The Proof of Main Theorem

In order to prove the main result, we prepare several notions. First, we review the minimal extension property [4] or the optimal L^2-extension property [1, 2]. In this paper, we follow the formulation of Deng, Ning, Wang and Zhou.

Definition 2.1 ([1, Definition 1.1]). Let Ω be a domain in $\mathbb{C}^n$ and φ be an upper semi-continuous function. Then we say that φ satisfies *the optimal L^2-extension property* if for any $(a, r, s, A) \in \Omega_{\tilde{\delta}}$, there exists a holomorphic function f on $a + P_{r,s,A}$ such that $f(a) = 1$ and

$$\int_{a+P_{r,s,A}} |f|^2 e^{-\varphi} \leq |P_{r,s,A}| e^{-\varphi(a)}.$$

The following theorem is important in relation to this definition.

Theorem 2.2 ([1, Theorem 1.6]). *Keep the setting. Then φ is plurisubharmonic if and only if φ satisfies the optimal L^2-extension property.*

Next, we explain the L^2-extension index introduced by the author in [7]. Here we adopt a slightly extended definition.

Definition 2.3. Let φ be a function $\varphi\colon \Omega \to [-\infty, \infty]$. Then we define the *$L^2$-extension index* L_φ of φ on $\Omega_{\tilde{\delta}}$ as follows: for $(a, r, s, A) \in \Omega_{\tilde{\delta}}$, if $-\infty < \varphi(a) < +\infty$,

$$\begin{aligned} &L_\varphi(a, r, s, A) \\ &\quad := \frac{1}{|P_{r,s,A}| K_{P_{r,s,A},\varphi}(a)} \\ &\quad = \inf \left\{ \frac{\int_{a+P_{r,s,A}} |f|^2 e^{-\varphi}}{|P_{r,s,A}| e^{-\varphi(a)}} \;\middle|\; f \in A^2(a + P_{r,s,A}, \varphi) \;\&\; f(a) = 1 \right\}, \end{aligned}$$

if $\varphi(a) = +\infty$, $L_\varphi(a, r, s, A) = +\infty$ and if $\varphi(a) = -\infty$, $L_\varphi(a, r, s, A) = 0$. Here, $K_{P_{r,s,A},\varphi}$ is the weighted Bergman kernel on $P_{r,s,A}$ with respect to

φ, $A^2(a + P_{r,s,A}, \varphi) = \{f \in \mathcal{O}(a + P_{r,s,A}) \mid \int_{a+P_{r,s,A}} |f|^2 e^{-\varphi} < +\infty\}$ and $\mathcal{O}(a + P_{r,s,A})$ is the space of all holomorphic functions on $a + P_{r,s,A}$.

By using this notion, we can rephrase the optimal L^2-extension property as follows: if $L_\varphi \leq 1$ for an upper semi-continuous function, then φ is plurisubharmonic. Our goal is to prove the next proposition that is paired with the above result.

Proposition 2.4. *Let φ be a lower semi-continuous function on a domain $\Omega \subset \mathbb{C}^n$ with $\varphi \not\equiv +\infty$. If $L_\varphi \geq 1$, then φ is plurisuperharmonic.*

Proof. We use the proposition that for a non-negative function v with $v \not\equiv 0$, $\log v$ is plurisubharmonic if and only if $ve^{2\mathrm{Re} g}$ is plurisubharmonic for every polynomial g. We take any polynomial g and any $(a, r, s, A) \in \Omega_{\tilde{\delta}}$. We may assume that $\varphi(a) < +\infty$. If $L_\varphi \geq 1$, it holds that

$$\int_{a+P_{r,s,A}} |e^g|^2 e^{-\varphi} \geq |P_{r,s,A}||e^{g(a)}|^2 e^{-\varphi(a)}.$$

Since $-\varphi$ is upper semi-continuous, we can say that $e^{-\varphi}e^{2\mathrm{Re} g}$ is plurisubharmonic (see Lemma 3.1 in [1]). Hence, $-\varphi$ is plurisubharmonic. □

By using this proposition, we can prove Theorem 1.2.

Proof of Theorem 1.2. If φ is pluriharmonic, take a holomorphic function h on $a + P_{r,s,A}$ satisfying $2\mathrm{Re}(h) = \varphi$ and use the argument in [7, Section 5]. Then we only prove the if part. Note that the assumption in Theorem 1.2 says that $L_\varphi \equiv 1$. Since $L_\varphi \leq 1$ and φ is upper semi-continuous, Theorem 2.2 implies φ is plurisubharmonic. Also, since $L_\varphi \geq 1$ and φ is lower semi-continuous, Proposition 2.4 implies that φ is plurisuperharmonic. □

I have discussed Conjecture 1.1 with Wang Xu. On May 15, I sent him the above proof. Then, by using the linearity of certain minimal L^2-integrals, Xu showed that the assumption that φ is lower semi-continuous is not needed and sent me the proof on May 16. About a month later, with Zhuo Liu, Xu consequently obtained the result that the upper semi-continuity of φ is also unnecessary and sent the manuscript [8] to me on June

26. Although their result is literally a generalization of my theorem, they encouraged me to write this paper on my result as a first important step. I would like to thank them for their consideration and warm encouragement.

References

[1] Deng, F., Ning, J. and Z. Wang, Characterizations of plurisubharmonic functions, Sci. China Math. **64** (2021), 1959–1970.

[2] Deng, F., Ning, J., Wang, Z. and X. Zhou, Positivity of holomorphic vector bundles in terms of L^p-estimates for $\overline{\partial}$, Math. Ann. **385** (2023), 575–607.

[3] Guan, Q. and X. Zhou, A solution of an L^2 extension problem with an optimal estimate and applications, Ann. of Math. (2) **181** (2015), no. 3, 1139–1208.

[4] Hacon, C., Popa, M. and C. Schnell, Algebraic fiber spaces over Abelian varieties: around a recent theorem by Cao and Paun, Contemp. Math. **712** (2018), 143–195.

[5] Hörmander, L., L^2 estimates and existence theorems for the $\overline{\partial}$ operator, Acta Math. **113** (1965), 89–152.

[6] Inayama, T., Optimal L^2-extensions on tube domains and a simple proof of Prékopa's theorem, J. Geom. Anal. **32**, 32 (2022). `https://doi.org/10.1007/s12220-021-00796-w`.

[7] Inayama, T., L^2-extension indices, sharper estimates and curvature positivity, arXiv:2210.08456.

[8] Liu, Z. and W. Xu, Characterizations of Griffiths positivity, pluriharmonicity and flatness, preprint.

[9] Ohsawa, T. and K. Takegoshi, On the extension of L^2 holomorphic functions, Math. Z. **195** (1987), no. 2, 197–204.

(Received July 6, 2023)

Department of Mathematics
Faculty of Science and Technology
Tokyo University of Science
2641 Yamazaki, Noda
Chiba 278-8510, Japan
E-mail: inayama_takahiro@rs.tus.ac.jp
inayama570@gmail.com