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An analytic approach and applications to the Hodge decomposition on bounded domains in Euclidean space

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Abstract. An analytic proof of the Hodge decomposition on bounded domains in Euclidean space is given. This decomposition takes a form of the known fact in geometry concerning compact manifolds with boundary, and is also a natural generalization of recent results on three-dimensional vector fields. Applications to the theory of partial differential equations and their numerical methods are also given.

1. Introduction

Our purpose is to present an analytic proof of the Hodge decomposition on bounded domains in Euclidean space and its applications. This decomposition takes a form of the known result in geometry on compact manifolds with boundary (Morrey [16] p.310, Schwarz [21] p.3), and is a generalization and a refinement of recent results on three-space dimensional vector fields (Kozono-Yanagisawa [12]).

Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$ and $0 \leq k \leq n$ be an integer. Let D'_k be the set of k-forms with distributional coefficients in Ω, and

$$d : D'_k \to D'_{k+1}, \quad 0 \leq k \leq n-1\,,$$

be the exterior differentiation. The exterior product is denoted by

$$\wedge : D'_k \times D'_\ell \to D'_{k+\ell}, \quad k, \ell \geq 0,\ k+\ell \leq n\,,$$

and

$$\delta = (-1)^{n(k+1)+1} * d* : D'_k \to D'_{k-1}, \quad 1 \leq k \leq n\,, \tag{1}$$

stands for the co-differentiation ([17] p.155). Here,

$$* : D'_k \to D'_{n-k}$$

is the Hodge operator defined by

$$*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \text{sgn}\ \sigma \cdot dx_{i_{k+1}} \wedge \ldots \wedge dx_{i_n}\,,$$

with sgn σ denoting the sign of the permutation $\sigma : (i_1, \cdots, i_k, i_{k+1}, \ldots, i_n) \mapsto (1, \cdots, n)$. It holds that

$$** = (-1)^{k(n-k)}$$

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([17] p.151).

REMARK 1. We put $D'_k = \{0\}$ for $k \le -1$ or $k \ge n+1$, $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$ for $k \ge n+1$, $d = 0$ on D'_n, and $\delta = 0$ on D'_0.

Let Λ_k, $0 \le k \le n$, be the set of elements in D'_k of which coefficients are smooth on $\overline{\Omega}$. Henceforth, the unit outer normal vector $\vec{\nu} = (\nu^i)$ on $\partial\Omega$ is identified with 1-form on $\partial\Omega$ denoted by

$$\nu = \sum_i \nu^i dx_i \,. \tag{2}$$

Thus we write $\vec{\nu}$ and ν for the outer unit vector on $\partial\Omega$ and the 1-form identified with it, respectively. Our result is the following theorem.

THEOREM 1 (Hodge decomposition). *There arises the direct sum of vector spaces*

$$\Lambda_k = H_k \oplus \delta W_k \oplus dV_k, \quad 1 \le k \le n-1 \,, \tag{3}$$

where

$$\begin{aligned} H_k &= \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\} \\ W_k &= \{\omega \in \Lambda_{k+1} \mid d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0\} \\ V_k &= \{p \in \Lambda_{k-1} \mid \delta p = 0,\ \nu \wedge *p|_{\partial\Omega} = 0\} \,. \end{aligned}$$

Thus any $u \in \Lambda_k$ admits $(h, \omega, p) \in H_k \times W_k \times V_k$ such that

$$u = h + \delta\omega + dp \,. \tag{4}$$

Furthermore, H_k is isomorphic to the k-th de Rham cohomology, denoted by $H^k_{DR}(\Omega)$:

$$H_k \cong H^k_{DR}(\Omega) \,. \tag{5}$$

REMARK 2. It holds that

$$V_1 = \Lambda_0, \quad W_{n-1} = \{\omega \in \Lambda_n \mid \omega|_{\partial\Omega} = 0\}$$

by the definition.

REMARK 3. The decomposition (3) is orthogonal with respect to L^2-inner product.

REMARK 4. Analogous result holds in the context of L^r-theory (Theorem 30 in §4.2). There, the boundary conditions to (h, ω, p) are taken in the sense of weak traces first (Definition 14), and then that of traces in $W^{1,r}(\Omega)$ eventually, by Proposition 27.

REMARK 5. The de Rham theory guarantees the isomorphism

$$H^k_{DR}(\Omega) \cong H^k(\Omega, \mathbf{R}) \,, \tag{6}$$

where $H^k(\Omega, \mathbf{R})$ denotes the k-th singular cohomology of Ω ([17], p.113). Hence it holds that

$$\dim H_k = b_k\,, \tag{7}$$

where b_k is the k-th Betti number of Ω. Through the isomorphism

$$H^k(\Omega, \mathbf{R}) \to H^k_{DR}(\Omega)$$

([24] §6.2), the basis of H_k is constructed from that of $H^k(\Omega, \mathbf{R})$.

Remark 6. The decomposition (3) is a natural extension to arbitray dimension of the Hodge decompositions for three-dimensional vector fields established by [12] (Remark 12). In this case, b_1 and b_2 are the numbers of holes and cavities of Ω, respectively. If there is a family of N-smooth surfaces $\Sigma_1, \ldots, \Sigma_N$ transversal to $\partial\Omega$ such that $\Omega \setminus \cup_{j=1}^N \Sigma_j$ is simply-connected, therefore, it holds that $b_1 = N$. If $\partial\Omega$ is divided into $L+1$ connected components, denoted by Γ_j, $0 \le j \le L$, and $\Gamma_1 \ldots, \Gamma_L$'s are inside Γ_0, then we have $b_2 = L$. A family of b_1-harmonic functions, which forms a basis of H_1, is constructed by [8] for the study of inhomogeous stationary Navier-Stokes equation. A family of b_2-harmonic functions, which forms a basis of H_2, is constructed by [12] and is applied to the study of Leray's problem [13].

Remark 7. For the case that Ω is a compact manifold with boundary, the following decomposition is known ([16] p.310, [21] pp.2-3). That is

$$\Lambda_k = \tilde{H}_k \oplus \delta\tilde{W}_k \oplus d\tilde{V}_k \tag{8}$$

for

$$\begin{aligned}
\tilde{H}_k &= \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0\} \\
\tilde{W}_k &= \{\omega \in \Lambda_{k+1} \mid \mathbf{n}\omega = 0\} \\
\tilde{V}_k &= \{p \in \Lambda_{k-1} \mid \mathbf{t}p = 0\}\,,
\end{aligned}$$

where $\mathbf{t}p = j^*p$ and $\mathbf{n}\omega = \omega - \mathbf{t}\omega$ are tangential and normal components of p and ω, respectively, and $j : \partial\Omega \to \overline{\Omega}$ is a natural inclusion ([21] p.26). The harmonic part $\tilde{H}_k$ is of infinite dimension, decomposed furthermore into

$$\begin{aligned}
\tilde{H}_k &= \{h \in \tilde{H}_k \mid \mathbf{t}h = 0\} \oplus \{h \in \tilde{H}_k \mid h = \delta\gamma\} \\
\tilde{H}_k &= \{h \in \tilde{H}_k \mid \mathbf{n}h = 0\} \oplus \{h \in \tilde{H}_k \mid h = d\varepsilon\}\,.
\end{aligned}$$

If $\Omega \subset \mathbf{R}^n$ is a bounded domain and $u \in \Lambda_k$, we have $\mathbf{t}u = 0$ and $\mathbf{n}u = 0$ if and only if $\nu \wedge u|_{\partial\Omega} = 0$ and $\nu \wedge *u|_{\partial\Omega} = 0$, respectively (Remark 8), and therefore, the decomposition (3) is different from (8) for this case of Ω. We have, however, an analogous form to (3) for the compact manifold Ω with boundary (Remark 18). This paper thus provides an analytic proof of (3).

Remark 8. If Ω is the half space $\mathbf{R}^n_+ = \{(x', x_n) \mid x_n > 0\}$ and

$$u = \sum_{i_1 < \cdots < i_k} u^{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

is k-form, it holds that

$$\mathbf{t}u = \sum_{i_1<\cdots<i_k<n} u^{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

by the definition. In this case we have $\nu = (0, \cdots, 0, -1)^T$, and therefore,

$$\begin{aligned} \nu \wedge u &= (-dx_n) \wedge \sum_{i_1<\cdots<i_k} u^{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= - \sum_{i_1<\cdots<i_k<n} u^{i_1\cdots i_k} dx_n \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \,. \end{aligned}$$

Hence it holds that $\mathbf{t}u = 0$ on $\partial\mathbf{R}^n_+$ if and only if $\nu \wedge u|_{\partial\mathbf{R}^n_+} = 0$. If $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, we take a smooth closed set $\Gamma \subset \partial\Omega$ and diffeomorphism $X : D \to U \subset \mathbf{R}^n$ satisfying $X(D \cap \overline{\Omega}) \subset \overline{\mathbf{R}}^n_+$ and $X(\vec{\nu}) = (0, \cdots 0, -1)^T$, where D is a smooth open set containing Γ. Then we get

$$Y^*(\mathbf{t}u) = \mathbf{t}(Y^*u), \ Y^*(\nu \wedge u) = (Y^*\nu) \wedge (Y^*u) \quad \text{on } X(\Gamma)$$

for $Y = X^{-1}$ and $u \in \Lambda_k$, and therefore, $\mathbf{t}u = 0$ on Γ if and only if $\nu \wedge u|_\Gamma = 0$. We thus obtain $\mathbf{t}u = 0$ in $\partial\Omega$ if and only if $\nu \wedge u|_{\partial\Omega} = 0$. Since $*(\mathbf{n}u) = \mathbf{t}(*u)$ ([21] p.27), it holds also that $\mathbf{n}u = 0$ on $\partial\Omega$ if and only if $\nu \wedge *u|_{\partial\Omega} = 0$. This fact is noticed by [26] in the study of interface vaniching of differential forms subject to $d - \delta$ systems.

REMARK 9. If $n = 2$ and $h = h^1 dx_1 + h^2 dx_2 \in \Lambda_1$, it holds that

$$*h = h^1 dx_2 - h^2 dx_1 \,,$$

and therefore, this h belongs to H_1 if and only if

$$h^1_2 = h^2_1 \text{ in } \Omega, \quad h^1_1 + h^2_2 = 0 \text{ in } \Omega, \quad \nu \cdot h|_{\partial\Omega} = 0 \,, \tag{9}$$

where

$$h^i_j = \frac{\partial h^i}{\partial x_j}, \quad i, j = 1, 2 \,.$$

If Ω is simply-connected, furthermore, the first equality of (9) implies the existence of a scalar field u on Ω such that

$$(h_1, h_2)^T = \nabla u \,,$$

and then (9) is reduced to

$$\Delta u = 0 \text{ in } \Omega, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 \,.$$

It holds that $u \in \mathbf{R}$ or $h = 0$, which induces $\dim H_1 = b_1 = 0$. This reduction of H_1 for simply-connected domains, however, is valid even if Ω is only a Lipschitz domain. Details are not discussed here, but extension of Theorem 1 to this direction is expected in the context of H^1-theory.

REMARK 10. If $a, b \in \Lambda_k$, $0 \leq k \leq n$, there arises that

$$(*a, *b) = (a, b)\,, \tag{10}$$

and that

$$a \wedge *b = (a, b)\; dx_1 \wedge \cdots \wedge dx_n$$

([17] p.151). For $h \in \Lambda_1$, in particular, it holds that $\nu \wedge *h|_{\partial\Omega} = 0$ if and only if

$$(\nu, h)|_{\partial\Omega} = 0\,.$$

Recall (9) in Remark 9 concerning $n = 2$ and Remark 8 for $k = 1$.

Theorem 1 implies the following property.

COROLLARY 2 (Helmholtz decomposition). *Any $u \in \Lambda_k$, $1 \leq k \leq n-1$, admits $(v, p) \in \Lambda_k \times \Lambda_{k-1}$ such that*

$$u = v + dp, \quad \delta v = 0, \quad \nu \wedge *v|_{\partial\Omega} = 0\,. \tag{11}$$

If $\nu \wedge *u|_{\partial\Omega} = 0$, furthermore, there is $(z, \omega) \in \Lambda_k \times \Lambda_{k+1}$ such that

$$u = z + \delta\omega, \quad dz = 0, \quad \nu \wedge *z|_{\partial\Omega} = 0\,. \tag{12}$$

PROOF. Writing $v = h + \delta\omega$ in (4), we obtain

$$u = v + dp, \quad \delta v = 0\,.$$

Here, there arises that

$$\nu \wedge *v|_{\partial\Omega} = 0$$

by Lemma 3 below. Similarly, letting $z = h + dp$, we have

$$u = z + \delta\omega, \quad dz = 0$$

and furthermore,

$$\nu \wedge *u|_{\partial\Omega} = 0 \quad \Rightarrow \quad \nu \wedge *z|_{\partial\Omega} = 0\,. \qquad \square$$

LEMMA 3 (Tangential differentiation). *Each $\theta \in \Lambda_k$, $1 \leq k \leq n$, admits the property*

$$\nu \wedge *\theta|_{\partial\Omega} = 0 \quad \Rightarrow \quad \nu \wedge *\delta\theta|_{\partial\Omega} = 0\,.$$

REMARK 11. This lemma follows from Remark 8 if one uses the fact that $\mathbf{n}$ commutes with δ ([21] p.26). It thus holds that $\mathbf{n}\delta\theta = \delta\mathbf{n}\theta = 0$ if $\mathbf{n}\theta = 0$. An alternative proof of this lemma is given in §2.1.

REMARK 12. Any 1-form or $(n-1)$-form is identified with a vector field on $\overline{\Omega}$. If $n = 3$ and the vector field $u = (u^1, u^2, u^3)^T$ is identified with the 1-form

$$u = u^1 dx_1 + u^2 dx_2 + u^3 dx_3 \,,$$

Theorem 1 for $k = 1$ corresponds to the first decomposition of [12] presented as in the equality (2.15) there, in the context of L^r-theory. Thus any vector field u admits vector fields h, ω and scalar field p such that

$$\begin{aligned} &\text{div } h = 0, \quad \text{rot } h = 0, \quad \vec{\nu} \cdot h|_{\partial\Omega} = 0 \\ &\text{div } \omega = 0, \quad \vec{\nu} \times \omega|_{\partial\Omega} = 0 \end{aligned}$$

and

$$u = h + \text{rot } \omega + \text{grad } p \,, \tag{13}$$

where $\cdot$ and $\times$ denote the inner and outer products in $\mathbf{R}^3$, respectively. Then (13) implies the Helmholtz decomposition

$$u = v + \text{grad } p, \quad \text{div } v = 0, \quad \vec{\nu} \cdot v|_{\partial\Omega} = 0$$

by (11). Here, the boundary conditions to (h, ω, v) are taken in the sense of traces if u belongs only to $L^r(\Omega)$ (Proposition 27, Theorem 30, Theorem 31, and Remark 59). Similarly, if $u = (u_1, u_2, u_3)^T$ is identified with the 2-form

$$u = u^1 dx_2 \wedge dx_3 + u^2 dx_3 \wedge dx_1 + u^3 dx_1 \wedge dx_2 \,,$$

Theorem 1 for $k = 2$ represents the second decomposition of [12], the equality (2.16) there. Thus any vector field u admits vector fields h, p and scalar field ω such that

$$\begin{aligned} &\text{div } h = 0, \quad \text{rot } h = 0, \quad \vec{\nu} \times h|_{\partial\Omega} = 0 \\ &\omega|_{\partial\Omega} = 0, \quad \text{div } p = 0, \quad \vec{\nu} \cdot p|_{\partial\Omega} = 0 \,, \end{aligned}$$

and

$$u = h + \text{grad } \omega + \text{rot } p \,, \tag{14}$$

which implies the Helmholtz decomposition in the form of

$$u = z + \text{rot } p, \quad \text{rot } z = 0, \ \nu \times z|_{\partial\Omega} = 0 \,.$$

The classical Hodge theory is concerned on the case that Ω is a closed (i.e., compact, without boundary) manifold of any dimension $n \geq 2$. In this case it holds that

$$\dot{\Lambda}_k = \dot{H}_k \oplus \delta\dot{\Lambda}_{k+1} \oplus d\dot{\Lambda}_{k-1}, \quad 1 \leq k \leq n-1 \,, \tag{15}$$

where $\dot{\Lambda}_k$ denotes the set of k-th differential forms on Ω with smooth coefficients and

$$\dot{H}_k = \{h \in \dot{\Lambda}_k \mid dh = 0, \ \delta h = 0\} \,.$$

Furthermore, any $u \in \dot{\Lambda}_k$ admits $(h, \eta) \in \dot{H}_k \times \dot{\Lambda}_k$ such that

$$u = h + \delta d\eta + d\delta\eta \tag{16}$$

([17], p.160). This property is valid even when $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary.

THEOREM 4 (Structure theorem). *Any* $u \in \Lambda_k$ *admits* $(h, \eta) \in H_k \times (\Lambda_k \cap H_k^{\perp})$ *such that*

$$\nu \wedge *\eta\big|_{\partial\Omega} = 0, \quad \nu \wedge *d\eta\big|_{\partial\Omega} = 0 \tag{17}$$

and

$$u = h + \delta d\eta + d\delta\eta\,. \tag{18}$$

REMARK 13. We say that $u \in \Lambda_k$ is in $H_k^{\perp}$ if it is L^2-orthogonal to any $h \in H_k$. Similarly to Theorem 4, furthermore, we can take $\eta \in \dot{\Lambda}_k \cap \dot{H}_k^{\perp}$ in (16).

REMARK 14. It holds that

$$-\Delta = d\delta + \delta d \tag{19}$$

in (18) ([17] p.157).

REMARK 15. By Lemma 3, Theorem 4 implies (4) in Theorem 1 for

$$(\omega, p) = (d\eta, \delta\eta) \in W_k \times V_k\,.$$

REMARK 16. Theorem 4 for the case of $n = 3$ is stated as follows. First, any smooth vector field u on $\overline{\Omega}$ admits the smooth vector fields h, η such that

$$u = h + \mathrm{rot}\ \mathrm{rot}\ \eta - \mathrm{grad}\ \mathrm{div}\ \eta, \quad \mathrm{div}\ h = 0,\ \mathrm{rot}\ h = 0\,.$$

Second, the boundary conditions are provided to (h, η) as either

$$\vec{\nu} \cdot h\big|_{\partial\Omega} = 0, \quad \vec{\nu} \times \mathrm{rot}\ \eta\big|_{\partial\Omega} = 0$$

or

$$\vec{\nu} \times h\big|_{\partial\Omega} = 0, \quad \mathrm{div}\ \eta\big|_{\partial\Omega} = 0, \quad \vec{\nu} \cdot \mathrm{rot}\ \eta\big|_{\partial\Omega} = 0\,.$$

A numerical scheme for Helmholtz decomposition is provided by this formulation (§5.3).

REMARK 17. There is an L^r-version of Theorem 4, similarly to Theorem 1 (Theorem 31).

The Hodge operator $* : \dot{H}_k \to \dot{H}_{n-k}$ is an isomorphism in (15) concerning the closed manifold Ω. There arises, therefore, the Poincaré duality $b_{n-k} = b_k$ in this case. This duality does not hold any more if Ω has a boundary, as is described in Remark 6 for the case of $n = 3$. Through $* : \Lambda_k \to \Lambda_{n-k}$, actually, we obtain a dual form of (3). It thus holds that

$$\Lambda_k = H_k^* \oplus \delta W_k^* \oplus dV_k^*, \quad 1 \le k \le n-1 \tag{20}$$

where

$$H_k^* = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge h|_{\partial\Omega} = 0\}$$
$$W_k^* = \{\omega \in \Lambda_{k+1} \mid d\omega = 0,\ \nu \wedge \omega|_{\partial\Omega} = 0\}$$
$$V_k^* = \{p \in \Lambda_{k-1} \mid \delta p = 0,\ \nu \wedge p|_{\partial\Omega} = 0\}\,.$$

Furthermore, we have (18) for $(h, \eta) \in H_k^* \times (\Lambda_k \cap H_k^{*\perp})$ satisfying

$$\nu \wedge \eta|_{\partial\Omega} = 0, \quad \nu \wedge d\eta|_{\partial\Omega} = 0\,.$$

There is also the isomorphism

$$H_k^* \cong H_{DR}^{n-k}(\Omega)\,,$$

which results in

$$\dim H_k^* = b_{n-k}\,. \tag{21}$$

Equalities (7) and (21) indicate that the discrepancy between the Betti numbers b_k and b_{n-k} is due to the boundary conditions required to h in H_k and H_k^*, that is, $\nu \wedge *h|_{\partial\Omega} = 0$ and $\nu \wedge h|_{\partial\Omega} = 0$, respectively.

REMARK 18. An analogous fact to (20) is known even if Ω is a compact manifold with boundary. In fact, in this case any $v \in (H_k^*)^\perp$ admits $\eta \in \Lambda_k$ such that

$$v = \delta d\eta + d\delta\eta, \quad \mathbf{t}\eta = 0, \quad \mathbf{t}\delta\eta = 0\,,$$

where

$$H_k^* = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \mathbf{t}h = 0\}$$

([21] p.6). By this result, it holds that

$$\Lambda_k = H_k^* \oplus \delta W_k^* \oplus dV_k^*$$

for

$$H_k^* = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \mathbf{t}h = 0\}$$
$$W_k^* = \{\omega \in \Lambda_{k+1} \mid d\omega = 0,\ \mathbf{t}\omega = 0\}$$
$$V_k^* = \{p \in \Lambda_{k-1} \mid \delta p = 0,\ \mathbf{t}p = 0\}\,.$$

If Ω is a bounded domain in Euclidean space, this decomposition means (20). Then, taking the Hodge operator $*$, we obtain Theorem 1 for the case that Ω is a compact manifold with boundary by Remark 8. It thus holds that

$$\Lambda_k = H_k \oplus \delta W_k \oplus dV_k$$

for

$$H_k = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \mathbf{n}h = 0\}$$
$$W_k = \{\omega \in \Lambda_{k+1} \mid d\omega = 0,\ \mathbf{n}\omega = 0\}$$
$$V_k = \{p \in \Lambda_{k-1} \mid \delta p = 0,\ \mathbf{n}p = 0\}\,.$$

As is noticed in Remark 4 and Remark 17, Theorem 1 and Theorem 4 have L^r-versions (Theorem 30 and Theorem 31 in §4.2). If $n = 3$, $k = 1, 2$, Theorem 30 coincides with the result in [12], which induces the associated Helmholtz decomposition of [9] (Remark 59). The L^r-theory on compact manifolds with boundary is actually given in [21] p.81, p.93. As in this case, we derive Theorem 30 from Theorem 31. To prove Theorem 31, first, we show this theorem for $r = 2$, using the variational method and difference quotients, to derive H^2-regularity of η (§3.3). Then the elliptic regularity (Theorem 16) implies its smoothness, to establish Theorem 4. Finally we obtain Theorem 31 from the $W^{2,r}$-a priori estimate (Lemma 32 and Lemma 33).

Our proof of Theorem 4 thus follows the standard theory of elliptic equations based on the L^2-theory ([10] Chapter 8). To establish this L^2-theory, the H^1-regularity of $d - \delta$ systems (Theorem 13) is essential. It is derived from Gaffney's inequality (Lemma 9) and the Duvaut-Lions density (Lemma 15). Several other tools are constructed as the generalization of those in [12]. That is, Lemma 9, Theorem 13, and Theorem 16 in §2.2, §2.3, and §2.4, respectively. Our proof of Theorem 1, however, is rather different from [12]. It is actually executed through Theorem 4, as Theorem 30 follows from Theorem 31. Hence the terms $\delta\omega$ and dp in (4) are not determined individually, but simultaneously through η in Theorem 4, and by this argument we are able to derive the Hodge decomposition in any space dimension, independently of the Helmholtz decomposition in Corollary 2; compare the arguments in §3.3 and §5.2.

This paper is composed of five sections. Taking preliminaries in §2, Theorem 1 and Theorem 4 are proven in §3. Section 4 is devoted to applications; a priori estimats to $d - \delta$ systems, L^r-theory stated above after introducing several function spaces, compensated compactness lemma, and a form of Babuška-Brezzi-Kikuchi's inequality which leads to the $W^{1,r}$-regularity of the Poisson equation. Section 5 is concluding remarks. We describe a spectral property of $d - \delta$ operators, the relation between Hodge and Helmholtz decompositions, and a numerical scheme for the Helmholtz decomposition.

Henceforth, we use the following notation.

NOTATION 1. The standard L^p norm on Ω is denoted by $\| \cdot \|_p$, where $1 \leq p \leq \infty$.

NOTATION 2. The generic positive constant is denoted by C, which may vary from line to line.

NOTATION 3. Given a Banach space over $\mathbf{R}$, strong and weak convergences there are indicated by $\to$ and $\rightharpoonup$, respectively.

2. Preliminaries

This section is devoted to preliminaries. It is composed of four subsections dealing with the Gauss formula, H^1-estimate of $d - \delta$ systems, H^1-regularity of them, and an elliptic estimate. Meanwhile Lemma 3 is prove in §2.1.

2.1. Gauss formula

We begin with several definitions.

DEFINITION 1. Given $\theta, \omega \in \Lambda_{k'}$, $0 \le k' \le n$, we put

$$(\theta, \omega) = \sum_{i_1 < \cdots < i_{k'}} \theta^{i_1 \cdots i_{k'}} \omega^{i_1 \cdots i_{k'}}$$

for

$$\theta = \sum_{i_1 < \cdots < i_{k'}} \theta^{i_1 \cdots i_{k'}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k'}}$$
$$\omega = \sum_{i_1 < \cdots < i_{k'}} \omega^{i_1 \cdots i_{k'}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k'}} .$$

DEFINITION 2. Let

$$\omega = \sum_{i_1 < \cdots < i_{k'}} \omega^{i_1 \cdots i_{k'}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k'}} \in \Lambda_{k'}, \quad 0 \le k' \le n$$

and $1 \le i_1, \cdots, i_{k'} \le n$ be distinct integers. Then we define $i'_1 < \cdots < i'_{k'}$ by

$$\{i'_1, \cdots, i'_{k'}\} = \{i_1, \cdots, i_{k'}\}$$

and put

$$\tilde{\omega}^{i_1 \cdots i_{k'}} = \text{sgn } \sigma \cdot \omega^{i'_1 \cdots i'_{k'}} ,$$

where $\sigma : (i_1, \cdots, i_{k'}) \mapsto (i'_1, \cdots, i'_{k'})$ is the permutation. We put $\tilde{\omega}^{i_1 \cdots i_{k'}} = 0$ if $1 \le i_1, \cdots, i_{k'} \le n$ are not distinct.

REMARK 19. If $k' = 0$ we put $\tilde{\omega} = \omega$.

DEFINITION 3. Given $1 \le k' \le n$ and

$$\omega = \sum_{i_1 < \cdots < i_{k'}} \omega^{i_1 \cdots i_{k'}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k'}} \in \Lambda_{k'} ,$$

we put

$$\hat{\omega}^{i_2 \cdots i_{k'}} = \sum_{\ell} \tilde{\omega}^{i_2 \cdots i_{k'} \ell} dx_\ell \in \Lambda_1 \tag{22}$$

for $\tilde{\omega}^{i_1 \cdots i_{k'}}$ defined by Definition 2.

REMARK 20. If $k' = 1$, we put $\hat{\omega}^{i_2 \cdots i_{k'}} = \omega \in \Lambda_1$.

REMARK 21. Definitions 3 is introduced by [26] for the study of interface vanishing of $d - \delta$ systems.

In this subsection we show the Gauss formula in the following form.

THEOREM 5 (Gauss formula). *It holds that*

$$\int_\Omega (d\theta, \omega)\ dx = \int_\Omega (\theta, \delta\omega)\ dx + (-1)^k \int_{\partial\Omega} (*\theta, \nu \wedge *\omega)\ ds \tag{23}$$

for $(\theta, \omega) \in \Lambda_k \times \Lambda_{k+1}$ *and* $0 \leq k \leq n-1$, *where* ds *denotes the area element of* $\partial\Omega$.

We note the following facts in advance. First, given $(\theta, \omega) \in \Lambda_k \times \Lambda_{k+1}$, we have

$$d\theta \wedge *\omega = d(\theta \wedge *\omega) + \theta \wedge *\delta\omega$$

([17] p.155). Since

$$d\theta \wedge *\omega = (d\theta, \omega) dx_1 \wedge \cdots \wedge dx_n$$
$$\theta \wedge *\delta\omega = (\theta, \delta\omega) dx_1 \wedge \cdots \wedge dx_n$$

it holds that

$$\begin{aligned}\int_\Omega (d\theta, \omega)\ dx &= \int_\Omega d(\theta \wedge *\omega) + \int_\Omega (\theta, \delta\omega)\ dx \\ &= \int_{\partial\Omega} \theta \wedge *\omega + \int_\Omega (\theta, \delta\omega)\ dx\end{aligned}$$

by the Stokes formula ([17] p.107). Second, the $(n-1)$-differential form $*dx_i$ is associated with the surface element ds on $\partial\Omega$ as in

$$*dx_i = \nu^i ds \text{ on } \partial\Omega, \quad 1 \leq i \leq n\,, \tag{24}$$

which implies

$$ds = \sum_{i=1}^n \nu^i * dx_i\,.$$

REMARK 22. Equality (24) is used in [25] in the study of the transformation of area elements. It is consistent to the orientation

$$dx = dx_1 \wedge \cdots \wedge dx_n\,.$$

If $n = 3$, for example, it follows that

$$dx_1 \wedge dx_2 = \nu^3 ds,\ dx_2 \wedge dx_3 = \nu^1 ds,\ dx_3 \wedge dx_1 = \nu^2 ds \quad \text{on } \partial\Omega\,.$$

We begin with a counter part of (23).

LEMMA 6. *Each* $0 \leq k \leq n-1$ *admits*

$$\int_\Omega (d\theta, \omega)\ dx = \int_\Omega (\theta, \delta\omega)\ dx + \int_{\partial\Omega} (\nu \wedge \theta, \omega)\ ds, \quad (\theta, \omega) \in \Lambda_k \times \Lambda_{k+1}\,. \tag{25}$$

REMARK 23. If $(\theta, \omega) \in \Lambda_0 \times \Lambda_1$, it holds that

$$(\nu \wedge \theta, \omega) = \theta(\nu, \omega)\,,$$

and therefore, (25) is equivalent to the classical Gauss formula

$$\int_\Omega (\text{grad } q) \cdot v \, dx = \int_{\partial\Omega} q(\vec{\nu} \cdot v) \, ds - \int_\Omega q \text{ div } v \, dx \tag{26}$$

where $\cdot$ denotes the $\mathbf{R}^n$-inner product q and v are smooth scalar and vector fields on $\overline{\Omega}$, respectively. It holds also that

$$(\nu \wedge \theta, \omega) ds = \mathbf{t}\theta \wedge *(\mathbf{n}\omega) = \langle \theta, \mathbf{i}_N \omega \rangle_{\Lambda_k} \mu_\partial \,,$$

see [21] pp.26-27.

PROOF OF LEMMA 6: Letting

$$\begin{aligned} \theta &= \sum_{\ell_1 < \cdots < \ell_k} \theta^{\ell_1 \cdots \ell_k} dx_{\ell_1} \wedge \cdots \wedge dx_{\ell_k} \in \Lambda_k \\ \omega &= \sum_{i_1 < \cdots < i_{k+1}} \omega^{i_1 \cdots i_{k+1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \in \Lambda_{k+1} \,, \end{aligned}$$

we obtain

$$\begin{aligned} \theta \wedge *\omega &= \sum_{\ell_1 < \cdots < \ell_k} \sum_{i_1 < \cdots < i_{k+1}} \left(\theta^{\ell_1 \cdots \ell_k} dx_{\ell_1} \wedge \ldots \wedge dx_{\ell_k} \right) \\ &\quad \wedge (\omega^{i_1 \cdots i_{k+1}} * \left(dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \right)) \\ &= \sum_{i_1 < \cdots < i_{k+1}} \omega^{i_1 \cdots i_{k+1}} \sum_{\ell_1 < \cdots < \ell_k} \theta^{\ell_1 \cdots \ell_k} (dx_{\ell_1} \wedge \cdots \wedge dx_{\ell_k}) \\ &\qquad \wedge * \left(dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \right) . \end{aligned}$$

Here it holds that

$$(dx_{\ell_1} \wedge \cdots \wedge dx_{\ell_k}) \wedge *(dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}) = 0 \tag{27}$$

unless $\{\ell_1, \cdots, \ell_k\} \subset \{i_1, \cdots, i_{k+1}\}$, while if

$$\{i_1, \ldots, i_{k+1}\} \setminus \{\ell_1, \ldots, \ell_k\} = \{\ell_s\}, \quad 1 \le \exists s \le k+1 \,, \tag{28}$$

there arises that

$$\begin{aligned} &(dx_{\ell_1} \wedge \cdots \wedge dx_{\ell_k}) \wedge *(dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}) \\ &= (dx_{i_1} \wedge \cdots \wedge dx_{i_{s-1}} \wedge dx_{i_{s+1}} \wedge \cdots \wedge dx_{i_{k+1}}) \wedge *(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\ &= (-1)^{s-1} * dx_{i_s} \,. \end{aligned} \tag{29}$$

These relations, equality (27) unless $\{\ell_1, \cdots, \ell_k\} \subset \{i_1, \cdots, i_{k+1}\}$ and equality (29) for (28), ensure

$$\begin{aligned} &\sum_{\ell_1 < \cdots < \ell_k} \theta^{\ell_1 \cdots \ell_k} (dx_{\ell_1} \wedge \cdots \wedge dx_{\ell_k}) \wedge *(dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}) \\ &\quad = \theta^{i_2 \cdots i_{k+1}} * dx_{i_1} - \theta^{i_1 i_3 \cdots i_{k+1}} * dx_{i_2} + \cdots + (-1)^{k-1} \theta^{i_1 \cdots i_k} * dx_{i_{k+1}} \\ &\quad = \tilde{\theta}^{i_2 \cdots i_{k+1}} * dx_{i_1} + \tilde{\theta}^{i_3 \cdots i_{k+1} i_1} * dx_{i_2} + \cdots + \tilde{\theta}^{i_1 \cdots i_k} * dx_{i_{k+1}} \end{aligned}$$

by Definition 2, and therefore,

$$\begin{aligned}\theta \wedge *\omega = &\sum_{i_1<\cdots<i_{k+1}} \omega^{i_1\cdots i_{k+1}}(\tilde{\theta}^{i_2\cdots i_{k+1}} * dx_{i_1} \\ &+\tilde{\theta}^{i_3\cdots i_{k+1}i_1} * dx_{i_2} + \cdots + \tilde{\theta}^{i_1\cdots i_k} * dx_{i_{k+1}})\,. \end{aligned} \tag{30}$$

We have also

$$\begin{aligned}\nu \wedge \theta &= \sum_{\ell} \sum_{\ell_1<\cdots<\ell_k} (\nu^{\ell} dx_{\ell}) \wedge (\theta^{\ell_1\cdots\ell_k} dx_{\ell_1} \wedge \cdots \wedge dx_{\ell_k}) \\ &= \left(\sum_{\ell<\ell_1<\cdots<\ell_k} + \sum_{\ell_1<\ell<\ell_2<\cdots<\ell_k} + \sum_{\ell_1<\cdots<\ell_k<\ell} \right) \\ &\qquad \cdot\nu^{\ell}\theta^{\ell_1\cdots\ell_k} dx_{\ell} \wedge dx_{\ell_1} \wedge \cdots \wedge dx_{\ell_k} \\ &= \sum_{i_1<\cdots<i_{k+1}} \nu^{i_1}\theta^{i_2 i_3\cdots i_{k+1}} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{k+1}} \\ &\quad + \sum_{i_1<i_2<i_3<\cdots<i_{k+1}} \nu^{i_2}\theta^{i_1 i_3\cdots i_{k+1}} dx_{i_2} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} + \cdots \\ &\quad \cdots + \sum_{i_1<\cdots i_k<i_{k+1}} \nu^{i_{k+1}}\theta^{i_1 i_2\cdots i_k} dx_{i_{k+1}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1<i_2<\cdots<i_{k+1}} \big(\nu^{i_1}\theta^{i_2 i_3\cdots i_{k+1}} + (-1)\nu^{i_2}\theta^{i_1 i_3\cdots i_{k+1}} + \cdots \\ &\quad \cdots + (-1)^k \nu^{i_{k+1}}\theta^{i_1 i_2\cdots i_k}\big)\, dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \\ &= \sum_{i_1<\cdots<i_{k+1}} \zeta^{i_1\cdots i_{k+1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}\end{aligned}$$

for

$$\begin{aligned}\zeta^{i_1\cdots i_{k+1}} &= \sum_{s=1}^{k+1} \nu^{i_s}(-1)^{s-1}\theta^{i_1\cdots i_{s-1}i_{s+1}\cdots i_{k+1}} \\ &= \nu^{i_1}\tilde{\theta}^{i_2\cdots i_{k+1}} + \nu^{i_2}\tilde{\theta}^{i_3\cdots i_{k+1}i_1} + \cdots + \nu^{i_{k+1}}\tilde{\theta}^{i_1\cdots i_k}\,,\end{aligned} \tag{31}$$

and therefore,

$$\begin{aligned}(\nu \wedge \theta, \omega)ds &= \sum_{i_1<\cdots<i_{k+1}} \zeta^{i_1\cdots i_{k+1}}\omega^{i_1\cdots i_{k+1}} ds \\ &= \sum_{i_1<\cdots<i_{k+1}} \omega^{i_1\cdots i_{k+1}}(\tilde{\theta}^{i_2\cdots i_{k+1}}\nu^{i_1} ds + \cdots + \tilde{\theta}^{i_1\cdots i_k}\nu^{i_{k+1}} ds) \\ &= \sum_{i_1<\cdots<i_{k+1}} \omega^{i_1\cdots i_{k+1}}(\tilde{\theta}^{i_2\cdots i_{k+1}} * dx_{i_1} + \cdots + \tilde{\theta}^{i_1\cdots i_k} * dx_{i_{k+1}}) \\ &= \theta \wedge *\omega\end{aligned} \tag{32}$$

by (30)-(31). Equality (25) now follows from (24). □

The following lemma is Lemma 3.4 of [26], but the proof is provided for completeness. Recall Definition 3 of $\widehat{\omega}^{i_2\cdots i_{k'}}$.

LEMMA 7. *Each $\omega \in \Lambda_{k'}$, $1 \le k' \le n$, admits the equality*

$$\nu \wedge *\omega = *\beta\,, \tag{33}$$

where

$$\beta = \sum_{i_2<\cdots<i_{k'}} (\hat{\omega}^{i_2\cdots i_{k'}}, \nu)\ dx_{i_2} \wedge \cdots \wedge dx_{i_{k'}} \in \Lambda'_{k'-1}\,. \tag{34}$$

PROOF. Since $\Lambda_* = \oplus_{0\le k'\le n}\Lambda_{k'}$ forms an exterior algebra, this lemma is reduced to the case

$$\omega = dx_1 \wedge \cdots \wedge dx_{k'}\,. \tag{35}$$

Then we have

$$\tilde{\omega}^{i_1\cdots i_{k'}} = \mathrm{sgn}\ \sigma, \quad \sigma : (i_1, \cdots, i_{k'}) \mapsto (1, \cdots, k')$$

if $\{i_1, \dots, i_{k'}\} = \{1, \dots, k'\}$, and $\tilde{\omega}^{i_1\cdots i_{k'}} = 0$ otherwise.

The non-trivial case of equality (22),

$$\hat{\omega}^{i_2\cdots i_{k'}} = \sum_\ell \tilde{\omega}^{i_2\cdots i_{k'}\ell} dx_\ell\,,$$

therefore, is reduced to

$$\hat{\omega}^{1\cdots j-1, j+1\cdots k'} = (-1)^{k'-j+1} dx_j, \quad 1 \le j \le k'\,,$$

and otherwise it follows that $\hat{\omega}^{i_2\cdots i_{k'}} = 0$. We thus obtain

$$\begin{aligned}
\beta &= \sum_{i_2<\cdots<i_{k'}} (\hat{\omega}^{i_2\cdots i_{k'}}, \nu) dx_{i_2} \wedge \cdots \wedge dx_{i_{k'}} \\
&= \sum_{j=1}^{k'} (\hat{\omega}^{1\cdots j-1, j+1\cdots k'}, \nu) dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{k'} \\
&= \sum_{j=1}^{k'} \nu^j (-1)^{k'-j+1} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{k'}\,.
\end{aligned}$$

Here we use

$$*(dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{k'}) = (-1)^{k'-j+1} dx_j \wedge dx_{k'+1} \wedge \cdots \wedge dx_n\,,$$

to get

$$*\beta = \sum_{j=1}^{k'} \nu^j dx_j \wedge dx_{k'+1} \wedge \cdots \wedge dx_n\,. \tag{36}$$

Recalling (35), we notice

$$*\omega = dx_{k'+1} \wedge \cdots \wedge dx_n$$

and hence

$$\nu \wedge *\omega = \left(\sum_{j=1}^{n} \nu^j dx_j\right) \wedge (dx_{k'+1} \wedge \cdots \wedge dx_n)$$

$$= \sum_{j=1}^{k'} \nu^j dx_j \wedge dx_{k'+1} \wedge \cdots \wedge dx_n \,. \tag{37}$$

The result (33), for ω in the form of (35), follows from (36)-(37). □

We show the following lemma, which completes the proof of Theorem 5 by (25).

LEMMA 8. *Each* $(\theta, \omega) \in \Lambda_k \times \Lambda_{k+1}$, $0 \le k \le n-1$, *admits the equality*

$$(\nu \wedge \theta, \omega) = (-1)^k (*\theta, \nu \wedge *\omega) \tag{38}$$

on $\partial\Omega$.

PROOF. For $\beta \in \Lambda_k$ defined by (34) to $k' = k+1$, it holds that

$$(*\theta, \nu \wedge *\omega) = (*\theta, *\beta) = (\theta, \beta) \tag{39}$$

by (10) and (33). Then we obtain

$$\begin{aligned}(\theta, \beta) &= \sum_{i_2 < \cdots < i_{k+1}} (\widehat{\omega}^{i_2 \ldots i_{k+1}}, \nu) \theta^{i_2 \ldots i_{k+1}} \\ &= \sum_{\ell} \sum_{i_2 < \cdots < i_{k+1}} \nu^\ell \widetilde{\omega}^{i_2 \ldots i_{k+1} \ell} \theta^{i_2 \ldots i_{k+1}} \\ &= (-1)^k \sum_{\ell} \sum_{i_2 < \cdots < i_{k+1}} \nu^\ell \widetilde{\omega}^{\ell i_2 \ldots i_{k+1}} \theta^{i_2 \ldots i_{k+1}} \,. \end{aligned} \tag{40}$$

Here, it holds that

$$\begin{aligned}\sum_{\ell} \sum_{i_2 < \cdots < i_{k+1}} \nu^\ell \widetilde{\omega}^{\ell i_2 \ldots i_{k+1}} \theta^{i_2 \ldots i_{k+1}} &= \left(\sum_{\ell < i_2 < \cdots < i_{k+1}} + \sum_{i_2 < \ell < i_3 < \cdots < i_{k+1}} \right. \\ &\quad \left. + \cdots + \sum_{i_2 < \cdots < i_{k+1} < \ell} \right) \nu^\ell \widetilde{\omega}^{\ell i_2 \ldots i_{k+1}} \theta^{i_2 \ldots i_{k+1}} \\ &= \sum_{i_1 < i_2 < \cdots < i_{k+1}} \nu^{i_1} \widetilde{\omega}^{i_1 \ldots i_{k+1}} \theta^{i_2 \ldots i_{k+1}} \\ &\quad + \sum_{i_1 < i_2 < \cdots < i_{k+1}} \nu^{i_2} \widetilde{\omega}^{i_2 i_1 \ldots i_{k+1}} \theta^{i_1 i_3 \ldots i_{k+1}} + \cdots \\ &\quad \cdots + \sum_{i_1 < i_2 < \cdots < i_{k+1}} \nu^{i_{k+1}} \widetilde{\omega}^{i_{k+1} i_1 \ldots i_k} \theta^{i_1 \ldots i_k} \\ &= \sum_{i_1 < i_2 < \cdots < i_{k+1}} \omega^{i_1 \cdots i_{k+1}} \left(\nu^{i_1} \theta^{i_2 \ldots i_{k+1}} + (-1) \nu^{i_2} \theta^{i_1 i_3 \ldots i_{k+1}} + \right. \\ &\quad \left. \cdots + (-1)^k \nu^{i_{k+1}} \theta^{i_1 \ldots i_k} \right) \\ &= \sum_{i_1 < i_2 < \cdots < i_{k+1}} \omega^{i_1 \ldots i_{k+1}} \zeta^{i_1 \ldots i_{k+1}} \end{aligned}$$

for

$$\zeta^{i_1 i_2 \dots i_{k+1}} = \sum_{s=1}^{k+1} \nu^{i_s} (-1)^{s-1} \theta^{i_1 \dots i_{s-1} i_{s+1} \dots i_{k+1}}$$

defined by (31). It then follows that (38) from (39)-(40) and the first equality of (32), that is,

$$(\nu \wedge \theta, \omega) = \sum_{i_1 < \dots < i_{k+1}} \zeta^{i_1 \dots i_{k+1}} \omega^{i_1 \dots i_{k+1}} \,. \qquad \square$$

REMARK 24. Equality (38) holds for any $\nu \in \Lambda_1$.

We conclude this subsection with the following proof.

PROOF OF LEMMA 3: The case $k = 1$ is obvious because

$$\nu \wedge *\varphi|_{\partial\Omega} = 0$$

holds for any $\varphi \in \Lambda_0$. Assume $2 \le k \le n$, and take $\alpha \in \Lambda_{k-2}$ arbitrary. Then Theorem 5 ensures

$$\begin{aligned} (-1)^k \int_{\partial\Omega} (\nu \wedge *\delta\theta, *\alpha)\, ds + \int_\Omega (\delta^2\theta, \alpha)\, dx &= \int_\Omega (\delta\theta, d\alpha)\, dx \\ = \int_\Omega (\theta, d^2\alpha)\, dx - (-1)^{k-1} \int_{\partial\Omega} (\nu \wedge *\theta, *d\alpha)\, ds \,, \end{aligned}$$

and hence

$$\int_{\partial\Omega} (\nu \wedge *\delta\theta, *\alpha)\, ds = 0$$

by $d^2 = 0$, $\delta^2 = 0$, and $\nu \wedge *\theta|_{\partial\Omega} = 0$. We thus obtain

$$\nu \wedge *\delta\theta|_{\partial\Omega} = 0$$

because $\alpha \in \Lambda_{k-2}$ is arbitrary. $\square$

2.2. H^1-estimate of $d - \delta$ systems

We continute to deal with smooth differential forms on $\overline{\Omega}$, using the following definition.

DEFINITION 4. We put

$$\breve{\Lambda}_k = \{u \in \Lambda_k \mid \nu \wedge *u|_{\partial\Omega} = 0\}, \quad 0 \le k \le n \,.$$

DEFINITION 5. We write

$$\theta_\ell = \frac{\partial\theta}{\partial x_\ell}, \quad 1 \le \ell \le n \,,$$

for $\theta \in \Lambda_0$.

DEFINITION 6. The set of smooth vector fields on $\overline{\Omega}$ is denoted by X.

DEFINITION 7. The set $L^2_k = L^2_k(\Omega)$ stands for the totality of elements in D'_k of which coefficients are in $L^2(\Omega)$

Given

$$u = \sum_{i_1<\cdots<i_k} u^{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda_k, \quad 0 \le k \le n\,, \tag{41}$$

we define the vector-valued k-form ∇u by

$$\nabla u = \sum_{i_1<\cdots<i_k} \nabla u^{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}\,,$$

and put

$$\|\nabla u\|^2_{L^2_k} = \sum_{i_1<\cdots<i_k} \|\nabla u^{i_1\cdots i_k}\|^2_2\,.$$

We use also

$$\|u\|^2_{L^2_k} = \sum_{i_1<\cdots<i_k} \|u^{i_1\cdots i_k}\|^2_2\,.$$

The main result in this subsection is the following lemma. If $n = 3$ it is Lemma 3.8 of [12] in the context of L^r-theory.

LEMMA 9 ($d - \delta$ estimate). *It holds that*

$$\|\nabla u\|^2_{L^2_k} \le C\left(\|du\|^2_{L^2_{k+1}} + \|\delta u\|^2_{L^2_{k-1}} + \|u\|^2_{L^2_k}\right), \quad u \in \breve{\Lambda}_k\,, \tag{42}$$

where $1 \le k \le n-1$.

REMARK 25. It is known that inequality (42) holds for $u \in \Lambda_k$ if Ω is a compact manifold and $\mathbf{t}u = 0$, called Gaffney's inequality ([21] p.66). Replacing u by $*u$, we obtain the same result for u with $\mathbf{n}u = 0$, and Lemma 9 is nothing but this inequality applied to the smooth bounded domain $\Omega \subset \mathbf{R}^n$ by Remark 8. Here we provide an analytic proof.

PROOF OF LEMMA 9: Given

$$\varphi = \sum_{i_1<\cdots<i_k} \varphi^{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda_k\,, \tag{43}$$

let

$$(\nabla u, \nabla\varphi) = \sum_{i_1<\cdots<i_k} (\nabla u^{i_1\cdots i_k}, \nabla\varphi^{i_1\cdots i_k})\,,$$

where (,) denotes the standard L^2-inner product in the right-hand side. We obtain

$$\int_\Omega (\nabla u, \nabla \varphi)\ dx = \sum_{i_1<\cdots<i_k} \int_\Omega (\nabla u^{i_1\cdots i_k}, \nabla \varphi^{i_1\cdots i_k})\ dx$$
$$= \sum_{i_1<\cdots<i_k} \left\{ \int_\Omega u^{i_1\cdots i_k}(-\Delta \varphi^{i_1\cdots i_k})\ dx + \int_{\partial\Omega} u^{i_1\cdots i_k} \frac{\partial \varphi^{i_1\cdots i_k}}{\partial \nu}\ ds \right\}$$

by the classical Gauss formula (26), which implies

$$\int_\Omega (\nabla u, \nabla \varphi)\ dx = \int_\Omega (u, d\delta\varphi + \delta d\varphi)\ dx + \int_{\partial\Omega} \left(u, \frac{\partial \varphi}{\partial \nu}\right)\ ds \tag{44}$$

by (19), where

$$\frac{\partial \varphi}{\partial \nu} = \sum_{i_1<\cdots<i_k} \frac{\partial \varphi^{i_1\cdots i_k}}{\partial \nu}\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}\,.$$

Then, Theorem 5 ensures

$$\int_\Omega (u, d\delta\varphi)\ dx = \int_\Omega (\delta u, \delta\varphi)\ dx + (-1)^{k-1} \int_{\partial\Omega} (\nu \wedge *u, *\delta\varphi)\ ds$$
$$\int_\Omega (u, \delta d\varphi)\ dx = \int_\Omega (du, d\varphi)\ dx - (-1)^k \int_{\partial\Omega} (*u, \nu \wedge *d\varphi)\ ds\,,$$

and therefore,

$$\int_\Omega (\nabla u, \nabla \varphi)\ dx = \int_\Omega (du, d\varphi) + (\delta u, \delta\varphi)\ dx$$
$$+ \int_{\partial\Omega} \left(u, \frac{\partial \varphi}{\partial \nu}\right) + (-1)^{k+1}(*u, \nu \wedge *d\varphi)\ ds$$

by $u \in \breve{\Lambda}_k$. Here we use Lemma 8 to obtain

$$\int_{\partial\Omega} \left(u, \frac{\partial \varphi}{\partial \nu}\right) + (-1)^{k+1}(*u, \nu \wedge *d\varphi)\ ds = \int_{\partial\Omega} \left(u, \frac{\partial \varphi}{\partial \nu}\right) - (\nu \wedge u, d\varphi)\ ds\,,$$

or

$$\int_\Omega (\nabla u, \nabla \varphi)\ dx = \int_\Omega (du, d\varphi) + (\delta u, \delta\varphi)\ dx + \int_{\partial\Omega} \left(u, \frac{\partial \varphi}{\partial \nu}\right) - (\nu \wedge u, d\varphi)\ ds\,. \tag{45}$$

Inequality (42) is thus reduced to the following lemma, because any $\varepsilon > 0$ admits $C_\varepsilon > 0$ such that

$$\|u\|^2_{L^2_k(\partial\Omega)} \le \varepsilon \|\nabla u\|^2_{L^2_k} + C_\varepsilon \|u\|^2_{L^2_k}$$

by the trace and interpolation theorems ([15] p.337, [1] p.135; see also Remark 30 in the next subsection). In fact, putting $\varphi = u$ in (45), we have

$$\|\nabla u\|^2_{L^2_k} = \|du\|^2_{L^2_{k+1}} + \|\delta u\|^2_{L^2_{k-1}} + \int_{\partial\Omega} \left(u, \frac{\partial u}{\partial \nu}\right) - (\nu \wedge u, du)\ ds$$

with

$$\left|\int_{\partial\Omega}\left(u,\frac{\partial u}{\partial\nu}\right)-(\nu\wedge u,du)\;ds\right|\le C\|u\|^2_{L^2_k(\partial\Omega)}\,. \qquad \square$$

REMARK 26. Equality (45) for $n=3$, $k=1$ means

$$(\nabla u,\nabla\varphi)=(\mathrm{rot}\;u,\mathrm{rot}\;\varphi)+(\mathrm{div}\;u,\mathrm{div}\;\varphi)+\int_{\partial\Omega}u\cdot(\vec\nu\cdot\nabla\varphi+\vec\nu\times\mathrm{rot}\;\varphi)$$

for smooth vector fields u and φ on $\overline{\Omega}$ satisfying $u\cdot\vec\nu\big|_{\partial\Omega}=0$. This equality is noticed in [12] as (3.65).

LEMMA 10. *It holds that*

$$\left|\int_{\partial\Omega}\left(u,\frac{\partial\varphi}{\partial\nu}\right)-(\nu\wedge u,d\varphi)\;ds\right|\le C\|u\|_{L^2_k(\partial\Omega)}\cdot\|\varphi\|_{L^2_k(\partial\Omega)},\quad u,\varphi\in\breve\Lambda_k\,. \tag{46}$$

where $0\le k\le n-1$.

PROOF. Let $u\in\breve\Lambda_k$, $0\le k\le n-1$, be in the form of (41). Then we obtain

$$(\hat u^{i_2\cdots i_k},\nu)=0,\quad i_2<\cdots<i_k\,, \tag{47}$$

by Lemma 7, recalling Definition 3.

Here we identify the 1-form $\hat u^{i_2\cdots i_k}\in\Lambda_1$ with a smooth vector field on $\overline{\Omega}$, that is, $\vec u^{\,i_2\cdots i_k}\in X$. Then equality (47) means

$$\vec u^{\,i_2\cdots i_k}\perp\vec\nu\quad\text{on }\partial\Omega\,. \tag{48}$$

It holds also that

$$\vec\varphi^{\,i_2\cdots i_k}\perp\vec\nu\quad\text{on }\partial\Omega \tag{49}$$

with $\vec\varphi^{\,i_2\cdots i_k}\in X$, defined similarly to $\vec u^{\,i_2\cdots i_k}$ from $\varphi\in\Lambda_k$ in (43). The property (49) means

$$\vec\varphi^{\,i_2\cdots i_k}\cdot\vec\nu\big|_{\partial\Omega}=0\,,$$

and hence

$$\nabla(\vec\varphi^{\,i_2\cdots i_k}\cdot\vec\nu)\;\|\;\vec\nu\quad\text{on }\partial\Omega\,,$$

or

$$\vec u^{\,i_2\cdots i_k}\cdot\nabla(\vec\varphi^{\,j_2\cdots j_k}\cdot\vec\nu)\big|_{\partial\Omega}=0,\quad i_2<\cdots<i_k,\;j_2<\cdots<j_k$$

by (48). We thus obtain

$$\sum_\ell\tilde u^{i_2\cdots i_k\ell}\left(\sum_m\tilde\varphi^{j_2\cdots j_km}\nu^m\right)_\ell=0\,,$$

recalling

$$\left(\sum_m\tilde\varphi^{j_2\cdots j_km}\nu^m\right)_\ell=\frac{\partial}{\partial x_\ell}\sum_m\tilde\varphi^{j_2\cdots j_km}\nu^m\,,$$

and hence

$$\sum_{\ell,m} \tilde{u}^{i_2\cdots i_k\ell}\nu^m\tilde{\varphi}_\ell^{j_2\cdots j_km} = -\sum_{\ell,m} \tilde{u}^{i_2\cdots i_k\ell}\nu_\ell^m\tilde{\varphi}^{j_2\cdots j_km} \quad \text{on } \partial\Omega\,. \tag{50}$$

It holds also that

$$\begin{aligned}\frac{\partial\varphi}{\partial\nu} &= \sum_\ell \sum_{i_1<\cdots<i_k} \nu^\ell\varphi_\ell^{i_1\cdots i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k} \left(\sum_\ell \nu^\ell\varphi_\ell^{i_1\cdots i_k}\right) dx_{i_1}\wedge\cdots\wedge dx_{i_k}\,,\end{aligned}$$

and therefore,

$$\begin{aligned}\left(u, \frac{\partial\varphi}{\partial\nu}\right) &= \sum_{i_1<\cdots<i_k} \left(\sum_\ell \nu^\ell\varphi_\ell^{i_1\cdots i_k}\right) u^{i_1\cdots i_k} \\ &= \sum_\ell \nu^\ell \sum_{i_1<\cdots<i_k} \varphi_\ell^{i_1\cdots i_k} u^{i_1\cdots i_k} \\ &= \frac{1}{k!} \sum_{i_1,\cdots,i_{k+1}} \nu^{i_1}\tilde{\varphi}_{i_1}^{i_2\cdots i_{k+1}} \tilde{u}^{i_2\cdots i_{k+1}}\,. \end{aligned} \tag{51}$$

We have

$$\begin{aligned}\nu\wedge u &= \sum_\ell \sum_{i_1<\cdots<i_k} \nu^\ell u^{i_1\cdots i_k} dx_\ell\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \left(\sum_{\ell<i_1<i_2<\cdots<i_k} + \sum_{i_1<\ell<i_2<\cdots<i_k} + \right. \\ &\qquad \left. \cdots + \sum_{i_1<i_2<\cdots<i_k<\ell}\right) \nu^\ell u^{i_1\cdots i_k} dx_\ell\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \sum_{i_1<i_2<\cdots<i_{k+1}} \big(\nu^{i_1} u^{i_2 i_3\cdots i_{k+1}} dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_{k+1}} \\ &\qquad + \nu^{i_2} u^{i_1 i_3\cdots i_{k+1}} dx_{i_2}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i+1} + \\ &\qquad \cdots + \nu^{i_{k+1}} u^{i_1 i_2\cdots i_k} dx_{i_{k+1}}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\big) \\ &= \sum_{i_1<\cdots<i_{k+1}} \big(\nu^{i_1}\tilde{u}^{i_2 i_3\cdots i_k i_{k+1}} + \nu^{i_2}\tilde{u}^{i_3 i_1\cdots i_{k+1}} + \\ &\qquad \cdots + \nu^{i_{k+1}}\tilde{u}^{i_k i_1\cdots i_{k-1}}\big)\, dx_{i_1}\wedge\cdots\wedge dx_{i_{k+1}}\end{aligned}$$

and

$$
\begin{aligned}
d\varphi &= \sum_{\ell} \sum_{i_1<\cdots<i_k} \varphi_\ell^{i_1\cdots i_k} dx_\ell \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \left(\sum_{\ell<i_1<i_2<\cdots<i_k} + \sum_{i_1<\ell<i_2<\cdots i_k} + \right. \\
&\quad \left. \cdots + \sum_{i_1<i_2<\cdots<i_k<\ell} \right) \varphi_\ell^{i_1\cdots i_k} dx_\ell \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \\
&= \sum_{i_1<i_2<\cdots<i_{k+1}} \big(\varphi_{i_1}^{i_2 i_3 \cdots i_{k+1}} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{k+1}} \\
&\quad + \varphi_{i_2}^{i_3 i_1 \cdots i_{k+1}} dx_{i_2} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} + \\
&\quad \cdots + \varphi_{i_{k+1}}^{i_1 i_2 \cdots i_k} dx_{i_{k+1}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \big) \\
&= \sum_{i_1<i_2<\cdots<i_{k+1}} \big(\tilde{\varphi}_{i_1}^{i_2 \cdots i_k i_{k+1}} + \tilde{\varphi}_{i_2}^{i_3 i_1 \cdots i_{k+1}} + \\
&\quad \cdots + \tilde{\varphi}_{i_{k+1}}^{i_k i_1 \cdots i_{k-1}} \big) dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} , \qquad (52)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(\nu \wedge u, d\varphi) \\
&= \sum_{i_1<\cdots<i_{k+1}} \big(\nu^{i_1} \tilde{u}^{i_2 i_3 \cdots i_k i_{k+1}} + \nu^{i_2} \tilde{u}^{i_3 i_1 \cdots i_{k+1}} + \cdots + \nu^{i_{k+1}} \tilde{u}^{i_k i_1 \cdots i_{k-1}} \big) \\
&\quad \cdot \big(\tilde{\varphi}_{i_1}^{i_2 i_3 \cdots i_{k+1}} + \tilde{\varphi}_{i_2}^{i_3 i_1 \cdots i_{k+1}} + \cdots + \tilde{\varphi}_{i_{k+1}}^{i_k i_1 \cdots i_{k-1}} \big) .
\end{aligned}
$$

Here, each term of the right-hand side,

$$
\begin{aligned}
&\big(\nu^{i_1} \tilde{u}^{i_2 i_3 \cdots i_k i_{k+1}} + \nu^{i_2} \tilde{u}^{i_3 i_1 \cdots i_{k+1} i_1} + \cdots + \nu^{i_{k+1}} \tilde{u}^{i_k i_1 \cdots i_{k-1}} \big) \\
&\quad \cdot \big(\tilde{\varphi}_{i_1}^{i_2 i_3 \cdots i_{k+1}} + \tilde{\varphi}_{i_2}^{i_3 i_1 \cdots i_{k+1}} + \cdots + \tilde{\varphi}_{i_{k+1}}^{i_k i_1 \cdots i_{k-1}} \big) ,
\end{aligned}
$$

is invariant under the permutation of $(i_1, \cdots, i_{k+1})$, and hence it holds that

$$
\begin{aligned}
(\nu \wedge u, \varphi) = \frac{1}{(k+1)!} \sum_{i_1, \cdots, i_{k+1}} &\big(\nu^{i_1} \tilde{u}^{i_2 i_3 \cdots i_k i_{k+1}} + \nu^{i_2} \tilde{u}^{i_3 i_1 \cdots i_{k+1}} + \\
&\cdots + \nu^{i_{k+1}} \tilde{u}^{i_k i_1 \cdots i_{k-1}} \big) \\
&\cdot \big(\tilde{\varphi}_{i_1}^{i_2 i_3 \cdots i_{k+1}} + \tilde{\varphi}_{i_2}^{i_3 i_1 \cdots i_{k+1}} + \cdots + \tilde{\varphi}_{i_{k+1}}^{i_k i_1 \cdots i_{k-1}} \big) . \qquad (53)
\end{aligned}
$$

Expanding the right-hand side on (53), we have

$$
\begin{aligned}
\sum_{i_1, \cdots, i_{k+1}} \nu^{i_1} \tilde{u}^{i_2 i_3 \cdots i_{k+1}} \tilde{\varphi}_{i_1}^{i_2 i_3 \cdots i_{k+1}} &= \sum_{i_1, \cdots, i_{k+1}} \nu^{i_2} \tilde{u}^{i_3 i_1 \cdots i_{k+1}} \tilde{\varphi}^{i_3 i_1 \cdots i_{k+1}} \\
= \cdots\cdots &= \sum_{i_1, \cdots, i_{k+1}} \nu^{i_{k+1}} \tilde{u}^{i_k i_1 \cdots i_{k-1}} \tilde{\varphi}_{i_{k+1}}^{i_k i_1 \cdots i_{k-1}} ,
\end{aligned}
$$

while the other terms

$$\nu^m \tilde{u}^{i_1 \cdots i_k} \tilde{\varphi}_\ell^{j_1 \cdots j_k}, \quad m \in \{j_1, j_2, \cdots, j_k\}, \quad \ell \in \{i_1, i_2, \cdots, i_k\}$$

are treated by (50), that is,

$$\begin{aligned} &\frac{1}{(k+1)!} \sum_{i_1, \cdots, i_k} \sum_{j_1, \cdots j_k} \sum_{\ell \in \{i_1, \cdots, i_k\}} \sum_{m \in \{j_1, \cdots, j_k\}} \nu^m \tilde{u}^{i_1 \cdots i_k} \tilde{\varphi}_\ell^{j_1 \cdots j_k} \\ &\quad = -\frac{1}{(k+1)!} \sum_{i_1, \cdots, i_k} \sum_{j_1, \cdots, j_k} \sum_{\ell \in \{i_1, \cdots, i_k\}} \sum_{m \in \{j_1, \cdots, j_k\}} \tilde{u}^{i_1 \cdots i_k} \nu_\ell^m \tilde{\varphi}^{j_1 \cdots j_k} \equiv R \,. \end{aligned}$$

We thus reach

$$\begin{aligned} (\nu \wedge u, d\varphi) &= \frac{k+1}{(k+1)!} \sum_{i_1, \cdots, i_{k+1}} \nu^{i_1} \tilde{u}^{i_2 \cdots i_{k+1}} \tilde{\varphi}_{i_1}^{i_2 \cdots i_{k+1}} + R \\ &= \left(u, \frac{\partial \varphi}{\partial \nu} \right) + R \end{aligned}$$

by (51) with

$$|R| \le C |u|_2 \cdot |\varphi|_2 \,,$$

where

$$|u|_2^2 = \sum_{i_1 < \cdots < i_k} (u^{i_1 \cdots i_k})^2, \quad |\varphi|_2^2 = \sum_{i_1 < \cdots < i_k} (\varphi^{i_1 \cdots i_k})^2 \,.$$

Then inequality (46) follows. □

REMARK 27. By the proof of Lemma 9 and Lemma 10 it holds that

$$\begin{aligned} &\left| \int_\Omega (\nabla u, \nabla \varphi) \, dx \right| \le C \left(\|du\|_{L_{k+1}^r} + \|\delta u\|_{L_{k-1}^r} + \varepsilon \|\nabla u\|_{L_k^r} + C_\varepsilon \|u\|_{L_k^r} \right) \\ &\qquad \cdot \left(\|\nabla \varphi\|_{L_k^{r'}} + \|\varphi\|_{L_k^{r'}} \right), \quad u, \varphi \in \breve{\Lambda}_k \,, \end{aligned} \tag{54}$$

for each $1 < r < \infty$, where L_k^r denotes the set of elements in D_k' of which coefficients are in $L^r = L^r(\Omega)$ and

$$\frac{1}{r} + \frac{1}{r'} = 1 \,. \tag{55}$$

Inequality (3.66) of [12] is (54) for $n = 3$, $k = 1$.

2.3. H^1-regularity of $d - \delta$ systems

Here we extend the result in the previous subsections to differential forms with their coefficients belonging to Sobolev spaces, where Duvaut-Lions density [7] takes a role (Lemma 15). We use the following function spaces in the framework of L^2-theory.

DEFINITION 8. Let $s \ge 0$.

1. H^s_k denotes the set of elements in D'_k of which coefficients are in H^s.
2. $H^s_k(\partial\Omega)$ denotes the set of k-th differential forms on $\partial\Omega$ of which coefficients are in $H^s(\partial\Omega)$.

REMARK 28. The norms of H^s_k and $H^s_k(\partial\Omega)$ are induced from those of $H^s = W^{s,2}(\Omega)$ and $H^s(\partial\Omega) = W^{s,2}(\partial\Omega)$, respectively.

REMARK 29. It holds that $H^0_k = L^2_k$.

REMARK 30. From the result on Sobolev spaces, the trace operator

$$\omega \in H^s_k \mapsto \omega|_{\partial\Omega} \in H^{s-1/2}_k(\partial\Omega) \tag{56}$$

is well-defined for $s > \frac{1}{2}$ ([1] p.164, [15] p.316).

We use also the following function spaces.

DEFINITION 9. We put

$$\begin{aligned} H^1_{k,d} &= \{\theta \in L^2_k \mid d\theta \in L^2_{k+1}\}, \quad 0 \le k \le n-1 \\ H^1_{k,\delta} &= \{\theta \in L^2_k \mid \delta\theta \in L^2_{k-1}\}, \quad 1 \le k \le n \\ H^1_{k,d\delta} &= H^1_{k,d} \cap H^1_{k,\delta}, \qquad\qquad 1 \le k \le n-1\,. \end{aligned}$$

where the differentiations d and δ are taken in the sense of distributions in Ω.

REMARK 31. The spaces $H^1_{k,d}$, $H^1_{k,\delta}$, and $H^1_{k,d\delta}$ form Hilbert spaces under the norms

$$\left(\|\omega\|^2_{L^2_k} + \|d\omega\|^2_{L^2_{k+1}}\right)^{1/2}, \quad \left(\|\omega\|^2_{L^2_k} + \|\delta\omega\|^2_{L^2_{k-1}}\right)^{1/2},$$

and

$$\left(\|\omega\|^2_{L^2_k} + \|d\omega\|^2_{L^2_{k+1}} + \|\delta\omega\|^2_{L^2_{k-1}}\right)^{1/2},$$

respectively.

First, we confirm the following lemma.

LEMMA 11. *The set Λ_k, $0 \le k \le n-1$, is dense in $H^1_{k,d}$. Similarly, the set Λ_k, $1 \le k \le n$, is dense in $H^1_{k,\delta}$.*

PROOF. We take smooth bounded domains $\Omega_\ell \subset \mathbf{R}^n$, $1 \le \ell \le L$, such that

$$\partial\Omega \subset \bigcup_{\ell=1}^{\ell} \Omega_\ell$$

and the diffeomorphisms $X_\ell : \Omega_\ell \to U_\ell \subset \mathbf{R}^n$ such that

$$X_\ell(\Omega \cap \Omega_\ell) \subset \mathbf{R}^n_+ = \{(x', x_n) \mid x_n > 0\},\ x' = (x_1, \dots, x_{n-1})$$
$$X_\ell(\partial\Omega \cap \Omega_\ell) \subset \partial\mathbf{R}^n_+ . \tag{57}$$

We also take a domain Ω_0 such that $\overline{\Omega}_0 \subset \Omega$ and

$$\Omega \subset \bigcup_{\ell=0}^{L} \Omega_\ell . \tag{58}$$

Then we use the partition of unity, $0 \le \varphi_\ell \in C^\infty(\mathbf{R}^n)$, $0 \le \ell \le L$, subject to the covering (58), that is,

$$\sum_{\ell=0}^{L} \varphi_\ell = 1 \quad \text{on } \overline{\Omega}, \quad \text{supp } \varphi_\ell \subset \Omega_\ell,\ 0 \le \ell \le L . \tag{59}$$

Let $0 \le k \le n-1$. Given $f \in H^1_{k,d} = H^1_{k,d}(\Omega)$, we put $f_\ell = f \cdot \varphi_\ell$. Confirm that f_ℓ here does not mean $\frac{\partial f}{\partial x_\ell}$. Using

$$f_\ell \in H^1_{k,d}, \quad \text{supp } f_\ell \subset \overline{\Omega} \cap \Omega_\ell , \tag{60}$$

we will find

$$f^j_\ell \in \Lambda_k, \quad j = 1, 2, \dots ,$$

such that

$$\lim_{j\to\infty} (\|f^j_\ell - f_\ell\|_{L^2_k} + \|df^j_\ell - df_\ell\|_{L^2_{k+1}}) = 0 \tag{61}$$

for each $0 \le \ell \le L$. If such f^j_ℓ, $j = 1, 2, \dots$, are obtained, we get

$$\lim_{j\to\infty} (\|f^j - f\|_{L^2_k} + \|df^j - df\|_{L^2_{k+1}}) = 0$$

for

$$f^j = \sum_{\ell=0}^{L} f^j_\ell \in \Lambda_k ,$$

and therefore, Λ_k is dense in $H^1_{k,d}$.

For $\ell = 0$ we take

$$0 \le \rho \in C^\infty_0(\mathbf{R}^n), \quad \int_{\mathbf{R}^n} \rho(x)\ dx = 1$$

and put $f^j_0 = f_0 * \varphi_j$, where $\varphi_j(x) = j^n \varphi(jx)$ and

$$(f * g)(x) = \int_{\mathbf{R}^n} f(x-y) g(y) dy .$$

It then follows that

$$f^j_0 \in \Lambda_k, \quad \text{supp } f^j_0 \subset \Omega ,$$

for $j \gg 1$, and we obtain (61) for $\ell = 0$ by $f_0 \in L^2_k$ and $df_0 \in L^2_{k+1}$.

For $1 \leq \ell \leq L$ we put

$$g_\ell = Y_\ell^* f_\ell, \quad Y_\ell = X_\ell^{-1}\big|_{U_\ell^+}, \quad U_\ell^+ = U_\ell \cap \mathbf{R}^n_+ .$$

Here and henceforth, Y_ℓ^* denotes the pull-back of Y_ℓ; $g_\ell = f_\ell \circ Y_\ell$. Then we obtain

$$g_\ell \in L^2_k(U_\ell^+), \quad dg_\ell \in L^2_{k+1}(U_\ell^+), \quad \text{supp } g_\ell \subset U_\ell^+$$

by (60) and

$$d \circ Y_\ell^* = Y_\ell^* \circ d \tag{62}$$

([17] p.61).

We may suppose that the reflection of supp g_ℓ in $\mathbf{R}^n$ with respect to $x_n = 0$ is in U_ℓ. Let $\tilde{g}_\ell$ be the extension of g_ℓ on U_ℓ^+ to U_ℓ, obtained by the even reflection with respect to $x_n = 0$. It then holds that

$$\tilde{g}_\ell \in L^2_k(U_\ell), \quad d\tilde{g}_\ell \in L^2_{k+1}(U_\ell), \quad \text{supp } \tilde{g}_\ell \subset U_\ell .$$

Similarly to the case of $\ell = 0$, we obtain

$$\tilde{g}_\ell^j \in \Lambda_k(U_\ell), \quad \text{supp } \tilde{g}_\ell^j \subset U_\ell, \quad j = 1, 2, \dots ,$$

such that

$$\lim_{j\to\infty} \left(\|\tilde{g}_\ell^j - \tilde{g}_\ell\|_{L^2_k(U_\ell)} + \|d\tilde{g}_\ell^j - d\tilde{g}_\ell\|_{L^2_{k+1}(U_\ell)} \right) = 0 ,$$

Then we put

$$\tilde{f}_\ell^j = X_\ell^* \tilde{g}_\ell^j \in \Lambda_k(\Omega_\ell), \quad \tilde{f}_\ell = X_\ell^* \tilde{g}_\ell \in H^1_{k,d}(\Omega_\ell) ,$$

to obtain

$$\text{supp } \tilde{f}_\ell^j \subset \Omega_\ell, \quad \text{supp } \tilde{f}_\ell \subset \Omega_\ell ,$$

and

$$\lim_{j\to\infty} \left(\|\tilde{f}_\ell^j - \tilde{f}_\ell\|_{L^2_k(\Omega_\ell)} + \|d\tilde{f}_\ell^j - d\tilde{f}_\ell\|_{L^2_{k+1}(\Omega_\ell)} \right) = 0$$

by $X_\ell^* \circ d = d \circ X_\ell^*$.

Letting $f_\ell^j \in \Lambda_k(\Omega)$ be the zero extension of

$$\tilde{f}_\ell^j\big|_{\overline{\Omega}\cap\Omega_\ell}$$

to $\overline{\Omega}$, we obtain (61). Therefore, $\Lambda_k(\Omega)$, $0 \leq k \leq n-1$, is dense in $H^1_{k,d}(\Omega)$. Through the Hodge operator $*$, the space $\Lambda_k = \Lambda_k(\Omega)$, $1 \leq k \leq n$, is also dense in $H^1_{k,\delta}(\Omega)$. □

Remark 32. Similar results to Lemma 11 hold in the context of L^r-theory by the above argument (Remark 54 in §4.2).

We extend the trace operator (56) to a weak form, because this operator does not work on $H^1_{k,d}$ or $H^1_{k,\delta}$ directly.

PROPOSITION 12 (Trace form). Each $0 \le k' \le n-1$ admits the bounded bilinear form

$$J_{k'} = J_{k'}(\theta,\omega) : H^1_{k',d} \times H^1_{k'+1,\delta} \to \mathbf{R} \tag{63}$$

such that

$$\int_\Omega (d\theta,\omega)\ dx = \int_\Omega (\theta,\delta\omega)\ dx + (-1)^{k'} J_{k'}(\theta,\omega), \quad (\theta,\omega) \in H^1_{k',d} \times H^1_{k'+1,\delta}\,. \tag{64}$$

It holds also that

$$J_{k'}(\theta,\omega) = \int_{\partial\Omega} (*\theta, \nu\wedge *\omega)\ ds, \quad (\theta,\omega) \in H^1_{k'} \times H^1_{k'+1}\,, \tag{65}$$

where the trace operator (56) is taken into account.

PROOF. Theorem 5 implies

$$\begin{aligned}\left|\int_{\partial\Omega} (*\theta, \nu\wedge *\omega)\ ds\right| &= \left|\int_\Omega (d\theta,\omega) - (\theta,\delta\omega)\ dx\right| \\ &\le \left(\|\theta\|_{L^2_{k'}} + \|d\theta\|_{L^2_{k'+1}}\right)\left(\|\omega\|_{L^2_{k'+1}} + \|\delta\omega\|_{L^2_{k'}}\right)\end{aligned}$$

for $(\theta,\omega) \in \Lambda_{k'} \times \Lambda_{k'+1}$, and therefore, the result follows from Lemma 11. □

Henceforth, we use the following definition.

DEFINITION 10 (Weak trace). We say that $\omega \in H^1_{k,\delta}$, $1 \le k \le n$, satisfies

$$\nu \wedge *\omega|_{\partial\Omega} = 0 \tag{66}$$

if and only if $J_k(\theta,\omega) = 0$ for any $\theta \in H^1_{k-1,d}$.

DEFINITION 11. We put

$$\begin{aligned}&\breve{H}^1_{k,\delta} = \{\omega \in H^1_{k,\delta} \mid \nu\wedge *\omega|_{\partial\Omega} = 0\},\ 1 \le k \le n \\ &\breve{H}^1_{k,d\delta} = \breve{H}^1_{k,\delta} \cap H^1_{k,d}, \qquad\qquad 1 \le k \le n-1\,,\end{aligned}$$

recalling Definition 9.

REMARK 33. By Lemma 11, the property (66) to $\omega \in H^1_{k,\delta}$ is reduced to

$$\int_\Omega (d\theta,\omega)\ dx = \int_\Omega (\theta,\delta\omega)\ dx, \quad \forall\theta \in \Lambda_{k-1}\,.$$

REMARK 34. If $\omega \in \Lambda_k$, the condition (66) in Definition 10 implies its pointwise equality. If $\omega \in H^1_k$, furthermore, the condition (66) in Definition 10 implies that in the sense of H^1-trace of ω, (56).

The main result in this subsection is the following theorem.

THEOREM 13 (H^1-regularity). *Any $u \in \breve{H}^1_{k,d\delta}$, $1 \le k \le n-1$, belongs to H^1_k and satisfies inequality (42).*

REMARK 35. By this theorem, the condition $\nu \wedge *\omega|_{\partial\Omega} = 0$ in Definition 11, used to define $\omega \in \breve{H}^1_{k,d\delta}$, is replaced in the sense of usual trace, (56) on H^1_k.

REMARK 36. This theorem is used to derive H^2-regularity of the solution obtained by the variational method. See the proof of Proposition 20 and Lemma 22.

For the proof, we show that the propery (66) of $\omega \in H^1_{k,\delta}$ is localized.

LEMMA 14 (Localization). *Any $(p,\omega) \in \Lambda_0 \times \breve{H}^1_{k,\delta}$, $1 \le k \le n-1$, admits*

$$p\omega \in \breve{H}^1_{k,\delta}\,.$$

REMARK 37. If $p \in \Lambda_0$ satisfies supp $p \subset \Omega$, any $\omega \in \Lambda_k$ admits

$$\nu \wedge *(p\omega)|_{\partial\Omega} = 0\,.$$

Hence we obtain

$$p\omega \in \breve{H}^1_{k,\delta}, \quad \omega \in H^1_{k,\delta}$$

by Lemma 11.

REMARK 38. Let $G \subset \mathbf{R}^n$ be an open set satisfying $\partial\Omega \cap G \neq \emptyset$ and $\omega \in H^k_{k,\delta}$ be given. Then we say

$$\nu \wedge *\omega|_{\partial\Omega\cap G} = 0$$

if $p\omega \in \breve{H}^1_{k,\delta}$ holds to any $p \in \Lambda_0$ with supp $p \subset \overline{\Omega} \cap G$. This property is invariant under the transformation of variables.

PROOF OF LEMMA 14: Given $(p,\omega) \in \Lambda_0 \times \breve{H}^1_{k,\delta}$, $1 \le k \le n-1$, we show

$$\int_\Omega (d\theta, p\omega)\, dx = \int_\Omega (\theta, \delta(p\omega))\, dx, \quad \forall\theta \in \Lambda_{k-1}\,. \tag{67}$$

Since

$$p\omega = p \wedge \omega$$

it holds that

$$d(p\omega) = d(p\wedge\omega) = dp\wedge\omega + p\wedge d\omega = dp\wedge\omega + pd\omega$$

([17] p.66). Having

$$*(p\omega) = p * \omega = p \wedge *\omega\,,$$

similarly, we obtain

$$d * (p\omega) = d(p \wedge *\omega) = dp \wedge *\omega + pd * \omega\,.$$

and hence

$$\delta(p\omega) = (-1)^{n(k+1)+1} * (dp \wedge *\omega) + p\delta\omega \in L^2_{k-1} \tag{68}$$

by (1).

First, equality (68) implies

$$\int_\Omega (\theta, \delta(p\omega))\ dx = \int_\Omega (-1)^{n(k+1)+1}(\theta, *(dp \wedge *\omega)) + (\theta, p\delta\omega)\ dx\,,$$

where it holds that

$$\int_\Omega (\theta, p\delta\omega)\ dx = \int_\Omega (p\theta, \delta\omega)\ dx = \int_\Omega (d(p\theta), \omega)\ dx$$

by $\omega \in \breve{H}^1_{k,d\delta}$. We have, second,

$$d(p\theta) = d(p \wedge \theta) = dp \wedge \theta + pd\theta\,,$$

and hence equality (67) follows from

$$\int_\Omega (-1)^{n(k+1)+1}(\theta, *(dp \wedge *\omega)) + (dp \wedge \theta, \omega)\ dx = 0\,. \tag{69}$$

Noting $\theta \in \Lambda_{k-1}$, here, we see

$$\theta = (-1)^{(k-1)(n-k+1)} * *\theta\,,$$

which results in

$$(\theta, *(dp \wedge *\omega)) = (-1)^{(k-1)(n-k+1)}(*\theta, dp \wedge *\omega)$$

([17] p.151). Since

$$\begin{aligned} n(k+1) + 1 + (k-1)(n-k+1) &= 2kn + 1 - (k-1)^2 \\ &\equiv -k(k-1) + k \equiv k \pmod 2\,, \end{aligned}$$

equality (69) is reduced to

$$(dp \wedge \theta, \omega) = (-1)^{k-1}(*\theta, dp \wedge *\omega)\,,$$

which follows from $\nu = dp$ in (38). In fact we have Remark 24 in §2.1 and $(\theta, \omega) \in \Lambda_{k-1} \times \Lambda_k$. □

By Lemma 14, the property (66) for $\omega \in H^1_{k,\delta}$ is localized through the partition of unity, φ_ℓ, $0 \le \ell \le L$, in (59), and then the method of tangential regularization used for three dimensional vector fields ([7] p.363) is applicable. Thus we obtain the following lemma.

LEMMA 15 (Duvaut-Lions density). *The space* $\breve{\Lambda}_k$, $1 \le k \le n-1$, *is dense in* $\breve{H}^1_{k,d\delta}$.

PROOF. Given $f \in \breve{H}^1_{k,d\delta}$, $1 \le k \le n-1$, we put

$$f_\ell = f \cdot \varphi_\ell \in \breve{H}^1_{k,d\delta}\,.$$

We now approximate f_ℓ, $0 \le \ell \le L$, by $f^j_\ell \in \breve{\Lambda}_k$, $j = 1, 2, ...$, satisfying

$$\lim_{j\to\infty} \left(\|f^j_\ell - f_\ell\|_{L^2_k} + \|\delta f^j_\ell - \delta f_\ell\|_{L^2_{k-1}} + \|df^j_\ell - df_\ell\|_{L^2_{k+1}} \right) = 0\,. \tag{70}$$

Then we obtain the result as in the proof of Lemma 11.

The proof for $\ell = 0$ is the same as that in Lemma 11. Let $1 \le \ell \le L$. We can assume

$$X_\ell(\text{supp } \varphi_\ell) \subset Q_\ell \cap \overline{\mathbf{R}^n_+} \subset U_\ell\,, \tag{71}$$

where $X_\ell : \Omega_\ell \to U_\ell \subset \mathbf{R}^n$, $1 \le \ell \le L$, are the diffeomorphisms satisfying (57), and

$$Q_\ell = \Sigma \times (-\varepsilon_0, \varepsilon_0), \quad \Sigma = (0, \varepsilon_0)^{n-1},\ \varepsilon_0 > 0\,, \tag{72}$$

is a hemi-cube. We may assume also that

$$\vec{\tilde{\nu}} \equiv X_\ell(\vec{\nu}) = (0, \dots, 0, -1)^T \quad \text{on } \Sigma \times \{0\} \tag{73}$$

to obtain

$$\tilde{\nu} \wedge *g_\ell\big|_{\Sigma\times\{0\}} = 0 \tag{74}$$

by (62), where $g_\ell = Y^*_\ell f_\ell$ for

$$Y_\ell = X^{-1}_\ell\big|_{U^+_\ell}\,, \quad U^+_\ell = U_\ell \cap \mathbf{R}^n_+\,.$$

Henceforth, we write $y = (y', y_n) \in Q$, $y' = (y_1, \dots, y_{n-1})$, to take the tangential regularization of g_ℓ denoted by

$$g^j_\ell,\ j = 1, 2, \cdots,$$

that is,

$$g^j_\ell(y', y_n) = \int_{\mathbf{R}^{n-1}} \tilde{\rho}_j(y' - z') g_\ell(z', y_n) dz'$$

for $\tilde{\rho}_j(y') = j^{n-1}\tilde{\rho}(jy')$ with

$$0 \le \tilde{\rho} \in C^\infty_0(\mathbf{R}^{n-1}), \quad \int_{\mathbf{R}^{n-1}} \tilde{\rho}(y')\ dy' = 1\,.$$

Under this operation, the sequence $\hat{f}^j_\ell$, $j = 1, 2, ...$, defined by zero extensions of $X^*_\ell g^j_\ell$, $j = 1, 2, \cdots$, on $\overline{\Omega} \cap \Omega_\ell$ to $\overline{\Omega}$, satisfies (70) for $f^j_\ell = \hat{f}^j_\ell$. There arises also that

$$\partial^\alpha_{y'} g^j_\ell \in L^2_k(U^+_\ell)$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, as well as

$$\tilde{\nu} \wedge *g^j_\ell\big|_{\Sigma\times\{0\}} = 0\,. \tag{75}$$

For the moment, let

$$u = \sum_{i_1<\cdots<i_k} u^{i_1\cdots i_k} dy_{i_1} \wedge \cdots \wedge dy_{i_k} \in H^1_{k,d}(U^+_\ell)$$
$$\text{supp } u \subset Q^+_\ell \equiv \Sigma \times [0, \varepsilon_0)$$

be given. Then it holds that

$$du = \sum_{i_1<\cdots<i_{k+1}} \left(\tilde{u}^{i_2\cdots i_k i_{k+1}}_{i_1} + \tilde{u}^{i_1 i_3\cdots i_{k+1}}_{i_2} + \cdots + \tilde{u}^{i_k i_1\cdots i_{k-1}}_{i_{k+1}} \right) dy_{i_1} \wedge \cdots \wedge dy_{i_{k+1}} \tag{76}$$

as in (52), and therefore, if

$$\frac{\partial}{\partial y_j} \tilde{u}^{i_1 i_2\cdots i_k}, \quad 1 \le \forall j \le n-1, \ \forall i_1, i_2, \cdots, i_k$$

are smooth on Q^+_ℓ, we obtain

$$\frac{\partial}{\partial y_n} \tilde{u}^{i'_1 i'_2\cdots i'_k} \in L^2(U^+_\ell), \quad \forall i'_1, \cdots, i'_k$$

by $du \in L^2_{k+1}(U^+_\ell)$ in (76).

From this fact, the above g^j_ℓ, satisfying

$$\text{supp } g^j_\ell \subset Q^+_\ell \,,$$

belongs to $H^1_k(U^+_\ell)$, and therefore, this lemma is reduced to approximating each coefficient of the k-form g^j_ℓ, furthermore, by smooth functions with their supports in Q^+_ℓ.

Henceforth we fix j and write

$$g^j_\ell = \sum_{i_1<\cdots<i_k} g^{i_1\cdots i_k}_\ell dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad g^{i_1\cdots i_k}_\ell \in H^1(U^+_\ell)$$

for simplicity. Then, the condition (75) means, in the sense of trace,

$$\tilde{g}^{i_2\cdots i_k n}_\ell \big|_{\Sigma\times\{0\}} = 0, \quad \forall i_2, \cdots, i_k \tag{77}$$

by (73). We can thus approximate such $g^{i_2\cdots i_k n}_\ell$ by smooth functions with compact supports in $\Sigma \times (0, \varepsilon_0)$ ([15] p.325).

The other components, $g^{i_1\cdots i_k}_\ell$ with $i_k < n$, are free from boundary conditions, and therefore, approximating them by smooth functions on Q^+_ℓ, we obtain $f^{(m)}_\ell \in \breve{\Lambda}_k$, $m = 1, 2, \cdots$, with supports on $\overline{\Omega} \cap \Omega_\ell$, such that

$$\lim_{m\to\infty} \|f^{(m)}_\ell - f^j_\ell\|_{H^1_k} = 0 \,.$$

Then the desired property follows that $\breve{\Lambda}_k$ is dense in $\breve{H}^1_{k,d\delta}$. □

REMARK 39. We take odd and even reflections with respect to $y_n = 0$, for $i_k = n$ and $i_k \ne n$, respectively, of

$$\tilde{g}^{i_1\cdots i_{k-1} i_k}_\ell \in H^1(U^+_\ell) \,,$$

denoted by $\hat{g}_\ell^{i_1\cdots i_{k-1}i_k} \in H^1(U_\ell)$. Then, since (74) is equivalent to (77) under (73), the associated k-form

$$\hat{g}_\ell = \sum_{i_1<\cdots<i_k} \hat{g}_\ell^{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in H^1_k(Q_\ell)$$

satisfies

$$\nu \wedge *\hat{g}_\ell|_{\Sigma\times\{0\}} = 0\,.$$

We now reach the following proof.

Proof of Theorem 13: By Lemma 15, any $u \in \breve{H}^1_{k,d\delta}$ admits $u^j \in \breve{\Lambda}_k$, $j = 1, 2, \cdots$, such that

$$u^j \to u \quad \text{in } H^1_{k,d\delta}\,,$$

Lemma 9 then ensures

$$\|\nabla u^j\|_{L^2_k} \le C\left(\|du^j\|^2_{L^2_{k+1}} + \|\delta u^j\|_{L^2_{k-1}} + \|u^j\|_{L^2_k}\right)\,,$$

and therefore, the weak convergence

$$u^j \rightharpoonup u \quad \text{in } H^1_k$$

arises up to a subsequence. Then we obtain $u \in H^1_k$ with inequality (42). □

2.4. Elliptic estimate

Here we show an elliptic estimate to

$$-\Delta\theta = \alpha \text{ in } \Omega, \quad \nu \wedge *\theta = *\beta \text{ on } \partial\Omega, \quad \nu \wedge *d\theta = *\gamma \text{ on } \partial\Omega \tag{78}$$

valid to $\theta \in H^2_k$, by confirming the complementing boundary condition of [3] p.42.

Theorem 16 (Elliptic estimate). *Each $s \ge 0$ admits $C >$ such that if $\theta \in H^{2+s}_k$ satisfies (78), then it follows that*

$$\|\theta\|_{H^{s+2}_k} \le C\left(\|\alpha\|_{H^s_k} + \|\beta\|_{H^{s+3/2}_{k-1}(\partial\Omega)} + \|\gamma\|_{H^{s+1/2}_k(\partial\Omega)} + \|\theta\|_{H^s_k}\right)\,, \tag{79}$$

where $1 \le k \le n-1$,

We note several facts before proceeding to the proof.

Remark 40. If $n = 3$, $k = 1$, and $\theta \in H^2_1$ is identified with the vector field u as in Remark 12, equation (78) means

$$-\Delta u = \alpha \text{ in } \Omega, \quad \nu \cdot u = \beta \text{ on } \partial\Omega, \quad \nu \times \mathrm{rot}\, u = \gamma \text{ on } \partial\Omega\,,$$

where α, β, and γ are regarded as a vector field on Ω, a scalar filed on $\partial\Omega$, and a vector field on $\partial\Omega$, respectively. Then, inequality (79) is the second case of Lemma 4.4 for $r = 2$,

in [12]. Similarly, if $n = 3$, $k = 2$, and $\theta \in H^2_2$ is identified with the vector field u, equation (78) means

$$-\Delta u = \alpha \text{ in } \Omega, \quad \nu \times u = \beta \text{ on } \partial\Omega, \quad \text{div } u = \gamma \text{ on } \partial\Omega\,,$$

where α, β, and γ are regarded as a vector field on Ω, a vector field on $\partial\Omega$, and a scalar field on $\partial\Omega$, respectively. Then, inequality (79) is the first case of Lemma 4.4 for $r = 2$ in [12].

REMARK 41. The unknown

$$\theta = \sum_{i_1<\cdots<i_k} \theta^{i_1\cdots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_n} \tag{80}$$

has ${}_nC_k$ components in (78), and this system is formally over-determined because β and γ have ${}_nC_{k-1}$ and ${}_nC_k$ components, respectively. The operation $\nu \wedge \cdot$, however, reduces the number of the components of θ essentially used in these boundary conditions. Thus, any $x \in \partial\Omega$ admits the vector spaces $L(x)$ and $M(x)$ such that $\dim L(x) = {}_{n-1}C_{k-1}$, $\dim M(x) = {}_{n-1}C_k$, and

$$\beta(x) \in L(x), \quad \gamma(x) \in M(x)\,.$$

Since

$$\begin{aligned} {}_{n-1}C_{k-1} + {}_{n-1}C_k &= \frac{(n-1)!}{(n-k)!k!} + \frac{(n-1)!}{(n-1-k)!k!} = \frac{(n-1)!}{(n-k)!k!}(k+n-k) \\ &= \frac{n!}{(n-k)!k!} = {}_nC_k\,, \end{aligned}$$

the number of unknown components of θ coincides with that of boundary conditions. This fact is clarified in the proof of Theorem 16 below.

REMARK 42. Dual form of (78) is given by

$$-\Delta\zeta = u \text{ in } \Omega, \quad \mathbf{t}\zeta = v, \ \mathbf{t}\delta\zeta = w \text{ on } \partial\Omega\,,$$

of which ellipticity is shown in [21] §1.6. Here we provide with an analytic description of the proof.

REMARK 43. Inequality (79) is an a priori estimate and does not ensure the existence of H^2-solution to (78) by itself. More precisely, Theorem 16 says that if $\theta \in H^2_k$ solves (78) then it satisfies (79) for $s = 0$. If its right-hand side is finite for $s > 0$, furthermore, this θ belongs to H^{s+2}, and satisfies (79). Here we use the method of difference quotient to derive H^2-regularity of the H^1-solution (Proposition 20 and Lemma 22).

To proceed to the proof of Theorem 16, we write, first, $(\theta, \alpha) \in H^{s+2}_k \times H^s_k$ as in (80) and

$$\alpha = \sum_{i_1<\cdots<i_k} \alpha^{i_1\cdots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}\,.$$

Then the first equality of (78) is equivalent to

$$-\Delta\theta^{i_1\cdots i_k} = \alpha^{i_1\cdots i_k} \text{ in } \Omega, \quad 1 \le i_1 < \cdots < i_k \le n\,. \tag{81}$$

Second, we put $\tilde{\theta}^{i_1\cdots i_k}$, $1 \le i_1, \dots, i_k \le n$, and

$$\hat{\theta}^{i_2\cdots i_k} = \sum_{\ell} \tilde{\theta}^{i_2\cdots i_k\ell} dx_\ell\,,$$

as in Definition 2 and Definition 3 in §2.1. Then it holds that

$$\nu \wedge *\theta = \sum_{i_2<\cdots<i_k} (\hat{\theta}^{i_2\cdots i_k}, \nu) * (dx_{i_2} \wedge \dots \wedge dx_{i_k}) \in H^{s+3/2}_{n-k+1}(\partial\Omega)$$

similarly to Lemma 8, and therefore, the second equality of (78) is equivalent to

$$(\hat{\theta}^{i_2\cdots i_k}, \nu) = \beta^{i_2\cdots i_k} \text{ on } \partial\Omega, \quad 1 \le i_2 < \cdots < i_k \le n \tag{82}$$

for

$$\beta = \sum_{i_2<\cdots<i_k} \beta^{i_2\cdots i_k} dx_{i_2} \wedge \dots \wedge dx_{i_k} \in H^{s+3/2}_{k-1}(\partial\Omega)\,.$$

We have, third,

$$d\theta = \sum_{\ell} \sum_{i_1<\cdots<i_k} \theta^{i_1\cdots i_k}_\ell dx_\ell \wedge (dx_{i_1} \wedge \dots \wedge dx_{i_k})\,,$$

recalling

$$\theta^{i_1\cdots i_k}_\ell = \frac{\partial}{\partial x_\ell}\theta^{i_1\cdots i_k}\,,$$

and hence

$$\begin{aligned} d\theta &= \left(\sum_{\ell<i_1<\cdots<i_k} + \sum_{i_1<\ell<\cdots<i_k} + \right. \\ &\qquad \left. \cdots + \sum_{i_1<\cdots<i_k<\ell} \right) \theta^{i_1\cdots i_k}_\ell dx_\ell \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_{k+1}} \mu^{i_1\cdots i_{k+1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k+1}} \end{aligned}$$

for

$$\mu^{i_1\cdots i_{k+1}} = \sum_{\ell<i_1} \tilde{\theta}^{i_2\cdots i_{k+1}}_\ell + \sum_{i_1<\ell<i_2} \tilde{\theta}^{i_1 i_3\cdots i_{k+1}}_\ell + \cdots + \sum_{i_{k+1}<\ell} \tilde{\theta}^{i_1 i_2\cdots i_k}_\ell\,.$$

If we define $\tilde{\mu}^{i_1 i_2\cdots i_{k+1}}$, $1 \le i_1, \dots, i_{k+1} \le n$, and

$$\hat{\mu}^{i_2\cdots i_{k+1}} = \sum_{\ell} \tilde{\mu}^{i_2\cdots i_{k+1}\ell} dx_\ell$$

as in Definition 2 and Definition 3, it holds that

$$\nu \wedge *d\theta = \sum_{i_2<\cdots<i_{k+1}} (\hat{\mu}^{i_2\cdots i_{k+1}}, \nu) * (dx_{i_2} \wedge \dots \wedge dx_{i_{k+1}})$$

and therefore, the third equality of (78) is equivalent to

$$(\hat{\mu}^{i_2\cdots i_{k+1}}, \nu) = \gamma^{i_2\cdots i_{k+1}} \text{ on } \partial\Omega \tag{83}$$

for

$$\gamma = \sum_{i_2<\cdots<i_{k+1}} \gamma^{i_2\cdots i_{k+1}} dx_{i_2} \wedge \ldots \wedge dx_{i_{k+1}} \in H_k^{s+1/2}(\partial\Omega)\,.$$

Theorem 16 follows from the complementing boudary condition to (81)-(83) formulated in [3]. This condition, however, is algebraic and pointwise for each $x \in \partial\Omega$. Given $x_0 \in \partial\Omega$, therefore, first, we can assume $x_0 = 0$ by the translation of x. It is obvious that the system (78) is invariant under this operation. We take, second, an orthogonal transformation of variables, denoted by $x = Qy$ for $Q \in M_{n,n}(\mathbf{R})$ satisfying $Q^TQ = QQ^T = I_n$, where $I_n \in M_{n,n}(\mathbf{R})$ stands for the unit matrix. It is well-known that Δ is invariant under such orthogonal transformation. We note also that exterior product and exterior differentiation commute with the transformation of variables ([17] pp.60-61). Writing

$$\tilde{\theta} = Q^*\theta, \quad \tilde{\alpha} = Q^*\alpha, \quad \tilde{\nu} = Q^*\nu, \quad \tilde{\beta} = Q^*\beta, \quad \tilde{\gamma} = Q^*\gamma\,,$$

therefore, we obtain

$$-\Delta\tilde{\theta} = \tilde{\alpha} \text{ in } \tilde{\Omega} = Q\Omega, \quad \tilde{\nu} \wedge *\tilde{\theta} = *\tilde{\beta} \text{ on } \partial\tilde{\Omega}, \quad \tilde{\nu} \wedge *d\tilde{\theta} = *\tilde{\gamma} \text{ on } \partial\tilde{\Omega}\,,$$

recalling $Q^*\theta = \theta \circ Q$. Equality (2), also, implies

$$\tilde{\nu} = \sum_{ij} \nu^i q^{ij} dy_j, \quad Q = (q^{ij})\,,$$

and therefore, the 1-form $\tilde{\nu}$ corresponds to the vector field

$$\vec{\tilde{\nu}} = \vec{\nu}Q \quad \text{on } \partial\tilde{\Omega}\,.$$

This $\vec{\tilde{\nu}}$ is outer unit normal to $\partial\tilde{\Omega}$ at $x_0 = 0$ because Q is an orthogonal matrix, and therefore, we can assume also

$$\vec{\nu} = (1, 0, \cdots, 0)^T \tag{84}$$

to confirm the complementing boundary condition to (78) at $x_0 \in \partial\Omega$.

Putting (84) to (81)-(83), we see that Lemma 16 is obtained if the following proposition is shown.

PROPOSITION 17. The system

$$\begin{aligned}
&-\Delta\theta^{i_1\cdots i_k} = \alpha^{i_1\cdots i_k} \text{ in } \Omega, && 1 \le i_1 < \cdots < i_k \le n\\
&\theta^{1i_2\cdots i_k} = \beta^{i_2\cdots i_k} \quad \text{on } \partial\Omega, && 2 \le i_2 < \cdots < i_k \le n\\
&\mu^{1i_2\cdots i_{k+1}} = \gamma^{i_2\cdots i_{k+1}} \text{ on } \partial\Omega, && 2 \le i_2 < \cdots < i_{k+1} \le n
\end{aligned} \tag{85}$$

is elliptic in the sense of [3], where

$$\mu^{1i_2\cdots i_{k+1}} = \sum_{j=1}^{k+1} \sum_{i_j<\ell<i_{j+1}} \tilde{\theta}_\ell^{1i_2\cdots i_j i_{j+2}\cdots i_{k+1}}\,.$$

The numbers of the first, the second, and the third equations of (85) are equal to ${}_nC_k$, ${}_{n-1}C_{k-1}$, and ${}_{n-1}C_k$, respectively, and therefore, those of unknowns and the boudary conditions coincide as is noticed in Remark 41.

Now we represent system (85) in the canonical form, introducing the unknown vector and the coefficient matrix. For this purpose we put

$$u = (u^1, u^2, \cdots, u^{n-k+1})^T$$

for

$$\begin{aligned} u^1 &= (\theta^{12\ldots k}, \theta^{13\ldots k+1}, \cdots, \theta^{1,n-k+2,\ldots n})^T \\ u^2 &= (\theta^{23\ldots k+1}, \theta^{24\ldots k+2}, \cdots, \theta^{2,n-k+2,\ldots n})^T \\ &\vdots \\ u^{n-k+1} &= (\theta^{n-k+1,\ldots n})\,. \end{aligned}$$

Then, the coefficient matrix of the first equation of (85), with the differentiation ∂_x^α replaced by the symbol $(\sqrt{-1}\xi)^\alpha$, is equal to

$$L \equiv (\ell_{ij}) = (\ell'_{ij}) = -|\xi|^2 I_N, \quad N = {}_nC_k\,,$$

where α is the multi-index,

$$|\xi|^2 = \sum_i (\xi^i)^2, \quad \xi = (\xi^i)\,,$$

and (ℓ'_{ij}) is the principal part of (ℓ_{ij}). Here, for

$$s_i = 0,\ t_j = 2, \quad 1 \le i, j \le N\,,$$

it holds that $\deg \ell_{ij} \le s_i + t_j = 2$. We have, furthermore,

$$L = \det\,(\ell'_{ij}) = (-|\xi|^2)^N\,,$$

and the co-factor matrix of L is given by

$$(L^{jk}) = (-|\xi|^2)^{N-1} I_N\,.$$

First, the complementing boundary condition of [3] is associated with the non-trivial tangential vector denoted by $\Theta = (\Theta^i)$. Hence it holds that

$$\vec{\nu} \cdot \Theta = 0, \quad \Theta^1 = 0 \tag{86}$$

by (84). Since

$$|\Theta + \tau\vec{\nu}|^2 = \sum_i (\Theta^i + \tau\nu^i)^2 = \tau^2 + \sum_i (\Theta^i)^2\,,$$

the root of τ with positive imaginary part to

$$L(\Theta + \tau\vec{\nu}) = 0$$

is given by $\tau = |\Theta|\sqrt{-1}$ with multiplicity N. Then we define the polynomial $M_+ = M_+(\tau)$, $\tau \in \mathbf{C}$, by

$$M_+ = (\tau - |\Theta|\sqrt{-1})^N\,.$$

Second, the coefficient matrix of the boundary conditions in (85) is given by

$$B \equiv (B_{kj}) = (B'_{kj}) = \begin{pmatrix} I_{N_1} & 0 \\ * & \xi^1 I_{N_2} \end{pmatrix}, \quad N_1 = {}_{n-1}C_{k-1},\ N_2 = {}_{n-1}C_k\,, \tag{87}$$

which results in

$$\det\ B = (\xi^1)^{N_2}\,. \tag{88}$$

The complementing boundary condition of [3] to (85) is now formulated by

$$\begin{aligned} &\left(\sum_h c_h \sum_j (B'_{kj} L^{jk})(\Theta + \tau\vec{\nu}) \right)_k \equiv 0 \pmod{M_+} \\ &\Rightarrow \quad c_1 = \cdots = c_N = 0\,. \end{aligned} \tag{89}$$

Theorem 16 is thus reduced to (89). We now show the following lemma.

LEMMA 18. *It holds that*

$$(\Theta^i + \tau\nu^i)^m |\Theta + \tau\vec{\nu}|^{2(N-1)} \equiv (\Theta^i + |\Theta|\sqrt{-1}\nu^i)^m |\Theta + \tau\vec{\nu}|^{2(N-1)} \pmod{M_+}$$

for any $1 \le i \le N$ *and* $m \ge 1$.

PROOF. Letting $\alpha = |\Theta|\sqrt{-1}$, $a = \Theta$, we obtain

$$\vec{\nu} \cdot a = 0, \quad |\vec{\nu}| = 1, \quad M_+(\tau) = (\tau - \alpha)^N, \quad |a + \tau\vec{\nu}|^2 = \tau^2 - \alpha^2\,,$$

and hence

$$\begin{aligned} |a + \tau\vec{\nu}|^{2(N-1)} &= (\tau^2 - \alpha^2)^{N-1} = (\tau - \alpha)^{N-1}((\tau - \alpha) + 2\alpha)^{N-1} \\ &\equiv (2\alpha)^{N-1}(\tau - \alpha)^{N-1} \pmod{M_+}\,. \end{aligned}$$

It thus follows that

$$\begin{aligned} (a^i + \tau\nu^i)^m |a + \tau\vec{\nu}|^{2(N-1)} &\equiv (2\alpha)^{N-1}(a^i + \tau\nu^i)^m (\tau - \alpha)^{N-1} \\ &= (2\alpha)^{N-1}(a^i + \alpha\nu^i + (\tau - \alpha)\nu^i)^m (\tau - \alpha)^{N-1} \\ &\equiv (2\alpha)^{N-1}(a^i + \alpha\nu^i)^m (\tau - \alpha)^{N-1} \\ &\equiv (a^i + \alpha\nu^i)^m |a + \tau\vec{\nu}|^{2(N-1)} \pmod{M_+}\,, \end{aligned}$$

and hence the result. □

We are ready to give the following proof.

PROOF OF THEOREM 16: It suffices to confirm (89). In fact, we have

$$\left(\sum_h c_h \sum_j (B'_{kj} L^{jk})(\xi) \right)_k = (-|\xi|^2)^{N-1} \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} B$$

for $B = B(\xi)$ given in (87). Then Lemma 18 implies

$$\begin{aligned}|\xi|^{2(N-1)}B(\xi)\big|_{\xi=\Theta+\tau\vec{\nu}} &= |\Theta+\tau\vec{\nu}|^{2(N-1)}(\Theta^1+\tau\nu^1)^{N_2}\\ &\equiv |\Theta+\tau\vec{\nu}|^{2(N-1)}(\Theta^1+|\Theta|\sqrt{-1}\nu^1)^{N_2}\\ &= |\Theta+\tau\vec{\nu}|^{2(N-1)}B(\Theta+|\Theta|\sqrt{-1}\vec{\nu}) \pmod{M_+}\,.\end{aligned}$$

Here, it holds that

$$\xi = \Theta + |\Theta|\sqrt{-1}\vec{\nu} \quad \Rightarrow \quad \xi^1 = |\Theta|\sqrt{-1}$$

by (84) and (86). Hence we obtain

$$\det B(\Theta+|\Theta|\sqrt{-1}\vec{\nu}) = (\sqrt{-1})^{N_2}|\Theta|^{N_2} \neq 0\,,$$

and therefore, (89). □

3. Proof of Theorem 1 and Theorem 4

We divide this section into four subsections, which deal with the orthogonality in (3), the regularity of the harmonic part, decomposition (18), and isomorphism (5). We use an inequality of Poincaré type, (96) in §3.3, to get η in Theorem 4 as an H^1 solution of a variational problem. Then its H^2-regularity is assured by the method of Nirenberg [18, 19], to reach its smoothness by the ellitpic estimate of Agmon-Douglis-Nirenberg [3], that is, Theorem 16 in §2.4.

3.1. Orthogonality

Here we show the following fact noticed by Remark 3.

Proposition 19. The decomposition (3) is L^2-orthogonal.

Proof. Let $1 \le k \le n-1$. Given $(h,\omega) \in H_k \times W_k$, first, we have

$$\int_\Omega (h, \delta\omega)\ dx = \int_\Omega (dh, \omega)\ dx - (-1)^k \int_{\partial\Omega} (*h, \nu\wedge *\omega)\ ds = 0$$

by Theorem 5, and therefore, H_k is orthogonal to δW_k. For $(\omega, p) \in W_k \times V_k$, second, it holds that

$$\int_\Omega (\delta\omega, dp)\ dx = \int_\Omega (\omega, d^2p)dx - (-1)^k \int_{\partial\Omega} (*dp, \nu\wedge *\omega)\ ds = 0\,,$$

and therefore, δW_k is orthogonal to dV_k. If $(p,h) \in V_k \times H_k$, finally, we have

$$\int_\Omega (dp, h)\ dx = \int_\Omega (p, \delta h)\ dx + (-1)^{k-1} \int_{\partial\Omega} (*p, \nu\wedge *h)\ ds = 0\,,$$

and therefore, dV_k is orthogonal to H_k. □

Remark 44. The boundary condition $\nu \wedge *p|_{\partial\Omega} = 0$ is not used in the above proof.

REMARK 45. By Proposition 19, Theorem 1 is reduced to Theorem 4 and (5).

3.2. Regularity of the harmonic part

Here we show the following fact.

PROPOSITION 20 (Elliptic regularity). It holds that

$$\tilde{H}_k = H_k, \quad 1 \le k \le n-1\,,$$

where

$$\tilde{H}_k = \{h \in L^2_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}\,.$$

PROOF. It suffices to show $\tilde{H}_k \subset H_k$. Given $h \in \tilde{H}_k$, first, we have

$$h \in H^1_k \tag{90}$$

by Theorem 13. We show, second,

$$h \in H^2_k\,. \tag{91}$$

For this purpose we take the open covering of $\overline{\Omega}$ and the associated partition of unity, $(\Omega_\ell, \varphi_\ell)$, $0 \le \ell \le L$, defined by (57)-(58). Letting

$$h_\ell = h \cdot \varphi_\ell \in H^1_k, \quad \text{supp } h_\ell \subset \overline{\Omega} \cap \Omega_\ell\,,$$

we show

$$h_\ell \in H^2_k, \quad 0 \le \ell \le L\,. \tag{92}$$

For $\ell = 0$, we take the difference quotient

$$\Delta_{s,j} h_0(x) = \frac{1}{s}\{h_0(x + se_j) - h_0(x)\}, \quad 1 \le j \le n,\ 0 < |s| \ll 1\,,$$

where

$$e_j = (0, \cdots, \overset{j}{\widehat{1}}, \cdots, 0)^T\,.$$

It then follows that

$$\|d\Delta_{s,j} h_0\|_{L^2_{k+1}} \le C, \quad \|\delta\Delta_{s,j} h_0\|_{L^2_{k-1}} \le C, \quad \|\Delta_{s,j} h_0\|_{L^2_k} \le C$$

from $dh = 0$, $\delta h = 0$, and (90), while

$$\nu \wedge *(\Delta_{s,j} h_0)\big|_{\partial\Omega} = 0$$

holds by supp $h_0 \subset \Omega_0$. Then we obtain

$$\|\Delta_{s,j} h_0\|_{H^1_k} \le C$$

by Theorem 13, which implies

$$\frac{\partial}{\partial x_j} h_0 \in H^1_k, \quad 1 \le j \le n$$

in the sense of distributions. We thus end up with (92) for $\ell = 0$.

To treat $1 \leq \ell \leq L$, we use the diffeomorphism $X_\ell : \Omega_\ell \to U_\ell \subset \mathbf{R}^n$ in (57) and the set Q_ℓ in (71)-(72). We put

$$g_\ell = Y_\ell^* h_\ell \in H_k^1(U_\ell^+), \quad Y_\ell = X_\ell^{-1}\big|_{U_\ell^+}, \quad U_\ell^+ = U_\ell \cap \mathbf{R}_+^n$$

and take

$$\Delta_{s,j} g_\ell(y) = \frac{1}{s}\{g_\ell(y + se_j) - g_\ell(y)\}, \quad 1 \leq j \leq n-1,\ 0 < |s| \ll 1\,.$$

It then holds that

$$\begin{aligned} \|dX_\ell^* \Delta_{s,j} g_\ell\|_{L^2_{k+1}(\tilde{\Omega}_\ell)} &\leq C \\ \|\delta X_\ell^* \Delta_{s,j} g_\ell\|_{L^2_{k-1}(\tilde{\Omega}_\ell)} &\leq C \\ \|X_\ell^* \Delta_{s,j} g_\ell\|_{L^2_k(\tilde{\Omega}_\ell)} &\leq C\,, \end{aligned}$$

for $\tilde{\Omega}_\ell = \Omega \cap \Omega_\ell$, similarly, while

$$\nu \wedge *(X_\ell^* \Delta_{s,j} g_\ell)\big|_{\partial\Omega \cap \Omega_\ell} = 0\,,$$

holds by Lemma 14. Since

$$\text{supp } g_\ell \subset Q_\ell^+ = \Sigma \times [0, \varepsilon_0)$$

we obtain

$$\|X_\ell^* \Delta_{s,j} g_\ell\|_{H_k^1(\tilde{\Omega}_\ell)} \leq C\,,$$

by Theorem 13, or

$$\|\Delta_{s,j} g_\ell\|_{H_k^1(U_\ell^+)} \leq C\,,$$

and therefore,

$$\frac{\partial}{\partial y_j} g_\ell \in H_k^1(U_\ell^+), \quad 1 \leq j \leq n-1\,.$$

It thus holds that

$$\frac{\partial^2}{\partial y_i \partial y_j} g_\ell \in L_k^2(U_\ell^+), \quad 1 \leq i \leq n,\ 1 \leq j \leq n-1\,. \tag{93}$$

It holds also that

$$\Delta h_\ell = \alpha_\ell \in L_k^2\,,$$

by $dh = 0$, $\delta h = 0$, and (90), and therefore,

$$\Delta_{Y_\ell} g_\ell = Y_\ell^* \alpha_\ell \in L_k^2(U_\ell^+)\,,$$

where Δ_{Y_ℓ} is the Laplacian associated with the transformation of variables, $x = Y_\ell y$, that is,

$$\Delta_{Y_\ell} f = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_i (g^{ij} \sqrt{g} \partial_j f))$$
$$\partial_j = \frac{\partial}{\partial y_j} \quad g = \det(g_{ij}), \quad (g^{ij}) = (g_{ij})^{-1}$$
$$\sum_{ij} g_{ij} dy_i \otimes dy_j = ds^2 \equiv \sum_i dx_i \otimes dx_i \,.$$

Then it holds that

$$\frac{\partial^2}{\partial y_n^2} g_\ell \in L^2_k(U^+_\ell) \tag{94}$$

by (93), and hence $g_\ell \in H^2_k(U^+_\ell)$, We thus obtain (92) for $0 \le \ell \le L$, which implies $\tilde{H}_k \subset H^2_k$.

We use, third,

$$\int_\Omega (dh, d\theta) + (\delta h, \delta\theta)\, dx = 0, \quad \forall \theta \in \Lambda_k \,,$$

to deduce

$$\int_\Omega (\delta dh, \theta) + (d\delta h, \theta)\, dx + (-1)^k \int_{\partial\Omega} -(\nu \wedge *dh, *\theta) + (\nu \wedge *h, *\delta\theta)\, ds = 0 \tag{95}$$

for any $\theta \in \Lambda_k$, by $h \in H^2_k$. It then holds that

$$-\Delta h = 0, \quad \nu \wedge *h|_{\partial\Omega} = 0, \quad \nu \wedge *dh|_{\partial\Omega} = 0$$

because $\theta \in \Lambda_k$ is arbitrary in (95) and $\nu \wedge *h|_{\partial\Omega} = 0$ holds by $h \in \tilde{H}_k$. Then we obtain $h \in H^{s+2}_k$ for any $s > 0$ by Theorem 16 and Remark 43, and hence $h \in \Lambda_k$. □

3.3. Proof of Theorem 4

Here we show the decomposition (18). Given $1 \le k \le n-1$, let

$$Z_k = \left[\breve{\Lambda}_k \cap H^\perp_k\right]^{H^1_k} .$$

REMARK 46. It holds that $Z_k = \breve{H}^1_{k,d\delta} \cap H^\perp_k$ by Lemma 15.

We have an inequality of Poincaré type on Z_k.

LEMMA 21 (Poincaré inequality). *It holds that*

$$\|\theta\|_{L^2_k} \le C \left(\|d\theta\|_{L^2_{k+1}} + \|\delta\theta\|_{L^2_{k-1}} \right), \quad \theta \in Z_k \,. \tag{96}$$

PROOF. It suffices to show (96) for $\theta \in \breve{\Lambda}_k \cap H^\perp_k$. Assume the contrary, and let $\theta^j \in \breve{\Lambda}_k \cap H^\perp_k$, $j = 1, 2, \cdots$, be such that

$$\|\theta^j\|_{L^2_k} = 1, \quad \varepsilon_j = \|d\theta^j\|_{L^2_{k+1}} + \|\delta\theta^j\|_{L^2_{k-1}} \to 0 \,.$$

Then, Lemma 9 in §2.2 ensures

$$\|\theta^j\|_{H^1_k} \le C$$

and therefore, passing to a subsequence we have $\theta \in H^1_k$ such that

$$\theta^j \rightharpoonup \theta \text{ in } H^1_k, \quad \theta^j \to \theta \text{ in } L^2_k\,.$$

It follows that $\theta \in H^\perp_k$, $\|\theta\|_2 = 1$, and

$$\nu \wedge *\theta|_{\partial\Omega} = 0, \quad \|d\theta\|_{L^2_{k+1}} + \|\delta\theta\|_{L^2_{k-1}} \le \liminf_{j\to\infty} \varepsilon_j = 0\,,$$

which induces $\theta \in \tilde{H}_k = H_k$. Then we obtain $\theta = 0$ by $\theta \in H^\perp_k$, a contradiction. □

LEMMA 22. *Any $v \in \Lambda_k \cap H^\perp_k$ admits $\eta \in \Lambda_k \cap H^\perp_k$ such that*

$$-\Delta\eta = v, \quad \nu \wedge *\eta|_{\partial\Omega} = 0, \quad \nu \wedge *d\eta|_{\partial\Omega} = 0\,. \tag{97}$$

PROOF. We observe, first, that

$$a(\theta_1, \theta_2) = \int_\Omega (d\theta_1, d\theta_2) + (\delta\theta_1, \delta\theta_2)\, dx, \quad \theta_1, \theta_2 \in Z_k \tag{98}$$

is an inner product, which makes Z_k a Hilbert space. Second,

$$\theta \in Z_k \to \ell(\theta) = \int_\Omega (v, \theta)\, dx$$

is a bounded linear operator by Lemma 21, and therefore, there is $\eta \in Z_k$ such that

$$\int_\Omega (d\eta, d\theta) + (\delta\eta, \delta\theta)\, dx = \int_\Omega (v, \theta)\, dx, \quad \theta \in Z_k$$

by the representation theorem of Riesz. It holds, in particular, that

$$\int_\Omega (d\eta, d\theta_0) + (\delta\eta, \delta\theta_0)\, dx = \int_\Omega (v, \theta_0)\, dx, \quad \theta_0 \in \breve{\Lambda}_k \cap H^\perp_k\,.$$

Given $\theta \in \breve{\Lambda}_k$, we decompose it as in

$$\theta = \theta_0 + h, \quad \theta_0 \in H^\perp_k,\ h \in H_k\,.$$

Since

$$\nu \wedge *h|_{\partial\Omega} = 0, \quad h \in \Lambda_k, \quad dh = 0, \quad \delta h = 0\,,$$

it holds that

$$\theta_0 \in \breve{\Lambda}_k, \quad d\theta_0 = d\theta, \quad \delta\theta_0 = \delta\theta\,,$$

while

$$\int_\Omega (v, h)\, dx = 0$$

follows from $v \in H_k^\perp$. The above $\eta \in Z_k$, therefore, satisfies

$$\int_\Omega (d\eta, d\theta) + (\delta\eta, \delta\theta)\ dx = \int_\Omega (v, \theta)\ dx, \quad \theta \in \breve{\Lambda}_k\,. \tag{99}$$

Now we show

$$\eta \in H_k^2, \quad -\Delta\eta = v, \quad \nu \wedge *\eta|_{\partial\Omega} = 0, \quad \nu \wedge *d\eta|_{\partial\Omega} = 0\,.$$

First, we have

$$\eta \in H_k^1 \tag{100}$$

by Theorem 13. We take, second, the open covering of $\overline{\Omega}$ and the associated partition of unity, $(\Omega_\ell, \varphi_\ell)$, $0 \le \ell \le L$, to put

$$\eta_\ell = \eta \cdot \varphi_\ell\,.$$

If $\ell = 0$, we use

$$\Delta_{s,j}\eta(x) = \frac{1}{s}\{\eta(x + se_j) - \eta(x)\}, \quad 1 \le j \le n,\ 0 < |s| \ll 1$$

to deduce

$$\|\varphi_0 d\Delta_{s,j}\eta\|_2 \le C, \quad \|\varphi_0 \delta\Delta_{s,j}\eta\|_2 \le C$$

from (99) as in the classcial elliptic theory ([11] pp.87-90, [10] pp.183-187), which implies

$$\|d\Delta_{s,j}\eta_0\|_2 \le C, \quad \|\delta\Delta_{s,j}\eta_0\|_2 \le C$$

by (100).

Here we obtain

$$\Delta_{s,j}\eta_0 \in \breve{H}^1_{k,d\delta}$$

by supp $\eta_0 \in \Omega_0$, while(100) implies

$$\|\Delta_{s,j}\eta_0\|_{L_k^2} \le C\,.$$

It thus holds that

$$\|\Delta_{s,j}\eta_0\|_{H_k^1} \le C$$

by Theorem 13, and therefore,

$$\frac{\partial}{\partial x_j}\eta_0 \in H_k^1, \quad 1 \le j \le n\,,$$

or

$$\eta_0 \in H_k^2\,.$$

If $1 \le \ell \le L$, we use the diffeomorphism $X_\ell : \Omega_\ell \to U_\ell$ as in the proof of Proposition 20. We put

$$\zeta_\ell = Y_\ell^* \eta_\ell \in H_k^1(U_\ell^+)$$

and take

$$\Delta_{s,j}\zeta_\ell(y) = \frac{1}{s}\{\zeta_\ell(y+se_j) - \zeta_\ell(y)\}, \quad 1 \le j \le n-1,\ 0 < |s| \ll 1\,.$$

Then we obtain

$$\|dX_\ell^*\Delta_{s,j}\zeta_\ell\|_{L^2_{k+1}(\tilde\Omega_\ell)} \le C, \quad \|\delta X_\ell^*\Delta_{s,j}\zeta_\ell\|_{L^2_{k-1}(\tilde\Omega_\ell)} \le C\,,$$

similarly. It holds also that

$$\nu \wedge *(X_\ell^*\Delta_{s,j}\zeta_\ell)\big|_{\partial\Omega\cap\Omega_\ell} = 0$$

and supp $\Delta_{s,j}\zeta_\ell \subset Q_\ell^+$. Since

$$\|X_\ell^*\Delta_{s,j}\zeta_\ell\|_{L^2_k(\Omega_\ell)} \le C$$

holds by (100), furthermore, we obtain

$$\|X_\ell^*\Delta_{s,j}\zeta_\ell\|_{H^1_k(\Omega_\ell)} \le C\,,$$

by Theorem 13, and therefore,

$$X_\ell^*\frac{\partial}{\partial y_j}\zeta_\ell \in H^1_k(\Omega_\ell),\ 1 \le j \le n-1\,,$$

or

$$\frac{\partial^2}{\partial y_i \partial y_j}\zeta_\ell \in L^2_k(U_\ell^+), \quad 1 \le i \le n,\ 1 \le j \le n-1\,.$$

Here we use

$$-\Delta_{Y_\ell}\zeta_\ell = Y_\ell^* v \quad \text{in } U_\ell^+, \quad v \in \Lambda_k\,,$$

to obtain

$$\frac{\partial^2}{\partial y_n^2}\zeta_\ell \in H^1_\ell(U_\ell^+)$$

similarly to (94), and then $\zeta_\ell \in H^2_k(U_\ell^+)$ follows. We thus end up with $\eta_\ell \in H^2_k$, $0 \le \ell \le L$, or, $\eta \in H^2_k$.

Having (99) and $\eta \in H^2_k$, we obtain

$$\begin{aligned}\int_\Omega (v,\theta)\ dx &= \int_\Omega (d\eta, d\theta) + (\delta\eta, \delta\theta)\ dx \\ &= \int_\Omega (\delta d\eta + d\delta\eta, \theta)\ dx + (-1)^k \int_{\partial\Omega} (\nu \wedge *d\eta, *\theta) - (*\delta\eta, \nu \wedge *\theta)\ ds\end{aligned}$$

for any $\theta \in \breve\Lambda_k$ or $\theta \in \breve H^1_{k,d\delta}$, which implies

$$\delta d\eta + \delta d\eta = v, \quad \int_{\partial\Omega} (\nu \wedge *d\eta, *\theta) = 0,\ \forall\theta \in \breve H^1_{k,d\delta} \tag{101}$$

by $\nu \wedge *\theta|_{\partial\Omega} = 0$. Since

$$\nu \wedge (\nu \wedge *d\eta)|_{\partial\Omega} = 0$$

it holds that $\tilde{\nu} \wedge *d\eta \in \breve{H}^1_{n-k,d\delta}$ with $\tilde{\nu} \in \Lambda_1$ satisfying $\tilde{\nu}|_{\partial\Omega} = \nu$, and therefore,

$$\nu \wedge *d\eta|_{\partial\Omega} = 0$$

because $\beta = *\theta$ in (101) is arbitrary such as $\beta \in H^1_{n-k,d\delta}$ and $\nu \wedge \beta|_{\partial\Omega} = 0$. We thus obtain (97) with $\eta \in H^2_k \cap \Lambda_k \cap H^\perp_k$ and $v \in \Lambda_k$, and hence $\eta \in \Lambda_k$ by Remark 43. □

REMARK 47. Putting $\theta = \eta$ in (99), we get

$$\|d\eta\|^2_{L^2_{k+1}} + \|\delta\eta\|^2_{L^2_{k-1}} \le \|v\|_{L^2_k}\|\eta\|_{L^2_k}$$

and hence

$$\|\eta\|_{L^2_k} \le C\|v\|_{L^2_k} \tag{102}$$

by (96). Then it follows that

$$\|\eta\|_{H^2_k} \le C\|v\|_{L^2_k}$$

from (79). See Lemm 33 below.

REMARK 48. Using

$$a_\lambda(\theta_1, \theta_2) = \int_\Omega (d\theta_1, d\theta_2) + (\delta\theta_1, \delta\theta_2) + \lambda(\theta_1, \theta_2)\, dx, \quad \theta_1, \theta_2 \in H^1_{k,d\delta}$$

instead of $a(\theta_1, \theta_2)$, $\theta_1, \theta_2 \in Z_k$ in (98), we obtain the following proposition, where $\lambda > 0$ is a constant.

PROPOSITION 23. Any $u \in \Lambda_k$ admits $\theta \in \Lambda_k$ such that

$$(-\Delta + \lambda)\theta = u, \quad \nu \wedge *\theta|_{\partial\Omega} = 0, \quad \nu \wedge *d\theta|_{\partial\Omega} = 0\,.$$

We now complete the following proof.

PROOF OF THEOREM 4: If $1 \le k \le n-1$ and $u = v \in \Lambda_k \cap H^\perp_k$, we have $\eta \in \Lambda_k \cap H^\perp_k$ such that (97) by Lemma 22. Then we obtain (18) with $h = 0$.

This decomposition, on the other hand, is obvious if $u = h \in H_k$, as we have only to take $\eta = 0$. Then we obtain the result by combining these two cases. □

3.4. Isomorphic to the de Rham cohomology

Here we complete the proof of Theorem 1.

PROOF OF (5): We recall that the de Rham cohomology is defined by

$$H^k_{DR}(\Omega) = Z^k(\Omega)/B^k(\Omega), \quad 1 \le k \le n-1\,,$$

where

$$Z^k(\Omega) = \{\omega \in \Lambda_k \mid d\omega = 0\}, \quad B^k(\Omega) = d\Lambda_{k-1}\,.$$

Henceforth, the equivalent class of $h \in Z^k(\Omega)$ in $H^k_{DR}(\Omega)$ is denoted by $[h]$. This $H^k_{DR}(\Omega)$ is isomorphic to the k-th singular cohomology of Ω, denoted by $H^k(\Omega, \mathbf{R})$, and hence

$$\dim\, H^k_{DR}(\Omega) = b_k$$

follows, where b_k is the k-th Betti number of Ω ([17] p.114). We now show that the canonical mapping

$$T : h \in H_k \;\mapsto\; [h] \in H^k_{DR}(\Omega) \tag{103}$$

is an isomorphism.

To confirm the injectivity of T, let $h \in H_k$ be in the form of

$$h = dp, \quad \exists p \in \Lambda_{k-1}\,.$$

Then we obtain

$$\int_\Omega (h,h)\; dx = \int_\Omega (dp,h)\; dx = \int_\Omega (p,\delta h)\; dx + (-1)^k \int_{\partial\Omega} (*p, \nu \wedge *h)\; ds = 0$$

by Theorem 5, and hence $h = 0$. The mapping $T : H_k \to H^k_{DR}(\Omega)$ is thus injective.

For surjectivity we assume $du = 0$ for $u \in \Lambda_k$ to show

$$[h] = [u], \quad \exists h \in H_k\,.$$

We have only to find $p \in \Lambda_{k-1}$ such that

$$h \equiv u - dp \in H_k\,,$$

which means

$$\delta dp = \delta u, \quad \nu \wedge *(dp - u)\big|_{\partial\Omega} = 0\,.$$

In fact, we have readily shown (17)-(18) in the previous subsection. Then we take $p = \delta\eta$, which satisfies

$$\delta u = \delta dp, \quad \nu \wedge *p\big|_{\partial\Omega} = 0, \quad \nu \wedge *(u - dp)\big|_{\partial\Omega} = \nu \wedge *(h - \delta d\eta)\big|_{\partial\Omega} = 0$$

by Lemma 3. □

4. Applications

This section is devoted to applications of Theorem 1 and Theorem 4. It is composed of four subsections. We show a priori estimates to $d-\delta$ systems, general L^r-theory developed by [12] for $n = 3$, compensated compactness lemma, and inequality of Babuška-Brezzi-Kikuchi type which implies the $W^{1,r}$-estimate to the Poisson equation. Here, the L^r-theory is derived from the structure theorem, Theorem 4, and an a priori estimate.

4.1. A priori estimates to $d-\delta$ systems

Lemma 9 in §2.2 is concerned on $u \in \breve{\Lambda}_k$, while we have the following estimate for general $u \in \Lambda_k$.

THEOREM 24. *Each $s \geq 0$ admits*

$$\|u\|_{H_k^{s+2}} \leq C \left(\|du\|_{H_{k+1}^{s+1}} + \|\delta u\|_{H_{k-1}^{s+1}} + \|u\|_{H_k^s} + \|\nu \wedge *u\|_{H_{n-k+1}^{s+3/2}(\partial\Omega)} \right) \tag{104}$$

for any $u \in \Lambda_k$, where $1 \leq k \leq n-1$.

REMARK 49. Inequality (42) concerning $u \in \breve{\Lambda}_k$ is regarded as inequality (104) to $s = -1$, with $\|u\|_{H_k^{-1}}$ on the right-hand side replaced by $\|u\|_{L_k^2}$.

REMARK 50. Several modifications of (104) arise such as the L^r-estimate or eliminating the term $\|u\|_{H_k^s}$ in the right-hand side of (104) for $u \in H_k^\perp$, as is described in (2.28) and (2.30) of [12] for $n = 3$.

REMARK 51. The case $k = 1$ is studied by [4] p.349, where the exponent $s \geq 0$ in (104) is extended to $s > \frac{n}{r} - 1$ in the context of L^r-theory.

PROOF OF THEOREM 24: We apply Theorem 16 in §2.4 for

$$\alpha = \delta du + d\delta u, \quad *\beta = \nu \wedge *u, \quad *\gamma = \nu \wedge *du\,,$$

to obtain

$$\|u\|_{H_k^{s+2}} \leq C \left(\|\alpha\|_{H_{k+1}^s} + \|\beta\|_{H_{k-1}^{s+3/2}(\partial\Omega)} + \|\gamma\|_{H_k^{s+1/2}(\partial\Omega)} + \|u\|_{H_k^s} \right)$$

Since

$$\|\alpha\|_{H_k^s} \leq \|du\|_{H_{k+1}^{s+1}} + \|\delta u\|_{H_{k-1}^{s+1}}, \quad \|\gamma\|_{H_k^{s+1/2}(\partial\Omega)} \leq C\|du\|_{H_{k+1}^{s+1}}\,,$$

it holds that (104). □

4.2. L^r-theory

Here we develop the L^r-theory, using Theorem 4 and the L^r-estimate derived from a duality argument. We introduce the following function spaces.

DEFINITION 12. Let $0 \leq k \leq n$, $s \geq 0$, $1 \leq r \leq \infty$.

1. $W_k^{s,r}$ denotes the set of elements in $D_k'(\Omega)$ of which coefficients are in $W^{s,r} = W^{s,r}(\Omega)$.
2. $W_k^{s,r}(\partial\Omega)$ denotes the set of k-th differential forms on $\partial\Omega$ of which coefficients are in $W^{s,r}(\partial\Omega)$.

Definition 13. We put

$$W^{1,r}_{k,d} = \{\theta \in L^r_k \mid d\theta \in L^r_{k+1}\}, \quad 0 \le k \le n-1$$
$$W^{1,r}_{k,\delta} = \{\theta \in L^2_k \mid \delta\theta \in L^2_{k+1}\}, \quad 1 \le k \le n$$
$$W^{1,r}_{k,d\delta} = W^{1,r}_{k,d} \cap W^{1,r}_{k,\delta}, \qquad 1 \le k \le n-1\,.$$

Remark 52. It holds that $L^r_k = W^{0,r}_k$.

Remark 53. The trace operator is defined as in

$$\theta \in W^{s,r}_k \;\mapsto\; \theta|_{\partial\Omega} \in W^{s-1/r,r}_k(\partial\Omega)$$

for $s > \frac{1}{r}$, $1 < r < \infty$ ([15] p.337).

Remark 54. Similarly to the L^2-theory, Λ_k is dense in $W^{1,r}_{k,d}$, $0 \le k \le n-1$, $1 < r < \infty$, and so is in $W^{1,r}_{k,\delta}$, $1 \le k \le n$, $1 < r < \infty$.

The elliptic estimate (79) to (78) is now extended as follows from the complementing boundary condition examined in the proof of Theorem 16.

Proposition 25 (Elliptic estimate). Let $\theta \in W^{2,r}_k$, $1 \le k \le n-1$, $1 < r < \infty$, satisfy (78):

$$-\Delta\theta = \alpha \text{ in } \Omega, \quad \nu \wedge *\theta = *\beta \text{ on } \partial\Omega, \quad \nu \wedge *d\theta = *\gamma \text{ on } \partial\Omega\,.$$

Then it holds that

$$\|\theta\|_{W^{s+2,r}_k} \le C\Big(\|\alpha\|_{W^{s,r}_k} + \|\beta\|_{W^{s+2-1/r,r}_{k-1}(\partial\Omega)} + \|\gamma\|_{W^{s+1-1/r,r}_k(\partial\Omega)} + \|\theta\|_{W^{s,r}_k}\Big) \tag{105}$$

for $s = 0$. If $s > 0$ and the right-hand side on (105) is finite, furthermore, we have $\theta \in W^{s+2,r}_k$ and this inequality holds.

Similarly to Proposition 12 in §2.3, there is a bounded bilinear operator

$$J^r_k = J^r_k(\theta,\omega) : W^{1,r'}_{k,d} \times W^{1,r}_{k+1,\delta} \to \mathbf{R}$$

for $0 \le k \le n-1$, $1 < r < \infty$, such that

$$\int_\Omega (d\theta,\omega)\; dx = \int_\Omega (\theta,\delta\omega)\; dx + (-1)^k J^r_k(\theta,\omega)\,,$$

where $1 < r' < \infty$ is the dual exponent of r defined by (55):

$$\frac{1}{r'} + \frac{1}{r} = 1\,. \tag{106}$$

It thus holds that

$$J^r_k(\theta,\omega) = \int_{\partial\Omega} (*\theta, \nu \wedge *\omega)\; ds, \quad (\theta,\omega) \in W^{1,r'}_k \times W^{1,r}_{k+1}\,,$$

DEFINITION 14 (Weak trace). We say

$$\nu \wedge *\omega|_{\partial\Omega} = 0, \quad \omega \in W^{1,r}_{k+1,\delta}$$

if $J^r_k(\theta, \omega) = 0$ for any $\theta \in W^{1,r'}_{k,d}$.

DEFINITION 15. We put

$$\begin{aligned} \breve{W}^{1,r}_{k,\delta} &= \{\theta \in W^{1,r}_{k,\delta} \mid \nu \wedge *\theta|_{\partial\Omega} = 0\}, \quad 1 \le k \le n \\ \breve{W}^{1,r}_{k,d\delta} &= \breve{W}^{1,r}_{k,\delta} \cap W^{1,r}_{k,d}, \qquad\qquad 1 \le k \le n-1\,. \end{aligned}$$

The following lemma is obtained similarly to the L^2-theory.

PROPOSITION 26 (Duvaut-Lions density). The set $\breve{\Lambda}_k$ is dense in $\breve{W}^{1,r}_{k,d\delta}$.

We have also the following fact comparable to Theorem 13 for $r = 2$. This proposition for $n = 3$ is Lemma 3.9 of [12].

PROPOSITION 27 ($W^{1,r}$-regularity). It holds that $\breve{W}^{1,r}_{k,d\delta} \subset W^{1,r}_k$ and

$$\|\nabla u\|_{L^r_k} \le C\left(\|du\|_{L^r_{k+1}} + \|\delta u\|_{L^r_{k-1}} + \|u\|_{L^r_k}\right), \quad u \in \breve{W}^{1,r}_{k,d\delta}\,, \tag{107}$$

where $1 < r < \infty$, $1 \le k \le n-1$.

PROOF. This proposition is reduced to inequality (107) for $u \in \breve{\Lambda}_k$ by Proposition 26. This inequality is a direct consequence of (54) in §2.2 and an inequality of Babuška-Brezzi-Kikuchi type, see §4.4, that is,

$$\inf_{u\in\breve{\Lambda}_k\setminus\{0\}} \sup_{\varphi\in\breve{\Lambda}_k\setminus\{0\}} \frac{\left|\int_\Omega (\nabla u, \nabla\varphi) + (u, \varphi)\ dx\right|}{\left(\|\nabla u\|_{L^r_k} + \|u\|_{L^r_k}\right)\left(\|\nabla\varphi\|_{L^{r'}_k} + \|\varphi\|_{L^{r'}_k}\right)} > 0\,. \tag{108}$$

Inequality (108), then, is proven similarly to Lemma 3.1 of [12] for $n = 3$ through the local chart and reflection described in Remark 39 in §2.3. □

Prosition 27 implies the following properties as in the L^2-theory.

PROPOSITION 28 (Elliptic regularity). It holds that $\tilde{H}^r_k = H_k$ for

$$\tilde{H}^r_k = \{h \in L^r_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}\,.$$

We have also the following proposition simlarly to Lemma 21.

PROPOSITION 29 (Poincaré inequaltity). It holds that

$$\|\theta\|_{L^r_k} \le C\left(\|d\theta\|_{L^r_{k+1}} + \|\delta\theta\|_{L^r_{k-1}}\right), \quad \theta \in \breve{W}^{1,r}_{k,d\delta} \cap H^\perp_k \tag{109}$$

where $1 < r < \infty$, $1 \le k \le n-1$.

The main resuls in this subsection are Theorem 30 and Theorem 31 below. We recall $1 < r < \infty$, $1 \le k \le n-1$.

Definition 16. Let

$$W_k^r = \{\omega \in W^{1,r}_{k+1,d\delta} \mid d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0\} \tag{110}$$
$$V_k^r = \{p \in W^{1,r}_{k-1,d\delta} \mid \delta p = 0,\ \nu \wedge *p|_{\partial\Omega} = 0\}.$$

Remark 55. Above $W_k^{1,r}$, $V_k^{1,r}$ are Banach spaces under the norms

$$\|\omega\|_{L^r_{k+1}} + \|\delta\omega\|_{L^r_k}, \quad \|p\|_{L^r_{k-1}} + \|dp\|_{L^r_k},$$

respectively. They are closed subspaces of the reflexive Banach space

$$W^{1,r}(\Omega) \hookrightarrow L^r(\Omega) \times L^r(\Omega)^n,$$

and therefore, reflexive ([5] p.45).

Remark 56. We obtain $V_1^r = W^{1,r}(\Omega)$ and $W^{1,r}_{n-1} \cong W_0^{1,r}(\Omega)$.

Theorem 30 (L^r-theory). *It holds that*

$$L^r_k = H_k \oplus \delta W_k^r \oplus dV_k^r, \quad 1 < r < \infty,\ 1 \le k \le n-1. \tag{111}$$

Theorem 31 (Structure theorem). *Any $u \in L^r_k$ admits $(h, \eta) \in H_k \times (W_k^{2,r} \cap H_k^{\perp})$ such that*

$$u = h + \delta d\eta + d\delta\eta, \quad \nu \wedge *\eta\big|_{\partial\Omega} = 0, \quad \nu \wedge *d\eta\big|_{\partial\Omega} = 0 \tag{112}$$

in Theorem 30. Hence there arises that

$$(h, d\eta, \delta\eta) \in H_k \times W_k^r \times V_k^r. \tag{113}$$

It holds, furthermore, that

$$\|h\|_{L^r_k} + \|\eta\|_{W_k^{2,r}} \le C\|u\|_{L^r_k}. \tag{114}$$

Remark 57. Equality (112) and inequality (114) ensure the direct sum of the vector spaces in (111). Then Theorem 31 implies Theorem 30 because the tangential differentiation, Lemma 3, is valid even to $\theta \in W^{1,r}_{k,\delta}$.

Remark 58. Theorem 30 for $n = 3$, $k = 1, 2$, is the result obtained by [12].

Remark 59. The decomposition (111) with (110) implies the Helmholtz decomposition in L^r_k for $1 < r < \infty$, $1 \le k \le n-1$. First, any $u \in L^r_k$ admits $(v, p) \in L^r_k \times V_k^r$ such that

$$u = v + dp, \quad \delta v = 0, \quad \nu \wedge *v\big|_{\partial\Omega} = 0.$$

This result for $n = 3$, $k = 1, 2$ is due to [9]. Second, any $u \in L^r_k$ admits $(z, \omega) \in L^r_k \times W^r_k$ such that

$$u = z + \delta\omega, \quad dz = 0\,. \tag{115}$$

If

$$u \in W^{1,r}_{k,\delta}, \quad \nu \wedge *u|_{\partial\Omega} = 0\,,$$

furthermore, it holds that

$$z \in W^{1,r}_{k,\delta}, \quad \nu \wedge *z|_{\partial\Omega} = 0\,.$$

Theorem 31 is obtained by Theorem 4 in §3.3 and an a priori estimate. To derive it, we use the decomposition (18) to $u \in \Lambda_k$,

$$u = h + \delta d\eta + d\delta\eta, \quad \nu \wedge *\eta|_{\partial\Omega} = 0,\ \ \nu \wedge *d\eta|_{\partial\Omega} = 0\,. \tag{116}$$

with

$$(h, \eta) \in H_k \times (\Lambda_k \cap H_k^{\perp})\,. \tag{117}$$

First, we show the following lemma.

LEMMA 32. *Each $1 < r < \infty$ admits*

$$\|h\|_{L^r_k} \le C\|u\|_{L^r_k}\,. \tag{118}$$

PROOF. The case $r = 2$ is obvious because the decomposition (116) is L^2-orthogonal:

$$\|u\|^2_{L^2_k} = \|h\|^2_{L^2_k} + \|\delta d\eta\|^2_{L^2_k} + \|d\delta\eta\|^2_{L^2_k}\,.$$

We thus obtain

$$\|h\|_{L^2_k} \le \|u\|_{L^2_k}\,.$$

Since $\dim H_k = b_k$ is finite, we have the equivalence of the norms, $\|h\|_{L^2_k} \approx \|h\|_{L^r_k}$ for $1 < r < \infty$, and therefore, if $r > 2$ we obtain (118) as in

$$\|h\|_{L^r_k} \approx \|h\|_{L^2_k} \le \|u\|_{L^2_k} \le C\|u\|_{L^r_k}\,.$$

For $1 < r < 2$ we use $2 < r' < \infty$ in (106). Given $u_1, u_2 \in \Lambda_k$, we take

$$(h_1, \omega_1, p_1),\ (h_2, \omega_2, p_2) \in H_k \times W_k \times V_k \tag{119}$$

satisfying

$$u_1 = h_1 + \delta\omega_1 + dp_1, \quad u_2 = h_2 + \delta\omega_2 + dp_2\,. \tag{120}$$

Since

$$\|h_2\|_{L^{r'}_k} \le C\|u_2\|_{L^{r'}_k}$$

by $2 < r' < \infty$, we obtain

$$\begin{aligned}\|h_1\|_{L^r_k} &= \sup_{\|u_2\|_{L^{r'}_k}=1} (h_1, u_2) = \sup_{\|u_2\|_{L^{r'}_k}=1} (h_1, h_2) \le C \sup_{\|h_2\|_{L^{r'}_k}=1} (h_1, h_2) \\ &= C \sup_{\|h_2\|_{L^{r'}_k}=1} (u_1, h_2) \le C\|u_1\|_{L^r_k}\,,\end{aligned}$$

by Proposition 19 in §3.1, and hence (118). □

Second, we show the following lemma.

LEMMA 33. *Each* $1 < r < \infty$ *admits*

$$\|\eta\|_{W^{2,r}_k} \le C\|u\|_{L^r_k}\,. \tag{121}$$

PROOF. Since

$$-\Delta\eta = u - h, \quad \nu \wedge *\eta|_{\partial\Omega} = 0, \quad \nu \wedge *d\eta|_{\partial\Omega} = 0$$

with $\eta \in \Lambda_k$, inequality (105) holds as

$$\|\eta\|_{W^{2,r}_k} \le C_1(\|u - h\|_{L^r_k} + \|\eta\|_{L^r_k}) \le C_2(\|u\|_{L^r_k} + \|\eta\|_{L^r_k})\,. \tag{122}$$

If $r = 2$, we have (102),

$$\|\eta\|_{L^2_k} \le C_3\|v\|_{L^2_k}$$

and hence (121).

If $2 < r < \infty$, we apply the interpolation inequality ([1] p.139) to obtain

$$\begin{aligned}\|\eta\|_{L^r_k} &\le \varepsilon\|\eta\|_{W^{2,r}_k} + C_{4,\varepsilon}\|\eta\|_{L^2_k} \\ &\le \varepsilon\|\eta\|_{W^{2,r}_k} + C_3 \cdot C_{4,\varepsilon}\|u\|_{L^2_k} \\ &\le \varepsilon\|\eta\|_{W^{2,r}_k} + C_{5.\varepsilon}\|u\|_{L^r_k}\end{aligned}$$

for any $\varepsilon > 0$. Thus it follows that (121).

If $1 < r < 2$ we define $2 < r' < \infty$ by (106). Given $u_1, u_2 \in \Lambda_k$, we take (h_1, η_1), (h_2, η_2), (ω_1, p_1), and (ω_2, p_2) as in (116)-(117) and (119)-(120);

$$\begin{aligned}u_1 &= h_1 + \delta\omega_1 + dp_1, \quad \omega_1 = d\eta_1,\ p_1 = \delta\eta_1 \\ u_2 &= h_2 + \delta\omega_2 + dp_2, \quad \omega_2 = d\eta_2,\ p_2 = \delta\eta_2\,.\end{aligned}$$

It holds that

$$\|\delta\omega_2\|_{L^{r'}_k} \le C_6\|\eta_2\|_{W^{2,r'}_k} \le C_7\|u_2\|_{L^{r'}_k}$$

by $2 < r' < \infty$. Then we obtain

$$\begin{aligned}\|\delta\omega_1\|_{L^r_k} &= \sup_{\|u_2\|_{L^{r'}_k}=1} (\delta\omega_1, u_2) = \sup_{\|u_2\|_{L^{r'}_k}=1} (\delta\omega_1, \delta\omega_2) \\ &\le C_7 \sup_{\|\delta\omega_2\|_{L^{r'}_k}=1} (\delta\omega_1, \delta\omega_2) \\ &= C_7 \sup_{\|\delta\omega_2\|_{L^{r'}_k}=1} (u_1, \delta\omega_2) \le C_7\|u_1\|_r\end{aligned}$$

by Proposition 19, and similarly, it holds that

$$\|dp_1\|_{L^r_k} \le C_8\|u_1\|_{L^r_k}\,.$$

It thus follows that

$$\|h\|_{L^r_k} + \|\delta d\eta\|_{L^r_k} + \|d\delta\eta\|_{L^r_k} \le C_9\|u\|_{L^r_k} \tag{123}$$

for any $1 < r < \infty$ in (116)-(117).

Here, we have

$$\theta = d\eta \in \breve{\Lambda}_{k+1} \cap H^\perp_{k+1}\,.$$

In fact, $\theta \in \breve{\Lambda}_{k+1}$ is obvious, while $\theta \in H^\perp_{k+1}$ follows from

$$\int_\Omega (h, d\eta)\,dx = \int_\Omega (\delta h, \eta) + (-1)^k \int_{\partial\Omega} (\nu \wedge *h, *\eta)\,ds = 0, \quad \forall h \in H_{k+1}\,.$$

Hence it holds that

$$\|d\eta\|_{L^r_{k+1}} = \|\theta\|_{L^r_{k+1}} \le C_{10}\|\delta\theta\|_{L^r_k} = C_{10}\|\delta d\eta\|_{L^r_k} \tag{124}$$

by (109) with $C = C_{10}$.

Similarly, we obtain

$$\delta\eta \in \breve{\Lambda}_{k-1} \cap H^\perp_{k-1}$$

by $\nu \wedge *\delta\eta|_{\partial\Omega} = 0$ and

$$\int_\Omega (h, \delta\eta)\,dx = \int_\Omega (dh, \eta)\,dx - (-1)^{k-1} \int_\Omega (*h, \nu \wedge *\eta)\,ds = 0, \quad \forall h \in H_{k-1}\,.$$

It thus holds that

$$\|\delta\eta\|_{L^r_{k-1}} \le C_{10}\|d\delta\eta\|_{L^r_k}\,. \tag{125}$$

Finally, we have $\eta \in \breve{\Lambda}_k \cap H^\perp_k$ and hence

$$\|\eta\|_{L^r_k} \le C_{10}(\|d\eta\|_{L^r_{k+1}} + \|\delta\eta\|_{L^r_{k-1}})\,. \tag{126}$$

Inequalities (123)-(126) imply

$$\|\eta\|_{L^r_k} \le C_{11}\|u\|_{L^r_k}\,,$$

and then we obtain the result by (122). □

REMARK 60. In the above proof, we use the following property: if $\eta \in \Lambda_k \cap H_k^\perp$ satisfies

$$\nu \wedge *\eta|_{\partial\Omega} = 0, \quad \nu \wedge *d\eta|_{\partial\Omega} = 0\,,$$

it holds that

$$p = \delta\eta \in \breve{\Lambda}_{k-1} \cap H_{k-1}^\perp, \quad \omega = d\eta \in \breve{\Lambda}_{k+1} \cap H_{k+1}^\perp\,.$$

We are able to give the following proof.

PROOF OF THEOREM 30: Any $u \in L_k^r$ admits $u^j \in \Lambda_k$ such that

$$\lim_{j\to\infty} \|u^j - u\|_{L_k^r} = 0\,.$$

Then we obtain $(h^j, \eta^j) \in H_k \times (\Lambda_k \cap H_k^\perp)$ such that

$$u^j = h^j + d\delta\eta^j + \delta d\eta^j, \quad \nu \wedge *\eta^j\big|_{\partial\Omega} = 0, \quad \nu \wedge *d\eta^j\big|_{\partial\Omega} = 0 \tag{127}$$

and

$$\|h^j\|_{L_k^r} + \|\eta^j\|_{W_k^{2,r}} \le C\|u^j\|_{L_k^r}$$

by Theorem 4, Lemma 32, and Lemma 33. Passing to a subsequence, we get $(h, \eta) \in H_k \times (W_k^{2,r} \cap H_k^\perp)$ such that

$$h^j \to h \text{ in } H_k, \quad \eta^j \rightharpoonup \eta \text{ in } W_k^{2,r} \tag{128}$$

by $\dim H_k < +\infty$. Equality (112) follows and the proof is complete. □

4.3. Compensated compactness lemma

Here we show a generalized form of Tartar's theorem [28] on compensated compactness, or div-curl lemma, following the argument of [14] for $n = 3$.

THEOREM 34 (div-curl lemma). *Let $1 < r, r' < \infty$ be as in (106) and put $r_1 = r$, $r_2 = r'$. Let $u_i^j \in L_k^{r_i}$, $i = 1, 2$, $j = 1, 2, \cdots$, be such that*

$$u_i^j \rightharpoonup u_i \text{ in } L_k^{r_i}, \quad i = 1, 2$$

with $u_1^j \in W_{k,d}^{1,r_1}$, $u_2^j \in W_{k,\delta}^{1,r_2}$, and

$$\|du_1^j\|_{L_{k+1}^{t_1}} + \|\delta u_2^j\|_{L_{k-1}^{t_2}} \le C, \quad \exists t_i > \frac{nr_i}{n + r_i},\ i = 1, 2\,.$$

Then it holds that

$$(u_1^j, u_2^j) \to (u_1, u_2) \quad \text{in } D_0'\,.$$

We begin with the following lemma.

LEMMA 35 (Orthogonality). *If $1 < r, r' < \infty$ satisfies (106), it holds that*

$$\int_\Omega (u, u')\ dx = \int_\Omega (h, h') + (\delta\omega, \delta\omega') + (dp, dp')\ dx, \quad (u, u') \in L^r_k \times L^{r'}_k\,, \tag{129}$$

where

$$\begin{aligned} &u = h + \delta\omega + dp, \qquad (h, \omega, p) \in H_k \times W^r_k \times V^r_k \\ &u' = h' + \delta\omega' + dp', \quad (h', \omega', p') \in H_k \times W^{r'}_k \times V^{r'}_k \end{aligned}$$

are the decompositions subject to (111).

PROOF. First, we have (129) for $(u, u') \in \Lambda_k \times \Lambda_k$ by Proposition 19. Second, we fix $u' \in \Lambda_k$ and approximate u in L^r_k by $u^j \in \Lambda_k$, $j = 1, 2, \cdots$, recalling Remark 54 in §4.2. Then Lemma 32 and Lemma 33 admit the limit (128) with the decomposition (127), which ensures (129) to $(u, u') \in L^r_k \times \Lambda_k$. We finally take this process of approximation to u'-component and get the result for $(u, u') \in L^r_k \times L^{r'}_k$. □

Here is the key lemma for the proof of Theorem 34.

LEMMA 36. *Let $u \in L^r_k$, $1 < r < \infty$, $1 \le k \le n-1$, be given and*

$$u = h + \delta\omega + dp, \quad (h, \omega, p) \in H_k \times W^r_k \times V^r_k \tag{130}$$

Let, furthermore, $1 < t < r$ and assume $u \in W^{1,t}_{k,d} \cap L^r_k$. Then it holds that

$$\|\delta\omega\|_{W^{1,t}_k} \le C \left(\|du\|_{L^t_{k+1}} + \|u\|_{L^r_k} \right)\,. \tag{131}$$

Similarly, if $u \in W^{1,t}_{k,\delta} \cap L^r_k$ it holds that

$$\|dp\|_{W^{1,t}_k} \le C \left(\|\delta u\|_{L^t_{k-1}} + \|u\|_{L^r_k} + \|\nu \wedge {*u}\|_{W^{1-1/t,t}_{n-k+1}(\partial\Omega)} \right)\,. \tag{132}$$

PROOF. It suffices to assume $u \in \Lambda_k$ to derive these inequalities. We may suppose also

$$(\omega, p) = (d\eta, \delta\eta)$$

for $\eta \in \Lambda_k \cap H^\perp_k$ satisfying (112). Then Lemma 33 ensures

$$\|\eta\|_{W^{2,r}_k} \le C_1 \|u\|_{L^r_k}\,,$$

which results in

$$\|\delta\omega\|_{L^r_k} + \|dp\|_{L^r_k} \le C_2 \|\eta\|_{W^{2,r}_k} \le C_1 C_2 \|u\|_{L^r_k}\,.$$

Since

$$(\omega, p) \in (\breve{\Lambda}_{k+1} \cap H^\perp_{k+1}) \times (\breve{\Lambda}_{k-1} \cap H^\perp_{k-1})$$

holds by Remark 60 in §4.2, we apply Poincaré's inequality (96), to obtain

$$\|\omega\|_{L^r_{k+1}} + \|p\|_{L^r_{k-1}} \le C_3 (\|\delta\omega\|_{L^r_k} + \|dp\|_{L^r_k}) \le C_1 C_2 C_3 \|u\|_{L^r_k}\,. \tag{133}$$

By (130), on the other hand, it holds that

$$
\begin{aligned}
&-\Delta\omega = d\delta\omega = du, \quad \nu\wedge *\omega|_{\partial\Omega} = 0, \quad \nu\wedge *d\omega|_{\partial\Omega} = 0\\
&-\Delta p = \delta dp = \delta u, \quad \nu\wedge *p|_{\partial\Omega} = 0, \quad \nu\wedge *dp|_{\partial\Omega} = \nu\wedge *u|_{\partial\Omega}\ ,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\|\omega\|_{W^{2,t}_{k+1}} \le C_4\left(\|du\|_{L^t_{k+1}} + \|\omega\|_{L^r_{k+1}}\right)\\
&\|p\|_{W^{2,t}_{k-1}} \le C_4\left(\|\delta u\|_{L^t_{k-1}} + \|p\|_{L^r_{k-1}} + \|\nu\wedge *u\|_{W^{1-1/t,t}_{n-k+1}(\partial\Omega)}\right)
\end{aligned}
\tag{134}
$$

by the elliptic estimate, Proposition 25 in the previous subsection, and $t < r$. Then inequalities (131)-(132) follow from (133)-(134). □

LEMMA 37. *Under the assumption of Theorem 34, suppose, furthermore,*

$$\|\nu\wedge *u_2^j\|_{W^{1-1/t_2,t_2}_{n-k+1}(\partial\Omega)} \le C\,.$$

Then it holds that

$$\lim_{j\to\infty}\int_\Omega (u_1^j, u_2^j)\ dx = \int_\Omega (u_1, u_2)\ dx\,.$$

PROOF. We have

$$u_i^j = h_i^j + \delta\omega_i^j + dp_i^j,\ u_i = h_i + \delta\omega_i + dp_i, \quad i = 1,2$$

with $(\omega_i^j, p_i^j) = (d\eta_i^j, \delta\eta_i^j)$ and

$$\|h_i^j\|_{L_k} + \|\eta_i^j\|_{W^{2,r_i}_k} \le C, \quad i = 1,2\,.$$

Therefore, we may assume

$$(h_i^j, \delta\omega_i^j, dp_i^j) \rightharpoonup (h_i, \delta\omega_i, dp_i) \quad \text{in } L^{r_i}, \quad i = 1,2\,. \tag{135}$$

We may assume also

$$\frac{nr_i}{n+r_i} < t_i < r_i, \quad i = 1,2$$

and hence it follows that

$$\|\delta\omega_1^j\|_{W^{1,t_1}_k} + \|dp_2^j\|_{W^{1,t_2}_k} \le C$$

by the previous lemma.

From Rellich-Kondrachov's theorem ([5] p.169), therefore, it follows that

$$\delta\omega_1^j \to \delta\omega_1 \text{ in } L^{r_1}_k, \quad dp_2^j \to dp_2 \text{ in } L^{r_2}_k\,, \tag{136}$$

while

$$h_1^j \to h_1,\ h_2^j \to h_2 \quad \text{in } H_k \tag{137}$$

holds because $\dim H_k < +\infty$. Using (129), we obtain

$$\int_\Omega (u_1^j, u_2^j)\ dx = \int_\Omega (h_1^j, h_2^j) + (\delta\omega_1^j, \delta\omega_2^j) + (dp_1^j, dp_2^j)\ dx$$
$$\to \int_\Omega (h_1, h_2) + (\delta\omega_1, \delta\omega_2) + (dp_1, dp_2)\ dx = \int_\Omega (u_1, u_2)\ dx$$

by (135)-(137). □

We now complete the following proof.

PROOF OF THEOREM 34: By Remark 54, the property $\varphi\omega \in \check{W}^{1,r}_{k,\delta}$ arises for any $(\varphi, \omega) \in C_0^\infty(\Omega) \times W^{1,r}_{k,\delta}$. In this theorem, therefore, it follows that

$$\tilde{u}_2^j \equiv \varphi u_2^j \in \check{W}^{1,r_2}_{k,\delta}\,.$$

Then we obtain

$$\lim_{j\to\infty} \int_\Omega \varphi(u_1^j, u_2^j)\ dx = \lim_{j\to\infty} \int_\Omega (u_1^j, \tilde{u}_2^j)\ dx = \int_\Omega (u_1, \varphi u_2)\ dx = \int_\Omega (u_1, u_2)\ dx$$

by the previous lemma, and the result follows. □

4.4. $W^{1,r}$-regularity of the Poisson equation

The argument in this subsection is associated with an abstract theorem stated below. See [20] for the proof.

Let Y, Z be Banach spaces over $\mathbf{R}$, and let Z be reflexive. Let, furthermore,

$$a = a(y, z) : Y \times Z \to \mathbf{R}$$

be a bounded bilinear form.

THEOREM 38 (Banach-Nečas-Babuška). *The following three conditions are equivalent.*

1. *Any $\ell \in Z'$ admits a unique $y \in Y$ such that*
$$a(y, z) = \langle z, \ell\rangle, \quad \forall z \in Z\,.$$
2. *It holds that*
$$\inf_{y\in Y\setminus\{0\}} \sup_{z\in Z\setminus\{0\}} \frac{a(y,z)}{\|y\|_Y\|z\|_Z} = b > 0 \tag{138}$$
and furthermore, if $z \in Z$ satisfies $a(y, z) = 0$ for any $y \in Y$, then it holds that $z = 0$.
3. *It holds that (138) and*
$$\inf_{z\in Z\setminus\{0\}} \sup_{y\in Y\setminus\{0\}} \frac{a(y,z)}{\|y\|_Y\|z\|_Z} = B > 0\,. \tag{139}$$

If inequalities (138) and (139) are valid, then it holds that $b = B$. It holds, furthermore, that

$$\|y\|_Y \le C\|\ell\|_{Z'},\ C = B^{-1} = b^{-1}\,.$$

REMARK 61. Inequalities (138)-(139) are called Babuška-Brezzi-Kikuchi's inequality.

We show above type of inequalities to $d-\delta$ systems, where $1<r,r'<\infty$ are dual exponents defined by (106). Recall the Banach spaces $W_k^{1,r}$ and W_k^r, V_k^r, defined by Definition 12 and Definition 16 in §4.2, respectively.

THEOREM 39. *Let*

$$\tilde{W}_k^r = W_k^r \cap H_{k+1}^\perp, \quad \tilde{V}_k = V_k^r \cap H_{k-1}^\perp \,. \tag{140}$$

Then it holds that

$$\inf_{\omega\in\tilde{W}_k^r\setminus\{0\}} \sup_{\theta\in\tilde{W}_k^{r'}\setminus\{0\}} \frac{\int_\Omega(\delta\omega,\delta\theta)\ dx}{\|\omega\|_{W_k^r}\|\theta\|_{W_k^{r'}}} > 0 \tag{141}$$

$$\inf_{q\in\tilde{V}_k^r\setminus\{0\}} \sup_{q\in\tilde{V}_k^{r'}\setminus\{0\}} \frac{\int_\Omega(dp,dq)\ dx}{\|p\|_{V_k^r}\|q\|_{V_k^{r'}}} > 0 \tag{142}$$

for $1<r,r'<\infty$ in (106), where $1\le k\le n-1$,

REMARK 62. By Theorem 38, Theorem 39 is equivalent to the following facts.

1. Any $f\in\tilde{W}_k^r$ admits a unique $\omega\in\tilde{W}_k^r$ such that
$$\int_\Omega(\delta\omega,\delta\theta)\ dx = \int_\Omega(f,\delta\omega), \quad \forall\theta\in\tilde{W}_k^{1,r'}\,.$$
2. Any $g\in\tilde{V}_k^r$ admits a unique $p\in\tilde{V}_k^r$ such that
$$\int_\Omega(dp,dq)\ dx = \int_\Omega(g,dq)\ dx, \quad \forall q\in\tilde{V}_k^{1,r'}\,.$$

REMARK 63. By classical Poincaré-Wirtinger's inequality

$$\|f\|_r \le C\|\nabla f\|_r,\ f\in\dot{W}^{1,r}, \quad \dot{W}^{1,r}=\{f\in W^{1,r}(\Omega)\mid\int_\Omega f\ dx=0\}\,,$$

inequality (142) for $k=1$ is equivalent to

$$\inf_{f\in\dot{W}^{1,r}\setminus\{0\}} \sup_{g\in\dot{W}^{1,r'}\setminus\{0\}} \frac{\int_\Omega\nabla f\cdot\nabla g\ dx}{\|\nabla f\|_r\|\nabla g\|_{r'}} > 0\,. \tag{143}$$

By Theorem 38, on the other hand, there arises the unique solvability of

$$-\Delta u = \text{div } f \text{ in } \Omega, \quad \frac{\partial u}{\partial\nu} = \nu\cdot f \text{ on } \partial\Omega, \quad \int_\Omega u\ dx = 0\,,$$

which means that any $f\in L^r(\Omega)^n$ admits a unique $u\in W^{1,r}(\Omega)$ such that

$$\int_\Omega u\ dx = 0, \quad \int_\Omega\nabla u\cdot\nabla v\ dx = \int_\Omega f\cdot\nabla v\ dx, \quad \forall v\in W^{1,r'}(\Omega)\,, \tag{144}$$

and it holds that

$$\|u\|_{W^{1,r}} \le C\|f\|_r\,. \tag{145}$$

In particular, (142) implies the unique solvability of (144) with (145). This is the result obtained by [23]. Similarly, by the Poincaré inequality

$$\|f\|_r \le \|\nabla f\|_{W^{1,r}}, \quad f \in W_0^{1,r} = W_0^{1,r}(\Omega)\,,$$

inequality (141) for $k = n-1$ is equivalent to

$$\inf_{f\in W_0^{1,r}\setminus\{0\}} \sup_{g\in W_0^{1,r'}\setminus\{0\}} \frac{\int_\Omega \nabla f\cdot\nabla g\ dx}{\|\nabla f\|_r\|\nabla g\|_{r'}} > 0\,,$$

and therefore, any $f \in L^r(\Omega)^n$ admits a unique $u \in W^{1,r}(\Omega)$ such that

$$-\Delta u = \text{div } f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0\,.$$

It holds also that (145). This result is due to [22]. See also [6] and the references therein.

By Theorem 31, any $u \in L^r_k$ admits $(h,\eta) \in H_k \times (W^{2,r}_k \cap H^\perp_k)$ such that (112)-(114). Then we obtain the following fact, recalling (140), where $(\omega, p) = (d\eta, \delta\eta)$.

LEMMA 40. *It holds that*

$$\omega \in \tilde W^r_k, \quad \int_\Omega (\delta\omega, \delta\theta)\ dx = \int_\Omega (u, \delta\theta)\ dx, \quad \forall\theta \in \tilde W^{r'}_k \tag{146}$$

$$p \in \tilde V^r_k, \quad \int_\Omega (dp, dq)\ dx = \int_\Omega (u, dq)\ dx, \quad \forall q \in \tilde V^{r'}_k\,. \tag{147}$$

PROOF. We have readily confirmed that $(\omega, p) \in \tilde W^r_k \times \tilde V^r_k$. We have also

$$(h^j, \eta^j) \in H_k \times (\Lambda_k \cap H^\perp_k), \quad j = 1, 2, \cdots,$$

such that

$$\nu\wedge\eta^j\big|_{\partial\Omega} = 0,\ \ \nu\wedge *d\eta^j\big|_{\partial\Omega} = 0, \quad h^j \to h \text{ in } H_k, \quad \eta^j \rightharpoonup \eta \text{ in } W^{2,r}_k\,.$$

Let

$$(\omega^j, p^j) = (d\eta^j, \delta\eta^j)\,.$$

Given $q \in \breve\Lambda_{k-1} \cap H^\perp_{k-1}$ we have

$$\int_\Omega (h + \delta\omega, dq)\ dx = \lim_{j\to\infty} \int_\Omega (h^j + \delta\omega^j, dq)\ dx$$

and

$$\int_\Omega (h^j + \delta\omega^j, dq)\ dx = \int_\Omega (\delta h^j, q) + (\omega^j, d^2 q)\ dx$$
$$+(-1)^{k-1}\int_{\partial\Omega} (\nu\wedge *h^j, *q) - (\nu\wedge *\omega^j, *dq)\ ds = 0\,,$$

which results in

$$\int_\Omega (h + \delta\omega, dq)\ dx = 0\,,$$

or

$$\int_\Omega (dp, dq)\ dx = \int_\Omega (u, dq)\ dx \tag{148}$$

for any $q \in \breve{\Lambda}_k \cap H_k^\perp$, and hence for any $q \in \breve{\Lambda}_k$. Equatlity (148) thus holds for any $q \in \breve{W}_k^{1,r}$, and therefore, for any $q \in V_k^r$. We thus obtain (147). The proof of (146) is similar. □

We are ready to give the following proof.

Proof of Theorem 39: We show the uniqueness of p in (147) to apply Theorem 38 because the existence of such p is confirmed in Lemma 40. We thus assume $u = 0$ to derive $p = 0$. In fact, then we have

$$\int_\Omega (dp, dq)\ dx = 0, \quad \forall q \in \breve{\Lambda}_{k-1} \cap H_{k-1}^\perp$$

and hence

$$\int_\Omega (dp, dq)\ dx = 0, \quad \forall q \in \breve{\Lambda}_{k-1}\,.$$

If $r \geq 2$, we have

$$p \in \breve{W}_{k-1,d\delta}^{1,r} \subset \breve{H}_{k-1,d\delta}^{1}\,,$$

and therefore, there is a sequence $q^j \in \breve{\Lambda}_{k-1}$ such that $q^j \to p \in H_{k-1}^1$. Making $j \to \infty$ in

$$\int_\Omega (dp, dq^j)\ dx = 0\,,$$

we obtain

$$\int_\Omega (dp, dp)\ dx = 0\,,$$

or $dp = 0$, while

$$\delta p = 0, \quad \nu \wedge *p\big|_{\partial\Omega} = 0$$

follows from $p \in \tilde{V}_k^r$. We thus obtain $p = 0$ by $p \in H_k \cap H_k^\perp$, and the solution p to (147) is unique for this case of $r \geq 2$.

By Theorem 38, therefore, each $r \geq 2$ admits

$$\inf_{p \in \tilde{V}_k^r \setminus \{0\}} \sup_{q \in \tilde{V}_k^{r'} \setminus \{0\}} \frac{\int_\Omega (dp, dq)\ dx}{\|p\|_{V_k^r} \|q\|_{V_k^{r'}}} > 0$$

$$\inf_{q \in \tilde{V}_k^{r'} \setminus \{0\}} \sup_{p \in \tilde{V}_k^{r} \setminus \{0\}} \frac{\int_\Omega (dp, dq)\ dx}{\|p\|_{V_k^r} \|q\|_{V_k^{r'}}} > 0\,.$$

The second inequality for $r \geq 2$ means (142) for $1 < r \leq 2$ and consequently, inequality (142) holds for any $1 < r < \infty$. Inequality (141) for $1 < r < \infty$ is proven simiarly. $\square$

5. Concluding remarks

This section is devoted to the concluding remarks, divided into three subsections. We describe the spectral property of $d - \delta$ operators, derivation of Hodge decomposition via Helmholtz decomposition, and a numerical scheme for the Helmholtz decomposition.

5.1. Spectral property of $d - \delta$ operators

Here we review the previous argument from the view point of the operator theory. Three topics are dealt with; L^2 theory, L^r theory, and d-operator in L^2-space for $n = 3$.

To state the L^2-theory, first, let

$$X = L^2_k, \quad V = \breve{H}^1_{k,d\delta} \subset H^1_k, \quad 1 \leq k \leq n-1\,.$$

The inclusion $V \hookrightarrow X$ is continuous, compact, and dense. The symmetric bilinear form

$$a(\theta_1, \theta_2) = \int_\Omega (d\theta_1, d\theta_2) + (\delta\theta_1, \delta\theta_2)\, dx, \quad \theta_1, \theta_2 \in V$$

is coercive on $V_0 \times V_0$ for $V_0 = V \cap H^\perp_k$, by Lemma 21 in §3.3. Since $\dim H_k < \infty$, we have $c, C > 0$ such that

$$a(\theta, \theta) \geq c\|\theta\|^2_V - C\|\theta\|^2_X, \quad \forall \theta \in V\,.$$

Then, there is an associated self-adjoint operator in X, denoted by $\hat{A}$, ([27] p.29). By the argument in §3.2, it holds that

$$D(\hat{A}) = \{u \in H^2_k \mid \nu \wedge *u|_{\partial\Omega} = 0,\ \nu \wedge *du|_{\partial\Omega} = 0\}$$
$$\hat{A}u = -\Delta u, \quad u \in D(\hat{A})\,.$$

REMARK 64. Above result is valid even to $k = 0$ with

$$D(\hat{A}) = \{u \in H^2(\Omega) \mid \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0\}$$

and $k = n$ with

$$D(\hat{A}) = \{u \in H^2(\Omega) \mid u|_{\partial\Omega} = 0\}\,.$$

For the L^r-theory, $1 < r < \infty$, second, we note that the complementing boundary condition holds also to the extended system,

$$(-\Delta - \partial_t^2)\theta = \alpha \text{ in } Q, \quad \nu \wedge *\theta = *\beta \text{ on } \partial\Omega \times (-\infty, +\infty)$$
$$\nu \wedge *d\theta = *\gamma \text{ on } \partial\Omega \times (-\infty, +\infty) \tag{149}$$

for $Q = \Omega \times (-\infty, +\infty)$ as in the proof of Theorem 16 in §2.4 by replacing n by $n+1$. Then inequality (105) in §4.2 is refined as

$$\begin{aligned} \|\theta\|_{W_k^{2+s,r}} \le C \Big(& \|(-\Delta+\lambda)\theta\|_{W_k^{s,r}} + \|\nu \wedge *\theta\|_{W_{n-k+1}^{s+3/2,r}(\partial\Omega)} \\ & + \|\nu \wedge *d\theta\|_{W_{n-k}^{s+1,r}(\partial\Omega)} \Big) \end{aligned} \tag{150}$$

for $\lambda > 0$ sufficiently large ([2], see also [27] pp.82-87). Let T be a densely-defined operator in $X = L_k^r$ defined by

$$\begin{aligned} & D(T) = \{\theta \in W_k^{2,r} \mid \nu \wedge *\theta|_{\partial\Omega} = 0,\ \nu \wedge *d\theta|_{\partial\Omega} = 0\} \\ & T\theta = (-\Delta+\lambda)\theta, \quad \theta \in D(T)\,. \end{aligned}$$

It is a closed operator in X and the kernel and the range denoted by $N(T)$ and $R(T)$, respectively, satisfy

$$N(T) = \{0\}, \quad \overline{R(T)} = R(T)$$

by (150). Let $1 < r' < \infty$ be the dual exponent of r defined by (106) and assume that $\omega \in L_k^{r'}$ satisfies

$$\int_\Omega (T\theta, \omega)\, dx = 0, \quad \forall \theta \in D(T)\,.$$

Then, since Proposition 23 in §3.3 ensures $\Lambda_k \subset R(T)$, it follows that $\omega = 0$. We thus obtain $R(T)^\perp = \{0\}$ and hence $R(T) = \overline{R(T)} = X$. Thus T is invertible and T^{-1} is a bounded linear opertor on X.

Remark 65. Taking $-\Delta - e^{\sqrt{-1}\phi}\partial_t^2$, $|\phi| < \frac{\pi}{2}$ in (149) for $-\Delta - \partial_t^2$, we can show that T, and hence $\widehat{A}$, is a generator of an analytic semigroup in $X = L_k^r$ ([27] p.88).

If $n = 3$, finally, $A = d$ is realized as a self-adjoint operator in

$$X = \{\delta\omega \mid \omega \in H^1_{2,d\delta},\ d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0\} \subset L_1^2$$

with the domain

$$D(A) = \{p \in X \mid dp \in L_2^2\} \subset L_1^2\,.$$

This result is due to [29] Theorem 1, where the identification

$$dp \in L_2^2 \cong L_1^2, \quad p \in D(A)$$

is adopted using $n = 3$.

5.2. Hodge decomposition via Helmholtz decomposition

Here we examine the method of [12] for the proof of Theorem 1 for $n = 3$, and confirm that it is efficient up to $n = 4$. For this purpose we introduce the following notation.

Notation 4. We put $(H)_k$, $(Hd)_k$, and $(H\delta)_k$ as follows.

1. The Hodge decomposition (3) is denoted by $(H)_k$, $1 \le k \le n-1$,
2. The Helmholz decompositions (11), $0 \le k \le n-1$, and (12), $1 \le k \le n$, are denoted by $(Hd)_k$ and $(H\delta)_k$, respectively.

We have readily confirmed the following fact in §1.

PROPOSITION 41. The property $(H)_k$, $1 \le k \le n-1$, implies $(Hd)_k$ and $(H\delta)_k$.

Above properties, $(H)_k$, $(Hd)_k$, $(H\delta)_k$, are, actually, valid to any $1 \le k \le n-1$ by Theorem 1. Besides, the decompositions $(Hd)_0$ and $(H\delta)_n$ are obvious, under the agreement of $\Lambda_{-1} = \Lambda_{n+1} = \emptyset$.

The properties $(H\delta)_{n-1}$ and $(Hd)_1$, however, are proven directly. Thus we give an alternative proof to them.

LEMMA 42. *It holds that* $(H\delta)_{n-1}$ *and* $(Hd)_1$.

PROOF. These cases are associated with the Poisson equation treated in Remark 63 in §4.2. First, we cofirm $(H\delta)_{n-1}$. Given

$$\begin{aligned} u = {} & u^1 dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n + u^2 dx_3 \wedge \cdots \wedge dx_n \wedge dx_1 \\ & + \cdots + u^n dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1} \in \Lambda_{n-1}\,, \end{aligned} \tag{151}$$

such that

$$\nu \wedge *u|_{\partial\Omega} = 0\,, \tag{152}$$

we find $(\alpha, \omega) \in \Lambda_{n-1} \times \Lambda_n$ such that

$$u = \alpha + \delta\omega,\ d\alpha = 0, \quad \nu \wedge *\alpha|_{\partial\Omega} = 0\,. \tag{153}$$

For this purpose it suffices to find $\omega \in \Lambda_n$ such that

$$d\delta\omega = du, \quad \nu \wedge *\omega|_{\partial\Omega} = 0\,. \tag{154}$$

Once such $\omega \in \Lambda_k$ is obtained, we see that $\alpha = u - \delta\omega$ satisfies $d\alpha = 0$ and also

$$\nu \wedge *\alpha = \nu \wedge *u - \nu \wedge *\delta\omega = 0 \text{ on } \partial\Omega$$

by Lemma 3 proven in §2.1. Then (153) follows.

Putting $z = *\omega \in \Lambda_0$, now, we see that (154) is equivalent to

$$-\Delta z = \operatorname{div} v, \quad z|_{\partial\Omega} = 0$$

for $v = *u$, which admits a unique solution. Thus we obtain the property $(H\delta)_{n-1}$ from the classical result on the Poisson equation.

To confirm $(Hd)_1$, second, given

$$u = u^1 dx_1 + \cdots + u^n dx_n \in \Lambda_1\,,$$

we find $p \in \Lambda_0$ satisfying

$$\delta dp = \delta u, \quad \nu \wedge *dp|_{\partial\Omega} = \nu \wedge *u|_{\partial\Omega} \,. \tag{155}$$

If such p is found, we define $w = u - dp$ to obtain

$$\delta w = 0, \quad \nu \wedge *w|_{\partial\Omega} = 0 \,,$$

which results in $(Hd)_1$.

In fact, if $u \in \Lambda_1$ in (151) is identified with the vector field $v = (u^i) \in X$, equation (155) means

$$\Delta p = \text{div } v, \quad \frac{\partial p}{\partial \nu}\Big|_{\partial\Omega} = \vec{\nu} \cdot v|_{\partial\Omega} \,, \tag{156}$$

of which solvability is assured by the well-known criterion

$$\int_\Omega -\text{div } v \, dx + \int_{\partial\Omega} \vec{\nu} \cdot v \, ds = 0 \,.$$

Hence $(Hd)_1$ arises again by the classical result on the Poisson equation. □

We have a counter part of Proposition 41 in the following form.

Proposition 43. The properties $(Hd)_{k-1}$ and $(H\delta)_{k+1}$ imply $(H)_k$ for $1 \le k \le n-1$.

Remark 66. We obtain an alternative proof of Theorem 1 for $n = 2, 3, 4$, using Proposition 41, Lemma 42, and Proposition 43. First, if $n = 2$, $(Hd)_0$ and $(H\delta)_2$ are obvious, and hence $(H)_1$ follows. Second, if $n = 3$, $(Hd)_0$ is obvious, while $(H\delta)_2$ holds by Lemm 42. Then we obtain $(H)_1$ by Proposition 43, which implies $(Hd)_1$ and $(H\delta)_1$ by Proposition 41. It thus follows that $(H)_2$ from $(Hd)_1$ and $(H\delta)_3$, the latter being obvious. If $n = 4$, finally, we have $(H)_2$ by $(Hd)_1$ and $(H\delta)_3$, and hence $(Hd)_2$ and $(H\delta)_2$. Then it holds that $(H)_1$ by $(Hd)_0$ and $(H\delta)_2$. Now $(H)_3$ is obtained by $(Hd)_2$ and $(H\delta)_4$. This approach is taken by [12] for $n = 3$, which, however, does not ensure Theorem 1 for $n \ge 5$.

The rest of this subsection is devoted to the proof of Proposition 43. It is derived from the following lemmas.

Lemma 44. *Let $u \in \Lambda_k$, $1 \le k \le n-1$. If $(Hd)_{k-1}$ holds, there is $p \in \Lambda_{k-1}$ such that*

$$\delta dp = \delta u, \ \delta p = 0, \quad \nu \wedge *p|_{\partial\Omega} = 0, \quad \nu \wedge *dp|_{\partial\Omega} = \nu \wedge *u|_{\partial\Omega} \,. \tag{157}$$

Lemma 45. *Let $u \in \Lambda_k$, $1 \le k \le n-1$. If $(H\delta)_{k+1}$ holds, there is $\omega \in \Lambda_{k+1}$ such that*

$$d\delta\omega = du, \ d\omega = 0, \quad \nu \wedge *\omega|_{\partial\Omega} = 0 \,. \tag{158}$$

Proof of Proposition 43: Once equalities (157)-(158) are obtained, it follows that $(\omega, p) \in W_k \times V_k$, and furthermore,

$$h = u - \delta\omega - dp$$

satisfies

$$dh = du - d\delta\omega = 0,\ \delta h = \delta u - \delta dp = 0 \ \text{ in } \Omega$$

as well as

$$\nu \wedge *h = \nu \wedge *(u - \delta\omega - dp) = \nu \wedge *(u - dp) = 0 \ \text{ on } \partial\Omega\,.$$

We thus obtain (18) with $(h, \omega, p) \in H_k \times W_k \times V_k$, to complete the proof of Proposition 43. □

For the proof of Lemma 44, we use Theorem 38 for $Y = Z = X_{k'}$, $k' = k - 1$, where

$$X_{k'} = [V_{k'+1} \cap H_{k'}^{\perp}]^{H^1_{k'}}\,.$$

This $X_{k'}$ is a closed subspace of $H^1_{k',d\delta}$ and is provided with the norm

$$\|p\|_{X_{k'}} = \left(\|dp\|^2_{L^2_{k'+1}} + \|p\|^2_{L^2_{k'}}\right)^{1/2}\,.$$

Babuška-Brezzi-Kikuchi's inequality, to be used in the proof of Lemma 44, is as follows. It is nothing but inequality (142) for $r = 2$ in Theorem 39, but is proven directly, without using Theorem 31 in §4.2.

LEMMA 46. *Any $0 \le k' \le n - 1$ admits*

$$\inf_{p \in X_{k'} \setminus \{0\}} \sup_{q \in X_{k'} \setminus \{0\}} \frac{\int_\Omega (dp, dq)\ dx}{\|p\|_{X_{k'}} \|q\|_{X_{k'}}} = b > 0\,. \tag{159}$$

PROOF. Given $p \in L^2_{k'}$, we write

$$\|p\|_2 = \|p\|_{L^2_{k'}}, \quad \|dp\|_2 = \|dp\|_{L^2_{k'+1}}$$

for simplicity. Since

$$\begin{aligned}\|dp\|_2\,(\|dp\|_2 + \|p\|_2) &\le \|dp\|_2^2 + \frac{1}{2}\,(\|dp\|_2^2 + \|p\|_2^2) \\ &= \frac{3}{2}\int_\Omega (dp, dp)\ dx + \frac{1}{2}\|p\|_2^2\,,\end{aligned}$$

and therefore, if $p \in X_{k'} \setminus \{0\}$ there arises that

$$\begin{aligned}\|dp\|_2 &\le \frac{3}{2}\frac{\int_\Omega (dp, dp)\ dx}{\|dp\|_2 + \|p\|_2} + \frac{1}{2}\frac{\|p\|_2^2}{\|dp\|_2 + \|p\|_2} \\ &\le \frac{3}{2}\sup_{q \in X_{k'} \setminus \{0\}} \frac{\int_\Omega (dp, dq)\ dx}{\|dq\|_2 + \|q\|_2} + \frac{1}{2}\|p\|_2\,.\end{aligned} \tag{160}$$

Now we show the existence of $C_1 > 0$ such that

$$\|p\|_2 \le C_1 \sup_{q \in X_{k'} \setminus \{0\}} \frac{\int_\Omega (dp, dq)\ dx}{\|dq\|_2 + \|q\|_2} \tag{161}$$

for any $p \in X_{k'}$. Then (159) follows from (160).

The proof of (161) is quite similar to Lemma 21 in §3.3. Assume, the contrary, the existence of $p_j \in X_{k'}$, $j = 1, 2, \cdots$, such that

$$\|p_j\|_2 = 1, \quad \varepsilon_j = \sup_{q \in X_{k'} \setminus \{0\}} \frac{\int_\Omega (dp_j, dq)\ dx}{\|dq\|_2 + \|q\|_2} \to 0\,.$$

Then we obtain

$$\|dp_j\|_2 \leq C_2$$

by (160), and therefore,

$$\|p_j\|_{H^1_{k'}} \leq C_3$$

by Theorem 13 in §2.3. Passing to a subsequence, we thus get

$$p_j \rightharpoonup p \text{ in } H^1_{k-1}, \quad p_j \to p \text{ in } L^2_{k-1}\,,$$

which ensures $\|p\|_2 = 1$.

It holds, therefore, that

$$\int_\Omega (dp, dq)\ dx = 0, \quad \forall q \in X_{k'},\ \|dq\|_2 + \|q\|_2 = 1$$

by

$$\varepsilon_j \geq \int_\Omega (dp_j, dq)\ dx\,,$$

and hence $dp = 0$. We thus obtain $p \in H_{k'}$, and therefore, $p = 0$ by $p \in H^\perp_{k'}$, which is impossible by $\|p\|_2 = 1$. □

Now we give the following proof.

Proof of Lemma 44: By Lemma 46 for $k' = k-1$ and Theorem 38 in §4.4, there is $p \in X_{k-1}$ such that

$$\int_\Omega (dp, dq)\ dx = \int_\Omega (u, dq)\ dx, \quad \forall q \in X_{k-1}\,,$$

which implies

$$\int_\Omega (dp, dq)\ dx = \int_\Omega (u, , dq)\ dx, \quad \forall q \in V_k \cap H^\perp_{k-1}\,.$$

Given $q \in V_k$ we decompose it as in

$$q = q_0 + h, \quad q_0 \in H^\perp_{k-1},\ h \in H_{k-1}$$

to obtain $q_0 \in V_k$ by $h \in V_k$. Since $dq = dq_0$, therefore, we obtain

$$\int_\Omega (dp, dq)\ dx = \int_\Omega (u, dq)\ dx, \quad \forall q \in V_k\,,$$

which means

$$\int_\Omega (dp, dq)\ dx = \int_\Omega (u, dq)\ dx, \quad \forall q \in \breve{\Lambda}_{k-1},\ \delta q = 0\,. \tag{162}$$

Here we use $(Hd)_{k-1}$, taking $\theta \in \Lambda_{k-1}$ arbitrary. There is thus $(q, z) \in \Lambda_{k-1} \times \Lambda_k$ such that

$$\delta q = 0, \quad \nu \wedge *q|_{\partial\Omega} = 0, \quad \theta = q + dz\,.$$

Then we obtain

$$q \in \breve{\Lambda}_{k-1},\ \delta q = 0,\ d\theta = dq\,,$$

and therefore,

$$\int_\Omega (dp, d\theta)\ dx = \int_\Omega (u, d\theta)\ dx, \quad \theta \in \Lambda_{k-1}\,. \tag{163}$$

It thus follows that

$$\int_\Omega (dp, d\theta) + (\delta p, \delta\theta) + (p, \theta)\ dx = \int_\Omega (u, d\theta) + (\alpha, \theta)\ dx, \quad \forall \theta \in \Lambda_{k-1}$$

for $\alpha = p \in L^2_{k-1}$, which is valid to any $\theta \in H^1_{k-1}$. Then we obtain

$$p \in H^2_{k-1} \tag{164}$$

as in the proof of Proposition 20 in §3.2. Equality (163) is now reduced to

$$\begin{aligned}&\int_\Omega (\delta dp, \theta)\ dx + (-1)^{k-1} \int_{\partial\Omega} (\nu \wedge *dp, *\theta)\ ds \\ &\quad = \int_\Omega (\delta u, \theta)\ dx + (-1)^{k-1} \int_{\partial\Omega} (\nu \wedge *u, *\theta)\ ds, \quad \forall \theta \in \Lambda_{k-1}\,,\end{aligned}$$

and hence

$$\delta dp = \delta u \text{ in } \Omega, \quad \nu \wedge *p|_{\partial\Omega} = 0, \quad \nu \wedge *dp|_{\partial\Omega} = \nu \wedge *u|_{\partial\Omega}\,,$$

because θ is arbitrary.

There thus arises that (157) with (164). Then Theorem 16 in §2.4 ensures $p \in H^{s+2}_{k-1}$ for any $s \geq 0$ by $u \in \Lambda_k$, and hence $p \in \Lambda_{k-1}$. □

We conclude this subsection with the following proof.

PROOF OF LEMMA 45: The proof is parallel to that of Lemma 44. We put

$$Y_{k'} = \left[W_{k'-1} \cap H^\perp_{k'}\right]^{H^1_{k'}}$$

for $k' = k + 1$, which is a closed subspace of $H^1_{k',d\delta}$ provided with the norm

$$\|\omega\|_{Y_{k'}} = \left(\|\delta\omega\|^2_{L^2_{k'-1}} + \|\omega\|^2_{L^2_{k'}}\right)^{1/2}\,.$$

Then we derive Babuška-Brezzi-Kikuchi's inequality

$$\inf_{\omega\in Y_{k'}\setminus\{0\}}\sup_{\theta\in Y_{k'}\setminus\{0\}}\frac{\int_\Omega(\delta\omega,\delta\theta)\,dx}{\|\omega\|_{Y_{k'}}\|\theta\|_{Y_{k'}}}=B>0$$

similarly to (159), to assure the existence of the solution $\omega\in Y_{k'}$ to

$$\int_\Omega(\delta\omega,\delta\theta)\,dx=\int_\Omega(u,\delta\theta),\quad \theta\in Y_{k'}\,,$$

and hence

$$\int_\Omega(\delta\omega,\delta\theta)\,dx=\int_\Omega(u,\delta\theta)\,dx,\quad \theta\in\breve{\Lambda}_{k+1},\ d\theta=0\,. \tag{165}$$

Here we use the assumption $(H\delta)_{k+1}$ for given $z\in\breve{\Lambda}_{k+1}$. There is $(\theta,\varphi)\in\breve{\Lambda}_{k+1}\times\Lambda_{k+2}$ such that

$$z=\theta+\delta\varphi,\quad d\theta=0\,,$$

Since $dz=d\theta$, equality (165) is improved as

$$\int_\Omega(\delta\omega,\delta z)\,dx=\int_\Omega(u,\delta z)\,dx,\quad z\in\breve{\Lambda}_{k+1}\,, \tag{166}$$

which is valid to any $z\in\breve{H}^1_{k+1,d\delta}$.

Then we get $\omega\in H^2_{k+1}$ similarly, and then equality (166) is reduced to

$$\int_\Omega(d\delta\omega,z)\,dx=\int_\Omega(du,z)\,dx,\quad z\in\breve{\Lambda}_{k+1}$$

by $\nu\wedge{*z}|_{\partial\Omega}=0$. It thus holds that (158) and then $\omega\in\breve{\Lambda}_{k+1}$ follows from Theorem 16. □

5.3. Numerical scheme for Helmholtz decompositions

Theorem 4 induces a variational formulation to the Helmholtz decomposition, which may be applied to construct its numerical scheme through the process of discretizations. Here, we describe it for $n=3$ in short, because this decomposition is used in engineering particularly.

First, we construct a basis of H_k, as is noticed in Remark 6. This basis is actually obtained by solving elliptic boundary value problems on Ω, following [8] and [12] for $k=1$ and $k=2$, respectively. If X_k denotes the vector space generated by this basis, it holds that $\dim X_k=b_k$, and therefore, this process is not necessary if $b_k=0$.

The set of smooth vector fields on $\overline{\Omega}$ is denoted by X. Given $\vec{u}\in X$, we take its orthogonal projection to $H_k^\perp$, denoted by $\vec{y}_k\in X_k^\perp$, using the above basis:

$$\vec{u}=\vec{y}_k+\vec{h}_k,\quad \vec{y}_k\in H_k^\perp,\ \vec{h}_k\in H_k\,.$$

Then we put

$$\breve{X}_1=\{\vec{\theta}\in X\mid \vec{\nu}\cdot\vec{\theta}\big|_{\partial\Omega}=0\},\quad \breve{X}_2=\{\vec{\theta}\in X\mid \vec{\nu}\times\vec{\theta}\big|_{\partial\Omega}=0\}\,,$$

to find $\vec{\eta}_k \in \breve{X}_k \cap X_k^\perp$, $k = 1, 2$, such that

$$\int_\Omega (\text{rot } \vec{\eta}_k, \text{rot } \vec{\theta}) + (\text{div } \vec{\eta}_k, \text{div } \vec{\theta})\ dx = \int_\Omega (\vec{y}_k, \vec{\theta})\ dx, \quad \vec{\theta} \in \breve{X}_k \cap X_k^\perp\,. \tag{167}$$

By the proof of Theorem 4 this variational problem (167) has a unique solution.

If $k = 1$ we put

$$\vec{v} = \vec{h}_1 + \text{rot rot } \vec{\eta}_1, \quad p = -\text{div } \vec{\eta}_1\,,$$

to obtain

$$\vec{u} = \vec{v} + \text{grad } p, \quad \text{div } \vec{v} = 0, \quad \vec{\nu} \cdot \vec{v}|_{\partial\Omega} = 0\,.$$

For $k = 2$ we take

$$\vec{z} = \vec{h}_2 - \text{grad div } \vec{\eta}_2, \quad \vec{w} = \text{rot } \vec{\eta}_2\,.$$

Then it follows that

$$\vec{u} = \vec{z} + \text{rot } \vec{w}, \quad \text{rot } \vec{z} = 0, \quad \vec{\nu} \times \vec{z}|_{\partial\Omega} = 0\,.$$

The advantage of this formulation is that the constraction of divergence free or rotation free spaces as in (163) or (165) in the previous section is not necessary. We must take, however, second derivatives of η to reach these decompositions, and consequently, trial functions of higher degrees are necessary in the discretization.

Construction of the harmonic part, H_k, is the other difficulty, where solving inhomogenous elliptic boudary value problems is required. This process, however, is skipped if the topology of Ω is simple as we have noticed above, that is, $b_1 = 0$ or $b_2 = 0$. Using the penalty method to (167) as in finding $\eta_k \in \breve{X}_k$ such that

$$\int_\Omega (\text{rot } \vec{\eta}_k, \text{rot } \vec{\theta}) + (\text{div } \vec{\eta}_k, \text{div } \vec{\theta}) + \lambda(\vec{\eta}_k, \vec{\theta})\ dx = \int_\Omega (\vec{y}_k, \vec{\theta})\ dx, \quad \vec{\theta} \in \breve{X}_k$$

may be also applicable, where $0 < \lambda \ll 1$.

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References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, second edition, Academic Press, Amsterdam, 2003.

[2] S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962) 119–147.

[3] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math. **17** (1964) 35–92.

[4] J. P. Bourguignon and H. Brezis, *Remarks on the Euler equation*, J. Funct. Anal. **15** (1974) 341–363.
[5] H. Brezis, *Analyse Fonctionnele*, Masson, Paris, 1983.
[6] H. Brezis and J. van Schaftingen, *Boundary estimates for elliptic system with L^1-data*, Calc. Var. **30** (2007) 369–398.
[7] G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.
[8] C. Foias and R. Temam, *Remarques sur les equation de Navier-Stokes stationaires et les phénomenès successifs de bufurcation*, Ann. Scuola Norm. Sup. Pisa Cl. Ser. (4) **5** (1978) 28–63.
[9] D. Fujiwara and H. Morimoto, *An L_r-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **24** (1977) 685–700.
[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer-Veralag, Berlin, 1983.
[11] P. Grisvard, *Elliptic Problems in Non-Smooth Domains*, Pitmann, Boston, 1985.
[12] H. Kozono and T. Yanagisawa, *L^r-variational inequality for vector fields and the Helmholtz-Wyle decomposition in bounded domains*, Indiana Univ. Math. J. **58** (2009) 1853–1920.
[13] H. Kozono and T. Yanagisawa, *Leray's problem on the stationary Navier-Stokes equations with inhomogeneous boundary data*, Math. Z. **262** (2009) 27–39.
[14] H. Kozono and T. Yanagisawa, *Global div-curl lemma on bounded domains in $\mathbf{R}^3$*, J. Funct. Anal. **256** (2009) 3847–3859.
[15] A. Kufner (ed.), *Function Spaces*, Noordhoff Ineternational Puplishing, Leyden, 1977.
[16] C. B. Morrey, *Multiple Integrals in the Calculus of Variation*, Springer-Verlag, New York, 1966.
[17] S. Morita, *Geometry of Differential Forms*, Amer. Math. Soc., Providence, RI, 2001.
[18] L. Nirenberg, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **8** (1955) 648–674.
[19] L. Nirenberg, *On elliptic partial differential equations*, Annali. Scoula Norm. Sup. Pisa (3), **13** (1959) 115–162.
[20] N. Saito, *Notes on the Banach-Nečas-Babuška theorem and Kato's minimum modulus of operators*, arXiv:1711.01533v4, 2018.
[21] G. Schwarz, *Hodge Decomposition - A Method for Solving Boundary Value Problems*, Lecture Notes in Math. 1607, Springer-Verlag, Berlin, 1995.
[22] C. G. Simader, *On Dirichlet's boundary value problem*, Lecture Notes in Math. 268, Springer-Verlag, Berlin, 1972.
[23] C. G. Simader and H. Sohr, *A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded exterior domains*, In; *Mathematical Problems Relating to the Navier-Stokes Equation*, Ser. Adv. Math. Appl. Scis. 11, World Sci. Publ., River Edge, NJ 1992, pp. 1–35.
[24] I. M. Singer and J. A. Thorpe, *Lecture Notes on Elementary Topology and Geometry*, Scott, Foresmann and Company, Glenview, Ill. 1967.
[25] T. Suzuki and T. Tsuchiya, *Liouville's formulae and Hadamard variation with respect to general domain perturbations*, J. Math. Soc. Japan **75** (2023) 983–1024.
[26] T. Suzuki and K. Watanabe, *Interface vanishing of $d-\delta$ systems*, AIMS Mathematics **9** (2024) 7848–7865.
[27] H. Tanabe, *Equations of Evolution*, Pitman, London, 1975.
[28] L. Tartar, *Compensated compactness and applications to partial differential equations*, In; *Nonlinear Analysis and Mechanics*, Herriot-Watt Symposium vol.4 (ed. R. J. Knops), Res Notes in Math. **30**, Pitmann, 1979, pp.136–212.
[29] Z. Yoshida and Y. Giga, *Remarks on spectra of operator rot*, Math. Z. **204** (1990) 235–245.

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