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Extension of the One-Sample Kolmogorov-Smirnov Test

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Abstract. We propose here a new goodness-of-fit test, named the one-sample OVL-q test ($q = 1, 2, \dots$), which can be considered an extension of the one-sample Kolmogorov-Smirnov test and the Kuiper test (equivalent to the one-sample OVL-1 test and OVL-2 test, respectively). We described the asymptotic distribution of the one-sample OVL-q test statistic ($q = 1, 2, \dots$) using a Brownian bridge. We further conducted numerical experiments and demonstrated that the one-sample OVL-3 test can sometimes exceed the detection power of conventional goodness-of-fit tests, including the one-sample KS test and the Kuiper test.

1. Introduction

The Kolmogorov-Smirnov (KS) test is a nonparametric method used to determine whether a sample originates from a specific probability distribution (one-sample KS test) or to assess whether two samples come from the same distribution (two-sample KS test). In our previous study, we devised an extended version of the two-sample KS test, named the (two-sample) OVL-q test ($q = 1, 2, \dots$) [10].

The two-sample OVL-q test can in fact be regarded as a test based on the distribution metric D_q, which we propose in Section 2 of this paper. Notably, D_1 in the infinite sequence of metrics $D_1, D_2, \dots$ coincides with the KS metric. Furthermore, D_2 corresponds to the two-sample Kuiper test. This naturally leads to the idea that the one-sample KS test and Kuiper test, like their two-sample counterparts, could also be extended using the same D_q framework. The present study aims to realize this idea by proposing the one-sample OVL-q test. Our motivation lies in reinterpreting the one-sample KS test and Kuiper test within the framework of D_q, with particular

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focus on understanding the characteristics of the one-sample OVL-q ($q \geq 3$): how these tests behave and how their statistical powers compare with those of the one-sample KS test (OVL-1 test) and Kuiper test (OVL-2 test).

As a result, we have demonstrated the theoretical existence of an infinite family of tests—the one-sample OVL-q test ($q = 1, 2, \dots$)—based on D_q. We have not only provided a theoretical formula for calculating the p-value of the one-sample OVL-q test (6), but also described the asymptotic distribution of the one-sample OVL-q test statistic using a Brownian bridge (11).

Furthermore, we experimentally confirmed the existence of cases in which the one-sample OVL-3 test outperforms conventional goodness-of-fit tests, including the one-sample KS test and the Kuiper test, in terms of statistical power under certain alternative hypotheses.

In this paper, we describe the analytical framework in Section 2. Experimental results are shown in Section 3. Discussion follows in Section 4. The proofs of Theorems 2.7, 2.9 and 2.10 are given in Section 5. The source code for the experiments in Section 3 is provided in the Supplementary Material.

General notation

We denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Q}$, and $\mathbb{R}$ the sets of integers, nonnegative integers, positive integers, rational numbers, and real numbers, respectively. If $-\infty \leq a \leq b \leq \infty$ and if there is no confusion, we write $[a, b] := \{x : a \leq x \leq b\}$, $[a, b) := \{x : a \leq x < b\}$, $(a, b] := \{x : a < x \leq b\}$, and $(a, b) := \{x : a < x < b\}$ as (extended) real intervals. For $n \in \mathbb{N}_+$, let $\mathbb{R}^n$ be the Euclidean n-dimensional space and $\mathbb{R}^n_{\leq} := \{(v_1, \dots, v_n) \in \mathbb{R}^n : v_1 \leq \cdots \leq v_n\}$. For a topological space A, we denote by $\mathcal{B}(A)$ the σ-algebra of Borel sets in A. For a set A, $\#A$ denotes the cardinality of A. For a real function f on a set A and $x, y \in A$, we write $f|_x^y = f(y) - f(x)$. We denote by $\mathbb{1}_A$ the indicator function of a set A. For a random variable X, $\mathbb{E}[X]$ denotes its expectation.

2. Analytical Framework

Let $\mathcal{F}$ be the set of distribution functions on $\mathbb{R}$, where each $F \in \mathcal{F}$ is nondecreasing, is continuous from the right, and satisfies $F(-\infty) := \lim_{x \to -\infty} F(x) = 0$ and $F(\infty) := \lim_{x \to \infty} F(x) = 1$. For $F, G \in \mathcal{F}$ and

$q \in \mathbb{N}_+$,

$$D_q(F, G) := 1 - \inf_{\boldsymbol{v} \in \mathbb{R}^q_{\leq}} r_{F,G}(\boldsymbol{v}), \tag{1}$$

where

$$r_{F,G}(\boldsymbol{v}) := \sum_{i=0}^{q} \min\{F|_{v_i}^{v_{i+1}}, G|_{v_i}^{v_{i+1}}\} \tag{2}$$

for $\boldsymbol{v} = (v_1, \ldots, v_q) \in \mathbb{R}^q_{\leq}$, $v_0 = -\infty$, and $v_{q+1} = \infty$. Note that $r_{F,G}(\boldsymbol{v}) \in [0, 1]$ for all $\boldsymbol{v} \in \mathbb{R}^q_{\leq}$, so that $D_q(F, G) \in [0, 1]$. We sometimes denote $\inf_{\boldsymbol{v} \in \mathbb{R}^q_{\leq}} r_{F,G}(\boldsymbol{v})$ by $\mathrm{OVL}_q(F, G)$, according to the overlap coefficient (OVL) (*cf.* [9, page 22] and [10, Definition 2.2, Remark 2.5, and Theorem 2.6]). Using this notation, (1) becomes the following.

$$D_q(F, G) = 1 - \mathrm{OVL}_q(F, G). \tag{3}$$

Theorem 2.1 (See [12] for reference). *The KS metric on $\mathcal{F}$ equals D_1, that is,*

$$\sup_{v \in \mathbb{R}} |F(v) - G(v)| = D_1(F, G) \qquad (F, G \in \mathcal{F}).$$

Theorem 2.2. *For each $q \in \mathbb{N}_+$, $(\mathcal{F}, D_q)$ is a complete metric space.*

Theorem 2.1 can be proved similarly as in the proof of [10, Proposition 2.7]. Theorem 2.2 will be proved in Section 5.1. By these theorems, we can see that D_q are extension of the KS metric. Furthermore, by Theorem 5.11, any D_q and D_1 generate the same topology on $\mathcal{F}$.

For calculating $D_q(F, G)$, we introduce the following definition.

Definition 2.3. For a bounded function $H : \mathbb{R} \to \mathbb{R}$ and $v_1, \ldots, v_q \in \mathbb{R}$, we define the alternating sum

$$\begin{aligned}\mathcal{A}(H)(v_1, v_2, \ldots, v_q) &= \sum_{i=1}^{q} (-1)^{i+1} H(v_i) \\ &= H(v_1) - H(v_2) + \cdots + (-1)^{q+1} H(v_q).\end{aligned}$$

Note that

$$\mathcal{A}(-H)(v_1, v_2, \dots, v_q) = -H(v_1) + H(v_2) - \cdots + (-1)^q H(v_q).$$

For $q \in \mathbb{N}_+$, we can define

$$\mathcal{A}_q(H) = \sup_{v_1 \leq v_2 \leq \cdots \leq v_q} \mathcal{A}(H)(v_1, v_2, \dots, v_q).$$

Since $F - G$ is a bounded function, $\mathcal{A}_q(F - G)$ and $\mathcal{A}_q(G - F)$ can be defined for $F, G \in \mathcal{F}$.

THEOREM 2.4. *For $F, G \in \mathcal{F}$,*

$$D_q(F, G) = \max\{\mathcal{A}_q(F - G), \mathcal{A}_q(G - F)\}.$$

REMARK 2.5. For $q = 2$,

$$D_2(F, G) = \sup_{x \in \mathbb{R}}\{F(x) - G(x)\} - \inf_{x \in \mathbb{R}}\{F(x) - G(x)\}. \tag{4}$$

Theorem 2.4 and (4) in Remark 2.5 will be proved in Section 5.2.

2.1. One-sample KS test and its extension

As a null hypothesis H_0, we assume that $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathfrak{A}, P)$ with a given distribution function $F \in \mathcal{F}$. Let F_n be the corresponding empirical distribution function, i.e.,

$$F_n(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty, x]}(X_i) \qquad (x \in \mathbb{R}). \tag{5}$$

Here we propose $D_q(F_n, F)\colon \Omega \to \mathbb{R}$ ($q \in \mathbb{N}_+$) as an extension of the one-sample KS test statistic, which equals $D_1(F_n, F)$ by Theorem 2.1. The p-value (function) of the extended test is given by

$$p_{q,n}(x) := P(x \leq D_q(F_n, F)) \qquad (x \in \mathbb{R}), \tag{6}$$

and the upper limit of a $100(1-\alpha)\%$ confidence interval ($0 < \alpha < 1$) of $D_q(F_n, F)$ is

$$u_{q,n}(\alpha) := \inf\{x \in \mathbb{R} : p_{q,n}(x) < \alpha\}. \tag{7}$$

This can be regarded as the one-sample OVL-q test since the (two-sample) OVL-q test statistic $\rho_{q,m,n}$ in [10, Definition 2.2] equals $1 - D_q(F_{0,m}, F_{1,n})$ (see [10, Definition 2.1]), whose p-value is equal to that of $D_q(F_{0,m}, F_{1,n})$ as described in [10, Section 2.3].

REMARK 2.6. The one-sample OVL-1 test is equivalent to the one-sample KS test, because the KS metric equals D_1 by Theorem 2.1. The one-sample OVL-2 test is equivalent to the one-sample Kuiper test [11] by Remark 2.5. The one-sample Kuiper test is a previously established goodness-of-fit test particularly effective for analyzing periodic or circular data.

THEOREM 2.7. *For each $q \in \mathbb{N}_+$, $D_q(F_n, F)$ converges completely to* 0 *as $n \to \infty$, i.e.,*

$$\sum_{n=1}^{\infty} P(D_q(F_n, F) > \epsilon) < \infty$$

for any $\epsilon > 0$.

Note that complete convergence implies almost sure convergence, as described in [8] and [9, Remark 4.4].

COROLLARY 2.8. *For $F, G \in \mathcal{F}$,*

$$D_q(F_n, G) \to D_q(F, G) \ \textit{a.s.}$$

as $n \to \infty$.

THEOREM 2.9. *For each $q \in \mathbb{N}_+$, the distribution of $D_q(F_n, F)$ is the same for all continuous $F \in \mathcal{F}$.*

THEOREM 2.10. *Let $Y : [0,1] \times \Omega \to \mathbb{R}$ be a Brownian bridge (see Definition 5.28 for definition). If $q \in \mathbb{N}_+$ and $F \in \mathcal{F}$ is continuous on $\mathbb{R}$,*

$$\lim_{n\to\infty} P(D_q(F_n, F) \geq \frac{a}{\sqrt{n}}) = P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a) \tag{8}$$

for every $a > 0$ *at which* $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a)$ *is continuous.*

If $q = 2$, then we obtain the formula for the one-sample Kuiper test.

Corollary 2.11 (cf. [11, (3.3)]). *If* $F \in \mathcal{F}$ *is continuous on* $\mathbb{R}$,

$$\lim_{n\to\infty} P\left(D_2(F_n, F) \geq \frac{a}{\sqrt{n}}\right) = 2\sum_{i=1}^{\infty}(4i^2a^2 - 1)\exp(-2i^2a^2), \tag{9}$$

$$\lim_{n\to\infty} P\left(D_2(F_n, F) \leq \frac{a}{\sqrt{n}}\right) = 1 - 2\sum_{i=1}^{\infty}(4i^2a^2 - 1)\exp(-2i^2a^2) \tag{10}$$

for any $a > 0$.

Note that $P(D_q(F_n, F) \geq a/\sqrt{n})$ in (8) is independent of any continuous $F \in \mathcal{F}$ by Theorem 2.9. Similarly to this theorem, we have the asymptotic distribution function of $\sqrt{n}D_q(F_n, F)$:

$$\lim_{n\to\infty} P(\sqrt{n}D_q(F_n, F) \leq a) = P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \leq a) \tag{11}$$

for every $a > 0$ at which $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} = a) = 0$ (see Remark 5.42).

See Sections 5.3 to 5.5 for the proofs of Theorem 2.7 and Corollary 2.8, Theorem 2.9, and Theorem 2.10 and Corollary 2.11, respectively.

3. Numerical Experiments

We conducted a computer-based experiment to compare the statistical power of the one-sample OVL-3 test with those of conventional statistical tests, including the one-sample KS test (OVL-1 test) and the Kuiper test (OVL-2 test).

Beforehand, we computed $p_{q,n}$ ($q = 2, 3$), as defined in (6), using the Monte Carlo method for $n = 2^3, 2^4, \ldots, 2^{12}$, because exact p-values for the one-sample OVL-q test ($q = 2, 3$) could not be computed. More specifically, instead of $p_{q,n}$ ($q = 2, 3$), we used the empirical distribution functions of $D_2(U_n, U)$ and $D_3(U_n, U)$ computed from 100,000 samples of size n drawn from the standard uniform distribution, whose distribution function is defined as

$$U(x) = \max\{0, \min\{x, 1\}\} \qquad (x \in \mathbb{R}). \tag{12}$$

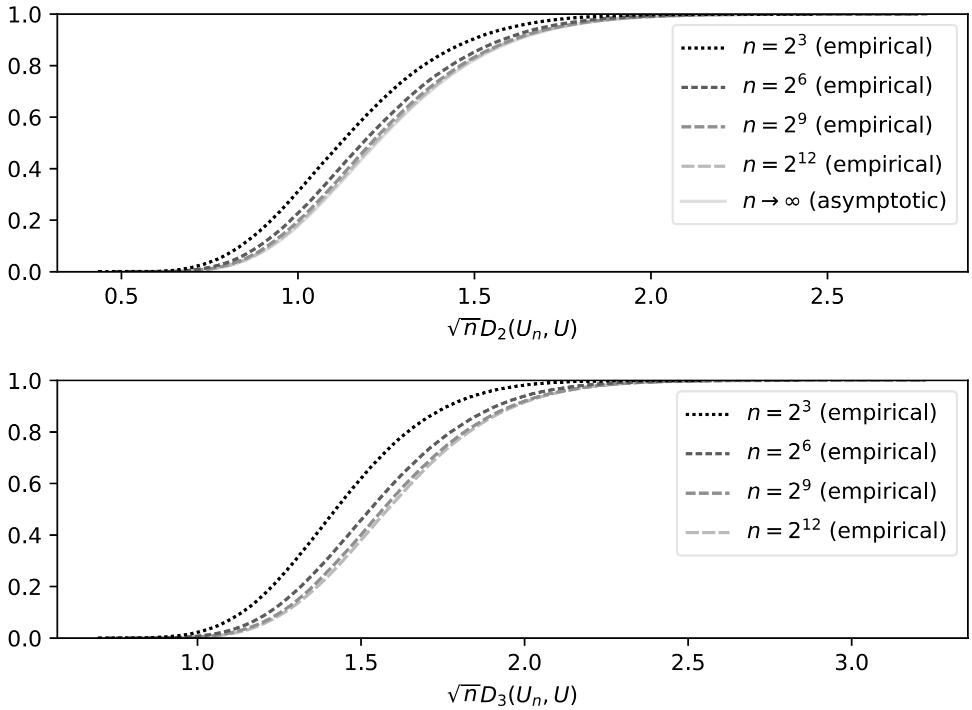

Fig. 1. For each $n = 2^3, 2^6, 2^9, 2^{12}$, we generated $100,000$ samples of size n following the standard uniform distribution U, and computed $D_2(U_n, U)$ and $D_3(U_n, U)$ for each sample. The upper subplot shows the empirical distribution function based on the $100,000$ values of $\sqrt{n}D_2(U_n, U)$, along with the right hand side of (10), which represents the asymptotic distribution function where $n \to \infty$. The lower subplot shows the empirical distribution function based on the $100,000$ values of $\sqrt{n}D_3(U_n, U)$.

The empirical distribution functions for $n = 2^3, 2^6, 2^9, 2^{12}$ are shown in Fig. 1 together with the theoretical asymptotic distribution function given in (10).

The probability density functions used in the experiments are defined as follows:

$$\text{Normal}(\mu, \sigma)(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$$(\mu \in \mathbb{R},\ \sigma > 0,\ x \in \mathbb{R}),$$

$$\mathrm{Trapezoidal}(x) = \begin{cases} (x+2)/2 & \text{if } -2 \leq x \leq -\sqrt{2}, \\ (2-\sqrt{2})/2 & \text{if } -\sqrt{2} < x \leq \sqrt{2}, \\ (-x+2)/2 & \text{if } \sqrt{2} < x \leq 2, \\ 0 & \text{if } x < -2 \text{ or } 2 < x, \end{cases} \qquad (x \in \mathbb{R}),$$

$$\mathrm{SkewNormal}(a,\xi)(x) = 2\,\mathrm{Normal}(0,1)(x-\xi) \int_{-\infty}^{a(x-\xi)} \mathrm{Normal}(0,1)(t)dt \qquad (a,\xi \in \mathbb{R}),$$

and they are illustrated in Fig. 2. (For details on the skew-normal distribution, see [1] and [14].)

First, we chose two different distributions as the sampling distribution and the reference distribution. Then we repeated the following trial 100,000 times for each sample size $n = 2^3, 2^4, \ldots, 2^{12}$: a random sample of size n was drawn from the sampling distribution, and tested under the null hypothesis that it were drawn from the reference distribution. We performed the one-sample OVL-3 test, the one-sample KS test (OVL-1 test), the one-sample Kuiper test (OVL-2 test), and the Cramér-von Mises test for each sample, and counted the number of times that the null hypothesis was rejected at 0.05 level of significance. The rate of rejection out of 100,000 trials was assumed to represent the statistical power of the test. The entire source code for the experiment, written in Python 3.11.8, is provided as the Supplementary Material on pages 155–159.

The result is shown in Fig. 2. When the sampling distribution was $\mathrm{Normal}(0.2, 1)$ and the reference distribution was $\mathrm{Normal}(0, 1)$, the powers of the Cramér-von Mises test and the one-sample KS test were respectively the first and second highest of the four, while the power of the one-sample Kuiper test was the lowest. When the sampling distribution was $\mathrm{Normal}(0, 1.1)$ and the reference distribution was $\mathrm{Normal}(0, 1)$, the highest statistical power was observed for the one-sample Kuiper test, followed by the one-sample OVL-3 test and the Cramér-von Mises test, while the one-sample KS-test showed the lowest power. When the sampling distribution was Trapezoidal and the reference distribution was $\mathrm{Normal}(0, 1)$, the power of the one-sample OVL-3 test was the highest among the four, followed by that of the one-sample Kuiper test, and those of the other two were almost equally lower. When the sampling distribution was $\mathrm{SkewNormal}(1.6, -0.6)$

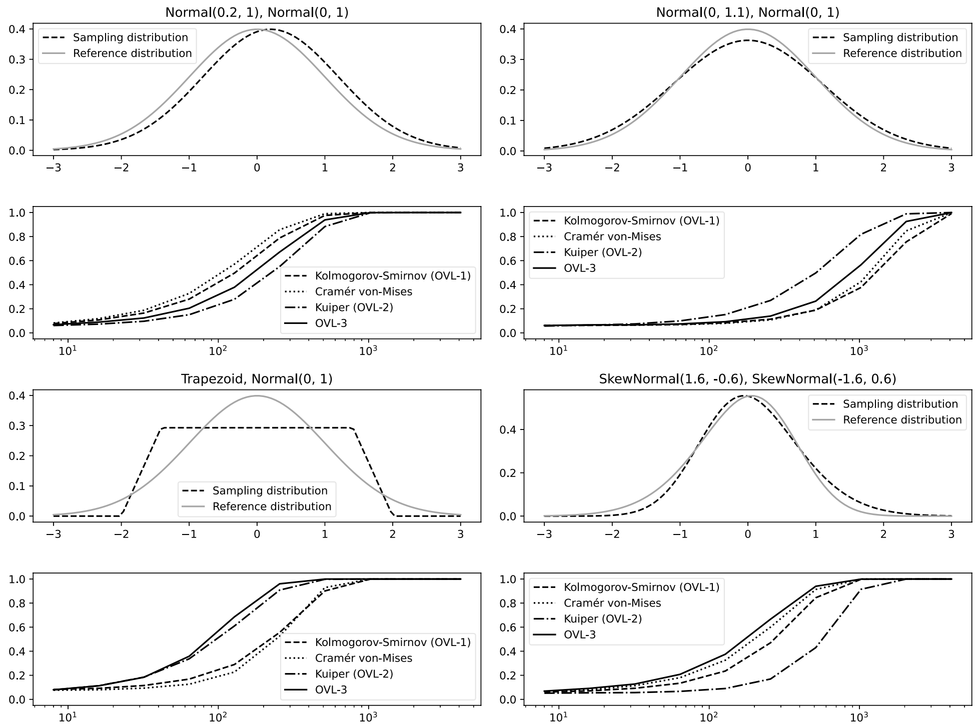

Fig. 2. Comparison of the statistical powers of the one-sample OVL-q test ($q = 1, 2, 3$) with the Cramér–von Mises test. This figure consists of 4 rows and 2 columns (8 subplots in total). The first and third rows show the probability density functions used in the experiments. For each column, the second row corresponds to the first row, and the fourth row corresponds to the third row. The second and fourth rows show the statistical power of the one-sample OVL-3 test to detect samples drawn from distributions that do not follow the corresponding reference distributions, compared with the statistical powers of the one-sample KS test (OVL-1 test), the Kuiper test (OVL-2 test), and the Cramér–von Mises test. The horizontal axis in all plots represents the sample size.

and the reference distribution was SkewNormal$(-1.6, 0.6)$, the highest statistical power was observed for the one-sample OVL-3 test, followed by the Cramér-von Mises test and the one-sample KS test, while the one-sample Kuiper test showed the lowest power.

4. Discussion

In this study, we have developed the one-sample OVL-q test ($q = 1, 2, \dots$) as a new goodness-of-fit test, which can also be considered an extended version of the one-sample KS test and the Kuiper test (because the one-sample KS test and the Kuiper test are equivalent to the one-sample OVL-1 test and the OVL-2 test, respectively). There are several possible directions for extension, and with regard to the KS test in particular, smoothed versions are another such example (for example, see [13]). The smoothed versions by Fan [7] and by Cao and Lugosi [3] are based on the L^2-metric and L^1-metric between $f_{n,h}$ and g for probability density functions f and g, respectively, where $f_{n,h}$ is a Kernel density estimator of f. On the other hand, the OVL-q test is based on the metric $D_q(F_n, G)$ for $F, G \in \mathcal{F}$, which is another extension of KS test.

We have provided a theoretical formula for calculating the p-value of the one-sample OVL-q test for all $q = 1, 2, \dots$ (see Equation (6)). We also described the asymptotic distribution of the one-sample OVL-q test statistic ($q = 1, 2, \dots$) using a Brownian bridge (see Equation (11)).

We conducted numerical experiments to compare the detection power of the one-sample OVL-3 test with conventional goodness-of-fit tests, including the one-sample KS test and the Kuiper test (OVL-2 test). In several instances, the one-sample OVL-q test ($q = 2, 3$) demonstrated superior performance, suggesting its potential utility.

The limitations of this study are as follows:

- We have not obtained the explicit form of the asymptotic formula (11) for the one-sample OVL-q test ($q \geq 3$).

- The asymptotic p-values for the one-sample OVL-q test statistic ($q \geq 3$) given by (11) are not applicable when the sample size is small. In such case, we have not proposed a method for obtaining exact p-values, while we can consider how to calculate p-values by means of the Monte Carlo method, as in [3, Section 4.1].

- We could not clarify why OVL-3 is superior to conventional goodness-of-fit tests, including the one-sample KS test (OVL-1 test) and the Kuiper test (OVL-2 test), in several cases.

To make the one-sample OVL-q test ($q \geq 3$) practical, these issues need to be addressed in future research.

5. Proofs

5.1. Proof of Theorem 2.2

Let us denote by $\mathcal{F}'$ the set of bounded right-continuous real functions on $\mathbb{R}$, and by $\|\cdot\|$ the supremum norm on $\mathcal{F}'$, i.e.,

$$\|\xi\| := \sup_{x\in\mathbb{R}}|\xi(x)| < \infty \qquad (\xi \in \mathcal{F}'). \tag{13}$$

REMARK 5.1. We can easily see that $\mathcal{F}'$ is a normed linear space with norm $\|\cdot\|$, and that $\mathcal{F}$ is a convex subset of $\mathcal{F}'$.

THEOREM 5.2. *$\mathcal{F}'$ is a Banach space with norm $\|\cdot\|$.*

PROOF. Suppose $\{\xi_n\}$ is a Cauchy sequence in $\mathcal{F}'$. For each $x \in \mathbb{R}$, $|\xi_m(x) - \xi_n(x)| \leq \|\xi_m - \xi_n\|$ implies that $\{\xi_n(x)\}$ is a Cauchy sequence, so that there exists $\xi(x) := \lim_{n\to\infty} \xi_n(x) \in \mathbb{R}$ by the completeness of the real line.

For any $\epsilon > 0$, there exists $N \in \mathbb{N}_+$ such that $m, n \geq N$ implies $\|\xi_m - \xi_n\| < \epsilon$, so that $|\xi(x) - \xi_n(x)| = \lim_{m\to\infty}|\xi_m(x) - \xi_n(x)| \leq \epsilon$ for all $x \in \mathbb{R}$, i.e., $\|\xi - \xi_n\| \leq \epsilon$, which also implies that $\|\xi\| \leq \|\xi_n\| + \epsilon < \infty$.

For any $\epsilon > 0$, there exists $n \in \mathbb{N}_+$ with $\|\xi - \xi_n\| \leq \epsilon$ by the argument above. For each $x \in \mathbb{R}$, there exists $\delta > 0$ such that $|\xi_n(x) - \xi_n(y)| < \epsilon$ for all $y \in (x, x + \delta)$ since ξ_n is right-continuous, so that

$$\begin{aligned}|\xi(x) - \xi(y)| &= |\xi(x) - \xi_n(x) + \xi_n(x) - \xi_n(y) + \xi_n(y) - \xi(y)| \\ &\leq |\xi(x) - \xi_n(x)| + |\xi_n(x) - \xi_n(y)| + |\xi_n(y) - \xi(y)| \\ &< 3\epsilon\end{aligned}$$

for all $y \in (x, x + \delta)$. Hence ξ is right-continuous.

Now we see that $\xi \in \mathcal{F}'$ and $\lim_{n\to\infty}\|\xi - \xi_n\| = 0$, and the proof is complete. □

LEMMA 5.3. *$\mathcal{F}$ is a closed subspace of $(\mathcal{F}', \|\cdot\|)$.*

PROOF. Let $\{\xi_n\}$ be a convergent sequence in $\mathcal{F} \subset \mathcal{F}'$ and $\xi := \lim_{n\to\infty} \xi_n \in \mathcal{F}'$. For each $x \in \mathbb{R}$, $|\xi_n(x) - \xi(x)| \leq \|\xi_n - \xi\| \to 0$ $(n \to \infty)$ implies that $\xi(x) = \lim_{n\to\infty} \xi_n(x)$. Hence $\xi(x) \leq \xi(y)$ for any $x < y$, since $\xi_n(x) \leq \xi_n(y)$ for all n. This means that ξ is nondecreasing.

For any $\epsilon > 0$, there exists $n \in \mathbb{N}_+$ with $\|\xi_n - \xi\| < \epsilon$. Since $\lim_{x\to-\infty} \xi_n(x) = 0$, there exists $M \in \mathbb{R}$ such that $|\xi_n(x)| < \epsilon$ for all $x < M$. Hence $|\xi(x)| = |\xi(x) - \xi_n(x) + \xi_n(x)| \leq |\xi(x) - \xi_n(x)| + |\xi_n(x)| < \|\xi - \xi_n\| + \epsilon < 2\epsilon$ for all $x < M$. Therefore, $\lim_{x\to-\infty} \xi(x) = 0$. The proof for $\lim_{x\to\infty} \xi(x) = 1$ is similar.

Taken together, we have shown that $\xi \in \mathcal{F}$. □

The following theorem follows immediately from Theorems 2.1 and 5.2 and Lemma 5.3.

THEOREM 5.4. *$(\mathcal{F}, D_1)$ is a complete metric space.*

REMARK 5.5. We can see that $(\mathcal{F}, D_1)$ is not separable. For example, the collection of open subsets

$$\{F \in \mathcal{F} : D_1(F, \mathbb{1}_{[a,\infty)}) < 1/2\} \qquad (a \in \mathbb{R})$$

is pairwise disjoint and uncountable. Here note that $D_1(\mathbb{1}_{[a,\infty)}, \mathbb{1}_{[b,\infty)}) = 1$ if $a \neq b$.

LEMMA 5.6. *For any $a, b, c \in \mathbb{R}$, $\min\{a, b\} + \min\{b, c\} \leq \min\{a, c\} + b$.*

PROOF. We can assume that $a \leq c$ without loss of generality. If $a \leq b$, then $\min\{a, b\} + \min\{b, c\} = \min\{a, c\} + \min\{b, c\} \leq \min\{a, c\} + b$. If $a \geq b$, then $\min\{a, b\} + \min\{b, c\} = b + \min\{b, c\} \leq b + \min\{a, c\}$. □

THEOREM 5.7. *For each $q \in \mathbb{N}_+$, $(\mathcal{F}, D_q)$ is a metric space.*

PROOF. We have to show that, for all $F, G, H \in \mathcal{F}$,

(a) $0 \leq D_q(F, G) < \infty$.

(b) $D_q(F, G) = 0$ if and only if $F = G$.

(c) $D_q(F, G) = D_q(G, F)$.

(d) $D_q(F,G) \le D_q(F,H) + D_q(H,G)$.

(a) and (c) follows from definition.

Let us start with (b). If $F = G$, then $r_{F,G}(\boldsymbol{v}) = 1$ for any $\boldsymbol{v} \in \mathbb{R}^q_{\le}$, so that $D_q(F,G) = 0$, by definition. If $D_q(F,G) = 0$, then $\inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} r_{F,G}(\boldsymbol{v}) = 1$, so that $r_{F,G}(\boldsymbol{v}) = 1$ for all $\boldsymbol{v} \in \mathbb{R}^q_{\le}$, which implies $F = G$ by the following arguments. If $F(x) < G(x)$ for some $x \in \mathbb{R}$, then for $\boldsymbol{v} = (x, \dots, x) \in \mathbb{R}^q_{\le}$, we have

$$\begin{aligned} r_{F,G}(\boldsymbol{v}) &= \min\bigl\{F|_{-\infty}^{x}, G|_{-\infty}^{x}\bigr\} + \min\bigl\{F|_{x}^{\infty}, G|_{x}^{\infty}\bigr\} \\ &= \min\{F(x), G(x)\} + \min\{1 - F(x), 1 - G(x)\} \\ &= F(x) - G(x) + 1 \\ &< 1. \end{aligned}$$

Hence $F \ge G$ if $D_q(F,G) = 0$. Similarly, $F \le G$ if $D_q(F,G) = 0$, proving (b).

As for (d), we have

$$\begin{aligned} &\inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} r_{F,H}(\boldsymbol{v}) + \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} r_{H,G}(\boldsymbol{v}) \\ &= \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} \sum_{i=0}^{q} \min\bigl\{F|_{v_i}^{v_{i+1}}, H|_{v_i}^{v_{i+1}}\bigr\} + \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} \sum_{i=0}^{q} \min\bigl\{H|_{v_i}^{v_{i+1}}, G|_{v_i}^{v_{i+1}}\bigr\} \\ &\le \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} \sum_{i=0}^{q} \bigl(\min\bigl\{F|_{v_i}^{v_{i+1}}, H|_{v_i}^{v_{i+1}}\bigr\} + \min\bigl\{H|_{v_i}^{v_{i+1}}, G|_{v_i}^{v_{i+1}}\bigr\}\bigr) \\ &\le \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} \sum_{i=0}^{q} \bigl(\min\bigl\{F|_{v_i}^{v_{i+1}}, G|_{v_i}^{v_{i+1}}\bigr\} + H|_{v_i}^{v_{i+1}}\bigr) \\ &= \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} r_{F,G}(\boldsymbol{v}) + 1 \end{aligned}$$

by (2) and Lemma 5.6, so that $D_q(F,G) \le D_q(F,H) + D_q(H,G)$. This completes the proof. □

LEMMA 5.8. *For any $F, G \in \mathcal{F}$, $D_q(F,G) \le D_{q'}(F,G)$ if $q < q'$.*

PROOF. Since $\inf_{\boldsymbol{v}\in\mathbb{R}^{q'}_{\le}} r_{F,G}(\boldsymbol{v}) \le \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\le}} r_{F,G}(\boldsymbol{v})$ by definition, $D_q(F,G) \le D_{q'}(F,G)$ holds. □

LEMMA 5.9. *For each $q \in \mathbb{N}_+$, $D_q(F,G) \le qD_1(F,G)$ for all $F,G \in \mathcal{F}$.*

PROOF. It follows from definition that

$$\begin{aligned} D_q(F,G) &= 1 - \inf_{\boldsymbol{v} \in \mathbb{R}^q_\le} \sum_{i=0}^{q} \min\bigl\{F|_{v_i}^{v_{i+1}}, G|_{v_i}^{v_{i+1}}\bigr\} \\ &= 1 - \inf_{\boldsymbol{v} \in \mathbb{R}^q_\le} \sum_{i=0}^{q} \frac{1}{2}\bigl(F|_{v_i}^{v_{i+1}} + G|_{v_i}^{v_{i+1}} - \bigl|F|_{v_i}^{v_{i+1}} - G|_{v_i}^{v_{i+1}}\bigr|\bigr) \\ &= 1 - \inf_{\boldsymbol{v} \in \mathbb{R}^q_\le} \left(1 - \sum_{i=0}^{q} \frac{1}{2}\bigl|F|_{v_i}^{v_{i+1}} - G|_{v_i}^{v_{i+1}}\bigr|\right) \\ &= \sup_{\boldsymbol{v} \in \mathbb{R}^q_\le} \sum_{i=0}^{q} \frac{1}{2}\bigl|F|_{v_i}^{v_{i+1}} - G|_{v_i}^{v_{i+1}}\bigr| \\ &\le \sup_{\boldsymbol{v} \in \mathbb{R}^q_\le} \sum_{i=1}^{q} |F(v_i) - G(v_i)| \\ &\le q \sup_{x \in \mathbb{R}} |F(x) - G(x)| \\ &= qD_1(F,G), \end{aligned}$$

and the proof is complete. □

REMARK 5.10. As shown in the proof above, we obtain the equation

$$D_q(F,G) = \frac{1}{2} \sup_{\boldsymbol{v} \in \mathbb{R}^q_\le} \sum_{i=0}^{q} \bigl|F|_{v_i}^{v_{i+1}} - G|_{v_i}^{v_{i+1}}\bigr| \qquad (F,G \in \mathcal{F}).$$

The following theorem follows immediately from Lemmas 5.8 and 5.9.

THEOREM 5.11. *For each $q \in \mathbb{N}_+$, $D_1(F,G) \le D_q(F,G) \le qD_1(F,G)$ for all $F,G \in \mathcal{F}$.*

Now Theorem 2.2 follows from Theorems 5.4, 5.7 and 5.11.

5.2. Proof of Theorem 2.4

THEOREM 5.12. *For $F, G \in \mathcal{F}$,*

$$D_q(F, G) = \max\{\mathcal{A}_q(F - G), \mathcal{A}_q(G - F)\}.$$

PROOF. Since the equality is trivial when $F = G$, we consider the case $F \neq G$. Set $H = F - G \neq 0$. By Remark 5.10,

$$\begin{aligned}
&\mathcal{A}(H)(v_1, \dots, v_q) \\
&= H(v_1) - H(v_2) + \cdots + (-1)^{q+1} H(v_q) \\
&= \frac{1}{2} H \,|_{v_0}^{v_1} - \frac{1}{2} H \,|_{v_1}^{v_2} + \frac{1}{2} H \,|_{v_2}^{v_3} - \cdots + (-1)^{q-1} \frac{1}{2} H \,|_{v_{q-1}}^{v_q} + (-1)^q \frac{1}{2} H \,|_{v_q}^{v_{q+1}} \\
&\leq \frac{1}{2} \left| H \,|_{v_0}^{v_1} \right| + \frac{1}{2} \left| H \,|_{v_1}^{v_2} \right| + \frac{1}{2} \left| H \,|_{v_2}^{v_3} \right| + \cdots + \frac{1}{2} \left| H \,|_{v_{q-1}}^{v_q} \right| + \frac{1}{2} \left| H \,|_{v_q}^{v_{q+1}} \right| \\
&\leq D_q(F, G)
\end{aligned}$$

for any $(v_1, \dots, v_q) \in \mathbb{R}^q_{\leq}$, which implies that $\mathcal{A}_q(F - G) \leq D_q(F, G)$. Similarly, $\mathcal{A}_q(G - F) \leq D_q(F, G)$. Thus, we have $D_q(F, G) \geq \max\{\mathcal{A}_q(F - G), \mathcal{A}_q(G - F)\}$.

Conversely, let us prove that $D_q(F, G) \leq \max\{\mathcal{A}_q(F - G), \mathcal{A}_q(G - F)\}$ when $F \neq G$. It suffices to show that $D_q(F, G) - \epsilon < \max\{\mathcal{A}_q(F - G), \mathcal{A}_q(G - F)\}$ for any $0 < \epsilon < D_q(F, G)$. Choose $(v_1, \dots, v_q) \in \mathbb{R}^q_{\leq}$ such that

$$\frac{1}{2} \sum_{i=0}^{q} \left| (F - G) \,|_{v_i}^{v_{i+1}} \right| > D_q(F, G) - \epsilon > 0.$$

We may assume that $(F - G)(v_1) \geq 0$ without loss of generality. Set $H = F - G$ again. Let us consider the signs of $H \,|_{v_0}^{v_1}, H \,|_{v_1}^{v_2}, \dots, H \,|_{v_q}^{v_{q+1}}$. There exist $1 \leq j_1 < j_2 < \cdots < j_r \leq q$ such that

$$\begin{gathered}
H \,|_{v_0}^{v_1} \geq 0, H \,|_{v_1}^{v_2} \geq 0, \dots, H \,|_{v_{j_1 - 1}}^{v_{j_1}} \geq 0, \\
H \,|_{v_{j_1}}^{v_{j_1+1}} \leq 0, H \,|_{v_{j_1+1}}^{v_{j_1+2}} \leq 0, \dots, H \,|_{v_{j_2 - 1}}^{v_{j_2}} \leq 0, \\
H \,|_{v_{j_2}}^{v_{j_2+1}} \geq 0, H \,|_{v_{j_2+1}}^{v_{j_2+2}} \geq 0, \dots, H \,|_{v_{j_3 - 1}}^{v_{j_3}} \geq 0, \\
\cdots \\
H \,|_{v_{j_r}}^{v_{j_r+1}}, H \,|_{v_{j_r+1}}^{v_{j_r+2}}, \dots, H \,|_{v_q}^{v_{q+1}} \ \text{ have the same sign.}
\end{gathered}$$

Then

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=0}^{q}\left|(F-G)\,\Big|_{v_i}^{v_{i+1}}\right| \\
&= \frac{1}{2}\left|H\,\Big|_{v_0}^{v_1}\right| + \frac{1}{2}\left|H\,\Big|_{v_1}^{v_2}\right| + \frac{1}{2}\left|H\,\Big|_{v_2}^{v_3}\right| + \cdots + \frac{1}{2}\left|H\,\Big|_{v_{q-1}}^{v_q}\right| + \frac{1}{2}\left|H\,\Big|_{v_q}^{v_{q+1}}\right| \\
&= (\frac{1}{2}H\,\Big|_{v_0}^{v_1} + \frac{1}{2}H\,\Big|_{v_1}^{v_2} + \cdots + \frac{1}{2}H\,\Big|_{v_{j_1-1}}^{v_{j_1}}) - (\frac{1}{2}H\,\Big|_{v_{j_1}}^{v_{j_1+1}} + \cdots + \frac{1}{2}H\,\Big|_{v_{j_2-1}}^{v_{j_2}}) \\
&\qquad + \cdots + (-1)^r(\frac{1}{2}H\,\Big|_{v_{j_r}}^{v_{j_r+1}} + \cdots + \frac{1}{2}H\,\Big|_{v_q}^{v_{q+1}}) \\
&= \mathcal{A}(H)(v_{j_1},\dots,v_{j_r}),
\end{aligned}
$$

and hence $\mathcal{A}(H)(v_{j_1},\dots,v_{j_r}) > D_q(F,G)-\epsilon$. Since

$$\lim_{x\to\infty} H(x) = \lim_{x\to\infty}(F-G)(x) = 0,$$

there exists sufficiently large $M > v_{j_r}$ such that

$$\mathcal{A}(H)(v_{j_1},\dots,v_{j_r},\underbrace{M,M,\dots,M}_{(q-r)\text{ terms}}) > D_q(F,G)-\epsilon.$$

This implies that $\mathcal{A}_q(F-G) > D_q(F,G)-\epsilon$, and hence $D_q(F,G) \le \max\{\mathcal{A}_q(F-G),\mathcal{A}_q(G-F)\}$. This completes the proof. □

Thus, we have proved Theorem 2.4.

Definition 5.13. For $x\in\mathbb{R}$, we define

$$\mathcal{A}_{q,x}(H) = \sup_{v_1\le v_2\le\cdots\le v_q = x}\mathcal{A}(H)(v_1,v_2,\dots,v_q).$$

Note that

$$\mathcal{A}_q(H) = \sup_{x\in\mathbb{R}}\mathcal{A}_{q,x}(H).$$

Remark 5.14. For $q=1$ and $x\in\mathbb{R}$, we have

$$\mathcal{A}_{1,x}(H) = \mathcal{A}(H)(x) = H(x).$$

For $q=2$ and $x\in\mathbb{R}$, we have

$$
\begin{aligned}
\mathcal{A}_{2,x}(H) &= \sup_{v_1\le v_2 = x}\mathcal{A}(H)(v_1,v_2) \\
&= \sup_{v_1\le x}\mathcal{A}(H)(v_1) - H(x)
\end{aligned}
$$

$$= \sup_{v_1 \le x} \mathcal{A}_{1,v_1}(H) - H(x)$$
$$= \sup_{v_1 \le x} H(v_1) - H(x).$$

Proposition 5.15. *For a bounded function H on $\mathbb{R}$,*

$$\max\{\mathcal{A}_1(H), \mathcal{A}_1(-H)\} = \sup_{v\in\mathbb{R}} |H(v)|,$$
$$\max\{\mathcal{A}_2(H), \mathcal{A}_2(-H)\} = \sup_{v\in\mathbb{R}} H(v) - \inf_{v\in\mathbb{R}} H(v).$$

In particular,

$$D_2(F, G) = \sup_{v\in\mathbb{R}}(F - G)(v) - \inf_{v\in\mathbb{R}}(F - G)(v),$$

for $F, G \in \mathcal{F}$.

Proof. As the first formula is straightforward to prove, let us prove the second one. Since $\mathcal{A}_2(H) = \sup_{x\in\mathbb{R}} \mathcal{A}_{2,x}(H)$,

$$\mathcal{A}_2(H) = \sup_{x\in\mathbb{R}}\{\sup_{v\le x} H(v) - H(x)\}$$
$$\le \sup_{v\in\mathbb{R}} H(v) - \inf_{x\in\mathbb{R}} H(x).$$

Similarly, $\mathcal{A}_2(-H) \le \sup_{v\in\mathbb{R}} H(v) - \inf_{v\in\mathbb{R}} H(v)$. Hence,

$$\max\{\mathcal{A}_2(H), \mathcal{A}_2(-H)\} \le \sup_{v\in\mathbb{R}} H(v) - \inf_{v\in\mathbb{R}} H(v).$$

Conversely, choose $v_1, v_2 \in \mathbb{R}$ such that

$$\sup_{v\in\mathbb{R}} H(v) - \epsilon/2 < H(v_1) \le \sup_{v\in\mathbb{R}} H(v),$$
$$\inf_{v\in\mathbb{R}} H(v) \le H(v_2) < \inf_{v\in\mathbb{R}} H(v) + \epsilon/2$$

for $\epsilon > 0$. If $v_1 \le v_2$, then

$$\sup_{v\in\mathbb{R}} H(v) - \inf_{v\in\mathbb{R}} H(v) - \epsilon < \mathcal{A}(H)(v_1, v_2) \le \mathcal{A}_2(H).$$

If $v_1 > v_2$, then

$$\sup_{v\in\mathbb{R}} H(v) - \inf_{v\in\mathbb{R}} H(v) - \epsilon < \mathcal{A}(-H)(v_2, v_1) \leq \mathcal{A}_2(-H).$$

For any $\epsilon > 0$, $\sup_{v\in\mathbb{R}} H(v) - \inf_{v\in\mathbb{R}} H(v) - \epsilon < \max\{\mathcal{A}_2(H), \mathcal{A}_2(-H)\}$, and hence $\sup_{v\in\mathbb{R}} H(v) - \inf_{v\in\mathbb{R}} H(v) \leq \max\{\mathcal{A}_2(H), \mathcal{A}_2(-H)\}$. □

Proposition 5.15 implies (4) in Remark 2.5. The following proposition is useful for computing $\mathcal{A}_q(H)$ inductively.

PROPOSITION 5.16. *For $x \in \mathbb{R}$,*

$$\mathcal{A}_{q+1,x}(H) = \sup_{v_q \leq x} \mathcal{A}_{q,v_q}(H) + (-1)^q H(x).$$

PROOF. We have verified the equality for $q = 1$. For $q > 1$, we have

$$\begin{aligned}\mathcal{A}_{q+1,x}(H) &= \sup_{v_1\leq v_2\leq\cdots\leq v_{q+1}=x} \mathcal{A}(H)(v_1, v_2, \ldots, v_{q+1}) \\ &= \sup_{v_1\leq v_2\leq\cdots\leq v_q\leq x} \mathcal{A}(H)(v_1, v_2, \ldots, v_q, x) \\ &= \sup_{v_1\leq v_2\leq\cdots\leq v_q\leq x} \mathcal{A}(H)(v_1, v_2, \ldots, v_q) + (-1)^q H(x) \\ &= \sup_{v_q\leq x} \mathcal{A}_{q,v_q}(H) + (-1)^q H(x),\end{aligned}$$

which completes the proof. □

5.3. Proof of Theorem 2.7

THEOREM 5.17 (The Glivenko-Cantelli theorem. See the proof of [15, Section 2.1.4, Theorem A]). *For $F \in \mathcal{F}$, $\sup_{x\in\mathbb{R}}|F_n(x) - F(x)|$ converges completely to 0 as $n \to \infty$, i.e.,*

$$\sum_{n=1}^{\infty} P\left(\sup_{x\in\mathbb{R}}|F_n(x) - F(x)| > \epsilon\right) < \infty$$

for any $\epsilon > 0$.

It follows from Lemma 5.9 and Theorem 5.17 that

$$\sum_{n=1}^{\infty} P(D_q(F_n, F) > \epsilon) \leq \sum_{n=1}^{\infty} P\left(q \sup_{x\in\mathbb{R}}|F_n(x) - F(x)| > \epsilon\right)$$

$$= \sum_{n=1}^{\infty} P\left(\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| > \epsilon/q \right)$$
$$< \infty$$

for any $\epsilon > 0$. This proves Theorem 2.7.

Corollary 2.8 follows from the following:

COROLLARY 5.18. *For $F, G \in \mathcal{F}$,*

$$D_q(F_n, G) \to D_q(F, G) \ a.s.$$

as $n \to \infty$.

PROOF. The statement follows from that

$$|D_q(F_n, G) - D_q(F, G)| \le D_q(F_n, F) \to 0 \text{ a.s.}$$

as $n \to \infty$ by Theorem 5.17. □

5.4. Proof of Theorem 2.9

Let us denote by F^- the quantile function of $F \in \mathcal{F}$, i.e.,

$$F^-(y) := \inf\{x \in \mathbb{R} : F(x) \ge y\} \qquad (y \in [0,1]) \tag{14}$$

with the convention that $\inf \emptyset := \infty$ and $\inf \mathbb{R} := -\infty$. Let U be the standard uniform distribution function defined in (12).

THEOREM 5.19 (See [5, Proposition 1.1] or [6, Propositions 1 and 2] for reference).

(a) For any $y \in (0,1)$, $F^-(y)$ is a finite real number.

(b) For any $y \in (0,1)$, $F(F^-(y)) \ge y$.

(c) For any $x \in \mathbb{R}$ and $y \in (0,1)$, $F(x) \ge y$ if and only if $x \ge F^-(y)$.

(d) If the distribution function of Z is U, then the distribution function of $F^-(Z)$ is F.

Let $W_1, \dots, W_n$ be i.i.d. random variables on a probability space $(\Omega', \mathfrak{A}', P')$ with U, $X'_i := F^-(W_i)$ for $i = 1, \dots, n$, and

$$U_n(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty,x]}(W_i) \qquad (x \in \mathbb{R}),$$

$$F'_n(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty,x]}(X'_i) \qquad (x \in \mathbb{R}).$$

As described in Section 2.1, $X_1, \dots, X_n$ are i.i.d. random variables on $(\Omega, \mathfrak{A}, P)$ with F and

$$F_n(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty,x]}(X_i) \qquad (x \in \mathbb{R}).$$

The following corollaries are immediate consequences of Theorem 5.19.

COROLLARY 5.20. *The random variables $X'_1, \dots, X'_n$ are i.i.d. with the same distribution function F. The probability measure on $\mathcal{B}(\mathbb{R}^n)$ induced by $(X'_1, \dots, X'_n)$ and that by $(X_1, \dots, X_n)$ are the same, i.e.,*

$$P'((X'_1, \dots, X'_n) \in A) = P((X_1, \dots, X_n) \in A) \qquad (A \in \mathcal{B}(\mathbb{R}^n)).$$

COROLLARY 5.21. *It holds almost surely that $F'_n = U_n \circ F$.*

Note that $F = U \circ F$ holds obviously.

THEOREM 5.22. *For each $q \in \mathbb{N}_+$, $D_q(F'_n, F) \le D_q(U_n, U)$ almost surely. If F is continuous on $\mathbb{R}$, $D_q(F'_n, F) = D_q(U_n, U)$ almost surely.*

PROOF. For $\boldsymbol{v} = (v_1, \dots, v_q) \in \mathbb{R}^q_{\le}$, it follows from (2) and Corollary 5.21 that

$$\begin{aligned} r_{F'_n,F}(\boldsymbol{v}) &= \sum_{i=0}^{q} \min\bigl\{F'_n|_{v_i}^{v_{i+1}}, F|_{v_i}^{v_{i+1}}\bigr\} \\ &= \sum_{i=0}^{q} \min\bigl\{U_n \circ F|_{v_i}^{v_{i+1}}, U \circ F|_{v_i}^{v_{i+1}}\bigr\} \\ &= \sum_{i=0}^{q} \min\Bigl\{U_n|_{F(v_i)}^{F(v_{i+1})}, U|_{F(v_i)}^{F(v_{i+1})}\Bigr\}, \end{aligned}$$

where $F(v_0) = F(-\infty) = 0$ and $F(v_{q+1}) = F(\infty) = 1$. Since $U_n(0) = 0 = U_n(-\infty)$ and $U_n(1) = 1 = U_n(\infty)$ with probability 1, we have

$$r_{F'_n,F}(\boldsymbol{v}) = r_{U_n,U}(F(\boldsymbol{v})), \qquad F(\boldsymbol{v}) := (F(v_1), \dots, F(v_n)) \in \mathbb{R}^q_{\leq}$$

almost surely. Hence

$$\inf_{\boldsymbol{v}\in\mathbb{R}^q_{\leq}} r_{F'_n,F}(\boldsymbol{v}) = \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\leq}} r_{U_n,U}(F(\boldsymbol{v})) \geq \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\leq}} r_{U_n,U}(\boldsymbol{v})$$

and $D_q(F'_n, F) \leq D_q(U_n, U)$ almost surely. If F is continuous on $\mathbb{R}$, we have $F(\mathbb{R}) \supset (0, 1)$, so that

$$\inf_{\boldsymbol{v}\in\mathbb{R}^q_{\leq}} r_{U_n,U}(F(\boldsymbol{v})) = \inf_{\boldsymbol{v}\in\mathbb{R}^q_{\leq}} r_{U_n,U}(\boldsymbol{v})$$

and $D_q(F'_n, F) = D_q(U_n, U)$ almost surely. $\square$

Let us define, for each $(t_1, \dots, t_n) \in \mathbb{R}^n$,

$$\Phi_{(t_1,\dots,t_n)}(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty,x]}(t_i) \qquad (x \in \mathbb{R}).$$

We also define, for each $q \in \mathbb{N}_+$ and $\xi \in \mathcal{F}$,

$$\widetilde{D}_{q,\xi}(\boldsymbol{t}) := D_q(\Phi_{\boldsymbol{t}}, \xi) \qquad (\boldsymbol{t} \in \mathbb{R}^n).$$

REMARK 5.23. We see that $D_q(U_n, U) = \widetilde{D}_{q,U} \circ (W_1, \dots, W_n)$ and $D_q(F'_n, F) = \widetilde{D}_{q,F} \circ (X'_1, \dots, X'_n)$ on $(\Omega', \mathfrak{A}', P')$, and $D_q(F_n, F) = \widetilde{D}_{q,F} \circ (X_1, \dots, X_n)$ on $(\Omega, \mathfrak{A}, P)$.

THEOREM 5.24. *For each $q \in \mathbb{N}_+$ and $\xi \in \mathcal{F}$, $\widetilde{D}_{q,\xi}$ is a Borel measurable function on $\mathbb{R}^n$.*

PROOF. Let us put $\mathbb{Q}^q_{\leq} := \mathbb{R}^q_{\leq} \cap \mathbb{Q}^q$. It is obvious that

$$\inf_{\boldsymbol{v}\in\mathbb{R}^q_{\leq}} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{v}) \leq \inf_{\boldsymbol{v}\in\mathbb{Q}^q_{\leq}} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{v}) \qquad (\boldsymbol{t} \in \mathbb{R}^n). \tag{15}$$

For any $\epsilon > 0$, there exists $\boldsymbol{x} := (x_1, \ldots, x_q) \in \mathbb{R}^q_{\leq}$ such that $r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{x}) < \inf_{\boldsymbol{v} \in \mathbb{R}^q_{\leq}} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{v}) + \epsilon$. Let $\{\boldsymbol{a}_i := (a_{i,1}, \ldots, a_{i,q})\}$ be a sequence in $\mathbb{Q}^q_{\leq}$ such that for each $j \in \{1, \ldots, q\}$, $a_{i,j}$ converges to x_j from the right as $i \to \infty$. Since $\Phi_{\boldsymbol{t}}$ and ξ are right-continuous on $\mathbb{R}$, $\lim_{i\to\infty} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{a}_i) = r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{x})$. Hence $r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{b}) < \inf_{\boldsymbol{v} \in \mathbb{R}^q_{\leq}} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{v}) + \epsilon$ for some $\boldsymbol{b} \in \mathbb{Q}^q_{\leq}$. With (15), we have

$$\inf_{\boldsymbol{v} \in \mathbb{R}^q_{\leq}} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{v}) = \inf_{\boldsymbol{v} \in \mathbb{Q}^q_{\leq}} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{v}) \qquad (\boldsymbol{t} \in \mathbb{R}^n), \tag{16}$$

so that

$$\widetilde{D}_{q,\xi}(\boldsymbol{t}) = 1 - \inf_{\boldsymbol{v} \in \mathbb{Q}^q_{\leq}} r_{\Phi_{\boldsymbol{t}},\xi}(\boldsymbol{v}) \qquad (\boldsymbol{t} \in \mathbb{R}^n) \tag{17}$$

by definition.

It is immediate from definition that $\Phi_{\bullet}(x) \colon \mathbb{R}^n \to \mathbb{R}$ and $r_{\Phi_{\bullet},\xi}(\boldsymbol{v}) \colon \mathbb{R}^n \to \mathbb{R}$ are Borel measurable for each $x \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^q_{\leq}$, respectively. Hence $\widetilde{D}_{q,\xi} \colon \mathbb{R}^n \to \mathbb{R}$ can be described by the countable infimum of Borel measurable functions by (17). This implies the claim. □

The next corollary follows from Remark 5.23 and Theorem 5.24.

Corollary 5.25. *$D_q(U_n, U)$ and $D_q(F'_n, F)$ are random variables on $(\Omega', \mathfrak{A}', P')$, and $D_q(F_n, F)$ is a random variable on $(\Omega, \mathfrak{A}, P)$.*

Theorem 5.26. *For each $q \in \mathbb{N}_+$, the probability measure on $\mathcal{B}(\mathbb{R})$ induced by $D_q(U_n, U)$ and that by $D_q(F_n, F)$ are the same if F is continuous on $\mathbb{R}$.*

Proof. For any $A \in \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned} P'\big(D_q(U_n, U)^{-1}(A)\big) &= P'\big(D_q(F'_n, F)^{-1}(A)\big) \\ &= \big(P' \circ (X'_1, \ldots, X'_n)^{-1}\big)\Big(\widetilde{D}^{-1}_{q,F}(A)\Big) \\ &= \big(P \circ (X_1, \ldots, X_n)^{-1}\big)\Big(\widetilde{D}^{-1}_{q,F}(A)\Big) \\ &= P\big(D_q(F_n, F)^{-1}(A)\big) \end{aligned}$$

by Corollaries 5.20 and 5.25, Theorems 5.22 and 5.24, and Remark 5.23. □

This theorem implies Theorem 2.9.

5.5. Proof of Theorem 2.10

Definition 5.27 (See [4, pages 353 and 443] for reference). Let T be a set and $(\Omega, \mathfrak{A}, P)$ a probability space. A mapping $Y\colon T \times \Omega \to \mathbb{R}$ is called a *stochastic process* if $Y_t := Y(t, \cdot)\colon \Omega \to \mathbb{R}$ is measurable for each $t \in T$. We say that Y is Gaussian if $(Y_{t_1}, \ldots, Y_{t_m})\colon \Omega \to \mathbb{R}^m$ is Gaussian for any $t_1, \ldots, t_m \in T$.

Definition 5.28 (See [4, page 445] for reference). Let $Y\colon T \times \Omega \to \mathbb{R}$ be a Gaussian stochastic process with $T = [0, 1]$. If the following conditions hold:

- $\mathbb{E}[Y_t] = 0$ for any $t \in T$,
- $\mathbb{E}[Y_s Y_t] = s(1 - t)$ for any $s, t \in T$ with $s \le t$,
- Y is *sample continuous*, i.e., $Y(\cdot, \omega)\colon T \to \mathbb{R}$ is continuous for any $\omega \in \Omega$,

then Y is called a *Brownian bridge.*

Theorem 5.29 ([4, Proposition 12.3.4]). *For a Brownian bridge Y and any $a > 0$,*

$$P\Bigl(\sup_{t\in[0,1]} |Y_t| \ge a\Bigr) = 2\sum_{i=1}^{\infty}(-1)^{i-1}\exp(-2i^2a^2).$$

Theorem 5.30 ([4, Proposition 12.3.6]). *For a Brownian bridge Y and any $a > 0$,*

$$P\Bigl(\sup_{t\in[0,1]} Y_t - \inf_{t\in[0,1]} Y_t \ge a\Bigr) = 2\sum_{i=1}^{\infty}(4i^2a^2 - 1)\exp(-2i^2a^2).$$

Definition 5.31 (See [2, Section 12] for reference). Let $D[0, 1]$ be the space of real functions on $[0, 1]$ that are right-continuous and have left-hand limits (such functions are called càdlàg functions). Let Λ be the set of strictly increasing, continuous mappings of $[0, 1]$ onto itself. For $g, h \in D[0, 1]$, define

$$\|g\| := \sup_{t\in[0,1]} |g(t)| < \infty,$$

$$d(g,h) := \inf_{\lambda\in\Lambda} \max\{\|\lambda - I\|, \|g - h\circ\lambda\|\} < \infty,$$

where I denotes the identity map on $[0,1]$. The function d is a metric on $D[0,1]$, which defines the Skorohod topology.

Lemma 5.32 (See [2, page 124] for reference). *For an element g and a sequence $\{g_n\}$ in $D[0,1]$, $\lim_{n\to\infty} g_n = g$ if and only if $\lim_{n\to\infty}\|\lambda_n - I\| = 0$ and $\lim_{n\to\infty}\|g - g_n\circ\lambda_n\| = 0$ for some sequence $\{\lambda_n\}$ in Λ.*

Proof. If $\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} d(g, g_n) = \lim_{n\to\infty}\inf_{\lambda\in\Lambda} \max\{\|\lambda - I\|, \|g - g_n\circ\lambda\|\} = 0$. Since there exists $\lambda_n \in \Lambda$ for each $n \in \mathbb{N}_+$ such that

$$\inf_{\lambda\in\Lambda} \max\{\|\lambda - I\|, \|g - g_n\circ\lambda\|\} \le \max\{\|\lambda_n - I\|, \|g - g_n\circ\lambda_n\|\} < \inf_{\lambda\in\Lambda} \max\{\|\lambda - I\|, \|g - g_n\circ\lambda\|\} + \frac{1}{n},$$

$\lim_{n\to\infty}\|\lambda_n - I\| = 0$ and $\lim_{n\to\infty}\|g - g_n\circ\lambda_n\| = 0$ hold.

On the other hand, suppose $\lim_{n\to\infty}\|\lambda_n - I\| = 0$ and $\lim_{n\to\infty}\|g - g_n\circ\lambda_n\| = 0$ for some sequence $\{\lambda_n\}$ in Λ. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\|\lambda_n - I\| < \epsilon$ and $\|g - g_n\circ\lambda_n\| < \epsilon$, so that $\inf_{\lambda\in\Lambda}\max\{\|\lambda - I\|, \|g - g_n\circ\lambda\|\} < \epsilon$. Hence $\lim_{n\to\infty} g_n = g$. □

Definition 5.33 (See [2, pages 7 and 15–16] for reference). Suppose P is a Borel probability measure and $\{P_n\}$ a sequence of Borel probability measures on a metric space S. We say that P_n *converges weakly* to P (denoted by $P_n \Rightarrow P$) if $\lim_{n\to\infty} P_n g = Pg$ for all bounded continuous functions $g\colon S \to \mathbb{R}$, where $Pg := \int_S g\,\mathrm{d}P$. A set $A \subset S$ whose boundary ∂A satisfies $P(\partial A) = 0$ is called a *P-continuity set*.

Theorem 5.34 ([2, Theorem 2.1]). *Suppose P is a Borel probability measure and $\{P_n\}$ a sequence of Borel probability measures on a metric space S. Then these five conditions are equivalent:*

(i) *$P_n \Rightarrow P$.*

(ii) *$\lim_{n\to\infty} P_n g = Pg$ for all bounded, uniformly continuous functions $g\colon S \to \mathbb{R}$.*

(iii) $\limsup_{n\to\infty} P_n(K) \le P(K)$ *for any closed set* $K \subset S$.

(iv) $\liminf_{n\to\infty} P_n(V) \ge P(V)$ *for any open set* $V \subset S$.

(v) $\lim_{n\to\infty} P_n(A) = P(A)$ *for any* P*-continuity set* $A \subset S$.

Definition 5.35 (See [2, pages 24–26] for reference). We call a map from a probability space $(\Omega, \mathfrak{A}, P)$ to a metric space S a *random element* if it is Borel measurable. (As is customary, we call it a random variable if, in addition, $S = \mathbb{R}$.) Suppose Z is a random element and $\{Z_n\}$ a sequence of random elements from $(\Omega, \mathfrak{A}, P)$ to S. The *law* of Z is the Borel probability measure $L_Z := P \circ Z^{-1}$ on S. We say that Z_n *converges in distribution* to Z (denoted by $Z_n \Rightarrow Z$) if $L_{Z_n} \Rightarrow L_Z$. A set $A \subset S$ with $P(Z^{-1}(\partial A)) = 0$ is called a Z*-continuity set.*

Theorem 5.36 ([2, page 26]). *Suppose* Z *is a random element and* $\{Z_n\}$ *a sequence of random elements from a probability space* $(\Omega, \mathfrak{A}, P)$ *to a metric space* S. *Then these five conditions are equivalent:*

(i) $Z_n \Rightarrow Z$.

(ii) $\lim_{n\to\infty} \mathbb{E}[g(Z_n)] = \mathbb{E}[g(Z)]$ *for all bounded, uniformly continuous functions* $g\colon S \to \mathbb{R}$.

(iii) $\limsup_{n\to\infty} P(Z_n \in K) \le P(Z \in K)$ *for any closed set* $K \subset S$.

(iv) $\liminf_{n\to\infty} P(Z_n \in V) \ge P(Z \in V)$ *for any open set* $V \subset S$.

(v) $\lim_{n\to\infty} P(Z_n \in A) = P(Z \in A)$ *for any* Z*-continuity set* $A \subset S$.

Theorem 5.37 ([2, page 20]). *Suppose* P *is a Borel probability measure and* $\{P_n\}$ *a sequence of Borel probability measures on a metric space* S. *Let* S' *be another metric space and* $h\colon S \to S'$ *a continuous map. If* $P_n \Rightarrow P$, *then* $P_n \circ h^{-1} \Rightarrow P \circ h^{-1}$.

Remark 5.38 (See [2, page 135] for reference). Note that $Y\colon (\Omega, \mathcal{A}, P) \to D[0,1]$ is a random element if and only if $Y_\bullet\colon [0,1] \times \Omega \to \mathbb{R}$ is a stochastic process (i.e., $Y_t\colon \Omega \to \mathbb{R}$ is a random variable for all $t \in [0,1]$).

For $H \in D[0,1]$, $\mathcal{A}_q(H) = \sup_{0 \le t_1 \le \cdots \le t_q \le 1} \mathcal{A}(H)(t_1, \ldots, t_q)$ can be defined, since H is a bounded function on $[0,1]$.

Proposition 5.39. *The map $\mathcal{A}_q : D[0,1] \to \mathbb{R}$ is continuous.*

Proof. Let $\{H_n\}$ be a sequence converging to $H \in D[0,1]$. By Lemma 5.32, there exist $\lambda_n \in \Lambda$ $(n = 1, 2, \ldots)$ such that $\max\{||H - H_n \circ \lambda_n||, ||\lambda_n - I||\} \to 0$ as $n \to \infty$. For $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $||H - H_n \circ \lambda_n|| < \epsilon/2q$ for $n > N$. For any $(v_1, \ldots, v_q) \in [0,1]^q_{\le} := \mathbb{R}^q_{\le} \cap [0,1]^q$,

$$|\mathcal{A}(H)(v_1, \ldots, v_q) - \mathcal{A}(H_n)(\lambda_n(v_1), \ldots, \lambda_n(v_q))| < \epsilon/2 \tag{18}$$

for $n > N$. Thus,

$$\mathcal{A}(H_n)(\lambda_n(v_1), \ldots, \lambda_n(v_q)) < \mathcal{A}(H)(v_1, \ldots, v_q) + \epsilon/2 \le \mathcal{A}_q(H) + \epsilon/2$$

and hence

$$\mathcal{A}_q(H_n) \le \mathcal{A}_q(H) + \epsilon/2 < \mathcal{A}_q(H) + \epsilon.$$

On the other hand, take $(v_1, \ldots, v_q) \in [0,1]^q_{\le}$ such that

$$\mathcal{A}_q(H) - \epsilon/2 < \mathcal{A}(H)(v_1, \ldots, v_q).$$

For $n > N$,

$$\begin{aligned} \mathcal{A}_q(H) - \epsilon &< \mathcal{A}(H)(v_1, \ldots, v_q) - \epsilon/2 \\ &< \mathcal{A}(H_n)(\lambda_n(v_1), \ldots, \lambda_n(v_q)) \le \mathcal{A}_q(H_n) \end{aligned}$$

by (18). Thereby, we have

$$|\mathcal{A}_q(H_n) - \mathcal{A}_q(H)| < \epsilon$$

for $n > N$, which implies the continuity of $\mathcal{A}_q$. □

Definition 5.40. For $q \in \mathbb{N}$, we define $\phi_q : D[0,1] \to \mathbb{R}$ by $\phi_q(H) = \max\{\mathcal{A}_q(H), \mathcal{A}_q(-H)\}$. By Proposition 5.39, ϕ_q is continuous.

Theorem 5.41. *Let $Y : [0,1] \times \Omega \to \mathbb{R}$ be a Brownian bridge. Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables on a probability space $(\Omega, \mathfrak{A}, P)$*

with a continuous distribution function $F \in \mathcal{F}$, and $F_n(t)$ is an empirical distribution function with respect to $X_1, \dots, X_n$. Then

$$\lim_{n\to\infty} P(D_q(F_n, F) \geq \frac{a}{\sqrt{n}}) = P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a)$$

for every $a > 0$ at which $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a)$ is continuous.

Proof. Let U_n be defined as in Section 5.4 on $(\Omega, \mathfrak{A}, P)$, $Y_t^n = \sqrt{n}(U_n(t) - U(t))$ for $t \in [0,1]$. Then Y^n and Y are random elements on $(\Omega, \mathfrak{A}, P)$ to $D[0,1]$, and $Y^n \Rightarrow Y$ (which means $L_{Y^n} \Rightarrow L_Y$) by [2, Theorem 14.3]. Since

$$L_{Y^n} \circ \phi_q^{-1}([a,\infty)) = P\Big(\max\{\mathcal{A}_q(Y^n), \mathcal{A}_q(-Y^n)\} \geq a\Big),$$
$$L_Y \circ \phi_q^{-1}([a,\infty)) = P\Big(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a\Big),$$

and $\lim_{n\to\infty}(L_{Y^n} \circ \phi_q^{-1})([a,\infty)) = (L_Y \circ \phi_q^{-1})([a,\infty))$ for $a \in \mathbb{R}$ at which $(L_Y \circ \phi_q^{-1})([a,\infty))$ is continuous by Theorems 5.34 and 5.37 and Proposition 5.39, we obtain

$$\lim_{n\to\infty} P\Big(\max\{\mathcal{A}_q(Y^n), \mathcal{A}_q(-Y^n)\} \geq a\Big) = P\Big(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a\Big) \tag{19}$$

for $a \in \mathbb{R}$ at which $(L_Y \circ \varphi^{-1})([a,\infty))$ is continuous. By Theorem 5.12,

$$\begin{aligned}\max\{\mathcal{A}_q(Y^n), \mathcal{A}_q(-Y^n)\} &= \sqrt{n}\max\{\mathcal{A}_q(U_n - U), \mathcal{A}_q(U_n - U)\}\\ &= \sqrt{n} D_q(U_n, U).\end{aligned}$$

Since the distributions of $D_q(U_n, U)$ and $D_q(F_n, F)$ are same for continuous $F \in \mathcal{F}$ by Theorem 5.26,

$$\lim_{n\to\infty} P\Big(D_q(F_n, F) \geq \frac{a}{\sqrt{n}}\Big) = P\Big(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a\Big) \tag{20}$$

for every $a > 0$ at which $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a)$ is continuous. □

This theorem is equivalent to Theorem 2.10.

Remark 5.42. For $a > 0$, $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a)$ is continuous at a if and only if $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} = a) = 0$, which are also equivalent to the condition that $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \leq a)$ is continuous at a. Under these equivalent conditions, we see that

$$\begin{aligned}(21) \qquad \lim_{n\to\infty} P(D_q(F_n, F) \leq \frac{a}{\sqrt{n}}) &= P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \leq a) \\ &= 1 - P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a).\end{aligned}$$

Remark 5.43. We conjecture that $P(\max\{\mathcal{A}_q(Y), \mathcal{A}_q(-Y)\} \geq a)$ is a continuous function of $a > 0$. We leave the proof of this conjecture and the determination of the explicit form of the function $P(\mathcal{A}_q(Y) \geq a)$ to future work.

Example 5.44. For a Brownian bridge Y,

$$\begin{aligned}\max\{\mathcal{A}_1(Y), \mathcal{A}_1(-Y)\} &= \sup_{t\in[0,1]} |Y_t|, \\ \max\{\mathcal{A}_2(Y), \mathcal{A}_2(-Y)\} &= \sup_{t\in[0,1]} Y_t - \inf_{t\in[0,1]} Y_t\end{aligned}$$

by Proposition 5.15. By Theorems 5.29, 5.30, and 5.41,

$$(22) \qquad \lim_{n\to\infty} P\left(D_1(F_n, F) \geq \frac{a}{\sqrt{n}}\right) = 2\sum_{i=1}^{\infty}(-1)^{i-1}\exp(-2i^2a^2),$$

$$(23) \qquad \lim_{n\to\infty} P\left(D_2(F_n, F) \geq \frac{a}{\sqrt{n}}\right) = 2\sum_{i=1}^{\infty}(4i^2a^2 - 1)\exp(-2i^2a^2)$$

for any $a > 0$, since $P(\mathcal{A}_q(Y) \geq a)$ $(q = 1, 2)$ is continuous for every $a > 0$. These are the well-known asymptotic formula for the p-value of the one-sample Kolmogorov–Smirnov test and the Kuiper test, respectively.

Corollary 2.11 follows from (23).

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Supplementary Material

The source code of the experiment in Section 3, which requires Python 3.11.8 and the following packages:

- `numpy==1.26.4`
- `pandas==2.2.2`
- `scipy==1.13.0`
- `matplotlib==3.8.4`


```
# Copyright (c) 2024-2025 Atsushi Komaba

import math
import numpy as np
import pandas as pd
from scipy import stats
from matplotlib import pyplot as plt

q_values = [2, 3]

# The number of D_3 values computed to obtain
# its empirical distribution
num_d_q_values_under_h0 = 100_000

# Repetition time for the experiments
num_repeat = 100_000

# Sample sizes used in the experiments
sample_sizes = [2**i for i in range(3, 13)]

# Significance level
significance_level = 0.05


dists = {
  "Normal(0, 1)": stats.norm(),
  "Normal(0.2, 1)": stats.norm(0.2, 1),
```

```
  "Normal(0, 1.1)": stats.norm(0, 1.1),
  "Trapezoid": stats.trapezoid(
    (2 - math.sqrt(2)) / 4, (2 + math.sqrt(2)) / 4, -2, 4
  ),
  "SkewNormal(1.6, -0.6)": stats.skewnorm(1.6, -0.6),
  "SkewNormal(-1.6, 0.6)": stats.skewnorm(-1.6, 0.6),
}

dist_pairs = [
  ["Normal(0.2, 1)", "Normal(0, 1)"],
  ["Normal(0, 1.1)", "Normal(0, 1)"],
  ["Trapezoid", "Normal(0, 1)"],
  ["SkewNormal(1.6, -0.6)", "SkewNormal(-1.6, 0.6)"],
]


def d_q(data, cdf, q, axis=0):
  sample_size = data.shape[axis]
  extended_length = (sample_size + 1) * 2
  data.sort(axis)
  cdf_values = np.repeat(cdf(data), 2, axis)
  zero_pad = np.zeros_like(np.take(data, [0], axis))
  one_pad = np.ones_like(zero_pad)
  padded_cdf = np.concatenate((zero_pad, cdf_values, one_pad), axis)
  del cdf_values, zero_pad, one_pad
  assert padded_cdf.shape[axis] == extended_length
  ecdf_steps = np.linspace(0, 1, sample_size + 1)
  repeated_ecdf = np.repeat(ecdf_steps, 2)
  broadcast_shape = (extended_length,) + (1,) * (data.ndim - axis - 1)
  reshaped_ecdf = np.reshape(repeated_ecdf, broadcast_shape)
  diff = padded_cdf - reshaped_ecdf
  del data, padded_cdf, reshaped_ecdf
  diff_max = diff_min = diff
  for i in range(1, q):
    sign = 1 if i % 2 == 0 else -1
    diff_max = np.maximum.accumulate(diff_max, axis) + sign * diff
    diff_min = np.minimum.accumulate(diff_min, axis) + sign * diff
  return np.maximum(np.max(diff_max, axis), -np.min(diff_min, axis))


def main():
  rng = np.random.default_rng(0)

  uniform_distribution = stats.uniform()
  d_q_values_under_h0 = {}
  for sample_size in sample_sizes:
    print(f"Sample size: {sample_size:,}")
    x = uniform_distribution.rvs(
      (sample_size, num_d_q_values_under_h0), rng
    )
    d_q_values_under_h0[sample_size] = {
```

```
    q: np.sort(d_q(x, uniform_distribution.cdf, q))
    for q in q_values
  }

fig_ecdf = plt.figure()
for q_index, q in enumerate(q_values):
  ax_dist = fig_ecdf.add_subplot(2, 1, q_index + 1)
  if q == 2:
    a = np.linspace(0.5, 2.5, 1000)
    i = np.arange(1, 10)
    sq = np.outer(a, i) ** 2
    asymp_cdf = 1 - 2 * np.sum(
      (4 * sq - 1) * np.exp(-2 * sq), axis=1
    )
    asymp_cdf[a < 0.4] = 0
    ax_dist.plot(
      a,
      asymp_cdf,
      color=str(1 - 1 / 5),
      label=r"$n \rightarrow \infty$ (asymptotic)",
    )
  for sample_size_index, log2_sample_size in enumerate(
    [12, 9, 6, 3]
  ):
    sample_size = 2**log2_sample_size
    ax_dist.ecdf(
      d_q_values_under_h0[sample_size][q] * math.sqrt(sample_size),
      label=f"$n = 2^{{{log2_sample_size}}}$ (empirical)",
      color=str((3 - sample_size_index) / 5),
      linestyle=(0, (4 - sample_size_index, 1)),
    )
  handles, labels = ax_dist.get_legend_handles_labels()
  ax_dist.legend(handles[::-1], labels[::-1])
  ax_dist.set_xlabel(rf"$\sqrt{{n}} D_{q}(U_n, U)$")
fig_ecdf.tight_layout()
fig_ecdf.savefig("D_q_eCDF.pdf")

fig_powers = plt.figure(figsize=[12.8, 9.6])
for dist_index, (sampling_dist_name, ref_dist_name) in enumerate(
  dist_pairs
):
  print(
    f"Sampling distribution: {sampling_dist_name}, "
    f"reference distribution: {ref_dist_name}"
  )
  sampling_dist = dists[sampling_dist_name]
  ref_dist = dists[ref_dist_name]
  ax_index = dist_index + (dist_index // 2) * 2
  ax_pdf = fig_powers.add_subplot(4, 2, ax_index + 1)
  x_pdf = np.linspace(-3, 3, 100)
  ax_pdf.plot(
```

```
  x_pdf,
  sampling_dist.pdf(x_pdf),
  color="black",
  linestyle="--",
  label="Sampling distribution",
)
ax_pdf.plot(
  x_pdf,
  ref_dist.pdf(x_pdf),
  color="gray",
  linestyle="-",
  label="Reference distribution",
)
ax_pdf.legend()
ax_pdf.set_title(f"{sampling_dist_name}, {ref_dist_name}")

powers = pd.DataFrame()
for sample_size in sample_sizes:
  print(f"Sample size: {sample_size:,}")
  x = sampling_dist.rvs((sample_size, num_repeat), rng)

  ks_p_values = stats.ks_1samp(x, ref_dist.cdf).pvalue
  ks_num_rejected = np.count_nonzero(
    ks_p_values < significance_level
  )
  powers.loc[sample_size, "Kolmogorov-Smirnov (OVL-1)"] = (
    ks_num_rejected / num_repeat
  )

  cvm_p_values = stats.cramervonmises(x, ref_dist.cdf).pvalue
  cvm_num_rejected = np.count_nonzero(
    cvm_p_values < significance_level
  )
  powers.loc[sample_size, "Cram\xe9r von-Mises"] = (
    cvm_num_rejected / num_repeat
  )

  for q in q_values:
    d_q_statistics = d_q(x, ref_dist.cdf, q)
    d_q_p_values = (
      1
      - np.searchsorted(
        d_q_values_under_h0[sample_size][q], d_q_statistics
      )
      / num_d_q_values_under_h0
    )
    d_q_num_rejected = np.count_nonzero(
      d_q_p_values < significance_level
    )
    label = "Kuiper (OVL-2)" if q == 2 else f"OVL-{q}"
    powers.loc[sample_size, label] = d_q_num_rejected / num_repeat
```

```
    ax_powers = fig_powers.add_subplot(
      4, 2, ax_index + 3, xscale="log"
    )
    ax_powers.set_ylim(-0.05, 1.05)
    powers.plot(ax=ax_powers, style=["--k", ":k", "-.k", "-k"])
  fig_powers.tight_layout()
  fig_powers.savefig("power_comparison.pdf")


if __name__ == "__main__":
  main()
```

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