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Superconducting Phase in the BCS Model with Imaginary Magnetic Field

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Abstract. We prove that in the reduced quartic BCS model with an imaginary external magnetic field a spontaneous $U(1)$-symmetry breaking (SSB) and an off-diagonal long range order (ODLRO) occur. The system is defined on a hyper-cubic lattice with periodic boundary conditions at positive temperature. In the free part of the Hamiltonian we assume the nearest-neighbor hopping. The chemical potential is fixed so that the free Fermi surface does not degenerate. The term representing the interaction between electrons' spin and the imaginary external magnetic field is the z-component of the spin operator multiplied by a pure imaginary parameter. The SSB and the ODLRO are shown in the infinite-volume limit of the thermal average over the full Fermionic Fock space. The magnitude of the negative coupling constant must be larger than a certain value so that the gap equation is solvable. The gap equation is different from that of the conventional mean field BCS model because of the presence of the imaginary magnetic field. By adjusting the imaginary magnetic field this model shows the SSB and the ODLRO in high temperature, weak coupling regimes where the conventional reduced BCS model does not show these phenomena. The proof is based on Grassmann Gaussian integral formulations and a double-scale integration scheme to analytically control the formulations.

Contents

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1. Introduction

1.1. Introductory remarks

In 1957 ([1]) Bardeen, Cooper and Schrieffer proposed a microscopic theory of superconductivity which is widely known as the BCS theory today. As the importance of the BCS theory was recognized, many began to mathematically verify effective approximations made in the theory. Proving a superconductivity within the fundamental principles proposed by Bardeen, Cooper and Schrieffer has been a stimulating topic in mathematical physics until today. See e.g. the review article [8] for a recent trend of the subject. Many papers concerning mathematical physics of the BCS theory have been published since the early stage. The historical review [3] reported in 100th year after the discovery of superconductivity is enlightening. However, if we focus our attention on a basic simple question whether the BCS model shows superconductivity characterized by spontaneous $U(1)$-symmetry breaking (SSB) and off-diagonal long range order (ODLRO), we notice that there are unexpectedly few mathematical results answering this question. Here the BCS model is meant to be the Hamiltonian consisting of a kinetic part, quadratic in Fermionic operators, describing free movements of electrons and an interacting part, quartic in Fermionic operators, describing a long range interaction between Cooper pairs. We also require SSB and ODLRO to be shown in the infinite-volume limit of the thermal average over the full Fermionic Fock space.

In the strong coupling limit of the reduced BCS model, where the free part is the number operator multiplied by the chemical potential only and the interacting part is a product of the Cooper pair operators, a SSB and an ODLRO in the above sense were proved by a C^*-algebraic approach by Bru and de Siqueire Pedra in [4]. The model considered in [4] is allowed to contain the Hubbard type on-site interaction as well. The same authors also extended their C^*-algebraic framework to be applicable to the BCS model having a non-constant kinetic term and gave a mathematical sense of ODLRO in a limit of the finite systems under periodic boundary conditions in [5]. Before [4] many researchers had continued their efforts to analyze the BCS model in the quasi-spin formulation at positive temperature. The achievements of these authors are listed in the references of [3]. Here we refer to the original article [20] where the equivalence between correlation functions in the strong coupling limit of the reduced BCS model and those in the mean field BCS model was proved in the quasi-spin formulation. See also [6] for an analysis of the BCS model with non-constant free dispersion relations in the quasi-spin formulation at positive temperature. It should be remarked that the thermal average in the quasi-spin formulation amounts to the average over a proper subspace of the full Fermionic Fock space. There were also attempts to demonstrate SSB in the BCS model in Grassmann integral formulations. When the grand canonical partition function of the reduced BCS model is formulated into a Grassmann Gaussian integral, a quartic Grassmann polynomial resembling the BCS interaction appears to be integrated with a time-variable in its action. By artificially dividing the single time-integral into a double time-integral and thus deriving the so-called doubly reduced BCS model, Lehmann showed that a SSB occurs in a form of the Schwinger function in [14]. In [16] Mastropietro extended the approach based on the Grassmann integral formulation and showed that a SSB occurs in the Schwinger function where the interaction is of the doubly reduced BCS type tempered by time-integration with a Kac potential. The insertion of the Kac potential is in effect an interpolation between the doubly reduced BCS interaction and the reduced BCS interaction in the Grassmannian level. The gap equation in these studies is equal to that of the mean field BCS model.

Despite a long history of the research we can hardly find a thoroughly explicit demonstration of SSB and ODLRO in the full BCS model. In this situation this paper is devoted to demonstrating them in the BCS model in a non-standard parameter region of the complex plane. We will prove that a SSB and an ODLRO occur in the reduced BCS model in a way fulfilling the above-mentioned requirements, provided a term representing the interaction between electrons' spin and an imaginary external magnetic field is added to the Hamiltonian. More precisely, the interacting term with the imaginary magnetic field is given by the z-component of the spin operator multiplied by a pure imaginary parameter. The model is initially defined on a finite hyper-cubic lattice with periodic boundary conditions. In the free part of the Hamiltonian we assume the nearest-neighbor hopping. We restrict the range of the chemical potential so that the free Fermi surface does not degenerate. The magnitude of the negative coupling constant must be larger than a certain value so that the gap equation has a positive solution. At the same time it must be smaller than a certain value so that our perturbative treatments make sense. Thus, there are two kinds of constraint on the magnitude of the negative coupling constant. It is due to a fine tuning of the imaginary magnetic field that we can actually choose a coupling constant satisfying both the constraints. The gap equation is different from that of the conventional mean field BCS model because of the insertion of the imaginary magnetic field. Consequently it turns out that the SSB and the ODLRO can occur in high temperature, weak coupling regimes where these phenomena do not show up in the conventional reduced BCS model. The presence of the imaginary magnetic field breaks the hermiticity of the whole Hamiltonian. However, it will be proved as a part of our main results that the grand canonical partition function takes a real positive value in parameter regions where our analytical methods are valid.

From a technical view point this paper is seen to be a continuation of the Grassmann integration approach by Lehmann ([14]) and Mastropietro ([16]). In the Grassmannian level we divide the reduced BCS interaction into the doubly reduced BCS interaction and the correction term, transform the doubly reduced interaction into an integral of quadratic Grassmann polynomials by means of the Hubbard-Stratonovich transformation and estimate the Grassmann Gaussian integral having the quartic correction term in its exponent by the tree expansion. The use of the Hubbard-Stratonovich transformation was motivated by [14], [16] and an important division technique of truncated expectations which works to produce an extra inverse volume factor was influenced by [16]. However, there are notable differences between the conclusions of this paper and those of the preceding articles. This paper starts with Fermionic operators defined on the Fock space and concludes the SSB and the ODLRO in the infinite-volume limit of the full trace thermal expectations, while the conclusions of [14], [16] concern limit values of the Schwinger functions on Grassmann algebras. In fact it is not yet known how to realize the doubly reduced BCS interaction with or without the Kac potential in a concrete form of Fermionic operators. Moreover, since our gap equation is different from that of the mean field BCS model, the SSB and the ODLRO take place in a parameter region where the conventional BCS gap equation is not solvable and thus where SSB and ODLRO do not emerge in the sense of [14], [16]. As another new technical aspect we show the convergence of the infinite-volume limit of the finite-volume thermal expectations by sending the box size to infinity without taking a subsequence. As the result the SSB and the ODLRO can be claimed in the limit, not only in some accumulation points of the finite-volume formulations. To prove this, it is essential to establish the full convergent property of the Grassmann Gaussian integral of the correction term which does not simply follow from a uniform boundedness of the Grassmann integration and requires a detailed analysis of the tree expansion of truncated expectations. The analysis is performed in Subsection 4.3.

The technical core of our construction is the estimation of the correction to the doubly reduced BCS model. The estimation is completed by a double-scale integration process over the Matsubara frequency. The first integration involves a covariance with all but one time-momentum, while the covariance in the second integration contains only one time-momentum.

We should declare in this remark that we use Pedra-Salmhofer's type determinant bound (PS bound, [17]) to bound the determinant of the first covariance with large Matsubara frequencies. In general the application of the PS bound is a very efficient alternative to a multi-scale integration procedure over large Matsubara frequencies. By applying it one can prepare the input to the succeeding infrared integration process by a simple single-scale integration. In this paper, which aims at providing the first convincing proof of SSB and ODLRO in the BCS model with an imaginary magnetic field, we decide to make the construction simple and thus choose to apply the PS bound rather than go through a self-contained but lengthy multi-scale Matsubara ultra-violet integration. Also, in the interest of simplicity we do not perform a multi-scale infrared integration to improve the dependency of possible magnitude of the coupling constant on the temperature. As a consequence, this paper does not ensure that one can take the temperature close to zero while keeping the magnitude of the coupling constant positive. This may be seen as a shortcoming of this paper's results. Since it needs to classify Grassmann polynomials at each scale, the proof based on a multi-scale integration would be substantially longer. This thought together with the hope that a simpler construction must be more convincing led us to conclude our construction only by the double-scale integration. A qualitative improvement of the temperature-dependency of the coupling constant by means of a multi-scale infrared integration should be performed elsewhere.

The main novelty of this paper is to reveal the mathematical fact that the insertion of the imaginary magnetic field in the BCS model makes it possible to prove SSB and ODLRO in wide parameter regions. We should note that the extension of the external magnetic field to the complex plane in many-body systems has been an important subject of mathematical physics since the pioneering study by Lee and Yang ([24], [13]). At the same time the presence of the imaginary magnetic field admittedly makes it difficult to find a conventional physical meaning of the model. It is interesting and encouraging to know that the Lee-Yang zeros of the partition function of the Ising model are being experimentally realized through a time domain measurement of the spin system ([18]). What appears particularly interesting to us in [18] is that the partition function of the ferromagnetic Ising model with long range interaction under an imaginary magnetic field was identified with the coherence of a central spin coupled to the spin bath, which was experimentally measured. See also [22], [21] for the idea behind the experiment [18]. A message from [18] is that if the partition function with an imaginary magnetic field is measurable, then extending the magnetic field into the complex plane is not only a way to solve a mathematical problem but could be an analysis of a model of the real world. Amid the latest progress of physical experiments our hope from a mathematical side is that the superconducting phase in the BCS model with an imaginary magnetic field should be experimentally realized someday.

The contents of this paper are outlined as follows. In the rest of this section we define the model and officially state the main results of this paper. In Section 2 we formulate the grand canonical partition function of the model Hamiltonian by means of the Hubbard-Stratonovich transformation and the Grassmann Gaussian integration. In Section 3 we construct a double-scale integration process in a generalized setting with the aim of applying it to estimate the Grassmann Gaussian integral of the correction term in the following section. In Section 4 we apply the general results obtained in the previous section to the actual model problem and derive necessary bound properties. Then we show necessary convergent properties of the Grassmann Gaussian integral of the correction term in the time-continuum, infinite-volume limit. After these preparations we complete the proof of the main theorem. In Appendix A we provide a short proof to Pedra-Salmhofer's type determinant bound used in our construction for completeness. We also list notations which are used over multiple sections for readers' convenience at the end of the paper.

1.2. The model and the main results

Throughout the paper the spatial dimension is denoted by d. With $L(\in \mathbb{N} = \{1, 2, \cdots\})$ we define the spatial lattice Γ by $\Gamma := \{0, 1, 2, \cdots, L-1\}^d$. For $(\mathbf{x}, \sigma) \in \Gamma \times \{\uparrow, \downarrow\}$ let $\psi^*_{\mathbf{x}\sigma}$, $\psi_{\mathbf{x}\sigma}$ denote the Fermionic creation / annihilation operator respectively. We impose periodic boundary conditions on the finite-volume system. To describe the periodicity, it is convenient to use the map $r_L : \mathbb{Z}^d \to \Gamma$ which satisfies $r_L(\mathbf{x}) = \mathbf{x}$ in $(\mathbb{Z}/L\mathbb{Z})^d$ for any $\mathbf{x} \in \mathbb{Z}^d$. For $(\mathbf{x}, \sigma) \in \mathbb{Z}^d \times \{\uparrow, \downarrow\}$ we identify $\psi^{(*)}_{\mathbf{x}\sigma}$ with $\psi^{(*)}_{r_L(\mathbf{x})\sigma}$. The free part

H_0 of our Hamiltonian is defined by

$$\mathsf{H}_0 := \sum_{\mathbf{x}\in\Gamma}\sum_{\sigma\in\{\uparrow,\downarrow\}}\left((-1)^{hop}\sum_{j=1}^{d}(\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}+\mathbf{e}_j\sigma}+\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}-\mathbf{e}_j\sigma})-\mu\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}\right),$$

where $hop \in \{0,1\}$, $\mathbf{e}_j$ $(j=1,2,\cdots,d)$ are the standard basis of $\mathbb{R}^d$ and the real parameter μ is the chemical potential. For simplicity we adopt the unit where the hopping amplitude is scaled to be 1. We use the parameter *hop* to treat the positive hopping and the negative hopping at once. Moreover, we include the number operator multiplied by the chemical potential in the free Hamiltonian. Also we restrict the hopping of electrons to be only between nearest-neighbor sites. Throughout the paper except Remark 1.9 we assume that

$$\mu \in (-2d, 2d)$$

so that the free Fermi surface $\{\mathbf{k}\in[0,2\pi)^d \mid (-1)^{hop}2\sum_{j=1}^{d}\cos k_j-\mu=0\}$ does not degenerate. Only in Remark 1.9 we consider the degenerate case. In [1] the complex phonon-electron interaction was reduced into a sum of product of 2 Cooper pair operators. We consider the reduced BCS interaction with constant matrix element defined as follows.

$$\mathsf{V} := \frac{U}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\uparrow},$$

where U is a real negative parameter controlling the strength of non-local attraction between Cooper pairs. The BCS model H is defined by

$$\begin{aligned}\mathsf{H} &:= \mathsf{H}_0+\mathsf{V}\\ &= \sum_{\mathbf{x}\in\Gamma}\sum_{\sigma\in\{\uparrow,\downarrow\}}\left((-1)^{hop}\sum_{j=1}^{d}(\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}+\mathbf{e}_j\sigma}+\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}-\mathbf{e}_j\sigma})-\mu\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}\right)\\ &\quad+\frac{U}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\uparrow},\end{aligned}$$

which is a self-adjoint operator on the Fermionic Fock space $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$.

In this paper we focus on two characteristics of superconductivity and try to prove their existence in the infinite-volume limit of the system. One characteristic is spontaneous symmetry breaking (SSB). The other characteristic of our interest is off-diagonal long range order (ODLRO). A mathematical description of SSB is the following. We add a $U(1)$-symmetry breaking external field

$$\mathsf{F} = \gamma \sum_{\mathbf{x}\in\Gamma} (\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow} + \psi_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\uparrow}), \quad \gamma \in \mathbb{R}$$

to the system and observe the thermal expectation value of the pairing operator in the limit $\gamma \to 0$ after taking the limit $L \to \infty$,

$$\lim_{\gamma\to 0} \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+\mathsf{F})}\psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+\mathsf{F})}}.$$

Here the trace operation is taken over the Fermionic Fock space and $\beta(\in \mathbb{R}_{>0})$ is the inverse temperature. If the expectation value converges to a nonzero value, it is said that a SSB occurs in the system. This is because the $U(1)$-gauge symmetry which the original system possesses remains broken even after removing the symmetry-breaking external field. A long range correlation between Cooper pairs is explained by the behavior of the 4-point correlation function in the infinite-volume limit,

$$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta\mathsf{H}}\psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow}\psi_{\hat{\mathbf{y}}\downarrow}\psi_{\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta\mathsf{H}}}.$$

If the correlation function in the infinite-volume limit converges to a nonzero value as the distance between $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ goes to infinity, the system is said to exhibit an ODLRO (see [23]). These phenomena have been desired to be proven in the BCS model. Despite many years of research after [1], the full rigorous demonstration of SSB and ODLRO in the BCS model seems unexpectedly scarce. Amid this situation this paper is devoted to revealing a new fact of the BCS model that a SSB and an ODLRO are present under an external imaginary magnetic field. More precise explanation of our plan is that we add the operator $i\theta\mathsf{S}_z$ ($\theta \in \mathbb{R}$) to the BCS model H and prove the existence of a SSB and an ODLRO. Here S_z is the z-component of the spin operator.

$$\mathsf{S}_z := \frac{1}{2}\sum_{\mathbf{x}\in\Gamma}(\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow} - \psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow}).$$

The term $i\theta \mathsf{S}_z$ is formally interpreted as an interaction between the imaginary magnetic field $(0,0,i\theta)$ and the electrons' spin.

Since adding $i\theta \mathsf{S}_z$ to the Hamiltonian breaks hermiticity, we do not know whether the partition function $\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}$ remains non-zero. Thus, even the well-definedness of the thermal expectation is unclear. We know at least the following. Set

$$\mathsf{A}_1 := \psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow}, \quad \mathsf{A}_2 := \psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow}\psi_{\hat{\mathbf{y}}\downarrow}\psi_{\hat{\mathbf{y}}\uparrow}.$$

LEMMA 1.1.

$$\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})},\ \mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1),\ \mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_2) \in \mathbb{R}$$

and

$$\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1) = \mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1^*).$$

PROOF. Let us define the transforms $\mathcal{U}_1, \mathcal{U}_2$ on $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$ by

$$\begin{aligned}
&\mathcal{U}_1\Omega := \Omega,\\
&\mathcal{U}_1\psi^*_{\mathbf{x}_1\sigma_1}\psi^*_{\mathbf{x}_2\sigma_2}\cdots\psi^*_{\mathbf{x}_n\sigma_n}\Omega := \psi^*_{\mathbf{x}_1-\sigma_1}\psi^*_{\mathbf{x}_2-\sigma_2}\cdots\psi^*_{\mathbf{x}_n-\sigma_n}\Omega,\\
&\mathcal{U}_2\Omega := \Omega,\\
&\mathcal{U}_2\psi^*_{\mathbf{x}_1\sigma_1}\psi^*_{\mathbf{x}_2\sigma_2}\cdots\psi^*_{\mathbf{x}_n\sigma_n}\Omega := i^n\psi^*_{\mathbf{x}_1\sigma_1}\psi^*_{\mathbf{x}_2\sigma_2}\cdots\psi^*_{\mathbf{x}_n\sigma_n}\Omega,\\
&(\forall n\in\mathbb{N},\ (\mathbf{x}_j,\sigma_j)\in\Gamma\times\{\uparrow,\downarrow\}\ (j=1,2,\cdots,n))
\end{aligned}$$

and by linearity, where Ω is the vacuum of the Fock space. The transforms $\mathcal{U}_1, \mathcal{U}_2$ are unitary. Moreover,

$$\begin{aligned}
\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})} &= \mathrm{Tr}\, e^{-\beta\mathcal{U}_1(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})\mathcal{U}_1^*} = \mathrm{Tr}\, e^{-\beta(\mathsf{H}-i\theta\mathsf{S}_z-\mathsf{F})}\\
&= \mathrm{Tr}\, e^{-\beta\mathcal{U}_2(\mathsf{H}-i\theta\mathsf{S}_z-\mathsf{F})\mathcal{U}_2^*} = \mathrm{Tr}\, e^{-\beta(\mathsf{H}-i\theta\mathsf{S}_z+\mathsf{F})}\\
&= \overline{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}.
\end{aligned}$$

Thus, $\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\in\mathbb{R}$. The other claims can be checked in the same way. □

It will be proved as a part of the main theorem that $\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})} > 0$ for sufficiently large L. The next lemma tells us that it suffices to analyze the system for $\theta\in[0,2\pi/\beta]$.

LEMMA 1.2. *Assume that $\theta \in \mathbb{R}$, $\theta' \in (-2\pi/\beta, 2\pi/\beta]$ and $\theta = \theta'$ in $\mathbb{R}/\frac{4\pi}{\beta}\mathbb{Z}$. Then,*

$$\begin{aligned}&\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})} = \mathrm{Tr}\, e^{-\beta(\mathsf{H}+i|\theta'|\mathsf{S}_z+\mathsf{F})},\\ &\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_j) = \mathrm{Tr}(e^{-\beta(\mathsf{H}+i|\theta'|\mathsf{S}_z+\mathsf{F})}\mathsf{A}_j),\ (j=1,2).\end{aligned}$$

PROOF. Note that the operator S_z commutes with H, F, $\psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow}$, $\psi_{\hat{\mathbf{x}}\downarrow}\psi_{\hat{\mathbf{x}}\uparrow}$ for any $\hat{\mathbf{x}} \in \mathbb{Z}^d$. The trace operation over the Fock space can be decomposed into the sum of the trace over each eigenspace of S_z. Since each eigenvalue of S_z belongs to $\frac{1}{2}\mathbb{Z}$,

$$\begin{aligned}\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})} &= \mathrm{Tr}(e^{-\beta(\mathsf{H}+\mathsf{F})}e^{-i\beta\theta\mathsf{S}_z}) = \mathrm{Tr}(e^{-\beta(\mathsf{H}+\mathsf{F})}e^{-i\beta\theta'\mathsf{S}_z})\\ &= \mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta'\mathsf{S}_z+\mathsf{F})}.\end{aligned}$$

Moreover, by Lemma 1.1,

$$\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta'\mathsf{S}_z+\mathsf{F})} = \mathrm{Tr}\, e^{-\beta(\mathsf{H}+i|\theta'|\mathsf{S}_z+\mathsf{F})}.$$

Thus, the first equality is obtained. The other equalities can be derived in the same way. □

From here we always assume that

$$\theta \in \left[0, \frac{2\pi}{\beta}\right).$$

In this paper we do not treat the case $\theta = 2\pi/\beta$. As in this case the free partition function vanishes (see Lemma 2.1), we are unable to define the free covariance which plays a central role in our analysis.

In order to officially state the main results of this paper, we should make clear notations used in the statements. Let $\|\cdot\|_{\mathbb{R}^d}$ denote the euclidean norm of $\mathbb{R}^d$. For a function $f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ and $c \in \mathbb{C}$ we write

$$\lim_{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d}\to\infty} f(\mathbf{x},\mathbf{y}) = c$$

if for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $r \in \mathbb{R}_{>0}$ such that $|f(\mathbf{x},\mathbf{y}) - c| < \varepsilon$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ satisfying $\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d} \geq r$. For a proposition P let 1_P be 1 if P is true, 0 otherwise. We define the function $e : \mathbb{R}^d \to \mathbb{R}$ by

$$e(\mathbf{k}) := (-1)^{hop} 2 \sum_{j=1}^{d} \cos k_j - \mu, \quad \mathbf{k} = (k_1, k_2, \cdots, k_d) \in \mathbb{R}^d,$$

which is in fact the free dispersion relation. To estimate possible magnitude of the coupling constant, we use the function $g_d : (0, \infty) \to \mathbb{R}$ defined as follows.

$$g_d(x) := 1_{d \ge 2} (\log(x^{-1} + 1))^{\frac{d}{d+1}} x^{-\frac{1}{d+1}} + 1_{d=1} (4 - \mu^2)^{-\frac{1}{2}} \log(x^{-1} + 1). \tag{1.1}$$

The main result of this paper is the following.

THEOREM 1.3. *Assume that $\beta \in \mathbb{R}_{>0}$, $U \in \mathbb{R}_{<0}$. There exist constants $c_1(d) \in \mathbb{R}_{>0}$, $c_2(d) \in (0, 1]$ depending only on d such that the following statements hold true.*

(i) *Assume that $\theta \in [0, 2\pi/\beta)$ and*

$$|U| < c_2(d) \left(1 + \beta^{d+3} + (1 + \beta^{-1}) g_d \left(\left| \frac{\theta}{2} - \frac{\pi}{\beta} \right| \right) \right)^{-2}.$$

Then, there exists $L_0 \in \mathbb{N}$ such that

$$\mathrm{Tr}\, e^{-\beta(\mathsf{H} + i\theta \mathsf{S}_z + \mathsf{F})} \in \mathbb{R}_{>0}, \quad (\forall L \in \mathbb{N} \text{ with } L \ge L_0,\ \gamma \in [0, 1]).$$

(ii) *Assume that $\theta \in [0, 2\pi/\beta)$ and*

$$\begin{aligned} & c_1(d)(2d - |\mu|)^{1-d} \beta \left| \frac{\theta}{2} - \frac{\pi}{\beta} \right| \\ & \cdot \left(1_{|\frac{\theta}{2} - \frac{\pi}{\beta}| \le \frac{1}{2}(2d - |\mu|)} + 1_{|\frac{\theta}{2} - \frac{\pi}{\beta}| > \frac{1}{2}(2d - |\mu|)} (2d - |\mu|)^{-1} \left| \frac{\theta}{2} - \frac{\pi}{\beta} \right| \right) \\ & < |U| < c_2(d) \left(1 + \beta^{d+3} + (1 + \beta^{-1}) g_d \left(\left| \frac{\theta}{2} - \frac{\pi}{\beta} \right| \right) \right)^{-2}. \end{aligned} \tag{1.2}$$

Then, there uniquely exists $\Delta \in \mathbb{R}$ such that $\Delta > 0$ and

(1.3)

$$-\frac{2}{|U|}+\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\frac{\sinh(\beta\sqrt{e(\mathbf{k})^2+\Delta^2})}{(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+\Delta^2}))\sqrt{e(\mathbf{k})^2+\Delta^2}}=0.$$

Moreover,

(1.4)

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})\right)$$
$$=\frac{\Delta^2}{|U|}-\frac{1}{\beta(2\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\log\Bigg(2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta e(\mathbf{k})}+e^{\beta(\sqrt{e(\mathbf{k})^2+\Delta^2}-e(\mathbf{k}))}+e^{-\beta(\sqrt{e(\mathbf{k})^2+\Delta^2}+e(\mathbf{k}))}\Bigg).$$

(1.5)

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}=\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi_{\hat{\mathbf{x}}\downarrow}\psi_{\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}=-\frac{\Delta}{|U|}.$$

(1.6)

$$\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow}\psi_{\hat{\mathbf{y}}\downarrow}\psi_{\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}=\frac{\Delta^2}{U^2}.$$

(iii) *Assume that* $\theta\in[0,\pi/\beta)$ *and*

$$|U|<c_2(d)\left(1+\beta^{d+3}+(1+\beta^{-1})g_d\left(\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right|\right)\right)^{-2}.$$

Then, for any $\Delta\in\mathbb{R}$ *the equation* (1.3) *does not hold. Moreover, the statements* (1.4), (1.5), (1.6) *hold with* $\Delta=0$.

(iv) *For any* $\beta\in\mathbb{R}_{>0}$ *there exists* $\delta\in\mathbb{R}_{>0}$ *such that*

$$c_1(d)(2d-|\mu|)^{1-d}\beta\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right| \cdot\left(1_{|\frac{\theta}{2}-\frac{\pi}{\beta}|\le\frac{1}{2}(2d-|\mu|)}+1_{|\frac{\theta}{2}-\frac{\pi}{\beta}|>\frac{1}{2}(2d-|\mu|)}(2d-|\mu|)^{-1}\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right|\right) < c_2(d)\left(1+\beta^{d+3}+(1+\beta^{-1})g_d\left(\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right|\right)\right)^{-2} \tag{1.7}$$

for any $\theta\in[2\pi/\beta-\delta,2\pi/\beta)$. *Thus, for any* $\theta\in[2\pi/\beta-\delta,2\pi/\beta)$ *there exists* $U\in\mathbb{R}_{<0}$ *such that* (1.2) *holds.*

(v) *Assume that* $\theta\in[\pi/\beta,2\pi/\beta)$ *and* (1.2) *holds. Then,*

$$|U|<c_2(d)\left(1+\beta^{d+3}+(1+\beta^{-1})g_d\left(\left|\frac{\eta}{2}-\frac{\pi}{\beta}\right|\right)\right)^{-2}$$

for any $\eta\in[0,\pi/\beta)$ *and thus the conclusions of* (iii) *hold with these* β, U *and* η *in place of* θ.

The claims (iv), (v) can be proved here. Set $\Theta:=|\theta/2-\pi/\beta|$. If $\Theta\le\frac{1}{2}(2d-|\mu|)$, the inequality (1.7) is equivalently written as follows.

$$\begin{aligned}&(1+\beta^{d+3})\Theta^{\frac{1}{2}}+1_{d=1}(1+\beta^{-1})(4-\mu^2)^{-\frac{1}{2}}(\log(\Theta^{-1}+1))\Theta^{\frac{1}{2}}\\&+1_{d\ge2}(1+\beta^{-1})(\log(\Theta^{-1}+1))^{\frac{d}{d+1}}\Theta^{\frac{1}{2}-\frac{1}{d+1}}\\&<(c_1(d)^{-1}(2d-|\mu|)^{d-1}\beta^{-1}c_2(d))^{\frac{1}{2}}.\end{aligned}$$

Since the left-hand side converges to 0 as $\Theta\searrow 0$, the claim (iv) holds true. The claim (v) follows from the fact that $g_d:(0,\infty)\to\mathbb{R}$ is decreasing.

REMARK 1.4. The implication of the claim (iv) is that at any temperature we can choose $\theta\in[\pi/\beta,2\pi/\beta)$ and the negative coupling constant U so that SSB and ODLRO occur in the system. Moreover, since

$$\lim_{\theta\nearrow\frac{2\pi}{\beta}}c_2(d)\left(1+\beta^{d+3}+(1+\beta^{-1})g_d\left(\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right|\right)\right)^{-2}=0,$$

for any small $U_0\in\mathbb{R}_{>0}$ and at any positive temperature we can choose $\theta\in[\pi/\beta,2\pi/\beta)$ and the negative coupling constant U so that $|U|\le U_0$ and

SSB and ODLRO occur in the system. In other words, at arbitrarily high temperature, for arbitrarily weak coupling SSB and ODLRO take place in the system with an imaginary magnetic field.

REMARK 1.5. The implication of the claim (v) is the following. Assume that (1.2) holds with some $(U,\beta,\theta) \in \mathbb{R}_{<0} \times \mathbb{R}_{>0} \times [\pi/\beta, 2\pi/\beta)$. Then, SSB and ODLRO occur in the system with (U,β,θ) by the claim (ii), while SSB and ODLRO do not occur in the system with (U,β,η) for any $\eta \in [0,\pi/\beta)$. By the claim (iv) we can always choose $(U,\beta,\theta) \in \mathbb{R}_{<0} \times \mathbb{R}_{>0} \times [\pi/\beta, 2\pi/\beta)$ such that (1.2) holds with (U,β,θ). By fixing these U, β and taking η to be 0 we can conclude in other words that superconductivity characterized by SSB and ODLRO emerges in the BCS model with an imaginary magnetic field and it does not emerge in the BCS model without an imaginary magnetic field.

REMARK 1.6. The condition

$$|U| < c_2(d)\left(1+\beta^{d+3}+(1+\beta^{-1})g_d\left(\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right|\right)\right)^{-2}$$

is necessary to ensure that our double-scale integration converges. Thus, the claim (iii) especially means that within the analytical framework of the present paper we cannot prove the existence of superconductivity in the conventional reduced BCS Hamiltonian, which is the case $\theta = 0$.

REMARK 1.7. There is no essential reason to choose the spatial lattice to be $\{0,1,\cdots,L-1\}^d$. One can prove that all the partition functions and the thermal expectations in the theorem are equivalent to those defined in the system on the spatial lattice $\{0,1,\cdots,L-1\}^d + \mathbf{a}$ with the periodic boundary conditions for any $\mathbf{a} \in \mathbb{Z}^d$.

REMARK 1.8. We introduce the parameter $hop(\in \{0,1\})$ to treat the model with positive hopping and the model with negative hopping at the same time. However, if $L \in 2\mathbb{N}$, by the unitary transform $\psi^*_{\mathbf{x}\sigma} \to (-1)^{\sum_{j=1}^d x_j}\psi^*_{\mathbf{x}\sigma}$ $(\mathbf{x} = (x_1,\cdots,x_d) \in \Gamma)$ we can change the sign of hopping by keeping all the other terms unchanged. Thus, the role of the parameter *hop* seems not essential. We add it for completeness. It causes no technical complication.

REMARK 1.9. The reason why we only consider the nearest-neighbor hopping in the free Hamiltonian is that in this case the free dispersion relation takes the relatively simple form $e(\mathbf{k})$ which allows us to make explicit the condition (1.2). We made this choice to claim the main results of this paper simply and explicitly. In fact the condition

$$\begin{aligned}|U| >& c_1(d)(2d-|\mu|)^{1-d}\beta\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right| \\ &\cdot\left(1_{|\frac{\theta}{2}-\frac{\pi}{\beta}|\le\frac{1}{2}(2d-|\mu|)}+1_{|\frac{\theta}{2}-\frac{\pi}{\beta}|>\frac{1}{2}(2d-|\mu|)}(2d-|\mu|)^{-1}\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right|\right)\end{aligned} \tag{1.8}$$

in (1.2) is a sufficient condition for the inequality

$$-\frac{2}{|U|}+\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|}>0,$$

which is a necessary and sufficient condition for the existence of a positive solution to the gap equation (1.3). The theorem can be claimed under the condition

$$|U|>2\left(\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|}\right)^{-1}$$

in place of the condition (1.8). We should also remark that apart from a multiplication of irrelevant positive constant, the term $g_d(|\theta/2-\pi/\beta|)$ is derived as an upper bound on the integral

$$\int_{[0,2\pi]^d}d\mathbf{k}\frac{1}{\sqrt{|\theta/2-\pi/\beta|^2+e(\mathbf{k})^2}}. \tag{1.9}$$

In fact we can replace the term $g_d(|\theta/2-\pi/\beta|)$ in the condition

$$|U|<c_2(d)\left(1+\beta^{d+3}+(1+\beta^{-1})g_d\left(\left|\frac{\theta}{2}-\frac{\pi}{\beta}\right|\right)\right)^{-2}$$

by the integral (1.9). The validity of these modifications would be clearly seen after completing the proof of Theorem 1.3. Since it is important in this approach to guarantee the solvability of the gap equation and the convergence of Grassmann integrations at the same time, the sufficient conditions should be explicitly comparable. We decide not to pursue the issue of generalization of the dispersion relation or the whole Hamiltonian in this paper. Here we list the lemmas which use the specific form of the free dispersion relation and eventually lead to the condition (1.2). These are Lemma 4.8, Lemma 4.17 and Lemma 4.18.

However, based on the above modifications of the crucial conditions, let us see that the results hold for the degenerate case $\mu \in \{2d, -2d\}$ as the least extension of the theorem. By letting Θ, $c(d)$ denote $|\theta/2 - \pi/\beta|$, a positive constant depending only on d respectively we observe in this case that

$$\begin{aligned}
&\int_{[0,2\pi]^d} d\mathbf{k} \frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2) + \cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|} \\
&\ge c(d)\beta^{-1} \int_{[0,2\pi]^d} d\mathbf{k} \frac{1_{|e(\mathbf{k})| \le \beta^{-1}}}{e(\mathbf{k})^2 + \Theta^2} \ge c(d)\beta^{-1} \int_0^1 dr r^{d-1} \frac{1_{r^2 \le \beta^{-1}}}{r^4 + \Theta^2} \\
&\ge c(d)\beta^{-1}\big(1_{\Theta \le \min\{1,\beta^{-1}\}} \\
&\qquad \cdot (1_{d \le 3}\Theta^{\frac{d}{2}-2} + 1_{d=4}\log(\min\{1,\beta^{-1}\}\Theta^{-1}) + 1_{d \ge 5}(\min\{1,\beta^{-1}\})^{\frac{d}{2}-2}) \\
&\quad + 1_{\Theta > \min\{1,\beta^{-1}\}}(\min\{1,\beta^{-1}\})^{\frac{d}{2}}\Theta^{-2}\big), \\
&\int_{[0,2\pi]^d} d\mathbf{k} \frac{1}{\sqrt{\Theta^2 + e(\mathbf{k})^2}} \le c(d) \int_0^1 dr \frac{r^{d-1}}{\sqrt{\Theta^2 + r^4}} \\
&\le c(d)\big(1_{\Theta \le 1}(1_{d=1}\Theta^{-\frac{1}{2}} + 1_{d=2}\log(1+\Theta^{-1}) + 1_{d \ge 3}) + 1_{\Theta > 1}\Theta^{-1}\big).
\end{aligned}$$

Define the function $g_{d,s} : (0,\infty) \to \mathbb{R}$ by

$$g_{d,s}(x) := 1_{x \le 1}(1_{d=1}x^{-\frac{1}{2}} + 1_{d=2}(\log 2)^{-1}\log(1+x^{-1}) + 1_{d \ge 3}) + 1_{x > 1}x^{-1}.$$

Here we inserted $\log 2$ in order to make the function decreasing. Then, by using the lower, upper bounds obtained above the condition (1.2) is modified as follows.

$$\begin{aligned}
&c_1(d)\beta\big(1_{\Theta \le \min\{1,\beta^{-1}\}} \\
&\qquad \cdot (1_{d \le 3}\Theta^{2-\frac{d}{2}} + 1_{d=4}\log(\min\{1,\beta^{-1}\}\Theta^{-1})^{-1} + 1_{d \ge 5}(\min\{1,\beta^{-1}\})^{2-\frac{d}{2}}) \\
&\quad + 1_{\Theta > \min\{1,\beta^{-1}\}}(\min\{1,\beta^{-1}\})^{-\frac{d}{2}}\Theta^2\big) \\
&< |U| < c_2(d)(1 + \beta^{d+3} + (1+\beta^{-1})g_{d,s}(\Theta))^{-2}.
\end{aligned} \tag{1.10}$$

Since with positive constants $c_1(d,\beta)$, $c_2(d,\beta)$ depending only on d,β,

$$\begin{aligned}&(\text{L.H.S of } (1.10)) \le c_1(d,\beta)(1_{d\le 3}\Theta^{2-\frac{d}{2}} + 1_{d=4}|\log\Theta|^{-1} + 1_{d\ge 5}),\\&(\text{R.H.S of } (1.10)) \ge c_2(d,\beta)(1_{d=1}\Theta + 1_{d=2}|\log\Theta|^{-2} + 1_{d\ge 3})\end{aligned}$$

for small Θ, we can find $U \in \mathbb{R}_{<0}$ satisfying the condition (1.10) for small Θ in the case $d = 1,2,3,4$. We can expect that the claims parallel to those of Theorem 1.3 hold for $\mu \in \{-2d, 2d\}$ in the case $d = 1,2,3,4$. We should note that in the case $d = 3,4$ the upper bound on $|U|$ can be independent of Θ and thus we can take Θ arbitrarily close to zero.

2. Formulation

In this section we will derive a finite-dimensional Grassmann integral formulation of the grand canonical partition function. Then, by means of the Hubbard-Stratonovich transformation we will transform it into a Gaussian integral formulation involving both Grassmann variables and real variables, which will be analyzed as a central object in the following sections.

2.1. Grassmann algebra

To begin with, let us recall some basics of finite-dimensional Grassmann integration. Let S_0 be a finite set and let $S := S_0 \times \{1,-1\}$. In practice we will need to change the index set S_0 several times during the construction. Here we do not fix any detail of S_0. Let R be the complex vector space spanned by the abstract basis $\{\psi_X \mid X \in S\}$. We should remark that ψ_X ($X \in S$) are not operators on the Fock space, though we use the same symbol as the Fermionic annihilation operator. For any $X \in S_0$ we let $\overline{\psi}_X$, ψ_X denote $\psi_{(X,1)}$, $\psi_{(X,-1)}$ respectively. For $n \in \mathbb{N}$, $\bigwedge^n R$ denotes the n-fold anti-symmetric tensor product of R. We set $\bigwedge^0 R := \mathbb{C}$ by convention. The Grassmann algebra $\bigwedge R$ generated by $\{\psi_X \mid X \in S\}$ is the direct sum of $\bigwedge^n R$.

$$\bigwedge R := \bigoplus_{n=0}^{\sharp S} \bigwedge^n R.$$

We will often work in a situation where R is the direct sum $\bigoplus_{p=1}^m R^p$ of other vector spaces R^p ($p = 1,2,\cdots,m$). We assume that the basis of R^p is $\{\psi_X^p \mid X \in S\}$. For a function $D : S_0^2 \to \mathbb{C}$ the Grassmann Gaussian integral

$\int \cdot d\mu_D(\psi^1)$ is a linear map from $\bigwedge\left(\bigoplus_{p=1}^m R^p\right)$ to $\bigwedge\left(\bigoplus_{p=2}^m R^p\right)$ defined as follows. For $f \in \bigwedge\left(\bigoplus_{p=2}^m R^p\right)$, $X_1, X_2, \cdots, X_a$, $Y_1, Y_2, \cdots, Y_b \in S_0$,

$$\int f\overline{\psi}^1_{X_1}\overline{\psi}^1_{X_2}\cdots\overline{\psi}^1_{X_a}\psi^1_{Y_b}\cdots\psi^1_{Y_2}\psi^1_{Y_1}d\mu_D(\psi^1)$$
$$:= \begin{cases} \det(D(X_i,Y_j))_{1\le i,j\le a}f & \text{if } a=b, \\ 0 & \text{otherwise,} \end{cases}$$
$$\int f d\mu_D(\psi^1) := f.$$

Then by linearity and anti-symmetry the value $\int g d\mu_D(\psi^1)$ is uniquely determined for any $g \in \bigwedge\left(\bigoplus_{p=1}^m R^p\right)$. We can define $\int \cdot d\mu_D(\psi)$ as a linear functional on $\bigwedge R$ in the same way.

Exponential and logarithm of a Grassmann polynomial appear in many parts of this paper. Let us recall their definitions. For $f \in \bigwedge R$ with the constant part $f_0(\in \mathbb{C})$

$$e^f := e^{f_0}\sum_{n=0}^{\sharp S}\frac{1}{n!}(f-f_0)^n.$$

If $f_0 \in \mathbb{C}\backslash\mathbb{R}_{\le 0}$,

$$\log f := \log f_0 + \sum_{n=1}^{\sharp S}\frac{(-1)^{n-1}}{n}\left(\frac{f-f_0}{f_0}\right)^n.$$

Throughout the paper $\log\alpha$ for $\alpha \in \mathbb{C}\backslash\mathbb{R}_{\le 0}$ is assumed to be representing the principal value $\log|\alpha| + i\theta$, where $\theta \in (-\pi,\pi)$ satisfies $\alpha = |\alpha|e^{i\theta}$. See e.g. [7] for more properties of Grassmann algebra.

2.2. One-band formulation

It is systematic to introduce artificial parameters $\lambda_1, \lambda_2 \in \mathbb{C}$ and deal with the normalized partition function

$$\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A})}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}, \tag{2.1}$$

where $\mathsf{A} := \lambda_1\mathsf{A}_1 + \lambda_2\mathsf{A}_2$. From now we always assume that

$$r_L(\hat{\mathbf{x}}) \neq r_L(\hat{\mathbf{y}}).$$

We can assume this condition to prove Theorem 1.3, since the theorem concerns the limit $\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty$ and for any $\hat{\mathbf{x}},\hat{\mathbf{y}}\in\mathbb{Z}^d$ with $\hat{\mathbf{x}}\neq\hat{\mathbf{y}}$ there exists $N_0\in\mathbb{N}$ such that $r_L(\hat{\mathbf{x}})\neq r_L(\hat{\mathbf{y}})$ for any $L\in\mathbb{N}$ with $L\geq N_0$. We will derive the thermal expectation values of our interest by differentiating (2.1) with λ_1,λ_2. We are going to formulate (2.1) into a limit of finite-dimensional Grassmann integration. First of all we should make sure that the denominator is non-zero. Let us define the momentum lattice Γ^* by

$$\Gamma^*:=\left\{0,\frac{2\pi}{L},\frac{2\pi}{L}\cdot 2,\cdots,\frac{2\pi}{L}(L-1)\right\}^d.$$

LEMMA 2.1.

(2.2)

$$\begin{aligned}\operatorname{Tr} e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}&=\prod_{\mathbf{k}\in\Gamma^*}\left(1+2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta e(\mathbf{k})}+e^{-2\beta e(\mathbf{k})}\right)\\&=e^{-\beta\sum_{\mathbf{k}\in\Gamma^*}e(\mathbf{k})}2^{L^d}\prod_{\mathbf{k}\in\Gamma^*}\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh(\beta e(\mathbf{k}))\right)\neq 0.\end{aligned}$$

PROOF. For $\sigma\in\{\uparrow,\downarrow\}$ we consider the Fermionic Fock space $F_f(L^2(\Gamma\times\{\sigma\}))$ as a subspace of $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$. Set

$$\begin{aligned}\mathsf{H}_{0,\sigma}&:=\sum_{\mathbf{x}\in\Gamma}\left((-1)^{hop}\sum_{j=1}^{d}(\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}+\mathbf{e}_j\sigma}+\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}-\mathbf{e}_j\sigma})-\mu\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}\right),\\ \mathsf{N}_\sigma&:=\sum_{\mathbf{x}\in\Gamma}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}.\end{aligned}$$

By letting Tr_σ mean the trace operation over $F_f(L^2(\Gamma\times\{\sigma\}))$ we have that

$$\begin{aligned}\operatorname{Tr} e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}&=\operatorname{Tr}_\uparrow e^{-\beta(\mathsf{H}_{0,\uparrow}+i\frac{\theta}{2}\mathsf{N}_\uparrow)}\operatorname{Tr}_\downarrow e^{-\beta(\mathsf{H}_{0,\downarrow}-i\frac{\theta}{2}\mathsf{N}_\downarrow)}\\&=|\operatorname{Tr}_\uparrow e^{-\beta(\mathsf{H}_{0,\uparrow}+i\frac{\theta}{2}\mathsf{N}_\uparrow)}|^2\\&=\prod_{\mathbf{k}\in\Gamma^*}|1+e^{-\beta(e(\mathbf{k})+i\frac{\theta}{2})}|^2,\end{aligned}$$

which is the right-hand side of (2.2). □

The covariance in our Grassmann Gaussian integral formulation of (2.1) is equal to a restriction of the following free two-point correlation function. For $(\mathbf{x},\sigma,s)$, $(\mathbf{y},\tau,t)\in\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta)$, set

$$G(\mathbf{x}\sigma s,\mathbf{y}\tau t):=\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}(1_{s\ge t}\psi^*_{\mathbf{x}\sigma}(s)\psi_{\mathbf{y}\tau}(t)-1_{s<t}\psi_{\mathbf{y}\tau}(t)\psi^*_{\mathbf{x}\sigma}(s)))}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}},$$

where $\psi^{(*)}_{\mathbf{x}\sigma}(s):=e^{s(\mathsf{H}_0+i\theta\mathsf{S}_z)}\psi^{(*)}_{\mathbf{x}\sigma}e^{-s(\mathsf{H}_0+i\theta\mathsf{S}_z)}$. We introduce the finite index set of Grassmann algebra by discretizing the time interval. With $h\in\frac{2}{\beta}\mathbb{N}$ we set

$$[0,\beta)_h:=\left\{0,\frac{1}{h},\frac{2}{h},\cdots,\beta-\frac{1}{h}\right\},$$

which is a discretization of $[0,\beta)$. We take the parameter h from $\frac{2}{\beta}\mathbb{N}$ rather than from $\frac{1}{\beta}\mathbb{N}$, since it is technically convenient according to the earlier study [9, Appendix C]. The index sets of Grassmann algebra for our one-band model are defined by

$$J_0:=\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)_h,\quad J:=J_0\times\{1,-1\}.$$

The restriction $G|_{J_0^2}$ is the covariance of our Grassmann Gaussian integral formulation. For simplicity let us omit the notation $\cdot|_{J_0^2}$ in the following.

Let $\mathcal{W}$ be the complex vector space spanned by the basis $\{\psi_X\mid X\in J\}$. Set $N:=4L^d\beta h$ so that $\sharp J=N$. Here we state the Grassmann integral formulation of (2.1) in the Grassmann algebra $\bigwedge\mathcal{W}$. For $r\in\mathbb{R}_{>0}$ let $D(r)$ denote the open disk $\{z\in\mathbb{C}\mid |z|<r\}$. Set

$$\begin{aligned}
\mathsf{V}(\psi)&:=\frac{U}{hL^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{\mathbf{x}\uparrow s}\overline{\psi}_{\mathbf{x}\downarrow s}\psi_{\mathbf{y}\downarrow s}\psi_{\mathbf{y}\uparrow s},\\
\mathsf{F}(\psi)&:=\frac{\gamma}{h}\sum_{\mathbf{x}\in\Gamma}\sum_{s\in[0,\beta)_h}(\overline{\psi}_{\mathbf{x}\uparrow s}\overline{\psi}_{\mathbf{x}\downarrow s}+\psi_{\mathbf{x}\downarrow s}\psi_{\mathbf{x}\uparrow s}),\\
\mathsf{A}^1(\psi)&:=\frac{1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{r_L(\hat{\mathbf{x}})\uparrow s}\overline{\psi}_{r_L(\hat{\mathbf{x}})\downarrow s},\\
\mathsf{A}^2(\psi)&:=\frac{1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{r_L(\hat{\mathbf{x}})\uparrow s}\overline{\psi}_{r_L(\hat{\mathbf{x}})\downarrow s}\psi_{r_L(\hat{\mathbf{y}})\downarrow s}\psi_{r_L(\hat{\mathbf{y}})\uparrow s},\\
\mathsf{A}(\psi)&:=\lambda_1\mathsf{A}^1(\psi)+\lambda_2\mathsf{A}^2(\psi).
\end{aligned}$$

We let $\boldsymbol{\lambda}$ denote (λ_1,λ_2) $(\in \mathbb{C}^2)$.

Lemma 2.2. *For any* $r \in \mathbb{R}_{>0}$,

(2.3)
$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2}\left|\int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)}d\mu_G(\psi)-\frac{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A})}}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}\right|=0.$$

Proof. The proof is close to the Grassmann integral formulation process in [9], [10], [11]. However, we sketch the procedure for readers' convenience. For any objects $\alpha_1,\alpha_2,\cdots,\alpha_n$ we let $\prod_{j=1}^n\alpha_j$ denote $\alpha_1\alpha_2\cdots\alpha_n$. This definition should be reminded especially when α_j $(j=1,2,\cdots,n)$ are non-commutative. For $\mathbf{x},\mathbf{y}\in\Gamma$, $a\in\{0,-1,1\}$ set

$$\begin{aligned}V(\mathbf{x},\mathbf{y},a):=1_{a=0}\left(\frac{U}{L^d}+\lambda_2 1_{(\mathbf{x},\mathbf{y})=(r_L(\hat{\mathbf{x}}),r_L(\hat{\mathbf{y}}))}\right)\\ +1_{a=1}\left(\frac{\gamma}{L^d}+\frac{\lambda_1}{L^d}1_{\mathbf{x}=r_L(\hat{\mathbf{x}})}\right)+1_{a=-1}\frac{\gamma}{L^d}.\end{aligned}$$

The partition function (2.1) can be expanded as follows (see e.g. [9, Lemma B.3]).

(2.4)
$$\begin{aligned}&\frac{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A})}}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}\\ &=1+\sum_{n=1}^{\infty}(-1)^n\prod_{j=1}^n\left(\sum_{\mathbf{x}_j,\mathbf{y}_j\in\Gamma}\int_0^\beta ds_j\sum_{a_j\in\{0,1,-1\}}V(\mathbf{x}_j,\mathbf{y}_j,a_j)\right)\\ &\qquad\cdot 1_{s_1>s_2>\cdots>s_n}1_{\sum_{j=1}^n a_j=0}\\ &\qquad\cdot\langle\prod_{j=1}^n(1_{a_j=0}\psi^*_{\mathbf{x}_j\uparrow}(s_j)\psi^*_{\mathbf{x}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\uparrow}(s_j)\\ &\qquad\qquad+1_{a_j=1}\psi^*_{\mathbf{x}_j\uparrow}(s_j)\psi^*_{\mathbf{x}_j\downarrow}(s_j)+1_{a_j=-1}\psi_{\mathbf{y}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\uparrow}(s_j))\rangle_0,\end{aligned}$$

where

$$\langle\mathcal{O}\rangle_0:=\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}\mathcal{O})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}$$

for any operator $\mathcal{O}$ on $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$. The constraint $1_{\sum_{j=1}^n a_j=0}$ is due to the fact that $\mathsf{H}_0+i\theta\mathsf{S}_z$ conserves the particle number.

Assume that $\beta>s_1>s_2>\cdots>s_n>0$ and $\sum_{j=1}^n a_j=0$. We can choose $\{i_p\}_{p=1}^l$, $\{j_p\}_{p=1}^m$, $\{k_p\}_{p=1}^m\subset\{1,2,\cdots,n\}$ so that $l+2m=n$ and

$$\begin{aligned}
&i_1<i_2<\cdots<i_l,\quad a_{i_p}=0\ (\forall p\in\{1,2,\cdots,l\}),\\
&j_1<j_2<\cdots<j_m,\quad a_{j_p}=1\ (\forall p\in\{1,2,\cdots,m\}),\\
&k_1<k_2<\cdots<k_m,\quad a_{k_p}=-1\ (\forall p\in\{1,2,\cdots,m\}).
\end{aligned}$$

Then, let us set

$$\begin{aligned}
&\mathbf{X}^0:=((\mathbf{x}_{i_1},\uparrow,s_{i_1}),(\mathbf{x}_{i_1},\downarrow,s_{i_1}),\cdots,(\mathbf{x}_{i_l},\uparrow,s_{i_l}),(\mathbf{x}_{i_l},\downarrow,s_{i_l}))\\
&\quad\in(\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta))^{2l},\\
&\mathbf{X}^1:=((\mathbf{x}_{j_1},\uparrow,s_{j_1}),(\mathbf{x}_{j_1},\downarrow,s_{j_1}),\cdots,(\mathbf{x}_{j_m},\uparrow,s_{j_m}),(\mathbf{x}_{j_m},\downarrow,s_{j_m}))\\
&\quad\in(\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta))^{2m},\\
&\mathbf{Y}^0:=((\mathbf{y}_{i_1},\uparrow,s_{i_1}),(\mathbf{y}_{i_1},\downarrow,s_{i_1}),\cdots,(\mathbf{y}_{i_l},\uparrow,s_{i_l}),(\mathbf{y}_{i_l},\downarrow,s_{i_l}))\\
&\quad\in(\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta))^{2l},\\
&\mathbf{Y}^1:=((\mathbf{y}_{k_1},\uparrow,s_{k_1}),(\mathbf{y}_{k_1},\downarrow,s_{k_1}),\cdots,(\mathbf{y}_{k_m},\uparrow,s_{k_m}),(\mathbf{y}_{k_m},\downarrow,s_{k_m}))\\
&\quad\in(\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta))^{2m}.
\end{aligned}$$

By giving a number to each component we write as follows.

$$(\mathbf{X}^0,\mathbf{X}^1)=(X_j)_{1\le j\le 2l+2m},\quad(\mathbf{Y}^0,\mathbf{Y}^1)=(Y_j)_{1\le j\le 2l+2m}.$$

Because of the assumption $s_1>\cdots>s_n$, the operator

$$\begin{aligned}
&\prod_{j=1}^n(1_{a_j=0}\psi^*_{\mathbf{x}_j\uparrow}(s_j)\psi^*_{\mathbf{x}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\uparrow}(s_j)+1_{a_j=1}\psi^*_{\mathbf{x}_j\uparrow}(s_j)\psi^*_{\mathbf{x}_j\downarrow}(s_j)\\
&\quad+1_{a_j=-1}\psi_{\mathbf{y}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\uparrow}(s_j))
\end{aligned}$$

is already ordered with respect to the standard lexicographical order in the product set $[0,\beta)\times\{\text{particle},\text{hole}\}$. Thus,

$$\begin{aligned}
&\langle\prod_{j=1}^n(1_{a_j=0}\psi^*_{\mathbf{x}_j\uparrow}(s_j)\psi^*_{\mathbf{x}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\uparrow}(s_j)+1_{a_j=1}\psi^*_{\mathbf{x}_j\uparrow}(s_j)\psi^*_{\mathbf{x}_j\downarrow}(s_j)\\
&\quad+1_{a_j=-1}\psi_{\mathbf{y}_j\downarrow}(s_j)\psi_{\mathbf{y}_j\uparrow}(s_j))\rangle_0
\end{aligned}$$

$$= \det(G(X_i, Y_j))_{1\le i,j\le 2l+2m}.$$

See e.g. [9, Lemma B.7, Lemma B.8, Lemma B.9] for a proof of the above equality. By substituting this into (2.4) we have

$$\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A})}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}} = 1 + \sum_{n=1}^{\infty}(-1)^n \prod_{j=1}^{n}\left(\sum_{\mathbf{x}_j,\mathbf{y}_j\in\Gamma}\int_0^\beta ds_j \sum_{a_j\in\{0,1,-1\}} V(\mathbf{x}_j,\mathbf{y}_j,a_j)\right) \cdot 1_{s_1>s_2>\cdots>s_n} 1_{\sum_{j=1}^n a_j=0} \det(G(X_i,Y_j))_{1\le i,j\le 2l+2m}. \tag{2.5}$$

Define the function $P(\boldsymbol{\lambda})$ ($\boldsymbol{\lambda}\in\mathbb{C}^2$) by the right-hand side of (2.5). We also define its discrete analogue $P_h(\boldsymbol{\lambda})$ by

$$P_h(\boldsymbol{\lambda}) := 1 + \sum_{n=1}^{\infty}(-1)^n \prod_{j=1}^{n}\left(\frac{1}{h}\sum_{\mathbf{x}_j,\mathbf{y}_j\in\Gamma}\sum_{s_j\in[0,\beta)_h} \sum_{a_j\in\{0,1,-1\}} V(\mathbf{x}_j,\mathbf{y}_j,a_j)\right) \cdot 1_{s_1>s_2>\cdots>s_n} 1_{\sum_{j=1}^n a_j=0} \det(G(X_i,Y_j))_{1\le i,j\le 2l+2m}. \tag{2.6}$$

Based on the fact that the function

$$\mathbf{s}\mapsto 1_{s_1>s_2>\cdots>s_n}\det(G(X_i,Y_j))_{1\le i,j\le 2l+2m} : [0,\beta)^n\to\mathbb{C}$$

is continuous almost everywhere in $[0,\beta)^n$ and the uniform bounds

$$\prod_{j=1}^{n}\left(\int_0^\beta ds_j\right)1_{s_1>s_2>\cdots>s_n} \le \frac{\beta^n}{n!},$$

$$|\det(G(W_i,Z_j))_{1\le i,j\le n}| \le \frac{2^{2L^d} e^{(2n+1)\beta\|\mathsf{H}_0+i\theta\mathsf{S}_z\|_{\mathcal{B}(F_f)}}}{|\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}|}, \tag{2.7}$$
$$(\forall n\in\mathbb{N},\ W_j,Z_j\in\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)\ (j=1,2,\cdots,n)),$$

where $\|\cdot\|_{\mathcal{B}(F_f)}$ is the operator norm of operators on $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$, we can prove that for any $r\in\mathbb{R}_{>0}$, $n\in\mathbb{N}$,

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2}\left|\prod_{j=1}^{n}\left(\frac{1}{h}\sum_{\mathbf{x}_j,\mathbf{y}_j\in\Gamma}\sum_{s_j\in[0,\beta)_h}\sum_{a_j\in\{0,1,-1\}}V(\mathbf{x}_j,\mathbf{y}_j,a_j)\right)\right.$$

$$\cdot 1_{s_1>s_2>\cdots>s_n} 1_{\sum_{j=1}^n a_j=0} \det(G(X_i,Y_j))_{1\le i,j\le 2l+2m}$$
$$-\prod_{j=1}^n\left(\sum_{\mathbf{x}_j,\mathbf{y}_j\in\Gamma}\int_0^\beta ds_j\sum_{a_j\in\{0,1,-1\}}V(\mathbf{x}_j,\mathbf{y}_j,a_j)\right)$$
$$\left.\cdot 1_{s_1>s_2>\cdots>s_n} 1_{\sum_{j=1}^n a_j=0} \det(G(X_i,Y_j))_{1\le i,j\le 2l+2m}\right| = 0$$

and by the dominated convergence theorem that

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2}|P_h(\boldsymbol{\lambda})-P(\boldsymbol{\lambda})|=0. \tag{2.8}$$

By the definition of the Grassmann Gaussian integral, it holds inside (2.6) that

$$\begin{aligned}&\det(G(X_i,Y_j))_{1\le i,j\le 2l+2m}\\ &=\int\prod_{j=1}^n(1_{a_j=0}\overline{\psi}_{\mathbf{x}_j\uparrow s_j}\overline{\psi}_{\mathbf{x}_j\downarrow s_j}\psi_{\mathbf{y}_j\downarrow s_j}\psi_{\mathbf{y}_j\uparrow s_j}+1_{a_j=1}\overline{\psi}_{\mathbf{x}_j\uparrow s_j}\overline{\psi}_{\mathbf{x}_j\downarrow s_j}\\ &\qquad +1_{a_j=-1}\psi_{\mathbf{y}_j\downarrow s_j}\psi_{\mathbf{y}_j\uparrow s_j})d\mu_G(\psi).\end{aligned}$$

By substituting this we observe that

$$\begin{aligned}P_h(\boldsymbol{\lambda})=1+\sum_{n=1}^{2L^2\beta h}\frac{(-1)^n}{n!}\prod_{j=1}^n&\left(\frac{1}{h}\sum_{\mathbf{x}_j,\mathbf{y}_j\in\Gamma}\sum_{s_j\in[0,\beta)_h}\sum_{a_j\in\{0,1,-1\}}V(\mathbf{x}_j,\mathbf{y}_j,a_j)\right)\\ &\cdot 1_{j\neq k\to s_j\neq s_k}\\ &\cdot\int\prod_{j=1}^n(1_{a_j=0}\overline{\psi}_{\mathbf{x}_j\uparrow s_j}\overline{\psi}_{\mathbf{x}_j\downarrow s_j}\psi_{\mathbf{y}_j\downarrow s_j}\psi_{\mathbf{y}_j\uparrow s_j}\\ &\qquad +1_{a_j=1}\overline{\psi}_{\mathbf{x}_j\uparrow s_j}\overline{\psi}_{\mathbf{x}_j\downarrow s_j}\\ &\qquad +1_{a_j=-1}\psi_{\mathbf{y}_j\downarrow s_j}\psi_{\mathbf{y}_j\uparrow s_j})d\mu_G(\psi).\end{aligned}$$

Note that if we drop the constraint $1_{j\neq k\to s_j\neq s_k}$, the right-hand side is equal to $\int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)}d\mu_G(\psi)$. By using the estimate

$$\prod_{j=1}^n\left(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\right)1_{\exists j\exists k(j\neq k\wedge s_j=s_k)}\le 1_{n\ge 2}\begin{pmatrix}n\\2\end{pmatrix}\frac{\beta^{n-1}}{h}$$

and the uniform bound (2.7) we can prove that for any $r \in \mathbb{R}_{>0}$

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2} \left| P_h(\boldsymbol{\lambda}) - \int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)} d\mu_G(\psi) \right| = 0.$$

This convergence property and (2.8) imply (2.3). □

As the second step we decompose the quartic Grassmann polynomial $\mathsf{V}(\psi)$ into quadratic polynomials and a quartic correction term by means of the Hubbard-Stratonovich transformation. Let us define $\mathsf{V}_+(\psi)$, $\mathsf{V}_-(\psi)$, $\mathsf{W}(\psi) \in \bigwedge \mathcal{W}$ by

$$\begin{aligned}
\mathsf{V}_+(\psi) &:= \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}} h} \sum_{\mathbf{x}\in\Gamma} \sum_{s\in[0,\beta)_h} \overline{\psi}_{\mathbf{x}\uparrow s} \overline{\psi}_{\mathbf{x}\downarrow s}, \\
\mathsf{V}_-(\psi) &:= \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}} h} \sum_{\mathbf{x}\in\Gamma} \sum_{s\in[0,\beta)_h} \psi_{\mathbf{x}\downarrow s} \psi_{\mathbf{x}\uparrow s}, \\
\mathsf{W}(\psi) &:= \frac{U}{\beta L^d h^2} \sum_{\mathbf{x},\mathbf{y}\in\Gamma} \sum_{s,t\in[0,\beta)_h} \overline{\psi}_{\mathbf{x}\uparrow s} \overline{\psi}_{\mathbf{x}\downarrow s} \psi_{\mathbf{y}\downarrow t} \psi_{\mathbf{y}\uparrow t}.
\end{aligned}$$

LEMMA 2.3.

(2.9)
$$\begin{aligned}
&\int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)} d\mu_G(\psi) \\
&= \frac{1}{\pi} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-|\phi|^2} \int e^{-\mathsf{V}(\psi)+\mathsf{W}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)} d\mu_G(\psi),
\end{aligned}$$

where $\phi := \phi_1 + i\phi_2$, $|\phi| := \sqrt{\phi_1^2 + \phi_2^2}$.

PROOF. For $f_j(\psi) \in \bigwedge \mathcal{W}$, $g_j \in L^1(\mathbb{R}^2)$ $(j = 1, 2, \cdots, n)$ we can define the Grassmann polynomial $\int_{\mathbb{R}^2} d\phi_1 d\phi_2 \sum_{j=1}^n g_j(\phi_1,\phi_2) f_j(\psi)$ by

$$\int_{\mathbb{R}^2} d\phi_1 d\phi_2 \sum_{j=1}^n g_j(\phi_1,\phi_2) f_j(\psi) := \sum_{j=1}^n \left(\int_{\mathbb{R}^2} d\phi_1 d\phi_2 g_j(\phi_1,\phi_2) \right) f_j(\psi).$$

Bearing this definition in mind, the Hubbard-Stratonovich transformation gives that

$$e^{\mathsf{V}_+(\psi)\mathsf{V}_-(\psi)} = \frac{1}{\pi}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-|\phi|^2+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)}. \tag{2.10}$$

This equality can be confirmed without difficulty. In fact,

$$\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-|\phi|^2+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)}\\
&= \sum_{m,n=0}^{N/2} \frac{1}{\pi(2m)!(2n)!}(\mathsf{V}_+(\psi)+\mathsf{V}_-(\psi))^{2m}(i\mathsf{V}_+(\psi)-i\mathsf{V}_-(\psi))^{2n}\\
&\quad\cdot\int_{\mathbb{R}} d\phi_1 e^{-\phi_1^2}\phi_1^{2m}\int_{\mathbb{R}} d\phi_2 e^{-\phi_2^2}\phi_2^{2n}\\
&= \sum_{m,n=0}^{N/2} \frac{2^{-2m-2n}}{m!n!}(\mathsf{V}_+(\psi)+\mathsf{V}_-(\psi))^{2m}(i\mathsf{V}_+(\psi)-i\mathsf{V}_-(\psi))^{2n}\\
&= e^{\frac{1}{4}(\mathsf{V}_+(\psi)+\mathsf{V}_-(\psi))^2+\frac{1}{4}(i\mathsf{V}_+(\psi)-i\mathsf{V}_-(\psi))^2} = e^{\mathsf{V}_+(\psi)\mathsf{V}_-(\psi)}.
\end{aligned}$$

By substituting the equality $\mathsf{V}_+(\psi)\mathsf{V}_-(\psi) = -\mathsf{W}(\psi)$ and (2.10) we can derive the result. □

2.3. Two-band formulation

To complete the formulation, we will include the quadratic terms $\mathsf{V}_+(\psi)$, $\mathsf{V}_-(\psi)$, $\mathsf{F}(\psi)$ in the covariance. This procedure leads to another Grassmann integration where Grassmann algebra is indexed by the band index $\{1,2\}$ rather than the spin $\{\uparrow,\downarrow\}$. To this end, let us introduce some notations. We define the new index sets I_0, I by

$$I_0 := \{1,2\}\times\Gamma\times[0,\beta)_h, \quad I := I_0\times\{1,-1\}.$$

Let $\mathcal{V}$ be the complex vector space spanned by the basis $\{\psi_X \mid X\in I\}$. Then, define the Grassmann polynomials $V(\psi)$, $W(\psi)$, $A^1(\psi)$, $A^2(\psi)$, $A(\psi)\in\bigwedge\mathcal{V}$ by

$$V(\psi) := \frac{U}{L^d h}\sum_{\mathbf{x}\in\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\mathbf{x}s}\psi_{1\mathbf{x}s} \tag{2.11}$$

$$+\frac{U}{L^d h}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\mathbf{x}s}\psi_{2\mathbf{x}s}\overline{\psi}_{2\mathbf{y}s}\psi_{1\mathbf{y}s},$$

$$W(\psi):=\frac{U}{\beta L^d h^2}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{s,t\in[0,\beta)_h}\overline{\psi}_{1\mathbf{x}s}\psi_{2\mathbf{x}s}\overline{\psi}_{2\mathbf{y}t}\psi_{1\mathbf{y}t}, \tag{2.12}$$

$$A^1(\psi):=\frac{1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1r_L(\hat{\mathbf{x}})s}\psi_{2r_L(\hat{\mathbf{x}})s}, \tag{2.13}$$

$$A^2(\psi):=\frac{1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1r_L(\hat{\mathbf{x}})s}\psi_{2r_L(\hat{\mathbf{x}})s}\overline{\psi}_{2r_L(\hat{\mathbf{y}})s}\psi_{1r_L(\hat{\mathbf{y}})s},$$

$$A(\psi):=\lambda_1 A^1(\psi)+\lambda_2 A^2(\psi). \tag{2.14}$$

In order to introduce the Grassmann Gaussian integral formulation, we need to define its covariance. To define the covariance as a free 2-point correlation function, first we need to introduce a free Hamiltonian on the Fermionic Fock space $F_f(L^2(\{1,2\}\times\Gamma))$. For $\phi\in\mathbb{C}$ set

(2.15)

$$H_0(\phi) := \frac{1}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\left\langle\begin{pmatrix}\psi^*_{1\mathbf{x}}\\ \psi^*_{2\mathbf{x}}\end{pmatrix},\begin{pmatrix}i\frac{\theta}{2}+e(\mathbf{k}) & \phi\\ \overline{\phi} & i\frac{\theta}{2}-e(\mathbf{k})\end{pmatrix}\begin{pmatrix}\psi_{1\mathbf{y}}\\ \psi_{2\mathbf{y}}\end{pmatrix}\right\rangle,$$

where $\psi^*_{\rho\mathbf{x}}$ ($\psi_{\rho\mathbf{x}}$) is the creation (annihilation) operator in $F_f(L^2(\{1,2\}\times\Gamma))$. Because of the presence of $i\frac{\theta}{2}$, $H_0(\phi)$ is not self-adjoint. Therefore, it may not be appropriate to call $H_0(\phi)$ Hamiltonian. We included the imaginary magnetic field inside only for conciseness. The covariance of the 2-band formulation is the restriction of the free 2-point correlation function $C(\phi):(\{1,2\}\times\mathbb{Z}^d\times[0,\beta))^2\to\mathbb{C}$ defined by

(2.16)

$$C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t):=\frac{\mathrm{Tr}(e^{-\beta H_0(\phi)}(1_{s\ge t}\psi^*_{\rho\mathbf{x}}(s)\psi_{\eta\mathbf{y}}(t)-1_{s<t}\psi_{\eta\mathbf{y}}(t)\psi^*_{\rho\mathbf{x}}(s)))}{\mathrm{Tr}\,e^{-\beta H_0(\phi)}},$$

where $\psi^{(*)}_{\rho\mathbf{x}}(s):=e^{sH_0(\phi)}\psi^{(*)}_{\rho\mathbf{x}}e^{-sH_0(\phi)}$. Here again we identify $\psi^{(*)}_{\rho\mathbf{x}}$ with $\psi^{(*)}_{\rho r_L(\mathbf{x})}$ for $\mathbf{x}\in\mathbb{Z}^d$. Since $H_0(\phi)$ is not self-adjoint, the denominator could be zero. We have to make sure that this is not the case.

Lemma 2.4.

(i)

$$\begin{aligned}\mathrm{Tr}\, e^{-\beta H_0(\phi)} &= \prod_{\mathbf{k}\in\Gamma^*}\prod_{\delta\in\{1,-1\}}\left(1+e^{-\beta(i\frac{\theta}{2}+\delta\sqrt{e(\mathbf{k})^2+|\phi|^2})}\right)\\ &= e^{-i\frac{\beta\theta}{2}L^d}2^{L^d}\prod_{\mathbf{k}\in\Gamma^*}\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh\left(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}\right)\right)\neq 0.\end{aligned}$$

(ii) For any $(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in\{1,2\}\times\mathbb{Z}^d\times[0,\beta)$,

(2.17)

$$\begin{aligned}&C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t)\\ &=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}e^{(s-t)(i\frac{\theta}{2}I_2+E(\phi)(\mathbf{k}))}\\ &\quad\cdot(1_{s\ge t}(I_2+e^{\beta(i\frac{\theta}{2}I_2+E(\phi)(\mathbf{k}))})^{-1}-1_{s<t}(I_2+e^{-\beta(i\frac{\theta}{2}I_2+E(\phi)(\mathbf{k}))})^{-1})(\rho,\eta),\end{aligned}$$

where I_2 *is the* 2×2 *unit matrix and*

$$E(\phi)(\mathbf{k}):=\begin{pmatrix} e(\mathbf{k}) & \overline{\phi}\\ \phi & -e(\mathbf{k})\end{pmatrix}. \tag{2.18}$$

Proof. (i): Since the materials will be used later, we describe the derivation in some detail. Define the $(\phi,\mathbf{k})$-dependent 2×2 matrix $U(\phi)(\mathbf{k})$ as follows.

$$U(\phi)(\mathbf{k}):=1_{\phi=0}I_2+1_{\phi\neq 0}\left(\frac{\mathbf{X}(\phi)(\mathbf{k})}{\|\mathbf{X}(\phi)(\mathbf{k})\|_{\mathbb{C}^2}},\frac{\mathbf{Y}(\phi)(\mathbf{k})}{\|\mathbf{Y}(\phi)(\mathbf{k})\|_{\mathbb{C}^2}}\right), \tag{2.19}$$

where

$$\mathbf{X}(\phi)(\mathbf{k}):=\begin{pmatrix}\overline{\phi}\\ \sqrt{e(\mathbf{k})^2+|\phi|^2}-e(\mathbf{k})\end{pmatrix},$$
$$\mathbf{Y}(\phi)(\mathbf{k}):=\begin{pmatrix}-\overline{\phi}\\ \sqrt{e(\mathbf{k})^2+|\phi|^2}+e(\mathbf{k})\end{pmatrix}$$

and $\|\cdot\|_{\mathbb{C}^2}$ is the norm of $\mathbb{C}^2$ induced by the hermitian inner product. Moreover, set

$$e(\phi)(\mathbf{k}) := 1_{\phi=0}e(\mathbf{k}) + 1_{\phi\neq 0}\sqrt{e(\mathbf{k})^2+|\phi|^2}. \tag{2.20}$$

One can check that $U(\phi)(\mathbf{k})$ is unitary and

$$U(\phi)(\mathbf{k})^* E(\phi)(\mathbf{k}) U(\phi)(\mathbf{k}) = \begin{pmatrix} e(\phi)(\mathbf{k}) & 0 \\ 0 & -e(\phi)(\mathbf{k}) \end{pmatrix}. \tag{2.21}$$

Note that

$$H_0(\phi) = \frac{1}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\langle \begin{pmatrix}\psi^*_{1\mathbf{x}}\\ \psi^*_{2\mathbf{x}}\end{pmatrix}, \left(i\frac{\theta}{2}I_2 + E(\overline{\phi})(\mathbf{k})\right)\begin{pmatrix}\psi_{1\mathbf{y}}\\ \psi_{2\mathbf{y}}\end{pmatrix}\rangle.$$

With the matrix $U(\phi)(\mathbf{k})$ one can define a unitary transform $\mathcal{U}(\phi)$ on $F_f(L^2(\{1,2\}\times\Gamma))$ satisfying that

(2.22)

$$\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^* = \frac{1}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle} \cdot\langle \begin{pmatrix}\psi^*_{1\mathbf{x}}\\ \psi^*_{2\mathbf{x}}\end{pmatrix}, \begin{pmatrix} i\frac{\theta}{2}+e(\phi)(\mathbf{k}) & 0\\ 0 & i\frac{\theta}{2}-e(\phi)(\mathbf{k})\end{pmatrix}\begin{pmatrix}\psi_{1\mathbf{y}}\\ \psi_{2\mathbf{y}}\end{pmatrix}\rangle,$$

(2.23)

$$\mathcal{U}(\phi)\psi^*_{\rho\mathbf{x}}\mathcal{U}(\phi)^* = \frac{1}{L^d}\sum_{\mathbf{y}\in\Gamma}\sum_{\eta\in\{1,2\}}\sum_{\mathbf{k}\in\Gamma^*} e^{-i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\overline{U(\overline{\phi})(\mathbf{k})(\rho,\eta)}\psi^*_{\eta\mathbf{y}}.$$

Since $\mathrm{Tr}\, e^{-\beta H_0(\phi)} = \mathrm{Tr}\, e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}$ and $\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*$ is diagonalized with the band index, the result follows.

(ii): By the periodicity of both sides of (2.17) we can restrict the spatial variables to Γ during the proof. By (2.22), (2.23) and $\overline{U(\overline{\phi})(\mathbf{k})} = U(\phi)(\mathbf{k})$,

$$\begin{aligned} &C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t)\\ &= \frac{1}{L^{2d}}\sum_{\mathbf{k},\mathbf{p}\in\Gamma^*}\sum_{\mathbf{x}',\mathbf{y}'\in\Gamma}\sum_{\rho',\eta'\in\{1,2\}}\\ &\qquad \cdot e^{-i\langle \mathbf{k},\mathbf{x}-\mathbf{x}'\rangle + i\langle \mathbf{p},\mathbf{y}-\mathbf{y}'\rangle}U(\phi)(\mathbf{k})(\rho,\rho')\overline{U(\phi)(\mathbf{p})(\eta,\eta')}\end{aligned}$$

$$\cdot \frac{\mathrm{Tr}(e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}(1_{s\ge t}\tilde{\psi}^*_{\rho'\mathbf{x}'}(s)\tilde{\psi}_{\eta'\mathbf{y}'}(t) - 1_{s<t}\tilde{\psi}_{\eta'\mathbf{y}'}(t)\tilde{\psi}^*_{\rho'\mathbf{x}'}(s)))}{\mathrm{Tr}\, e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}},$$

where $\tilde{\psi}^{(*)}_{\rho\mathbf{x}}(s) := e^{s\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}\psi^{(*)}_{\rho\mathbf{x}}e^{-s\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}$. Since $\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*$ is diagonalized with the band index, it can be derived by a standard procedure (see e.g. [9, Appendix B]) that

$$\begin{aligned}
&\frac{\mathrm{Tr}(e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}(1_{s\ge t}\tilde{\psi}^*_{\rho'\mathbf{x}'}(s)\tilde{\psi}_{\eta'\mathbf{y}'}(t) - 1_{s<t}\tilde{\psi}_{\eta'\mathbf{y}'}(t)\tilde{\psi}^*_{\rho'\mathbf{x}'}(s)))}{\mathrm{Tr}\, e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}}\\
&= \frac{1_{\rho'=\eta'}}{L^d}\sum_{\mathbf{k}'\in\Gamma^*} e^{i\langle\mathbf{k}',\mathbf{x}'-\mathbf{y}'\rangle}e^{(s-t)(i\frac{\theta}{2}+(-1)^{1_{\rho'=2}}e(\phi)(\mathbf{k}'))}\\
&\qquad\cdot\left(\frac{1_{s\ge t}}{1+e^{\beta(i\frac{\theta}{2}+(-1)^{1_{\rho'=2}}e(\phi)(\mathbf{k}'))}} - \frac{1_{s<t}}{1+e^{-\beta(i\frac{\theta}{2}+(-1)^{1_{\rho'=2}}e(\phi)(\mathbf{k}'))}}\right).
\end{aligned}$$

Then by using the equality $e(\phi)(\mathbf{k}) = e(\phi)(-\mathbf{k})$ we obtain that

(2.24)

$$\begin{aligned}
&C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t)\\
&= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\xi\in\{1,2\}} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}U(\phi)(\mathbf{k})(\rho,\xi)\overline{U(\phi)(\mathbf{k})(\eta,\xi)}\\
&\qquad\cdot e^{(s-t)(i\frac{\theta}{2}+(-1)^{1_{\xi=2}}e(\phi)(\mathbf{k}))}\\
&\qquad\cdot\left(\frac{1_{s\ge t}}{1+e^{\beta(i\frac{\theta}{2}+(-1)^{1_{\xi=2}}e(\phi)(\mathbf{k}))}} - \frac{1_{s<t}}{1+e^{-\beta(i\frac{\theta}{2}+(-1)^{1_{\xi=2}}e(\phi)(\mathbf{k}))}}\right).
\end{aligned}$$

Then by using (2.21) again we reach the claimed equality. □

The following lemma will form the basis of our analysis which eventually leads to the proof of Theorem 1.3.

Lemma 2.5. *The following statements hold true for any $r \in \mathbb{R}_{>0}$.*

(i) For any $\phi \in \mathbb{C}$,

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi)$$

converges in $C(\overline{D(r)}^2)$ *as a sequence of function with the variable* $\boldsymbol{\lambda}(\in \overline{D(r)}^2)$.

(ii) The $C(\overline{D(r)}^2)$*-valued function*

$$(\phi_1,\phi_2)\mapsto e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2}\frac{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))} \cdot \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi)$$

belongs to $L^1(\mathbb{R}^2, C(\overline{D(r)}^2))$.

(iii)

(2.25)

$$\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A})}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}} = \frac{\beta L^d}{\pi|U|}\int_{\mathbb{R}^2}d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2}\frac{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))} \cdot \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi).$$

(2.26)

$$\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_j)}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}} = \frac{L^d}{\pi|U|}\int_{\mathbb{R}^2}d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2}\frac{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))} \cdot \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}A^j(\psi)d\mu_{C(\phi)}(\psi),$$

$(j=1,2)$.

REMARK 2.6. At this point we do not prove that we can change the order of the integration over $\mathbb{R}^2$ and the limit operation $h\to\infty$ in (2.25), (2.26). It suffices to establish a suitable uniform bound on

$$\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi),$$
$$\int e^{-V(\psi)+W(\psi)}A^j(\psi)d\mu_{C(\phi)}(\psi)\quad (j=1,2)$$

with ϕ, h in order to ensure that these operations are exchangeable. Later we will prove the uniform bound (4.78) and thus we will be able to exchange these operations in (2.25) with $\boldsymbol{\lambda}=(0,0)$ and in (2.26). It is also possible to use Pedra-Salmhofer's type determinant bound Proposition 4.2 to directly establish a desirable uniform boundedness of these Grassmann integrals.

Proof of Lemma 2.5. We decompose the Grassmann polynomial $\mathsf{W}(\psi)$ in the right-hand side of (2.9) temporarily by the Hubbard-Stratonovich transformation. Set

$$\mathsf{W}_+(\psi):=\frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}h}\sum_{\mathbf{x}\in\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{\mathbf{x}\uparrow s}\overline{\psi}_{\mathbf{x}\downarrow s},$$
$$\mathsf{W}_-(\psi):=\frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}h}\sum_{\mathbf{x}\in\Gamma}\sum_{s\in[0,\beta)_h}\psi_{\mathbf{x}\downarrow s}\psi_{\mathbf{x}\uparrow s}.$$

For the same reason as the equality (2.10) holds, the following equality holds true.

$$\begin{aligned}(2.27)\quad &(\text{R.H.S of (2.9)})\\ &=\frac{1}{\pi^2}\int_{\mathbb{R}^2}d\phi_1d\phi_2\int_{\mathbb{R}^2}d\xi_1d\xi_2e^{-|\phi|^2-|\xi|^2}\\ &\quad\cdot\int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)+\xi\mathsf{W}_+(\psi)+\overline{\xi}\mathsf{W}_-(\psi)}d\mu_G(\psi),\end{aligned}$$

where we set $\xi:=\xi_1+i\xi_2$. Let us transform the Grassmann integral inside the Gaussian integral. By expanding each exponential of the Grassmann polynomials and using the determinant bound (2.7) we can derive that

$$(2.28)\quad \left|\int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)+\xi\mathsf{W}_+(\psi)+\overline{\xi}\mathsf{W}_-(\psi)}d\mu_G(\psi)\right|$$

$$\le \max\left\{1, \frac{2^{2L^d}e^{\beta\|\mathsf{H}_0+i\theta\mathsf{S}_z\|_{\mathcal{B}(F_f)}}}{|\operatorname{Tr} e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}|}\right\}$$
$$\cdot e^{|U|\beta L^d D^2+2|\gamma|\beta L^d D+|\lambda_1|\beta D+|\lambda_2|\beta D^2+2|\phi||U|^{\frac{1}{2}}\beta^{\frac{1}{2}}L^{\frac{d}{2}}D+2|\xi||U|^{\frac{1}{2}}\beta^{\frac{1}{2}}L^{\frac{d}{2}}D},$$

where $D := e^{2\beta\|\mathsf{H}_0+i\theta\mathsf{S}_z\|_{\mathcal{B}(F_f)}}$. The same argument as in the proof of Lemma 2.2 proves that for any $r \in \mathbb{R}_{>0}$

(2.29)
$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2}\Bigg|\int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)+\xi\mathsf{W}_+(\psi)+\overline{\xi}\mathsf{W}_-(\psi)}d\mu_G(\psi)$$
$$-\frac{\operatorname{Tr} e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A}-\phi\mathsf{V}_+-\overline{\phi}\mathsf{V}_--\xi\mathsf{W}_+-\overline{\xi}\mathsf{W}_-)}}{\operatorname{Tr} e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}\Bigg| = 0,$$

where V_+, V_-, W_+, W_- are operators on $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$ defined by

$$\mathsf{V}_+ := \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}}\sum_{\mathbf{x}\in\Gamma}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}, \quad \mathsf{V}_- := \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}}\sum_{\mathbf{x}\in\Gamma}\psi_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\uparrow},$$
$$\mathsf{W}_+ := i\mathsf{V}_+, \quad \mathsf{W}_- := i\mathsf{V}_-.$$

Here we introduce the band index $\{1,2\}$ and relate the partition function in (2.29) to a partition function in the Fermionic Fock space $F_f(L^2(\{1,2\}\times\Gamma))$. Let us give a number to each $\mathbf{x}\in\Gamma$ so that we can write $\Gamma = \{\mathbf{x}_j\}_{j=1}^{L^d}$. Define the linear map $\mathcal{U}$ from $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$ to $F_f(L^2(\{1,2\}\times\Gamma))$ by

$$\mathcal{U}\Omega := \prod_{j=1}^{L^d}\psi^*_{2\mathbf{x}_j}\Omega_2,$$
$$\mathcal{U}(\psi^*_{\mathbf{x}_{i_1}\uparrow}\psi^*_{\mathbf{x}_{i_2}\uparrow}\cdots\psi^*_{\mathbf{x}_{i_l}\uparrow}\psi^*_{\mathbf{x}_{j_1}\downarrow}\psi^*_{\mathbf{x}_{j_2}\downarrow}\cdots\psi^*_{\mathbf{x}_{j_m}\downarrow}\Omega)$$
$$:= \psi^*_{1\mathbf{x}_{i_1}}\psi^*_{1\mathbf{x}_{i_2}}\cdots\psi^*_{1\mathbf{x}_{i_l}}\psi_{2\mathbf{x}_{j_1}}\psi_{2\mathbf{x}_{j_2}}\cdots\psi_{2\mathbf{x}_{j_m}}\mathcal{U}\Omega,$$
$$(\forall i_1, i_2, \cdots, i_l, j_1, j_2, \cdots, j_m \in \{1,2,\cdots,L^d\})$$

and by linearity. Here Ω, Ω_2 are the vacuum of $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$, $F_f(L^2(\{1,2\}\times\Gamma))$ respectively. We can see that the map $\mathcal{U}$ is unitary and

(2.30)
$$\mathcal{U}\psi^*_{\mathbf{x}\uparrow}\mathcal{U}^* = \psi^*_{1\mathbf{x}},\ \mathcal{U}\psi_{\mathbf{x}\uparrow}\mathcal{U}^* = \psi_{1\mathbf{x}},\ \mathcal{U}\psi^*_{\mathbf{x}\downarrow}\mathcal{U}^* = \psi_{2\mathbf{x}},\ \mathcal{U}\psi_{\mathbf{x}\downarrow}\mathcal{U}^* = \psi^*_{2\mathbf{x}},\ (\forall\mathbf{x}\in\Gamma).$$

Let us define the operators V, A, W_+, W_- on $F_f(L^2(\{1,2\}\times\Gamma))$ by

$$V := \frac{U}{L^d}\sum_{\mathbf{x}\in\Gamma}\psi^*_{1\mathbf{x}}\psi_{1\mathbf{x}} - \frac{U}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\psi^*_{1\mathbf{x}}\psi^*_{2\mathbf{y}}\psi_{2\mathbf{x}}\psi_{1\mathbf{y}},$$
$$A := \lambda_1\psi^*_{1\hat{\mathbf{x}}}\psi_{2\hat{\mathbf{x}}} - \lambda_2\psi^*_{1\hat{\mathbf{x}}}\psi^*_{2\hat{\mathbf{y}}}\psi_{2\hat{\mathbf{x}}}\psi_{1\hat{\mathbf{y}}},$$
$$W_+ := \frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}}\sum_{\mathbf{x}\in\Gamma}\psi^*_{1\mathbf{x}}\psi_{2\mathbf{x}}, \quad W_- := \frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}}\sum_{\mathbf{x}\in\Gamma}\psi^*_{2\mathbf{x}}\psi_{1\mathbf{x}}.$$

We can see from (2.15), (2.30) that

$$\mathcal{U}(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A}-\phi\mathsf{V}_+-\overline{\phi}\mathsf{V}_--\xi\mathsf{W}_+-\overline{\xi}\mathsf{W}_-)\mathcal{U}^*$$
$$= H_0(\phi') + V + A - \xi W_+ - \overline{\xi}W_- + \sum_{\mathbf{k}\in\Gamma^*}e(\mathbf{k}) - i\frac{\theta}{2}L^d,$$

where we set $\phi' := \gamma - |U|^{\frac{1}{2}}\beta^{-\frac{1}{2}}L^{-\frac{d}{2}}\phi$. Therefore,

$$\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A}-\phi\mathsf{V}_+-\overline{\phi}\mathsf{V}_--\xi\mathsf{W}_+-\overline{\xi}\mathsf{W}_-)}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}$$
$$= \frac{e^{-\beta(\sum_{\mathbf{k}\in\Gamma^*}e(\mathbf{k})-i\frac{\theta}{2}L^d)}\,\mathrm{Tr}\, e^{-\beta H_0(\phi')}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}\frac{\mathrm{Tr}\, e^{-\beta(H_0(\phi')+V+A-\xi W_+-\overline{\xi}W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(\phi')}}.$$

For conciseness, set

$$B(\phi) := \frac{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))}.$$

Recalling Lemma 2.1 and Lemma 2.4, we observe that

$$\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}+\mathsf{A}-\phi\mathsf{V}_+-\overline{\phi}\mathsf{V}_--\xi\mathsf{W}_+-\overline{\xi}\mathsf{W}_-)}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}} = B(\phi')\frac{\mathrm{Tr}\, e^{-\beta(H_0(\phi')+V+A-\xi W_+-\overline{\xi}W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(\phi')}}. \tag{2.31}$$

The normalized partition function for the 2-band Hamiltonian $H_0(\phi') + V + A - \xi W_+ - \overline{\xi}W_-$ can be formulated into the time-continuum limit of a Grassmann Gaussian integral in $\bigwedge\mathcal{V}$ in the same way as the proof of Lemma 2.2. Here we especially need to make sure that the creation operators are on the left of the annihilation operators in $V+A-\xi W_+-\overline{\xi}W_-$. The result is that for any $r\in\mathbb{R}_{>0}$

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2}\left|\int e^{-V(\psi)-A(\psi)+\xi W_+(\psi)+\overline{\xi}W_-(\psi)}d\mu_{C(\phi')}(\psi)\right.$$
$$\left.-\frac{\mathrm{Tr}\, e^{-\beta(H_0(\phi')+V+A-\xi W_+-\overline{\xi}W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(\phi')}}\right|=0, \tag{2.32}$$

where

$$W_+(\psi):=\frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}h}\sum_{\mathbf{x}\in\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\mathbf{x}s}\psi_{2\mathbf{x}s},$$
$$W_-(\psi):=\frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}L^{\frac{d}{2}}h}\sum_{\mathbf{x}\in\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{2\mathbf{x}s}\psi_{1\mathbf{x}s}.$$

By considering the original definition (2.16) we can see that $C(\phi)$ has a determinant bound like (2.7). We can expand each exponential of the Grassmann polynomials to derive that

$$\left|\int e^{-V(\psi)-A(\psi)+\xi W_+(\psi)+\overline{\xi}W_-(\psi)}d\mu_{C(\phi')}(\psi)\right|$$
$$\le\max\left\{1,\frac{2^{2L^d}e^{\beta\|H_0(\phi')\|_{\mathcal{B}(F_{f,2})}}}{|\,\mathrm{Tr}\, e^{-\beta H_0(\phi')}|}\right\}$$
$$\cdot e^{|U|\beta D_2+|U|\beta L^dD_2^2+|\lambda_1|\beta D_2+|\lambda_2|\beta D_2^2+2|\xi||U|^{\frac{1}{2}}\beta^{\frac{1}{2}}L^{\frac{d}{2}}D_2}, \tag{2.33}$$

where $\|\cdot\|_{\mathcal{B}(F_{f,2})}$ denotes the operator norm of operators on $F_f(L^2(\{1,2\}\times\Gamma))$ and $D_2:=e^{2\beta\|H_0(\phi')\|_{\mathcal{B}(F_{f,2})}}$.

Here let us put these pieces together. By (2.28), (2.29) and (2.31) we can apply the dominated convergence theorem in $L^1(\mathbb{R}^2,C(\overline{D(r)}^2))$, $L^1(\mathbb{R}^4,C(\overline{D(r)}^2))$ to prove that

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2}$$
$$\cdot\left|\frac{1}{\pi}\int_{\mathbb{R}^2}d\xi_1d\xi_2e^{-|\xi|^2}\right. \tag{2.34}$$

$$\cdot \int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)+\xi\mathsf{W}_+(\psi)+\overline{\xi}\mathsf{W}_-(\psi)} d\mu_G(\psi)$$
$$- \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} B(\phi')$$
$$\left.\cdot \frac{\operatorname{Tr} e^{-\beta(H_0(\phi')+V+A-\xi W_+-\overline{\xi}W_-)}}{\operatorname{Tr} e^{-\beta H_0(\phi')}}\right| = 0, \quad (\forall \phi \in \mathbb{C}),$$

$$(2.35) \quad \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2}$$
$$\cdot \left| \frac{1}{\pi^2}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 \int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\phi|^2-|\xi|^2} \right.$$
$$\cdot \int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)+\xi\mathsf{W}_+(\psi)+\overline{\xi}\mathsf{W}_-(\psi)} d\mu_G(\psi)$$
$$- \frac{1}{\pi^2}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 \int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\phi|^2-|\xi|^2} B(\phi')$$
$$\left.\cdot \frac{\operatorname{Tr} e^{-\beta(H_0(\phi')+V+A-\xi W_+-\overline{\xi}W_-)}}{\operatorname{Tr} e^{-\beta H_0(\phi')}}\right| = 0.$$

By (2.32), (2.33) the dominated convergence theorem in $L^1(\mathbb{R}^2, C(\overline{D(r)}^2))$ ensures that

$$\lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2} \left| \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \int e^{-V(\psi)-A(\psi)+\xi W_+(\psi)+\overline{\xi}W_-(\psi)} d\mu_{C(\phi')}(\psi) \right.$$
$$\left. - \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \frac{\operatorname{Tr} e^{-\beta(H_0(\phi')+V+A-\xi W_+-\overline{\xi}W_-)}}{\operatorname{Tr} e^{-\beta H_0(\phi')}} \right|$$
$$= 0, \ (\forall \phi \in \mathbb{C}),$$

or by using the Hubbard-Stratonovich transformation again,

$$(2.36) \quad \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \sup_{\boldsymbol{\lambda}\in\overline{D(r)}^2} \left| \int e^{-V(\psi)+W(\psi)-A(\psi)} d\mu_{C(\phi')}(\psi) \right.$$
$$\left. - \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \frac{\operatorname{Tr} e^{-\beta(H_0(\phi')+V+A-\xi W_+-\overline{\xi}W_-)}}{\operatorname{Tr} e^{-\beta H_0(\phi')}} \right|$$

$$= 0, \ (\forall \phi \in \mathbb{C}),$$

which implies the claim (i). We can deduce from (2.28), (2.34) that the $C(\overline{D(r)}^2)$-valued function

$$(\phi_1, \phi_2) \mapsto \frac{e^{-|\phi|^2}}{\pi} \int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} B(\phi') \frac{\mathrm{Tr}\, e^{-\beta(H_0(\phi') + V + A - \xi W_+ - \overline{\xi} W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(\phi')}}$$

belongs to $L^1(\mathbb{R}^2, C(\overline{D(r)}^2))$. By combining this fact with (2.36) we see that the $C(\overline{D(r)}^2)$-valued function

$$(\phi_1, \phi_2) \mapsto \frac{e^{-|\phi|^2}}{\pi} B(\phi') \lim_{\substack{h \to \infty \\ h \in \frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi) + W(\psi) - A(\psi)} d\mu_{C(\phi')}(\psi)$$

belongs to $L^1(\mathbb{R}^2, C(\overline{D(r)}^2))$. Then, by changing ϕ to $|U|^{-\frac{1}{2}} \beta^{\frac{1}{2}} L^{\frac{d}{2}} (\gamma - \phi)$ we see that the claim (ii) holds. Moreover, by (2.3), (2.9), (2.27), (2.35), (2.36) and changing ϕ to $|U|^{-\frac{1}{2}} \beta^{\frac{1}{2}} L^{\frac{d}{2}} (\gamma - \phi)$,

(2.37)

$$\begin{aligned}
&\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H} + i\theta \mathsf{S}_z + \mathsf{F} + \mathsf{A})}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0 + i\theta \mathsf{S}_z)}} \\
&= \frac{1}{\pi} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-|\phi|^2} B(\phi') \lim_{\substack{h \to \infty \\ h \in \frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi) + W(\psi) - A(\psi)} d\mu_{C(\phi')}(\psi) \\
&= \frac{\beta L^d}{\pi |U|} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi - \gamma|^2} B(\phi) \lim_{\substack{h \to \infty \\ h \in \frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi) + W(\psi) - A(\psi)} d\mu_{C(\phi)}(\psi),
\end{aligned}$$

which is (2.25). Furthermore, by Cauchy's integral formula,

(2.38)

$$\begin{aligned}
&\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H} + i\theta \mathsf{S}_z + \mathsf{F})} \mathsf{A}_j)}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0 + i\theta \mathsf{S}_z)}} = -\frac{1}{2\pi i \beta} \oint_{|\lambda_j| = r} d\lambda_j \frac{1}{\lambda_j^2} \frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H} + i\theta \mathsf{S}_z + \mathsf{F} + \lambda_j \mathsf{A}_j)}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0 + i\theta \mathsf{S}_z)}} \\
&= -\frac{L^d}{\pi |U|} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 \frac{1}{2\pi i} \oint_{|\lambda_j| = r} d\lambda_j \frac{1}{\lambda_j^2} e^{-\frac{\beta L^d}{|U|}|\phi - \gamma|^2} B(\phi) \\
&\quad \cdot \lim_{\substack{h \to \infty \\ h \in \frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi) + W(\psi) - \lambda_j A^j(\psi)} d\mu_{C(\phi)}(\psi)
\end{aligned}$$

$$= -\frac{L^d}{\pi|U|}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2} B(\phi)$$
$$\cdot \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \frac{1}{2\pi i}\oint_{|\lambda_j|=r} d\lambda_j \frac{1}{\lambda_j^2}\int e^{-V(\psi)+W(\psi)-\lambda_j A^j(\psi)} d\mu_{C(\phi)}(\psi)$$
$$= \frac{L^d}{\pi|U|}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2} B(\phi) \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)} A^j(\psi) d\mu_{C(\phi)}(\psi),$$

which is (2.26). Note that the claim (ii) justifies the change of order of the integrals in the 2nd equality. The uniform convergence property claimed in (i) justifies the change of order of the integral and the limit operation in the 3rd equality. □

3. Estimation of Grassmann Integration

Thanks to Lemma 2.5, our objective is set to analyze the Grassmann integral

$$\int e^{-V(\psi)+W(\psi)-A(\psi)} d\mu_{C(\phi)}(\psi)$$

with $\phi(\in \mathbb{C})$ fixed. We especially need to find out which term will remain relevant after taking the limit $h \to \infty$, $L \to \infty$. To achieve this aim, it is efficient to generalize the problem to some extent so that we can describe the basic mechanism of convergence without taking care of a bunch of physical parameters. In the next section we will substitute the physical parameters into the general results obtained in this section.

In (2.11), (2.12) the Grassmann polynomials $V(\psi)$, $W(\psi)$ were defined with the coupling constant $U(\in \mathbb{R}_{<0})$. It is convenient for our analysis to extend the coupling constant to be a complex parameter. To avoid confusion, let us use the notation $V(u)(\psi)$, $W(u)(\psi)$ when we consider a complex parameter u in place of U. More precisely, we set for $u \in \mathbb{C}$

$$V(u)(\psi) := \frac{u}{L^d h}\sum_{\mathbf{x}\in\Gamma}\sum_{s\in[0,\beta)_h} \overline{\psi}_{1\mathbf{x}s}\psi_{1\mathbf{x}s} + \frac{u}{L^d h}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{s\in[0,\beta)_h} \overline{\psi}_{1\mathbf{x}s}\psi_{2\mathbf{x}s}\overline{\psi}_{2\mathbf{y}s}\psi_{1\mathbf{y}s},$$
$$W(u)(\psi) := \frac{u}{\beta L^d h^2}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{s,t\in[0,\beta)_h} \overline{\psi}_{1\mathbf{x}s}\psi_{2\mathbf{x}s}\overline{\psi}_{2\mathbf{y}t}\psi_{1\mathbf{y}t}$$

so that $V(U)(\psi) = V(\psi)$, $W(U)(\psi) = W(\psi)$.

3.1. Preliminaries

In addition to the brief introduction of Grassmann algebra in Subsection 2.1 here we need to define more notations, notational conventions and other tools necessary in the forthcoming analysis. We keep using I_0 and I defined in Subsection 2.3 as the index sets of Grassmann algebra. Let us admit that for any set S, $n \in \mathbb{N}$ and $\mathbf{X}$ belonging to the product set S^n, X_j denotes the j-th component of $\mathbf{X}$. Thus, $\mathbf{X}$ is equal to $(X_1, X_2, \cdots, X_n)$. We will use this notational rule, which helps to shorten formulas, without any additional comment. In many occasions we will apply this rule to the sets I_0^n, I^n. Size of a Grassmann polynomial can be measured through norms on its kernel functions. Thus, it is important to organize various notions concerning kernel functions. For $n \in \mathbb{N}$ let $\mathbb{S}_n$ denote the set of permutations over $\{1, 2, \cdots, n\}$. For $\mathbf{X} \in I^n$ and $\sigma \in \mathbb{S}_n$ we let $\mathbf{X}_\sigma$ denote $(X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(n)})$. For a function $f : I^n \to \mathbb{C}$ we call it anti-symmetric if

$$f(\mathbf{X}) = \mathrm{sgn}(\sigma) f(\mathbf{X}_\sigma), \quad (\forall \mathbf{X} \in I^n,\ \sigma \in \mathbb{S}_n).$$

For a function $g : I^m \times I^n \to \mathbb{C}$ we call it bi-anti-symmetric if

$$\begin{aligned}&g(\mathbf{X}, \mathbf{Y}) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) g(\mathbf{X}_\sigma, \mathbf{Y}_\tau),\\ &(\forall (\mathbf{X}, \mathbf{Y}) \in I^m \times I^n,\ \sigma \in \mathbb{S}_m,\ \tau \in \mathbb{S}_n).\end{aligned}$$

For any function $f : I^n \to \mathbb{C}$ $(n \in \mathbb{N}_{\geq 2})$ we define the norms $\|f\|_{1,\infty}$, $\|f\|_1$ by

$$\begin{aligned}\|f\|_{1,\infty} &:= \sup_{j \in \{1,2,\cdots,n\}} \sup_{X_0 \in I} \left(\frac{1}{h}\right)^{n-1} \sum_{\mathbf{X} \in I^{j-1}} \sum_{\mathbf{Y} \in I^{n-j}} |f(\mathbf{X}, X_0, \mathbf{Y})|,\\ \|f\|_1 &:= \left(\frac{1}{h}\right)^n \sum_{\mathbf{X} \in I^n} |f(\mathbf{X})|.\end{aligned}$$

Since anti-symmetric functions on I^2 play special roles in our analysis, we need to define other kinds of norm on them. For this purpose as well as other later use let us introduce a few notational conventions. For $\mathbf{X} =$

$(\rho_1\mathbf{x}_1s_1\xi_1, \rho_2\mathbf{x}_2s_2\xi_2, \cdots, \rho_n\mathbf{x}_ns_n\xi_n) \in (\{1,2\}\times\mathbb{Z}^d\times\frac{1}{h}\mathbb{Z}\times\{1,-1\})^n$, $s\in\frac{1}{h}\mathbb{Z}$, we set

$$\mathbf{X}+s := (\rho_1\mathbf{x}_1(s_1+s)\xi_1, \rho_2\mathbf{x}_2(s_2+s)\xi_2, \cdots, \rho_n\mathbf{x}_n(s_n+s)\xi_n). \tag{3.1}$$

Similarly for $\mathbf{X} = (\rho_1\mathbf{x}_1s_1, \cdots, \rho_n\mathbf{x}_ns_n) \in (\{1,2\}\times\mathbb{Z}^d\times\frac{1}{h}\mathbb{Z})^n$, $s\in\frac{1}{h}\mathbb{Z}$ we set $\mathbf{X}+s := (\rho_1\mathbf{x}_1(s_1+s), \cdots, \rho_n\mathbf{x}_n(s_n+s))$. Define the index set I^0 by

$$I^0 := \{1,2\}\times\Gamma\times\{0\}\times\{1,-1\}.$$

It follows that for $\mathbf{X}\in(I^0)^n$, $s\in[0,\beta)_h$, $\mathbf{X}+s\in I^n$. With these notational rules we define the norms $\|\cdot\|'_{1,\infty}$, $\|\cdot\|$ on anti-symmetric functions on I^2 as follows. For any anti-symmetric function $g : I^2\to\mathbb{C}$,

$$\|g\|'_{1,\infty} := \sup_{\substack{X_0\in I\\ s\in[0,\beta)_h}} \sum_{X\in I^0} |g(X_0, X+s)|, \quad \|g\| := \|g\|'_{1,\infty} + \beta^{-1}\|g\|_{1,\infty}.$$

We will also deal with bi-anti-symmetric functions on the product set $I^m\times I^n$. By considering that these functions are defined on I^{m+n} the norms $\|\cdot\|_{1,\infty}$, $\|\cdot\|_1$ can be defined on them. We will need to measure these functions coupled with another anti-symmetric function on I^2. The measurement will be carried out in terms of the following quantities. For a bi-anti-symmetric function $f_{m,n} : I^m\times I^n\to\mathbb{C}$ $(m,n\in\mathbb{N}_{\geq 2})$ and an anti-symmetric function $g : I^2\to\mathbb{C}$ we set

$$\begin{aligned}
&[f_{m,n}, g]_{1,\infty}\\
&:= \max\Bigg\{ \sup_{X_0\in I}\left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{X}\in I^{m-1}}\\
&\qquad\cdot\left\{\sup_{Y_0\in I}\left(\frac{1}{h}\right)^{n} \sum_{\mathbf{Y}\in I^{n}} |f_{m,n}((X_0,\mathbf{X}),\mathbf{Y})||g(Y_0,Y_1)|\right\},\\
&\qquad \sup_{Y_0\in I}\left(\frac{1}{h}\right)^{n-1} \sum_{\mathbf{Y}\in I^{n-1}}\\
&\qquad\cdot\left\{\sup_{X_0\in I}\left(\frac{1}{h}\right)^{m} \sum_{\mathbf{X}\in I^{m}} |f_{m,n}(\mathbf{X},(Y_0,\mathbf{Y}))||g(X_0,X_1)|\right\}\Bigg\},\\
&[f_{m,n}, g]_1 := \left(\frac{1}{h}\right)^{m+n} \sum_{\substack{\mathbf{X}\in I^m\\ \mathbf{Y}\in I^n}} |f_{m,n}(\mathbf{X},\mathbf{Y})||g(X_1,Y_1)|.
\end{aligned}$$

For $\mathbf{X} \in I^n$ let $\psi_{\mathbf{X}}$ denote $\psi_{X_1}\psi_{X_2}\cdots\psi_{X_n}$. Anti-symmetry of Grassmann variables implies that for any $f(\psi) \in \bigwedge \mathcal{V}$ there uniquely exist $f_0 \in \mathbb{C}$ and anti-symmetric functions $f_n : I^n \to \mathbb{C}$ $(n = 1, 2, \cdots, N)$ such that

$$f(\psi) = \sum_{n=0}^{N} \left(\frac{1}{h}\right)^n \sum_{\mathbf{X} \in I^n} f_n(\mathbf{X})\psi_{\mathbf{X}}.$$

Based on this fact we admit that for $f(\psi) \in \bigwedge \mathcal{V}$, f_n $(n = 0, 1, \cdots, N)$ denote the unique anti-symmetric kernels of $f(\psi)$. A norm can be defined in the vector space $\bigwedge \mathcal{V}$ by defining a norm in every space of anti-symmetric kernels. Finite dimensionality of $\bigwedge \mathcal{V}$ implies that $\bigwedge \mathcal{V}$ is a Banach space with the norm. Then, by considering as a Banach-space-valued function the standard notions such as continuity, differentiability and analyticity of a Grassmann polynomial parameterized by real or complex variables are defined. Since we introduce various norms on anti-symmetric kernels, it is clearer to define these notions without specifying a norm on $\bigwedge \mathcal{V}$. We say that a sequence of elements of $\bigwedge \mathcal{V}$, $f^m(\psi)$ $(m = 1, 2, \cdots)$ converges in $\bigwedge \mathcal{V}$ if each anti-symmetric kernel function of $f^m(\psi)$ converges point-wise, or more precisely $\lim_{m\to\infty} f_n^m(\mathbf{X})$ converges in $\mathbb{C}$ for any $n \in \{0, 1, \cdots, N\}$, $\mathbf{X} \in I^n$. For a domain O of $\mathbb{R}^m$ or $\mathbb{C}^m$ and $f(\mathbf{z})(\psi) \in \bigwedge \mathcal{V}$ parameterized by $\mathbf{z} \in \overline{O}$ we say that $f(\mathbf{z})(\psi)$ is continuous with $\mathbf{z}$ in $\overline{O}$, differentiable with $\mathbf{z}$ in O and analytic with $\mathbf{z}$ in O if so is $f(\mathbf{z})_n(\mathbf{X})$ for any $n \in \{0, 1, \cdots, N\}$, $\mathbf{X} \in I^n$. Moreover, when it is differentiable, for $j \in \{1, 2, \cdots, m\}, \mathbf{z} = (z_1, z_2, \cdots, z_m) \in O$ we define the Grassmann polynomial $\frac{\partial}{\partial z_j} f(\mathbf{z})(\psi) \in \bigwedge \mathcal{V}$ by

$$\frac{\partial}{\partial z_j} f(\mathbf{z})(\psi) := \sum_{n=0}^{N} \left(\frac{1}{h}\right)^n \sum_{\mathbf{X} \in I^n} \frac{\partial}{\partial z_j} f(\mathbf{z})_n(\mathbf{X})\psi_{\mathbf{X}}.$$

The single-scale integration is well-described in terms of trees. We refer to the clear statement of the tree formula with a self-contained proof presented in [19, Theorem 3, Appendix A]. We should also lead the readers to the references of [19] for more original versions of such expansion techniques known as the Brydges-Battle-Federbush formula. To state the formula, we need to recall the definition of Grassmann left-derivatives. Let $\mathcal{V}^j$ be the vector space spanned by the basis $\{\psi_X^j \mid X \in I\}$ for $j = 1, 2, \cdots, n$. For $p \in \{1, \cdots, n\}$, $X \in I$ the Grassmann left-derivative $\partial/\partial\psi_X^p$ is a linear transform on $\bigwedge(\mathcal{V}^1 \oplus \cdots \oplus \mathcal{V}^n)$ defined by

$$\frac{\partial}{\partial \psi_X^p}(\psi_{X_1}^{p_1} \cdots \psi_{X_j}^{p_j} \psi_X^p \psi_{X_{j+1}}^{p_{j+1}} \cdots \psi_{X_m}^{p_m}) := (-1)^j \psi_{X_1}^{p_1} \cdots \psi_{X_j}^{p_j} \psi_{X_{j+1}}^{p_{j+1}} \cdots \psi_{X_m}^{p_m},$$
$$\frac{\partial}{\partial \psi_X^p}(\psi_{X_1}^{p_1} \cdots \psi_{X_j}^{p_j} \psi_{X_{j+1}}^{p_{j+1}} \cdots \psi_{X_m}^{p_m}) := 0$$

for any $(p_j, X_j) \in \{1, 2, \cdots, n\} \times I$ satisfying $(p_j, X_j) \neq (p, X)$ $(j = 1, 2, \cdots, m)$ and by linearity. The Grassmann left-derivative $\partial/\partial \psi_X$ $(X \in I)$ can be defined as a linear transform on $\bigwedge(\mathcal{V} \oplus \mathcal{V}^1 \oplus \cdots \oplus \mathcal{V}^n)$ in the same way.

Let $\bigwedge_{even} \mathcal{V}$ denote a subspace of $\bigwedge \mathcal{V}$ consisting of even polynomials. More precisely,

$$\bigwedge_{even} \mathcal{V} := \bigoplus_{n=0}^{N/2} \bigwedge^{2n} \mathcal{V}.$$

For a covariance $\mathcal{C} : I_0^2 \to \mathbb{C}$ and $f^j(\psi) \in \bigwedge_{even} \mathcal{V}$ $(j = 1, 2, \cdots, n)$ the Grassmann polynomial

$$\log\left(\int e^{\sum_{j=1}^n z_j f^j(\psi + \psi^1)} d\mu_{\mathcal{C}}(\psi^1)\right)$$

is well-defined and analytic with $(z_1, z_2, \cdots, z_n)$ in a neighborhood of the origin. The strategy of single-scale analysis is to expand the logarithm into the Taylor series around the origin and estimate each order term. To this end we need to know a formula for

$$\frac{1}{n!} \prod_{j=1}^n \left(\frac{\partial}{\partial z_j}\right) \log\left(\int e^{\sum_{j=1}^n z_j f^j(\psi + \psi^1)} d\mu_{\mathcal{C}}(\psi^1)\right)\Bigg|_{\substack{z_j = 0 \\ (\forall j \in \{1, 2, \cdots, n\})}}.$$

The formula for $n = 1$ can be derived from the definition as follows.

$$\begin{aligned}
&\frac{d}{dz} \log\left(\int e^{z f^1(\psi + \psi^1)} d\mu_{\mathcal{C}}(\psi^1)\right)\Bigg|_{z=0} \\
&= \int f^1(\psi + \psi^1) d\mu_{\mathcal{C}}(\psi^1) \\
&= e^{-\sum_{\mathbf{X} \in I_0^2} \mathcal{C}(\mathbf{X}) \frac{\partial}{\partial \overline{\psi}^1_{X_1}} \frac{\partial}{\partial \psi^1_{X_2}}} f^1(\psi + \psi^1)\Big|_{\psi^1 = 0}.
\end{aligned}$$

The tree formula characterizes the derivative for $n \in \mathbb{N}_{\ge 2}$.

$$
\begin{aligned}
&\frac{1}{n!}\prod_{j=1}^{n}\left(\frac{\partial}{\partial z_j}\right)\log\left(\int e^{\sum_{j=1}^{n} z_j f^j(\psi+\psi^1)}d\mu_{\mathcal{C}}(\psi^1)\right)\Bigg|_{\substack{z_j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=\frac{1}{n!}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n\})}\prod_{\{p,q\}\in T}(\Delta_{p,q}(\mathcal{C})+\Delta_{q,p}(\mathcal{C}))\\
&\quad\cdot\int_{[0,1]^{n-1}}d\mathbf{s}\sum_{\sigma\in\mathbb{S}_n(T)}\varphi(T,\sigma,\mathbf{s})e^{\sum_{a,b=1}^{n}M(T,\sigma,\mathbf{s})_{a,b}\Delta_{a,b}(\mathcal{C})}\\
&\quad\cdot\prod_{j=1}^{n}f^j(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}},
\end{aligned}
$$

where $\mathbb{T}(\{1,2,\cdots,n\})$ is the set of all trees over the vertices $\{1,2,\cdots,n\}$,

$$
\Delta_{p,q}(\mathcal{C}) := -\sum_{\mathbf{X}\in I_0^2}\mathcal{C}(\mathbf{X})\frac{\partial}{\partial\overline{\psi}^p_{X_1}}\frac{\partial}{\partial\psi^q_{X_2}},
$$

$\mathbb{S}_n(T)$ is a T-dependent subset of $\mathbb{S}_n$, $\varphi(T,\sigma,\cdot)$ is a real non-negative function on $[0,1]^{n-1}$ depending on $T \in \mathbb{T}(\{1,2,\cdots,n\})$, $\sigma \in \mathbb{S}_n(T)$ and $(M(T,\sigma,\mathbf{s})_{a,b})_{1\le a,b\le n}$ is a $(T,\sigma,\mathbf{s})$-dependent real symmetric non-negative matrix which satisfies that

$$
\begin{aligned}
&M(T,\sigma,\mathbf{s})_{a,a}=1,\\
&(\forall a\in\{1,\cdots,n\},\ T\in\mathbb{T}(\{1,\cdots,n\}),\ \sigma\in\mathbb{S}_n(T),\ \mathbf{s}\in[0,1]^{n-1}).\\
&\mathbf{s}\mapsto M(T,\sigma,\mathbf{s})_{a,b}\text{ is continuous in }[0,1]^{n-1},\\
&(\forall a,b\in\{1,\cdots,n\},\ T\in\mathbb{T}(\{1,\cdots,n\}),\ \sigma\in\mathbb{S}_n(T)).
\end{aligned}
$$

Moreover, the function $\varphi(T,\sigma,\cdot)$ satisfies that

$$
\int_{[0,1]^{n-1}}d\mathbf{s}\sum_{\sigma\in\mathbb{S}_n(T)}\varphi(T,\sigma,\mathbf{s})=1,\quad(\forall T\in\mathbb{T}(\{1,2,\cdots,n\})). \tag{3.2}
$$

Because of the property (3.2) the function $\varphi(T,\sigma,\cdot)$ does not affect our estimation of Grassmann polynomials in practice. We can deduce from the fact that the matrix $M(T,\sigma,\mathbf{s})$ is real symmetric non-negative and all the diagonal elements are 1 that there are $\mathbf{v}_1,\cdots,\mathbf{v}_n\in\mathbb{R}^n$ such that $\|\mathbf{v}_i\|_{\mathbb{R}^n}=1$ $(i=1,\cdots,n)$ and

$$M(T,\sigma,\mathbf{s})_{a,b}=\langle\mathbf{v}_a,\mathbf{v}_b\rangle,\quad(\forall a,b\in\{1,\cdots,n\}).\tag{3.3}$$

Thus,

$$\begin{aligned}&|M(T,\sigma,\mathbf{s})_{a,b}|\le 1,\\&(\forall a,b\in\{1,\cdots,n\},\ T\in\mathbb{T}(\{1,\cdots,n\}),\ \sigma\in\mathbb{S}_n(T),\ \mathbf{s}\in[0,1]^{n-1}).\end{aligned}\tag{3.4}$$

To systematize our estimation, let us define operators on Grassmann algebras which are slight generalization of the above formulas. For $p,q\in\mathbb{Z}$, set

$$\Delta_{\{p,q\}}(\mathcal{C}):=\sum_{\mathbf{X}\in I^2}\tilde{\mathcal{C}}(\mathbf{X})\frac{\partial}{\partial\psi^p_{X_1}}\frac{\partial}{\partial\psi^q_{X_2}},$$

where $\tilde{\mathcal{C}}:I^2\to\mathbb{C}$ is the anti-symmetric extension of $\mathcal{C}$ defined by

$$\begin{aligned}&\tilde{\mathcal{C}}((X,\xi),(Y,\zeta)):=\frac{1}{2}(1_{(\xi,\zeta)=(1,-1)}\mathcal{C}(X,Y)-1_{(\xi,\zeta)=(-1,1)}\mathcal{C}(Y,X)),\\&(\forall X,Y\in I_0,\ \xi,\zeta\in\{1,-1\}).\end{aligned}\tag{3.5}$$

We can see that

$$\begin{aligned}&-2\Delta_{\{p,q\}}(\mathcal{C})=\Delta_{p,q}(\mathcal{C})+\Delta_{q,p}(\mathcal{C}),\\&-\sum_{p,q=1}^n M(T,\sigma,\mathbf{s})_{p,q}\Delta_{\{p,q\}}(\mathcal{C})=\sum_{p,q=1}^n M(T,\sigma,\mathbf{s})_{p,q}\Delta_{p,q}(\mathcal{C}).\end{aligned}$$

For $S=\{s_1,s_2,\cdots,s_n\}(\subset\mathbb{Z})$ with $\sharp S=n\ge 2$ let $\mathbb{T}(S)$ denote the set of all trees over the vertices $\{s_1,s_2,\cdots,s_n\}$. Using these notations we set for $S=\{s_1,s_2,\cdots,s_n\}(\subset\mathbb{Z})$, if $n=1$,

$$Tree(S,\mathcal{C}):=e^{\Delta_{\{s_1,s_1\}}(\mathcal{C})},$$

if $n\ge 2$,

$$\begin{aligned}Tree(S,\mathcal{C}):=(-2)^{n-1}\sum_{T\in\mathbb{T}(S)}\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\int_{[0,1]^{n-1}}d\mathbf{s}\sum_{\sigma\in\mathbb{S}_n(T)}\varphi(T,\sigma,\mathbf{s})\\ \cdot e^{-\sum_{a,b=1}^n M(T,\sigma,\mathbf{s})_{a,b}\Delta_{\{s_a,s_b\}}(\mathcal{C})}.\end{aligned}$$

It follows that for any $n\in\mathbb{N}$, $f^j(\psi)\in\bigwedge_{even}\mathcal{V}$ $(j=1,2,\cdots,n)$,

$$\frac{1}{n!}\prod_{j=1}^{n}\left(\frac{\partial}{\partial z_j}\right)\log\left(\int e^{\sum_{j=1}^{n}z_jf^j(\psi+\psi^1)}d\mu_{\mathcal{C}}(\psi^1)\right)\Bigg|_{\substack{z_j=0\\(\forall j\in\{1,2,\cdots,n\})}} = \frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^{n}f^j(\psi+\psi^j)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}. \tag{3.6}$$

When $f^j(\psi)=f(\psi)$ for any $j\in\{1,2,\cdots,n\}$, the formula (3.6) implies that

$$\frac{1}{n!}\left(\frac{d}{dz}\right)^{n}\log\left(\int e^{zf(\psi+\psi^1)}d\mu_{\mathcal{C}}(\psi^1)\right)\Bigg|_{z=0} = \frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^{n}f(\psi+\psi^j)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}. \tag{3.7}$$

The following inequalities will be crucially important.

$$\left|e^{\Delta_{\{s_1,s_1\}}(\mathcal{C})}\psi^{s_1}_{\mathbf{X}}\Big|_{\psi^{s_1}=0}\right| \le \begin{cases}0 & \text{if } m \text{ is odd},\\ \sup_{\substack{Y_j,Z_j\in I_0\\(j=1,2,\cdots,\frac{m}{2})}}\left|\det(\mathcal{C}(Y_i,Z_j))_{1\le i,j\le\frac{m}{2}}\right| & \text{if } m \text{ is even},\end{cases} \tag{3.8}$$

$$(\forall m\in\mathbb{N},\ \mathbf{X}\in I^m).$$

$$\left|e^{-\sum_{a,b=1}^{n}M(T,\sigma,\mathbf{s})_{a,b}\Delta_{\{s_a,s_b\}}(\mathcal{C})}\prod_{j=1}^{n}\psi^{s_j}_{\mathbf{X}_j}\Bigg|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,n\})}}\right| \tag{3.9}$$

$$\le\begin{cases}0 & \text{if } m \text{ is odd},\\ \sup_{\substack{\mathbf{u}_j,\mathbf{v}_j\in\mathbb{C}^n\text{ with }\|\mathbf{u}_j\|_{\mathbb{C}^n},\|\mathbf{v}_j\|_{\mathbb{C}^n}\le 1\\(j=1,\cdots,\frac{m}{2})}}\ \sup_{\substack{Y_j,Z_j\in I_0\\(j=1,\cdots,\frac{m}{2})}} & \\ \quad\cdot\left|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^n}\mathcal{C}(Y_i,Z_j))_{1\le i,j\le\frac{m}{2}}\right| & \text{if } m \text{ is even},\end{cases}$$

$$(\forall m_j\in\{0,1,\cdots,N\},\ \mathbf{X}_j\in I^{m_j}\ (j=1,2,\cdots,n),$$
$$T\in\mathbb{T}(\{s_1,s_2,\cdots,s_n\}),\ \sigma\in\mathbb{S}_n(T),\ \mathbf{s}\in[0,1]^{n-1}),$$

where $m:=\sum_{j=1}^{n}m_j$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ is the hermitian inner product and $\|\cdot\|_{\mathbb{C}^n}$ is the norm induced by $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$. The inequality (3.9) is based on the Gram representation (3.3) of the matrix $M(T,\sigma,\mathbf{s})$. See e.g. [9, Lemma 4.5] for details of how to derive an inequality of this kind.

3.2. General estimation

Here we estimate Grassmann polynomials produced by applying the operator $Tree(S,\mathcal{C})$ to given Grassmann polynomials. As explained in the beginning of the section our purpose here is to summarize generic structures of single-scale integrations. Let us introduce some notions which are necessary to describe properties of the Grassmann input and the covariances of the single-scale integrations. To describe periodicity and translation invariance with the time variable, we define the map $r_\beta : \frac{1}{h}\mathbb{Z} \to [0,\beta)_h$ by the condition that $r_\beta(s) \in [0,\beta)_h$ and $r_\beta(s) = s$ in $\frac{1}{h}\mathbb{Z}/\beta\mathbb{Z}$ for $s \in \frac{1}{h}\mathbb{Z}$. Then we define the map $\mathcal{R}_\beta$ from $(\{1,2\}\times\Gamma\times\frac{1}{h}\mathbb{Z}\times\{1,-1\})^n$ to I^n by

$$\mathcal{R}_\beta(\rho_1\mathbf{x}_1 s_1\xi_1,\cdots,\rho_n\mathbf{x}_n s_n\xi_n) := (\rho_1\mathbf{x}_1 r_\beta(s_1)\xi_1,\cdots,\rho_n\mathbf{x}_n r_\beta(s_n)\xi_n).$$

We will sometimes consider $\mathcal{R}_\beta$ as the map from $(\{1,2\}\times\Gamma\times\frac{1}{h}\mathbb{Z})^n$ to I_0^n satisfying that

$$\mathcal{R}_\beta(\rho_1\mathbf{x}_1 s_1,\cdots,\rho_n\mathbf{x}_n s_n) = (\rho_1\mathbf{x}_1 r_\beta(s_1),\cdots,\rho_n\mathbf{x}_n r_\beta(s_n)),$$

by admitting the notational abuse. The meaning of the map $\mathcal{R}_\beta$ should be understood from the context.

We assume that the covariance $\mathcal{C} : I_0^2 \to \mathbb{C}$ satisfies that

$$\mathcal{C}(\mathcal{R}_\beta(\mathbf{X}+s)) = \mathcal{C}(\mathbf{X}), \quad \left(\forall \mathbf{X}\in I_0^2,\ s\in\frac{1}{h}\mathbb{Z}\right), \tag{3.10}$$

$$\begin{aligned}&|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}\mathcal{C}(X_i,Y_j))_{1\le i,j\le n}| \le D^n,\\ &(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,\\ &\quad X_i,Y_i\in I_0\ (i=1,2,\cdots,n)),\end{aligned} \tag{3.11}$$

where D is a fixed positive constant. The condition (3.10) might appear unnatural if $\mathcal{C}$ is thought to be a sum over the Matsubara frequency. However, one can modify such a covariance to satisfy (3.10) by a simple gauge transform.

One implication of the property (3.10) is that

(3.12)

$$Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^n \psi^j_{\mathcal{R}_\beta(\mathbf{X}_j+s)}\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

$$= Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^{n}\psi^j_{\mathbf{X}_j}\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}},$$

$$\left(\forall n\in\mathbb{N},\ m_j\in\{0,1,\cdots,N\},\ \mathbf{X}_j\in I^{m_j}\ (j=1,2,\cdots,n),\ s\in\frac{1}{h}\mathbb{Z}\right).$$

For $j\in\mathbb{N}$ let $F^j(\psi)\in\bigwedge_{even}\mathcal{V}$ be such that its anti-symmetric kernels $F^j_m : I^m\to\mathbb{C}$ $(m=2,4,\cdots,N)$ satisfy

$$F^j_m(\mathcal{R}_\beta(\mathbf{X}+s)) = F^j_m(\mathbf{X}),\quad \left(\forall\mathbf{X}\in I^m,\ s\in\frac{1}{h}\mathbb{Z}\right). \tag{3.13}$$

In this subsection we will give the Grassmann polynomials $F^j(\psi)$ $(j\in\mathbb{N})$ as the input to the single-scale integrations.

For $n\in\mathbb{N}$ we define $A^{(n)}(\psi)\in\bigwedge_{even}\mathcal{V}$ by

$$A^{(n)}(\psi) := Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^{n}F^j(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.$$

Since $Tree(S,\mathcal{C})$ consists of Grassmann left-derivatives of even degree, it is clear that the output belongs to $\bigwedge_{even}\mathcal{V}$ if so does the input.

For conciseness of formulas we let $\|f_0\|_{1,\infty}=\|f_0\|_1 := |f_0|$ for the constant term f_0 of $f(\psi)\in\bigwedge\mathcal{V}$. We admit this notational convention throughout this section. The next lemma is the simplest among other lemmas in this subsection.

Lemma 3.1. *For any $m\in\{2,4,\cdots,N\}$, $n\in\mathbb{N}$ the anti-symmetric kernel $A^{(n)}_m(\cdot)$ satisfies* (3.13). *Moreover the following inequalities hold for any $m\in\{0,2,\cdots,N\}$, $n\in\mathbb{N}_{\ge 2}$.*

$$\|A^{(1)}_m\|_{1,\infty} \le \sum_{p=m}^{N}\left(\frac{N}{h}\right)^{1_{m=0\wedge p\neq 0}}\begin{pmatrix} p\\ m\end{pmatrix}D^{\frac{p-m}{2}}\|F^1_p\|_{1,\infty}. \tag{3.14}$$

$$\|A^{(1)}_m\|_1 \le \sum_{p=m}^{N}\begin{pmatrix} p\\ m\end{pmatrix}D^{\frac{p-m}{2}}\|F^1_p\|_1. \tag{3.15}$$

$$\|A^{(n)}_m\|_{1,\infty} \le \left(\frac{N}{h}\right)^{1_{m=0}}(n-2)!D^{-n+1-\frac{m}{2}}2^{-2m}\|\tilde{\mathcal{C}}\|^{n-1}_{1,\infty} \tag{3.16}$$

$$\cdot\prod_{j=1}^{n}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|F^j_{p_j}\|_{1,\infty}\right)1_{\sum_{j=1}^{n}p_j-2(n-1)\ge m}.$$

$$\|A^{(n)}_m\|_1\le(n-2)!D^{-n+1-\frac{m}{2}}2^{-2m}\|\tilde{\mathcal{C}}\|^{n-1}_{1,\infty}\sum_{p_1=2}^{N}2^{3p_1}D^{\frac{p_1}{2}}\|F^1_{p_1}\|_1 \tag{3.17}$$
$$\cdot\prod_{j=2}^{n}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|F^j_{p_j}\|_{1,\infty}\right)1_{\sum_{j=1}^{n}p_j-2(n-1)\ge m}.$$

Here $\tilde{\mathcal{C}}(:I^2\to\mathbb{C})$ *is the anti-symmetric extension of* $\mathcal{C}$ *defined as in* (3.5).

PROOF. By anti-symmetry,

$$A^{(1)}(\psi)=\sum_{p=0}^{N}\sum_{m=0}^{p}\begin{pmatrix}p\\m\end{pmatrix}\left(\frac{1}{h}\right)^p\sum_{\substack{\mathbf{X}\in I^m\\ \mathbf{Y}\in I^{p-m}}}F^1_p(\mathbf{Y},\mathbf{X})Tree(\{1\},\mathcal{C})\psi^1_{\mathbf{Y}}\Big|_{\psi^1=0}\psi_{\mathbf{X}}.$$

Thus, by the uniqueness of anti-symmetric kernels, for any $m\in\{0,2,\cdots,N\}$, $\mathbf{X}\in I^m$,

$$A^{(1)}_m(\mathbf{X})=\sum_{p=m}^{N}\begin{pmatrix}p\\m\end{pmatrix}\left(\frac{1}{h}\right)^{p-m}\sum_{\mathbf{Y}\in I^{p-m}}F^1_p(\mathbf{Y},\mathbf{X})Tree(\{1\},\mathcal{C})\psi^1_{\mathbf{Y}}\Big|_{\psi^1=0}. \tag{3.18}$$

By using the translation invariant properties (3.12), (3.13) we see that for any $\mathbf{X}\in I^m$, $s\in\frac{1}{h}\mathbb{Z}$

$$\begin{aligned}
&A^{(1)}_m(\mathcal{R}_\beta(\mathbf{X}+s))\\
&=\sum_{p=m}^{N}\begin{pmatrix}p\\m\end{pmatrix}\left(\frac{1}{h}\right)^{p-m}\sum_{\mathbf{Y}\in I^{p-m}}F^1_p(\mathbf{Y},\mathcal{R}_\beta(\mathbf{X}+s))Tree(\{1\},\mathcal{C})\psi^1_{\mathbf{Y}}\Big|_{\psi^1=0}\\
&=\sum_{p=m}^{N}\begin{pmatrix}p\\m\end{pmatrix}\\
&\quad\cdot\left(\frac{1}{h}\right)^{p-m}\sum_{\mathbf{Y}\in I^{p-m}}F^1_p(\mathcal{R}_\beta((\mathbf{Y},\mathbf{X})+s))Tree(\{1\},\mathcal{C})\psi^1_{\mathcal{R}_\beta(\mathbf{Y}+s)}\Big|_{\psi^1=0}\\
&=A^{(1)}_m(\mathbf{X}).
\end{aligned}$$

Thus, $A_m^{(1)}$ satisfies (3.13). The inequalities (3.14), (3.15) can be derived from (3.18) by using (3.8), (3.11).

Next let us consider the case $n \geq 2$. By anti-symmetry,

$$\begin{aligned}
&A^{(n)}(\psi)\\
&= \prod_{j=1}^{n}\left(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\left(\begin{array}{c} p_j \\ m_j \end{array}\right)\left(\frac{1}{h}\right)^{p_j}\sum_{\substack{\mathbf{X}_j\in I^{m_j}\\ \mathbf{Y}_j\in I^{p_j-m_j}}} F_{p_j}^{j}(\mathbf{Y}_j,\mathbf{X}_j)\right)\\
&\quad\cdot Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^{n}\psi_{\mathbf{Y}_j}^{j}\Big|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n\})}}(-1)^{\sum_{j=1}^{n-1}m_j\sum_{k=j+1}^{n}(p_k-m_k)}\\
&\quad\cdot\prod_{k=1}^{n}\psi_{\mathbf{X}_k}.
\end{aligned}$$

Thus, for $m \in \{0, 2, \cdots, N\}$, $\mathbf{X} \in I^m$,

(3.19)

$$\begin{aligned}
&A_m^{(n)}(\mathbf{X})\\
&= \frac{1}{m!}\sum_{\sigma\in\mathbb{S}_m}\mathrm{sgn}(\sigma)\prod_{j=1}^{n}\left(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\left(\begin{array}{c} p_j \\ m_j \end{array}\right)\right.\\
&\qquad\qquad\left.\cdot\left(\frac{1}{h}\right)^{p_j-m_j}\sum_{\mathbf{Y}_j\in I^{p_j-m_j}} F_{p_j}^{j}(\mathbf{Y}_j,\mathbf{X}_j')\right)\\
&\quad\cdot Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^{n}\psi_{\mathbf{Y}_j}^{j}\Big|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n\})}}(-1)^{\sum_{j=1}^{n-1}m_j\sum_{k=j+1}^{n}(p_k-m_k)}\\
&\quad\cdot 1_{(\mathbf{X}_1',\mathbf{X}_2',\cdots,\mathbf{X}_n')=\mathbf{X}_\sigma}1_{\sum_{j=1}^{n}m_j=m}1_{\sum_{j=1}^{n}p_j-2(n-1)\geq m}.
\end{aligned}$$

The constraint $\sum_{j=1}^{n}p_j - 2(n-1) \geq m$ is added since the operator $\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})$ inside $Tree(\{1,2,\cdots,n\},\mathcal{C})$ erases $2(n-1)$ Grassmann variables. Again by using (3.12), (3.13) we can check that $A_m^{(n)} : I^m \to \mathbb{C}$ satisfies (3.13).

To establish upper bounds on the norms of $A_m^{(n)}$ we need to replace the sum over trees by a sum over possible degrees of trees. For $j \in \{1, 2, \cdots, n\}$,

$T \in \mathbb{T}(\{1,2,\cdots,n\})$ let $d_j(T)$ denote the degree of the vertex j in T. The following calculation, which we will frequently refer to during this subsection, is based on Cayley's theorem on the number of trees with fixed degrees. For any $k_1, k_2, \cdots, k_n \in \mathbb{N}$,

$$\begin{aligned}(3.20)\quad &\sum_{T\in\mathbb{T}(\{1,2,\cdots,n\})}\prod_{j=1}^{n}\left(\begin{pmatrix} k_j \\ d_j(T)\end{pmatrix} d_j(T)!\right)\\ &=\prod_{j=1}^{n}\left(\sum_{d_j=1}^{k_j}\begin{pmatrix} k_j \\ d_j\end{pmatrix} d_j!\right)\frac{(n-2)!}{\prod_{k=1}^{n}(d_k-1)!}1_{\sum_{j=1}^{n} d_j=2(n-1)}\\ &\le (n-2)!\prod_{j=1}^{n}(k_j 2^{k_j-1}) \le (n-2)!2^{-n}2^{2\sum_{j=1}^{n}k_j}.\end{aligned}$$

Since we will need to deal with the case $n=1$ at the same time, let us give a meaning to the left-hand side of (3.20) for $n=1$ and generalize the combinatorial estimate (3.20) to be valid for any $n \in \mathbb{N}$. We assume that $\mathbb{T}(\{1\}) = \{\{1\}\}$ and $d_1(\{1\}) = 0$ so that the left-hand side is 1. It follows from this convention that for any $n \in \mathbb{N}$

$$\begin{aligned}(3.21)\quad &\sum_{T\in\mathbb{T}(\{1,2,\cdots,n\})}\prod_{j=1}^{n}\left(\begin{pmatrix} k_j \\ d_j(T)\end{pmatrix} d_j(T)!\right)\\ &\le (1_{n=1}+1_{n\ge 2}(n-2)!2^{-n})2^{2\sum_{j=1}^{n}k_j}.\end{aligned}$$

By (3.2), (3.9), (3.11),

$$\begin{aligned}&|A_m^{(n)}(\mathbf{X})|\\ &\le \frac{2^{n-1}}{m!}\sum_{\sigma\in\mathbb{S}_m}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n\})}\prod_{j=1}^{n}\\ &\quad\cdot\left(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\begin{pmatrix} p_j \\ m_j\end{pmatrix}\begin{pmatrix} p_j-m_j \\ d_j(T)\end{pmatrix}\left(\frac{1}{h}\right)^{p_j-m_j}\right.\\ &\quad\quad\cdot\left.\sum_{\substack{\mathbf{X}'_j\in I^{m_j},\mathbf{Y}_j\in I^{p_j-m_j-d_j(T)}\\ \mathbf{Z}_j\in I^{d_j(T)}}}|F_{p_j}^{j}(\mathbf{Y}_j,\mathbf{Z}_j,\mathbf{X}'_j)|\right)D^{\frac{1}{2}(\sum_{j=1}^{n}p_j-2(n-1)-m)}\end{aligned}$$

$$\cdot\left|\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{n}\psi^{j}_{\mathbf{Z}_j}\right|$$
$$\cdot 1_{(\mathbf{X}'_1,\mathbf{X}'_2,\cdots,\mathbf{X}'_n)=\mathbf{X}_\sigma}1_{\sum_{j=1}^{n}m_j=m}1_{\sum_{j=1}^{n}p_j-2(n-1)\ge m}.$$

Note that

$$\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{n}\psi^{j}_{\mathbf{Z}_j}$$

creates at most $\prod_{j=1}^{n}d_j(T)!$ terms, since

$$\left(\sum_{X\in I}\frac{\partial}{\partial\psi^{j}_{X}}\right)^{d_j(T)}\psi^{j}_{\mathbf{Z}_j}$$

creates $d_j(T)!$ terms for $j\in\{1,2,\cdots,n\}$. For every $T\in\mathbb{T}(\{1,2,\cdots,n\})$ we consider the vertex 1 as the root. Then, by recursively estimating along the lines of T from younger branches to the root and using (3.21) for $k_j=p_j-m_j$ $(j=1,2,\cdots,n)$ we observe that

$$\begin{aligned}
&\|A^{(n)}_m\|_1\\
&\le 2^{n-1}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n\})}\sum_{p_1=2}^{N}\sum_{m_1=0}^{p_1-1}\begin{pmatrix}p_1\\m_1\end{pmatrix}\begin{pmatrix}p_1-m_1\\d_1(T)\end{pmatrix}d_1(T)!\|F^1_{p_1}\|_1\\
&\quad\cdot\prod_{j=2}^{n}\Bigg(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\begin{pmatrix}p_j\\m_j\end{pmatrix}\begin{pmatrix}p_j-m_j\\d_j(T)\end{pmatrix}d_j(T)!\\
&\qquad\cdot\sup_{X_0\in I}\left(\frac{1}{h}\right)^{p_j}\sum_{\mathbf{X}\in I^{p_j}}|F^j_{p_j}(\mathbf{X})||\tilde{\mathcal{C}}(X_0,X_1)|\Bigg)D^{\frac{1}{2}(\sum_{j=1}^{n}p_j-2(n-1)-m)}\\
&\quad\cdot 1_{\sum_{j=1}^{n}m_j=m}1_{\sum_{j=1}^{n}p_j-2(n-1)\ge m}\\
&\le(n-2)!D^{-n+1-\frac{m}{2}}\|\tilde{\mathcal{C}}\|^{n-1}_{1,\infty}\sum_{p_1=2}^{N}\sum_{m_1=0}^{p_1-1}\begin{pmatrix}p_1\\m_1\end{pmatrix}2^{2(p_1-m_1)}D^{\frac{p_1}{2}}\|F^1_{p_1}\|_1\\
&\quad\cdot\prod_{j=2}^{n}\left(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\begin{pmatrix}p_j\\m_j\end{pmatrix}2^{2(p_j-m_j)}D^{\frac{p_j}{2}}\|F^j_{p_j}\|_{1,\infty}\right)\\
&\quad\cdot 1_{\sum_{j=1}^{n}m_j=m}1_{\sum_{j=1}^{n}p_j-2(n-1)\ge m},
\end{aligned}$$

which is bounded from above by the right-hand side of (3.17).

We can estimate $\|A_m^{(n)}\|_{1,\infty}$ from (3.19) in a way parallel to the above argument. In the case $m \ge 2$, first we fix a component of $\mathbf{X}(\in I^m)$. For fixed $\sigma \in \mathbb{S}_m$ there uniquely exists $j_1 \in \{1, 2, \cdots, n\}$ such that the fixed component is one component of the variable $\mathbf{X}'_{j_1}(\in I^{m_{j_1}})$. Then we consider j_1 as the root of each tree and repeat the same recursive calculation as above to reach the claimed inequality (3.16). The inequality (3.16) for $m = 0$ follows from (3.17) for $m = 0$ and $\|F^1_{p_1}\|_1 \le \frac{N}{h}\|F^1_{p_1}\|_{1,\infty}$. □

In addition to $F^j(\psi)$ $(j = 1, 2, \cdots, n)$ we give a Grassmann polynomial having bi-anti-symmetric kernels as one piece of the input. Assume that we have bi-anti-symmetric functions $F_{p,q} : I^p \times I^q \to \mathbb{C}$ $(p, q \in \{2, 4, \cdots, N\})$ satisfying (3.13) and the following property. For any function $g : [0, \beta)_h^p \to \mathbb{C}$, $h : [0, \beta)_h^q \to \mathbb{C}$ satisfying

$$
\begin{aligned}
(3.22)\qquad & g(r_\beta(s_1 + s), r_\beta(s_2 + s), \cdots, r_\beta(s_p + s)) = g(s_1, s_2, \cdots, s_p) \\
& \left(\forall (s_1, s_2, \cdots, s_p) \in [0, \beta)_h^p,\ s \in \frac{1}{h}\mathbb{Z}\right), \\
& h(r_\beta(s_1 + s), r_\beta(s_2 + s), \cdots, r_\beta(s_q + s)) = h(s_1, s_2, \cdots, s_q) \\
& \left(\forall (s_1, s_2, \cdots, s_q) \in [0, \beta)_h^q,\ s \in \frac{1}{h}\mathbb{Z}\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.23)\qquad & \sum_{(s_1,\cdots,s_p)\in[0,\beta)_h^p} F_{p,q}((\rho_1\mathbf{x}_1 s_1\xi_1, \cdots, \rho_p\mathbf{x}_p s_p\xi_p), \mathbf{Y}) g(s_1, \cdots, s_p) = 0, \\
& (\forall \mathbf{Y} \in I^q,\ (\rho_j, \mathbf{x}_j, \xi_j) \in \{1, 2\} \times \Gamma \times \{1, -1\}\ (j = 1, 2, \cdots, p)), \\
& \sum_{(t_1,\cdots,t_q)\in[0,\beta)_h^q} F_{p,q}(\mathbf{X}, (\eta_1\mathbf{y}_1 t_1\zeta_1, \cdots, \eta_q\mathbf{y}_q t_q\zeta_q)) h(t_1, \cdots, t_q) = 0, \\
& (\forall \mathbf{X} \in I^p,\ (\eta_j, \mathbf{y}_j, \zeta_j) \in \{1, 2\} \times \Gamma \times \{1, -1\}\ (j = 1, 2, \cdots, q)).
\end{aligned}
$$

We are going to analyze the Grassmann polynomial $B^{(n)}(\psi) \in \bigwedge_{even} \mathcal{V}$ $(n \in \mathbb{N})$ defined by

$$
\begin{aligned}
& B^{(n)}(\psi) \\
& := \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{p+q} \sum_{\mathbf{X}\in I^p, \mathbf{Y}\in I^q} F_{p,q}(\mathbf{X}, \mathbf{Y}) Tree(\{1, 2, \cdots, n+1\}, \mathcal{C})
\end{aligned}
$$

$$\cdot(\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\prod_{j=3}^{n+1}F^j(\psi^j+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n+1\})}}.$$

Lemma 3.2. *For any $m\in\{2,4,\cdots,N\}$, $n\in\mathbb{N}$ the anti-symmetric kernel $B_m^{(n)}(\cdot)$ satisfies (3.13). Moreover, for any $m\in\{0,2,\cdots,N\}$, $n\in\mathbb{N}_{\ge 2}$,*

(3.24)

$$\|B_m^{(1)}\|_{1,\infty}\le\left(\frac{N}{h}\right)^{1_{m=0}}D^{-1-\frac{m}{2}}\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{2p_1+2p_2}D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty}1_{p_1+p_2-2\ge m}.$$

(3.25)

$$\|B_m^{(1)}\|_1\le D^{-1-\frac{m}{2}}\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{2p_1+2p_2}D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_1 1_{p_1+p_2-2\ge m}.$$

(3.26)

$$\begin{aligned}\|B_m^{(n)}\|_{1,\infty}\le&\left(\frac{N}{h}\right)^{1_{m=0}}(n-1)!D^{-n-\frac{m}{2}}2^{-2m}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\\&\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{3p_1+3p_2}D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty}\\&\cdot\prod_{j=3}^{n+1}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|F_{p_j}^j\|_{1,\infty}\right)1_{\sum_{j=1}^{n+1}p_j-2n\ge m}.\end{aligned}$$

(3.27)

$$\begin{aligned}\|B_m^{(n)}\|_1\le&(n-1)!D^{-n-\frac{m}{2}}2^{-2m}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\\&\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{3p_1+3p_2}D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty}\\&\cdot\prod_{j=3}^{n}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|F_{p_j}^j\|_{1,\infty}\right)\end{aligned}$$

$$\cdot \sum_{p_{n+1}=2}^{N} 2^{3p_{n+1}} D^{\frac{p_{n+1}}{2}} \|F^{n+1}_{p_{n+1}}\|_1 1_{\sum_{j=1}^{n+1} p_j - 2n \ge m}.$$

Proof. By anti-symmetry,

$$\begin{aligned}
&B^{(n)}(\psi)\\
&= \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} \sum_{m_1=0}^{p_1-1}\sum_{m_2=0}^{p_2-1} \left(\begin{array}{c} p_1 \\ m_1 \end{array}\right)\left(\begin{array}{c} p_2 \\ m_2 \end{array}\right)\\
&\quad\cdot \left(\frac{1}{h}\right)^{p_1+p_2} \sum_{\substack{\mathbf{X}_1\in I^{m_1},\mathbf{Y}_1\in I^{p_1-m_1}\\ \mathbf{X}_2\in I^{m_2},\mathbf{Y}_2\in I^{p_2-m_2}}} F_{p_1,p_2}((\mathbf{Y}_1,\mathbf{X}_1),(\mathbf{Y}_2,\mathbf{X}_2))\\
&\quad\cdot \prod_{j=3}^{n+1}\left(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\left(\begin{array}{c} p_j \\ m_j \end{array}\right)\left(\frac{1}{h}\right)^{p_j} \sum_{\substack{\mathbf{X}_j\in I^{m_j}\\ \mathbf{Y}_j\in I^{p_j-m_j}}} F^j_{p_j}(\mathbf{Y}_j,\mathbf{X}_j)\right)\\
&\quad\cdot Tree(\{1,2,\cdots,n+1\},\mathcal{C})\\
&\quad\cdot \prod_{j=1}^{n+1}\psi^j_{\mathbf{Y}_j}\Bigg|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n+1\})}} (-1)^{\sum_{j=1}^{n} m_j \sum_{k=j+1}^{n+1}(p_k-m_k)}\\
&\quad\cdot \prod_{k=1}^{n+1}\psi_{\mathbf{X}_k}.
\end{aligned}$$

Then, the uniqueness of anti-symmetric kernels ensures that for $m \in \{0,2,\cdots,N\}$, $\mathbf{X}\in I^m$,

$$\begin{aligned}
&B^{(n)}_m(\mathbf{X})\\
&= \frac{1}{m!}\sum_{\sigma\in\mathbb{S}_m}\mathrm{sgn}(\sigma)\sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} \sum_{m_1=0}^{p_1-1}\sum_{m_2=0}^{p_2-1} \left(\begin{array}{c} p_1 \\ m_1 \end{array}\right)\left(\begin{array}{c} p_2 \\ m_2 \end{array}\right)\\
&\quad\cdot \left(\frac{1}{h}\right)^{p_1+p_2-m_1-m_2} \sum_{\mathbf{Y}_1\in I^{p_1-m_1},\mathbf{Y}_2\in I^{p_2-m_2}} F_{p_1,p_2}((\mathbf{Y}_1,\mathbf{X}'_1),(\mathbf{Y}_2,\mathbf{X}'_2))\\
&\quad\cdot \prod_{j=3}^{n+1}\left(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\left(\begin{array}{c} p_j \\ m_j \end{array}\right)\left(\frac{1}{h}\right)^{p_j-m_j} \sum_{\mathbf{Y}_j\in I^{p_j-m_j}} F^j_{p_j}(\mathbf{Y}_j,\mathbf{X}'_j)\right)
\end{aligned}$$

$$\cdot Tree(\{1,2,\cdots,n+1\},\mathcal{C})$$
$$\cdot \prod_{j=1}^{n+1}\psi^j_{\mathbf{Y}_j}\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n+1\})}}(-1)^{\sum_{j=1}^{n}m_j\sum_{k=j+1}^{n+1}(p_k-m_k)}$$
$$\cdot 1_{\sum_{j=1}^{n+1}m_j=m}1_{\sum_{j=1}^{n+1}p_j-2n\ge m}1_{(\mathbf{X}'_1,\mathbf{X}'_2,\cdots,\mathbf{X}'_{n+1})=\mathbf{X}_\sigma}.$$

The property (3.13) of $B_m^{(n)}$ follows from (3.12) and the property (3.13) of the input.

By (3.2), (3.9), (3.11),

(3.28)

$$\begin{aligned}
&|B_m^{(n)}(\mathbf{X})|\\
&\le \frac{2^n}{m!}\sum_{\sigma\in\mathbb{S}_m}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n+1\})}\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}\\
&\quad\cdot\sum_{m_1=0}^{p_1-1}\sum_{m_2=0}^{p_2-1}\begin{pmatrix}p_1\\m_1\end{pmatrix}\begin{pmatrix}p_2\\m_2\end{pmatrix}\\
&\quad\cdot\begin{pmatrix}p_1-m_1\\d_1(T)\end{pmatrix}\begin{pmatrix}p_2-m_2\\d_2(T)\end{pmatrix}\left(\frac{1}{h}\right)^{p_1+p_2-m_1-m_2}\\
&\quad\cdot\sum_{\substack{\mathbf{Y}_1\in I^{p_1-m_1-d_1(T)},\mathbf{Y}_2\in I^{p_2-m_2-d_2(T)}\\ \mathbf{Z}_1\in I^{d_1(T)},\mathbf{Z}_2\in I^{d_2(T)}}}|F_{p_1,p_2}((\mathbf{Y}_1,\mathbf{Z}_1,\mathbf{X}'_1),(\mathbf{Y}_2,\mathbf{Z}_2,\mathbf{X}'_2))|\\
&\quad\cdot\prod_{j=3}^{n+1}\Bigg(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\begin{pmatrix}p_j\\m_j\end{pmatrix}\begin{pmatrix}p_j-m_j\\d_j(T)\end{pmatrix}\\
&\qquad\qquad\cdot\left(\frac{1}{h}\right)^{p_j-m_j}\sum_{\substack{\mathbf{Y}_j\in I^{p_j-m_j-d_j(T)}\\ \mathbf{Z}_j\in I^{d_j(T)}}}|F^j_{p_j}(\mathbf{Y}_j,\mathbf{Z}_j,\mathbf{X}'_j)|\Bigg)\\
&\quad\cdot D^{\frac{1}{2}(\sum_{j=1}^{n+1}p_j-2n-m)}\left|\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{n+1}\psi^j_{\mathbf{Z}_j}\right|\\
&\quad\cdot 1_{\sum_{j=1}^{n+1}m_j=m}1_{\sum_{j=1}^{n+1}p_j-2n\ge m}1_{(\mathbf{X}'_1,\mathbf{X}'_2,\cdots,\mathbf{X}'_{n+1})=\mathbf{X}_\sigma}.
\end{aligned}$$

Let us derive the inequality (3.25) from (3.28). For any $m \in$

$\{0, 2, \cdots, N\}$,

$$\|B_m^{(1)}\|_1 \le 2 \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} \sum_{m_1=0}^{p_1-1}\sum_{m_2=0}^{p_2-1} \left(\begin{array}{c} p_1 \\ m_1 \end{array}\right)\left(\begin{array}{c} p_2 \\ m_2 \end{array}\right)(p_1-m_1)(p_2-m_2)$$
$$\cdot [F_{p_1,p_2}, \tilde{\mathcal{C}}]_1 D^{\frac{1}{2}(\sum_{j=1}^{2} p_j - 2 - m)} 1_{m_1+m_2=m} 1_{p_1+p_2-2\ge m},$$

which is less than or equal to the right-hand side of (3.25). The inequality (3.24) can be derived in the same way.

Let us consider the case that $n \ge 2$. We decompose (3.28) as follows.

$$|B_m^{(n)}(\mathbf{X})| \le \sum_{T\in\mathbb{T}(\{1,2,\cdots,n+1\})} B_m^{(n)}(T)(\mathbf{X}), \tag{3.29}$$

(3.30)

$$\begin{aligned}
&B_m^{(n)}(T)(\mathbf{X})\\
&:= \frac{2^n}{m!}\sum_{\sigma\in\mathbb{S}_m}\sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}}\\
&\quad\cdot \sum_{m_1=0}^{p_1-1}\sum_{m_2=0}^{p_2-1}\left(\begin{array}{c} p_1 \\ m_1 \end{array}\right)\left(\begin{array}{c} p_2 \\ m_2 \end{array}\right)\\
&\quad\cdot \left(\begin{array}{c} p_1-m_1 \\ d_1(T) \end{array}\right)\left(\begin{array}{c} p_2-m_2 \\ d_2(T) \end{array}\right)\left(\frac{1}{h}\right)^{p_1+p_2-m_1-m_2}\\
&\quad\cdot \sum_{\substack{\mathbf{Y}_1\in I^{p_1-m_1-d_1(T)},\mathbf{Y}_2\in I^{p_2-m_2-d_2(T)}\\ \mathbf{Z}_1\in I^{d_1(T)},\mathbf{Z}_2\in I^{d_2(T)}}} |F_{p_1,p_2}((\mathbf{Y}_1,\mathbf{Z}_1,\mathbf{X}_1'),(\mathbf{Y}_2,\mathbf{Z}_2,\mathbf{X}_2'))|\\
&\quad\cdot \prod_{j=3}^{n+1}\Bigg(\sum_{p_j=2}^{N}\sum_{m_j=0}^{p_j-1}\left(\begin{array}{c} p_j \\ m_j \end{array}\right)\left(\begin{array}{c} p_j-m_j \\ d_j(T) \end{array}\right)\\
&\qquad\quad\cdot \left(\frac{1}{h}\right)^{p_j-m_j}\sum_{\substack{\mathbf{Y}_j\in I^{p_j-m_j-d_j(T)}\\ \mathbf{Z}_j\in I^{d_j(T)}}} |F_{p_j}^{j}(\mathbf{Y}_j,\mathbf{Z}_j,\mathbf{X}_j')|\Bigg)\\
&\quad\cdot D^{\frac{1}{2}(\sum_{j=1}^{n+1} p_j - 2n - m)}\left|\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{n+1}\psi_{\mathbf{Z}_j}^{j}\right|
\end{aligned}$$

$$\cdot 1_{\sum_{j=1}^{n+1} m_j = m} 1_{\sum_{j=1}^{n+1} p_j - 2n \ge m} 1_{(\mathbf{X}'_1, \mathbf{X}'_2, \cdots, \mathbf{X}'_{n+1}) = \mathbf{X}_\sigma}.$$

Let us estimate $\|B_m^{(n)}(T)\|_1$. For $T \in \mathbb{T}(\{1, 2, \cdots, n+1\})$ we consider the vertex $n+1$ as the root of T. Without loss of generality we can assume that

(3.31) the distance between 1 and $n+1$ is shorter than or equal to that between 2 and $n+1$ in T.

We can derive the same inequality by assuming otherwise. For $j \in \{1, 2, \cdots, n+1\}$ let us introduce the conditions P_j, Q as follows.

P_j : The vertex 1 is on the shortest path between j and 2 in T.
Q : The distance between 1 and 2 in T is 1.

Then, only one of the following cases occurs.

$$P_{n+1} \wedge Q \qquad P_{n+1} \wedge \neg Q \qquad \neg P_{n+1}$$

By recursively estimating along the lines of T from younger branches to the root $n+1$ we obtain from (3.30) that for any $m \in \{0, 2, \cdots, N\}$

(3.32)

$$\begin{aligned}
&\|B_m^{(n)}(T)\|_1 \\
&\le 2^n \sum_{p_1, p_2 = 2}^{N} 1_{p_1, p_2 \in 2\mathbb{N}} \\
&\quad \cdot \sum_{m_1=0}^{p_1-1} \sum_{m_2=0}^{p_2-1} \begin{pmatrix} p_1 \\ m_1 \end{pmatrix} \begin{pmatrix} p_2 \\ m_2 \end{pmatrix} \begin{pmatrix} p_1 - m_1 \\ d_1(T) \end{pmatrix} d_1(T)! \begin{pmatrix} p_2 - m_2 \\ d_2(T) \end{pmatrix} d_2(T)! \\
&\quad \cdot \prod_{j=3}^{n} \Bigg(\sum_{p_j=2}^{N} \sum_{m_j=0}^{p_j-1} \begin{pmatrix} p_j \\ m_j \end{pmatrix} \begin{pmatrix} p_j - m_j \\ d_j(T) \end{pmatrix} d_j(T)! \\
&\qquad \cdot \sup_{X_0 \in I} \left(\frac{1}{h}\right)^{p_j} \sum_{\mathbf{X} \in I^{p_j}} |F_{p_j}^j(\mathbf{X})| |\tilde{\mathcal{C}}(X_0, X_1)| \Bigg) \\
&\quad \cdot \sum_{p_{n+1}=2}^{N} \sum_{m_{n+1}=0}^{p_{n+1}-1} \begin{pmatrix} p_{n+1} \\ m_{n+1} \end{pmatrix} \begin{pmatrix} p_{n+1} - m_{n+1} \\ d_{n+1}(T) \end{pmatrix} d_{n+1}(T)! \|F_{p_{n+1}}^{n+1}\|_1
\end{aligned}$$

$$
\begin{aligned}
&\cdot D^{\frac{1}{2}(\sum_{j=1}^{n+1}p_j-2n-m)}\\
&\cdot \Bigg(1_{P_{n+1}\wedge Q}\sup_{X_0\in I}\left(\frac{1}{h}\right)^{p_1+p_2}\sum_{\substack{\mathbf{X}\in I^{p_1}\\ \mathbf{Y}\in I^{p_2}}}|F_{p_1,p_2}(\mathbf{X},\mathbf{Y})||\tilde{\mathcal{C}}(X_0,X_1)||\tilde{\mathcal{C}}(X_2,Y_1)|\\
&\quad+(1_{P_{n+1}\wedge\neg Q}+1_{\neg P_{n+1}})\sup_{X_0\in I}\Bigg(\left(\frac{1}{h}\right)^{p_1+p_2}\sum_{\mathbf{X}\in I^{p_1}}\\
&\qquad\qquad\cdot\sup_{Y_0\in I}\sum_{\mathbf{Y}\in I^{p_2}}|F_{p_1,p_2}(\mathbf{X},\mathbf{Y})||\tilde{\mathcal{C}}(X_0,X_1)||\tilde{\mathcal{C}}(Y_0,Y_1)|\Bigg)\Bigg)\\
&\cdot 1_{\sum_{j=1}^{n+1}m_j=m}1_{\sum_{j=1}^{n+1}p_j-2n\ge m}\\
&\le 2^nD^{-n-\frac{m}{2}}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\\
&\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}D^{\frac{p_1+p_2}{2}}\sum_{m_1=0}^{p_1-1}\sum_{m_2=0}^{p_2-1}\begin{pmatrix}p_1\\ m_1\end{pmatrix}\begin{pmatrix}p_2\\ m_2\end{pmatrix}\\
&\quad\cdot\begin{pmatrix}p_1-m_1\\ d_1(T)\end{pmatrix}d_1(T)!\begin{pmatrix}p_2-m_2\\ d_2(T)\end{pmatrix}d_2(T)![F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty}\\
&\cdot\prod_{j=3}^{n}\left(\sum_{p_j=2}^{N}D^{\frac{p_j}{2}}\sum_{m_j=0}^{p_j-1}\begin{pmatrix}p_j\\ m_j\end{pmatrix}\begin{pmatrix}p_j-m_j\\ d_j(T)\end{pmatrix}d_j(T)!\|F_{p_j}^j\|_{1,\infty}\right)\\
&\cdot\sum_{p_{n+1}=2}^{N}D^{\frac{p_{n+1}}{2}}\sum_{m_{n+1}=0}^{p_{n+1}-1}\begin{pmatrix}p_{n+1}\\ m_{n+1}\end{pmatrix}\begin{pmatrix}p_{n+1}-m_{n+1}\\ d_{n+1}(T)\end{pmatrix}d_{n+1}(T)!\|F_{p_{n+1}}^{n+1}\|_1\\
&\cdot 1_{\sum_{j=1}^{n+1}m_j=m}1_{\sum_{j=1}^{n+1}p_j-2n\ge m}.
\end{aligned}
$$

When P_{n+1} holds, the first inequality above can be derived smoothly. The point of the derivation of the first inequality when P_{n+1} does not hold is to complete the recursive estimation along the branch containing the vertex 2 before the recursive estimation along the branch containing the vertex 1. Here we use the assumption (3.31) to exclude that the vertex 2 is on the shortest path between $n+1$ and 1 in T. By applying (3.21) we can derive (3.27) from (3.29), (3.32). Note that (3.27) for $m=0$ implies (3.26) for $m=0$, since $\|B_0^{(n)}\|_1=\|B_0^{(n)}\|_{1,\infty}=|B_0^{(n)}|$ and $\|F_{p_{n+1}}^{n+1}\|_1\le\frac{N}{h}\|F_{p_{n+1}}^{n+1}\|_{1,\infty}$.

In order to derive the claimed upper bound on $\|B_m^{(n)}\|_{1,\infty}$ for $m\ge 2$ from (3.30), we fix $T\in\mathbb{T}(\{1,2,\cdots,n+1\})$ and the first component X_1 of the variable $\mathbf{X}(\in I^m)$. For any $\sigma \in \mathbb{S}_m$ there uniquely exists $j_1 \in \{1, 2, \cdots, n+1\}$ such that X_1 is a component of $\mathbf{X}'_{j_1}$. Then, we consider the vertex j_1 as the root of the tree T. Again without loss of generality we can assume that

(3.33) the distance between 1 and j_1 is shorter than or equal to that between 2 and j_1 in T.

Only one of the following cases happens.

$$P_{j_1} \wedge Q \wedge j_1 = 1 \quad P_{j_1} \wedge Q \wedge j_1 \neq 1 \quad P_{j_1} \wedge \neg Q \wedge j_1 = 1 \quad P_{j_1} \wedge \neg Q \wedge j_1 \neq 1 \quad \neg P_{j_1}$$

By the recursive estimation from younger branches to the root j_1 we deduce that

$$\begin{aligned}
&\|B_m^{(n)}(T)\|_{1,\infty} \\
&\le \frac{2^n}{m!} \sum_{\sigma \in \mathbb{S}_m} \sum_{p_1, p_2 = 2}^{N} 1_{p_1, p_2 \in 2\mathbb{N}} \\
&\quad \cdot \sum_{m_1=0}^{p_1-1} \sum_{m_2=0}^{p_2-1} \begin{pmatrix} p_1 \\ m_1 \end{pmatrix} \begin{pmatrix} p_2 \\ m_2 \end{pmatrix} \begin{pmatrix} p_1 - m_1 \\ d_1(T) \end{pmatrix} d_1(T)! \begin{pmatrix} p_2 - m_2 \\ d_2(T) \end{pmatrix} d_2(T)! \\
&\quad \cdot \prod_{j=3}^{n+1} \left(\sum_{p_j=2}^{N} \sum_{m_j=0}^{p_j-1} \begin{pmatrix} p_j \\ m_j \end{pmatrix} \begin{pmatrix} p_j - m_j \\ d_j(T) \end{pmatrix} d_j(T)! \right) D^{\frac{1}{2}(\sum_{j=1}^{n+1} p_j - 2n - m)} \\
&\quad \cdot 1_{\sum_{j=1}^{n+1} m_j = m} 1_{\sum_{j=1}^{n+1} p_j - 2n \ge m} \\
&\quad \cdot \Bigg(1_{P_{j_1} \wedge Q \wedge j_1 = 1} \\
&\qquad \cdot \sup_{X_0 \in I} \left(\left(\frac{1}{h}\right)^{p_1+p_2-1} \sum_{\substack{\mathbf{X} \in I^{p_1-1} \\ \mathbf{Y} \in I^{p_2}}} |F_{p_1,p_2}((X_0, \mathbf{X}), \mathbf{Y})| |\tilde{\mathcal{C}}(X_1, Y_1)| \right) \\
&\qquad \cdot \prod_{j=3}^{n+1} \left(\sup_{X_0 \in I} \left(\frac{1}{h}\right)^{p_j} \sum_{\mathbf{X} \in I^{p_j}} |F_{p_j}^j(\mathbf{X})| |\tilde{\mathcal{C}}(X_0, X_1)| \right) \\
&\quad + 1_{P_{j_1} \wedge Q \wedge j_1 \neq 1} \\
&\qquad \cdot \sup_{X_0 \in I} \left(\left(\frac{1}{h}\right)^{p_1+p_2} \sum_{\substack{\mathbf{X} \in I^{p_1} \\ \mathbf{Y} \in I^{p_2}}} |F_{p_1,p_2}(\mathbf{X}, \mathbf{Y})| |\tilde{\mathcal{C}}(X_0, X_1)| |\tilde{\mathcal{C}}(X_2, Y_1)| \right)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \|F^{j_1}_{p_{j_1}}\|_{1,\infty} \prod_{\substack{j=3\\ j\neq j_1}}^{n+1} \left(\sup_{X_0\in I} \left(\frac{1}{h}\right)^{p_j} \sum_{\mathbf{X}\in I^{p_j}} |F^j_{p_j}(\mathbf{X})||\tilde{\mathcal{C}}(X_0,X_1)| \right)\\
&+ 1_{P_{j_1}\wedge\neg Q\wedge j_1=1} \sup_{X_0\in I} \Bigg(\left(\frac{1}{h}\right)^{p_1+p_2-1} \sum_{\mathbf{X}\in I^{p_1-1}}\\
&\qquad \cdot \sup_{Y_0\in I} \left(\sum_{\mathbf{Y}\in I^{p_2}} |F_{p_1,p_2}((X_0,\mathbf{X}),\mathbf{Y})||\tilde{\mathcal{C}}(Y_0,Y_1)| \right)\Bigg)\\
&\cdot \prod_{j=3}^{n+1} \left(\sup_{X_0\in I} \left(\frac{1}{h}\right)^{p_j} \sum_{\mathbf{X}\in I^{p_j}} |F^j_{p_j}(\mathbf{X})||\tilde{\mathcal{C}}(X_0,X_1)| \right)\\
&+ (1_{P_{j_1}\wedge\neg Q\wedge j_1\neq 1} + 1_{\neg P_{j_1}}) \sup_{X_0\in I} \Bigg(\left(\frac{1}{h}\right)^{p_1+p_2} \sum_{\mathbf{X}\in I^{p_1}}\\
&\qquad \cdot \sup_{Y_0\in I} \left(\sum_{\mathbf{Y}\in I^{p_2}} |F_{p_1,p_2}(\mathbf{X},\mathbf{Y})||\tilde{\mathcal{C}}(X_0,X_1)||\tilde{\mathcal{C}}(Y_0,Y_1)| \right)\\
&\cdot \|F^{j_1}_{p_{j_1}}\|_{1,\infty} \prod_{\substack{j=3\\ j\neq j_1}}^{n+1} \left(\sup_{X_0\in I} \left(\frac{1}{h}\right)^{p_j} \sum_{\mathbf{X}\in I^{p_j}} |F^j_{p_j}(\mathbf{X})||\tilde{\mathcal{C}}(X_0,X_1)| \right)\Bigg)\\
&\le 2^n D^{-n-\frac{m}{2}} \|\tilde{\mathcal{C}}\|^{n-1}_{1,\infty}\\
&\quad \cdot \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} D^{\frac{p_1+p_2}{2}} \sum_{m_1=0}^{p_1-1}\sum_{m_2=0}^{p_2-1} \begin{pmatrix} p_1\\ m_1\end{pmatrix}\begin{pmatrix} p_2\\ m_2\end{pmatrix}\\
&\quad \cdot \begin{pmatrix} p_1-m_1\\ d_1(T)\end{pmatrix} d_1(T)! \begin{pmatrix} p_2-m_2\\ d_2(T)\end{pmatrix} d_2(T)! [F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty}\\
&\quad \cdot \prod_{j=3}^{n+1} \left(\sum_{p_j=2}^{N} D^{\frac{p_j}{2}} \sum_{m_j=0}^{p_j-1} \begin{pmatrix} p_j\\ m_j\end{pmatrix}\begin{pmatrix} p_j-m_j\\ d_j(T)\end{pmatrix} d_j(T)! \|F^j_{p_j}\|_{1,\infty} \right)\\
&\quad \cdot 1_{\sum_{j=1}^{n+1} m_j = m} 1_{\sum_{j=1}^{n+1} p_j - 2n\ge m}.
\end{aligned}
$$

Again when P_{j_1} does not hold, the assumption (3.33) excludes that the vertex 2 is on the shortest path between j_1 and 1 in T so that we can carry out the recursive estimation along the branch containing the vertex 2 before that along the branch containing the vertex 1. Combining this inequality with (3.29) and (3.21) results in (3.26) for $m \ge 2$. □

Next we consider the Grassmann polynomials $E^{(n)}(\psi) \in \bigwedge_{even} \mathcal{V}$ $(n \in \mathbb{N})$ defined as follows.

$$\begin{aligned}
E^{(n)}(\psi) :=& \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p \\ \mathbf{Y}\in I^q}} F_{p,q}(\mathbf{X},\mathbf{Y}) \\
&\cdot Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{C})(\psi^1+\psi)_{\mathbf{X}} \prod_{j=2}^{m+1} F^{s_j}(\psi^{s_j}+\psi)\Big|_{\substack{\psi^{s_j}=0 \\ (\forall j\in\{1,2,\cdots,m+1\})}} \\
&\cdot Tree(\{t_k\}_{k=1}^{n-m}, \mathcal{C})(\psi^1+\psi)_{\mathbf{Y}} \prod_{k=2}^{n-m} F^{t_k}(\psi^{t_k}+\psi)\Big|_{\substack{\psi^{t_k}=0 \\ (\forall k\in\{1,2,\cdots,n-m\})}},
\end{aligned}$$

where the functions $F_{p,q} : I^p \times I^q \to \mathbb{C}$ $(p, q \in \{2, 4, \cdots, N\})$ are bi-anti-symmetric and satisfy (3.13), (3.23) and

$$\begin{aligned}
&m \in \{0, 1, \cdots, n-1\}, \\
&1 = s_1 < s_2 < \cdots < s_{m+1} \le n, \quad 1 = t_1 < t_2 < \cdots < t_{n-m} \le n, \\
&\{s_j\}_{j=2}^{m+1} \cup \{t_k\}_{k=2}^{n-m} = \{2, 3, \cdots, n\}, \quad \{s_j\}_{j=2}^{m+1} \cap \{t_k\}_{k=2}^{n-m} = \emptyset.
\end{aligned}$$

Here we assume that $\{s_j\}_{j=2}^{m+1} = \emptyset$ if $m = 0$, $\{t_k\}_{k=2}^{n-m} = \emptyset$ if $m = n-1$.

LEMMA 3.3. *For any $n \in \mathbb{N}$, $a, b \in \{2, 4, \cdots, N\}$ there exists a function $E^{(n)}_{a,b} : I^a \times I^b \to \mathbb{C}$ such that $E^{(n)}_{a,b}$ is bi-anti-symmetric, satisfies* (3.13), (3.23) *and*

$$E^{(n)}(\psi) = \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{a+b} \sum_{\substack{\mathbf{X}\in I^a \\ \mathbf{Y}\in I^b}} E^{(n)}_{a,b}(\mathbf{X},\mathbf{Y}) \psi_{\mathbf{X}} \psi_{\mathbf{Y}}.$$

Moreover, the following inequalities hold for any $a, b \in \{2, 4, \cdots, N\}$, $n \in \mathbb{N}_{\ge 2}$.

(3.34)
$$\|E^{(1)}_{a,b}\|_{1,\infty} \le \sum_{p=a}^{N} \sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \begin{pmatrix} p \\ a \end{pmatrix} \begin{pmatrix} q \\ b \end{pmatrix} D^{\frac{1}{2}(p+q-a-b)} \|F_{p,q}\|_{1,\infty}.$$

(3.35)
$$\|E^{(1)}_{a,b}\|_{1} \le \sum_{p=a}^{N} \sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \begin{pmatrix} p \\ a \end{pmatrix} \begin{pmatrix} q \\ b \end{pmatrix} D^{\frac{1}{2}(p+q-a-b)} \|F_{p,q}\|_{1}.$$

(3.36)

$$\begin{aligned}
&\|E_{a,b}^{(n)}\|_{1,\infty}\\
&\le (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\quad\cdot 2^{-2a-2b}D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\sum_{p_1,q_1=2}^{N}1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}D^{\frac{p_1+q_1}{2}}\|F_{p_1,q_1}\|_{1,\infty}\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|F_{p_j}^{s_j}\|_{1,\infty}\right)\prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N}2^{3q_k}D^{\frac{q_k}{2}}\|F_{q_k}^{t_k}\|_{1,\infty}\right)\\
&\quad\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\ge a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b}.
\end{aligned}$$

(3.37)

$$\begin{aligned}
&\|E_{a,b}^{(n)}\|_{1}\\
&\le (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\quad\cdot 2^{-2a-2b}D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\sum_{p_1,q_1=2}^{N}1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}D^{\frac{p_1+q_1}{2}}\|F_{p_1,q_1}\|_{1,\infty}\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}(1_{s_j\neq n}\|F_{p_j}^{s_j}\|_{1,\infty}+1_{s_j=n}\|F_{p_j}^{s_j}\|_{1})\right)\\
&\quad\cdot\prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N}2^{3q_k}D^{\frac{q_k}{2}}(1_{t_k\neq n}\|F_{q_k}^{t_k}\|_{1,\infty}+1_{t_k=n}\|F_{q_k}^{t_k}\|_{1})\right)\\
&\quad\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\ge a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b}.
\end{aligned}$$

Proof. By using anti-symmetry we can transform $E^{(n)}(\psi)$ as follows.

(3.38)

$$\begin{aligned}
&E^{(n)}(\psi)\\
&=\sum_{p_1,q_1=2}^{N}1_{p_1,q_1\in 2\mathbb{N}}\sum_{u_1=0}^{p_1}(1_{m=0}+1_{m\neq 0}1_{u_1\le p_1-1})\binom{p_1}{u_1}\left(\frac{1}{h}\right)^{u_1}\sum_{\mathbf{X}_1\in I^{u_1}}\\
&\quad\cdot\sum_{v_1=0}^{q_1}(1_{m=n-1}+1_{m\neq n-1}1_{v_1\le q_1-1})\binom{q_1}{v_1}\left(\frac{1}{h}\right)^{v_1}\sum_{\mathbf{Y}_1\in I^{v_1}}
\end{aligned}$$

$$
\cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N}\sum_{u_j=0}^{p_j-1}\begin{pmatrix} p_j \\ u_j \end{pmatrix}\left(\frac{1}{h}\right)^{u_j}\sum_{\mathbf{X}_j\in I^{u_j}}\right)
$$
$$
\cdot \prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N}\sum_{v_k=0}^{q_k-1}\begin{pmatrix} q_k \\ v_k \end{pmatrix}\left(\frac{1}{h}\right)^{v_k}\sum_{\mathbf{Y}_k\in I^{v_k}}\right)
$$
$$
\cdot f_m^n((p_j)_{1\le j\le m+1},(u_j)_{1\le j\le m+1},(q_j)_{1\le j\le n-m},(v_j)_{1\le j\le n-m})
$$
$$
((\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_{m+1}),(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_{n-m}))
$$
$$
\cdot \psi_{\mathbf{X}_1}\psi_{\mathbf{X}_2}\cdots\psi_{\mathbf{X}_{m+1}}\psi_{\mathbf{Y}_1}\psi_{\mathbf{Y}_2}\cdots\psi_{\mathbf{Y}_{n-m}},
$$

where the function

$$
f_m^n((p_j)_{1\le j\le m+1},(u_j)_{1\le j\le m+1},(q_j)_{1\le j\le n-m},(v_j)_{1\le j\le n-m})
$$
$$
:\prod_{j=1}^{m+1} I^{u_j}\times\prod_{k=1}^{n-m} I^{v_k}\to\mathbb{C}
$$

is defined by

(3.39)
$$
f_m^n((p_j)_{1\le j\le m+1},(u_j)_{1\le j\le m+1},(q_j)_{1\le j\le n-m},(v_j)_{1\le j\le n-m})
$$
$$
((\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_{m+1}),(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_{n-m}))
$$
$$
:=\left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1}\sum_{\mathbf{W}_1\in I^{p_1-u_1}}\sum_{\mathbf{Z}_1\in I^{q_1-v_1}}F_{p_1,q_1}((\mathbf{W}_1,\mathbf{X}_1),(\mathbf{Z}_1,\mathbf{Y}_1))
$$
$$
\cdot\prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j-u_j}\sum_{\mathbf{W}_j\in I^{p_j-u_j}}F_{p_j}^{s_j}(\mathbf{W}_j,\mathbf{X}_j)\right)
$$
$$
\cdot\prod_{k=2}^{n-m}\left(\left(\frac{1}{h}\right)^{q_k-v_k}\sum_{\mathbf{Z}_k\in I^{q_k-v_k}}F_{q_k}^{t_k}(\mathbf{Z}_k,\mathbf{Y}_k)\right)
$$
$$
\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C})\prod_{j=1}^{m+1}\psi_{\mathbf{W}_j}^{s_j}\Big|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}
$$
$$
\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C})\prod_{k=1}^{n-m}\psi_{\mathbf{Z}_k}^{t_k}\Big|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}}
$$
$$
\cdot(-1)^{\sum_{j=1}^{m}u_j\sum_{i=j+1}^{m+1}(p_i-u_i)+\sum_{k=1}^{n-m-1}v_k\sum_{i=k+1}^{n-m}(q_i-v_i)}.
$$

For simplicity, set $\mathbf{p} := (p_j)_{1\le j\le m+1}$, $\mathbf{u} := (u_j)_{1\le j\le m+1}$, $\mathbf{q} := (q_j)_{1\le j\le n-m}$, $\mathbf{v} := (v_j)_{1\le j\le n-m}$. Since the kernel of $E^{(n)}(\psi)$ inherits many properties from the function $f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})$, we should study $f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})$ first.

It follows from the property (3.13) of F_{p_1,q_1}, F^j and (3.12) that $f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\cdot)$ satisfies (3.13). Assume that $\mathbf{u}=\mathbf{0}$ and define the function $g : I^{p_1} \to \mathbb{C}$ by

$$g(\mathbf{X}) := \prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j}\sum_{\mathbf{W}_j\in I^{p_j}} F_{p_j}^{s_j}(\mathbf{W}_j)\right) \cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C})\psi_{\mathbf{X}}^{s_1}\prod_{j=2}^{m+1}\psi_{\mathbf{W}_j}^{s_j}\Bigg|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}.$$

By (3.12) and the property (3.13) of F^j the function g satisfies (3.13) too. Then, the property (3.23) of F_{p_1,q_1} implies that

$$\begin{aligned}
&f_m^n(\mathbf{p},\mathbf{0},\mathbf{q},\mathbf{v})(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_{n-m})\\
&=\left(\frac{1}{h}\right)^{p_1+q_1-v_1}\sum_{\mathbf{W}_1\in I^{p_1}}\sum_{\mathbf{Z}_1\in I^{q_1-v_1}} F_{p_1,q_1}(\mathbf{W}_1,(\mathbf{Z}_1,\mathbf{Y}_1))g(\mathbf{W}_1)\\
&\quad\cdot\prod_{k=2}^{n-m}\left(\left(\frac{1}{h}\right)^{q_k-v_k}\sum_{\mathbf{Z}_k\in I^{q_k-v_k}} F_{q_k}^{t_k}(\mathbf{Z}_k,\mathbf{Y}_k)\right)\\
&\quad\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C})\prod_{k=1}^{n-m}\psi_{\mathbf{Z}_k}^{t_k}\Bigg|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}}(-1)^{\sum_{k=1}^{n-m-1}v_k\sum_{i=k+1}^{n-m}(q_i-v_i)}\\
&=0,\quad\left(\forall(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_{n-m})\in\prod_{k=1}^{n-m}I^{v_k}\right).
\end{aligned}$$

Similarly we can check that $f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{0})\equiv 0$.

To confirm that $f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\cdot,\cdot)$ satisfies (3.23), let us take a function $h : \prod_{j=1}^{m+1}[0,\beta)_h^{u_j} \to \mathbb{C}$ satisfying (3.22). Here let us temporarily extend the notational rule defined in (3.1) as follows. For $\mathbf{X} = (\rho_1\mathbf{x}_1 s_1\xi_1,\cdots,\rho_n\mathbf{x}_n s_n\xi_n) \in (\{1,2\}\times\mathbb{Z}^d\times\frac{1}{h}\mathbb{Z}\times\{1,-1\})^n$, $\mathbf{t}=(t_1,\cdots,t_n)\in(\frac{1}{h}\mathbb{Z})^n$, set

$$\mathbf{X}+\mathbf{t} := (\rho_1\mathbf{x}_1(s_1+t_1)\xi_1,\cdots,\rho_n\mathbf{x}_n(s_n+t_n)\xi_n).$$

Then, we see that for any $\mathbf{X}_j \in (I^0)^{u_j}$ $(j = 1, 2, \cdots, m+1)$, $\mathbf{Y}_k \in I^{v_k}$ $(k = 1, 2, \cdots, n-m)$,

$$\begin{aligned}(3.40)\quad &\prod_{j=1}^{m+1}\left(\sum_{\mathbf{s}_j \in [0,\beta)_h^{u_j}}\right) h(\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_{m+1})\\ &\cdot f_m^n(\mathbf{p}, \mathbf{u}, \mathbf{q}, \mathbf{v})((\mathbf{X}_1 + \mathbf{s}_1, \mathbf{X}_2 + \mathbf{s}_2, \cdots, \mathbf{X}_{m+1} + \mathbf{s}_{m+1}),\\ &\qquad (\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_{n-m}))\\ &= \left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1} \sum_{\substack{\mathbf{W}_1 \in (I^0)^{p_1-u_1}\\ \mathbf{Z}_1 \in I^{q_1-v_1}}} \sum_{\substack{\mathbf{s}_1 \in [0,\beta)_h^{u_1}\\ \mathbf{t}_1 \in [0,\beta)_h^{p_1-u_1}}}\\ &\cdot F_{p_1,q_1}((\mathbf{W}_1 + \mathbf{t}_1, \mathbf{X}_1 + \mathbf{s}_1), (\mathbf{Z}_1, \mathbf{Y}_1)) h'(\mathbf{W}_1, \mathbf{Z}_1)(\mathbf{t}_1, \mathbf{s}_1),\end{aligned}$$

where

$$\begin{aligned} h'(\mathbf{W}_1, \mathbf{Z}_1)(\mathbf{t}_1, \mathbf{s}_1) := &\prod_{j=2}^{m+1}\left(\sum_{\mathbf{s}_j \in [0,\beta)_h^{u_j}}\right) h(\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_{m+1})\\ &\cdot \prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j-u_j} \sum_{\mathbf{W}_j \in I^{p_j-u_j}} F_{p_j}^{s_j}(\mathbf{W}_j, \mathbf{X}_j + \mathbf{s}_j)\right)\\ &\cdot \prod_{k=2}^{n-m}\left(\left(\frac{1}{h}\right)^{q_k-v_k} \sum_{\mathbf{Z}_k \in I^{q_k-v_k}} F_{q_k}^{t_k}(\mathbf{Z}_k, \mathbf{Y}_k)\right)\\ &\cdot Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{C}) \psi_{\mathbf{W}_1+\mathbf{t}_1}^{s_1} \prod_{j=2}^{m+1} \psi_{\mathbf{W}_j}^{s_j}\Bigg|_{\substack{\psi^{s_j}=0\\ (\forall j \in \{1,2,\cdots,m+1\})}}\\ &\cdot Tree(\{t_k\}_{k=1}^{n-m}, \mathcal{C}) \prod_{k=1}^{n-m} \psi_{\mathbf{Z}_k}^{t_k}\Bigg|_{\substack{\psi^{t_k}=0\\ (\forall k \in \{1,2,\cdots,n-m\})}}\\ &\cdot (-1)^{\sum_{j=1}^{m} u_j \sum_{i=j+1}^{m+1}(p_i-u_i) + \sum_{k=1}^{n-m-1} v_k \sum_{i=k+1}^{n-m}(q_i-v_i)}.\end{aligned}$$

The equality (3.12), the property (3.13) of F^j and the property (3.22) of h imply that

$$h'(\mathbf{W}_1, \mathbf{Z}_1)(r_\beta(t_1 + s), \cdots, r_\beta(t_{p_1-u_1} + s), r_\beta(s_1 + s), \cdots, r_\beta(s_{u_1} + s))$$

$$= h'(\mathbf{W}_1, \mathbf{Z}_1)(t_1, \cdots, t_{p_1-u_1}, s_1, \cdots, s_{u_1}),$$
$$\Big(\forall t_j \in [0,\beta)_h \ (j = 1, \cdots, p_1 - u_1),\ s_k \in [0,\beta)_h \ (k = 1, \cdots, u_1),$$
$$s \in \frac{1}{h}\mathbb{Z}\Big).$$

Thus, the property (3.23) of F_{p_1,q_1} ensures that the right-hand side of (3.40) vanishes. By the same procedure as above we can check that for any function $\phi : \prod_{j=1}^{n-m}[0,\beta)_h^{v_j} \to \mathbb{C}$ satisfying (3.22), $\mathbf{X}_j \in I^{u_j}$ $(j = 1, 2, \cdots, m+1)$, $\mathbf{Y}_k \in (I^0)^{v_k}$ $(k = 1, 2, \cdots, n-m)$,

$$\begin{aligned}&\prod_{k=1}^{n-m}\left(\sum_{\mathbf{t}_k \in [0,\beta)_h^{v_k}}\right)\phi(\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_{n-m})\\ &\cdot f_m^n(\mathbf{p}, \mathbf{u}, \mathbf{q}, \mathbf{v})((\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{m+1}),\\ &\qquad (\mathbf{Y}_1 + \mathbf{t}_1, \mathbf{Y}_2 + \mathbf{t}_2, \cdots, \mathbf{Y}_{n-m} + \mathbf{t}_{n-m})) = 0.\end{aligned}$$

After these preparations we define the functions $E_{a,b}^{(n)} : I^a \times I^b \to \mathbb{C}$ $(a, b \in \{0, 2, 4, \cdots, N\})$ by

$$\begin{aligned}(3.41)\quad & E_{a,b}^{(n)}(\mathbf{X}, \mathbf{Y})\\ &:= \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} \sum_{u_1=0}^{p_1}(1_{m=0} + 1_{m\neq 0}1_{u_1\le p_1-1})\begin{pmatrix} p_1 \\ u_1\end{pmatrix}\\ &\quad\cdot \sum_{v_1=0}^{q_1}(1_{m=n-1} + 1_{m\neq n-1}1_{v_1\le q_1-1})\begin{pmatrix} q_1 \\ v_1\end{pmatrix}\\ &\quad\cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N}\sum_{u_j=0}^{p_j-1}\begin{pmatrix} p_j \\ u_j\end{pmatrix}\right)\prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N}\sum_{v_k=0}^{q_k-1}\begin{pmatrix} q_k \\ v_k\end{pmatrix}\right)\\ &\quad\cdot f_m^n((p_j)_{1\le j\le m+1}, (u_j)_{1\le j\le m+1}, (q_j)_{1\le j\le n-m}, (v_j)_{1\le j\le n-m})\\ &\qquad ((\mathbf{X}_1', \mathbf{X}_2', \cdots, \mathbf{X}_{m+1}'), (\mathbf{Y}_1', \mathbf{Y}_2', \cdots, \mathbf{Y}_{n-m}'))\\ &\quad\cdot 1_{\sum_{j=1}^{m+1}u_j=a}1_{\sum_{k=1}^{n-m}v_k=b}1_{\sum_{j=1}^{m+1}p_j-2m\ge a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b}\\ &\quad\cdot \frac{1}{a!b!}\sum_{\substack{\sigma\in\mathbb{S}_a\\ \tau\in\mathbb{S}_b}}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)1_{(\mathbf{X}_1',\mathbf{X}_2',\cdots,\mathbf{X}_{m+1}')=\mathbf{X}_\sigma}1_{(\mathbf{Y}_1',\mathbf{Y}_2',\cdots,\mathbf{Y}_{n-m}')=\mathbf{Y}_\tau}.\end{aligned}$$

By definition $E^{(n)}_{a,b}$ is bi-anti-symmetric. Moreover, it follows from the above study on the function $f^n_m(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})$ that $E^{(n)}_{a,b}$ satisfies (3.13), (3.23) and that $E^{(n)}_{a,b}\equiv 0$ if $a=0$ or $b=0$. Thus, by (3.38)

$$E^{(n)}(\psi)=\sum_{a,b=2}^{N}1_{a,b\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{a+b}\sum_{\substack{\mathbf{X}\in I^a\\ \mathbf{Y}\in I^b}}E^{(n)}_{a,b}(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.$$

To establish upper bounds on the integrals of $E^{(n)}_{a,b}$, let us study bound properties of $f^n_m(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})$. First we consider the case $n=1$. In this case it simply follows from (3.8), (3.11) that for any $\mathbf{X}_1\in I^{u_1}$, $\mathbf{Y}_1\in I^{v_1}$

$$\begin{aligned}&|f^n_m(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\mathbf{X}_1,\mathbf{Y}_1)|\\ &\le\left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1}\sum_{\substack{\mathbf{W}_1\in I^{p_1-u_1}\\ \mathbf{Z}_1\in I^{q_1-v_1}}}|F_{p_1,q_1}((\mathbf{W}_1,\mathbf{X}_1),(\mathbf{Z}_1,\mathbf{Y}_1))|D^{\frac{1}{2}(p_1+q_1-u_1-v_1)},\end{aligned}$$

and thus

$$\|f^n_m(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})\|_{1,\infty}\le D^{\frac{1}{2}(p_1+q_1-u_1-v_1)}\|F_{p_1,q_1}\|_{1,\infty},\tag{3.42}$$

$$\|f^n_m(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})\|_{1}\le D^{\frac{1}{2}(p_1+q_1-u_1-v_1)}\|F_{p_1,q_1}\|_{1}.\tag{3.43}$$

Next let us assume that $n\ge 2$. By (3.2), (3.9), (3.11), for any $\mathbf{X}_j\in I^{u_j}$ $(j=1,2,\cdots,m+1)$, $\mathbf{Y}_j\in I^{v_j}$ $(j=1,2,\cdots,n-m)$,

$$\begin{aligned}&|f^n_m(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})((\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_{m+1}),(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_{n-m}))|\\ &\le 2^{n-1}\sum_{S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})}\sum_{T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})}\left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1}\\ &\quad\cdot\sum_{\substack{\mathbf{W}_1\in I^{p_1-u_1-d_1(S)}\\ \mathbf{W}'_1\in I^{d_1(S)}}}\sum_{\substack{\mathbf{Z}_1\in I^{q_1-v_1-d_1(T)}\\ \mathbf{Z}'_1\in I^{d_1(T)}}}\\ &\quad\cdot\begin{pmatrix}p_1-u_1\\ d_1(S)\end{pmatrix}\begin{pmatrix}q_1-v_1\\ d_1(T)\end{pmatrix}|F_{p_1,q_1}((\mathbf{W}_1,\mathbf{W}'_1,\mathbf{X}_1),(\mathbf{Z}_1,\mathbf{Z}'_1,\mathbf{Y}_1))|\\ &\quad\cdot\prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j-u_j}\sum_{\substack{\mathbf{W}_j\in I^{p_j-u_j-d_{s_j}(S)}\\ \mathbf{W}'_j\in I^{d_{s_j}(S)}}}\begin{pmatrix}p_j-u_j\\ d_{s_j}(S)\end{pmatrix}|F^{s_j}_{p_j}(\mathbf{W}_j,\mathbf{W}'_j,\mathbf{X}_j)|\right)\end{aligned}$$

$$
\begin{aligned}
&\cdot \prod_{k=2}^{n-m} \left(\left(\frac{1}{h}\right)^{q_k - v_k} \sum_{\substack{\mathbf{Z}_k \in I^{q_k - v_k - d_{t_k}(T)} \\ \mathbf{Z}'_k \in I^{d_{t_k}(T)}}} \begin{pmatrix} q_k - v_k \\ d_{t_k}(T) \end{pmatrix} |F^{t_k}_{q_k}(\mathbf{Z}_k, \mathbf{Z}'_k, \mathbf{Y}_k)| \right) \\
&\cdot D^{\frac{1}{2}(\sum_{j=1}^{m+1} p_j - 2m - \sum_{j=1}^{m+1} u_j) + \frac{1}{2}(\sum_{k=1}^{n-m} q_k - 2(n-m-1) - \sum_{k=1}^{n-m} v_k)} \\
&\cdot \left| \prod_{\{p,q\} \in S} \Delta_{\{p,q\}}(\mathcal{C}) \prod_{j=1}^{m+1} \psi^{s_j}_{\mathbf{W}'_j} \right| \left| \prod_{\{p,q\} \in T} \Delta_{\{p,q\}}(\mathcal{C}) \prod_{k=1}^{n-m} \psi^{t_k}_{\mathbf{Z}'_k} \right| \\
&= 2^{n-1} D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1} u_j + \sum_{k=1}^{n-m} v_k)} \sum_{S \in \mathbb{T}(\{s_j\}_{j=1}^{m+1})} \sum_{T \in \mathbb{T}(\{t_k\}_{k=1}^{n-m})} \\
&\cdot \left(\frac{1}{h}\right)^{p_1 + q_1 - u_1 - v_1} \sum_{\substack{\mathbf{W}_1 \in I^{p_1 - u_1 - d_1(S)} \\ \mathbf{W}'_1 \in I^{d_1(S)}}} \sum_{\substack{\mathbf{Z}_1 \in I^{q_1 - v_1 - d_1(T)} \\ \mathbf{Z}'_1 \in I^{d_1(T)}}} \\
&\cdot \begin{pmatrix} p_1 - u_1 \\ d_1(S) \end{pmatrix} \begin{pmatrix} q_1 - v_1 \\ d_1(T) \end{pmatrix} D^{\frac{1}{2}(p_1 + q_1)} \\
&\cdot |F_{p_1, q_1}((\mathbf{W}_1, \mathbf{W}'_1, \mathbf{X}_1), (\mathbf{Z}_1, \mathbf{Z}'_1, \mathbf{Y}_1))| \\
&\cdot \prod_{j=2}^{m+1} \left(\left(\frac{1}{h}\right)^{p_j - u_j} \sum_{\substack{\mathbf{W}_j \in I^{p_j - u_j - d_{s_j}(S)} \\ \mathbf{W}'_j \in I^{d_{s_j}(S)}}} \begin{pmatrix} p_j - u_j \\ d_{s_j}(S) \end{pmatrix} D^{\frac{p_j}{2}} |F^{s_j}_{p_j}(\mathbf{W}_j, \mathbf{W}'_j, \mathbf{X}_j)| \right) \\
&\cdot \prod_{k=2}^{n-m} \left(\left(\frac{1}{h}\right)^{q_k - v_k} \sum_{\substack{\mathbf{Z}_k \in I^{q_k - v_k - d_{t_k}(T)} \\ \mathbf{Z}'_k \in I^{d_{t_k}(T)}}} \begin{pmatrix} q_k - v_k \\ d_{t_k}(T) \end{pmatrix} D^{\frac{q_k}{2}} |F^{t_k}_{q_k}(\mathbf{Z}_k, \mathbf{Z}'_k, \mathbf{Y}_k)| \right) \\
&\cdot \left| \prod_{\{p,q\} \in S} \Delta_{\{p,q\}}(\mathcal{C}) \prod_{j=1}^{m+1} \psi^{s_j}_{\mathbf{W}'_j} \right| \left| \prod_{\{p,q\} \in T} \Delta_{\{p,q\}}(\mathcal{C}) \prod_{k=1}^{n-m} \psi^{t_k}_{\mathbf{Z}'_k} \right|,
\end{aligned}
$$

where for consistency we admit that

$$
\left| \prod_{\{p,q\} \in S} \Delta_{\{p,q\}}(\mathcal{C}) \prod_{j=1}^{m+1} \psi^{s_j}_{\mathbf{W}'_j} \right| := 1 \text{ if } m = 0,
$$

$$
\left| \prod_{\{p,q\} \in T} \Delta_{\{p,q\}}(\mathcal{C}) \prod_{k=1}^{n-m} \psi^{t_k}_{\mathbf{Z}'_k} \right| := 1 \text{ if } m = n - 1,
$$

since in these cases S (or T) has no line. For each $S \in \mathbb{T}(\{s_j\}_{j=1}^{m+1})$, $T \in \mathbb{T}(\{t_k\}_{k=1}^{n-m})$ we can draw a line between the vertex $s_1(=1)$ of S and the vertex $t_1(=1)$ of T to form a tree containing both S and T. To estimate $\|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})\|_1$, we consider the vertex n as the root of this large tree. Then, by repeating the recursive estimation from younger branches to the root n and using the inequality (3.21) we deduce that

(3.44)

$$\begin{aligned}
&\|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})\|_1\\
&\le 2^{n-1}D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\sum_{S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})}\sum_{T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})}\\
&\quad\cdot\begin{pmatrix}p_1-u_1\\ d_1(S)\end{pmatrix}\begin{pmatrix}q_1-v_1\\ d_1(T)\end{pmatrix}d_1(S)!d_1(T)!D^{\frac{1}{2}(p_1+q_1)}\\
&\quad\cdot\Bigg(1_{n\in\{s_j\}_{j=2}^{m+1}}\sup_{X_0\in I}\left(\frac{1}{h}\right)^{p_1+q_1}\sum_{\substack{\mathbf{X}\in I^{p_1}\\ \mathbf{Y}\in I^{q_1}}}|F_{p_1,q_1}(\mathbf{X},\mathbf{Y})||\tilde{\mathcal{C}}(X_0,X_1)|\\
&\qquad+1_{n\in\{t_k\}_{k=2}^{n-m}}\sup_{Y_0\in I}\left(\frac{1}{h}\right)^{p_1+q_1}\sum_{\substack{\mathbf{X}\in I^{p_1}\\ \mathbf{Y}\in I^{q_1}}}|F_{p_1,q_1}(\mathbf{X},\mathbf{Y})||\tilde{\mathcal{C}}(Y_0,Y_1)|\Bigg)\\
&\quad\cdot\prod_{j=2}^{m+1}\Bigg(\begin{pmatrix}p_j-u_j\\ d_{s_j}(S)\end{pmatrix}d_{s_j}(S)!D^{\frac{p_j}{2}}\\
&\qquad\cdot\Bigg(1_{s_j\neq n}\sup_{X_0\in I}\left(\frac{1}{h}\right)^{p_j}\sum_{\mathbf{X}\in I^{p_j}}|F_{p_j}^{s_j}(\mathbf{X})||\tilde{\mathcal{C}}(X_0,X_1)|+1_{s_j=n}\|F_{p_j}^{s_j}\|_1\Bigg)\Bigg)\\
&\quad\cdot\prod_{k=2}^{n-m}\Bigg(\begin{pmatrix}q_k-v_k\\ d_{t_k}(T)\end{pmatrix}d_{t_k}(T)!D^{\frac{q_k}{2}}\\
&\qquad\cdot\Bigg(1_{t_k\neq n}\sup_{X_0\in I}\left(\frac{1}{h}\right)^{q_k}\sum_{\mathbf{X}\in I^{q_k}}|F_{q_k}^{t_k}(\mathbf{X})||\tilde{\mathcal{C}}(X_0,X_1)|+1_{t_k=n}\|F_{q_k}^{t_k}\|_1\Bigg)\Bigg)\\
&\le(1_{m+1\ge 2}(m-1)!+1_{m+1=1})(1_{n-m\ge 2}(n-m-2)!+1_{n-m=1})\\
&\quad\cdot 2^{-2\sum_{j=1}^{m+1}u_j-2\sum_{k=1}^{n-m}v_k}D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\\
&\quad\cdot 2^{2p_1+2q_1}D^{\frac{1}{2}(p_1+q_1)}\|F_{p_1,q_1}\|_{1,\infty}
\end{aligned}$$

$$\cdot \prod_{j=2}^{m+1} (2^{2p_j} D^{\frac{p_j}{2}} (1_{s_j \neq n} \|F_{p_j}^{s_j}\|_{1,\infty} + 1_{s_j=n} \|F_{p_j}^{s_j}\|_1))$$
$$\cdot \prod_{k=2}^{n-m} (2^{2q_k} D^{\frac{q_k}{2}} (1_{t_k \neq n} \|F_{q_k}^{t_k}\|_{1,\infty} + 1_{t_k=n} \|F_{q_k}^{t_k}\|_1)).$$

We should remark that the inequality

$$\begin{aligned}&2^{n-1}(1_{m+1\geq 2}(m-1)!2^{-m-1} + 1_{m+1=1})\\ &\quad \cdot (1_{n-m\geq 2}(n-m-2)!2^{-n+m} + 1_{n-m=1})\\ &\leq (1_{m+1\geq 2}(m-1)! + 1_{m+1=1})(1_{n-m\geq 2}(n-m-2)! + 1_{n-m=1})\end{aligned}$$

was used to derive the second inequality.

Estimation of $\|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})\|_{1,\infty}$ can be done similarly. In this case first we fix a component of $((\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_{m+1}),(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_{n-m}))$. Then, there uniquely exists a vertex of the enlarged tree containing both S and T such that the fixed component is a variable of the function F^j or $F_{p,q}$ on the vertex. We consider the vertex as the root of the enlarged tree and repeat the same recursive estimation as above. The result is that

$$\begin{aligned}&\|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})\|_{1,\infty}\\ &\leq (1_{m+1\geq 2}(m-1)! + 1_{m+1=1})(1_{n-m\geq 2}(n-m-2)! + 1_{n-m=1})\\ &\quad \cdot 2^{-2\sum_{j=1}^{m+1}u_j - 2\sum_{k=1}^{n-m}v_k} D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\\ &\quad \cdot 2^{2p_1+2q_1}D^{\frac{1}{2}(p_1+q_1)}\|F_{p_1,q_1}\|_{1,\infty}\\ &\quad \cdot \prod_{j=2}^{m+1}(2^{2p_j}D^{\frac{p_j}{2}}\|F_{p_j}^{s_j}\|_{1,\infty})\prod_{k=2}^{n-m}(2^{2q_k}D^{\frac{q_k}{2}}\|F_{q_k}^{t_k}\|_{1,\infty}).\end{aligned} \tag{3.45}$$

It follows from the definition (3.41) that

(3.46)

$$\|E_{a,b}^{(n)}\|_{norm}$$
$$\leq \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} \sum_{u_1=0}^{p_1} (1_{m=0} + 1_{m\neq 0}1_{u_1\leq p_1-1}) \begin{pmatrix} p_1 \\ u_1 \end{pmatrix}$$

$$\cdot \sum_{v_1=0}^{q_1} (1_{m=n-1} + 1_{m\neq n-1} 1_{v_1 \le q_1 - 1}) \begin{pmatrix} q_1 \\ v_1 \end{pmatrix}$$
$$\cdot \prod_{j=2}^{m+1} \left(\sum_{p_j=2}^{N} \sum_{u_j=0}^{p_j-1} \begin{pmatrix} p_j \\ u_j \end{pmatrix} \right) \prod_{k=2}^{n-m} \left(\sum_{q_k=2}^{N} \sum_{v_k=0}^{q_k-1} \begin{pmatrix} q_k \\ v_k \end{pmatrix} \right) \| f_m^n(\mathbf{p}, \mathbf{u}, \mathbf{q}, \mathbf{v}) \|_{norm}$$
$$\cdot 1_{\sum_{j=1}^{m+1} u_j = a} 1_{\sum_{k=1}^{n-m} v_k = b} 1_{\sum_{j=1}^{m+1} p_j - 2m \ge a} 1_{\sum_{k=1}^{n-m} q_k - 2(n-m-1) \ge b},$$

where $norm = `1, \infty'$ or $norm = 1$. When $n = 1$, by substituting (3.42) into (3.46) with $norm = `1, \infty'$ we obtain (3.34). By substituting (3.43) into (3.46) with $norm = 1$ we obtain (3.35). Assume that $n \ge 2$. By inserting (3.45) into (3.46) with $norm = `1, \infty'$ we see that

$$\| E_{a,b}^{(n)} \|_{1,\infty} \le \sum_{p_1, q_1 = 2}^{N} 1_{p_1, q_1 \in 2\mathbb{N}} \sum_{u_1=0}^{p_1} \begin{pmatrix} p_1 \\ u_1 \end{pmatrix} \sum_{v_1=0}^{q_1} \begin{pmatrix} q_1 \\ v_1 \end{pmatrix}$$
$$\cdot \prod_{j=2}^{m+1} \left(\sum_{p_j=2}^{N} \sum_{u_j=0}^{p_j-1} \begin{pmatrix} p_j \\ u_j \end{pmatrix} \right) \prod_{k=2}^{n-m} \left(\sum_{q_k=2}^{N} \sum_{v_k=0}^{q_k-1} \begin{pmatrix} q_k \\ v_k \end{pmatrix} \right)$$
$$\cdot (1_{m+1 \ge 2} (m-1)! + 1_{m+1=1})(1_{n-m \ge 2}(n-m-2)! + 1_{n-m=1})$$
$$\cdot 2^{-2a-2b} D^{-n+1-\frac{1}{2}(a+b)} \|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1} 2^{2p_1+2q_1} D^{\frac{1}{2}(p_1+q_1)} \|F_{p_1,q_1}\|_{1,\infty}$$
$$\cdot \prod_{j=2}^{m+1} (2^{2p_j} D^{\frac{p_j}{2}} \|F_{p_j}^{s_j}\|_{1,\infty}) \prod_{k=2}^{n-m} (2^{2q_k} D^{\frac{q_k}{2}} \|F_{q_k}^{t_k}\|_{1,\infty})$$
$$\cdot 1_{\sum_{j=1}^{m+1} p_j - 2m \ge a} 1_{\sum_{k=1}^{n-m} q_k - 2(n-m-1) \ge b},$$

which gives (3.36). By combining (3.44) with (3.46) with $norm = 1$ we obtain (3.37). □

3.3. Generalized covariances

To construct a double-scale integration process in a generalized setting, here we list the assumptions on a couple of generalized covariances. Let $c_0 \in \mathbb{R}_{\ge 1}$, $D_c \in \mathbb{R}_{>0}$. We assume that covariances $\mathcal{C}_0$, $\mathcal{C}_1 : I_0^2 \to \mathbb{C}$ satisfy the following properties.

- $\mathcal{C}_1$ satisfies (3.10).

•

(3.47)
$$\mathcal{C}_0(\rho\mathbf{x}s,\eta\mathbf{y}t)=\mathcal{C}_0(\rho\mathbf{x}0,\eta\mathbf{y}0),\ (\forall\rho,\eta\in\{1,2\},\ \mathbf{x},\mathbf{y}\in\Gamma,\ s,t\in[0,\beta)_h).$$

•

$$|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}\mathcal{C}_l(X_i,Y_j))_{1\le i,j\le n}|\le c_0^n,\quad (3.48)$$
$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,$$
$$X_i,Y_i\in I_0\ (i=1,2,\cdots,n),\ l\in\{0,1\}).$$

•

$$\|\tilde{\mathcal{C}}_1\|_{1,\infty}\le c_0. \quad (3.49)$$
$$\|\tilde{\mathcal{C}}_1\|\le c_0. \quad (3.50)$$
$$\|\tilde{\mathcal{C}}_0\|_{1,\infty}\le c_0D_c. \quad (3.51)$$

Here $\tilde{\mathcal{C}}_l(: I^2\to\mathbb{C})$ is the anti-symmetric extension of $\mathcal{C}_l$ defined as in (3.5). In practice $\mathcal{C}_1$ will be replaced by the free covariance with many Matsubara frequencies and the covariance $\mathcal{C}_0$ will be the free covariance containing only one Matsubara frequency closest to the parameter $\theta/2$. The condition (3.47) requires $\mathcal{C}_0$ to be independent of the time variables, which may be seen as a strong assumption at this point. If a covariance sums over only one time-momentum, then by a gauge transform the covariance can be made independent of the time variables. It will turn out that because of the time-independence of $\mathcal{C}_0$, only negligibly small data bounded by the inverse volume factor remain after the double-scale integration of the correction term.

3.4. The first integration without the artificial term

Our purpose here is to develop a single-scale analysis concerning the single-scale integration

$$\log\left(\int e^{-V(u)(\psi+\psi^1)+W(u)(\psi+\psi^1)}d\mu_{\mathcal{C}_1}(\psi^1)\right).$$

In fact what we will analyze is an analytic continuation of the above Grassmann polynomial which a priori makes sense only if the coupling constant is sufficiently small. The analytically continued polynomial will be related to the Grassmann integral of the correction term by the identity theorem in Subsection 4.2. In the next subsection we will add the artificial term $-A(\psi)$ to the input $-V(\psi)+W(\psi)$.

To describe properties of the output of the above integration, we introduce a couple of sets of $\bigwedge\mathcal{V}$-valued functions. For sets O, O' let $\text{Map}(O, O')$ denote the set of maps from O to O'. From now we use a parameter $\alpha\in\mathbb{R}_{\ge 1}$ in many situations. In this subsection kernels of Grassmann polynomials are parameterized by $u\in\overline{D(r)}$. To describe uniform convergent properties of the kernels, let us modify the norm $\|\cdot\|_{1,\infty}$ defined in Subsection 3.1 as follows. For $f\in\text{Map}(\overline{D(r)},\text{Map}(I^m,\mathbb{C}))$ we set

$$\|f\|_{1,\infty,r} := \sup_{u\in\overline{D(r)}}\|f(u)\|_{1,\infty}.$$

For notational consistency we set

$$\|f\|_{1,\infty,r} := \sup_{u\in\overline{D(r)}}|f(u)|$$

for $f\in\text{Map}(\overline{D(r)},\mathbb{C})$ as well.

With these notations, for $r\in\mathbb{R}_{>0}$ we define the subset $\mathcal{Q}(r)$ of $\text{Map}(\overline{D(r)},\bigwedge_{even}\mathcal{V})$ as follows. f belongs to $\mathcal{Q}(r)$ if and only if the following statements hold.

- $f\in\text{Map}\left(\overline{D(r)},\bigwedge_{even}\mathcal{V}\right)$.
- $u\mapsto f(u)(\psi):\overline{D(r)}\to\bigwedge\mathcal{V}$ is continuous in $\overline{D(r)}$ and analytic in $D(r)$.
- For any $u\in\overline{D(r)}$ the anti-symmetric kernels $f(u)_m: I^m\to\mathbb{C}$ $(m=2,4,\cdots,N)$ satisfy (3.13) and

$$\begin{aligned}&\frac{h}{N}\alpha^2\|f_0\|_{1,\infty,r}\le L^{-d},\\ &\sum_{m=2}^{N}c_0^{\frac{m}{2}}\alpha^m\|f_m\|_{1,\infty,r}\le L^{-d}.\end{aligned}\tag{3.52}$$

In short the set $\mathcal{Q}(r)$ gathers Grassmann data bounded by L^{-d}.

Next we define the subset $\mathcal{R}(r)$ of $\mathrm{Map}(\overline{D(r)}, \bigwedge_{even}\mathcal{V})$ as follows. f belongs to $\mathcal{R}(r)$ if and only if the following statements hold.

- $f \in \mathrm{Map}\left(\overline{D(r)}, \bigwedge_{even}\mathcal{V}\right)$.
- $u \mapsto f(u)(\psi) : \overline{D(r)} \to \bigwedge\mathcal{V}$ is continuous in $\overline{D(r)}$ and analytic in $D(r)$.
- There exist $f_{p,q} \in \mathrm{Map}(\overline{D(r)}, \mathrm{Map}(I^p \times I^q, \mathbb{C}))$ $(p, q \in \{2, 4, \cdots, N\})$ such that for any $u \in \overline{D(r)}$, $p, q \in \{2, 4, \cdots, N\}$, $f_{p,q}(u) : I^p \times I^q \to \mathbb{C}$ is bi-anti-symmetric and satisfies (3.13), (3.23) and

$$f(u)(\psi) = \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} f_{p,q}(u)(\mathbf{X}, \mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}},$$

$$\sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} c_0^{\frac{1}{2}(p+q)} \alpha^{p+q} \|f_{p,q}\|_{1,\infty,r} \le 1. \tag{3.53}$$

In short the set $\mathcal{R}(r)$ collects Grassmann data whose kernels have the good property (3.23).

With fixed $r \in \mathbb{R}_{>0}$ let us define $V^{0-1,1}, V^{0-2,1}, V^{0,1} \in \mathrm{Map}(\overline{D(r)}, \bigwedge_{even}\mathcal{V})$ as follows.

$$\begin{aligned}
&V^{0-1,1}(u)(\psi) := \left(\frac{1}{h}\right)^2 \sum_{\mathbf{X}\in I^2} V_2^{0-1,1}(u)(\mathbf{X})\psi_{\mathbf{X}},\\
&V^{0-2,1}(u)(\psi) := \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,1}(u)(\mathbf{X}, \mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}},\\
&V^{0,1}(u)(\psi) := V^{0-1,1}(u)(\psi) + V^{0-2,1}(u)(\psi), \quad (u \in \overline{D(r)}),
\end{aligned}$$

where

$$\begin{aligned}
&V_2^{0-1,1}(u)(\rho_1\mathbf{x}_1 s_1\xi_1, \rho_2\mathbf{x}_2 s_2\xi_2)\\
&:= -\frac{1}{2}uL^{-d}h 1_{(\rho_1,\mathbf{x}_1,s_1)=(\rho_2,\mathbf{x}_2,s_2)} 1_{\rho_1=1}(1_{(\xi_1,\xi_2)=(1,-1)} - 1_{(\xi_1,\xi_2)=(-1,1)}),
\end{aligned} \tag{3.54}$$

(3.55)

$$
\begin{aligned}
&V_{2,2}^{0-2,1}(u)(\rho_1\mathbf{x}_1 s_1\xi_1,\rho_2\mathbf{x}_2 s_2\xi_2,\eta_1\mathbf{y}_1 t_1\zeta_1,\eta_2\mathbf{y}_2 t_2\zeta_2)\\
&:= -\frac{1}{4}uL^{-d}h^2 1_{(\mathbf{x}_1,s_1,\mathbf{y}_1,t_1)=(\mathbf{x}_2,s_2,\mathbf{y}_2,t_2)}(h1_{s_1=t_1}-\beta^{-1})\\
&\qquad\cdot\sum_{\sigma,\tau\in\mathbb{S}_2}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)1_{(\rho_{\sigma(1)},\rho_{\sigma(2)},\eta_{\tau(1)},\eta_{\tau(2)})=(1,2,2,1)}\\
&\qquad\qquad\cdot 1_{(\xi_{\sigma(1)},\xi_{\sigma(2)},\zeta_{\tau(1)},\zeta_{\tau(2)})=(1,-1,1,-1)}.
\end{aligned}
$$

One can check that $V_2^{0-1,1}(u): I^2\to\mathbb{C}$ is anti-symmetric, $V_{2,2}^{0-2,1}(u): I^2\times I^2\to\mathbb{C}$ is bi-anti-symmetric and $V^{0,1}(u)(\psi)$ is equal to the initial data $-V(u)(\psi)+W(u)(\psi)$. Then, we define $V^{0-1-1,0},V^{0-1-2,0},V^{0-2,0}\in\mathrm{Map}(\overline{D(r)},\bigwedge_{even}\mathcal{V})$ as follows. For any $n\in\mathbb{N}$, $u\in\overline{D(r)}$,

$$
\begin{aligned}
&V^{0-1-1,0,(n)}(u)(\psi)\\
&:=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_1)\\
&\quad\cdot\prod_{j=1}^{n}\left(\sum_{b_j\in\{1,2\}}V^{0-b_j,1}(u)(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}1_{\exists j(b_j=1)},\\
&V^{0-1-2,0,(n)}(u)(\psi)\\
&:=\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V_{2,2}^{0-2,1}(u)(\mathbf{X},\mathbf{Y})\frac{1}{n!}Tree(\{1,2,\cdots,n+1\},\mathcal{C}_1)\\
&\quad\cdot(\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\prod_{j=3}^{n+1}V^{0-2,1}(u)(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n+1\})}},\\
&V^{0-2,0,(n)}(u)(\psi)\\
&:=\frac{1}{n!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V_{2,2}^{0-2,1}(u)(\mathbf{X},\mathbf{Y})\\
&\quad\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C}_1)(\psi^{s_1}+\psi)_{\mathbf{X}}\prod_{j=2}^{m+1}V^{0-2,1}(u)(\psi^{s_j}+\psi)\Bigg|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}\\
&\quad\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C}_1)(\psi^{t_1}+\psi)_{\mathbf{Y}}\prod_{k=2}^{n-m}V^{0-2,1}(u)(\psi^{t_k}+\psi)\Bigg|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}},
\end{aligned}
$$

where

$$S(n,m) := \left\{ (\{s_j\}_{j=1}^{m+1}, \{t_k\}_{k=1}^{n-m}) \ \middle| \ \begin{array}{l} 1 = s_1 < s_2 < \cdots < s_{m+1} \le n, \\ 1 = t_1 < t_2 < \cdots < t_{n-m} \le n, \\ \{s_j\}_{j=2}^{m+1} \cup \{t_k\}_{k=2}^{n-m} = \{2,3,\cdots,n\}, \\ \{s_j\}_{j=2}^{m+1} \cap \{t_k\}_{k=2}^{n-m} = \emptyset. \end{array} \right\}.$$

Then, set

$$\begin{aligned} V^{0-1-j,0}(u)(\psi) &:= \sum_{n=1}^{\infty} V^{0-1-j,0,(n)}(u)(\psi), \ (j=1,2), \\ V^{0-1,0}(u)(\psi) &:= V^{0-1-1,0}(u)(\psi) + V^{0-1-2,0}(u)(\psi), \\ V^{0-2,0}(u)(\psi) &:= \sum_{n=1}^{\infty} V^{0-2,0,(n)}(u)(\psi), \end{aligned}$$

on the assumption that these series converge in $\bigwedge \mathcal{V}$. The reason why we use the label $0-1$, $0-2$ as the 1st superscript is that these Grassmann data are independent of the artificial parameters λ_1, λ_2 and thus are classified as the data of degree 0 with λ_1, λ_2. In the next subsection we will introduce the data V^{1-j} $(j = 1,2,3)$ and V^2 which are of degree 1 and of degree at least 2 with the parameters λ_1, λ_2 respectively. The 2nd superscripts 1, 0 indicate the scale of integration. The data being integrated with the covariance $\mathcal{C}_1$ have the 2nd superscript 1, while the data to be integrated with the covariance $\mathcal{C}_0$ have the 2nd superscript 0. Thus, it can be read that $V^{0,1}$ is independent of λ_1, λ_2 and to be integrated with $\mathcal{C}_1$, $V^{0-1,0}$ is independent of λ_1, λ_2 and to be integrated with $\mathcal{C}_0$ and so on.

We should explain the structure of the above definitions. The idea of the following transformation is essentially same as the equalities [16, (3.38)], [15, (IV.15)]. It follows from the general formulas (3.6), (3.7) that

(3.56)

$$\begin{aligned} &\frac{1}{n!}\left(\frac{d}{dz}\right)^n \log\left(\int e^{zV^{0,1}(u)(\psi^1+\psi)} d\mu_{\mathcal{C}_1}(\psi^1)\right)\Bigg|_{z=0} \\ &= V^{0-1-1,0,(n)}(u)(\psi) \\ &\quad + \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,1}(u)(\mathbf{X},\mathbf{Y}) \frac{1}{n!} Tree(\{1,2,\cdots,n\},\mathcal{C}_1) \end{aligned}$$

$$
\begin{aligned}
&\cdot(\psi^1+\psi)_{\mathbf{X}}(\psi^1+\psi)_{\mathbf{Y}}\prod_{j=2}^{n}V^{0-2,1}(u)(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=V^{0-1-1,0,(n)}(u)(\psi)\\
&\quad+\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V^{0-2,1}_{2,2}(u)(\mathbf{X},\mathbf{Y})\frac{1}{n!}\prod_{j=1}^{n}\left(\frac{\partial}{\partial z_j}\right)\\
&\quad\cdot\log\left(\int e^{z_1(\psi^1+\psi)_{\mathbf{X}}(\psi^1+\psi)_{\mathbf{Y}}+\sum_{j=2}^{n}z_jV^{0-2,1}(u)(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\right)\Bigg|_{\substack{z_j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=V^{0-1-1,0,(n)}(u)(\psi)\\
&\quad+\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V^{0-2,1}_{2,2}(u)(\mathbf{X},\mathbf{Y})\\
&\quad\cdot\frac{1}{n!}\prod_{j=2}^{n}\left(\frac{\partial}{\partial z_j}\right)\int(\psi^1+\psi)_{\mathbf{X}}(\psi^1+\psi)_{\mathbf{Y}}e^{\sum_{j=2}^{n}z_jV^{0-2,1}(u)(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\\
&\quad\cdot\left(\int e^{\sum_{j=2}^{n}z_jV^{0-2,1}(u)(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\right)^{-1}\Bigg|_{\substack{z_j=0\\(\forall j\in\{2,3,\cdots,n\})}}\\
&=V^{0-1-1,0,(n)}(u)(\psi)\\
&\quad+\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V^{0-2,1}_{2,2}(u)(\mathbf{X},\mathbf{Y})\frac{1}{n!}\prod_{j=0}^{n}\left(\frac{\partial}{\partial z_j}\right)\\
&\quad\cdot\Bigg(\log\left(\int e^{z_0(\psi^1+\psi)_{\mathbf{X}}+z_1(\psi^1+\psi)_{\mathbf{Y}}+\sum_{j=2}^{n}z_jV^{0-2,1}(u)(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\right)\\
&\qquad+\log\left(\int e^{z_0(\psi^1+\psi)_{\mathbf{X}}+\sum_{j=2}^{n}z_jV^{0-2,1}(u)(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\right)\\
&\qquad\cdot\log\left(\int e^{z_1(\psi^1+\psi)_{\mathbf{Y}}+\sum_{j=2}^{n}z_jV^{0-2,1}(u)(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\right)\Bigg)\Bigg|_{\substack{z_j=0\\(\forall j\in\{0,1,\cdots,n\})}}\\
&=V^{0-1-1,0,(n)}(u)(\psi)+V^{0-1-2,0,(n)}(u)(\psi)+V^{0-2,0,(n)}(u)(\psi).
\end{aligned}
$$

Remark that for any $f^j(\psi)\in\bigwedge_{even}\mathcal{V}$ $(j=1,2,\cdots,n)$ the maps

$$
\begin{aligned}
&(z_1,z_2,\cdots,z_n)\mapsto\log\left(\int e^{\sum_{j=1}^{n}z_jf^j(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\right),\\
&(z_1,z_2,\cdots,z_n)\mapsto\left(\int e^{\sum_{j=1}^{n}z_jf^j(\psi^1+\psi)}d\mu_{\mathcal{C}_1}(\psi^1)\right)^{-1}
\end{aligned}
$$

are analytic in a neighborhood of the origin and thus the above transformation holds true. See e.g. [7] for properties of inverse and logarithm of even Grassmann polynomials.

In the rest of this subsection we prove the following lemma.

LEMMA 3.4. *For any* $\alpha \in [2^3, \infty)$,

$$V^{0-1,0} \in \mathcal{Q}(2^{-9}c_0^{-2}\alpha^{-4}), \quad V^{0-2,0} \in \mathcal{R}(2^{-9}c_0^{-2}\alpha^{-4}).$$

REMARK 3.5. The reason why we introduce the norm $\|\cdot\|_{1,\infty,r}$ is that we want to make use of the following fact. If $f^n \in \mathrm{Map}(\overline{D(r)}, \bigwedge_{even}\mathcal{V})$ ($n \in \mathbb{N}$) satisfy that $u \mapsto f^n(u)(\psi)$ is continuous in $\overline{D(r)}$, analytic in $D(r)$ ($\forall n \in \mathbb{N}$) and $\sum_{n=1}^{\infty}\|f_m^n\|_{1,\infty,r} < \infty$ ($\forall m \in \{0, 2, \cdots, N\}$), then $\sum_{n=1}^{\infty} f^n(u)(\psi)$ converges for any $u \in \overline{D(r)}$. Moreover, $u \mapsto \sum_{n=1}^{\infty} f^n(u)(\psi)$ is continuous in $\overline{D(r)}$ and analytic in $D(r)$.

PROOF OF LEMMA 3.4. We can derive from (3.50), (3.54), (3.55) and the uniqueness of anti-symmetric kernel that

$$\|V_2^{0-1,1}\|_{1,\infty,r} \le rL^{-d}, \tag{3.57}$$

$$\|V_4^{0-2,1}\|_{1,\infty,r} \le \|V_{2,2}^{0-2,1}\|_{1,\infty,r} \le r, \tag{3.58}$$

$$\begin{aligned}&\sup_{u\in\overline{D(r)}}[V_{2,2}^{0-2,1}(u), \tilde{\mathcal{C}}_1]_{1,\infty}\\ &\le \sup_{s\in[0,\beta)_h}\sup_{Y_0\in I}\frac{1}{h}\sum_{(\eta,\mathbf{y},t,\zeta)\in I} rL^{-d}(h1_{s=t}+\beta^{-1})|\tilde{\mathcal{C}}_1(Y_0,\eta\mathbf{y}t\zeta)|\\ &\le rL^{-d}\|\tilde{\mathcal{C}}_1\| \le c_0 rL^{-d}.\end{aligned} \tag{3.59}$$

In the following we assume that $\alpha \ge 2^3$ and

$$2^9 c_0^2\alpha^4 r \le 1. \tag{3.60}$$

We can use Lemma 3.1 to estimate $V^{0-1-1,0,(n)}$. The lemma ensures that the anti-symmetric kernel $V(u)_m^{0-1-1,0,(n)}(\cdot)$ satisfies (3.13). Moreover, by using (3.14), (3.48), (3.57) we have that

$$\|V_m^{0-1-1,0,(1)}\|_{1,\infty,r} \le \left(\frac{N}{h}\right)^{1_{m=0}} c_0^{1-\frac{m}{2}} rL^{-d}1_{2\ge m}.$$

Thus,

$$\|V_0^{0-1-1,0,(1)}\|_{1,\infty,r} \le \frac{N}{h} c_0 r L^{-d}, \tag{3.61}$$

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{0-1-1,0,(1)}\|_{1,\infty,r} \le c_0 \alpha^2 r L^{-d}. \tag{3.62}$$

Also, by (3.16), (3.48), (3.49), (3.57), (3.58), for any $n \in \mathbb{N}_{\ge 2}$, $m \in \{0, 2, \cdots, N\}$,

$$\begin{aligned}
&\|V_m^{0-1-1,0,(n)}\|_{1,\infty,r}\\
&\le \left(\frac{N}{h}\right)^{1_{m=0}} 2^{-2m} c_0^{-\frac{m}{2}} \prod_{j=1}^{n} \left(\sum_{b_j \in \{1,2\}} \sum_{p_j=2}^{4} 2^{3p_j} c_0^{\frac{p_j}{2}} \|V_{p_j}^{0-b_j,1}\|_{1,\infty,r} \right)\\
&\quad \cdot 1_{\sum_{j=1}^{n} p_j - 2(n-1) \ge m} 1_{\exists j (b_j = 1)}\\
&\le \left(\frac{N}{h}\right)^{1_{m=0}} 2^{-2m} c_0^{-\frac{m}{2}}\\
&\quad \cdot \sum_{l=1}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) (2^6 c_0 \|V_2^{0-1,1}\|_{1,\infty,r})^l (2^{12} c_0^2 \|V_4^{0-2,1}\|_{1,\infty,r})^{n-l}\\
&\quad \cdot 1_{2l+4(n-l)-2(n-1) \ge m}\\
&\le \left(\frac{N}{h}\right)^{1_{m=0}} 2^{-2m} c_0^{-\frac{m}{2}} \sum_{l=1}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) (2^6 c_0 r L^{-d})^l (2^{12} c_0^2 r)^{n-l} 1_{2n-2l+2 \ge m}.
\end{aligned}$$

Therefore, by $c_0 \ge 1$,

$$\|V_0^{0-1-1,0,(n)}\|_{1,\infty,r} \le \frac{N}{h} (2^{13} c_0^2 r)^n L^{-d}, \tag{3.63}$$

$$\begin{aligned}
&\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{0-1-1,0,(n)}\|_{1,\infty,r}\\
&\le \sum_{l=1}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) (2^6 c_0 r L^{-d})^l (2^{12} c_0^2 r)^{n-l} 2 (2^{-2}\alpha)^{2n-2l+2}\\
&\le 2(2^{-2}\alpha)^2 \sum_{l=1}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) (2^6 c_0 r L^{-d})^l (2^8 c_0^2 \alpha^2 r)^{n-l}\\
&\le \alpha^2 (2^9 c_0^2 \alpha^2 r)^n L^{-d},
\end{aligned} \tag{3.64}$$

where we used $\alpha \geq 2^3$ so that $2^{-2}\alpha/(2^{-2}\alpha - 1) \leq 2$.

Lemma 3.2 is the tool to estimate $V^{0-1-2,0,(n)}$. According to the lemma, the anti-symmetric kernel $V(u)_m^{0-1-2,0,(n)}(\cdot)$ satisfies (3.13). By substituting (3.48), (3.59) into (3.24) we obtain that

$$\|V_m^{0-1-2,0,(1)}\|_{1,\infty,r} \leq 2^8 \left(\frac{N}{h}\right)^{1_{m=0}} c_0^{2-\frac{m}{2}} r L^{-d} 1_{2\geq m},$$

or

$$\|V_0^{0-1-2,0,(1)}\|_{1,\infty,r} \leq 2^8 \frac{N}{h} c_0^2 r L^{-d}, \tag{3.65}$$

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{0-1-2,0,(1)}\|_{1,\infty,r} \leq 2^8 c_0^2 \alpha^2 r L^{-d}. \tag{3.66}$$

Also, by (3.26), (3.48), (3.49), (3.58) and (3.59), for $n \in \mathbb{N}_{\geq 2}$, $m \in \{0, 2, \cdots, N\}$,

$$\|V_m^{0-1-2,0,(n)}\|_{1,\infty,r} \leq \left(\frac{N}{h}\right)^{1_{m=0}} 2^{-2m} c_0^{-\frac{m}{2}} (2^{12} c_0^2 r)^n L^{-d} 1_{2n\geq m}.$$

Thus,

$$\|V_0^{0-1-2,0,(n)}\|_{1,\infty,r} \leq \frac{N}{h} (2^{12} c_0^2 r)^n L^{-d}, \tag{3.67}$$

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{0-1-2,0,(n)}\|_{1,\infty,r} \leq 2(2^8 c_0^2 \alpha^2 r)^n L^{-d}, \tag{3.68}$$

where we used $\alpha \geq 2^3$ so that $2^{-2}\alpha/(2^{-2}\alpha - 1) \leq 2$. Then, we see from (3.60), (3.61), (3.62), (3.63), (3.64), (3.65), (3.66), (3.67), (3.68) and $\alpha \geq 2^3$ that

$$\begin{aligned}
&\frac{h}{N}\alpha^2 \sum_{n=1}^{\infty}\sum_{j=1}^{2} \|V_0^{0-1-j,0,(n)}\|_{1,\infty,r} \\
&\leq \left(2^{-9}\alpha^{-2} + \alpha^2 \sum_{n=2}^{\infty}(2^4\alpha^{-4})^n + 2^{-1}\alpha^{-2} + \alpha^2 \sum_{n=2}^{\infty}(2^3\alpha^{-4})^n\right) L^{-d} \leq L^{-d}, \\
&\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \sum_{n=1}^{\infty}\sum_{j=1}^{2} \|V_m^{0-1-j,0,(n)}\|_{1,\infty,r}
\end{aligned}$$

$$\leq \left(2^{-9}\alpha^{-2}+\alpha^{2}\sum_{n=2}^{\infty}\alpha^{-2n}+2^{-1}\alpha^{-2}+2\sum_{n=2}^{\infty}(2^{-1}\alpha^{-2})^{n}\right)L^{-d}\leq L^{-d}.$$

This implies that $V^{0-1,0}\in\mathcal{Q}(2^{-9}c_0^{-2}\alpha^{-4})$.

Let us consider $V^{0-2,0,(n)}$. By Lemma 3.3, for $n\in\mathbb{N}$, $m\in\{0,1,\cdots,n-1\}$, $(S,T)\in S(n,m)$, $a,b\in\{2,4,\cdots,N\}$, $u\in\overline{D(r)}$ there exists a function $E_{a,b}^{(n,m,S,T)}(u):I^a\times I^b\to\mathbb{C}$ such that $E_{a,b}^{(n,m,S,T)}(u)$ is bi-anti-symmetric, satisfies (3.13), (3.23) and

$$\begin{aligned}&V^{0-2,0,(n)}(u)(\psi)\\&=\frac{1}{n!}\sum_{m=0}^{n-1}\sum_{(S,T)\in S(n,m)}\sum_{a,b=2}^{N}1_{a,b\in2\mathbb{N}}\left(\frac{1}{h}\right)^{a+b}\sum_{\substack{\mathbf{X}\in I^a\\ \mathbf{Y}\in I^b}}E_{a,b}^{(n,m,S,T)}(u)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.\end{aligned}$$

Define the function $V_{a,b}^{0-2,0,(n)}(u):I^a\times I^b\to\mathbb{C}$ by

$$V_{a,b}^{0-2,0,(n)}(u)(\cdot,\cdot):=\frac{1}{n!}\sum_{m=0}^{n-1}\sum_{(S,T)\in S(n,m)}E_{a,b}^{(n,m,S,T)}(u)(\cdot,\cdot).$$

Then, $V_{a,b}^{0-2,0,(n)}(u)(\cdot,\cdot)$ is bi-anti-symmetric, satisfies (3.13), (3.23) and

$$V^{0-2,0,(n)}(u)(\psi)=\sum_{a,b=2}^{N}1_{a,b\in2\mathbb{N}}\left(\frac{1}{h}\right)^{a+b}\sum_{\substack{\mathbf{X}\in I^a\\ \mathbf{Y}\in I^b}}V_{a,b}^{0-2,0,(n)}(u)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.$$

It is also clear from the construction that $u\mapsto V_{a,b}^{0-2,0,(n)}(u)(\mathbf{X},\mathbf{Y})$ is continuous in $\overline{D(r)}$ and analytic in $D(r)$, ($\forall\mathbf{X}\in I^a,\mathbf{Y}\in I^b$). Let us prove bound properties of $V_{a,b}^{0-2,0,(n)}(u)$. The inequalities proved in Lemma 3.3 support our analysis. Note that

$$\sharp S(n,m)=\begin{pmatrix}n-1\\ m\end{pmatrix}. \tag{3.69}$$

By (3.34), (3.58),

$$\|V_{2,2}^{0-2,0,(1)}\|_{1,\infty,r}\leq r. \tag{3.70}$$

Combination of (3.36), (3.48), (3.49), (3.58), (3.69) yields that for $n \in \mathbb{N}_{\geq 2}$, $a, b \in \{2, 4, \cdots, N\}$,

$$\begin{aligned}\|V_{a,b}^{0-2,0,(n)}\|_{1,\infty,r} \leq \frac{1}{n!} \sum_{m=0}^{n-1} \binom{n-1}{m} &(1_{m\neq 0}(m-1)! + 1_{m=0}) \\ &\cdot (1_{m\neq n-1}(n-m-2)! + 1_{m=n-1}) \\ &\cdot 2^{-2a-2b} c_0^{-\frac{1}{2}(a+b)} (2^{12} c_0^2 r)^n 1_{2+2m\geq a} 1_{2n-2m\geq b}.\end{aligned}$$

Note that

(3.71)

$$\begin{aligned}&\sum_{m=0}^{n-1} \binom{n-1}{m} (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1}) \\ &\leq n!.\end{aligned}$$

By using (3.71) and $2^{-2}\alpha/(2^{-2}\alpha - 1) \leq 2$ we can derive that

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} c_0^{\frac{1}{2}(a+b)} \alpha^{a+b} \|V_{a,b}^{0-2,0,(n)}\|_{1,\infty,r} \leq \alpha^2 (2^8 c_0^2 \alpha^2 r)^n. \tag{3.72}$$

It follows from (3.60), (3.70), (3.72) and $\alpha \geq 2^3$ that

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} c_0^{\frac{1}{2}(a+b)} \alpha^{a+b} \sum_{n=1}^{\infty} \|V_{a,b}^{0-2,0,(n)}\|_{1,\infty,r} \leq 2^{-9} + 2^{-1}\alpha^{-2} \leq 1.$$

Thus we conclude that $V^{0-2,0} \in \mathcal{R}(2^{-9} c_0^{-2} \alpha^{-4})$. □

3.5. The first integration with the artificial term

In this subsection we perform a single-scale integration where Grassmann polynomials are dependent on the artificial parameter $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. To be specific, we are going to analyze an analytic continuation of the Grassmann polynomial

$$\log\left(\int e^{-V(u)(\psi+\psi^1)+W(u)(\psi+\psi^1)-A(\psi+\psi^1)} d\mu_{C_1}(\psi^1)\right).$$

For this purpose we need to introduce sets of Grassmann polynomials parameterized by $(u, \boldsymbol{\lambda})$. Bound properties of these Grassmann polynomials are measured in a variant of the L^1-norm $\|\cdot\|_1$, while polynomials belonging to $\mathcal{Q}(r)$, $\mathcal{R}(r)$ were measured in the norm $\|\cdot\|_{1,\infty,r}$. To prove uniform bounds with $(u, \boldsymbol{\lambda})$, we modify the norm $\|\cdot\|_1$ defined in Subsection 3.1 as follows. For $f \in \mathrm{Map}(\overline{D(r)} \times \overline{D(r')}^2, \mathrm{Map}(I^m, \mathbb{C}))$ let

$$\|f\|_{1,r,r'} := \sup_{\substack{u \in \overline{D(r)} \\ \boldsymbol{\lambda} \in \overline{D(r')}^2}} \|f(u, \boldsymbol{\lambda})\|_1.$$

Also for $f \in \mathrm{Map}(\overline{D(r)} \times \overline{D(r')}^2, \mathbb{C})$ we set

$$\|f\|_{1,r,r'} := \sup_{\substack{u \in \overline{D(r)} \\ \boldsymbol{\lambda} \in \overline{D(r')}^2}} |f(u, \boldsymbol{\lambda})|$$

for notational consistency.

For $r, r' \in \mathbb{R}_{>0}$ we define the subset $\mathcal{Q}'(r, r')$ of $\mathrm{Map}(\overline{D(r)} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V})$ as follows. f belongs to $\mathcal{Q}'(r, r')$ if and only if

- $$f \in \mathrm{Map}\left(\overline{D(r)} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V}\right).$$
- For any $u \in \overline{D(r)}$, $\boldsymbol{\lambda} \mapsto f(u, \boldsymbol{\lambda})(\psi) : \mathbb{C}^2 \to \bigwedge \mathcal{V}$ is linear.
- For any $\boldsymbol{\lambda} \in \mathbb{C}^2$, $u \mapsto f(u, \boldsymbol{\lambda})(\psi) : \overline{D(r)} \to \bigwedge \mathcal{V}$ is continuous in $\overline{D(r)}$ and analytic in $D(r)$.
- For any $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \mathbb{C}^2$ the anti-symmetric kernels $f(u, \boldsymbol{\lambda})_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (3.13) and

$$\begin{aligned} &\alpha^2 \|f_0\|_{1,r,r'} \le L^{-d}, \\ &\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|f_m\|_{1,r,r'} \le L^{-d}. \end{aligned} \tag{3.73}$$

In other words the set $\mathcal{Q}'(r, r')$ contains Grassmann polynomials which are linearly dependent on $\boldsymbol{\lambda}$ and become negligibly small as $L \to \infty$.

We also need a set containing Grassmann polynomials with bi-anti-symmetric kernels linearly depending on $\boldsymbol{\lambda}$. For $r, r' \in \mathbb{R}_{>0}$ the subset $\mathcal{R}'(r,r')$ of $\mathrm{Map}(\overline{D(r)} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V})$ is defined as follows. f belongs to $\mathcal{R}'(r,r')$ if and only if

- $$f \in \mathrm{Map}\left(\overline{D(r)} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V}\right).$$
- For any $u \in \overline{D(r)}$, $\boldsymbol{\lambda} \mapsto f(u,\boldsymbol{\lambda})(\psi) : \mathbb{C}^2 \to \bigwedge \mathcal{V}$ is linear.
- For any $\boldsymbol{\lambda} \in \mathbb{C}^2$, $u \mapsto f(u,\boldsymbol{\lambda})(\psi) : \overline{D(r)} \to \bigwedge \mathcal{V}$ is continuous in $\overline{D(r)}$ and analytic in $D(r)$.
- There exist $f_{p,q} \in \mathrm{Map}(\overline{D(r)} \times \mathbb{C}^2, \mathrm{Map}(I^p \times I^q, \mathbb{C}))$ $(p, q = 2, 4, \cdots, N)$ such that for any $(u,\boldsymbol{\lambda}) \in \overline{D(r)} \times \mathbb{C}^2$, $p, q \in \{2, 4, \cdots, N\}$, $f_{p,q}(u,\boldsymbol{\lambda}) : I^p \times I^q \to \mathbb{C}$ is bi-anti-symmetric, satisfies (3.13), (3.23) and

$$f(u,\boldsymbol{\lambda})(\psi) = \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p \\ \mathbf{Y}\in I^q}} f_{p,q}(u,\boldsymbol{\lambda})(\mathbf{X},\mathbf{Y}) \psi_{\mathbf{X}} \psi_{\mathbf{Y}},$$

$$\sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} c_0^{\frac{1}{2}(p+q)} \alpha^{p+q} \|f_{p,q}\|_{1,r,r'} \le 1. \tag{3.74}$$

We introduce another set of Grassmann polynomials with linear dependence on $\boldsymbol{\lambda}$, which is used to contain the offspring of the artificial term $A(\psi)$. For $r, r' \in \mathbb{R}_{>0}$, f belongs to $\mathcal{S}(r,r')$ if and only if

- $$f \in \mathrm{Map}\left(\overline{D(r)} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V}\right).$$
- For any $u \in \overline{D(r)}$, $\boldsymbol{\lambda} \mapsto f(u,\boldsymbol{\lambda})(\psi) : \mathbb{C}^2 \to \bigwedge \mathcal{V}$ is linear.
- For any $\boldsymbol{\lambda} \in \mathbb{C}^2$, $u \mapsto f(u,\boldsymbol{\lambda})(\psi) : \overline{D(r)} \to \bigwedge \mathcal{V}$ is continuous in $\overline{D(r)}$ and analytic in $D(r)$.
- For any $(u,\boldsymbol{\lambda}) \in \overline{D(r)} \times \mathbb{C}^2$ the anti-symmetric kernels $f(u,\boldsymbol{\lambda})_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (3.13) and

$$\alpha^2\|f_0\|_{1,r,r'} \le 1,$$
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m\|f_m\|_{1,r,r'} \le 1. \tag{3.75}$$

Finally we introduce a set of Grassmann polynomials whose degree with $\boldsymbol{\lambda}$ is more than 1. For $r, r' \in \mathbb{R}_{>0}$, f belongs to $\mathcal{W}(r,r')$ if and only if

- $$f \in \mathrm{Map}\left(\overline{D(r)} \times \overline{D(r')}^2, \bigwedge_{even} \mathcal{V}\right).$$
- $(u,\boldsymbol{\lambda}) \mapsto f(u,\boldsymbol{\lambda})(\psi)$ is continuous in $\overline{D(r)} \times \overline{D(r')}^2$ and analytic in $D(r) \times D(r')^2$.
- For any $u \in D(r)$, $j \in \{1,2\}$,
$$f(u,\mathbf{0})(\psi) = \frac{\partial}{\partial \lambda_j} f(u,\mathbf{0})(\psi) = 0.$$
- For any $(u,\boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r')}^2$ the anti-symmetric kernels $f(u,\boldsymbol{\lambda})_m : I^m \to \mathbb{C}$ $(m = 2,4,\cdots,N)$ satisfy (3.13) and

$$\alpha^2\|f_0\|_{1,r,r'} \le 1,$$
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m\|f_m\|_{1,r,r'} \le 1. \tag{3.76}$$

Here let us systematically define the input and the output of the single-scale integration. We admit the results of Lemma 3.4 claiming that

$$V^{0-1,0} \in \mathcal{Q}(2^{-9}c_0^{-2}\alpha^{-4}), \quad V^{0-2,0} \in \mathcal{R}(2^{-9}c_0^{-2}\alpha^{-4})$$

and define $V^{0,0} \in \mathrm{Map}\left(\overline{D(2^{-9}c_0^{-2}\alpha^{-4})}, \bigwedge_{even}\mathcal{V}\right)$ by

$$V^{0,0} := V^{0-1,0} + V^{0-2,0}.$$

We define $V^{1,1} \in \mathrm{Map}(\mathbb{C}^2, \bigwedge_{even}\mathcal{V})$ by

$$V^{1,1}(\boldsymbol{\lambda})(\psi) := -A(\psi),$$

where $A(\psi)$ is the Grassmann polynomial defined in (2.14). Then, by recalling the formula (3.7) let us observe the following expansion.

(3.77)

$$\begin{aligned}
&\frac{1}{n!}\left(\frac{d}{dz}\right)^n \log\left(\int e^{z(V^{0,1}(u)(\psi^1+\psi)+V^{1,1}(\boldsymbol{\lambda})(\psi^1+\psi))}d\mu_{\mathcal{C}_1}(\psi^1)\right)\Bigg|_{z=0}\\
&=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_1)\prod_{j=1}^{n}\left(\sum_{b=0}^{1}V^{b,1}(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_1)\prod_{j=1}^{n}V^{0,1}(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&\quad+1_{n=1}Tree(\{1\},\mathcal{C}_1)V^{1,1}(\psi^1+\psi)\Big|_{\psi^1=0}\\
&\quad+1_{n\ge 2}\frac{1}{(n-1)!}Tree(\{1,2,\cdots,n\},\mathcal{C}_1)\\
&\qquad\qquad\cdot V^{1,1}(\psi^1+\psi)\prod_{j=2}^{n}V^{0,1}(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&\quad+\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_1)\prod_{j=1}^{n}\left(\sum_{b_j=0}^{1}V^{b_j,1}(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&\qquad\cdot 1_{\sum_{j=1}^{n}b_j\ge 2}.
\end{aligned}$$

We further decompose or rename each term of this expansion from top to bottom. It follows from (3.56) that if we set for $n\in\mathbb{N}$

$$V^{0,0,(n)}(\psi):=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_1)\prod_{j=1}^{n}V^{0,1}(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}},$$

then $V^{0,0}(\psi)=\sum_{n=1}^{\infty}V^{0,0,(n)}(\psi)$. Let us set

$$V^{1-3,0}(\psi):=Tree(\{1\},\mathcal{C}_1)V^{1,1}(\psi^1+\psi)\Big|_{\psi^1=0}.$$

For $n\in\mathbb{N}_{\ge 2}$ we set

$$V^{1-1-1,0,(n)}(\psi)$$

$$:= \frac{1}{(n-1)!} Tree(\{1,2,\cdots,n\},\mathcal{C}_1)$$

$$\cdot \prod_{j=1}^{n-1}\left(\sum_{b_j=1}^{2} V^{0-b_j,1}(\psi^j+\psi)\right) V^{1,1}(\psi^n+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}} 1_{\exists j(b_j=1)},$$

$V^{1-1-2,0,(n)}(\psi)$

$$:= \frac{1}{(n-1)!} Tree(\{1,2,\cdots,n+1\},\mathcal{C}_1)\left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,1}(\mathbf{X},\mathbf{Y})$$

$$\cdot (\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}$$

$$\cdot \prod_{j=3}^{n} V^{0-2,1}(\psi^j+\psi)\cdot V^{1,1}(\psi^{n+1}+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n+1\})}},$$

$V^{1-2,0,(n)}(\psi)$

$$:= \frac{1}{(n-1)!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,1}(\mathbf{X},\mathbf{Y})$$

$$\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C}_1)(\psi^{s_1}+\psi)_{\mathbf{X}}$$

$$\cdot \prod_{j=2}^{m+1}(1_{s_j\neq n}V^{0-2,1}(\psi^{s_j}+\psi)+1_{s_j=n}V^{1,1}(\psi^{s_j}+\psi))\Big|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}$$

$$\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C}_1)(\psi^{t_1}+\psi)_{\mathbf{Y}}$$

$$\cdot \prod_{k=2}^{n-m}(1_{t_k\neq n}V^{0-2,1}(\psi^{t_k}+\psi)+1_{t_k=n}V^{1,1}(\psi^{t_k}+\psi))\Big|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}}.$$

By the same argument as in (3.56) we can derive that

$$\frac{1}{(n-1)!} Tree(\{1,2,\cdots,n\},\mathcal{C}_1) V^{1,1}(\psi^1+\psi)\prod_{j=2}^{n} V^{0,1}(\psi^j+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

$$= \frac{1}{(n-1)!} Tree(\{1,2,\cdots,n\},\mathcal{C}_1)$$

$$\cdot \prod_{j=1}^{n-1} V^{0,1}(\psi^j+\psi)\cdot V^{1,1}(\psi^n+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

$$= V^{1-1-1,0,(n)}(\psi)+V^{1-1-2,0,(n)}(\psi)+V^{1-2,0,(n)}(\psi).$$

Finally we set for $n \in \mathbb{N}_{\ge 2}$,

$$\begin{aligned}
&V^{2,0,(n)}(\psi)\\
&:= \frac{1}{n!} Tree(\{1,2,\cdots,n\},\mathcal{C}_1) \prod_{j=1}^{n} \left(\sum_{b_j=0}^{1} V^{b_j,1}(\psi^j + \psi) \right) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\cdots,n\})}}\\
&\quad \cdot 1_{\sum_{j=1}^{n} b_j \ge 2}.
\end{aligned}$$

Then, the expansion (3.77) can be equivalently written as follows.

$$\begin{aligned}
&\frac{1}{n!}\left(\frac{d}{dz}\right)^n \log\left(\int e^{z(V^{0,1}(u)(\psi^1+\psi)+V^{1,1}(\boldsymbol{\lambda})(\psi^1+\psi))} d\mu_{\mathcal{C}_1}(\psi^1) \right)\Bigg|_{z=0}\\
&= V^{0,0,(n)}(\psi) + 1_{n=1} V^{1-3,0}(\psi)\\
&\quad + 1_{n\ge 2}(V^{1-1-1,0,(n)}(\psi) + V^{1-1-2,0,(n)}(\psi) + V^{1-2,0,(n)}(\psi) + V^{2,0,(n)}(\psi)).
\end{aligned}$$

By assuming their convergence let us set

$$\begin{aligned}
&V^{1-1-j,0}(\psi) := \sum_{n=2}^{\infty} V^{1-1-j,0,(n)}(\psi),\ (j=1,2),\\
&V^{1-1,0}(\psi) := \sum_{j=1}^{2} V^{1-1-j,0}(\psi),\\
&V^{1-2,0}(\psi) := \sum_{n=2}^{\infty} V^{1-2,0,(n)}(\psi), \quad V^{2,0}(\psi) := \sum_{n=2}^{\infty} V^{2,0,(n)}(\psi).
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{d}{dz}\right)^n \log\left(\int e^{z(V^{0,1}(u)(\psi^1+\psi)+V^{1,1}(\boldsymbol{\lambda})(\psi^1+\psi))} d\mu_{\mathcal{C}_1}(\psi^1) \right)\Bigg|_{z=0}\\
&= V^{0,0}(\psi) + \sum_{j=1}^{3} V^{1-j,0}(\psi) + V^{2,0}(\psi).
\end{aligned}$$

Our purpose is to prove that these Grassmann polynomials are indeed convergent and they have desired invariant and bound properties. Not to confuse, we should keep in mind that the data $V^{0,j}$ ($j \in \{0,1\}$) are independent of the artificial parameter $\boldsymbol{\lambda}$, the data $V^{1,1}$, $V^{1-j,0}$ ($j \in \{1,2,3\}$) are linearly dependent on $\boldsymbol{\lambda}$ and the data $V^{2,0}$ depends on $\boldsymbol{\lambda}$ at least quadratically. The input have the 2nd superscript 1 and the output have the 2nd superscript 0 in this single-scale integration. More detailed properties of these Grassmann data are summarized in the following lemma.

LEMMA 3.6. *For any* $\alpha \in [2^3, \infty)$,

$$\begin{aligned}
&V^{1-1,0} \in \mathcal{Q}'(2^{-9}c_0^{-2}\alpha^{-4}, 2^{-9}\beta^{-1}c_0^{-2}\alpha^{-4}),\\
&V^{1-2,0} \in \mathcal{R}'(2^{-9}c_0^{-2}\alpha^{-4}, 2^{-9}\beta^{-1}c_0^{-2}\alpha^{-4}),\\
&V^{1-3,0} \in \mathcal{S}(2^{-9}c_0^{-2}\alpha^{-4}, 2^{-9}\beta^{-1}c_0^{-2}\alpha^{-4}),\\
&V^{2,0} \in \mathcal{W}(2^{-9}c_0^{-2}\alpha^{-4}, 2^{-9}\min\{1,\beta\}\beta^{-1}c_0^{-2}\alpha^{-4}).
\end{aligned}$$

REMARK 3.7. It is clear from the definition that $V^{1-3,0}$ is independent of the parameter u. The condition on the first variable assumed in the set $\mathcal{S}(r, r')$ is in fact unnecessary. However, we define the set in this way in accordance with the other sets.

PROOF OF LEMMA 3.6. During the proof we often hide the sign of dependency on the parameter $(u, \boldsymbol{\lambda})$ for conciseness. In the following we always assume that $\alpha \geq 2^3$, (3.60) and

$$2^9\beta c_0^2\alpha^4 r' \leq 1. \tag{3.78}$$

Let us start by estimating $V^{1,1}$ and $V^{1-3,0}$. Since $V_4^{1,1}(\psi) = -\lambda_2 A^2(\psi)$,

$$\begin{aligned}
&V_4^{1,1}(\rho_1\mathbf{x}_1s_1\xi_1, \rho_2\mathbf{x}_2s_2\xi_2, \rho_3\mathbf{x}_3s_3\xi_3, \rho_4\mathbf{x}_4s_4\xi_4)\\
&= -\frac{\lambda_2 h^3}{4!}1_{s_1=s_2=s_3=s_4}\\
&\quad\cdot\sum_{\sigma\in\mathbb{S}_4}\mathrm{sgn}(\sigma)1_{\substack{((\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}),(\rho_{\sigma(3)},\mathbf{x}_{\sigma(3)},\xi_{\sigma(3)}),(\rho_{\sigma(4)},\mathbf{x}_{\sigma(4)},\xi_{\sigma(4)}))\\=((1,r_L(\hat{\mathbf{x}}),1),(2,r_L(\hat{\mathbf{x}}),-1),(2,r_L(\hat{\mathbf{y}}),1),(1,r_L(\hat{\mathbf{y}}),-1))}},\\
&(\forall(\rho_j,\mathbf{x}_j,s_j,\xi_j)\in I\ (j=1,2,3,4)).
\end{aligned}$$

Thus,

$$\|V_4^{1,1}\|_{1,r,r'} = \|V_4^{1-3,0}\|_{1,r,r'} \leq \beta r'. \tag{3.79}$$

Also,

$$V_2^{1,1}(\rho_1\mathbf{x}_1s_1\xi_1, \rho_2\mathbf{x}_2s_2\xi_2)$$

$$= -\frac{\lambda_1 h}{2} 1_{s_1=s_2} \sum_{\sigma\in\mathbb{S}_2} \mathrm{sgn}(\sigma) 1_{\substack{((\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}))\\ =((1,r_L(\hat{\mathbf{x}}),1),(2,r_L(\hat{\mathbf{x}}),-1))}},$$

$$(\forall(\rho_j,\mathbf{x}_j,s_j,\xi_j)\in I\ (j=1,2)).$$

Thus,

$$\|V_2^{1,1}\|_{1,r,r'} \le \beta r'. \tag{3.80}$$

We can derive from the definition that

$$\begin{aligned} &V_2^{1-3,0}(\psi)\\ &= V_2^{1,1}(\psi)\\ &\quad + \left(\frac{1}{h}\right)^2 \sum_{\mathbf{X}\in I^2} \left(\begin{pmatrix} 4\\ 2\end{pmatrix} \left(\frac{1}{h}\right)^2 \sum_{\mathbf{Y}\in I^2} V_4^{1,1}(\mathbf{Y},\mathbf{X}) Tree(\{1\},\mathcal{C}_1)\psi^1_{\mathbf{Y}}\Big|_{\psi^1=0}\right)\psi_{\mathbf{X}}. \end{aligned}$$

By using (3.48), (3.79), (3.80) and $c_0\ge 1$ we have

$$\|V_2^{1-3,0}\|_{1,r,r'} \le \|V_2^{1,1}\|_{1,r,r'} + \begin{pmatrix} 4\\ 2\end{pmatrix} c_0 \|V_4^{1,1}\|_{1,r,r'} \le 7\beta c_0 r'. \tag{3.81}$$

It also follows from (3.48), (3.79), (3.80), $c_0\ge 1$ and the definition that

$$\|V_0^{1-3,0}\|_{1,r,r'} \le c_0 \|V_2^{1,1}\|_{1,r,r'} + c_0^2 \|V_4^{1,1}\|_{1,r,r'} \le 2\beta c_0^2 r'. \tag{3.82}$$

The inequalities (3.79), (3.81), (3.82) result in

$$\alpha^2 \|V_0^{1-3,0}\|_{1,r,r'} \le 2\beta c_0^2\alpha^2 r',\quad \sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{1-3,0}\|_{1,r,r'} \le 2^3\beta c_0^2 \alpha^4 r'.$$

Though we can see from the explicit characterization of the kernels, the statement of Lemma 3.1 ensures that $V_m^{1-3,0}: I^m\to\mathbb{C}$ $(m=2,4)$ satisfy (3.13). It is also clear from the definition that $\boldsymbol{\lambda}\mapsto V^{1-3,0}(\boldsymbol{\lambda})(\psi)$ is linear. Combined with these basic properties, the above inequalities and (3.78) imply that

$$V^{1-3,0} \in \mathcal{S}(2^{-9}c_0^{-2}\alpha^{-4}, 2^{-9}\beta^{-1}c_0^{-2}\alpha^{-4}). \tag{3.83}$$

Let us consider $V^{1-1-1,0,(n)}(\psi)$. Here we use Lemma 3.1. The lemma states that the anti-symmetric kernels of $V^{1-1-1,0,(n)}(\psi)$ satisfy (3.13). By definition, $\boldsymbol{\lambda} \mapsto V^{1-1-1,0,(n)}(\boldsymbol{\lambda})(\psi)$ is linear. Thus, $\sum_{n=2}^{\infty} V^{1-1-1,0,(n)}$ must satisfy these properties if it is convergent. Let us establish bound properties of the kernels. By applying (3.17) together with (3.48), (3.49), (3.57), (3.58), (3.79), (3.80) we observe that for any $n \in \mathbb{N}_{\geq 2}$, $m \in \{0, 2, \cdots, N\}$,

$$
\begin{aligned}
&\|V_m^{1-1-1,0,(n)}\|_{1,r,r'}\\
&\leq 2^{-2m} c_0^{-\frac{m}{2}} \prod_{j=1}^{n-1}\left(\sum_{b_j=1}^{2} \sum_{p_j \in \{2,4\}} 2^{3p_j} c_0^{\frac{p_j}{2}} \|V_{p_j}^{0-b_j,1}\|_{1,\infty,r}\right)\\
&\quad \cdot \sum_{p_n \in \{2,4\}} 2^{3p_n} c_0^{\frac{p_n}{2}} \|V_{p_n}^{1,1}\|_{1,r,r'} 1_{\sum_{j=1}^{n} p_j - 2(n-1) \geq m} 1_{\exists j (b_j=1)}\\
&\leq 2^{-2m} c_0^{-\frac{m}{2}} \sum_{l=1}^{n-1} \left(\begin{array}{c} n-1 \\ l \end{array}\right) (2^6 c_0 \|V_2^{0-1,1}\|_{1,\infty,r})^l (2^{12} c_0^2 \|V_4^{0-2,1}\|_{1,\infty,r})^{n-1-l}\\
&\quad \cdot \sum_{p_n \in \{2,4\}} 2^{3p_n} c_0^{\frac{p_n}{2}} \|V_{p_n}^{1,1}\|_{1,r,r'} 1_{2l+4(n-1-l)+p_n-2(n-1) \geq m}\\
&\leq 2^{-2m+13} c_0^{-\frac{m}{2}} \sum_{l=1}^{n-1} \left(\begin{array}{c} n-1 \\ l \end{array}\right) (2^6 c_0 r L^{-d})^l (2^{12} c_0^2 r)^{n-1-l} c_0^2 \beta r'\\
&\quad \cdot 1_{2(n-1-l)+4 \geq m}.
\end{aligned}
$$

Then, by (3.60), (3.78) and $\alpha \geq 2^3$,

(3.84)

$$
\begin{aligned}
&\|V_0^{1-1-1,0,(n)}\|_{1,r,r'}\\
&\leq 2^4 \sum_{l=1}^{n-1} \left(\begin{array}{c} n-1 \\ l \end{array}\right) (2^{-3}\alpha^{-4})^l (2^3 \alpha^{-4})^{n-1-l} L^{-d} \alpha^{-4} \leq (2^4 \alpha^{-4})^n L^{-d},
\end{aligned}
$$

(3.85)

$$
\begin{aligned}
&\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{1-1-1,0,(n)}\|_{1,r,r'}\\
&\leq 2^6 \sum_{l=1}^{n-1} \left(\begin{array}{c} n-1 \\ l \end{array}\right) (2^6 c_0 r L^{-d})^l (2^8 c_0^2 \alpha^2 r)^{n-1-l} c_0^2 \alpha^4 \beta r'\\
&\leq \sum_{l=1}^{n-1} \left(\begin{array}{c} n-1 \\ l \end{array}\right) (2^{-3}\alpha^{-4})^l (2^{-1}\alpha^{-2})^{n-1-l} L^{-d} \leq \alpha^{-2(n-1)} L^{-d}.
\end{aligned}
$$

Let us study properties of $V^{1-1-2,0}$. By Lemma 3.2 the anti-symmetric kernels of $V^{1-1-2,0,(n)}(\psi)$ satisfy (3.13). Thus, if $\sum_{n=2}^{\infty} V^{1-1-2,0,(n)}(\psi)$ converges, the anti-symmetric kernels of $V^{1-1-2,0}(\psi)$ must satisfy (3.13) as well. We can see from the definition that $\boldsymbol{\lambda} \mapsto V^{1-1-2,0,(n)}(\boldsymbol{\lambda})(\psi)$ is linear and thus so must be $V^{1-1-2,0}(\psi)$ if it converges. Let us find upper bounds on the norms of the kernels of $V^{1-1-2,0,(n)}(\psi)$. By substituting (3.48), (3.49), (3.58), (3.59), (3.79), (3.80) into (3.27) we have that for any $m \in \{0, 2, \cdots, N\}$, $n \in \mathbb{N}_{\geq 2}$,

$$\begin{aligned}\|V_m^{1-1-2,0,(n)}\|_{1,r,r'} &\leq 2^{-2m} c_0^{-\frac{m}{2}} L^{-d} (2^{12} c_0^2 r)^{n-1} \sum_{p\in\{2,4\}} 2^{3p} c_0^{\frac{p}{2}} \beta r' 1_{2n-4+p\geq m} \\ &\leq 2^{-2m+1} c_0^{-\frac{m}{2}} L^{-d} (2^{12} c_0^2 r)^{n-1} (2^{12} c_0^2 \beta r') 1_{2n\geq m}.\end{aligned}$$

Thus, by (3.60), (3.78) and the assumption $\alpha \geq 2^3$,

$$\|V_0^{1-1-2,0,(n)}\|_{1,r,r'} \leq 2L^{-d} (2^3 \alpha^{-4})^n, \tag{3.86}$$

$$\begin{aligned}\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{1-1-2,0,(n)}\|_{1,r,r'} &\leq 2^2 (2^8 c_0^2 \alpha^2 r)^{n-1} (2^8 c_0^2 \alpha^2 \beta r') L^{-d} \\ &\leq 2^2 (2^{-1} \alpha^{-2})^n L^{-d}.\end{aligned} \tag{3.87}$$

It follows from (3.84), (3.85), (3.86), (3.87) and $\alpha \geq 2^3$ that

$$\alpha^2 \sum_{n=2}^{\infty} \sum_{j=1}^{2} \|V_0^{1-1-j,0,(n)}\|_{1,r,r'} \leq L^{-d},$$

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \sum_{n=2}^{\infty} \sum_{j=1}^{2} \|V_m^{1-1-j,0,(n)}\|_{1,r,r'} \leq L^{-d}.$$

These uniform convergence properties imply the well-definedness of $V^{1-1,0}$ and its regularity with $(u, \boldsymbol{\lambda})$. Therefore, $V^{1-1,0} \in \mathcal{Q}'(r, r')$.

Next let us consider $V^{1-2,0}$. An application of Lemma 3.3 ensures that there exist bi-anti-symmetric functions $V_{a,b}^{1-2,0,(n)} : I^a \times I^b \to \mathbb{C}$ ($a, b \in \{2, 4, \cdots, N\}$) satisfying (3.13), (3.23) such that

$$V^{1-2,0,(n)}(\psi) = \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{a+b} \sum_{\substack{\mathbf{X}\in I^a \\ \mathbf{Y}\in I^b}} V_{a,b}^{1-2,0,(n)}(\mathbf{X}, \mathbf{Y}) \psi_{\mathbf{X}} \psi_{\mathbf{Y}}.$$

By definition, $\boldsymbol{\lambda} \mapsto V^{1-2,0,(n)}(u,\boldsymbol{\lambda})(\psi)$ is linear for any $u \in \overline{D(r)}$. Moreover, by construction, $(u,\boldsymbol{\lambda}) \mapsto V_{a,b}^{1-2,0,(n)}(u,\boldsymbol{\lambda})(\mathbf{X},\mathbf{Y})$ is continuous in $\overline{D(r)} \times \overline{D(r')}^2$ and analytic in $D(r) \times D(r')^2$, $(\forall \mathbf{X} \in I^a, \mathbf{Y} \in I^b)$. Let us establish bound properties of the bi-anti-symmetric kernels. By combining (3.48), (3.49), (3.58), (3.69), (3.79), (3.80) with (3.37) and using $c_0 \geq 1$ we observe that for any $a, b \in \{2, 4, \cdots, N\}$, $n \in \mathbb{N}_{\geq 2}$,

$$\begin{aligned}
&\|V_{a,b}^{1-2,0,(n)}\|_{1,r,r'}\\
&\leq \frac{1}{(n-1)!}\\
&\quad \cdot \sum_{m=0}^{n-1} \begin{pmatrix} n-1 \\ m \end{pmatrix} (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad \cdot 2^{-2a-2b} c_0^{-\frac{1}{2}(a+b)} (2^{12}c_0^2 r)^{n-1} (2^6 c_0 \beta r' + 2^{12} c_0^2 \beta r') 1_{2+2m\geq a} 1_{2n-2m\geq b}\\
&\leq \frac{1}{(n-1)!}\\
&\quad \cdot \sum_{m=0}^{n-1} \begin{pmatrix} n-1 \\ m \end{pmatrix} (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad \cdot 2^{-2a-2b+13} c_0^{-\frac{1}{2}(a+b)} (2^{12}c_0^2 r)^{n-1} c_0^2 \beta r' 1_{2+2m\geq a} 1_{2n-2m\geq b}.
\end{aligned}$$

Thus, by (3.60), (3.71), (3.78) and $\alpha \geq 2^3$,

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} c_0^{\frac{1}{2}(a+b)} \alpha^{a+b} \|V_{a,b}^{1-2,0,(n)}\|_{1,r,r'} \leq 2^3 n (2^{-2}\alpha)^2 (2^{-1}\alpha^{-2})^n \leq \alpha^{2-2n},$$

or

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} c_0^{\frac{1}{2}(a+b)} \alpha^{a+b} \sum_{n=2}^{\infty} \|V_{a,b}^{1-2,0,(n)}\|_{1,r,r'} \leq 2\alpha^{-2} \leq 1.$$

This means that $V^{1-2,0} \in \mathcal{R}'(r,r')$.

It remains to analyze $V^{2,0}$. By Lemma 3.1 the anti-symmetric kernels of $V^{2,0,(n)}(\psi)$ $(n \in \mathbb{N}_{\geq 2})$ satisfy (3.13). The constraint $1_{\sum_{j=1}^{n} b_j \geq 2}$ implies that $V^{2,0,(n)}(\psi)$ is of degree at least 2 with λ_1, λ_2. Thus,

$$V^{2,0,(n)}(u,\mathbf{0})(\psi) = \frac{\partial}{\partial \lambda_j} V^{2,0,(n)}(u,\mathbf{0})(\psi) = 0, \quad (\forall u \in D(r),\ j \in \{1,2\}).$$

Let us prove uniform bound properties of the anti-symmetric kernels. Here we need to measure $V^{1,1}$ with the $\|\cdot\|_{1,\infty}$-norm as well. We can see from the definition that for $m \in \{2,4\}$

$$\sup_{\lambda \in \overline{D(\beta r')}} \|V_m^{1,1}(\lambda)\|_{1,\infty} \le \beta r'. \tag{3.88}$$

By definition,

$$V^{2,0,(n)}(\psi) = \frac{1}{n!}\sum_{l=2}^{n}\begin{pmatrix} n \\ l \end{pmatrix} Tree(\{1,2,\cdots,n\},\mathcal{C}_1)\prod_{j=1}^{l} V^{1,1}(\psi^j+\psi) \cdot \prod_{k=l+1}^{n} V^{0,1}(\psi^k+\psi)\Big|_{\substack{\psi^j=0 \\ (\forall j\in\{1,2,\cdots,n\})}}.$$

Then, it follows from (3.17), (3.48), (3.49), (3.57), (3.58), (3.79), (3.80), (3.88) and $c_0 \ge 1$ that for any $m \in \{0,2,\cdots,N\}$, $n \in \mathbb{N}_{\ge 2}$,

$$\begin{aligned}
&\|V_m^{2,0,(n)}\|_{1,r,\min\{1,\beta\}r'} \\
&\le \frac{(n-2)!}{n!}\sum_{l=2}^{n}\begin{pmatrix} n \\ l \end{pmatrix} c_0^{-\frac{m}{2}}2^{-2m}\sum_{p_1\in\{2,4\}} 2^{3p_1}c_0^{\frac{p_1}{2}}\|V_{p_1}^{1,1}\|_{1,r,r'} \\
&\quad \cdot \prod_{j=2}^{l}\left(\sum_{p_j\in\{2,4\}} 2^{3p_j}c_0^{\frac{p_j}{2}}\sup_{\lambda\in\overline{D(\beta r')}}\|V_{p_j}^{1,1}(\lambda)\|_{1,\infty}\right) \\
&\quad \cdot (2^6c_0\|V_2^{0,1}\|_{1,\infty,r} + 2^{12}c_0^2\|V_4^{0,1}\|_{1,\infty,r})^{n-l} \\
&\quad \cdot 1_{\sum_{j=1}^{l} p_j + 4(n-l) - 2(n-1) \ge m} \\
&\le \frac{(n-2)!}{n!}\sum_{l=2}^{n}\begin{pmatrix} n \\ l \end{pmatrix} c_0^{-\frac{m}{2}}2^{-2m}(2^{13}c_0^2\beta r')^l(2^{13}c_0^2 r)^{n-l}1_{2n+2\ge m}.
\end{aligned}$$

Moreover by (3.60), (3.78),

$$\begin{aligned}
&\|V_m^{2,0,(n)}\|_{1,r,\min\{1,\beta\}r'} \\
&\le \frac{(n-2)!}{n!}\sum_{l=2}^{n}\begin{pmatrix} n \\ l \end{pmatrix} c_0^{-\frac{m}{2}}2^{-2m}(2^4\alpha^{-4})^l(2^4\alpha^{-4})^{n-l}1_{2n+2\ge m} \\
&\le c_0^{-\frac{m}{2}}2^{-2m}(2^5\alpha^{-4})^n 1_{2n+2\ge m}.
\end{aligned}$$

Thus by $\alpha \ge 2^3$,

$$\alpha^2 \sum_{n=2}^{\infty} \|V_0^{2,0,(n)}\|_{1,r,\min\{1,\beta\}r'} \le 2\alpha^2(2^5\alpha^{-4})^2 \le 1,$$
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|V_m^{2,0,(n)}\|_{1,r,\min\{1,\beta\}r'} \le 2(2^{-2}\alpha)^2(2\alpha^{-2})^n,$$
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \sum_{n=2}^{\infty} \|V_m^{2,0,(n)}\|_{1,r,\min\{1,\beta\}r'} \le 2^2(2^{-2}\alpha)^2(2\alpha^{-2})^2 \le 1.$$

This implies that $V^{2,0} \in \mathcal{W}(r, \min\{1,\beta\}r')$. The proof is complete. □

3.6. The second integration

Here we establish bound properties of the output of the single-scale integration with the covariance $\mathcal{C}_0$. The input to the integration is the Grassmann polynomials $V^{0-j,0}(\psi)$ $(j = 1,2)$, $V^{1-k,0}(\psi)$ $(k = 1,2,3)$, $V^{2,0}(\psi)$ whose properties were studied in Lemma 3.4 and Lemma 3.6. In fact the object we are going to analyze is an analytic continuation of

$$\log\left(\int e^{\sum_{j=1}^{2} V^{0-j,0}(\psi)+\sum_{k=1}^{3} V^{1-k,0}(\psi)+V^{2,0}(\psi)} d\mu_{\mathcal{C}_0}(\psi)\right),$$

which is also an analytic continuation of

$$\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)} d\mu_{\mathcal{C}_0+\mathcal{C}_1}(\psi)\right).$$

Set

$$r := 2^{-9}c_0^{-2}\alpha^{-4}, \quad r' := \beta^{-1}r, \quad r'' := \min\{1,\beta\}r'.$$

We define V^{end}, $V^{1-3,end} \in \mathrm{Map}(\overline{D(r)} \times \overline{D(r'')}^2, \mathbb{C})$ by

$$\begin{aligned} &V^{end,(n)} \\ &:= \frac{1}{n!} Tree(\{1,2,\cdots,n\}, \mathcal{C}_0) \\ &\quad \cdot \prod_{j=1}^{n} \left(\sum_{m=1}^{2} V^{0-m,0}(\psi^j) + \sum_{k=1}^{3} V^{1-k,0}(\psi^j) + V^{2,0}(\psi^j) \right) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\cdots,n\})}}. \end{aligned}$$

$$V^{end} := \sum_{n=1}^{\infty} V^{end,(n)},$$
$$V^{1-3,end} := Tree(\{1\}, \mathcal{C}_0) V^{1-3,0}(\psi^1)\Big|_{\psi^1=0}$$

by assuming its convergence. Our purpose here is to prove the following lemma.

LEMMA 3.8. *Assume that $h \geq 1$. Then, the following statements hold for any $\alpha \in [2^3, \infty)$, $L \in \mathbb{N}$ with $L^d \geq 2^2 D_c$.*

- V^{end} *is continuous in*
$$\overline{D(2^{-9}c_0^{-2}\alpha^{-4})} \times \overline{D(2^{-11}L^{-d}h^{-1}\beta^{-1}\min\{1,\beta\}c_0^{-2}\alpha^{-4})}^2,$$
analytic in
$$D(2^{-9}c_0^{-2}\alpha^{-4}) \times D(2^{-11}L^{-d}h^{-1}\beta^{-1}\min\{1,\beta\}c_0^{-2}\alpha^{-4})^2.$$

-
$$\frac{h}{N}|V^{end}(u,\mathbf{0})| \leq 2^8\alpha^{-2}L^{-d}, \quad (\forall u \in \overline{D(2^{-9}c_0^{-2}\alpha^{-4})}). \tag{3.89}$$

-
$$\left|\frac{\partial}{\partial \lambda_j}V^{end}(u,\mathbf{0}) - \frac{\partial}{\partial \lambda_j}V^{1-3,end}(u,\mathbf{0})\right| \leq 2^{10}\beta c_0^2\alpha^4(1+2D_c)L^{-d}, \tag{3.90}$$
$$(\forall u \in D(2^{-9}c_0^{-2}\alpha^{-4}),\ j \in \{1,2\}).$$

PROOF. First let us observe that $V^{0-2,0}(\psi)$, $V^{1-2,0}(\psi)$ do not contribute to the value of the integration. With the aim of proving this, let us take $f(\psi) \in \bigwedge \mathcal{V}$, $p, q \in \{2, 4, \cdots, N\}$, $\mathbf{X} \in (I^0)^p$, $\mathbf{Y} \in I^q$. If we define the function $g : [0,\beta)_h^p \to \mathbb{C}$ by

$$g(s_1, s_2, \cdots, s_p) := \int \prod_{j=1}^{p} (\psi_{X_j+s_j})\psi_{\mathbf{Y}} f(\psi) d\mu_{\mathcal{C}_0}(\psi),$$

the property (3.47) ensures that the function g satisfies (3.22). If we expand $\int V^{j-2,0}(\psi)f(\psi)d\mu_{\mathcal{C}_0}(\psi)$ $(j=0,1)$, we see that each kernel of $V^{j-2,0}$ is multiplied by a function of the same form as g and is integrated with respect to the time-variables. Thus, the property (3.23) of the bi-anti-symmetric kernels of $V^{j-2,0}(\psi)$ $(j=0,1)$ implies that

$$\int V^{j-2,0}(\psi)f(\psi)d\mu_{\mathcal{C}_0}(\psi)=0, \quad (j=0,1).$$

Arbitrariness of $f(\psi)$ implies that for any $z\in\mathbb{C}$

$$\begin{aligned}&\int e^{z(\sum_{k=1}^{2}V^{0-k,0}(\psi)+\sum_{k=1}^{3}V^{1-k,0}(\psi)+V^{2,0}(\psi))}d\mu_{\mathcal{C}_0}(\psi)\\&=\int e^{z(V^{0-1,0}(\psi)+V^{1-1,0}(\psi)+V^{1-3,0}(\psi)+V^{2,0}(\psi))}d\mu_{\mathcal{C}_0}(\psi).\end{aligned}$$

Therefore,

$$\begin{aligned}&V^{end,(n)}\\&=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_0)\\&\quad\cdot\prod_{j=1}^{n}(V^{0-1,0}(\psi^j)+V^{1-1,0}(\psi^j)+V^{1-3,0}(\psi^j)+V^{2,0}(\psi^j))\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.\end{aligned}$$

Note that

$$\begin{aligned}&(3.91)\qquad \|V_m^{a,0}(u,\varepsilon\boldsymbol{\lambda})\|_1\le\varepsilon\|V_m^{a,0}\|_{1,r,r''},\\&(3.92)\qquad \|V_m^{a,0}(u,\varepsilon\boldsymbol{\lambda})\|_{1,\infty}\le h\varepsilon\|V_m^{a,0}\|_{1,r,r''},\\&\qquad\qquad (\forall u\in\overline{D(r)},\ \boldsymbol{\lambda}\in\overline{D(r'')}^2,\ \varepsilon\in[0,1/2],\ a\in\{1-1,1-3,2\}).\end{aligned}$$

For $a=1-1,1-3$, (3.91) and (3.92) are clear. For $a=2$ we can use the following equality based on Cauchy's integral formula to derive (3.91), (3.92).

$$\begin{aligned}V_m^{2,0}(u,\varepsilon\boldsymbol{\lambda})&=\sum_{n=2}^{\infty}\frac{1}{2\pi i}\oint_{|z|=\delta}dz\frac{V_m^{2,0}(u,z\boldsymbol{\lambda})}{z^{n+1}}\varepsilon^n\\&=\frac{1}{2\pi i}\oint_{|z|=\delta}dzV_m^{2,0}(u,z\boldsymbol{\lambda})\frac{\varepsilon^2}{z^2(z-\varepsilon)},\end{aligned}$$

$$(\forall u \in \overline{D(r)},\ \boldsymbol{\lambda} \in \overline{D(r'')}^2,\ \varepsilon \in [0,1/2],\ \delta \in (1/2,1)).$$

In the following we let $\varepsilon = \frac{1}{3}L^{-d}h^{-1}$, $\alpha \geq 2^3$. The assumption $h \geq 1$ implies that $\varepsilon \in (0,1/2]$. Take any $u \in \overline{D(r)}$, $\boldsymbol{\lambda} \in \overline{D(r'')}^2$. By (3.15), (3.48), (3.52), (3.73), (3.75), (3.76), (3.91) we have that

$$\begin{aligned}
&|V^{end,(1)}(u,\varepsilon\boldsymbol{\lambda})|\\
&\leq \frac{N}{h}L^{-d}\alpha^{-2} + 3\varepsilon\alpha^{-2}\\
&\quad + \sum_{m=2}^{N} c_0^{\frac{m}{2}}\left(\frac{N}{h}\|V_m^{0-1,0}\|_{1,\infty,r} + \varepsilon \sum_{a\in\{1-1,1-3,2\}}\|V_m^{a,0}\|_{1,r,r''}\right)\\
&\leq 2\left(\frac{N}{h}L^{-d} + 3\varepsilon\right)\alpha^{-2} = 2(N+1)h^{-1}L^{-d}\alpha^{-2}.
\end{aligned}$$

Also by (3.17), (3.48), (3.51), (3.52), (3.73), (3.75), (3.76), (3.91), (3.92) and $\alpha \geq 2^3$, for $n \in \mathbb{N}_{\geq 2}$

$$\begin{aligned}
&|V^{end,(n)}(u,\varepsilon\boldsymbol{\lambda})|\\
&\leq D_c^{n-1}\left(\sum_{p=2}^{N} 2^{3p}c_0^{\frac{p}{2}}\left(\frac{N}{h}\|V_p^{0-1,0}\|_{1,\infty,r} + \varepsilon \sum_{a\in\{1-1,1-3,2\}}\|V_p^{a,0}\|_{1,r,r''}\right)\right)\\
&\quad \cdot \left(\sum_{q=2}^{N} 2^{3q}c_0^{\frac{q}{2}}\left(\|V_q^{0-1,0}\|_{1,\infty,r} + h\varepsilon \sum_{a\in\{1-1,1-3,2\}}\|V_q^{a,0}\|_{1,r,r''}\right)\right)^{n-1}\\
&\leq D_c^{n-1}\left(\frac{N}{h}L^{-d} + 3\varepsilon\right)(L^{-d} + 3h\varepsilon)^{n-1}(2^6\alpha^{-2})^n\\
&= (N+1)h^{-1}L^{-d}(2D_cL^{-d})^{n-1}(2^6\alpha^{-2})^n.
\end{aligned}$$

Thus, if $2D_cL^{-d} \leq 1/2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sup_{\substack{u\in\overline{D(r)}\\ \boldsymbol{\lambda}\in\overline{D(\varepsilon r'')}^2}} |V^{end,(n)}(u,\boldsymbol{\lambda})|\\
&\leq 2(N+1)h^{-1}L^{-d}\alpha^{-2} + 2^{12}(N+1)h^{-1}L^{-d}\alpha^{-4}\\
&\leq 2^7(N+1)h^{-1}L^{-d}\alpha^{-2} \leq 2^8Nh^{-1}L^{-d}\alpha^{-2}.
\end{aligned}$$

This estimation implies that V^{end} is continuous in $\overline{D(r)} \times \overline{D(\varepsilon r'')}^2$, analytic in $D(r) \times D(\varepsilon r'')^2$ and

$$\frac{h}{N}|V^{end}(u,\mathbf{0})| \le 2^8 L^{-d}\alpha^{-2}, \quad (\forall u \in \overline{D(r)}).$$

Moreover, observe that for any $u \in D(r)$, $j \in \{1,2\}$,

$$\begin{aligned}
&\frac{\partial}{\partial \lambda_j} V^{end}(u,\mathbf{0}) \\
&= \frac{1}{r'} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} Tree(\{1,2,\cdots,n\},\mathcal{C}_0) \\
&\quad \cdot \sum_{a \in \{1-1,1-3\}} V^{a,0}(u,r'\mathbf{e}_j)(\psi^1) \prod_{k=2}^{n} V^{0-1,0}(u)(\psi^k)\Big|_{\substack{\psi^k=0 \\ (\forall k \in \{1,2,\cdots,n\})}}.
\end{aligned}$$

Thus, by (3.15), (3.17), (3.48), (3.51), (3.52), (3.73), (3.75) and the assumptions $\alpha \ge 2^3$, $2^2 D_c L^{-d} \le 1$,

$$\begin{aligned}
&\left|\frac{\partial}{\partial \lambda_j} V^{end}(u,\mathbf{0}) - \frac{\partial}{\partial \lambda_j} V^{1-3,end}(u,\mathbf{0})\right| \\
&\le \frac{1}{r'} \left| Tree(\{1\},\mathcal{C}_0) V^{1-1,0}(u,r'\mathbf{e}_j)(\psi^1)\Big|_{\psi^1=0} \right| \\
&\quad + \frac{1}{r'} \Bigg| \sum_{n=2}^{\infty} \frac{1}{(n-1)!} Tree(\{1,2,\cdots,n\},\mathcal{C}_0) \\
&\qquad \cdot \sum_{a \in \{1-1,1-3\}} V^{a,0}(u,r'\mathbf{e}_j)(\psi^1) \prod_{k=2}^{n} V^{0-1,0}(u)(\psi^k)\Big|_{\substack{\psi^k=0 \\ (\forall k \in \{1,2,\cdots,n\})}} \Bigg| \\
&\le \frac{1}{r'} \sum_{m=0}^{N} c_0^{\frac{m}{2}} \|V_m^{1-1,0}\|_{1,r,r'} \\
&\quad + \frac{1}{r'} \sum_{n=2}^{\infty} D_c^{n-1} \sum_{m=2}^{N} 2^{3m} c_0^{\frac{m}{2}} \sum_{a \in \{1-1,1-3\}} \|V_m^{a,0}\|_{1,r,r'} \\
&\qquad \cdot \left(\sum_{p=2}^{N} 2^{3p} c_0^{\frac{p}{2}} \|V_p^{0-1,0}\|_{1,\infty,r} \right)^{n-1}
\end{aligned}$$

$$\le \frac{2}{r'}L^{-d} + \frac{2}{r'}\sum_{n=2}^{\infty}(D_cL^{-d})^{n-1} \le \frac{2}{r'}(1+2D_c)L^{-d}.$$

We can see from above that the claims of the lemma have been proved. □

Remark 3.9. There is no essential necessity to complete the generalized double-scale integration by explicitly estimating the combinatorial factors as in Lemma 3.4, Lemma 3.6, Lemma 3.8. We did so only to feature the explicitness of our construction. In fact the following statements, which are less explicit but are sufficient to achieve the main goal of this paper, can be proved by shorter arguments. There exists a positive constant c independent of any parameter such that if $h \ge 1$, $\alpha \ge c$, $L^d \ge cD_c$,

- $$\begin{aligned}
&V^{0-1,0} \in \mathcal{Q}(c^{-1}c_0^{-2}\alpha^{-4}), \quad V^{0-2,0} \in \mathcal{R}(c^{-1}c_0^{-2}\alpha^{-4}),\\
&V^{1-1,0} \in \mathcal{Q}'(c^{-1}c_0^{-2}\alpha^{-4}, c^{-1}\beta^{-1}c_0^{-2}\alpha^{-4}),\\
&V^{1-2,0} \in \mathcal{R}'(c^{-1}c_0^{-2}\alpha^{-4}, c^{-1}\beta^{-1}c_0^{-2}\alpha^{-4}),\\
&V^{1-3,0} \in \mathcal{S}(c^{-1}c_0^{-2}\alpha^{-4}, c^{-1}\beta^{-1}c_0^{-2}\alpha^{-4}),\\
&V^{2,0} \in \mathcal{W}(c^{-1}c_0^{-2}\alpha^{-4}, c^{-1}\beta^{-1}\min\{1,\beta\}c_0^{-2}\alpha^{-4}).
\end{aligned}$$

- V^{end} is continuous in

  $$\overline{D(c^{-1}c_0^{-2}\alpha^{-4})} \times \overline{D(c^{-1}L^{-d}h^{-1}\beta^{-1}\min\{1,\beta\}c_0^{-2}\alpha^{-4})}^2$$

  and analytic in

  $$D(c^{-1}c_0^{-2}\alpha^{-4}) \times D(c^{-1}L^{-d}h^{-1}\beta^{-1}\min\{1,\beta\}c_0^{-2}\alpha^{-4})^2.$$

- $$\frac{h}{N}|V^{end}(u,\mathbf{0})| \le c\alpha^{-2}L^{-d}, \quad (\forall u \in \overline{D(c^{-1}c_0^{-2}\alpha^{-4})}).$$

- $$\left|\frac{\partial}{\partial\lambda_j}V^{end}(u,\mathbf{0}) - \frac{\partial}{\partial\lambda_j}V^{1-3,end}(u,\mathbf{0})\right| \le c\beta c_0^2\alpha^4(1+D_c)L^{-d},$$
  $$(\forall u \in D(c^{-1}c_0^{-2}\alpha^{-4}),\ j \in \{1,2\}).$$

Remark 3.10. In practice D_c will be the biggest parameter as θ approaches to $2\pi/\beta$. The essential benefit of Lemma 3.8 is that the parameter D_c does not affect the domain of analyticity with the extended coupling constant u. This is because the heavy contribution from D_c was absorbed by the inverse of the volume factor.

4. Proof of the Theorem

In this section we will prove Theorem 1.3. In view of the formulation (2.25), (2.26) we must know to what the Grassmann Gaussian integrals converge inside the normal Gaussian integral as $h \to \infty$, $L \to \infty$. One part of this question will be answered by realizing the general results of the double-scale integration prepared in the previous section. To do so, we need to confirm that the actual covariances satisfy the properties required in the previous section. It follows from the double-scale integration, especially from the bound (3.89) that the spatial mean of logarithm of the Grassmann Gaussian integral converges to zero in the infinite-volume limit. However, it will turn out necessary to make sure that the Grassmann Gaussian integral itself, not the spatial mean, converges in the time-continuum, infinite-volume limit. To prove this, which cannot be deduced from the results of the previous section, we will study detailed convergent properties of each term of the perturbative expansion of logarithm of the Grassmann Gaussian integral. After these preparations we will move on to the proof of Theorem 1.3. To shorten formulas, we set

$$\Theta := \left|\frac{\theta}{2} - \frac{\pi}{\beta}\right|$$

throughout this section.

4.1. Application of Pedra-Salmhofer's determinant bound

Here we derive a uniform bound on the determinant of $C(\phi)$ by applying Pedra-Salmhofer's determinant bound ([17]). We especially use the general theorem [17, Theorem 1.3] which is a generalization of Gram's inequality to covariances with time-discontinuity typically caused by time-ordering. We restrict our attention to what is sufficient to solve the current problem. The following proposition, which is a specific version of [17, Theorem 1.3], is in fact sufficient.

PROPOSITION 4.1. *Let* $C : (\{1,2\}\times\Gamma\times[0,\beta))^2 \to \mathbb{C}$. *Assume that there is a complex Hilbert space* $\mathcal{H}$ *and* $f_j^{\geq}$, $g_j^{\geq}$, $f_j^{<}$, $g_j^{<} \in \mathrm{Map}(\{1,2\}\times\Gamma\times\mathbb{R},\mathcal{H})$ $(j = 1,2)$ *such that*

(4.1)

$$\begin{aligned}&C(\rho\mathbf{x}s,\eta\mathbf{y}t)\\&= 1_{s\geq t}\sum_{j\in\{1,2\}}\langle f_j^{\geq}(\rho\mathbf{x}s), g_j^{\geq}(\eta\mathbf{y}t)\rangle_{\mathcal{H}} + 1_{s<t}\sum_{j\in\{1,2\}}\langle f_j^{<}(\rho\mathbf{x}s), g_j^{<}(\eta\mathbf{y}t)\rangle_{\mathcal{H}},\\&(\forall(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in\{1,2\}\times\Gamma\times[0,\beta)),\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ *is the inner product of* $\mathcal{H}$. *Moreover, assume that there exists* $D \in \mathbb{R}_{>0}$ *such that*

$$\begin{aligned}&\|f_j^{\geq}(X)\|_{\mathcal{H}},\ \|g_j^{\geq}(X)\|_{\mathcal{H}},\ \|f_j^{<}(X)\|_{\mathcal{H}},\ \|g_j^{<}(X)\|_{\mathcal{H}} \leq D,\\&(\forall X\in\{1,2\}\times\Gamma\times\mathbb{R},\ j\in\{1,2\})\end{aligned}$$

and the maps $s \mapsto f_j^{\geq}(\rho\mathbf{x}s)$, $s \mapsto g_j^{\geq}(\rho\mathbf{x}s)$, $s \mapsto f_j^{<}(\rho\mathbf{x}s)$, $s \mapsto g_j^{<}(\rho\mathbf{x}s)$ $(j = 1,2)$ *are continuous in* $\mathbb{R}$ *for any* $\rho \in \{1,2\}$, $\mathbf{x} \in \Gamma$. *Then,*

$$\begin{aligned}&|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}C(X_i,Y_j))_{1\leq i,j\leq n}| \leq (4D)^{2n},\\&(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\leq 1,\\&\quad X_i,Y_i\in\{1,2\}\times\Gamma\times[0,\beta)\ (i=1,2,\cdots,n)).\end{aligned}$$

Proposition 4.1 is a direct implication of [17, Theorem 1.3]. For readers' convenience we provide a proof for this proposition in Appendix A. In fact we added the continuity condition of $f_j^{\geq}$, $g_j^{\geq}$, $f_j^{<}$, $g_j^{<}$ with the time variable, which is not assumed in the original [17, Theorem 1.3], to shorten the proof. By applying this proposition we obtain the following.

PROPOSITION 4.2.

(4.2)

$$\begin{aligned}&|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}C(\phi)(X_i,Y_j))_{1\leq i,j\leq n}|\\&\leq\left(\frac{2^4}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\left(1+2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}}+e^{-2\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}\right)^n,\end{aligned}$$

$$(\forall m, n \in \mathbb{N},\ \mathbf{u}_i, \mathbf{v}_i \in \mathbb{C}^m \ \textit{with}\ \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{v}_i\|_{\mathbb{C}^m} \le 1,\\ X_i, Y_i \in \{1,2\} \times \Gamma \times [0,\beta)\ (i = 1, 2, \cdots, n),\ \phi \in \mathbb{C}).$$

Remark 4.3. In the next subsection we will derive a ϕ-independent upper bound on

$$\frac{1}{L^d} \sum_{\mathbf{k} \in \Gamma^*} \left(1 + 2\cos\left(\frac{\beta\theta}{2}\right) e^{-\beta\sqrt{e(\mathbf{k})^2 + |\phi|^2}} + e^{-2\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}.$$

See (4.21).

Remark 4.4. We need to find a representation of the form (4.1). Such a representation was constructed for one-band models with a real-valued dispersion relation in [17, Subsection 4.1]. It is straightforward to modify the construction of [17, Subsection 4.1] to fit in our 2-band model with the complex-valued dispersion relation. We should also mention that an extension of the construction of [17, Subsection 4.1] to one-band models with a complex-valued dispersion relation was reported in [10, Subsection V.A]. Though it is close to both [17, Subsection 4.1] and [10, Subsection V.A], we will provide a concrete representation of the form (4.1) for our 2-band model for completeness of the paper.

Proof of Proposition 4.2. Define the functions $e_j : \Gamma^* \to \mathbb{C}$ ($j = 1, 2$) by $e_j(\mathbf{k}) := i\frac{\theta}{2} + (-1)^{1_{j=2}} e(\phi)(\mathbf{k})$, where $e(\phi)(\cdot)$ is the function defined in (2.20). Since Γ^* is the finite set, for any sufficiently small $\varepsilon \in \mathbb{R}_{>0}$, $e_j(\mathbf{k}) + \varepsilon \neq 0$ ($\forall \mathbf{k} \in \Gamma^*$). Set $e_{j,\varepsilon}(\mathbf{k}) := e_j(\mathbf{k}) + \varepsilon$ and

$$\begin{aligned}(4.3)\quad & C_\varepsilon(\rho\mathbf{x}s, \eta\mathbf{y}t)\\ &:= \frac{1}{L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{j \in \{1,2\}} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y}\rangle} U(\phi)(\mathbf{k})(\rho, j) U(\phi)(\mathbf{k})^*(j, \eta)\\ &\qquad \cdot e^{(s-t)e_{j,\varepsilon}(\mathbf{k})} \left(\frac{1_{s \ge t}}{1 + e^{\beta e_{j,\varepsilon}(\mathbf{k})}} - \frac{1_{s<t}}{1 + e^{-\beta e_{j,\varepsilon}(\mathbf{k})}}\right),\end{aligned}$$

where $U(\phi)(\mathbf{k})$ is the 2×2 matrix defined in (2.19). Let us find a determinant bound of C_ε and send $\varepsilon \searrow 0$ afterward. We can see from (2.24) that $\lim_{\varepsilon \searrow 0} C_\varepsilon(\mathbf{X}) = C(\phi)(\mathbf{X})$ ($\forall \mathbf{X} \in I_0^2$).

Remark that $L^2(\Gamma^*\times\mathbb{R})$ is the Hilbert space whose inner product $\langle\cdot,\cdot\rangle_{L^2(\Gamma^*\times\mathbb{R})}$ is defined by

$$\langle f,g\rangle_{L^2(\Gamma^*\times\mathbb{R})}:=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\int_{\mathbb{R}}dv\overline{f(\mathbf{k},v)}g(\mathbf{k},v).$$

For $(\rho,\mathbf{x},s)\in\{1,2\}\times\Gamma\times\mathbb{R}$, $j\in\{1,2\}$, $a\in\{1,-1\}$ we define $f^{j,a}_{\rho\mathbf{x}s}$, $g^{j,a}_{\rho\mathbf{x}s}\in L^2(\Gamma^*\times\mathbb{R})$ by

$$\begin{aligned}f^{j,a}_{\rho\mathbf{x}s}(\mathbf{k},v):=&1_{a\operatorname{Re}e_{j,\varepsilon}(\mathbf{k})>0}\overline{U(\phi)(\mathbf{k})(\rho,j)}e^{-i\langle\mathbf{k},\mathbf{x}\rangle-is(a\operatorname{Im}e_{j,\varepsilon}(\mathbf{k})-v)}\\&\cdot\frac{1+e^{-\beta a e_{j,\varepsilon}(\mathbf{k})}}{|1+e^{-\beta a e_{j,\varepsilon}(\mathbf{k})}|^{\frac{3}{2}}}\sqrt{\frac{|\operatorname{Re}e_{j,\varepsilon}(\mathbf{k})|}{\pi}}\frac{1}{iv+\operatorname{Re}e_{j,\varepsilon}(\mathbf{k})},\\g^{j,a}_{\rho\mathbf{x}s}(\mathbf{k},v):=&1_{a\operatorname{Re}e_{j,\varepsilon}(\mathbf{k})>0}\overline{U(\phi)(\mathbf{k})(\rho,j)}e^{-i\langle\mathbf{k},\mathbf{x}\rangle-is(a\operatorname{Im}e_{j,\varepsilon}(\mathbf{k})-v)}\\&\cdot\frac{1}{|1+e^{-\beta a e_{j,\varepsilon}(\mathbf{k})}|^{\frac{1}{2}}}\sqrt{\frac{|\operatorname{Re}e_{j,\varepsilon}(\mathbf{k})|}{\pi}}\frac{1}{iv+\operatorname{Re}e_{j,\varepsilon}(\mathbf{k})}.\end{aligned}$$

Then, let us define the maps $f_j^{\ge}$, $g_j^{\ge}$, $f_j^{<}$, $g_j^{<}\in\mathrm{Map}(\{1,2\}\times\Gamma\times\mathbb{R},L^2(\Gamma^*\times\mathbb{R}))$ $(j=1,2)$ by

$$\begin{aligned}&f_j^{\ge}(\rho,\mathbf{x},s)=f_j^{<}(\rho,\mathbf{x},s):=f^{j,1}_{\rho\mathbf{x}s}+f^{j,-1}_{\rho\mathbf{x}(-s)},\\&g_j^{\ge}(\rho,\mathbf{x},s):=g^{j,1}_{\rho\mathbf{x}(\beta+s)}+g^{j,-1}_{\rho\mathbf{x}(-s)},\quad g_j^{<}(\rho,\mathbf{x},s):=-g^{j,1}_{\rho\mathbf{x}s}-g^{j,-1}_{\rho\mathbf{x}(\beta-s)},\\&(\forall j\in\{1,2\},\ (\rho,\mathbf{x},s)\in\{1,2\}\times\Gamma\times\mathbb{R}).\end{aligned}$$

By using the formula

$$e^{-tA}=\frac{A}{\pi}\int_{\mathbb{R}}dv\frac{e^{itv}}{v^2+A^2},\quad(\forall t\in\mathbb{R}_{\ge0},\ A\in\mathbb{R}_{>0})$$

and the uniform bound $|U(\phi)(\mathbf{k})(\rho,\eta)|\le1$ $(\forall\mathbf{k}\in\Gamma^*,\ \rho,\eta\in\{1,2\})$ one can check that

(4.4)

$$\begin{aligned}C_\varepsilon(\rho\mathbf{x}s,\eta\mathbf{y}t)=&1_{s\ge t}\sum_{j\in\{1,2\}}\langle f_j^{\ge}(\rho\mathbf{x}s),g_j^{\ge}(\eta\mathbf{y}t)\rangle_{L^2(\Gamma^*\times\mathbb{R})}\\&+1_{s<t}\sum_{j\in\{1,2\}}\langle f_j^{<}(\rho\mathbf{x}s),g_j^{<}(\eta\mathbf{y}t)\rangle_{L^2(\Gamma^*\times\mathbb{R})},\end{aligned}$$

$(\forall(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in\{1,2\}\times\Gamma\times[0,\beta))$,

(4.5)

$$\|f_j^{\ge}(X)\|_{L^2(\Gamma^*\times\mathbb{R})},\ \|g_j^{\ge}(X)\|_{L^2(\Gamma^*\times\mathbb{R})},\ \|f_j^{<}(X)\|_{L^2(\Gamma^*\times\mathbb{R})},\ \|g_j^{<}(X)\|_{L^2(\Gamma^*\times\mathbb{R})}$$
$$\le\Bigg(\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}$$
$$\cdot\left(1+2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta|(-1)^{1_{j=2}}e(\phi)(\mathbf{k})+\varepsilon|}+e^{-2\beta|(-1)^{1_{j=2}}e(\phi)(\mathbf{k})+\varepsilon|}\right)^{-\frac{1}{2}}\Bigg)^{\frac{1}{2}},$$

$(\forall X\in\{1,2\}\times\Gamma\times\mathbb{R})$.

It is clear that $f_j^{\ge},g_j^{\ge},f_j^{<},g_j^{<}$ $(j=1,2)$ are continuous with respect to the time variable as the maps from $\mathbb{R}$ to $L^2(\Gamma^*\times\mathbb{R})$. Here we can apply Proposition 4.1 to the perturbed matrix C_ε. Then, by sending $\varepsilon\searrow 0$ we obtain the claimed bound. □

4.2. Completion of the double-scale integration

The analysis of the previous section was constructed on the basic assumptions on the two generalized covariances. We have to demonstrate that the actual full covariance can be decomposed into a sum of 2 covariances and each of them satisfies the required bound properties. Our plan is to reformulate the full covariance into a sum over the Matsubara frequency and let $\mathcal{C}_0$ be one portion with only one Matsubara frequency closest to $\theta/2$ and let $\mathcal{C}_1$ be the one with the rest of the Matsubara frequencies. Concerning the determinant bound, Gram's inequality applies to $\mathcal{C}_0$, while it does not to $\mathcal{C}_1$. However, since the Pedra-Salmhofer's type determinant bound obtained in the previous subsection applies to $\mathcal{C}_0+\mathcal{C}_1$, we can derive the determinant bound on $\mathcal{C}_1$ by decomposing $\mathcal{C}_1$ as $(\mathcal{C}_0+\mathcal{C}_1)-\mathcal{C}_0$. In order to derive the L^1-type norm bounds, we introduce a family of scale-dependent UV cut-off and estimate the norm of scale-dependent covariances with the Matsubara UV cut-off. This is a normal technique used in multi-scale analysis over the Matsubara frequency. Since the L^1-type norm bound of the covariance with UV cut-off is summable with the scale index, we can obtain an upper bound on the norm of $\mathcal{C}_1$.

The momentum variable dual to the time variable is the Matsubara frequency $\frac{\pi}{\beta}(2\mathbb{Z}+1)$. Since we discretized $[0,\beta)$ by the step size $\frac{1}{h}$, we automatically have a cut-off in the infinite set $\frac{\pi}{\beta}(2\mathbb{Z}+1)$. Set

$$\mathcal{M}_h := \left\{ \omega \in \frac{\pi}{\beta}(2\mathbb{Z}+1) \ \Big| \ |\omega| < \pi h \right\}.$$

To begin with, let us reformulate the restriction of $C(\phi)(\cdot)$ into a sum over $\mathcal{M}_h$.

LEMMA 4.5.

(4.6)

$$\begin{aligned} &C(\phi)(\rho \mathbf{x} s, \eta \mathbf{y} t) \\ &= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\mathcal{M}_h} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y}\rangle + i\omega(s-t)} h^{-1} (I_2 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1}(\rho, \eta), \\ &(\forall (\rho, \mathbf{x}, s), (\eta, \mathbf{y}, t) \in \{1,2\} \times \Gamma \times [0,\beta)_h). \end{aligned}$$

PROOF. One can derive (4.6) by using (2.21), (2.24) and the equality

$$e^{sA}\left(\frac{1_{s\ge 0}}{1+e^{\beta A}} - \frac{1_{s<0}}{1+e^{-\beta A}} \right) = \frac{1}{\beta} \sum_{\omega\in\mathcal{M}_h} \frac{e^{i\omega s}}{h(1-e^{-i\frac{\omega}{h}+\frac{A}{h}})},$$
$$\left(\forall s \in \left\{ -\beta, -\beta+\frac{1}{h}, \cdots, \beta-\frac{1}{h} \right\},\ A \in \mathbb{C}\backslash i\frac{\pi}{\beta}(2\mathbb{Z}+1) \right).$$

See [9, Appendix C] for the proof of the above formula. □

Let us take a function $\chi \in C^\infty(\mathbb{R})$ satisfying that

$$\begin{aligned} &\chi(x) = 1, \quad (\forall x \in (-\infty, 1]), \\ &\chi(x) = 0, \quad (\forall x \in [2, \infty)), \\ &\chi(x) \in (0,1), \quad (\forall x \in (1,2)), \\ &\frac{d}{dx}\chi(x) \le 0, \quad (\forall x \in \mathbb{R}). \end{aligned}$$

We do not need more detailed information on the function χ. See e.g. [7, Problem II.6. Solution] for an explicit construction of cut-off functions of this type. Let us take the parameter M from $[2\pi, \infty)$. With the aim of dealing with small as well as large β at the same time, we set the smallest scale of cut-off to be β-dependent, which is the idea implemented in [11, Section 3]. We define the function $\chi^M : \mathbb{R} \to \mathbb{R}$ by

$$\chi^M(x) := \chi\left(\frac{x - M}{M^2 - M} + 1\right).$$

Note that

$$\begin{aligned}
&\chi^M(x) = 1, \quad (\forall x \in (-\infty, M]),\\
&\chi^M(x) = 0, \quad (\forall x \in [M^2, \infty)),\\
&\chi^M(x) \in (0,1), \quad (\forall x \in (M, M^2)),\\
&\frac{d}{dx}\chi^M(x) \le 0, \quad (\forall x \in \mathbb{R}).
\end{aligned}$$

For $h \in \frac{2}{\beta}\mathbb{N}$, set

$$N_h := \left\lfloor \frac{\log(2h)}{\log M} \right\rfloor, \quad N_\beta := \max\left\{ \left\lfloor \frac{\log(1/\beta)}{\log M} \right\rfloor + 1, 1 \right\},$$

where $\lfloor x \rfloor$ denotes the largest integer which does not exceed x for $x \in \mathbb{R}$. We want N_h to be larger than N_β. One can find a sufficient condition as follows.

Lemma 4.6. *If* $h \ge \frac{1}{2}\max\{1, \beta^{-1}\}M^2$, $N_h \ge N_\beta + 1$.

Since we will need the condition $h \ge 4d$ later, let us assume from now that

$$h \ge \max\left\{ \frac{1}{2}\max\{1, \beta^{-1}\}M^2, 4d \right\}. \tag{4.7}$$

It follows that

$$1_{\beta \ge 1}M + 1_{\beta < 1}\beta^{-1} \le M^{N_\beta} \le \max\{1, \beta^{-1}\}M. \tag{4.8}$$

$$M^l \le 2h, \quad (\forall l \in \{N_\beta, N_\beta + 1, \cdots, N_h\}). \tag{4.9}$$

Then, let us define the cut-off functions $\chi_l \in C^\infty(\mathbb{R})$ ($l = N_\beta, N_\beta + 1, \cdots, N_h$) by

$$\chi_{N_\beta}(\omega) := \chi^M(M^{-N_\beta} h |1 - e^{i\frac{\omega}{h}}|),$$

$$\chi_l(\omega) := \chi^M(M^{-l}h|1-e^{i\frac{\omega}{h}}|) - \chi^M(M^{-(l-1)}h|1-e^{i\frac{\omega}{h}}|),$$
$$(\forall l \in \{N_\beta+1, N_\beta+2, \cdots, N_h\}).$$

It follows from the inequalities $h|1-e^{i\frac{\omega}{h}}| \le 2h \le M^{N_h+1}$ that $\chi^M(M^{-N_h}h|1-e^{i\frac{\omega}{h}}|) = 1$ $(\forall \omega \in \mathbb{R})$. Thus,

$$\sum_{l=N_\beta}^{N_h} \chi_l(\omega) = 1, \quad (\forall \omega \in \mathbb{R}). \tag{4.10}$$

The values of the cut-off functions are summarized as follows.

$$\chi_{N_\beta}(\omega) = \begin{cases} 1 & \text{if } h|1-e^{i\frac{\omega}{h}}| \le M^{N_\beta+1}, \\ \in (0,1) & \text{if } M^{N_\beta+1} < h|1-e^{i\frac{\omega}{h}}| < M^{N_\beta+2}, \\ 0 & \text{if } h|1-e^{i\frac{\omega}{h}}| \ge M^{N_\beta+2}, \end{cases} \tag{4.11}$$
$$\chi_l(\omega) = \begin{cases} 0 & \text{if } h|1-e^{i\frac{\omega}{h}}| \le M^{l}, \\ \in (0,1] & \text{if } M^{l} < h|1-e^{i\frac{\omega}{h}}| < M^{l+2}, \\ 0 & \text{if } h|1-e^{i\frac{\omega}{h}}| \ge M^{l+2}, \end{cases}$$
$$(\forall \omega \in \mathbb{R},\ l \in \{N_\beta+1, N_\beta+2, \cdots, N_h\}).$$

We show a couple of necessary properties in the following lemma. To be correct, we should remark that the lemma holds for any $\beta \in \mathbb{R}_{>0}$.

LEMMA 4.7.

(i) There exists a positive constant c independent of any parameter such that

$$\frac{1}{\beta}\sum_{\omega \in \mathcal{M}_h} 1_{\chi_l(\omega+x)\neq 0} \le cM^{l+2}, \quad (\forall l \in \{N_\beta, N_\beta+1, \cdots, N_h\},\ x \in \mathbb{R}).$$

(ii) If $\omega \in [-\pi h, \pi h] \cap \operatorname{supp} \chi_l(\cdot)$ for some $l \in \{N_\beta+1, N_\beta+2, \cdots, N_h\}$, then $|\omega - \frac{\theta}{2}| \ge \frac{1}{2}|\omega|$, $(\forall \theta \in [0, \frac{2\pi}{\beta}))$.

PROOF. (i): By the periodicity that $\chi_l(x+2\pi h) = \chi_l(x)$ $(\forall x \in \mathbb{R})$, (4.8) and (4.11),

$$\frac{1}{\beta}\sum_{\omega \in \mathcal{M}_h} 1_{\chi_l(\omega+x)\neq 0} \le \sup_{r \in [0, \frac{2\pi}{\beta})} \left(\frac{1}{\beta} \sum_{m=-\frac{\beta h}{2}}^{\frac{\beta h}{2}-1} 1_{\chi_l(\frac{2\pi}{\beta}m + r)\neq 0} \right)$$

$$\leq \sup_{r\in[0,\frac{2\pi}{\beta})}\left(\frac{1}{\beta}\sum_{m=-\frac{\beta h}{2}}^{\frac{\beta h}{2}-1}1_{|\frac{2\pi}{\beta}m+r|\leq cM^{l+2}}\right)\leq cM^{l+2}.$$

(ii): It follows from the assumption $M\geq 2\pi$ and (4.8), (4.11) that $|\omega|\geq h|1-e^{i\frac{\omega}{h}}|\geq M^{N_\beta+1}\geq\frac{2\pi}{\beta}$, which implies the result. □

Here we introduce the covariances with scale-dependent UV cut-off. In the following we fix $\phi\in\mathbb{C}$ unless otherwise stated. For $(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in I_0$, set

$$\begin{aligned}
&C_l(\rho\mathbf{x}s,\eta\mathbf{y}t)\\
&:=\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle+i(s-t)(\omega-\frac{\pi}{\beta})}\\
&\qquad\cdot\chi_{l+N_\beta-1}(\omega)h^{-1}(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}(\rho,\eta),\\
&(l\in\{2,3,\cdots,N_h-N_\beta+1\}).\\
&C_1(\rho\mathbf{x}s,\eta\mathbf{y}t)\\
&:=\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h\backslash\{\frac{\pi}{\beta}\}}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle+i(s-t)(\omega-\frac{\pi}{\beta})}\\
&\qquad\cdot\chi_{N_\beta}(\omega)h^{-1}(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}(\rho,\eta),\\
&C_0(\rho\mathbf{x}s,\eta\mathbf{y}t)\\
&:=\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}h^{-1}(I_2-e^{-\frac{i}{h}(\frac{\pi}{\beta}-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}(\rho,\eta).
\end{aligned}$$

We can deduce from (4.6), (4.10), (4.11) and the inequality $h|1-e^{i\frac{\pi}{\beta h}}|\leq\frac{\pi}{\beta}\leq M^{N_\beta+1}$ that

$$\sum_{l=0}^{N_h-N_\beta+1}C_l(\rho\mathbf{x}s,\eta\mathbf{y}t)=e^{-i\frac{\pi}{\beta}(s-t)}C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t),\quad(\forall(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in I_0).\tag{4.12}$$

We want to consider $\sum_{l=1}^{N_h-N_\beta+1}C_l$, C_0 as $\mathcal{C}_1$, $\mathcal{C}_0$ introduced in Subsection 3.3 respectively. For this purpose we are going to study properties of C_l $(l=0,1,\cdots,N_h-N_\beta+1)$.

Let us make an inequality which will be used in the estimation of C_l. Recall the function $g_d : (0,\infty) \to \mathbb{R}$ defined in (1.1).

LEMMA 4.8. *Let $K \in \mathbb{R}_{>0}$. There exists a positive constant $c(d)$ depending only on d such that for any $L \in \mathbb{N}$ satisfying $L \ge K^{-3} g_d(K)^{-1}$,*

$$\frac{1}{L^d} \sum_{\mathbf{k}\in\Gamma^*} \frac{1}{\sqrt{K^2 + e(\mathbf{k})^2}} \le c(d) g_d(K).$$

REMARK 4.9. The claimed inequality crucially affects the possible magnitude of the coupling constant in our double-scale integration process. In terms of the order with K as $K \searrow 0$, the claimed upper bound is better than the crude upper bound K^{-1}, which is out of use for our purpose of proving SSB and ODLRO. However, it is unlikely to be optimal especially in the case $d \ge 2$. More delicate analysis specifying d and μ can improve the result. In this paper we prefer to obtain an order with which the coupling constant can satisfy both the condition for the convergence of the Grassmann integration and the condition for the solvability of the gap equation under the minimum assumption on d and μ, rather than to obtain the optimal order with some complication.

PROOF OF LEMMA 4.8. Note that for any $\mathbf{k}' \in \{0, \frac{2\pi}{L}, \frac{2\pi}{L}\cdot 2, \cdots, 2\pi - \frac{2\pi}{L}\}^{d-1}$,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} dk \frac{1}{\sqrt{K^2 + e(k,\mathbf{k}')^2}} - \frac{1}{L} \sum_{l=0}^{L-1} \frac{1}{\sqrt{K^2 + e(\frac{2\pi}{L} l, \mathbf{k}')^2}} \right| \le c(d) K^{-3} L^{-1},$$

where we used the assumption $|\mu| \le 2d$ to suppress the dependency of the error on μ. By repeating this estimation for each coordinate we obtain that

$$\left| \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} d\mathbf{k} \frac{1}{\sqrt{K^2 + e(\mathbf{k})^2}} - \frac{1}{L^d} \sum_{\mathbf{k}\in\Gamma^*} \frac{1}{\sqrt{K^2 + e(\mathbf{k})^2}} \right| \le c(d) K^{-3} L^{-1}. \tag{4.13}$$

Take any $\varepsilon \in (0, \frac{\pi}{2})$ and set $I_\varepsilon := [0,\varepsilon] \cup [\pi-\varepsilon, \pi+\varepsilon] \cup [2\pi-\varepsilon, 2\pi]$. Note that $\inf_{\mathbf{k}\in[0,2\pi]^d \setminus I_\varepsilon^d} \|\nabla e(\mathbf{k})\|_{\mathbb{R}^d} \ge c\varepsilon$. It follows from the coarea formula and this inequality that

(4.14)

$$
\begin{aligned}
&\int_{[0,2\pi]^d} d\mathbf{k} \frac{1}{\sqrt{K^2+e(\mathbf{k})^2}} \\
&\le c\varepsilon^{-1}\int_{[0,2\pi]^d\setminus I_\varepsilon^d} d\mathbf{k} \frac{\|\nabla e(\mathbf{k})\|_{\mathbb{R}^d}}{\sqrt{K^2+e(\mathbf{k})^2}} + c(d)\varepsilon^d K^{-1} \\
&\le c\varepsilon^{-1}\int_{-2d-\mu}^{2d-\mu} d\eta \frac{\mathcal{H}^{d-1}(\{\mathbf{k}\in[0,2\pi]^d \mid e(\mathbf{k})=\eta\})}{\sqrt{K^2+\eta^2}} + c(d)\varepsilon^d K^{-1} \\
&\le c\varepsilon^{-1}\sup_{r\in\mathbb{R}}\mathcal{H}^{d-1}\left(\left\{\mathbf{k}\in[0,2\pi]^d \;\middle|\; \sum_{j=1}^d \cos k_j = r\right\}\right)\int_{-4d}^{4d} d\eta \frac{1}{\sqrt{K^2+\eta^2}} \\
&\quad + c(d)\varepsilon^d K^{-1} \\
&\le c(d)(\varepsilon^{-1}\log(K^{-1}+1)+\varepsilon^d K^{-1}),
\end{aligned}
$$

where $\mathcal{H}^{d-1}$ denotes the $d-1$ dimensional Hausdorff measure. One can check that the function $x\mapsto x^{-1}\log(K^{-1}+1)+x^dK^{-1} : (0,\infty)\to\mathbb{R}$ attains its minimum at

$$x_0 = (d^{-1}\log(K^{-1}+1)\cdot K)^{\frac{1}{d+1}}$$

and the minimum value is

$$(d^{\frac{1}{d+1}}+d^{-\frac{d}{d+1}})(\log(K^{-1}+1))^{\frac{d}{d+1}}K^{-\frac{1}{d+1}}.$$

Since $\log(K^{-1}+1)\le K^{-1}$, $x_0\in(0,\frac{\pi}{2})$. By taking ε to be x_0 we have from (4.14) that

$$\int_{[0,2\pi]^d} d\mathbf{k}\frac{1}{\sqrt{K^2+e(\mathbf{k})^2}} \le c(d)(\log(K^{-1}+1))^{\frac{d}{d+1}}K^{-\frac{1}{d+1}}. \tag{4.15}$$

Let us improve the upper bound in the case $d=1$.

$$\int_{[0,2\pi]} dk\frac{1}{\sqrt{K^2+e(k)^2}} \le c\int_0^{\frac{\pi}{2}} dk\frac{1}{\sqrt{K^2+(2\cos k-|\mu|)^2}}.$$

Let $\arccos : (-1,1)\to(0,\pi)$ be the inverse function of $\cos|_{(0,\pi)}$. Note that for $k\in[0,\frac{\pi}{2}]$,

$$\left|\cos k-\frac{|\mu|}{2}\right| = \left|\int_{\arccos(|\mu|/2)}^k dp\sin p\right| \ge \frac{2}{\pi}\left|\int_{\arccos(|\mu|/2)}^k dpp\right|$$

$$\geq \frac{1}{\pi}\arccos\left(\frac{|\mu|}{2}\right)\left|k-\arccos\left(\frac{|\mu|}{2}\right)\right|.$$

By substituting this inequality we have

$$\int_{[0,2\pi]} dk \frac{1}{\sqrt{K^2+e(k)^2}} \leq \frac{c}{\arccos(\frac{|\mu|}{2})}\int_0^{\frac{\pi}{2}} dk \frac{1}{\sqrt{K^2+k^2}} \leq \frac{c}{\arccos(\frac{|\mu|}{2})}\log(K^{-1}+1). \tag{4.16}$$

The claim follows from (4.13), (4.15), (4.16). □

In the following, unless otherwise stated, we assume that

$$L \geq \Theta^{-3} g_d(\Theta)^{-1} \tag{4.17}$$

so that Lemma 4.8 holds for $K=\Theta$. In the next lemma we collect bound properties of C_l. We should emphasize that the constant $c(d,M,\chi)$ appearing in the lemma is independent of ϕ.

Lemma 4.10. *There exists a positive constant $c(d,M,\chi)$ depending only on d,M,χ such that the following statements hold.*

(i)

$$\begin{aligned}&|\det(\langle \mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m} C_0(X_i,Y_j))_{1\leq i,j\leq n}|,\\ &\left|\det\left(\langle \mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}\sum_{l=1}^{N_h-N_\beta+1} C_l(X_i,Y_j)\right)_{1\leq i,j\leq n}\right|\\ &\leq (c(d,M,\chi)(1+\beta^{-1}g_d(\Theta)))^n,\\ &(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\leq 1,\\ &\quad X_i,Y_i\in I_0\ (i=1,2,\cdots,n)).\end{aligned}$$

(ii)

$$\begin{aligned}&\|\tilde{C}_l\|_{1,\infty}\leq c(d,M,\chi)\min\{1,\beta\}M^{-l},\quad (\forall l\in\{2,3,\cdots,N_h-N_\beta+1\}),\\ &\|\tilde{C}_1\|_{1,\infty}\leq c(d,M,\chi)\beta(1+\beta)^{d+1},\\ &\|\tilde{C}_0\|_{1,\infty}\leq c(d,M,\chi)\Theta^{-1}(1+\Theta^{-1})^d.\end{aligned}$$

(iii)

$$\left\| \sum_{l=1}^{N_h - N_\beta + 1} \tilde{C}_l \right\|_{1,\infty}' \le c(d, M, \chi)(\beta^{-1} g_d(\Theta) + (1+\beta)^{d+1}).$$

Here $\tilde{C}_l : I^2 \to \mathbb{C}$ *is the anti-symmetric extension of* C_l *defined as in* (3.5).

Proof. In the following 'c' denotes a generic positive constant and '$c(a_1, a_2, \cdots, a_n)$' denotes a positive constant depending only on parameters $a_1, a_2, \cdots, a_n$. First of all let us make some inequalities concerning the integrand inside the covariances. For $x \in [-\pi h, \pi h]$, $\delta \in \{1, -1\}$,

$$\begin{aligned}
&|h(1 - e^{-i\frac{x}{h} + \frac{\delta}{h}\sqrt{e(\mathbf{k})^2 + |\phi|^2}})|^2 \\
&\ge h^2(1 - e^{-\frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^2 + 4h^2 e^{-\frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}} \sin^2\left(\frac{x}{2h}\right) \\
&\ge 1_{\sqrt{e(\mathbf{k})^2+|\phi|^2} > h} h^2 (1 - e^{-1})^2 \\
&\quad + 1_{\sqrt{e(\mathbf{k})^2+|\phi|^2} \le h} \left(e^{-2} e(\mathbf{k})^2 + 4h^2 e^{-1} \sin^2\left(\frac{x}{2h}\right)\right) \\
&\ge c(e(\mathbf{k})^2 + x^2),
\end{aligned}$$

where we used that $h \ge 4d \ge \sup_{\mathbf{k} \in \mathbb{R}^d} |e(\mathbf{k})|$. Since the eigen values of $E(\phi)(\mathbf{k})$ are $\pm\sqrt{e(\mathbf{k})^2 + |\phi|^2}$, this implies that

$$\|h^{-1}(I_2 - e^{-i\frac{x}{h} I_2 + \frac{1}{h} E(\phi)(\mathbf{k})})^{-1}\|_{2\times 2} \le \frac{c}{\sqrt{x^2 + e(\mathbf{k})^2}}, \quad (\forall x \in [-\pi h, \pi h] \backslash \{0\},\ \mathbf{k} \in \mathbb{R}^d), \tag{4.18}$$

where $\|\cdot\|_{2\times 2}$ is the operator norm for 2×2 matrices. Similarly we can prove that

$$\|h^{-1}(I_2 - e^{-i\frac{x}{h} I_2 + \frac{1}{h} E(\phi)(\mathbf{k})})^{-1} e^{\frac{1}{h} E(\phi)(\mathbf{k})}\|_{2\times 2} \le \frac{c}{|x|}, \quad (\forall x \in [-\pi h, \pi h] \backslash \{0\},\ \mathbf{k} \in \mathbb{R}^d). \tag{4.19}$$

(i): We derive the claimed determinant bound on C_0 by means of Gram's inequality. Set $\mathcal{H} := L^2(\{1,2\} \times \Gamma^* \times \mathcal{M}_h)$, which is a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined by

$$\langle f,g\rangle_{\mathcal{H}} := \frac{1}{\beta L^d}\sum_{K\in\{1,2\}\times\Gamma^*\times\mathcal{M}_h}\overline{f(K)}g(K).$$

Let us define vectors $f_X, g_X \in \mathcal{H}$ $(X\in I_0)$ by

$$\begin{aligned}
f_{\rho\mathbf{x}s}(\eta,\mathbf{k},\omega) &:= e^{-i\langle\mathbf{k},\mathbf{x}\rangle - is(\omega-\frac{\pi}{\beta})}1_{\omega=\frac{\pi}{\beta}}\delta_{\rho,\eta}\left(\left(\omega-\frac{\theta}{2}\right)^2+e(\mathbf{k})^2\right)^{-\frac{1}{4}},\\
g_{\rho\mathbf{x}s}(\eta,\mathbf{k},\omega) &:= e^{-i\langle\mathbf{k},\mathbf{x}\rangle - is(\omega-\frac{\pi}{\beta})}1_{\omega=\frac{\pi}{\beta}}\left(\left(\omega-\frac{\theta}{2}\right)^2+e(\mathbf{k})^2\right)^{\frac{1}{4}}\\
&\qquad\cdot h^{-1}(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}(\eta,\rho).
\end{aligned}$$

We can deduce from Lemma 4.8 and (4.18) that

$$\|f_X\|_{\mathcal{H}},\ \|g_X\|_{\mathcal{H}} \le c(d)(\beta^{-1}g_d(\Theta))^{\frac{1}{2}},\quad (\forall X\in I_0).$$

Since $C_0(X,Y) = \langle f_X, g_Y\rangle_{\mathcal{H}}$ $(\forall X,Y\in I_0)$, Gram's inequality in the Hilbert space $\mathbb{C}^m\times\mathcal{H}$ ensures that

$$\begin{aligned}
(4.20)\qquad & |\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}C_0(X_i,Y_j))_{1\le i,j\le n}|\\
&\le \prod_{j=1}^n \|f_{X_j}\|_{\mathcal{H}}\|g_{Y_j}\|_{\mathcal{H}} \le (c(d)\beta^{-1}g_d(\Theta))^n,\\
&(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,\\
&\quad X_i,Y_i\in I_0\ (i=1,2,\cdots,n)),
\end{aligned}$$

which implies that C_0 satisfies the claimed determinant bound.

To derive the determinant bound on $\sum_{l=1}^{N_h-N_\beta+1}C_l$ we use Proposition 4.2. Note that by Lemma 4.8

$$\begin{aligned}
(4.21)\quad & \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\left(1+2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}} + e^{-2\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}\\
&\le \frac{c}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\left(1_{\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}>1} + 1_{\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}\le 1}\beta^{-1}(\Theta^2+e(\mathbf{k})^2)^{-\frac{1}{2}}\right)\\
&\le c(d)(1+\beta^{-1}g_d(\Theta)).
\end{aligned}$$

Thus, by (4.2) and (4.12)

(4.22)
$$\left|\det\left(\langle \mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}\sum_{l=0}^{N_h-N_\beta+1}C_l(X_i,Y_j)\right)_{1\le i,j\le n}\right|\le (c(d)(1+\beta^{-1}g_d(\Theta)))^n,$$
$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,$$
$$X_i,Y_i\in I_0\ (i=1,2,\cdots,n)).$$

Here we can apply Lemma A.1 with (4.20), (4.22) to derive the claimed determinant bound on $\sum_{l=1}^{N_h-N_\beta+1}C_l$.

(ii): We can deduce from (4.11), Lemma 4.7 (i), (ii) and (4.18) that

(4.23)
$$|C_l(\mathbf{X})|\le cM^2,\quad |C_1(\mathbf{X})|\le cM^{N_\beta+2}\beta,\quad |C_0(\mathbf{X})|\le c\beta^{-1}\Theta^{-1},$$
$$(\forall l\in\{2,3,\cdots,N_h-N_\beta+1\},\ \mathbf{X}\in I_0^2).$$

Let $n\in\mathbb{N}$, $j\in\{1,2,\cdots,d\}$. Note that by periodicity,

$$\left(\frac{L}{2\pi}\left(e^{-i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1\right)\right)^n C_l(\cdot\mathbf{x}s,\cdot\mathbf{y}t)$$
$$=\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle+i(s-t)(\omega-\frac{\pi}{\beta})}$$
$$\cdot\left(\chi_{l+N_\beta-1}(\omega)(1_{l\ge 2}+1_{l=1}1_{\omega\neq\frac{\pi}{\beta}})+1_{l=0}1_{\omega=\frac{\pi}{\beta}}\right)\prod_{m=1}^n\left(\frac{L}{2\pi}\int_0^{\frac{2\pi}{L}}dp_m\right)$$
$$\cdot\left(\frac{\partial}{\partial q_j}\right)^n h^{-1}(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{q})})^{-1}\Big|_{\mathbf{q}=\mathbf{k}+\sum_{m=1}^n p_m\mathbf{e}_j}.$$

We need to find a ϕ-independent upper bound on

$$\left\|\left(\frac{\partial}{\partial k_j}\right)^n h^{-1}(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\right\|_{2\times 2}$$

for $\omega\in\mathcal{M}_h$, $\mathbf{k}\in\mathbb{R}^d$. First let us consider the case that $\sqrt{e(\mathbf{k})^2+|\phi|^2}\le h$. Observe that

$$\left\|\left(\frac{\partial}{\partial k_j}\right)^n h^{-1}(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\right\|_{2\times 2}$$

$$\le c(n)h^{-1}\sum_{m=1}^{n}\prod_{u=1}^{m}\left(\sum_{l_u=1}^{n}\right)1_{\sum_{u=1}^{m}l_u=n}$$
$$\cdot\prod_{u=1}^{m}\left\|(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\left(\frac{\partial}{\partial k_j}\right)^{l_u}e^{\frac{1}{h}E(\phi)(\mathbf{k})}\right\|_{2\times 2}$$
$$\cdot\|(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2\times 2}.$$

See e.g. the formula [12, (C.1)] for derivatives of inverse of a matrix-valued function. Since

$$\left\|\left(\frac{\partial}{\partial k_j}\right)^{n}e^{\frac{1}{h}E(\phi)(\mathbf{k})}\right\|_{2\times 2}\le c(n)h^{-1}$$

in this case, we deduce from (4.18) and the above inequality that

$$\begin{aligned}(4.24)\qquad &\left\|\left(\frac{\partial}{\partial k_j}\right)^{n}h^{-1}(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\right\|_{2\times 2}\\ &\le c(n)\left(\left|\omega-\frac{\theta}{2}\right|^{-2}+\left|\omega-\frac{\theta}{2}\right|^{-n-1}\right).\end{aligned}$$

Next let us consider the case that $\sqrt{e(\mathbf{k})^2+|\phi|^2}>h$. It is convenient to use the equality

(4.25)

$$\begin{aligned}&(I_2-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\\ &=\prod_{\delta\in\{1,-1\}}(1-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})+\frac{\delta}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1}\\ &\cdot\begin{pmatrix}1-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})}e^{\frac{1}{h}E(\phi)(\mathbf{k})}(2,2) & e^{-\frac{i}{h}(\omega-\frac{\theta}{2})}e^{\frac{1}{h}E(\phi)(\mathbf{k})}(1,2)\\ e^{-\frac{i}{h}(\omega-\frac{\theta}{2})}e^{\frac{1}{h}E(\phi)(\mathbf{k})}(2,1) & 1-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})}e^{\frac{1}{h}E(\phi)(\mathbf{k})}(1,1)\end{pmatrix}.\end{aligned}$$

Let us estimate derivatives of each component. Remark that

$$\begin{aligned}(4.26)\qquad &|(1-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})-\frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1}|\\ &=|(1-e^{-\frac{i}{h}(\omega-\frac{\theta}{2})+\frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1}e^{\frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}}|\\ &\le c.\end{aligned}$$

Moreover, for any $m \in \mathbb{N} \cup \{0\}$, $\delta \in \{1, -1\}$,

$$\left| \left(\frac{\partial}{\partial k_j} \right)^m e^{\frac{\delta}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}} \right| \le c(m)(1_{m=0} + 1_{m\ge 1}h^{-2}) e^{\frac{\delta}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}}, \tag{4.27}$$

$$\left\| \left(\frac{\partial}{\partial k_j} \right)^m e^{\frac{1}{h}E(\phi)(\mathbf{k})} \right\|_{2\times 2} \le c(m)(1_{m=0} + 1_{m\ge 1}h^{-1}) e^{\frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}}. \tag{4.28}$$

To derive (4.28), the following formula can be repeatedly used.

$$\frac{\partial}{\partial k_j} e^{\frac{1}{h}E(\phi)(\mathbf{k})} = \frac{1}{h}\int_0^1 ds e^{\frac{1-s}{h}E(\phi)(\mathbf{k})} \frac{\partial}{\partial k_j} E(\phi)(\mathbf{k}) e^{\frac{s}{h}E(\phi)(\mathbf{k})}.$$

By (4.26) and (4.27)

$$\left| \left(\frac{\partial}{\partial k_j} \right)^m (1 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) - \frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1} \right| \le c(m)(1_{m=0} + 1_{m\ge 1}h^{-2}), \tag{4.29}$$

$$\begin{aligned} &\left| \left(\frac{\partial}{\partial k_j} \right)^m (1 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) + \frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1} \right| \\ &\le c(m)(1_{m=0} + 1_{m\ge 1}h^{-2}) |1 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) + \frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}}|^{-1}. \end{aligned} \tag{4.30}$$

Thus, we have that

$$\left| \left(\frac{\partial}{\partial k_j} \right)^n \prod_{\delta \in \{1,-1\}} (1 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) + \frac{\delta}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1} \right| \le c(n)h^{-2}. \tag{4.31}$$

Also, by (4.26), (4.28), (4.30)

$$\begin{aligned} &\left\| \left(\frac{\partial}{\partial k_j} \right)^m (1 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) + \frac{1}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1} e^{\frac{1}{h}E(\phi)(\mathbf{k})} \right\|_{2\times 2} \\ &\le c(m)(1_{m=0} + 1_{m\ge 1}h^{-1}), \end{aligned}$$

which combined with (4.29) implies that

$$\left\| \left(\frac{\partial}{\partial k_j} \right)^n \prod_{\delta \in \{1,-1\}} (1 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) + \frac{\delta}{h}\sqrt{e(\mathbf{k})^2+|\phi|^2}})^{-1} e^{\frac{1}{h}E(\phi)(\mathbf{k})} \right\|_{2\times 2} \le c(n)h^{-1}. \tag{4.32}$$

We can see from (4.25), (4.31), (4.32) and $\pi h \ge |\omega - \theta/2|$ that

$$\begin{aligned}(4.33)\qquad &\left\|\left(\frac{\partial}{\partial k_j}\right)^n h^{-1}(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\right\|_{2\times 2}\\ &\le c(n)h^{-2} \le c(n)\left(\left|\omega-\frac{\theta}{2}\right|^{-2}+\left|\omega-\frac{\theta}{2}\right|^{-n-1}\right).\end{aligned}$$

By using (4.11), Lemma 4.7 (i), (ii), (4.24) and (4.33) we can derive that

$$\begin{aligned}(4.34)\qquad &\left\|\left(\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1)\right)^n C_l(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\right\|_{2\times 2}\\ &\le \frac{c(n)}{\beta}\sum_{\omega\in\mathcal{M}_h}\Big(\chi_{l+N_\beta-1}(\omega)(1_{l\ge 2}+1_{l=1}1_{\omega\ne\frac{\pi}{\beta}})+1_{l=0}1_{\omega=\frac{\pi}{\beta}}\Big)\\ &\qquad\cdot\left(\left|\omega-\frac{\theta}{2}\right|^{-2}+\left|\omega-\frac{\theta}{2}\right|^{-n-1}\right)\\ &\le c(n)\Big(1_{l\ge 2}M^{-l-N_\beta+3}+1_{l=1}M^{N_\beta+2}(\beta^2+\beta^{n+1})\\ &\qquad + 1_{l=0}\beta^{-1}(\Theta^{-2}+\Theta^{-n-1})\Big).\end{aligned}$$

The inequality (4.34) coupled with (4.23) yields that

(4.35)

$$\begin{aligned}&|C_1(\rho\mathbf{x}s,\eta\mathbf{y}t)| \le \frac{c(d,M)M^{N_\beta}\beta}{1+(1+\beta)^{-d-1}\sum_{j=1}^d(\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1|)^{d+1}},\\ &|C_0(\rho\mathbf{x}s,\eta\mathbf{y}t)| \le \frac{c(d)\beta^{-1}\Theta^{-1}}{1+(1+\Theta^{-1})^{-d-1}\sum_{j=1}^d(\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1|)^{d+1}},\\ &(\forall(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in I_0).\end{aligned}$$

Thus, by using (4.8),

$$\begin{aligned}&\|\tilde{C}_1\|_{1,\infty}\le c(d,M)M^{N_\beta}\beta^2(1+\beta)^d \le c(d,M)\beta(1+\beta)^{d+1},\\ &\|\tilde{C}_0\|_{1,\infty}\le c(d)\Theta^{-1}(1+\Theta^{-1})^d.\end{aligned}$$

Let $l \in \{2,3,\cdots,N_h-N_\beta+1\}$. Let us estimate decay of the covariances with the time variable. By periodicity,

$$\left(\frac{\beta}{2\pi}\left(e^{-i\frac{2\pi}{\beta}(s-t)}-1\right)\right)^n C_l(\cdot\mathbf{x}s,\cdot\mathbf{y}t)$$

$$
\begin{aligned}
&= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\mathcal{M}_h} e^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(s-t)(\omega-\frac{\pi}{\beta})} \prod_{m=1}^{n} \left(\frac{\beta}{2\pi} \int_0^{\frac{2\pi}{\beta}} dr_m \frac{\partial}{\partial r_m} \right) \\
&\quad \cdot \chi_{l+N_\beta-1}\left(\omega + \sum_{m=1}^{n} r_m\right) h^{-1}(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2}+\sum_{m=1}^{n} r_m)I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\| \left(\frac{\beta}{2\pi}(e^{-i\frac{2\pi}{\beta}(s-t)} - 1) \right)^n C_l(\cdot\mathbf{x}s, \cdot\mathbf{y}t) \right\|_{2\times 2} \\
&\le \sup_{x\in\mathbb{R}} \left(\frac{1}{\beta} \sum_{\omega\in\mathcal{M}_h} 1_{\chi_{l+N_\beta-1}(\omega+x)\neq 0} \right) \\
&\quad \cdot \sup_{\substack{\omega\in[-\pi h,\pi h]\\ \mathbf{k}\in\mathbb{R}^d}} \left\| \left(\frac{\partial}{\partial\omega} \right)^n \chi_{l+N_\beta-1}(\omega) h^{-1}(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1} \right\|_{2\times 2}.
\end{aligned}
$$

Note that

(4.36)

$$
\begin{aligned}
&\left\| \left(\frac{\partial}{\partial\omega} \right)^n h^{-1}(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1} \right\|_{2\times 2} \\
&\le c(n) h^{-1} \sum_{m=1}^{n} \prod_{u=1}^{m} \left(\sum_{l_u=1}^{n} \right) 1_{\sum_{u=1}^{m} l_u = n} \\
&\quad \cdot \prod_{u=1}^{m} \left\| (I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1} \left(\frac{\partial}{\partial\omega} \right)^{l_u} e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})} \right\|_{2\times 2} \\
&\quad \cdot \|(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2\times 2} \\
&\le c(n) h^{-1-n} \sum_{m=1}^{n} \|(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1} e^{\frac{1}{h}E(\phi)(\mathbf{k})}\|_{2\times 2}^{m} \\
&\quad \cdot \|(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1}\|_{2\times 2}.
\end{aligned}
$$

Then by using (4.9), (4.11), Lemma 4.7 (i), (ii), (4.18), (4.19), (4.36) and the fact that

$$
\left| \left(\frac{d}{d\omega} \right)^n \chi_{l+N_\beta-1}(\omega) \right| \le c(n, M, \chi) M^{-(l+N_\beta)n}, \quad (\forall \omega \in \mathbb{R},\ n \in \mathbb{N}),
$$

we have that

$$
\begin{aligned}
(4.37)\quad & \left\| \left(\frac{\beta}{2\pi}(e^{-i\frac{2\pi}{\beta}(s-t)} - 1) \right)^n C_l(\cdot \mathbf{x}s, \cdot \mathbf{y}t) \right\|_{2\times 2} \\
&\le c(n,M) M^{l+N_\beta} \\
&\quad \cdot \Bigg(\sup_{\substack{\omega\in[-\pi h,\pi h] \\ \mathbf{k}\in\mathbb{R}^d}} \left| \left(\frac{d}{d\omega}\right)^n \chi_{l+N_\beta-1}(\omega) \right| \\
&\qquad\quad \cdot \| h^{-1}(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1} \|_{2\times 2} \\
&\qquad + \sum_{m=0}^{n-1} \sup_{\substack{\omega\in[-\pi h,\pi h] \\ \mathbf{k}\in\mathbb{R}^d}} \left| \left(\frac{d}{d\omega}\right)^m \chi_{l+N_\beta-1}(\omega) \right| \\
&\qquad\quad \cdot \left\| \left(\frac{\partial}{\partial\omega}\right)^{n-m} h^{-1}(I_2 - e^{-\frac{i}{h}(\omega-\frac{\theta}{2})I_2 + \frac{1}{h}E(\phi)(\mathbf{k})})^{-1} \right\|_{2\times 2} \Bigg) \\
&\le c(n,M,\chi) M^{l+N_\beta} \\
&\quad \cdot \Bigg(M^{-(l+N_\beta)n-(l+N_\beta)} \\
&\qquad + \sum_{m=0}^{n-1} M^{-(l+N_\beta)m} h^{-1-(n-m)} \sum_{u=1}^{n-m} h^{u+1} M^{-(l+N_\beta)(u+1)} \Bigg) \\
&\le c(n,M,\chi) M^{-(l+N_\beta)n}.
\end{aligned}
$$

By combining (4.23), (4.34) with (4.37) we reach the inequality

$$
\begin{aligned}
&(4.38) \\
&|C_l(\rho\mathbf{x}s, \eta\mathbf{y}t)| \\
&\le \frac{c(n,M,\chi)}{1 + M^{n(l+N_\beta)} |\frac{\beta}{2\pi}(e^{i\frac{2\pi}{\beta}(s-t)} - 1)|^n + M^{l+N_\beta} \sum_{j=1}^d |\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \mathbf{e}_j\rangle} - 1)|^n}, \\
&(\forall (\rho,\mathbf{x},s), (\eta,\mathbf{y},t) \in I_0,\ n\in\mathbb{N}).
\end{aligned}
$$

From (4.8), (4.9) and (4.38) we can deduce that

$$
\|\tilde{C}_l\|_{1,\infty} \le c(d,M,\chi) M^{-l-N_\beta} \le c(d,M,\chi) \min\{1,\beta\} M^{-l}.
$$

(iii): By using the result of (i) for $n = 1$, (4.35), (4.38) and (4.8) we have that for any $\rho, \eta \in \{1, 2\}$, $s, t \in [0, \beta)_h$, $\mathbf{y} \in \Gamma$,

$$\begin{aligned}
&\sum_{\mathbf{x}\in\Gamma}\left|\sum_{l=1}^{N_h-N_\beta+1} C_l(\rho\mathbf{x}s, \eta\mathbf{y}t)\right| = \sum_{\mathbf{x}\in\Gamma}\left|\sum_{l=1}^{N_h-N_\beta+1} C_l(\rho\mathbf{y}s, \eta\mathbf{x}t)\right| \\
&= \left|\sum_{l=1}^{N_h-N_\beta+1} C_l(\rho\mathbf{0}s, \eta\mathbf{0}t)\right| + \sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}}\left|\sum_{l=1}^{N_h-N_\beta+1} C_l(\rho\mathbf{x}s, \eta\mathbf{0}t)\right| \\
&\le c(d, M, \chi)(1 + \beta^{-1}g_d(\Theta)) \\
&\quad + \sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}} \frac{c(d, M)M^{N_\beta}\beta}{1 + (1+\beta)^{-d-1}\sum_{j=1}^d |\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle} - 1)|^{d+1}} \\
&\quad + \sum_{l=2}^{N_h-N_\beta+1}\sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}} \frac{c(d, M, \chi)M^{-l-N_\beta}}{\sum_{j=1}^d |\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle} - 1)|^{d+1}} \\
&\le c(d, M, \chi)(1 + \beta^{-1}g_d(\Theta) + M^{N_\beta}\beta(1+\beta)^d) \\
&\le c(d, M, \chi)(\beta^{-1}g_d(\Theta) + (1+\beta)^{d+1}),
\end{aligned}$$

which implies the result. □

By summarizing the results of Lemma 4.10 we can reach the following conclusion.

Corollary 4.11. *Set* $\mathcal{C}_1 := \sum_{l=1}^{N_h-N_\beta+1} C_l$, $\mathcal{C}_0 := C_0$. *There exists a constant* $c(d, M, \chi) \in \mathbb{R}_{\ge 1}$ *depending only on* d, M, χ *such that the following statements hold with* c_0, D_c *defined by*

$$c_0 := c(d, M, \chi)(1 + \beta^{d+2} + \beta^{-1}g_d(\Theta)), \quad D_c := \Theta^{-1}(1 + \Theta^{-1})^d.$$

- $\mathcal{C}_1$ *satisfies* (3.10), (3.48), (3.49), (3.50) *with* c_0.
- $\mathcal{C}_0$ *satisfies* (3.47), (3.48), (3.51) *with* c_0 *and* D_c.

Proof. The claims directly follow from Lemma 4.10. Since $\mathcal{M}_h - \frac{\pi}{\beta} \subset \frac{2\pi}{\beta}\mathbb{Z}$, $\mathcal{C}_1$ satisfies (3.10). Let us only show how to derive the claimed upper bounds on $\|\tilde{\mathcal{C}}_1\|_{1,\infty}$, $\|\tilde{\mathcal{C}}_1\|$ from the results of Lemma 4.10.

$$\begin{aligned}
\|\tilde{\mathcal{C}}_1\|_{1,\infty} &\le \|\tilde{C}_1\|_{1,\infty} + \sum_{l=2}^{N_h-N_\beta+1} \|\tilde{C}_l\|_{1,\infty} \\
&\le c(d,M,\chi)\beta(1+\beta)^{d+1} + c(d,M,\chi)\min\{1,\beta\} \sum_{l=2}^{N_h-N_\beta+1} M^{-l} \\
&\le c_0, \\
\|\tilde{\mathcal{C}}_1\| &\le \left\| \sum_{l=1}^{N_h-N_\beta+1} \tilde{C}_l \right\|_{1,\infty}' + \beta^{-1} \sum_{l=1}^{N_h-N_\beta+1} \|\tilde{C}_l\|_{1,\infty} \\
&\le c(d,M,\chi)(\beta^{-1}g_d(\Theta) + (1+\beta)^{d+1}) + c(d,M,\chi)(1+\beta)^{d+1} \\
&\quad + c(d,M,\chi) \sum_{l=2}^{N_h-N_\beta+1} M^{-l} \\
&\le c_0. \ \square
\end{aligned}$$

REMARK 4.12. Since we assumed the simple conditions on $\mathcal{C}_0$, $\mathcal{C}_1$ in Subsection 3.3, it became necessary to overestimate the covariances' bounds in terms of the order with β, Θ^{-1} in the derivation of Corollary 4.11 from Lemma 4.10. In this paper we choose to simplify the Grassmann integration process at the expense of the order with these parameters. The most important information in the statements of Corollary 4.11 is the dependency of c_0 on Θ, which ultimately decides whether SSB and ODLRO can be proven.

Since we have verified the necessary properties of the real covariances, we can apply the general results in the previous section to our model problem.

LEMMA 4.13. *There exists a positive constant $c(d)$ depending only on d such that the following statements hold with any $a \in \mathbb{R}_{\ge 1}$, $h \in \frac{2}{\beta}\mathbb{N}$ satisfying $h \ge c(d)\max\{1, \beta^{-1}\}$, $L \in \mathbb{N}$ satisfying*

$$L^d \ge c(d)\max\{\Theta^{-3d} g_d(\Theta)^{-d}, \Theta^{-1}(1+\Theta^{-1})^d\} \tag{4.39}$$

and $r, r' \in \mathbb{R}_{>0}$ *defined by*

$$r := c(d)^{-1}(1+\beta^{d+2}+\beta^{-1}g_d(\Theta))^{-2}a^{-4},$$
$$r' := c(d)^{-1}L^{-d}h^{-1}\beta^{-1}\min\{1,\beta\}(1+\beta^{d+2}+\beta^{-1}g_d(\Theta))^{-2}a^{-4}.$$

There exists a function $V^{end}:\mathbb{C}\times\overline{D(2r)}\times\overline{D(2r')}^2\to\mathbb{C}$ *such that for any* $\phi\in\mathbb{C}$, $(u,\boldsymbol{\lambda})\mapsto V^{end}(\phi,u,\boldsymbol{\lambda})$ *is continuous in* $\overline{D(2r)}\times\overline{D(2r')}^2$ *and analytic in* $D(2r)\times D(2r')^2$. *Moreover, for any* $(\phi,u,\boldsymbol{\lambda})\in\mathbb{C}\times\overline{D(r)}\times\overline{D(r')}^2$, $j=1,2$,

$$\frac{h}{N}|V^{end}(\phi,u,\mathbf{0})|\le a^{-2}L^{-d}, \tag{4.40}$$

$$\left|\frac{\partial}{\partial\lambda_j}V^{end}(\phi,u,\mathbf{0})+\int A^j(\psi)d\mu_{C(\phi)}(\psi)\right| \tag{4.41}$$
$$\le c(d)\beta(1+\beta^{d+2}+\beta^{-1}g_d(\Theta))^2(1+\Theta^{-1}(1+\Theta^{-1})^d)a^4L^{-d},$$

$$e^{V^{end}(\phi,u,\boldsymbol{\lambda})}=\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi). \tag{4.42}$$

Proof. By the division formula of Grassmann Gaussian integral (see e.g. [7, Proposition I.21]),

$$-\int A(\psi)d\mu_{\mathcal{C}_1+\mathcal{C}_0}(\psi)=\int\int V^{1,1}(\psi+\psi^1)d\mu_{\mathcal{C}_1}(\psi^1)d\mu_{\mathcal{C}_0}(\psi)$$
$$=\int V^{1-3,0}(\psi)d\mu_{\mathcal{C}_0}(\psi)=V^{1-3,end}.$$

As in Corollary 4.11 we set $\mathcal{C}_1=\sum_{l=1}^{N_h-N_\beta+1}C_l$, $\mathcal{C}_0=C_0$. Then, by (4.12) and the fact that $A(\psi)$ is invariant under the transform $\psi_{\rho\mathbf{x}s\xi}\mapsto e^{-i\xi\frac{\pi}{\beta}s}\psi_{\rho\mathbf{x}s\xi}$ $((\rho,\mathbf{x},s,\xi)\in I)$,

$$\int A(\psi)d\mu_{\mathcal{C}_1+\mathcal{C}_0}(\psi)=\int A(\psi)d\mu_{C(\phi)}(\psi).$$

Thus, $V^{1-3,end}=-\int A(\psi)d\mu_{C(\phi)}(\psi)$. By substituting this equality, c_0, D_c defined in Corollary 4.11 and $\alpha=2^4a$ we see that Lemma 3.8 implies the existence of the function V^{end} satisfying the claimed regularity and (4.40), (4.41).

The equality (4.42) can be derived by a basic argument close to [12, Proposition 6.4 (3)]. However, we provide the proof for completeness. We fix $\phi \in \mathbb{C}$ and hide the dependency on ϕ in the following. Since the expansion of $e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}$ terminates at a finite order, there exists a (β, L, h)-dependent positive constant $r(\beta, L, h)$ such that

(4.43)

$$\mathrm{Re} \int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)} d\mu_{\mathcal{C}_0+\mathcal{C}_1}(\psi) > 0, \quad (\forall (u, \boldsymbol{\lambda}) \in \overline{D(r(\beta, L, h))}^3).$$

Let us set

$$V^1(\psi) := \sum_{j=0}^{1} V^{j,1}(\psi),$$
$$V^0(\psi) := \sum_{j=1}^{2} V^{0-j,0}(\psi) + \sum_{j=1}^{3} V^{1-j,0}(\psi) + V^{2,0}(\psi).$$

By the results of Lemma 3.4, Lemma 3.6 and (3.57), (3.58), (3.79), (3.80) there exists a (β, L, h)-dependent positive constant $c(\beta, L, h)$ such that

$$\sum_{m=0}^{N} \|V_m^l(u, \boldsymbol{\lambda})\|_1 \le c(\beta, L, h)\alpha^{-2},$$
$$(\forall (u, \boldsymbol{\lambda}) \in \overline{D(c(\beta, L, h)^{-1}\alpha^{-4})}^3, \ \alpha \in [2^3, \infty), \ l \in \{0, 1\}).$$

This implies that we can choose a (β, L, h)-dependent positive constant $c(\beta, L, h)'$ such that if $\alpha \ge c(\beta, L, h)'$,

$$\mathrm{Re} \int e^{zV^l(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_l}(\psi) > 0, \tag{4.44}$$
$$(\forall (u, \boldsymbol{\lambda}) \in \overline{D(c(\beta, L, h)^{-1}\alpha^{-4})}^3, \ z \in \overline{D(2)}, \ l \in \{0, 1\}).$$

Let us take $\alpha_0 \in [c(\beta, L, h)', \infty)$ satisfying that $c(\beta, L, h)^{-1}\alpha_0^{-4} \le r(\beta, L, h)$ and fix $(u, \boldsymbol{\lambda}) \in \overline{D(c(\beta, L, h)^{-1}\alpha_0^{-4})}^3$ so that both (4.43) and (4.44) hold for this $(u, \boldsymbol{\lambda})$. By (4.44), for $l \in \{0, 1\}$,

$$z \mapsto \log \left(\int e^{zV^l(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_l}(\psi^1) \right)$$

is analytic in $D(2)$ and thus

(4.45)

$$\log \left(\int e^{V^1(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_1}(\psi^1) \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d}{dz}\right)^n \log \left(\int e^{zV^1(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_1}(\psi^1) \right) \Bigg|_{z=0} = V^0(u,\boldsymbol{\lambda})(\psi).$$

Similarly,

$$\log \left(\int e^{V^0(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_0}(\psi) \right) = V^{end}(u,\boldsymbol{\lambda}). \tag{4.46}$$

By (4.44) for $l = 1$, $z = 1$ and (4.45) we can apply the basic lemma [11, Lemma C.2] to ensure that

$$e^{V^0(u,\boldsymbol{\lambda})(\psi)} = \int e^{V^1(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_1}(\psi^1).$$

By substituting this equality into (4.46) and using the division formula again,

$$V^{end}(u,\boldsymbol{\lambda}) = \log \left(\int e^{V^1(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_0+\mathcal{C}_1}(\psi) \right). \tag{4.47}$$

Moreover, by (4.43),

$$e^{V^{end}(u,\boldsymbol{\lambda})} = \int e^{V^1(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_0+\mathcal{C}_1}(\psi).$$

By the identity theorem, this equality must hold for any $(u,\boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r')}^2$. By the gauge transform $\psi_{\rho\mathbf{x}s\xi} \mapsto e^{-i\xi\frac{\pi}{\beta}s}\psi_{\rho\mathbf{x}s\xi}$ $((\rho,\mathbf{x},s,\xi) \in I)$ the right-hand side of the above equality becomes that of (4.42). □

Lemma 4.13 can be reduced to the following explicit statements.

PROPOSITION 4.14. *There exists a positive constant* $c(d) \in \mathbb{R}_{\ge 1}$ *depending only on* d *such that the following statements hold with any* $h \in \frac{2}{\beta}\mathbb{N}$ *satisfying* $h \ge c(d)\max\{1,\beta^{-1}\}$, $L \in \mathbb{N}$ *satisfying* (4.39) *and* $r \in \mathbb{R}_{>0}$ *defined by*

$$r := c(d)^{-1}(1 + \beta^{d+3} + (1+\beta^{-1})g_d(\Theta))^{-2}.$$

$$\left| \int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{C(\phi)}(\psi) - 1 \right| \le \frac{1}{2}, \quad (\forall(\phi,u) \in \mathbb{C} \times \overline{D(r)}). \tag{4.48}$$

$$(4.49)\quad \left|\frac{\int e^{-V(u)(\psi)+W(u)(\psi)}A^j(\psi)d\mu_{C(\phi)}(\psi)}{\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)}-\int A^j(\psi)d\mu_{C(\phi)}(\psi)\right|$$
$$\le c(d)\beta(1+\beta^{d+3}+(1+\beta^{-1})g_d(\Theta))^2(1+\Theta^{-1}(1+\Theta^{-1})^d)L^{-d},$$
$$(\forall(\phi,u)\in\mathbb{C}\times\overline{D(r)},\ j\in\{1,2\}).$$

Proof. Take a to be

$$2\left(\log\left(\frac{3}{2}\right)\right)^{-\frac{1}{2}}(1+\beta)^{\frac{1}{2}},$$

which is larger than 1. Then, the r in Lemma 4.13 and the right-hand side of (4.41) are rescaled to be the r in this proposition and the right-hand side of (4.49) respectively. Moreover, (4.40) implies that

$$|V^{end}(\phi,u,\mathbf{0})|\le\log\left(\frac{3}{2}\right),\quad(\forall(\phi,u)\in\mathbb{C}\times\overline{D(r)}).$$

By combining this inequality with (4.42) we obtain (4.48). Also, by differentiating (4.42) the left-hand side of (4.41) is proved to be equal to that of (4.49). □

4.3. Existence of the infinite-volume limit of the correction term

According to (4.40), (4.47), (4.48), we know that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\limsup_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\sup_{(\phi,u)\in\mathbb{C}\times\overline{D(r)}}\frac{h}{N}\left|\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right)\right|=0.$$

However, this property does not imply a uniform convergence of

$$\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)$$

with (ϕ,u) as $h\to\infty$, $L\to\infty$. As we need such a stronger convergence property to complete the proof of the main theorem, let us prove it beforehand. Let us start by confirming spatial decay properties of $C(\phi)$.

Lemma 4.15. *There exists a positive constant $c(d,\beta,\theta)$ depending only on d,β,θ such that*

(4.50)
$$|C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t)|\le\frac{c(d,\beta,\theta)}{1+\sum_{j=1}^{d}|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1)|^{d+1}},$$
$$(\forall(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in\{1,2\}\times\mathbb{Z}^d\times[0,\beta),\ \phi\in\mathbb{C}).$$

(4.51)
$$|C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t)|\le\frac{c(d,\beta,\theta)}{1+(\frac{2}{\pi})^{d+1}\sum_{j=1}^{d}|\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle|^{d+1}},$$
$$(\forall(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in\{1,2\}\times\mathbb{Z}^d\times[0,\beta)\ \textit{with}\ \mathbf{x}-\mathbf{y}\in[-L/2,L/2]^d,\ \phi\in\mathbb{C}).$$

Proof. It follows from Lemma 4.10 (i) for $n=1$, (4.12) and (4.34) that the inequality (4.50) holds for any $(\rho,\mathbf{x},s),(\eta,\mathbf{y},t)\in\{1,2\}\times\mathbb{Z}^d\times[0,\beta)_h$, $\phi\in\mathbb{C}$. Since $(s,t)\mapsto C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t)$ is continuous in $\{(s,t)\in[0,\beta)^2\mid s\neq t\}$, (4.50) can be claimed for any $s,t\in[0,\beta)$ with $s\neq t$ by approximating s,t by converging sequences. The inequality (4.50) also holds in the case $s=t$, since $C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}s)=C(\phi)(\rho\mathbf{x}0,\eta\mathbf{y}0)$. The inequality (4.51) follows from (4.50). □

By definition, $\phi\mapsto C(\phi)(\mathbf{X})$ is continuous in $\mathbb{C}$ for any $\mathbf{X}\in(\{1,2\}\times\mathbb{Z}^d\times[0,\beta))^2$ and thus

$$(\phi,u)\mapsto\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)$$

is continuous in $\mathbb{C}^2$. In the rest of this subsection we prove the following proposition, which requires deeper analysis of the tree expansion than that performed in Subsection 3.2.

Proposition 4.16. *Let r be the radius set in Proposition 4.14. For any non-empty compact set Q of $\mathbb{C}$,*

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi),$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)$$

converge in $C(Q\times\overline{D(r/2)},\mathbb{C})$ *as sequences of function with the variable* (ϕ,u).

Proof. Recalling the definition (3.54), (3.55), let us define the anti-symmetric function $V_2 : I^2\to\mathbb{C}$ and the bi-anti-symmetric function $V_{2,2}$: $I^2\times I^2\to\mathbb{C}$ by $V_2(\mathbf{X}) := V_2^{0-1,1}(1)(\mathbf{X})$, $V_{2,2}(\mathbf{X},\mathbf{Y}) := V_{2,2}^{0-2,1}(1)(\mathbf{X},\mathbf{Y})$ and set

$$\hat{V}_2(\psi) := \left(\frac{1}{h}\right)^2\sum_{\mathbf{X}\in I^2}V_2(\mathbf{X})\psi_{\mathbf{X}},\quad \hat{V}_4(\psi) := \left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V_{2,2}(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}},$$

$$\hat{V}(\psi) := \hat{V}_2(\psi)+\hat{V}_4(\psi).$$

It follows that $u\hat{V}(\psi) = -V(u)(\psi)+W(u)(\psi)$. For $\phi\in\mathbb{C}$ we define the function $\mathcal{G}(\phi) : (\{1,2\}\times\mathbb{Z}^d\times[0,\beta))^2\to\mathbb{C}$ by $\mathcal{G}(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t) := e^{-i\frac{\pi}{\beta}(s-t)}C(\phi)(\rho\mathbf{x}s,\eta\mathbf{y}t)$. By the gauge transform $\psi_{\rho\mathbf{x}s\xi}\to e^{i\xi\frac{\pi}{\beta}s}\psi_{\rho\mathbf{x}s\xi}$ $((\rho,\mathbf{x},s,\xi)\in I)$,

$$\int e^{u\hat{V}(\psi)}d\mu_{\mathcal{G}(\phi)}(\psi) = \int e^{u\hat{V}(\psi)}d\mu_{C(\phi)}(\psi).$$

Though we often drop the sign of ϕ-dependency from $\mathcal{G}(\phi)$ for simplicity in the following, the dependency on ϕ should be reminded especially when we establish uniform bounds with ϕ. By (4.48),

$$(\phi,u)\mapsto\log\left(\int e^{u\hat{V}(\psi)}d\mu_{\mathcal{G}(\phi)}(\psi)\right)$$

is continuous in $\mathbb{C}\times\overline{D(r)}$ and

$$u\mapsto\log\left(\int e^{u\hat{V}(\psi)}d\mu_{\mathcal{G}(\phi)}(\psi)\right)$$

is analytic in $D(r)$ for any $\phi\in\mathbb{C}$. Take a non-empty compact set Q of $\mathbb{C}$. For $n\in\mathbb{N}$, $\phi\in Q$, set

$$\alpha_{n,L,h}(\phi) := \frac{1}{n!}\left(\frac{d}{du}\right)^n\log\left(\int e^{u\hat{V}(\psi)}d\mu_{\mathcal{G}}(\psi)\right)\Bigg|_{u=0}.$$

Let us prove that $\alpha_{n,L,h}$ converges in $C(Q,\mathbb{C})$ as a function of ϕ in the limit $h\to\infty$, $L\to\infty$. By the transformation close to (3.56) we have

$$
\begin{aligned}
&\alpha_{n,L,h}\\
&=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{G})\prod_{j=1}^{n}\hat{V}_2(\psi^j)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&\quad+\sum_{l=1}^{n}\begin{pmatrix} n\\ l\end{pmatrix}\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{G})\\
&\qquad\cdot\prod_{j=1}^{l}\hat{V}_4(\psi^j)\prod_{k=l+1}^{n}\hat{V}_2(\psi^k)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{G})\prod_{j=1}^{n}\hat{V}_2(\psi^j)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&\quad+\sum_{l=1}^{n}\begin{pmatrix} n\\ l\end{pmatrix}\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V_{2,2}(\mathbf{X},\mathbf{Y})\\
&\qquad\cdot\frac{1}{n!}Tree(\{1,2,\cdots,n+1\},\mathcal{G})\psi^1_{\mathbf{X}}\psi^2_{\mathbf{Y}}\\
&\qquad\cdot\prod_{j=3}^{l+1}\hat{V}_4(\psi^j)\prod_{k=l+2}^{n+1}\hat{V}_2(\psi^k)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n+1\})}}\\
&\quad+\sum_{l=1}^{n}\begin{pmatrix} n\\ l\end{pmatrix}\frac{1}{n!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\left(\frac{1}{h}\right)^4\sum_{\mathbf{X},\mathbf{Y}\in I^2}V_{2,2}(\mathbf{X},\mathbf{Y})\\
&\qquad\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{G})\psi^{s_1}_{\mathbf{X}}\\
&\qquad\cdot\prod_{j=2}^{m+1}(1_{s_j\le l}\hat{V}_4(\psi^{s_j})+1_{s_j>l}\hat{V}_2(\psi^{s_j}))\Big|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}\\
&\qquad\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{G})\psi^{t_1}_{\mathbf{Y}}\\
&\qquad\cdot\prod_{k=2}^{n-m}(1_{t_k\le l}\hat{V}_4(\psi^{t_k})+1_{t_k>l}\hat{V}_2(\psi^{t_k}))\Big|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}}.
\end{aligned}
$$

Note that the translation invariances (3.10), (3.13) are satisfied by $\mathcal{G}$ and the kernels of $\hat{V}_2, \hat{V}_4$. This implies that for any $b_j \in \{2,4\}$ $(j = 2, 3, \cdots, m+1)$,

$$\begin{aligned}
&\sum_{\mathbf{X}\in I^2} V_{2,2}(\mathbf{X},\mathbf{Y}) Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{G}) \psi_{\mathbf{X}}^{s_1} \prod_{j=2}^{m+1} \hat{V}_{b_j}(\psi^{s_j}) \Big|_{\substack{\psi^{s_j}=0 \\ (\forall j\in\{1,2,\cdots,m+1\})}} \\
&= \sum_{\substack{\mathbf{X}\in(I^0)^2 \\ s\in[0,\beta)_h}} V_{2,2}(\mathbf{X}+s,\mathbf{Y}) Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{G}) \psi_{\mathbf{X}}^{s_1} \\
&\quad \cdot \prod_{j=2}^{m+1} \hat{V}_{b_j}(\psi^{s_j}) \Big|_{\substack{\psi^{s_j}=0 \\ (\forall j\in\{1,2,\cdots,m+1\})}} = 0.
\end{aligned}$$

Thus,

$$\alpha_{n,L,h} = \alpha'_{n,L,h} + a_{n,L,h},$$

where

$$\begin{aligned}
\alpha'_{n,L,h} &:= \frac{1}{n!} Tree(\{1,2,\cdots,n\},\mathcal{G}) \prod_{j=1}^{n} \hat{V}_2(\psi^j) \Big|_{\substack{\psi^j=0 \\ (\forall j\in\{1,2,\cdots,n\})}} \\
&\quad + 1_{n\ge 2} \sum_{l=1}^{n-1} \begin{pmatrix} n \\ l \end{pmatrix} \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}(\mathbf{X},\mathbf{Y}) \\
&\quad \cdot \frac{1}{n!} Tree(\{1,2,\cdots,n+1\},\mathcal{G}) \psi_{\mathbf{X}}^1 \psi_{\mathbf{Y}}^2 \\
&\quad \cdot \prod_{j=3}^{l+1} \hat{V}_4(\psi^j) \prod_{k=l+2}^{n+1} \hat{V}_2(\psi^k) \Big|_{\substack{\psi^j=0 \\ (\forall j\in\{1,2,\cdots,n+1\})}}, \\
a_{n,L,h} &:= \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}(\mathbf{X},\mathbf{Y}) \\
&\quad \cdot \frac{1}{n!} Tree(\{1,2,\cdots,n+1\},\mathcal{G}) \psi_{\mathbf{X}}^1 \psi_{\mathbf{Y}}^2 \prod_{j=3}^{n+1} \hat{V}_4(\psi^j) \Big|_{\substack{\psi^j=0 \\ (\forall j\in\{1,2,\cdots,n+1\})}}.
\end{aligned}$$

Let us show that $a_{n,L,h}$ converges in $C(Q,\mathbb{C})$ as $h\to\infty$. Let us set

$$\begin{aligned}
&\nu(s,t) := \frac{1}{\beta} - h1_{s=t}, \quad (s,t\in[0,\beta)_h), \\
&V_{\mathbf{x}s1}(\psi) := \psi_{1\mathbf{x}s1}\psi_{2\mathbf{x}s-1}, \quad V_{\mathbf{x}s-1}(\psi) := \psi_{2\mathbf{x}s1}\psi_{1\mathbf{x}s-1}, \quad (\mathbf{x}\in\Gamma,\ s\in[0,\beta)_h).
\end{aligned}$$

Then, by the periodicity and the translation invariance with the spatial variable we observe that

$$
\begin{aligned}
(4.52)\qquad a_{n,L,h} &= \left(\frac{1}{h}\right)^2 \sum_{\substack{\mathbf{x}\in\Gamma\\ s,t\in[0,\beta)_h}} \nu(s,t)\\
&\quad \cdot \frac{1}{n!} Tree(\{1,2,\cdots,n+1\},\mathcal{G}) V_{\mathbf{0}s1}(\psi^1) V_{\mathbf{x}t-1}(\psi^2)\\
&\quad \cdot \prod_{j=3}^{n+1} \hat{V}_4(\psi^j)\Bigg|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n+1\})}}\\
&= L^{-d(n-1)} \sum_{a_1\in\{1,2\}} \sum_{\mathbf{x}_2\in\Gamma} \prod_{j=3}^{n+1} \left(\sum_{\mathbf{x}_j,\mathbf{y}_j\in\Gamma} \sum_{a_j\in\{1,2\}} \right)\\
&\quad \cdot \prod_{\substack{j=1\\ j\neq 2}}^{n+1} \left(1_{a_j=2}\frac{1}{\beta h^2} \sum_{\mathbf{s}_j\in[0,\beta)_h^{a_j}} - 1_{a_j=1}\frac{1}{h} \sum_{\mathbf{s}_j\in[0,\beta)_h^{a_j}} \right)\\
&\quad \cdot f(\phi)((a_1,(a_j)_{j=3}^{n+1}),(\mathbf{s}_1,(\mathbf{s}_j)_{j=3}^{n+1}),(\mathbf{x}_2,(\mathbf{x}_j,\mathbf{y}_j)_{j=3}^{n+1})),
\end{aligned}
$$

where we set

$$
\begin{aligned}
& f(\phi)((a_1,(a_j)_{j=3}^{n+1}),(\mathbf{s}_1,(\mathbf{s}_j)_{j=3}^{n+1}),(\mathbf{x}_2,(\mathbf{x}_j,\mathbf{y}_j)_{j=3}^{n+1}))\\
&:= \frac{1}{n!} Tree(\{1,2,\cdots,n+1\},\mathcal{G}(\phi))\\
&\quad \cdot (1_{a_1=1} V_{\mathbf{0}s_1 1}(\psi^1) V_{\mathbf{x}_2 s_1 -1}(\psi^2) + 1_{a_1=2} V_{\mathbf{0}s_{1,1}1}(\psi^1) V_{\mathbf{x}_2 s_{1,2}-1}(\psi^2))\\
&\quad \cdot \prod_{j=3}^{n+1} (1_{a_j=1} V_{\mathbf{x}_j s_j 1}(\psi^j) V_{\mathbf{y}_j s_j -1}(\psi^j) + 1_{a_j=2} V_{\mathbf{x}_j s_{j,1}1}(\psi^j) V_{\mathbf{y}_j s_{j,2}-1}(\psi^j))\\
&\quad \cdot \Bigg|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n+1\})}}.
\end{aligned}
$$

By recalling the definition of $Tree(\{1,2,\cdots,n+1\},\mathcal{G}(\phi))$ we see that f consists of a finite sum of products of $\mathcal{G}(\phi)$ and thus the domain of the function $\mathbf{s}\mapsto f(\phi)(\mathbf{a},\mathbf{s},\mathbf{X})$ is naturally extended to be $[0,\beta)^{a_1}\times\prod_{j=3}^{n+1}[0,\beta)^{a_j}$. For simplicity, set $[0,\beta)^{\mathbf{a}} := [0,\beta)^{a_1}\times\prod_{j=3}^{n+1}[0,\beta)^{a_j}$, $|\mathbf{a}| := a_1+\sum_{j=3}^{n+1}a_j$ for $\mathbf{a}=(a_1,(a_j)_{j=3}^{n+1})$. We can see from the definition that for any $s_0,t_0\in[0,\beta)$

with $s_0 \neq t_0$, $\rho, \eta \in \{1,2\}$, $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$,

$$\lim_{(s,t)\to(s_0,t_0)} \sup_{\phi\in Q} |\mathcal{G}(\phi)(\rho\mathbf{x}s, \eta\mathbf{y}t) - \mathcal{G}(\phi)(\rho\mathbf{x}s_0, \eta\mathbf{y}t_0)| = 0. \tag{4.53}$$

Define the subset S of $[0,\beta)^{\mathbf{a}}$ by

$$\begin{aligned} S := \{\mathbf{s} \in [0,\beta)^{\mathbf{a}} \mid \mathbf{s} = (s_1, s_2, \cdots, s_{|\mathbf{a}|}),& \\ (i,j \in \{1,\cdots,|\mathbf{a}|\}) \wedge i \neq j \to s_i \neq s_j\}.& \end{aligned}$$

Note that the Lebesgue measure of $[0,\beta)^{\mathbf{a}}\backslash S$ is zero. It follows from the properties (3.2), (3.4), (4.50), (4.53) that

$$\lim_{\substack{\mathbf{s}\to\mathbf{s}_0 \\ \mathbf{s}\in[0,\beta)^{\mathbf{a}}}} \sup_{\phi\in Q} |f(\phi)(\mathbf{a},\mathbf{s},\mathbf{X}) - f(\phi)(\mathbf{a},\mathbf{s}_0,\mathbf{X})| = 0, \quad (\forall \mathbf{s}_0 \in S), \tag{4.54}$$

$$\sup_{\mathbf{s}\in[0,\beta)^{\mathbf{a}}} \sup_{\phi\in Q} |f(\phi)(\mathbf{a},\mathbf{s},\mathbf{X})| < \infty. \tag{4.55}$$

We can consider $f(\cdot)(\mathbf{a},\cdot,\mathbf{X})$ as an element of $L^1([0,\beta)^{\mathbf{a}}, C(Q,\mathbb{C}))$. For any $s \in [0,\beta)$ there uniquely exists $s' \in [0,\beta)_h$ such that $s \in [s', s' + \frac{1}{h})$. Let us define the map $p_h : [0,\beta) \to [0,\beta)_h$ by $p_h(s) := s'$. Then, define the map $P_h : [0,\beta)^n \to [0,\beta)_h^n$ by $P_h(s_1,\cdots,s_n) := (p_h(s_1),\cdots,p_h(s_n))$. It follows from (4.54) that

$$\lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \sup_{\phi\in Q} |f(\phi)(\mathbf{a},P_h(\mathbf{s}),\mathbf{X}) - f(\phi)(\mathbf{a},\mathbf{s},\mathbf{X})| = 0, \quad (\forall \mathbf{s} \in S). \tag{4.56}$$

By (4.55), (4.56) we can apply the dominated convergence theorem in $L^1([0,\beta)^{\mathbf{a}}, C(Q,\mathbb{C}))$ to ensure that

$$\lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \int_{[0,\beta)^{\mathbf{a}}} d\mathbf{s} f(\cdot)(\mathbf{a},P_h(\mathbf{s}),\mathbf{X}) = \int_{[0,\beta)^{\mathbf{a}}} d\mathbf{s} f(\cdot)(\mathbf{a},\mathbf{s},\mathbf{X}) \text{ in } C(Q,\mathbb{C}).$$

By using this convergence property in (4.52) we can reach the conclusion that $a_{n,L,h}(\cdot)$ converges in $C(Q,\mathbb{C})$ as $h \to \infty$. In the same way as above we can prove that $\alpha'_{n,L,h}(\cdot)$ converges in $C(Q,\mathbb{C})$ as $h \to \infty$.

Next we will prove the convergence property as $L \to \infty$. Let us prove the convergence of $a_{n,L,h}$ first. The proof for the convergence of $\alpha'_{n,L,h}$ is much simpler because of the presence of the term $\hat{V}_2$. We will see it after completing the proof for $a_{n,L,h}$. For this purpose we need to disclose the operator $Tree(\{1,2,\cdots,n+1\},\mathcal{G})$. For $p,q\in\{1,2,\cdots,n+1\}$ set

$$B_{\{p,q\}}:=\sum_{\mathbf{X}\in I^2}\tilde{\mathcal{G}}(\mathbf{X})\frac{\partial}{\partial\psi^p_{X_1}}\frac{\partial}{\partial\psi^q_{X_2}},$$

where $\tilde{\mathcal{G}}$ is the anti-symmetric extension of $\mathcal{G}$ defined as in (3.5). Note that by anti-symmetry $B_{\{p,q\}}=B_{\{q,p\}}$. We can rewrite $a_{n,L,h}$ as follows.

$$a_{n,L,h}=\frac{2^n}{(n!)^2}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n+1\})}a_{n,L,h}(T),$$

where

(4.57)

$$\begin{aligned}
&a_{n,L,h}(T)\\
&:=L^{-d(n-1)}\int_{[0,1]^n}d\mathbf{s}\sum_{\sigma\in\mathbb{S}_{n+1}(T)}\varphi(T,\sigma,\mathbf{s})\left(\sum_{p,q=1}^{n+1}M(T,\sigma,\mathbf{s})_{p,q}B_{\{p,q\}}\right)^n\\
&\quad\cdot\prod_{\{p,q\}\in T}B_{\{p,q\}}\left(\frac{1}{h}\right)^2\sum_{\substack{\mathbf{x}\in\Gamma\\ s,t\in[0,\beta)_h}}\nu(s,t)V_{\mathbf{0}s1}(\psi^1)V_{\mathbf{x}t-1}(\psi^2)\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{\substack{\mathbf{x},\mathbf{y}\in\Gamma\\ s,t\in[0,\beta)_h}}\nu(s,t)V_{\mathbf{x}s1}(\psi^j)V_{\mathbf{y}t-1}(\psi^j)\right)\Bigg|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n+1\})}}.
\end{aligned}$$

We have to study case by case depending on the tree's configuration. Let us begin with the simplest case $n=1$. Set

$$\Gamma_L:=\left\{-\left\lfloor\frac{L}{2}\right\rfloor,-\left\lfloor\frac{L}{2}\right\rfloor+1,\cdots,-\left\lfloor\frac{L}{2}\right\rfloor+L-1\right\}^d.$$

Since $\mathbb{T}(\{1,2\})=\{1,2\}$ and $M(T,\sigma,\mathbf{s})$ is symmetric,

$$\begin{aligned}
&a_{1,L,h}(T)\\
&=\int_{[0,1]}ds\sum_{\sigma\in\mathbb{S}_2(T)}\varphi(T,\sigma,s)2M(T,\sigma,s)_{1,2}B^2_{\{1,2\}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\frac{1}{h}\right)^2 \sum_{\substack{\mathbf{x}\in\Gamma \\ u,t\in[0,\beta)_h}} \nu(u,t)V_{\mathbf{0}u1}(\psi^1)V_{\mathbf{x}t-1}(\psi^2)\Big|_{\psi^1=\psi^2=0} \\
&= \int_{[0,1]} ds \sum_{\sigma\in\mathbb{S}_2(T)} \varphi(T,\sigma,s)4M(T,\sigma,s)_{1,2} \sum_{\mathbf{x}\in\mathbb{Z}^d} 1_{\mathbf{x}\in\Gamma_L} \\
&\quad \cdot \Bigg(\frac{1}{\beta}\int_0^\beta du \int_0^\beta dt \tilde{\mathcal{G}}(2\mathbf{0}p_h(u)(-1), 2\mathbf{x}p_h(t)1)\tilde{\mathcal{G}}(1\mathbf{0}p_h(u)1, 1\mathbf{x}p_h(t)(-1)) \\
&\qquad - \beta\tilde{\mathcal{G}}(2\mathbf{0}0(-1), 2\mathbf{x}01)\tilde{\mathcal{G}}(1\mathbf{0}01, 1\mathbf{x}0(-1))\Bigg).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} a_{1,L,h}(T) \\
&\quad = \int_{[0,1]} ds \sum_{\sigma\in\mathbb{S}_2(T)} \varphi(T,\sigma,s)4M(T,\sigma,s)_{1,2} \sum_{\mathbf{x}\in\mathbb{Z}^d} 1_{\mathbf{x}\in\Gamma_L} \frac{1}{\beta}\int_0^\beta du \int_0^\beta dt \\
&\qquad \cdot \big(\tilde{\mathcal{G}}(2\mathbf{0}u(-1), 2\mathbf{x}t1)\tilde{\mathcal{G}}(1\mathbf{0}u1, 1\mathbf{x}t(-1)) \\
&\qquad\quad - \tilde{\mathcal{G}}(2\mathbf{0}0(-1), 2\mathbf{x}01)\tilde{\mathcal{G}}(1\mathbf{0}01, 1\mathbf{x}0(-1))\big).
\end{aligned}
$$

We can deduce from the definition that for any $\mathbf{X} \in (\{1,2\} \times \mathbb{Z}^d \times [0,\beta) \times \{1,-1\})^2$, $\lim_{L\to\infty} \tilde{\mathcal{G}}(\mathbf{X})$ converges in $C(Q,\mathbb{C})$. Moreover by (4.51),

$$
\begin{aligned}
&\sup_{\phi\in Q} 1_{\mathbf{x}\in\Gamma_L} |\tilde{\mathcal{G}}(2\mathbf{0}u(-1), 2\mathbf{x}t1)\tilde{\mathcal{G}}(1\mathbf{0}u1, 1\mathbf{x}t(-1)) \\
&\qquad\qquad - \tilde{\mathcal{G}}(2\mathbf{0}0(-1), 2\mathbf{x}01)\tilde{\mathcal{G}}(1\mathbf{0}01, 1\mathbf{x}0(-1))| \\
&\le \frac{c\cdot c(d,\beta,\theta)^2}{(1+(\frac{2}{\pi})^{d+1}\sum_{j=1}^d |\langle\mathbf{x},\mathbf{e}_j\rangle|^{d+1})^2}.
\end{aligned}
$$

Therefore, the dominated convergence theorem in $L^1(\mathbb{Z}^d \times [0,\beta)^2, C(Q,\mathbb{C}))$ guarantees that $\lim_{L\to\infty,L\in\mathbb{N}} \lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}} a_{1,L,h}$ converges in $C(Q,\mathbb{C})$.

Let us consider the case $n \in \mathbb{N}_{\ge 2}$. To make clear the structure, let us add the superscript $1,-1$ to the Grassmann variables and rewrite the formula

(4.57) as follows.

$$
\begin{aligned}
&a_{n,L,h}(T)\\
&=\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}ER^n\\
&\quad\cdot\prod_{\{p,q\}\in T}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)\sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})V_{\mathbf{x}t_2-1}(\psi^{2,-1})\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j 1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}},
\end{aligned}
$$

where

$$
\begin{aligned}
&E:=\int_{[0,1]^n}d\mathbf{s}\sum_{\sigma\in\mathbb{S}_{n+1}(T)}\varphi(T,\sigma,\mathbf{s}),\\
&B_{(p,f),(q,g)}:=\sum_{\mathbf{X}\in I^2}\tilde{\mathcal{G}}(\mathbf{X})\frac{\partial}{\partial\psi^{p,f}_{X_1}}\frac{\partial}{\partial\psi^{q,g}_{X_2}},\\
&R:=\sum_{a,b\in\{1,-1\}}\sum_{p,q=1}^{n+1}M(T,\sigma,\mathbf{s})_{p,q}B_{(p,a),(q,b)}.
\end{aligned}
$$

Since the integration and the summation with $\mathbf{s}$, σ are irrelevant in the following argument, we do not indicate the dependency of R on these variables. For $T\in\mathbb{T}(\{1,2,\cdots,n+1\})$ we consider the vertex 1 as the root of T. For $j\in\{1,2,\cdots,n+1\}$ let $\mathrm{dis}_T(1,j)$ denote the length of the shortest path between 1 and j in T. Let us consider the case that

$$
(4.58)\qquad \exists v\in\{3,4,\cdots,n+1\}(\{j,v\}\in T\to\mathrm{dis}_T(j,1)+1=\mathrm{dis}_T(v,1)).
$$

In this case v is the terminal of a branch of T and thus there uniquely exists $v'\in\{1,2,\cdots,n+1\}\backslash\{v\}$ such that $\{v',v\}\in T$. The operator $B_{(v,a),(v',b)}$ erases one Grassmann variable from $V_{\mathbf{x}s_v 1}(\psi^{v,1})V_{\mathbf{y}t_v-1}(\psi^{v,-1})$. The remaining 2 variables with the superscript '$v,-a$' must be erased by R^n. Thus, there is at least one operator, at most two operators among n of R such that they are to act on the Grassmann variables with the superscript '$v,-a$'. By decomposing these operators we have that

$$\begin{aligned}
&a_{n,L,h}(T)\\
&=\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}E\\
&\quad\cdot\prod_{\{p,q\}\in T\setminus\{\{v,v'\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)\sum_{a,b\in\{1,-1\}}B_{(v,a),(v',b)}\\
&\quad\cdot\Bigg(nR^{n-1}B_{(v,-a),(v,-a)}\\
&\qquad+\left(\begin{array}{c}n\\2\end{array}\right)R^{n-2}\left(2\sum_{c\in\{1,-1\}}\sum_{p=1}^{n+1}1_{(p,c)\neq(v,-a)}M(T,\sigma,\mathbf{s})_{p,v}B_{(p,c),(v,-a)}\right)^2\Bigg)\\
&\quad\cdot\sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})V_{\mathbf{x}t_2-1}(\psi^{2,-1})\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}\\
&=n\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}ER^{n-1}\\
&\quad\cdot\prod_{\{p,q\}\in T\setminus\{\{v,v'\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)\sum_{a,b\in\{1,-1\}}B_{(v,a),(v',b)}\\
&\quad\cdot\sum_{c\in\{1,-1\}}\sum_{p=1}^{n+1}M(T,\sigma,\mathbf{s})_{p,v}B_{(p,c),(v,-a)}\\
&\quad\cdot\sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})V_{\mathbf{x}t_2-1}(\psi^{2,-1})\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}},
\end{aligned}$$

where we used that

$$M(T,\sigma,\mathbf{s})_{p,q}=M(T,\sigma,\mathbf{s})_{q,p},\quad M(T,\sigma,\mathbf{s})_{p,p}=1,$$
$$B_{(p,f),(q,g)}=B_{(q,g),(p,f)}.$$

Remark that

(4.59)
$$\sum_{s\in[0,\beta)_h}\nu(s,t)B_{(v,1),(v,1)}V_{\mathbf{x}s1}(\psi^{v,1})=\sum_{t\in[0,\beta)_h}\nu(s,t)B_{(v,-1),(v,-1)}V_{\mathbf{y}t-1}(\psi^{v,-1})$$
$$=0.$$

Thus, the operator $B_{(v,-a),(v,-a)}$ in the above expansion can be eliminated. As the result,

$$\begin{aligned}
&a_{n,L,h}(T)\\
&=n\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}ER^{n-1}\\
&\quad\cdot\sum_{a\in\{1,-1\}}\left(\left(\sum_{c\in\{1,-1\}}\sum_{\substack{p=1\\p\neq v}}^{n+1}M(T,\sigma,\mathbf{s})_{p,v}B_{(p,c),(v,-a)}+B_{(v,1),(v,-1)}\right)\right.\\
&\qquad\left.\cdot\prod_{\{p,q\}\in T\backslash\{\{v,v'\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)\sum_{b\in\{1,-1\}}B_{(v,a),(v',b)}\right)\\
&\quad\cdot\sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_11}(\psi^{1,1})V_{\mathbf{x}t_2-1}(\psi^{2,-1})\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}.
\end{aligned}$$

Set for $a\in\{1,-1\}$

$$R(a):=\sum_{c\in\{1,-1\}}\sum_{\substack{p=1\\p\neq v}}^{n+1}M(T,\sigma,\mathbf{s})_{p,v}B_{(p,c),(v,-a)}+B_{(v,1),(v,-1)},$$

$$B(a) := \prod_{\{p,q\}\in T\backslash\{\{v,v'\}\}}\left(\sum_{f,g\in\{1,-1\}} B_{(p,f),(q,g)}\right)\sum_{b\in\{1,-1\}} B_{(v,a),(v',b)}$$

to simplify the following explanation. We carry out a recursive estimation along the tree lines from younger branches to the root 1. Here we need to estimate along the straight line whose terminal is the vertex v first of all. We apply $B(a)$ and then $R(a)$ to the given Grassmann polynomial. The rest of the Grassmann variables are erased by R^{n-1}. The application of $R(a)$ yields another $\tilde{\mathcal{G}}(\cdot)$ which together with $\tilde{\mathcal{G}}(\cdot)$ created by $B(a)$ are integrated with respect to the variables at the vertex v. The application of $B(a)$ combinatorially yields at most

$$\prod_{j=1}^{2}\left(\begin{array}{c}2\\ d_j(T)\end{array}\right)d_j(T)!\cdot\prod_{k=3}^{n+1}\left(\begin{array}{c}4\\ d_k(T)\end{array}\right)d_k(T)!$$

factors, which is bounded by c^n with a generic positive constant c. Recall that $d_j(T)$ is the degree of the vertex j in T. After applying $B(a)$ and $R(a)$ we have Grassmann monomials of degree $2(n-1)$. Applying R^{n-1} to each of the remaining monomials combinatorially gives at most $(2(n-1))!$ factors. By performing the recursive estimation as described above and using (3.2), (3.4), (4.50) we observe that

(4.60)

$$\begin{aligned}
&|a_{n,L,h}(T)|\\
&\le L^{-d(n-1)}(2(n-1))!c^n c(d,\beta,\theta)^{n-1}\\
&\quad\cdot\frac{1}{h}\sum_{s\in[0,\beta)_h}\sup_{\substack{X\in I,\eta\in\{1,2\}\\ \zeta\in\{1,-1\}}}\left(\frac{1}{h}\sum_{\substack{\mathbf{y}\in\Gamma\\ t\in[0,\beta)_h}}|\tilde{\mathcal{G}}(X,\eta\mathbf{y}t\zeta)||\nu(s,t)|\right)\\
&\quad\cdot\left(\sup_{\substack{X\in I,\eta\in\{1,2\}\\ \zeta\in\{1,-1\}}}\left(\frac{1}{h^2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Gamma\\ s,t\in[0,\beta)_h}}|\tilde{\mathcal{G}}(X,\eta\mathbf{y}t\zeta)||\nu(s,t)|\right)\right)^{n-2}\\
&\quad\cdot\left(\sup_{\substack{X,Z\in I,\rho,\eta\in\{1,2\}\\ \xi,\zeta\in\{1,-1\}}}\left(\frac{1}{h^2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Gamma\\ s,t\in[0,\beta)_h}}|\tilde{\mathcal{G}}(X,\rho\mathbf{x}s\xi)||\tilde{\mathcal{G}}(Z,\eta\mathbf{y}t\zeta)||\nu(s,t)|\right)\right.
\end{aligned}$$

$$
+\sup_{\substack{X\in I,\rho,\rho',\eta\in\{1,2\}\\ \xi,\xi',\zeta\in\{1,-1\}}}\left(\frac{1}{h^2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Gamma\\ s,t\in[0,\beta)_h}}|\tilde{\mathcal{G}}(X,\rho\mathbf{x}s\xi)||\tilde{\mathcal{G}}(\rho'\mathbf{x}s\xi',\eta\mathbf{y}t\zeta)||\nu(s,t)|\right)\Bigg)
$$

$$
\le L^{-d(n-1)}(2(n-1))!c^n c(d,\beta,\theta)^{2n}\beta\sum_{\mathbf{x}\in\Gamma}\frac{1}{1+\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}
$$

$$
\cdot\left(\beta L^d\sum_{\mathbf{x}\in\Gamma}\frac{1}{1+\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}\right)^{n-2}
$$

$$
\cdot\beta\left(\sum_{\mathbf{x}\in\Gamma}\frac{1}{1+\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}\right)^2
$$

$$
\le c^n(2n)!c(d,\beta,\theta)^{2n}\beta^nL^{-d}.
$$

Next let us consider the case that (4.58) does not hold. In this case the tree T is one straight line whose terminal is the vertex 2. By changing the numbers if necessary we may assume that

$$
T=\{\{1,n+1\},\{n+1,n\},\cdots,\{4,3\},\{3,2\}\}.
$$

The term $a_{n,L,h}(T)$ can be further decomposed as follows.

$$
a_{n,L,h}(T)=\sum_{a,b,c,d\in\{1,-1\}}a_{n,L,h}^{(a,b,c,d)}(T),
$$

where

$$
a_{n,L,h}^{(a,b,c,d)}(T)
$$

$$
:=\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}ER^n
$$

$$
\cdot\prod_{\{p,q\}\in T\setminus\{\{2,3\},\{3,4\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)B_{(2,a),(3,b)}B_{(3,c),(4,d)}
$$

$$
\cdot\sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_11}(\psi^{1,1})V_{\mathbf{x}t_2-1}(\psi^{2,-1})
$$

$$
\cdot\prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\ (\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}.
$$

Let us consider $a_{n,L,h}^{(a,1,1,d)}(T)$. In this case one Grassmann variable with the superscript '$2,-1'$ and two Grassmann variables with the superscript '$3,-1'$ are untouched by the derivatives along the tree lines and thus must be erased by the operator R^n. Set

$$\begin{aligned}R' &:= 2^2 M(T,\sigma,\mathbf{s})_{3,2}B_{(3,-1),(2,-1)}\\&\quad + 2\sum_{\delta\in\{1,-1\}}\sum_{p=1}^{n+1}1_{(p,\delta)\neq(3,-1)}M(T,\sigma,\mathbf{s})_{p,2}B_{(p,\delta),(2,-1)},\end{aligned}$$

$$R'' := \sum_{\delta\in\{1,-1\}}\sum_{p=1}^{n+1}1_{(p,\delta)\neq(2,-1),(3,-1)}M(T,\sigma,\mathbf{s})_{p,3}B_{(p,\delta),(3,-1)},$$

$$\begin{aligned}&B^{(a,1,1,d)}\\&:= \prod_{\{p,q\}\in T\backslash\{\{2,3\},\{3,4\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)B_{(2,a),(3,1)}B_{(3,1),(4,d)}\\&\quad\cdot\sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})V_{\mathbf{x}t_2 -1}(\psi^{2,-1})\prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j 1}(\psi^{j,1})V_{\mathbf{y}t_j -1}(\psi^{j,-1})\right).\end{aligned}$$

Let us observe that

$$\begin{aligned}&a_{n,L,h}^{(a,1,1,d)}(T)\\&= \left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}E\\&\quad\cdot nR^{n-1}\left(2\sum_{\delta\in\{1,-1\}}\sum_{p=1}^{n+1}M(T,\sigma,\mathbf{s})_{p,2}B_{(p,\delta),(2,-1)}\right)\\&\quad\cdot B^{(a,1,1,d)}\Big|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}\\&= n\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}E\\&\quad\cdot\left((n-1)R^{n-2}\left(2\sum_{\eta\in\{1,-1\}}\sum_{q=1}^{n+1}1_{(q,\eta)\neq(3,-1)}M(T,\sigma,\mathbf{s})_{q,3}B_{(q,\eta),(3,-1)}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&\quad\cdot 2M(T,\sigma,\mathbf{s})_{3,2}B_{(3,-1),(2,-1)}\\
&+\Bigg((n-1)R^{n-2}B_{(3,-1),(3,-1)}\\
&\quad+1_{n\ge 3}\begin{pmatrix} n-1\\ 2\end{pmatrix}R^{n-3}\\
&\qquad\cdot\Bigg(2\sum_{\eta\in\{1,-1\}}\sum_{q=1}^{n+1}1_{(q,\eta)\neq(3,-1)}M(T,\sigma,\mathbf{s})_{q,3}B_{(q,\eta),(3,-1)}\Bigg)^2\Bigg)\\
&\quad\cdot\Bigg(2\sum_{\delta\in\{1,-1\}}\sum_{p=1}^{n+1}1_{(p,\delta)\neq(3,-1)}M(T,\sigma,\mathbf{s})_{p,2}B_{(p,\delta),(2,-1)}\Bigg)\Bigg)\\
&\cdot B^{(a,1,1,d)}\Big|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}\\
&=n\Big(\frac{1}{h}\Big)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\Bigg(\Big(\frac{1}{h}\Big)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\Bigg)L^{-d(n-1)}E\\
&\cdot\Bigg((n-1)R^{n-2}\Bigg(2\sum_{\eta\in\{1,-1\}}\sum_{q=1}^{n+1}1_{(q,\eta)\neq(3,-1)}M(T,\sigma,\mathbf{s})_{q,3}B_{(q,\eta),(3,-1)}\Bigg)\\
&\quad\cdot 2M(T,\sigma,\mathbf{s})_{3,2}B_{(3,-1),(2,-1)}\\
&+(n-1)R^{n-2}\Bigg(\sum_{\eta\in\{1,-1\}}\sum_{q=1}^{n+1}M(T,\sigma,\mathbf{s})_{q,3}B_{(q,\eta),(3,-1)}\Bigg)\\
&\quad\cdot\Bigg(2\sum_{\delta\in\{1,-1\}}\sum_{p=1}^{n+1}1_{(p,\delta)\neq(3,-1)}M(T,\sigma,\mathbf{s})_{p,2}B_{(q,\delta),(2,-1)}\Bigg)\Bigg)\\
&\cdot B^{(a,1,1,d)}\Big|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}\\
&=n(n-1)\Big(\frac{1}{h}\Big)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\Bigg(\Big(\frac{1}{h}\Big)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\Bigg)\\
&\cdot L^{-d(n-1)}ER^{n-2}R'R''B^{(a,1,1,d)}\Big|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}.
\end{aligned}
$$

In the derivation of the last equality we used the fact that since (4.59) with $v = 3$ holds, the term with $B_{(3,-1),(3,-1)}$ does not contribute to the result. It is important that there is no link between the vertex 2 and the vertex 3 in the operator R''. By using (3.2), (3.4), (4.50) we perform the recursive estimation from the terminal 2 to the root 1. The important point is that the extra $\tilde{\mathcal{G}}(\cdot)$ produced by R'' is added to the integration on the vertex 3. We uniformly bound all the $\tilde{\mathcal{G}}(\cdot)$s produced by $R^{n-2}R'$, not integrating them. As the result,

(4.61)

$$\begin{aligned}
&|a_{n,L,h}^{(a,1,1,d)}(T)|\\
&\le L^{-d(n-1)}(2(n-1))!c^n c(d,\beta,\theta)^{n-1}\\
&\quad\cdot \frac{1}{h}\sum_{s\in[0,\beta)_h}\sup_{\substack{X\in I,\eta\in\{1,2\}\\ \zeta\in\{1,-1\}}}\left(\frac{1}{h}\sum_{\substack{\mathbf{y}\in\Gamma\\ t\in[0,\beta)_h}}|\tilde{\mathcal{G}}(X,\eta\mathbf{y}t\zeta)||v(s,t)|\right)\\
&\quad\cdot\left(\sup_{\substack{X\in I,\eta\in\{1,2\}\\ \zeta\in\{1,-1\}}}\left(\frac{1}{h^2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Gamma\\ s,t\in[0,\beta)_h}}|\tilde{\mathcal{G}}(X,\eta\mathbf{y}t\zeta)||v(s,t)|\right)\right)^{n-2}\\
&\quad\cdot\left(\sup_{\substack{X,Z\in I,\rho,\eta\in\{1,2\}\\ \xi,\zeta\in\{1,-1\}}}\left(\frac{1}{h^2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Gamma\\ s,t\in[0,\beta)_h}}|\tilde{\mathcal{G}}(X,\rho\mathbf{x}s\xi)||\tilde{\mathcal{G}}(Z,\eta\mathbf{y}t\zeta)||v(s,t)|\right)\right)\\
&\le (2n)!c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d}\left(\sum_{\mathbf{x}\in\Gamma}\frac{1}{1+\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}\right)^{n+1}\\
&\le (2n)!c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d}.
\end{aligned}$$

The same argument as above shows that

$$|a_{n,L,h}^{(a,-1,-1,d)}(T)| \le (2n)!c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d}.$$

Next let us consider $a_{n,L,h}^{(a,1,-1,d)}(T)$. In this case one Grassmann variable, which originally belongs to $V_{\mathbf{x}s1}(\psi^{3,1})$, remains after applying the operators along the tree lines. This Grassmann variable must be erased by R^n. Thus, inside $a_{n,L,h}^{(a,1,-1,d)}(T)$, the operator R^n can be decomposed as follows.

$$2nR^{n-1}M(T,\sigma,\mathbf{s})_{2,3}B_{(2,a),(3,1)} + 2nR^{n-1}B_{(3,-1),(3,1)}$$

$$+ 2nR^{n-1}\sum_{p=4}^{n+1}\sum_{\delta\in\{1,-1\}} M(T,\sigma,\mathbf{s})_{p,3}B_{(p,\delta),(3,1)}.$$

Let $\tilde{a}_{n,L,h}^{(a,1,-1,d)}(T)$ denote the following.

$$\begin{aligned}
&\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}E\\
&\cdot 2nR^{n-1}M(T,\sigma,\mathbf{s})_{2,3}B_{(2,a),(3,1)}\\
&\cdot\prod_{\{p,q\}\in T\backslash\{\{2,3\},\{3,4\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)B_{(2,a),(3,1)}B_{(3,-1),(4,d)}\\
&\cdot\sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})V_{\mathbf{x}t_2 -1}(\psi^{2,-1})\\
&\cdot\prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j 1}(\psi^{j,1})V_{\mathbf{y}t_j -1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}.
\end{aligned}$$

In fact $\tilde{a}_{n,L,h}^{(a,1,-1,d)}(T)$ is derived by replacing R^n by $2nR^{n-1}M(T,\sigma,\mathbf{s})_{2,3}$ $B_{(2,a),(3,1)}$ inside $a_{n,L,h}^{(a,1,-1,d)}(T)$. Since the application of $B_{(3,-1),(3,1)}$, $B_{(p,\delta),(3,1)}$ $(p\in\{4,5,\cdots,n+1\},\ \delta\in\{1,-1\})$ gives an additional free propagator at the vertex 3, the same calculation as that leading to (4.61) yields that

$$|a_{n,L,h}^{(a,1,-1,d)}(T)-\tilde{a}_{n,L,h}^{(a,1,-1,d)}(T)|\le(2n)!c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d}.$$

The term $a_{n,L,h}^{(a,-1,1,d)}(T)$ can be analyzed in the same way as above. The result is that

$$|a_{n,L,h}^{(a,-1,1,d)}(T)-\tilde{a}_{n,L,h}^{(a,-1,1,d)}(T)|\le(2n)!c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d},$$

where

$$\begin{aligned}
&\tilde{a}_{n,L,h}^{(a,-1,1,d)}(T)\\
&:=\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)L^{-d(n-1)}E
\end{aligned}$$

$$\cdot 2nR^{n-1}M(T,\sigma,\mathbf{s})_{2,3}B_{(2,a),(3,-1)}$$
$$\cdot \prod_{\{p,q\}\in T\backslash\{\{2,3\},\{3,4\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)B_{(2,a),(3,-1)}B_{(3,1),(4,d)}$$
$$\cdot \sum_{\mathbf{x}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})V_{\mathbf{x}t_2-1}(\psi^{2,-1})$$
$$\cdot \prod_{j=3}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j 1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}.$$

By combining these results we conclude that

$$|a_{n,L,h}(T)-\tilde{a}_{n,L,h}(T)|\le(2n)!c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d},$$

where

$$\tilde{a}_{n,L,h}(T):=\sum_{a,d\in\{1,-1\}}(\tilde{a}_{n,L,h}^{(a,1,-1,d)}(T)+\tilde{a}_{n,L,h}^{(a,-1,1,d)}(T)).$$

We can reform $\tilde{a}_{n,L,h}(T)$ as follows.

$$\tilde{a}_{n,L,h}(T)$$
$$=\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)$$
$$\cdot \sum_{b\in\{1,-1\}}\sum_{\mathbf{x}\in\Gamma}B_{(2,-1),(3,b)}B_{(2,-1),(3,b)}$$
$$\cdot V_{\mathbf{x}t_2-1}(\psi^{2,-1})(1_{b=1}V_{\mathbf{0}s_3 1}(\psi^{3,1})+1_{b=-1}V_{\mathbf{0}t_3-1}(\psi^{3,-1}))$$
$$\cdot 2nL^{-d(n-2)}EM(T,\sigma,\mathbf{s})_{2,3}R^{n-1}\prod_{\{p,q\}\in T\backslash\{\{2,3\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)$$
$$\cdot \sum_{\mathbf{z}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})(1_{b=1}V_{\mathbf{z}t_3-1}(\psi^{3,-1})+1_{b=-1}V_{\mathbf{z}s_3 1}(\psi^{3,1}))$$
$$\cdot \prod_{j=4}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j 1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}.$$

Let us observe that there is a recursive structure here. We can repeat the same procedure as above on the tree $T\backslash\{\{2,3\}\}$. The result is that

$$\left|a_{n,L,h}(T)-\tilde{\tilde{a}}_{n,L,h}(T)\right|\le(2n)!c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d},$$

where

$$
\begin{aligned}
&\tilde{\tilde{a}}_{n,L,h}(T)\\
&:=\left(\frac{1}{h}\right)^2\sum_{s_1,t_2\in[0,\beta)_h}\nu(s_1,t_2)\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j,t_j\in[0,\beta)_h}\nu(s_j,t_j)\right)\\
&\quad\cdot\sum_{b\in\{1,-1\}}\sum_{\mathbf{x}\in\Gamma}B_{(2,-1),(3,b)}B_{(2,-1),(3,b)}\\
&\qquad\cdot V_{\mathbf{x}t_2-1}(\psi^{2,-1})(1_{b=1}V_{\mathbf{0}s_3 1}(\psi^{3,1})+1_{b=-1}V_{\mathbf{0}t_3-1}(\psi^{3,-1}))\\
&\quad\cdot\sum_{c\in\{1,-1\}}\sum_{\mathbf{z}\in\Gamma}B_{(3,-b),(4,c)}B_{(3,-b),(4,c)}\\
&\qquad\cdot(1_{b=1}V_{\mathbf{z}t_3-1}(\psi^{3,-1})+1_{b=-1}V_{\mathbf{z}s_3 1}(\psi^{3,1}))\\
&\qquad\cdot(1_{c=1}V_{\mathbf{0}s_4 1}(\psi^{4,1})+1_{c=-1}V_{\mathbf{0}t_4-1}(\psi^{4,-1}))\\
&\quad\cdot 2^2n(n-1)L^{-d(n-3)}EM(T,\sigma,\mathbf{s})_{2,3}M(T,\sigma,\mathbf{s})_{3,4}R^{n-2}\\
&\quad\cdot\prod_{\{p,q\}\in T\setminus\{\{2,3\},\{3,4\}\}}\left(\sum_{f,g\in\{1,-1\}}B_{(p,f),(q,g)}\right)\\
&\quad\cdot\sum_{\mathbf{w}\in\Gamma}V_{\mathbf{0}s_1 1}(\psi^{1,1})(1_{c=1}V_{\mathbf{w}t_4-1}(\psi^{4,-1})+1_{c=-1}V_{\mathbf{w}s_4 1}(\psi^{4,1}))\\
&\quad\cdot\prod_{j=5}^{n+1}\left(\sum_{\mathbf{x},\mathbf{y}\in\Gamma}V_{\mathbf{x}s_j 1}(\psi^{j,1})V_{\mathbf{y}t_j-1}(\psi^{j,-1})\right)\Bigg|_{\substack{\psi^{j,\delta}=0\\(\forall j\in\{1,2,\cdots,n+1\},\delta\in\{1,-1\})}}.
\end{aligned}
$$

By repeating this procedure we eventually have

$$
|a_{n,L,h}(T)-b_{n,L,h}(T)|\le(2n)!c^n c(d,\beta,\theta)^{2n}\beta^nL^{-d}, \tag{4.62}
$$

where

$$
\begin{aligned}
&b_{n,L,h}(T):=\prod_{j=2}^{n+1}\left(\sum_{\mathbf{x}_j\in\Gamma}\right)g_{L,h}(\phi)(\mathbf{x}_2,\mathbf{x}_3,\cdots,\mathbf{x}_{n+1}),\\
&g_{L,h}(\phi)(\mathbf{x}_2,\mathbf{x}_3,\cdots,\mathbf{x}_{n+1})\\
&:=\left(\frac{1}{h}\right)^2\sum_{s_1,s_2^{-1}\in[0,\beta)_h}\nu(s_1,s_2^{-1})\prod_{j=3}^{n+1}\left(\left(\frac{1}{h}\right)^2\sum_{s_j^1,s_j^{-1}\in[0,\beta)_h}\nu(s_j^1,s_j^{-1})\right)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \sum_{b_3\in\{1,-1\}} B^2_{(2,-1),(3,b_3)} V_{\mathbf{x}_2 s_2^{-1} -1}(\psi^{2,-1}) V_{\mathbf{0} s_3^{b_3} b_3}(\psi^{3,b_3})\\
&\cdot \sum_{b_4\in\{1,-1\}} B^2_{(3,-b_3),(4,b_4)} V_{\mathbf{x}_3 s_3^{-b_3} -b_3}(\psi^{3,-b_3}) V_{\mathbf{0} s_4^{b_4} b_4}(\psi^{4,b_4})\\
&\vdots\\
&\cdot B^2_{(n+1,-b_{n+1}),(1,1)} V_{\mathbf{x}_{n+1} s_{n+1}^{-b_{n+1}} -b_{n+1}}(\psi^{n+1,-b_{n+1}}) V_{\mathbf{0} s_1 1}(\psi^{1,1})\\
&\cdot 2^n n! E \prod_{\{p,q\}\in T} M(T,\sigma,\mathbf{s})_{p,q}.
\end{aligned}
$$

Since $g_{L,h}(\phi)(\mathbf{X})$ is a finite sum of products of $\tilde{\mathcal{G}}$, we can naturally define $g_{L,h}$ as a map from $(\mathbb{Z}^d)^n$ to $C(Q,\mathbb{C})$. By the same argument as the proof of the convergence $\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}} a_{n,L,h}$ in $C(Q,\mathbb{C})$ we can prove that $\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}} g_{L,h}(\cdot)(\mathbf{X})$ converges in $C(Q,\mathbb{C})$ for any $\mathbf{X}\in(\mathbb{Z}^d)^n$ and so does $\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}} b_{n,L,h}(T)$. We can also deduce from the definition of $\tilde{\mathcal{G}}$ and $g_{L,h}$ that $\lim_{L\to\infty,L\in\mathbb{N}}\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}} g_{L,h}(\cdot)(\mathbf{X})$ converges in $C(Q,\mathbb{C})$ for any $\mathbf{X}\in(\mathbb{Z}^d)^n$. It follows from (3.2), (3.4), (4.51) that

$$
\begin{aligned}
&\sup_{\phi\in Q} |g_{L,h}(\phi)(\mathbf{x}_2,\mathbf{x}_3,\cdots,\mathbf{x}_{n+1})| 1_{\mathbf{x}_j\in\Gamma_L\ (j=2,3,\cdots,n+1)}\\
&\le n! c^n c(d,\beta,\theta)^{2n}\beta^n \prod_{l=2}^{n+1} \frac{1}{1+(\frac{2}{\pi})^{d+1}\sum_{j=1}^d |\langle \mathbf{x}_l,\mathbf{e}_j\rangle|^{d+1}},\\
&(\forall \mathbf{x}_j\in\mathbb{Z}^d\ (j=2,3,\cdots,n+1)).
\end{aligned}
$$

The right-hand side of the above inequality is summable over $(\mathbb{Z}^d)^n$. Since

$$
b_{n,L,h}(T) = \prod_{j=2}^{n+1}\left(\sum_{\mathbf{x}_j\in\mathbb{Z}^d}\right) 1_{\mathbf{x}_j\in\Gamma_L(j=2,3,\cdots,n+1)} g_{L,h}(\phi)(\mathbf{x}_2,\mathbf{x}_3,\cdots,\mathbf{x}_{n+1}),
$$

we can apply the dominated convergence theorem in $L^1((\mathbb{Z}^d)^n, C(Q,\mathbb{C}))$ to conclude that $\lim_{L\to\infty,L\in\mathbb{N}}\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}} b_{n,L,h}(T)$ converges in $C(Q,\mathbb{C})$. Observe that

$$
a_{n,L,h} = \frac{2^n}{(n!)^2}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n+1\})} 1_{(4.58)} a_{n,L,h}(T)
$$

$$+\frac{2^n}{(n!)^2}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n+1\})}1_{\neg(4.58)}(a_{n,L,h}(T)-b_{n,L,h}(T))$$
$$+\frac{2^n}{(n!)^2}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n+1\})}1_{\neg(4.58)}b_{n,L,h}(T).$$

By (4.60), (4.62) and the fact that $\sharp\mathbb{T}(\{1,2,\cdots,n+1\})\le c^n n!$,

$$\sup_{\phi\in Q}\left|a_{n,L,h}(\phi)-\frac{2^n}{(n!)^2}\sum_{T\in\mathbb{T}(\{1,2,\cdots,n+1\})}1_{\neg(4.58)}b_{n,L,h}(T)(\phi)\right|$$
$$\le\frac{(2n)!}{n!}c^n c(d,\beta,\theta)^{2n}\beta^n L^{-d}.$$

Since we have checked that $\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}}a_{n,L,h}$, $\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}}b_{n,L,h}(T)$, $\lim_{L\to\infty,L\in\mathbb{N}}\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}}b_{n,L,h}(T)$ converge in $C(Q,\mathbb{C})$, we can deduce from this inequality that $\lim_{L\to\infty,L\in\mathbb{N}}\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}}a_{n,L,h}$ converges in $C(Q,\mathbb{C})$.

Let us confirm the convergence of $\alpha'_{n,L,h}$. By definition,

$$\alpha'_{1,L,h}=-\beta\mathcal{G}(\phi)(\mathbf{1}\mathbf{0}0,\mathbf{1}\mathbf{0}0),$$

which converges in $C(Q,\mathbb{C})$ as $h\to\infty$, $L\to\infty$. Assume that $n\ge 2$. Let us estimate $|\alpha'_{n,L,h}|$ by using the general lemmas Lemma 3.1, Lemma 3.2, which is a simpler way than decomposing the operator $Tree(\{1,\cdots,n\},\mathcal{G})$ as above. By (4.12), $\mathcal{G}(\phi)(\mathbf{X})=\sum_{l=0}^{N_h-N_\beta+1}C_l(\mathbf{X})$, $(\forall\mathbf{X}\in I_0^2)$. Thus, by (4.22)

$$\begin{aligned}&|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}\mathcal{G}(\phi)(X_i,Y_j))_{1\le i,j\le n}|\\&\le(c(d)(1+\beta^{-1}g_d(\Theta)))^n\le c'(d,\beta,\theta)^n,\\&(\forall m,n\in\mathbb{N},\mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m\text{ with }\|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,\\&\quad X_i,Y_i\in I_0\ (i=1,2,\cdots,n)),\end{aligned}$$

where $c'(d,\beta,\theta)(\in\mathbb{R}_{\ge 1})$ is a positive constant depending only on d,β,θ. Moreover, by Lemma 4.15

$$\|\tilde{\mathcal{G}}(\phi)\|_{1,\infty},\|\tilde{\mathcal{G}}(\phi)\|\le\sum_{\mathbf{x}\in\Gamma}\frac{c'(d,\beta,\theta)}{1+\sum_{j=1}^d|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}.$$

Also, by definition

$$\|V_2\|_{1,\infty} \le L^{-d}, \quad \|V_{2,2}\|_{1,\infty} \le 1,$$
$$[V_{2,2}, \tilde{\mathcal{G}}(\phi)]_{1,\infty} \le L^{-d}\|\tilde{\mathcal{G}}(\phi)\| \le L^{-d}\sum_{\mathbf{x}\in\Gamma}\frac{c'(d,\beta,\theta)}{1+\sum_{j=1}^{d}|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}.$$

With these inequalities and the fact that the $\|\cdot\|_{1,\infty}$-norm of the anti-symmetric kernel of $\hat{V}_4$ is bounded by $\|V_{2,2}\|_{1,\infty}$ we can apply (3.16), (3.26) to derive that

$$\begin{aligned}
&|\alpha'_{n,L,h}| \\
&\le \frac{N}{h}\left(\sum_{\mathbf{x}\in\Gamma}\frac{1}{1+\sum_{j=1}^{d}|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}\right)^{n-1}(2^6c'(d,\beta,\theta)L^{-d})^n \\
&\quad+\sum_{l=1}^{n-1}\begin{pmatrix} n \\ l \end{pmatrix}\frac{N}{h}2^{12}c'(d,\beta,\theta)^2\left(\sum_{\mathbf{x}\in\Gamma}\frac{1}{1+\sum_{j=1}^{d}|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}\right)^{n} \\
&\quad\cdot L^{-d}(2^{12}c'(d,\beta,\theta)^2)^{l-1}(2^6c'(d,\beta,\theta)L^{-d})^{n-l} \\
&\le \frac{N}{h}L^{-2d}c^nc'(d,\beta,\theta)^{2n}\sum_{a=0}^{1}\left(\sum_{\mathbf{x}\in\Gamma}\frac{1}{1+\sum_{j=1}^{d}|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-1)|^{d+1}}\right)^{n-a},
\end{aligned}$$

which implies that $\lim_{L\to\infty,L\in\mathbb{N}}\lim_{h\to\infty,h\in\frac{2}{\beta}\mathbb{N}}\alpha'_{n,L,h}=0$ in $C(Q,\mathbb{C})$. Thus, we have seen that $\alpha_{n,L,h}$ converges in $C(Q,\mathbb{C})$ as $h\to\infty(h\in\frac{2}{\beta}\mathbb{N})$, $L\to\infty(L\in\mathbb{N})$ for any $n\in\mathbb{N}$.

Let us complete the proof of the proposition. The inequality (4.48) implies that

$$u\mapsto\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right)$$

is analytic in $D(r)$ for any $\phi\in\mathbb{C}$ and

$$\sup_{(\phi,u)\in\mathbb{C}\times\overline{D(r)}}\left|\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right)\right|\le 1.$$

Thus,

$$\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right)=\sum_{n=1}^{\infty}\alpha_{n,L,h}(\phi)u^n,$$

$$
\begin{aligned}
&(\forall(\phi,u)\in Q\times\overline{D(r/2)}),\\
&\sup_{(\phi,u)\in Q\times\overline{D(\frac{r}{2})}}|\alpha_{n,L,h}(\phi)u^n|\\
&\le\sup_{\phi\in Q}\left|\frac{1}{2\pi i}\oint_{|z|=(1+\varepsilon)\frac{r}{2}}dz\frac{1}{z^{n+1}}\log\left(\int e^{-V(z)(\psi)+W(z)(\psi)}d\mu_{C(\phi)}(\psi)\right)\right|\left(\frac{r}{2}\right)^n\\
&\le\frac{1}{(1+\varepsilon)^n},\quad(\forall\varepsilon\in(0,1)).
\end{aligned}
$$

Therefore we can use the dominated convergence theorem in $l^1(\mathbb{N},C(Q\times\overline{D(r/2)}))$ to ensure that

$$
\begin{aligned}
&\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right),\\
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right)
\end{aligned}
$$

converge in $C(Q\times\overline{D(r/2)})$. Since

$$
\begin{aligned}
&\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)=e^{\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right)},\\
&(\forall(\phi,u)\in Q\times\overline{D(r/2)}),
\end{aligned}
$$

the claims of the proposition follow. □

4.4. Completion of the proof of the main theorem

In this subsection we will complete the proof of Theorem 1.3. The main necessary tools have already been prepared. It remains to study the solvability of the gap equation which is different from the conventional BCS gap equation due to the presence of the imaginary magnetic field. Let us start by showing an inequality which will be used to give a sufficient condition for the solvability of our gap equation.

LEMMA 4.17. *Set* $K:=\frac{1}{2}(2d-|\mu|)$. *Then,*

$$
\inf_{\eta\in[-K,K]}\mathcal{H}^{d-1}(\{\mathbf{k}\in[0,2\pi]^d\mid e(\mathbf{k})=\eta\})\ge 1_{d=1}+1_{d\ge 2}\left(\frac{2d-|\mu|}{10(d-1)d}\right)^{d-1}.
$$

PROOF. Since $|\mu+\eta| < 2d$,

$$\{\mathbf{k} \in [0, 2\pi]^d \mid e(\mathbf{k}) = \eta\} \neq \emptyset, \quad (\forall \eta \in [-K, K]).$$

This implies the lower bound for $d = 1$. Let us assume that $d \geq 2$. Note that

$$\begin{aligned}&\mathcal{H}^{d-1}(\{\mathbf{k} \in [0, 2\pi]^d \mid e(\mathbf{k}) = \eta\})\\ &\geq \mathcal{H}^{d-1}\left(\left\{\mathbf{k} \in \left[0, \frac{\pi}{2}\right]^{d-1} \times [0, \pi] \,\Big|\, \sum_{j=1}^{d} \cos k_j = \frac{1}{2}|\eta + \mu|\right\}\right).\end{aligned}$$

In the following we assume that $\mathbf{k} \in [0, \frac{\pi}{2}]^{d-1} \times [0, \pi]$. Set

$$\varepsilon := \frac{\min\{1, 2d - |\mu|\}}{5(d-1)}.$$

Assume that $\frac{1}{2}|\eta + \mu| \in [l, l + \frac{1}{2})$ for some $l \in \{0, 1, \cdots, d-1\}$. If

$$\begin{aligned}&k_j \in [0, \varepsilon] \quad (\forall j \in \{1, 2, \cdots, l\}),\\ &k_j \in \left[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}\right] \quad (\forall j \in \{l+1, \cdots, d-1\}),\end{aligned} \tag{4.63}$$

then

$$\sum_{j=1}^{d-1} \cos k_j \in [l(1-\varepsilon), l + (d-1-l)\varepsilon] \subset \left[l - \frac{1}{5}, l + \frac{1}{5}\right].$$

Thus,

$$\frac{1}{2}|\eta + \mu| - \sum_{j=1}^{d-1} \cos k_j \in \left[-\frac{1}{5}, \frac{7}{10}\right].$$

Recall that we defined arccos as a map from $(-1, 1)$ to $(0, \pi)$ in the proof of Lemma 4.8. Then, we see that if (4.63) holds and $k_d \in [0, \pi]$, the equality $\sum_{j=1}^{d} \cos k_j = \frac{1}{2}|\eta + \mu|$ is equivalent to

$$k_d = \arccos\left(\frac{1}{2}|\eta + \mu| - \sum_{j=1}^{d-1} \cos k_j\right).$$

Thus,

$$\begin{aligned}&\mathcal{H}^{d-1}\left(\left\{\mathbf{k}\in\left[0,\frac{\pi}{2}\right]^{d-1}\times[0,\pi]\ \Big|\ \sum_{j=1}^{d}\cos k_j=\frac{1}{2}|\eta+\mu|\right\}\right)\\&\ge\prod_{i=1}^{l}\left(\int_0^{\varepsilon}dk_i\right)\prod_{j=l+1}^{d-1}\left(\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}dk_j\right)\\&\quad\cdot\left(1+\sum_{j=1}^{d-1}\frac{\sin^2 k_j}{1-(\frac{1}{2}|\eta+\mu|-\sum_{m=1}^{d-1}\cos k_m)^2}\right)^{\frac{1}{2}}\\&\ge\varepsilon^{d-1}.\end{aligned}$$

Assume that $\frac{1}{2}|\eta+\mu|\in[l+\frac{1}{2},l+1)$ for some $l\in\{0,1,\cdots,d-2\}$. If

$$\begin{aligned}&k_j\in[0,\varepsilon]\quad(\forall j\in\{1,2,\cdots,l+1\}),\\&k_j\in\left[\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}\right]\quad(\forall j\in\{l+2,\cdots,d-1\}),\end{aligned}\tag{4.64}$$

then

$$\sum_{j=1}^{d-1}\cos k_j\in[(l+1)(1-\varepsilon),l+1+(d-l-2)\varepsilon]\subset\left[l+\frac{4}{5},l+\frac{6}{5}\right].$$

Thus,

$$\frac{1}{2}|\eta+\mu|-\sum_{j=1}^{d-1}\cos k_j\in\left[-\frac{7}{10},\frac{1}{5}\right].$$

Therefore, if (4.64) and $k_d\in[0,\pi]$ hold, the equality $\sum_{j=1}^{d}\cos k_j=\frac{1}{2}|\eta+\mu|$ is equivalently written as

$$k_d=\arccos\left(\frac{1}{2}|\eta+\mu|-\sum_{j=1}^{d-1}\cos k_j\right).$$

Thus, by the same calculation as above we have that

$$\mathcal{H}^{d-1}\left(\left\{\mathbf{k}\in\left[0,\frac{\pi}{2}\right]^{d-1}\times[0,\pi]\ \Big|\ \sum_{j=1}^{d}\cos k_j=\frac{1}{2}|\eta+\mu|\right\}\right)\ge\varepsilon^{d-1}.\tag{4.65}$$

Assume that $\frac{1}{2}|\eta+\mu| \in [d-\frac{1}{2}, d)$. If $k_j \in [0,\varepsilon]$ ($\forall j \in \{1,2,\cdots,d-1\}$), then

$$\sum_{j=1}^{d-1} \cos k_j \in [(d-1)(1-\varepsilon), d-1] \subset \left[d-1-\frac{1}{5}(2d-|\mu|), d-1\right].$$

By assumption, $\frac{1}{2}|\eta+\mu| \le \frac{1}{2}|\mu| + \frac{1}{4}(2d-|\mu|)$. Thus,

$$\frac{1}{2}|\eta+\mu| - \sum_{j=1}^{d-1}\cos k_j \in \left[\frac{1}{2}, 1-\frac{1}{20}(2d-|\mu|)\right] \subset \left[\frac{1}{2}, 1\right).$$

Therefore, the same argument as above yields the estimate (4.65). Since $\min\{1, 2d-|\mu|\} \ge (2d-|\mu|)/(2d)$, the claimed inequality has been derived. □

Using Lemma 4.17, let us give a sufficient condition for the solvability and the non-solvability of the gap equation (1.3).

LEMMA 4.18. *The following statements hold true.*

(i) There exists a positive constant $c(d)$ depending only on d such that if

$$|U| > c(d)(2d-|\mu|)^{1-d}\beta\Theta\left(1_{\Theta\le\frac{1}{2}(2d-|\mu|)} + 1_{\Theta>\frac{1}{2}(2d-|\mu|)}(2d-|\mu|)^{-1}\Theta\right),$$

$$-\frac{2}{|U|} + \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} d\mathbf{k}\frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|} > 0.$$

(ii) Assume that $\theta \in [0, \frac{\pi}{\beta}]$. If $|U| < 2\beta^{-1}$,

$$-\frac{2}{|U|} + \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} d\mathbf{k}\frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|} < 0.$$

PROOF. Set $K := \frac{1}{2}(2d-|\mu|)$.
(i): Let us define the function $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x) := \frac{\sinh(\beta|x|)}{(\cos(\beta\theta/2)+\cosh(\beta x))|x|}.$$

We can see from the definition that $f \in C^\infty(\mathbb{R})$. Moreover, by the coarea formula and Lemma 4.17,

$$\begin{aligned}\int_{[0,2\pi]^d} d\mathbf{k} f(e(\mathbf{k})) &\ge \frac{1}{2\sqrt{d}}\int_{[0,2\pi]^d} d\mathbf{k} f(e(\mathbf{k}))\|\nabla e(\mathbf{k})\|_{\mathbb{R}^d}\\ &= \frac{1}{2\sqrt{d}}\int_{-\infty}^{\infty} d\eta f(\eta)\mathcal{H}^{d-1}(\{\mathbf{k}\in[0,2\pi]^d \mid e(\mathbf{k})=\eta\})\\ &\ge \frac{1}{2\sqrt{d}}\left(1_{d=1}+1_{d\ge 2}\left(\frac{2d-|\mu|}{10(d-1)d}\right)^{d-1}\right)\int_{-K}^{K} d\eta f(\eta).\end{aligned}$$

Note that

$$\begin{aligned}&\int_{-K}^{K} d\eta f(\eta)\\ &\ge \int_{-K}^{K} d\eta \frac{\beta}{\cosh(\beta\eta)-1+2\sin^2(\beta\Theta/2)}\\ &\ge c\beta^{-1}\int_0^{\min\{\Theta,K\}} d\eta\frac{1}{\eta^2+\Theta^2} = c\beta^{-1}\Theta^{-1}\arctan(\min\{1,K\Theta^{-1}\})\\ &\ge c\beta^{-1}\Theta^{-1}(1_{\Theta\le K}+1_{\Theta>K}K\Theta^{-1}).\end{aligned}$$

By combining this inequality with the above inequality we obtain

$$\int_{[0,2\pi]^d} d\mathbf{k} f(e(\mathbf{k})) \ge c(d)(2d-|\mu|)^{d-1}\beta^{-1}\Theta^{-1}(1_{\Theta\le K}+1_{\Theta>K}K\Theta^{-1}),$$

which implies the claim (i).

(ii): By the assumption on θ, $f(x)\le \tanh(\beta|x|)/|x|\le\beta$ for any $x\in\mathbb{R}$. The claim follows from this inequality. □

Before giving the proof of the main theorem, let us confirm a few more simple facts.

LEMMA 4.19. *Let* $\varepsilon\in(-1,1]$. *The function*

$$x\mapsto\frac{\sinh(x)}{x(\varepsilon+\cosh(x))}:[0,\infty)\to\mathbb{R}$$

is strictly monotone decreasing and converges to 0 *as* $x\to\infty$.

Proof. Observe that

$$\frac{\sinh(x)}{x(\varepsilon+\cosh(x))} = \frac{\tanh(x/2)}{x}\cdot\frac{1+\cosh(x)}{\varepsilon+\cosh(x)}.$$

One can check that the derivative of $\tanh(x/2)/x$, $(1+\cosh(x))/(\varepsilon+\cosh(x))$ ($\varepsilon\in(-1,1)$) are negative in $(0,\infty)$, which implies the strict monotone decreasing property of the function. The convergence property is clear. □

Lemma 4.20. *For any* $\mathbf{x},\mathbf{y}\in\Gamma$, $\phi\in\mathbb{C}$, $\rho,\eta\in\{1,2\}$, $\rho\neq\eta$,

$$\begin{aligned}
&C(\phi)(\rho\mathbf{x}0,\rho\mathbf{y}0)\\
&=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\Bigg(\frac{e^{-i\frac{\beta\theta}{2}}+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2})}{2(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}\\
&\qquad+\frac{(-1)^\rho\sinh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2})e(\mathbf{k})}{2\sqrt{e(\mathbf{k})^2+|\phi|^2}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}\Bigg),\\
&C(\phi)(\rho\mathbf{x}0,\eta\mathbf{y}0)\\
&=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\frac{-(1_{(\rho,\eta)=(1,2)}\overline{\phi}+1_{(\rho,\eta)=(2,1)}\phi)\sinh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2})}{2\sqrt{e(\mathbf{k})^2+|\phi|^2}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}.
\end{aligned}$$

Proof. By using the unitary matrix $U(\phi)(\mathbf{k})$ defined in (2.19), which diagonalizes $E(\phi)(\mathbf{k})$ as shown in (2.21), we can derive the claimed equalities. □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The claims (iv), (v) have been proved right after the statement of Theorem 1.3. Let us prove the claims (i), (ii), (iii). With the constant $c(d)(\in\mathbb{R}_{\geq 1})$ introduced in Proposition 4.14, set $c_2(d):=(2c(d))^{-1}$ and assume that

$$|U|<c_2(d)(1+\beta^{d+3}+(1+\beta^{-1})g_d(\Theta))^{-2}$$

throughout the proof. By assuming so the coupling constant U is inside a disk on which all the results of Proposition 4.14 and Proposition 4.16 hold.

The inequality (4.48) implies that

$$\mathrm{Re}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\ge\frac{1}{2},$$
$$(\forall\phi\in\mathbb{C},\ L\in\mathbb{N}\text{ satisfying (4.39)}).$$

Then, it follows from Lemma 2.1 and (2.25) that

$$\mathrm{Re}\,\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}>0,\quad(\forall L\in\mathbb{N}\text{ satisfying (4.39)},\ \gamma\in[0,1]).$$

Then, by taking into account Lemma 1.1 we observe that the claim (i) holds.

By considering Lemma 4.19 we see that the following statements hold. If

$$-\frac{2}{|U|}+\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|}>0, \tag{4.66}$$

there uniquely exists $\Delta\in(0,\infty)$ such that

$$\begin{aligned}&-\frac{2}{|U|}\\&+\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\frac{\sinh(\beta\sqrt{e(\mathbf{k})^2+\Delta^2})}{(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+\Delta^2}))\sqrt{e(\mathbf{k})^2+\Delta^2}}=0.\end{aligned} \tag{4.67}$$

If

$$-\frac{2}{|U|}+\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|}<0, \tag{4.68}$$

there is no positive solution to the equation (4.67). In this case we set $\Delta:=0$. During the proof we assume that either (4.66) or (4.68) occurs and $\Delta(\in\mathbb{R}_{\ge0})$ is defined as above.

Let us prove the claims concerning SSB. We assume that $\gamma\in(0,1]$ unless otherwise stated. Then, there uniquely exists $a(\gamma)\in(\Delta,\infty)$ such that

(4.69)

$$a(\gamma)\Bigg(-\frac{2}{|U|}$$

$$+\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} d\mathbf{k}\frac{\sinh(\beta\sqrt{e(\mathbf{k})^2+a(\gamma)^2})}{(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+a(\gamma)^2}))\sqrt{e(\mathbf{k})^2+a(\gamma)^2}}\Bigg)$$
$$=-\frac{2\gamma}{|U|}$$

and $\lim_{\gamma\searrow 0} a(\gamma)=\Delta$. Let us set $\mathbf{a}:=(a(\gamma),0)$. Here we define the function $F:\mathbb{R}^2\to\mathbb{R}$ by

$$\begin{aligned}F(\mathbf{x}) := &-\frac{1}{|U|}((x_1-\gamma)^2+x_2^2)\\ &+\frac{1}{\beta(2\pi)^d}\int_{[0,2\pi]^d} d\mathbf{k}\log\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh\left(\beta\sqrt{e(\mathbf{k})^2+\|\mathbf{x}\|_{\mathbb{R}^2}^2}\right)\right)\\ &-\frac{1}{\beta(2\pi)^d}\int_{[0,2\pi]^d} d\mathbf{k}\log\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh(\beta e(\mathbf{k}))\right).\end{aligned}$$

For $r\in\mathbb{R}_{>0}$, $\mathbf{b}\in\mathbb{R}^2$ we set $B_r(\mathbf{b}):=\{\mathbf{x}\in\mathbb{R}^2 \mid \|\mathbf{x}-\mathbf{b}\|_{\mathbb{R}^2}<r\}$. Some remarks concerning the function F are in order.

- $F\in C^\infty(\mathbb{R}^2)$.
- F takes its global maximum at and only at $\mathbf{x}=\mathbf{a}$.
-

(4.70) $$\frac{\partial^2 F}{\partial x_1^2}(\mathbf{a})\le -\frac{2\gamma}{|U|a(\gamma)},\quad \frac{\partial^2 F}{\partial x_1\partial x_2}(\mathbf{a})=0,\quad \frac{\partial^2 F}{\partial x_2^2}(\mathbf{a})=-\frac{2\gamma}{|U|a(\gamma)}.$$

- For any $r\in\mathbb{R}_{>0}$,

$$-\infty<\sup_{\mathbf{x}\in\mathbb{R}^2\backslash\overline{B_r(\mathbf{a})}}(F(\mathbf{x})-F(\mathbf{a}))<0.$$

Since these are the properties of the explicitly defined function, we omit the proof. It is also necessary to deal with the discrete analogue F_L of F. Set for $\mathbf{x}\in\mathbb{R}^2$

$$\begin{aligned}F_L(\mathbf{x}) := &-\frac{1}{|U|}((x_1-\gamma)^2+x_2^2)\\ &+\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\log\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh\left(\beta\sqrt{e(\mathbf{k})^2+\|\mathbf{x}\|_{\mathbb{R}^2}^2}\right)\right)\end{aligned}$$

$$- \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \log\left(\cos\left(\frac{\beta\theta}{2}\right) + \cosh(\beta e(\mathbf{k}))\right).$$

For sufficiently large L we can assume that

$$-\frac{2}{|U|} + \frac{1}{L^d} \sum_{\mathbf{k}\in\Gamma^*} \frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|} \neq 0. \tag{4.71}$$

Since the situation is parallel to that of $F(x)$, it follows that

- $F_L \in C^\infty(\mathbb{R}^2)$.
- F_L takes its global maximum at and only at $\mathbf{x} = \mathbf{a}_L = (a_L(\gamma), 0)$, where $a_L(\gamma) \in (0, \infty)$ and

$$a_L(\gamma)\Bigg(-\frac{2}{|U|} + \frac{1}{L^d} \sum_{\mathbf{k}\in\Gamma^*} \frac{\sinh(\beta\sqrt{e(\mathbf{k})^2 + a_L(\gamma)^2})}{(\cos(\beta\theta/2) + \cosh(\beta\sqrt{e(\mathbf{k})^2 + a_L(\gamma)^2}))\sqrt{e(\mathbf{k})^2 + a_L(\gamma)^2}}\Bigg) = -\frac{2\gamma}{|U|}.$$

Moreover, we observe that

- There exists a positive constant $c(\beta, d, \theta, |U|)$ depending only on $\beta, d, \theta, |U|$ such that

$$F_L(\mathbf{x}) \le -\frac{\|\mathbf{x}\|_{\mathbb{R}^2}^2}{|U|} + \left(\frac{2}{|U|} + 1\right) \|\mathbf{x}\|_{\mathbb{R}^2} + c(\beta, d, \theta, |U|), \ (\forall \mathbf{x} \in \mathbb{R}^2,\ L \in \mathbb{N}). \tag{4.72}$$

- For any compact set Q of $\mathbb{R}^2$ and $i, j \in \mathbb{N} \cup \{0\}$ with $i + j \le 2$,

$$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \sup_{\mathbf{x}\in Q} \left| \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} F_L(\mathbf{x}) - \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} F(\mathbf{x}) \right| = 0. \tag{4.73}$$

By making use of the properties (4.72), (4.73) we can prove that

$$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \mathbf{a}_L = \mathbf{a}. \tag{4.74}$$

Let $H(F)(\mathbf{x})$, $H(F_L)(\mathbf{x})$ denote the Hessian of F, F_L respectively. The property (4.70) implies that

$$H(F)(\mathbf{a}) \le -\frac{2\gamma}{|U|a(\gamma)}.$$

By applying (4.72), (4.73), (4.74) we can establish necessary basic properties as follows. There exist $\delta \in \mathbb{R}_{>0}$ and $L_0 \in \mathbb{N}$ such that the following statements hold true for any $L \in \mathbb{N}$ with $L \ge L_0$.

- For any $\mathbf{x} \in \overline{B_\delta(\mathbf{a}_L)}$,

(4.75)
$$F_L(\mathbf{x}) = F_L(\mathbf{a}_L) + \int_0^1 dt(1-t)\langle \mathbf{x} - \mathbf{a}_L, H(F_L)(t(\mathbf{x}-\mathbf{a}_L)+\mathbf{a}_L)(\mathbf{x}-\mathbf{a}_L)\rangle,$$
(4.76)
$$H(F_L)(t(\mathbf{x}-\mathbf{a}_L)+\mathbf{a}_L) \le \frac{1}{2}H(F)(\mathbf{a}) < 0, \quad (\forall t \in [0,1]).$$

- For any $\mathbf{x} \in \mathbb{R}^2 \backslash \overline{B_\delta(\mathbf{a}_L)}$,

$$F_L(\mathbf{x}) - F_L(\mathbf{a}_L) \le \frac{1}{2} \sup_{\mathbf{x} \in \mathbb{R}^2 \backslash \overline{B_{\delta/2}(\mathbf{a})}} (F(\mathbf{x}) - F(\mathbf{a})) < 0. \tag{4.77}$$

At this point we go back to the Grassmann integral formulations. By Proposition 4.14 and (4.22),

$$\begin{aligned}
&\sup_{\substack{L\in\mathbb{N}\\ \text{satisfying (4.39)}}} \sup_{\substack{h\in\frac{2}{\beta}\mathbb{N}\\ h\ge c(d)\max\{1,\beta^{-1}\}}} \sup_{\phi\in\mathbb{C}} \Bigg(\left| \int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi) \right| \\
&\quad + \sum_{j\in\{1,2\}} \left| \int e^{-V(\psi)+W(\psi)} A^j(\psi) d\mu_{C(\phi)}(\psi) \right| \\
&\quad + \sum_{j\in\{1,2\}} \left| \int A^j(\psi) d\mu_{C(\phi)}(\psi) \right| \Bigg) \\
&< \infty.
\end{aligned} \tag{4.78}$$

Since the functions inside the modulus above are continuous with ϕ over $\mathbb{C}$, the following transformation is justified.

(4.79)

$$\begin{aligned}
&\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2} \frac{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))}\\
&\quad\cdot\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(\phi)}(\psi)\\
&=\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{\beta L^d F_L(\boldsymbol{\phi})}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\int A^1(\psi)d\mu_{C(\phi)}(\psi)\\
&\quad+\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{\beta L^d F_L(\boldsymbol{\phi})}\bigg(\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(\phi)}(\psi)\\
&\qquad\qquad-\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\int A^1(\psi)d\mu_{C(\phi)}(\psi)\bigg)\\
&=e^{\beta L^d F_L(\mathbf{a}_L)}L^{-d}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\|\boldsymbol{\phi}\|_{\mathbb{R}^2}\le L^{\frac{d}{2}}\delta} e^{\beta\int_0^1 dt(1-t)\langle\boldsymbol{\phi},H(F_L)(tL^{-\frac{d}{2}}\boldsymbol{\phi}+\mathbf{a}_L)\boldsymbol{\phi}\rangle}\\
&\qquad\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)\int A^1(\psi)d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)\\
&\quad+e^{\beta L^d F_L(\mathbf{a}_L)}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\boldsymbol{\phi}\notin\overline{B_\delta(\mathbf{a}_L)}} e^{\beta L^d(F_L(\boldsymbol{\phi})-F_L(\mathbf{a}_L))}\\
&\qquad\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\int A^1(\psi)d\mu_{C(\phi)}(\psi)\\
&\quad+e^{\beta L^d F_L(\mathbf{a}_L)}L^{-d}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\|\boldsymbol{\phi}\|_{\mathbb{R}^2}\le L^{\frac{d}{2}}\delta} e^{\beta\int_0^1 dt(1-t)\langle\boldsymbol{\phi},H(F_L)(tL^{-\frac{d}{2}}\boldsymbol{\phi}+\mathbf{a}_L)\boldsymbol{\phi}\rangle}\\
&\qquad\cdot\bigg(\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)\\
&\qquad\quad-\int e^{-V(\psi)+W(\psi)}d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)\int A^1(\psi)d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)\bigg)\\
&\quad+e^{\beta L^d F_L(\mathbf{a}_L)}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\boldsymbol{\phi}\notin\overline{B_\delta(\mathbf{a}_L)}} e^{\beta L^d(F_L(\boldsymbol{\phi})-F_L(\mathbf{a}_L))}\\
&\qquad\cdot\bigg(\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(\phi)}(\psi)
\end{aligned}$$

$$-\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\int A^1(\psi)d\mu_{C(\phi)}(\psi)\Bigg),$$

where $\boldsymbol{\phi}=(\phi_1,\phi_2)$, $\phi=\phi_1+i\phi_2$ and $\delta(\in\mathbb{R}_{>0})$ is the parameter appearing in (4.75), (4.76), (4.77). It follows from Lemma 2.5 (i), (4.49), (4.72), (4.76), (4.77), (4.78) that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}L^d\int_{\mathbb{R}^2}d\phi_1 d\phi_2 1_{\boldsymbol{\phi}\notin\overline{B_\delta(\mathbf{a}_L)}}e^{\beta L^d(F_L(\boldsymbol{\phi})-F_L(\mathbf{a}_L))}\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\int A^1(\psi)d\mu_{C(\phi)}(\psi)=0,\tag{4.80}$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int_{\mathbb{R}^2}d\phi_1 d\phi_2 1_{\|\boldsymbol{\phi}\|_{\mathbb{R}^2}\le L^{\frac{d}{2}}\delta}e^{\beta\int_0^1dt(1-t)\langle\boldsymbol{\phi},H(F_L)(tL^{-\frac{d}{2}}\boldsymbol{\phi}+\mathbf{a}_L)\boldsymbol{\phi}\rangle}$$
$$\cdot\Bigg(\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)-\int e^{-V(\psi)+W(\psi)}d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)\cdot\int A^1(\psi)d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)\Bigg)=0,$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}L^d\int_{\mathbb{R}^2}d\phi_1 d\phi_2 1_{\boldsymbol{\phi}\notin\overline{B_\delta(\mathbf{a}_L)}}e^{\beta L^d(F_L(\boldsymbol{\phi})-F_L(\mathbf{a}_L))}$$
$$\cdot\Bigg(\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(\phi)}(\psi)-\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\int A^1(\psi)d\mu_{C(\phi)}(\psi)\Bigg)=0.$$

Moreover, we can apply Proposition 4.16, (4.73), (4.74), (4.76), (4.78) to conclude that

(4.81)

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int_{\mathbb{R}^2}d\phi_1 d\phi_2 1_{\|\boldsymbol{\phi}\|_{\mathbb{R}^2}\le L^{\frac{d}{2}}\delta}e^{\beta\int_0^1dt(1-t)\langle\boldsymbol{\phi},H(F_L)(tL^{-\frac{d}{2}}\boldsymbol{\phi}+\mathbf{a}_L)\boldsymbol{\phi}\rangle}$$

$$\cdot \int e^{-V(\psi)+W(\psi)} d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi) \int A^1(\psi) d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)$$
$$= \int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{\frac{\beta}{2}\langle \boldsymbol{\phi}, H(F)(\mathbf{a})\boldsymbol{\phi}\rangle} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi)+W(\psi)} d\mu_{C(a(\gamma))}(\psi)$$
$$\cdot \beta \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(a(\gamma))(1\mathbf{0}0, 2\mathbf{0}0).$$

Similarly we have that

(4.82)
$$\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{\beta L^d F_L(\phi)} \int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi)$$
$$= e^{\beta L^d F_L(\mathbf{a}_L)} L^{-d} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\|\boldsymbol{\phi}\|_{\mathbb{R}^2}\le L^{\frac{d}{2}}\delta} e^{\beta\int_0^1 dt(1-t)\langle \boldsymbol{\phi}, H(F_L)(tL^{-\frac{d}{2}}\boldsymbol{\phi}+\mathbf{a}_L)\boldsymbol{\phi}\rangle}$$
$$\cdot \int e^{-V(\psi)+W(\psi)} d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)$$
$$+ e^{\beta L^d F_L(\mathbf{a}_L)} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\boldsymbol{\phi}\notin \overline{B_\delta(\mathbf{a}_L)}} e^{\beta L^d (F_L(\boldsymbol{\phi})-F_L(\mathbf{a}_L))}$$
$$\cdot \int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi).$$

(4.83)
$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\|\boldsymbol{\phi}\|_{\mathbb{R}^2}\le L^{\frac{d}{2}}\delta} e^{\beta\int_0^1 dt(1-t)\langle \boldsymbol{\phi}, H(F_L)(tL^{-\frac{d}{2}}\boldsymbol{\phi}+\mathbf{a}_L)\boldsymbol{\phi}\rangle}$$
$$\cdot \int e^{-V(\psi)+W(\psi)} d\mu_{C(L^{-\frac{d}{2}}\phi+a_L(\gamma))}(\psi)$$
$$= \int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{\frac{\beta}{2}\langle \boldsymbol{\phi}, H(F)(\mathbf{a})\boldsymbol{\phi}\rangle} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi)+W(\psi)} d\mu_{C(a(\gamma))}(\psi),$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} L^d \int_{\mathbb{R}^2} d\phi_1 d\phi_2 1_{\boldsymbol{\phi}\notin \overline{B_\delta(\mathbf{a}_L)}} e^{\beta L^d (F_L(\boldsymbol{\phi})-F_L(\mathbf{a}_L))}$$
$$\cdot \int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi)$$
$$= 0.$$

The inequality (4.48) implies that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(a(\gamma))}(\psi)\neq 0. \tag{4.84}$$

Note that by Lemma 2.5 (i) and (4.78) we can change the order of the integral over $\mathbb{R}^2$ and the limit operation with h in (2.25) with $\boldsymbol{\lambda}=(0,0)$ and (2.26). Then, by using (4.79), (4.80), (4.81), (4.82), (4.83), (4.84) and (4.69), Lemma 4.20 we can derive from (2.25), (2.26) that

$$\begin{aligned}
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1)}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}\\
&=\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\lim_{h\to\infty, h\in\frac{2}{\beta}\mathbb{N}}\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{\beta L^dF_L(\boldsymbol{\phi})}\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(\phi)}(\psi)}{\lim_{h\to\infty, h\in\frac{2}{\beta}\mathbb{N}}\beta\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{\beta L^dF_L(\boldsymbol{\phi})}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)}\\
&=\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{\frac{\beta}{2}\langle\boldsymbol{\phi},H(F)(\mathbf{a})\boldsymbol{\phi}\rangle}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(a(\gamma))}(\psi)\\
&\quad\cdot\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}C(a(\gamma))(1\mathbf{0}0,2\mathbf{0}0)\\
&\quad\cdot\Big/\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{\frac{\beta}{2}\langle\boldsymbol{\phi},H(F)(\mathbf{a})\boldsymbol{\phi}\rangle}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(a(\gamma))}(\psi)\\
&=-\frac{a(\gamma)}{|U|}+\frac{\gamma}{|U|}.
\end{aligned}$$

Thus,

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1)}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}=-\frac{\Delta}{|U|}.$$

We let $c_1(d)$ be the constant $c(d)$ appearing in Lemma 4.18 (i). Then, if

$$|U|>c_1(d)(2d-|\mu|)^{1-d}\beta\Theta\left(1_{\Theta\le\frac{1}{2}(2d-|\mu|)}+1_{\Theta>\frac{1}{2}(2d-|\mu|)}(2d-|\mu|)^{-1}\Theta\right),$$

(4.66) holds and thus $\Delta>0$. This proves the claims (1.3), (1.5). Note that

$$c_2(d)(1+\beta^{d+3}+(1+\beta^{-1})g_d(\Theta))^{-2}\le(1+\beta^{d+3})^{-2}\le 2\beta^{-1},\quad(\forall\beta\in\mathbb{R}_{>0}).$$

Thus, if $\theta \in [0, \pi/\beta]$, Lemma 4.18 (ii) implies that (4.68) holds and thus $\Delta = 0$. Therefore, the first statement of (iii) and the claim concerning SSB in (iii) hold true.

Next let us prove the claim (1.6) and the claim concerning ODLRO in (iii). The proof is in fact close to the proof of SSB above. However we present it for completeness. Let us define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} &f(x) \\ &:= -\frac{x^2}{|U|} + \frac{1}{\beta(2\pi)^d} \int_{[0,2\pi]^d} d\mathbf{k} \log \left(\cos\left(\frac{\beta\theta}{2}\right) + \cosh(\beta\sqrt{e(\mathbf{k})^2 + x^2}) \right) \\ &\quad - \frac{1}{\beta(2\pi)^d} \int_{[0,2\pi]^d} d\mathbf{k} \log \left(\cos\left(\frac{\beta\theta}{2}\right) + \cosh(\beta e(\mathbf{k})) \right). \end{aligned}$$

Then, we see that

- $f \in C^\infty(\mathbb{R})$.
- $f|_{[0,\infty)} : [0,\infty) \to \mathbb{R}$ takes its global maximum at and only at $x = \Delta$, where $f|_{[0,\infty)}$ denotes the restriction of f on $[0,\infty)$.
- $$\frac{d^2 f}{dx^2}(\Delta) < 0.$$

The third statement above can be confirmed as follows.

$$\begin{aligned} &\frac{d^2 f}{dx^2}(\Delta) \\ &\le 1_{\Delta=0} \left(-\frac{2}{|U|} + \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} d\mathbf{k} \frac{\sinh(\beta|e(\mathbf{k})|)}{(\cos(\beta\theta/2) + \cosh(\beta e(\mathbf{k})))|e(\mathbf{k})|} \right) \\ &\quad + 1_{\Delta>0} \Delta \sup_{\eta \in [-2d-|\mu|, 2d+|\mu|]} \\ &\qquad \cdot \left(\frac{d}{dx} \left(\frac{\sinh(\beta\sqrt{\eta^2 + x^2})}{(\cos(\beta\theta/2) + \cosh(\beta\sqrt{\eta^2 + x^2}))\sqrt{\eta^2 + x^2}} \right) \Bigg|_{x=\Delta} \right) \\ &< 0. \end{aligned}$$

Again we need to introduce the L-dependent version of f as follows.

$$f_L(x) := -\frac{x^2}{|U|} + \frac{1}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \log \left(\cos\left(\frac{\beta\theta}{2}\right) + \cosh(\beta\sqrt{e(\mathbf{k})^2 + x^2}) \right)$$

$$-\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\log\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh(\beta e(\mathbf{k}))\right).$$

We may assume that (4.71) holds. When the left-hand side of (4.71) is positive, there uniquely exists $\Delta_L \in (0,\infty)$ such that

$$-\frac{2}{|U|}+\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\frac{\sinh(\beta\sqrt{e(\mathbf{k})^2+\Delta_L^2})}{(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+\Delta_L^2}))\sqrt{e(\mathbf{k})^2+\Delta_L^2}}=0.$$

If the left-hand side of (4.71) is negative, we set $\Delta_L := 0$. It follows that

- $f_L|_{[0,\infty)} : [0,\infty) \to \mathbb{R}$ takes its global maximum at and only at $x = \Delta_L$, where $f_L|_{[0,\infty)}$ is the restriction of f_L on $[0,\infty)$.

Based on this fact and that f_L and the derivatives of f_L locally uniformly converge to f and those of f respectively as $L \to \infty$, we can prove that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\Delta_L = \Delta. \tag{4.85}$$

Moreover, there exist $\delta \in \mathbb{R}_{>0}$ and $L_0 \in \mathbb{N}$ such that the following statements hold true for any $L \in \mathbb{N}$ with $L \geq L_0$.

- For any $x \in [\Delta_L - \delta, \Delta_L + \delta]$,

$$f_L(x) = f_L(\Delta_L) + \int_0^1 dt(1-t)\frac{d^2 f_L}{dx^2}(t(x-\Delta_L)+\Delta_L)(x-\Delta_L)^2, \tag{4.86}$$

$$\frac{d^2 f_L}{dx^2}(t(x-\Delta_L)+\Delta_L) \leq \frac{1}{2}\frac{d^2 f}{dx^2}(\Delta) < 0, \quad (\forall t \in [0,1]). \tag{4.87}$$

- For any $x \in [0,\infty)\backslash[\Delta_L - \delta, \Delta_L + \delta]$,

$$f_L(x) - f_L(\Delta_L) \leq \frac{1}{2}\sup_{x\in[0,\infty)\backslash[\Delta-\frac{\delta}{2},\Delta+\frac{\delta}{2}]}(f(x)-f(\Delta)) < 0. \tag{4.88}$$

Observe that

(4.89)

$$\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi|^2}\frac{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta\sqrt{e(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2)+\cosh(\beta e(\mathbf{k})))}$$

$$
\begin{aligned}
&\cdot\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C(\phi)}(\psi)\\
&=\int_0^{2\pi}d\xi\int_0^{\infty}drre^{\beta L^d f_L(r)}\int e^{-V(\psi)+W(\psi)}d\mu_{C(re^{i\xi})(\psi)}\int A^2(\psi)d\mu_{C(re^{i\xi})}(\psi)\\
&\quad+\int_0^{2\pi}d\xi\int_0^{\infty}drre^{\beta L^d f_L(r)}\Bigg(\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C(re^{i\xi})}(\psi)\\
&\qquad\qquad-\int e^{-V(\psi)+W(\psi)}d\mu_{C(re^{i\xi})}(\psi)\int A^2(\psi)d\mu_{C(re^{i\xi})}(\psi)\Bigg)\\
&=e^{\beta L^d f_L(\Delta_L)}L^{-\frac{d}{2}}\int_0^{2\pi}d\xi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta}dr(L^{-\frac{d}{2}}r+\Delta_L)\\
&\qquad\cdot e^{\beta\int_0^1dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}\\
&\qquad\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\int A^2(\psi)d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\\
&\quad+e^{\beta L^d f_L(\Delta_L)}\int_0^{2\pi}d\xi\int_{[0,\infty)\setminus[\Delta_L-\delta,\Delta_L+\delta]}drre^{\beta L^d(f_L(r)-f_L(\Delta_L))}\\
&\qquad\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(re^{i\xi})}(\psi)\int A^2(\psi)d\mu_{C(re^{i\xi})}(\psi)\\
&\quad+e^{\beta L^d f_L(\Delta_L)}L^{-\frac{d}{2}}\int_0^{2\pi}d\xi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta}dr(L^{-\frac{d}{2}}r+\Delta_L)\\
&\qquad\cdot e^{\beta\int_0^1dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}\\
&\qquad\cdot\Bigg(\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\\
&\qquad\quad-\int e^{-V(\psi)+W(\psi)}d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\\
&\qquad\qquad\cdot\int A^2(\psi)d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\Bigg)\\
&\quad+e^{\beta L^d f_L(\Delta_L)}\int_0^{2\pi}d\xi\int_{[0,\infty)\setminus[\Delta_L-\delta,\Delta_L+\delta]}drre^{\beta L^d(f_L(r)-f_L(\Delta_L))}\\
&\qquad\cdot\Bigg(\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C(re^{i\xi})}(\psi)
\end{aligned}
$$

$$- \int e^{-V(\psi)+W(\psi)} d\mu_{C(re^{i\xi})}(\psi) \int A^2(\psi) d\mu_{C(re^{i\xi})}(\psi) \Bigg),$$

where $\delta(\in \mathbb{R}_{>0})$ is that appearing in (4.86), (4.87), (4.88). To prove convergent properties, we need to multiply different volume factors depending on whether $\Delta > 0$ or $\Delta = 0$. Lemma 2.5 (i), the inequalities (4.78), (4.88) and a variant of the inequality (4.72) ensure that

$$\begin{aligned}
(4.90)\quad & \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} (1_{\Delta>0} L^{\frac{d}{2}} + 1_{\Delta=0} L^d) \int_0^{2\pi} d\xi \\
&\quad \cdot \int_{[0,\infty)\backslash[\Delta_L-\delta,\Delta_L+\delta]} dr r e^{\beta L^d (f_L(r)-f_L(\Delta_L))} \\
&\quad \cdot \int e^{-V(\psi)+W(\psi)} d\mu_{C(re^{i\xi})}(\psi) \int A^2(\psi) d\mu_{C(re^{i\xi})}(\psi) = 0, \\
& \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} (1_{\Delta>0} L^{\frac{d}{2}} + 1_{\Delta=0} L^d) \int_0^{2\pi} d\xi \\
&\quad \cdot \int_{[0,\infty)\backslash[\Delta_L-\delta,\Delta_L+\delta]} dr r e^{\beta L^d (f_L(r)-f_L(\Delta_L))} \\
&\quad \cdot \Bigg(\int e^{-V(\psi)+W(\psi)} A^2(\psi) d\mu_{C(re^{i\xi})}(\psi) \\
&\qquad - \int e^{-V(\psi)+W(\psi)} d\mu_{C(re^{i\xi})}(\psi) \int A^2(\psi) d\mu_{C(re^{i\xi})}(\psi) \Bigg) = 0.
\end{aligned}$$

Moreover, it follows from Lemma 2.5 (i), (4.49), Proposition 4.16, (4.78), (4.85), (4.87) and a variant of (4.73) that

$$\begin{aligned}
&(4.91) \\
&\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \int_0^{2\pi} d\xi \int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta} dr (1_{\Delta>0}(L^{-\frac{d}{2}} r + \Delta_L) + 1_{\Delta=0} r) \\
&\quad \cdot e^{\beta \int_0^1 dt (1-t) f_L''(tL^{-\frac{d}{2}} r + \Delta_L) r^2} \\
&\quad \cdot \int e^{-V(\psi)+W(\psi)} d\mu_{C((L^{-\frac{d}{2}} r+\Delta_L)e^{i\xi})}(\psi) \int A^2(\psi) d\mu_{C((L^{-\frac{d}{2}} r+\Delta_L)e^{i\xi})}(\psi)
\end{aligned}$$

$$= \left(1_{\Delta>0}\int_{-\infty}^{\infty}dr\Delta + 1_{\Delta=0}\int_0^{\infty}drr\right)e^{\frac{\beta}{2}f''(\Delta)r^2}\int_0^{2\pi}d\xi$$
$$\cdot \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\Delta e^{i\xi})}(\psi)$$
$$\cdot(-\beta)\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\det\begin{pmatrix} C(\Delta)(1\hat{\mathbf{x}}0,1\hat{\mathbf{y}}0) & C(\Delta)(1\mathbf{0}0,2\mathbf{0}0)\\ C(\Delta)(2\mathbf{0}0,1\mathbf{0}0) & C(\Delta)(2\hat{\mathbf{y}}0,2\hat{\mathbf{x}}0)\end{pmatrix},$$
$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int_0^{2\pi}d\xi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta}dr(1_{\Delta>0}(L^{-\frac{d}{2}}r+\Delta_L)+1_{\Delta=0}r)$$
$$\cdot e^{\beta\int_0^1dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}$$
$$\cdot\Bigg(\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)$$
$$-\int e^{-V(\psi)+W(\psi)}d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)$$
$$\cdot\int A^2(\psi)d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\Bigg)$$
$$=0.$$

For the same reason as above we have that

(4.92)

$$\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{\beta L^df_L(|\phi|)}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)$$
$$= e^{\beta L^df_L(\Delta_L)}L^{-\frac{d}{2}}\int_0^{2\pi}d\xi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta}dr(L^{-\frac{d}{2}}r+\Delta_L)$$
$$\cdot e^{\beta\int_0^1dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}\int e^{-V(\psi)+W(\psi)}d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)$$
$$+e^{\beta L^df_L(\Delta_L)}\int_0^{2\pi}d\xi\int_{[0,\infty)\backslash[\Delta_L-\delta,\Delta_L+\delta]}drre^{\beta L^d(f_L(r)-f_L(\Delta_L))}$$
$$\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(re^{i\xi})}(\psi)$$

and

(4.93)

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int_0^{2\pi}d\xi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta}dr(1_{\Delta>0}(L^{-\frac{d}{2}}r+\Delta_L)+1_{\Delta=0}r)$$
$$\cdot e^{\beta\int_0^1 dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}\int e^{-V(\psi)+W(\psi)}d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)$$
$$=\left(1_{\Delta>0}\int_{-\infty}^{\infty}dr\Delta+1_{\Delta=0}\int_0^{\infty}drr\right)e^{\frac{\beta}{2}f''(\Delta)r^2}\int_0^{2\pi}d\xi$$
$$\cdot\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\Delta e^{i\xi})}(\psi),$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}(1_{\Delta>0}L^{\frac{d}{2}}+1_{\Delta=0}L^d)\int_0^{2\pi}d\xi$$
$$\cdot\int_{[0,\infty)\backslash[\Delta_L-\delta,\Delta_L+\delta]}drre^{\beta L^d(f_L(r)-f_L(\Delta_L))}$$
$$\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(re^{i\xi})}(\psi)=0.$$

Furthermore, by (4.48)

$$\int_0^{2\pi}d\xi\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\Delta e^{i\xi})}(\psi)\neq 0. \tag{4.94}$$

To prove the claim (1.6), we first change the order of the integration over $\mathbb{R}^2$ and the limit operation $h\to\infty$ in (2.25), (2.26), which is justified by the uniform bound (4.78) and the dominated convergence theorem, and then apply (4.89), (4.90), (4.91), (4.92), (4.93), (4.94). As the result,

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\mathsf{A}_2)}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}$$
$$=-\left(1_{\Delta>0}\int_{-\infty}^{\infty}dr\Delta+1_{\Delta=0}\int_0^{\infty}drr\right)e^{\frac{\beta}{2}f''(\Delta)r^2}\int_0^{2\pi}d\xi$$
$$\cdot\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\Delta e^{i\xi})}(\psi)$$

$$
\begin{aligned}
&\cdot \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \det\begin{pmatrix} C(\Delta)(1\hat{\mathbf{x}}0,1\hat{\mathbf{y}}0) & C(\Delta)(1\mathbf{0}0,2\mathbf{0}0)\\ C(\Delta)(2\mathbf{0}0,1\mathbf{0}0) & C(\Delta)(2\hat{\mathbf{y}}0,2\hat{\mathbf{x}}0)\end{pmatrix}\\
&\cdot \Bigg/\Bigg(\bigg(1_{\Delta>0}\int_{-\infty}^{\infty}dr\Delta + 1_{\Delta=0}\int_0^{\infty}drr\bigg)e^{\frac{\beta}{2}f''(\Delta)r^2}\int_0^{2\pi}d\xi\\
&\qquad\cdot \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\Delta e^{i\xi})}(\psi)\Bigg)\\
&= -\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \det\begin{pmatrix} C(\Delta)(1\hat{\mathbf{x}}0,1\hat{\mathbf{y}}0) & C(\Delta)(1\mathbf{0}0,2\mathbf{0}0)\\ C(\Delta)(2\mathbf{0}0,1\mathbf{0}0) & C(\Delta)(2\hat{\mathbf{y}}0,2\hat{\mathbf{x}}0)\end{pmatrix},
\end{aligned}
$$

or by Lemma 4.20 and (4.67),

$$
\begin{aligned}
\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\mathsf{A}_2)}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}} &= \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(\Delta)(1\mathbf{0}0,2\mathbf{0}0)C(\Delta)(2\mathbf{0}0,1\mathbf{0}0)\\
&= \frac{\Delta^2}{U^2}.
\end{aligned}
$$

After reaching this equality we only need to repeat the same argument as in the end of the proof for SSB to complete the proof of the claim (1.6) and the claim concerning ODLRO in (iii).

It remains to prove the claim (1.4). Remark that by (4.48)

$$
\begin{aligned}
&\mathrm{Re}\int_0^{2\pi}d\xi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta}dr(1_{\Delta>0}(L^{-\frac{d}{2}}r+\Delta_L)+1_{\Delta=0}r)\\
&\quad\cdot e^{\beta\int_0^1 dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}\int e^{-V(\psi)+W(\psi)}d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\\
&\ge \pi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta}dr(1_{\Delta>0}(L^{-\frac{d}{2}}r+\Delta_L)+1_{\Delta=0}r)\\
&\quad\cdot e^{\beta\int_0^1 dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}\\
&>0,\\
&\mathrm{Re}\int_0^{2\pi}d\xi\int_{[0,\infty)\setminus[\Delta_L-\delta,\Delta_L+\delta]}drre^{\beta L^d(f_L(r)-f_L(\Delta_L))}\\
&\cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(re^{i\xi})}(\psi)
\end{aligned}
$$

$$\geq \pi \int_{[0,\infty)\backslash[\Delta_L-\delta,\Delta_L+\delta]} drre^{\beta L^d(f_L(r)-f_L(\Delta_L))} > 0$$

for sufficiently large L, h. The following transformation based on (4.92) is justified.

$$\begin{aligned}
&-\frac{1}{\beta L^d}\log\left(\frac{\beta L^d}{\pi|U|}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{\beta L^d f_L(|\phi|)}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\right)\\
&= -\frac{1}{\beta L^d}\log\left(\frac{\beta L^d}{\pi|U|}e^{\beta L^d f_L(\Delta_L)}(1_{\Delta>0}L^{-\frac{d}{2}}+1_{\Delta=0}L^{-d})\right)\\
&\quad -\frac{1}{\beta L^d}\log\Bigg(\int_0^{2\pi}d\xi\int_{-L^{d/2}\min\{\delta,\Delta_L\}}^{L^{d/2}\delta} dr(1_{\Delta>0}(L^{-\frac{d}{2}}r+\Delta_L)+1_{\Delta=0}r)\\
&\qquad\qquad \cdot e^{\beta\int_0^1 dt(1-t)f_L''(tL^{-\frac{d}{2}}r+\Delta_L)r^2}\int e^{-V(\psi)+W(\psi)}d\mu_{C((L^{-\frac{d}{2}}r+\Delta_L)e^{i\xi})}(\psi)\\
&\qquad + (1_{\Delta>0}L^{\frac{d}{2}}+1_{\Delta=0}L^{d})\int_0^{2\pi}d\xi\int_{[0,\infty)\backslash[\Delta_L-\delta,\Delta_L+\delta]} drre^{\beta L^d(f_L(r)-f_L(\Delta_L))}\\
&\qquad\qquad \cdot\int e^{-V(\psi)+W(\psi)}d\mu_{C(re^{i\xi})}(\psi)\Bigg).
\end{aligned}$$

Then, by (2.25), (4.93) and a variant of (4.73), (4.85),

$$\begin{aligned}
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left(-\frac{1}{\beta L^d}\log\left(\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}\right)\right)\\
&= \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\\
&\quad\cdot\left(-\frac{1}{\beta L^d}\log\left(\frac{\beta L^d}{\pi|U|}\int_{\mathbb{R}^2}d\phi_1 d\phi_2 e^{\beta L^d f_L(|\phi|)}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)\right)\right)\\
&= -\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} f_L(\Delta_L) = -f(\Delta).
\end{aligned}$$

By combining this with (2.2) we finally obtain the equality (1.4). □

Appendix A. Proof of Proposition 4.1

Here we provide a short proof of Proposition 4.1 for readers' convenience. We should remark that the proof below is essentially a digest of the general construction of [17]. First let us recall a simple fact based on the Cauchy-Binet formula.

LEMMA A.1. *Assume that* $n \times n$ *matrices* $A = (A(i,j))_{1 \le i,j \le n}$, $B = (B(i,j))_{1 \le i,j \le n}$ *satisfy that*

$$|\det(A(k_i,l_j))_{1 \le i,j \le m}| \le D_A^{2m}, \quad |\det(B(k_i,l_j))_{1 \le i,j \le m}| \le D_B^{2m}$$

with $D_A,\ D_B \in \mathbb{R}_{\ge 0}$ *for any* $\{k_i\}_{i=1}^m,\ \{l_i\}_{i=1}^m \subset \{1,2,\cdots,n\}$ *satisfying* $k_1 < \cdots < k_m$, $l_1 < \cdots < l_m$. *Then,*

$$|\det(A+B)| \le (D_A + D_B)^{2n}.$$

PROOF. By applying the Cauchy-Binet formula to the decomposition

$$A + B = \begin{pmatrix} A & I_n \end{pmatrix} \begin{pmatrix} I_n \\ B \end{pmatrix}$$

we observe that

$$\begin{aligned}
&|\det(A+B)| \\
&\le \sum_{\substack{\gamma:\{1,\cdots,n\}\to\{1,\cdots,2n\} \\ \text{with } \gamma(1)<\cdots<\gamma(n)}} \\
&\qquad \cdot |\det((\begin{matrix} A & I_n \end{matrix})(i,\gamma(j)))_{1\le i,j\le n}| \left|\det\left(\begin{pmatrix} I_n \\ B \end{pmatrix}(\gamma(i),j)\right)_{1\le i,j\le n}\right| \\
&\le \sum_{m=0}^{n} \sum_{\substack{\gamma:\{1,\cdots,n\}\to\{1,\cdots,2n\} \\ \text{with } \gamma(1)<\cdots<\gamma(n)}} 1_{\gamma(m)\le n<\gamma(m+1)} D_A^{2m} D_B^{2(n-m)} \\
&= \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix}^2 D_A^{2m} D_B^{2(n-m)} \le \sum_{m=0}^{n} \begin{pmatrix} 2n \\ 2m \end{pmatrix} D_A^{2m} D_B^{2(n-m)} \\
&\le (D_A + D_B)^{2n},
\end{aligned}$$

where we set $\gamma(0) := 0$, $\gamma(n+1) := n+1$. □

Proof of Proposition 4.1. Take any $m, n \in \mathbb{N}$, $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{C}^m$ with $\|\mathbf{u}_i\|_{\mathbb{C}^m}$, $\|\mathbf{v}_i\|_{\mathbb{C}^m} \le 1$, $(\rho_i, \mathbf{x}_i, s_i), (\eta_i, \mathbf{y}_i, t_i) \in \{1,2\} \times \Gamma \times [0,\beta)$ $(i = 1,2,\cdots,n)$, $j \in \{1,2\}$. Define the $n \times n$ matrix $M = (M_{k,l})_{1\le k,l\le n}$ by

$$M_{k,l} := \langle \mathbf{u}_k, \mathbf{v}_l\rangle_{\mathbb{C}^m} 1_{s_k \ge t_l} \langle f_j^{\ge}(\rho_k \mathbf{x}_k s_k), g_j^{\ge}(\eta_l \mathbf{y}_l t_l)\rangle_{\mathcal{H}}, \quad (l, k = 1, 2, \cdots, n).$$

Let us prove that $|\det M| \le D^{2n}$. By permutating rows and columns if necessary we may assume that $s_1 \ge \cdots \ge s_n$, $t_1 \ge \cdots \ge t_n$. By the assumption of continuity the function

$$\begin{aligned}&(\varepsilon_1, \cdots, \varepsilon_n, \delta_1, \cdots, \delta_n) \mapsto \\ &|\det(\langle \mathbf{u}_k, \mathbf{v}_l\rangle_{\mathbb{C}^m} 1_{s_k+\varepsilon_k \ge t_l - \delta_l} \langle f_j^{\ge}(\rho_k \mathbf{x}_k (s_k + \varepsilon_k)), g_j^{\ge}(\eta_l \mathbf{y}_l (t_l - \delta_l))\rangle_{\mathcal{H}})_{1\le k,l\le n}| \\ &: \mathbb{R}^{2n}_{\ge 0} \to \mathbb{R}\end{aligned}$$

is continuous at $\mathbf{0}$. Thus, we can choose real sequences $(s_k^p)_{p=1}^{\infty}$, $(t_k^p)_{p=1}^{\infty}$ $(k = 1,2,\cdots,n)$ such that $s_1^p > \cdots > s_n^p$, $t_1^p > \cdots > t_n^p$, $\{s_k^p\}_{k=1}^n \cap \{t_k^p\}_{k=1}^n = \emptyset$ for any $p \in \mathbb{N}$ and

$$\lim_{p\to\infty} |\det(\langle \mathbf{u}_k, \mathbf{v}_l\rangle_{\mathbb{C}^m} 1_{s_k^p \ge t_l^p} \langle f_j^{\ge}(\rho_k \mathbf{x}_k s_k^p), g_j^{\ge}(\eta_l \mathbf{y}_l t_l^p)\rangle_{\mathcal{H}})_{1\le k,l\le n}| = |\det M|.$$

Thus, by keeping in mind that we perform the limit operation in the end we may also assume that $s_1 > \cdots > s_n$, $t_1 > \cdots > t_n$, $\{s_k\}_{k=1}^n \cap \{t_k\}_{k=1}^n = \emptyset$.

Define the vectors $f(s_k)$, $g(t_k)$ $(k = 1,2,\cdots,n)$ of $\mathbb{C}^m \otimes \mathcal{H}$ by $f(s_k) := \mathbf{u}_k \otimes f_j^{\ge}(\rho_k \mathbf{x}_k s_k)$, $g(t_k) := \mathbf{v}_k \otimes g_j^{\ge}(\eta_k \mathbf{y}_k t_k)$. Let $\hat{\mathcal{H}}$ be the finite-dimensional subspace of $\mathbb{C}^m \otimes \mathcal{H}$ spanned by $f(s_k)$, $g(t_k)$ $(k = 1,2,\cdots,n)$. For $f \in \hat{\mathcal{H}}$ let $a(f)$ $(a(f)^*)$ be the annihilation (creation) operator on the Fermionic Fock space $F_f(\hat{\mathcal{H}})$. It is well-known (see e.g. [2, Subsection 5.2.1]) that

$$\{a(f), a(g)^*\} = \langle f, g\rangle_{\hat{\mathcal{H}}}, \tag{A.1}$$

$$\|a(f)\|_{\mathcal{B}(F_f(\hat{\mathcal{H}}))} = \|a(f)^*\|_{\mathcal{B}(F_f(\hat{\mathcal{H}}))} = \|f\|_{\hat{\mathcal{H}}}, \quad (\forall f, g \in \hat{\mathcal{H}}), \tag{A.2}$$

where $\|\cdot\|_{\mathcal{B}(F_f(\hat{\mathcal{H}}))}$ is the operator norm for operators on $F_f(\hat{\mathcal{H}})$. For $(b,\xi) \in (\{s_k\}_{k=1}^n \times \{-1\}) \cup (\{t_k\}_{k=1}^n \times \{1\})$ we set $a_{(b,\xi)} := a(f(b))$ if $\xi = -1$, $a(g(b))^*$ if $\xi = 1$. Let $l \in \{1,\cdots,2n\}$. For any distinct $(b_1,\xi_1),\cdots,(b_l,\xi_l) \in (\{s_k\}_{k=1}^n \times \{-1\}) \cup (\{t_k\}_{k=1}^n \times \{1\})$ there uniquely exists $\sigma \in \mathbb{S}_l$ such that $b_{\sigma(1)} > b_{\sigma(2)} > \cdots > b_{\sigma(l)}$. Then, we set

$$\mathbf{T}(a_{(b_1,\xi_1)} \cdots a_{(b_l,\xi_l)}) := \mathrm{sgn}(\sigma) a_{(b_{\sigma(1)},\xi_{\sigma(1)})} \cdots a_{(b_{\sigma(l)},\xi_{\sigma(l)})}.$$

Let us prove that

(A.3)

$$\det M = (-1)^{\frac{n(n-1)}{2}} \langle \hat{\Omega}, \mathbf{T}(a(f(s_1)) \cdots a(f(s_n)) a(g(t_1))^* \cdots a(g(t_n))^*) \hat{\Omega} \rangle_{F_f(\hat{\mathcal{H}})}$$

by induction with n, where $\hat{\Omega}$ denotes the vacuum and $\langle \cdot, \cdot \rangle_{F_f(\hat{\mathcal{H}})}$ is the inner product of $F_f(\hat{\mathcal{H}})$. It clearly holds for $n = 1$. Let us assume that it holds for $n - 1$ with $n \geq 2$. If $t_1 > s_1$,

$$\begin{aligned}
&(\text{R.H.S of (A.3)}) \\
&= (-1)^{\frac{n(n-1)}{2}+n} \\
&\quad \cdot \langle \hat{\Omega}, a(g(t_1))^* \mathbf{T}(a(f(s_1)) \cdots a(f(s_n)) a(g(t_2))^* \cdots a(g(t_n))^*) \hat{\Omega} \rangle_{F_f(\hat{\mathcal{H}})} \\
&= 0 = \det M.
\end{aligned}$$

Consider the case that $t_1 \leq s_1$. Then, there exists $k \in \{1, 2, \cdots, n\}$ such that $(k \leq n-1) \wedge (s_k > t_1 > s_{k+1})$ or $(k = n) \wedge (s_k > t_1)$. Then, by (A.1) and the induction hypothesis,

$$\begin{aligned}
&(\text{R.H.S of (A.3)}) \\
&= (-1)^{\frac{n(n-1)}{2}+n-k} \\
&\quad \cdot \langle \hat{\Omega}, a(f(s_1)) \cdots a(f(s_k)) a(g(t_1))^* \\
&\qquad \cdot \mathbf{T}(a(f(s_{k+1})) \cdots a(f(s_n)) a(g(t_2))^* \cdots a(g(t_n))^*) \hat{\Omega} \rangle_{F_f(\hat{\mathcal{H}})} \\
&= \sum_{l=1}^{k} (-1)^{\frac{n(n-1)}{2}+n+l} \langle f(s_l), g(t_1) \rangle_{\hat{\mathcal{H}}} \\
&\quad \cdot \langle \hat{\Omega}, a(f(s_1)) \cdots a(f(s_{l-1})) a(f(s_{l+1})) \cdots a(f(s_k)) \\
&\qquad \cdot \mathbf{T}(a(f(s_{k+1})) \cdots a(f(s_n)) a(g(t_2))^* \cdots a(g(t_n))^*) \hat{\Omega} \rangle_{F_f(\hat{\mathcal{H}})} \\
&= \sum_{l=1}^{k} (-1)^{\frac{n(n-1)}{2}+n+l} \langle f(s_l), g(t_1) \rangle_{\hat{\mathcal{H}}} \\
&\quad \cdot \langle \hat{\Omega}, \mathbf{T}(a(f(s_1)) \cdots a(f(s_{l-1})) a(f(s_{l+1})) \cdots a(f(s_n)) \\
&\qquad \cdot a(g(t_2))^* \cdots a(g(t_n))^*) \hat{\Omega} \rangle_{F_f(\hat{\mathcal{H}})}
\end{aligned}$$

$$= \sum_{l=1}^{k}(-1)^{l+1}M_{l,1}\det(M_{p,q})_{\substack{1\le p\le n, p\neq l\\ 2\le q\le n}} = \det M.$$

Here we used that

$$(-1)^{\frac{n(n-1)}{2}+n+l+\frac{(n-1)(n-2)}{2}} = (-1)^{l+1}.$$

Thus, by induction (A.3) holds for any $n \in \mathbb{N}$.

Then, by using (A.2) we can derive from (A.3) that

$$|\det M| \le \prod_{k=1}^{n} \|f_j^{\ge}(\rho_k \mathbf{x}_k s_k)\|_{\mathcal{H}} \|g_j^{\ge}(\eta_k \mathbf{y}_k t_k)\|_{\mathcal{H}} \le D^{2n}.$$

A parallel argument shows that

$$|\det(\langle \mathbf{u}_k, \mathbf{v}_l\rangle_{\mathbb{C}^m} 1_{s_k<t_l} \langle f_j^{<}(\rho_k \mathbf{x}_k s_k), g_j^{<}(\eta_l \mathbf{y}_l t_l)\rangle_{\mathcal{H}})_{1\le k,l\le n}| \le D^{2n}.$$

In fact in this case we may assume that $s_1 < \cdots < s_n$, $t_1 < \cdots < t_n$, $\{s_k\}_{k=1}^n \cap \{t_k\}_{k=1}^n = \emptyset$ by the continuity argument. Then we only need to define $\mathbf{T}$ to arrange in the opposite order. Now coming back to the decomposition (4.1), we can repeatedly apply Lemma A.1 to derive the claimed inequality. □

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Notation

Parameters and constants

Notation	Description	Reference
d	spatial dimension	Subsection 1.2
L	size of the spatial lattice	Subsection 1.2
hop	0 or 1, parameter to determine sign of hopping	Subsection 1.2
μ	chemical potential	Subsection 1.2
U	negative coupling constant	Subsection 1.2
γ	magnitude of symmetry breaking external field	Subsection 1.2
β	inverse temperature	Subsection 1.2
θ	magnitude of imaginary magnetic field	Subsection 1.2
λ_1, λ_2	artificial parameters	Subsection 2.2

h	inverse step size of time-discretization	Subsection 2.2
N	$4\beta h L^d$, cardinality of I	Subsection 2.2
Θ	$\lvert\theta/2-\pi/\beta\rvert$	beginning of Section 4

Sets and spaces

Notation	Description	Reference
Γ	$\{0,1,\cdots,L-1\}^d$	Subsection 1.2
Γ^*	$\{0,\frac{2\pi}{L},\frac{2\pi}{L}\cdot 2,\cdots,\frac{2\pi}{L}(L-1)\}^d$	Subsection 2.2
$[0,\beta)_h$	$\{0,\frac{1}{h},\frac{2}{h},\cdots,\beta-\frac{1}{h}\}$	Subsection 2.2
$D(r)$	$\{z\in\mathbb{C}\mid \lvert z\rvert<r\}$	Subsection 2.2
I_0	$\{1,2\}\times\Gamma\times[0,\beta)_h$	Subsection 2.3
I	$I_0\times\{1,-1\}$	Subsection 2.3
$\mathcal{V}$	complex vector space spanned by $\{\psi_X\}_{X\in I}$	Subsection 2.3
$\bigwedge\mathcal{V}$	Grassmann algebra generated by $\{\psi_X\}_{X\in I}$	Subsection 2.3
I^0	$\{1,2\}\times\Gamma\times\{0\}\times\{1,-1\}$	Subsection 3.1
$\bigwedge_{even}\mathcal{V}$	Subspace of $\bigwedge\mathcal{V}$ consisting of even polynomials	Subsection 3.1
$\mathrm{Map}(A,B)$	set of maps from A to B	Subsection 3.4

Functions and maps

Notation	Description	Reference
r_L	map from $\mathbb{Z}^d$ to Γ	Subsection 1.2
H_0	$\sum_{\mathbf{x}\in\Gamma,\sigma\in\{\uparrow,\downarrow\}}((-1)^{hop}\sum_{j=1}^{d}(\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}+\mathbf{e}_j\sigma}+\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}-\mathbf{e}_j\sigma})-\mu\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma})$	Subsection 1.2
V	$\frac{U}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\uparrow}$	Subsection 1.2
H	$\mathsf{H}_0+\mathsf{V}$	Subsection 1.2
F	$\gamma\sum_{\mathbf{x}\in\Gamma}(\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}+\psi_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\uparrow})$	Subsection 1.2
S_z	$\frac{1}{2}\sum_{\mathbf{x}\in\Gamma}(\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow}-\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow})$	Subsection 1.2
A_1	$\psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow}$	Subsection 1.2
A_2	$\psi^*_{\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\mathbf{x}}\downarrow}\psi_{\hat{\mathbf{y}}\downarrow}\psi_{\hat{\mathbf{y}}\uparrow}$	Subsection 1.2
$e(\cdot)$	$(-1)^{hop}2\sum_{j=1}^{d}\cos k_j-\mu$, free dispersion relation	Subsection 1.2
g_d	function to control possible magnitude of coupling constant	Equation (1.1)
$C(\phi)$	2-band free covariance parameterized by $\phi\in\mathbb{C}$	Equation (2.17)
$E(\phi)$	(2×2)-matrix-valued function parameterized by $\phi\in\mathbb{C}$	Equation (2.18)

$Tree(S,\mathcal{C})$	operator consisting of Grassmann left-derivatives	Subsection 3.1
r_β	map from $\frac{1}{h}\mathbb{Z}$ to $[0,\beta)_h$	Subsection 3.2
$\mathcal{R}_\beta$	map from $(\{1,2\}\times\Gamma\times\frac{1}{h}\mathbb{Z}\times\{1,-1\})^n$ to I^n or from $(\{1,2\}\times\Gamma\times\frac{1}{h}\mathbb{Z})^n$ to I_0^n	Subsection 3.2

Norms and semi-norms

Notation	Description	Reference
$\|\cdot\|_{1,\infty}$	integrating with all but one fixed variable	Subsection 3.1
$\|\cdot\|_1$	integrating with all variables	Subsection 3.1
$\|\cdot\|'_{1,\infty}$	norm defined on anti-symmetric function on I^2	Subsection 3.1
$\|\cdot\|$	$\|\cdot\|'_{1,\infty}+\beta^{-1}\|\cdot\|_{1,\infty}$	Subsection 3.1
$[\cdot,\cdot]_{1,\infty}$	measurement of function on $I^m\times I^n$ coupled with a function on I^2	Subsection 3.1
$[\cdot,\cdot]_1$	measurement of function on $I^m\times I^n$ coupled with a function on I^2	Subsection 3.1
$\|\cdot\|_{1,\infty,r}$	$\sup_{u\in\overline{D(r)}}\|f(u)\|_{1,\infty}$	Subsection 3.4
$\|\cdot\|_{1,r,r'}$	$\sup_{u\in\overline{D(r)},\boldsymbol{\lambda}\in\overline{D(r')}^2}\|f(u,\boldsymbol{\lambda})\|_1$	Subsection 3.5

Other notations

Notation	Description	Reference
$\mathbf{e}_j$ $(j=1,\cdots,d)$	standard basis of $\mathbb{R}^d$	Subsection 1.2
$V(\psi)$	sum of quadratic and quartic polynomials of $\bigwedge\mathcal{V}$	Equation (2.11)
$W(\psi)$	quartic polynomial of $\bigwedge\mathcal{V}$	Equation (2.12)
$A^1(\psi)$	quadratic polynomial of $\bigwedge\mathcal{V}$	Equation (2.13)
$A^2(\psi)$	quartic polynomial of $\bigwedge\mathcal{V}$	Equation (2.13)
$A(\psi)$	$\lambda_1A^1(\psi)+\lambda_2A^2(\psi)$	Equation (2.14)
$d_j(T)$	degree of vertex j in tree T	Subsection 3.2

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