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Holomorphic Lie Algebroid Connections on Holomorphic Principal Bundles on Compact Riemann Surfaces

By Indranil Biswas

Abstract. For a Γ–equivariant holomorphic Lie algebroid (V, ϕ), on a compact Riemann surface X equipped with an action of a finite group Γ, we investigate the equivariant holomorphic Lie algebroid connections on holomorphic principal G–bundles over X, where G is a connected affine complex reductive group. If (V, ϕ) is nonsplit, then it is proved that every holomorphic principal G–bundle admits an equivariant holomorphic Lie algebroid connection. If (V, ϕ) is split, then it is proved that the following four statements are equivalent:

(1) An equivariant principal G–bundle E_G admits an equivariant holomorphic Lie algebroid connection.

(2) The equivariant principal G–bundle E_G admits an equivariant holomorphic connection.

(3) The principal G–bundle E_G admits a holomorphic connection.

(4) For every triple $(P, L(P), \chi)$, where $L(P)$ is a Levi subgroup of a parabolic subgroup $P \subset G$ and χ is a holomorphic character of $L(P)$, and every Γ–equivariant holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$, the degree of the line bundle over X associated to $E_{L(P)}$ for χ is zero.

The correspondence between Γ–equivariant principal G–bundles over X and parabolic G–bundles on X/Γ translates the above result to the context of parabolic G–bundles.

Contents

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1. Introduction

A well-known theorem of Atiyah and Weil says the following: A holomorphic vector bundle E on a compact connected Riemann surface X admits a holomorphic connection if and only if the degree of each indecomposable component of E is zero [At], [We]. This criterion for the existence of holomorphic connections extends to holomorphic principal G–bundles over X, where G is a reductive affine algebraic group defined over $\mathbb{C}$ [AB].

The notion of a holomorphic connection on a holomorphic vector bundle E extends to the notion of holomorphic Lie algebroid connections on E, which we briefly recall.

A holomorphic Lie algebroid over X is a pair (V, ϕ), where V is a holomorphic vector bundle over X equipped with the structure of a $\mathbb{C}$–bilinear Lie algebra on its sheaf of holomorphic sections, and $\phi : V \longrightarrow TX$ is an $\mathcal{O}_X$–linear homomorphism satisfying the Leibniz rule

$$[s, ft] = f[s, t] + \phi(s)(f)\,t$$

for all locally defined holomorphic sections s, t of V and all locally defined holomorphic functions f.

We work with an equivariant set-up, meaning a finite subgroup $\Gamma \subset \mathrm{Aut}(X)$ is fixed, and all objects and structures on X are taken to be Γ–equivariant.

A Lie algebroid (V, ϕ) is called split if there is a holomorphic Γ–equivariant homomorphism $\eta : TX \longrightarrow V$ such that $\phi \circ \eta = \mathrm{Id}_{TX}$. If (V, ϕ) is not split, then it is called nonsplit. See Example 3.1 for nonsplit and split Lie algebroids.

Let E_G be a Γ–equivariant holomorphic principal G–bundle over X, where G, as before, is a reductive affine algebraic group defined over $\mathbb{C}$.

Using the Atiyah bundle for E_G and the pair $(V,\,\phi)$, a $\mathbb{C}$–Lie algebra bundle $\mathcal{A}(E_G)$ is constructed which fits in the following short exact sequence of Γ–equivariant vector bundles over X:

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathcal{A}(E_G) \xrightarrow{\rho} V \longrightarrow 0,$$

where $\mathrm{ad}(E_G)$ is the adjoint bundle for E_G. An equivariant holomorphic Lie algebroid connection on E_G is a Γ–equivariant holomorphic homomorphism $\delta : V \longrightarrow \mathcal{A}(E_G)$ such that $\rho \circ \delta = \mathrm{Id}_V$. In the special case where $(V,\,\phi) = (TX,\,\mathrm{Id}_{TX})$, a holomorphic Lie algebroid connection on E_G is a usual holomorphic connection on E_G.

There is a large body of research on Lie algebroids and Lie algebroid connections. Bruzzo and Rubtsov investigated the cohomology and moduli spaces of skew-holomorphic Lie algebroids [BR]. Tortella introduced modules over Lie algebroids and described moduli space of flat Lie algebroid connections which are also called Λ–modules [To1], [To2]. In [AO], Alfaya and Oliveira studied the moduli space of flat Lie algebroid connections and proved numerous properties of the moduli space [AO]. Bruzzo-Mencattini-Rubtsov-Tortella investigated extensions of Lie algebroids [BMRT]. Laurent-Gengoux, Stiénon and Xu investigate the relationships between holomorphic Lie algebroids and holomorphic Poisson structures.

Our aim here is to give a criterion for the existence of equivariant holomorphic Lie algebroid connections on an equivariant holomorphic principal G–bundle over X. We prove the following (see Theorem 6.1):

Theorem 1.1.

- *Let $(V,\,\phi)$ be a nonsplit Γ–equivariant Lie algebroid. Then any equivariant principal G–bundle over X admits an equivariant holomorphic Lie algebroid connection.*
- *Let $(V,\,\phi)$ be a split Γ–equivariant Lie algebroid. Let E_G be an equivariant principal G–bundle over X. The following four statements are equivalent:*
  1. *E_G over X admits an equivariant holomorphic Lie algebroid connection.*
  2. *E_G admits an equivariant holomorphic connection.*

(3) E_G *admits a holomorphic connection.*

(4) *For every triple* $(P, L(P), \chi)$*, where* $L(P)$ *is a Levi subgroup of a parabolic subgroup* $P \subset G$ *and* χ *is a holomorphic character of* $L(P)$*, and every* Γ*–equivariant holomorphic reduction of structure group* $E_{L(P)}$ *of* E_G *to* $L(P)$*, the degree of the line bundle over* X *associated to* $E_{L(P)}$ *for* χ *is zero.*

In [ABKS], Theorem 1.1 was proved under the assumption that $G = \mathrm{GL}(r, \mathbb{C})$.

There is a natural bijective correspondence between parabolic G–bundles on X/Γ and Γ–equivariant principal G–bundles over X. Using this correspondence, Theorem 1.1 translates into the following (see Theorem 6.3):

Theorem 1.2. *Let* Y *be a compact connected Riemann surface and* $\{s_1, \cdots, s_n\} \subset Y$ *a parabolic divisor. Fix an integer* $N_i \geq 2$ *for each* s_i*,* $1 \leq i \leq n$*.*

- *Let* (V_*, ϕ) *be a nonsplit parabolic Lie algebroid on* Y*. Then any parabolic* G*–bundle on* Y *admits a parabolic Lie algebroid connection.*
- *Let* (V_*, ϕ) *be a split parabolic Lie algebroid on* Y*. Let* $\mathcal{E}_G$ *be a parabolic* G*–bundle on* Y*. The following three statements are equivalent:*

  (1) $\mathcal{E}_G$ *admits a parabolic Lie algebroid connection.*

  (2) $\mathcal{E}_G$ *admits a parabolic holomorphic connection.*

  (3) *For every triple* $(P, L(P), \chi)$*, where* $L(P)$ *is a Levi subgroup of a parabolic subgroup* $P \subset G$ *and* χ *is a holomorphic character of* $L(P)$*, and every holomorphic reduction of structure group* $\mathcal{E}_{L(P)}$ *of* $\mathcal{E}_G$ *to* $L(P)$*, the parabolic line bundle over* X *associated to* $\mathcal{E}_{L(P)}$ *for* χ *has parabolic degree zero.*

2. Equivariant Lie Algebroids

Let X be a compact connected Riemann surface. Denote by $\mathrm{Aut}(X)$ the group of all holomorphic automorphisms of X. Fix a finite subgroup

$$\Gamma \subset \mathrm{Aut}(X). \tag{2.1}$$

So the group Γ has a tautological action on X.

Let $\mathbb{G}$ be a complex Lie group. Note that in the introduction G was a reductive affine algebraic group defined over $\mathbb{C}$. An *equivariant* principal $\mathbb{G}$–bundle over X is a holomorphic principal $\mathbb{G}$–bundle

$$p \,:\, E_{\mathbb{G}} \,\longrightarrow\, X \tag{2.2}$$

over X together with an action of Γ on $E_{\mathbb{G}}$ such that

(1) for every $\gamma \in \Gamma$, the automorphism of $E_{\mathbb{G}}$ given by the action of γ is holomorphic,

(2) the projection p in (2.2) is Γ–equivariant, and

(3) the actions of $\mathbb{G}$ and Γ on $E_{\mathbb{G}}$ commute.

A holomorphic vector bundle V of rank r over X is called equivariant if the corresponding holomorphic principal $\mathrm{GL}(r, \mathbb{C})$–bundle over X, given by the frames in the fibers of V, is equipped with an action of Γ that satisfies the above three conditions. This is equivalent to an action of Γ on V, via holomorphic vector bundle automorphisms, over the action of Γ on X.

The holomorphic tangent bundle of X will be denoted by TX, while the holomorphic cotangent bundle of X will be denoted by K_X. Using the action of Γ on X, both TX and K_X are equivariant line bundles.

The first jet bundle of a holomorphic vector bundle W over X will be denoted by $J^1(W)$. An equivariant $\mathbb{C}$–Lie algebra structure on an equivariant vector bundle V over X is a $\mathbb{C}$–bilinear pairing defined by a sheaf homomorphism

$$[-,\, -] \,:\, V \otimes_{\mathbb{C}} V \,\longrightarrow\, V,$$

which is given by a holomorphic homomorphism $J^1(V) \otimes J^1(V) \longrightarrow V$ of vector bundles, such that

(1) $[\gamma(s),\, \gamma(t)] \,=\, \gamma([s,\, t])$ for all $\gamma \in \Gamma$, and

(2) $[s,\, t] \,=\, -[t,\, s]$ and $[[s,\, t],\, u] + [[t,\, u],\, s] + [[u,\, s],\, t] \,=\, 0$ for all locally defined holomorphic sections s, t, u of V.

The Lie bracket operation on the sheaf of holomorphic vector fields on X gives the structure of an equivariant $\mathbb{C}$–Lie algebra on TX.

An equivariant Lie algebroid over X is a pair $(V,\, \phi)$, where

(1) V is an equivariant vector bundle over X equipped with the structure of an equivariant $\mathbb{C}$–Lie algebra, and

(2) $\phi : V \longrightarrow TX$ is a Γ–equivariant $\mathcal{O}_X$–linear homomorphism such that

$$[s, f \cdot t] = f \cdot [s, t] + \phi(s)(f) \cdot t \tag{2.3}$$

for all locally defined holomorphic sections s, t of V and all locally defined holomorphic functions f on X.

The above homomorphism ϕ is called the *anchor map* of the Lie algebroid. The two conditions in the definition of a Lie algebroid imply that

$$\phi([s, t]) = [\phi(s), \phi(t)] \tag{2.4}$$

for all locally defined holomorphic sections s, t of V; this is explained in Remark 2.1 below.

Remark 2.1. To show that (2.4) holds for (V, ϕ), note that for all holomorphic local sections s, t, u of V and each locally defined holomorphic function f in $\mathcal{O}_X$ we have

$$[[s, t], fu] = f[[s, t], u] + \phi([s, t])(f) \cdot u \tag{2.5}$$

(see (2.3)). On the other hand,

$$[[s, t], fu] = [[s, fu], t] + [s, [t, fu]] = [f[s, u] + \phi(s)(f)u, t] \tag{2.6}$$

$$+[s, f[t, u] + \phi(t)(f)u] = f[[s, u], t] - \phi(t)(f)[s, u] + \phi(s)(f)[u, t]$$

$$-\phi(t)(\phi(s)(f))u + f[s, [t, u]] + \phi(s)(f)[t, u] + \phi(t)(f)[s, u] + \phi(s)(\phi(t)(f))u$$

$$= f[[s, t], u] + (\phi(s)(\phi(t)(f)) - \phi(t)(\phi(s)(f)))\, u$$

$$= f[[s, t], u] + [\phi(s), \phi(t)](f) \cdot u.$$

Combining (2.5) and (2.6) we conclude that $\phi([s, t])(f) \cdot u = [\phi(s), \phi(t)](f) \cdot u$. Since this holds for all locally defined f and u, it follows that

$$\phi([s, t]) = [\phi(s), \phi(t)].$$

This proves (2.4).

Definition 2.2. An equivariant Lie algebroid $(V,\, \phi)$ over X will be called *split* if there is a Γ–equivariant $\mathcal{O}_X$–linear homomorphism

$$\rho \,:\, TX \,\longrightarrow\, V$$

such that $\phi\circ\rho \,=\, \mathrm{Id}_{TX}$. An equivariant Lie algebroid $(V,\, \phi)$ over X will be called *nonsplit* if it is not split.

Lemma 2.3. *Let $(V,\, \phi)$ be an equivariant Lie algebroid over X. If there is an $\mathcal{O}_X$–linear homomorphism*

$$\zeta \,:\, TX \,\longrightarrow\, V$$

such that $\phi\circ\zeta \,=\, \mathrm{Id}_{TX}$, then there is a Γ–equivariant $\mathcal{O}_X$–linear homomorphism

$$\widehat{\zeta} \,:\, TX \,\longrightarrow\, V$$

such that $\phi\circ\widehat{\zeta} \,=\, \mathrm{Id}_{TX}$.

Proof. Let $\zeta \,:\, TX \,\longrightarrow\, V$ be an $\mathcal{O}_X$–linear homomorphism such that $\phi\circ\zeta \,=\, \mathrm{Id}_{TX}$. For each $\gamma\,\in\,\Gamma$, let

$$\zeta_\gamma \,:\, TX \,\longrightarrow\, V$$

be the homomorphism given by the following composition of maps:

$$TX \,\xrightarrow{\,\gamma\cdot\,}\, TX \,\xrightarrow{\,\zeta\,}\, V \,\xrightarrow{\,\gamma^{-1}\cdot\,}\, V,$$

where $\gamma\cdot$ (respectively, $\gamma^{-1}\cdot$) is the action of γ (respectively, γ^{-1}) on TX (respectively, V). Then the homomorphism

$$\widehat{\zeta} \,:=\, \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma}\zeta_\gamma \,:\, TX \,\longrightarrow\, V,$$

where $\#\Gamma$ is the order of Γ, is clearly Γ–equivariant and it also satisfies the condition that $\phi\circ\widehat{\zeta} \,=\, \mathrm{Id}_{TX}$. $\square$

3. Lie Algebroid Connection on Principal Bundles

As before, $\mathbb{G}$ is a complex Lie group. Take an equivariant principal $\mathbb{G}$–bundle $p : E_{\mathbb{G}} \longrightarrow X$ (see (2.2)). The action of $\mathbb{G}$ on $E_{\mathbb{G}}$ produces an action of $\mathbb{G}$ on the direct image $p_*TE_{\mathbb{G}}$ of the holomorphic tangent bundle $TE_{\mathbb{G}}$. The invariant part

$$\psi : \mathrm{At}(E_{\mathbb{G}}) := (p_*TE_{\mathbb{G}})^{\mathbb{G}} \longrightarrow X \tag{3.1}$$

is the Atiyah bundle for $E_{\mathbb{G}}$ [At]. It fits in a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathrm{ad}(E_{\mathbb{G}}) \xrightarrow{\iota} \mathrm{At}(E_{\mathbb{G}}) \xrightarrow{\varpi} TX \longrightarrow 0, \tag{3.2}$$

where $\mathrm{ad}(E_{\mathbb{G}})$ is the adjoint vector bundle for $E_{\mathbb{G}}$ (see [At]); the projection ϖ in (3.2) is given by the differential $dp : TE_{\mathbb{G}} \longrightarrow p^*TX$ of the map p. The sequence in (3.2) is known as the Atiyah exact sequence for $E_{\mathbb{G}}$.

The Lie bracket operation on the sheaf of holomorphic vector fields on $E_{\mathbb{G}}$ produces a $\mathbb{C}$–Lie algebra structure on $\mathrm{At}(E_{\mathbb{G}})$. The homomorphism ϖ in (3.2) intertwines the $\mathbb{C}$–Lie algebra structures of $\mathrm{At}(E_{\mathbb{G}})$ and TX. In fact, $(\mathrm{At}(E_{\mathbb{G}}), \varpi)$ is a Lie algebroid.

A holomorphic connection on the principal $\mathbb{G}$–bundle $E_{\mathbb{G}}$ is a holomorphic splitting of the Atiyah exact sequence in (3.2) [At]. In other words, a holomorphic connection on $E_{\mathbb{G}}$ is a holomorphic $\mathcal{O}_X$–linear homomorphism $\mu : TX \longrightarrow \mathrm{At}(E_{\mathbb{G}})$ such that $\varpi \circ \mu = \mathrm{Id}_{TX}$, where ϖ is the homomorphism in (3.2).

Example 3.1. Assume that $E_{\mathbb{G}}$ does not admit any holomorphic connection. For example, set $\mathbb{G} = \mathrm{GL}(r, \mathbb{C})$ and take $E_{\mathbb{G}}$ to be the holomorphic principal $\mathrm{GL}(r, \mathbb{C})$–bundle over X associated to a holomorphic vector bundle of rank r and nonzero degree over X. Then the Lie algebroid $(\mathrm{At}(E_{\mathbb{G}}), \varpi)$ in (3.2) is nonsplit.

On the other hand, if $E_{\mathbb{G}}$ admits a holomorphic connection, then the Lie algebroid $(\mathrm{At}(E_{\mathbb{G}}), \varpi)$ in (3.2) is split. For example, take any indecomposable holomorphic vector bundle E over X of rank r with $\mathrm{degree}(E) = 0$. Then E admits a holomorphic connection [At], [We]. Hence the holomorphic principal $\mathrm{GL}(r, \mathbb{C})$–bundle $E_{\mathrm{GL}(r,\mathbb{C})}$ over X associated to E admits a holomorphic connection. Consequently, the Lie algebroid given by $\mathrm{At}(E_{\mathrm{GL}(r,\mathbb{C})})$ (see (3.2)) is split.

The action of Γ on $E_{\mathbb{G}}$ makes $\text{At}(E_{\mathbb{G}})$ an equivariant vector bundle. The homomorphism ϖ in (3.2) is Γ–equivariant. Thus $(\text{At}(E_{\mathbb{G}}),\, \varpi)$ is an equivariant Lie algebroid. The action of Γ on $E_{\mathbb{G}}$ produces an action of Γ on $\text{ad}(E_{\mathbb{G}})$, and the homomorphism ι in (3.2) is Γ–equivariant.

Take an equivariant Lie algebroid $(V,\, \phi)$ over X. Consider the homomorphism

$$\psi \;:\; V\oplus \text{At}(E_{\mathbb{G}}) \;\longrightarrow\; TX, \quad (v,\, w) \;\longmapsto\; \phi(v) - \varpi(w),$$

where ϖ is the homomorphism in (3.2). Note that ψ is surjective because ϖ is surjective. Define

$$\mathcal{A}(E_{\mathbb{G}}) \;:=\; \text{kernel}(\psi) \;\subset\; V\oplus \text{At}(E_{\mathbb{G}}). \tag{3.3}$$

The $\mathbb{C}$–Lie algebra structure on $V \oplus \text{At}(E_{\mathbb{G}})$, given by the $\mathbb{C}$–Lie algebra structures on V and $\text{At}(E_{\mathbb{G}})$, restricts to a $\mathbb{C}$–Lie algebra structure on $\mathcal{A}(E_{\mathbb{G}})$. Restricting the natural projection $V \oplus \text{At}(E_{\mathbb{G}}) \longrightarrow V$ to $\mathcal{A}(E_{\mathbb{G}}) \subset V\oplus \text{At}(E_{\mathbb{G}})$ we obtain a homomorphism

$$\rho \;:\; \mathcal{A}(E_{\mathbb{G}}) \;\longrightarrow\; V; \tag{3.4}$$

note that $\text{kernel}(\rho) = \text{kernel}(\varpi) = \text{ad}(E_{\mathbb{G}})$. Similarly, restricting the natural projection $V\oplus\text{At}(E_{\mathbb{G}}) \longrightarrow \text{At}(E_{\mathbb{G}})$ to $\mathcal{A}(E_{\mathbb{G}}) \subset V\oplus\text{At}(E_{\mathbb{G}})$ we obtain a homomorphism

$$\varphi \;:\; \mathcal{A}(E_{\mathbb{G}}) \;\longrightarrow\; \text{At}(E_{\mathbb{G}}). \tag{3.5}$$

The action of Γ on $V\oplus\text{At}(E_{\mathbb{G}})$, given by the actions of Γ on V and $\text{At}(E_{\mathbb{G}})$, preserves the subbundle $\mathcal{A}(E_{\mathbb{G}})$.

We have the commutative diagram of homomorphisms of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{ad}(E_{\mathbb{G}}) & \longrightarrow & \mathcal{A}(E_{\mathbb{G}}) & \stackrel{\rho}{\longrightarrow} & V & \longrightarrow & 0\\ & & \Vert & & \Big\downarrow\varphi & & \Big\downarrow\phi & & \\ 0 & \longrightarrow & \text{ad}(E_{\mathbb{G}}) & \stackrel{\iota}{\longrightarrow} & \text{At}(E_{\mathbb{G}}) & \stackrel{\varpi}{\longrightarrow} & TX & \longrightarrow & 0 \end{array} \tag{3.6}$$

where φ and ρ are constructed in (3.5) and (3.4) respectively. Note that every vector bundle in (3.6) is equipped with an action of Γ, and all the homomorphisms in (3.6) are Γ–equivariant.

Definition 3.2. An *equivariant holomorphic Lie algebroid connection* on $E_{\mathbb{G}}$ is a Γ–equivariant holomorphic homomorphism

$$\delta \,:\, V \,\longrightarrow\, \mathcal{A}(E_{\mathbb{G}})$$

such that $\rho\circ\delta \,=\, \mathrm{Id}_V$, where ρ is the homomorphism in (3.4).

Let δ be an equivariant holomorphic Lie algebroid connection on $E_{\mathbb{G}}$. For locally defined holomorphic sections s and t of V, consider

$$\alpha(s,\, t) \,:=\, [\delta(s),\, \delta(t)] - \delta([s,\, t]).$$

For a locally defined holomorphic function f on X,

$$f\cdot\alpha(s,\, t) \,=\, \alpha(fs,\, t) \,=\, \alpha(s,\, ft) \,=\, -\alpha(ft,\, s).$$

Also, $\rho(\alpha(s,\, t)) \,=\, 0$, where ρ is the homomorphism in (3.4); consequently, $\alpha(s,\, t)$ is a locally defined section of $\mathrm{ad}(E_{\mathbb{G}})$. From these it follows that α defines a Γ–invariant holomorphic section

$$\mathcal{K}(\delta) \,\in\, H^0(X,\, \mathrm{ad}(E_{\mathbb{G}})\otimes \bigwedge\nolimits^2 V^*)^{\Gamma}. \tag{3.7}$$

The section $\mathcal{K}(\delta)$ in (3.7) is the *curvature* of the equivariant holomorphic Lie algebroid connection δ.

When $V \,=\, TX$ and $\phi \,=\, \mathrm{Id}_{TX}$, an equivariant holomorphic Lie algebroid connection on $E_{\mathbb{G}}$ is a usual equivariant holomorphic connection on the principal $\mathbb{G}$–bundle $E_{\mathbb{G}}$.

When $\mathbb{G} \,=\, \mathrm{GL}(r, \mathbb{C})$, the notions of Lie algebroid connection and curvature coincide with those for holomorphic vector bundles.

4. Equivariant Holomorphic Connections and Split Lie Algebroids

Earlier the notation $\mathbb{G}$ was used to denote a complex Lie group. Nowonwards, we will consider principal bundles whose structure group is a connected reductive affine algebraic group defined over $\mathbb{C}$. To distinguish it from a general complex Lie group, the notation G will be used instead of $\mathbb{G}$.

4.1. Equivariant holomorphic connections

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. Take any parabolic subgroup $P \subset G$. Let $R_u(P) \subset P$ be the unipotent radical of P. A *Levi subgroup* of P is a connected reductive complex algebraic subgroup $L(P) \subset P$ such that the following composition of homomorphisms is an isomorphism:

$$L(P) \hookrightarrow P \longrightarrow P/R_u(P)$$

(see [Hu, p. 125], [Bo]).

Take a holomorphic principal G–bundle over X. Take a holomorphic character $\chi : L(P) \longrightarrow \mathbb{G}_m = \mathbb{C}^*$ of a Levi subgroup $L(P)$ of a parabolic subgroup P of G. Let $E_{L(P)} \subset E_G$ be a holomorphic reduction of structure group of E_G to $L(P) \subset G$. Let $E_{L(P)} \times^P \mathbb{C}^*$ be the holomorphic principal $\mathbb{C}^*$–bundle on X obtained by extending the structure group of $E_{L(P)}$ using the character χ. Using the standard multiplication action of $\mathbb{C}^*$ on $\mathbb{C}$, the principal $\mathbb{C}^*$–bundle $E_{L(P)} \times^P \mathbb{C}^*$ produces a holomorphic line bundle $\mathcal{L}(E_{L(P)}, \chi) \longrightarrow X$.

The principal G–bundle E_G admits a holomorphic connection if and only if for every triple $(P, L(P), \chi)$ as above, and every holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$, we have

$$\text{degree}(\mathcal{L}(E_{L(P)}, \chi)) = 0$$

[AB, Theorem 4.1].

Let E_G be an equivariant principal G–bundle over X. A reduction of structure group $E_{L(P)}$ of E_G to $L(P) \subset G$ is called *equivariant* if the action of Γ on E_G preserves the submanifold $E_{L(P)} \subset E_G$. The following lemma gives a criterion for the existence of an equivariant holomorphic connection on an equivariant principal G–bundle.

Lemma 4.1. *An equivariant principal G–bundle E_G over X admits an equivariant holomorphic connection if and only if for every triple $(P, L(P), \chi)$ as above, and every Γ–equivariant holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$,*

$$\text{degree}(\mathcal{L}(E_{L(P)}, \chi)) = 0. \tag{4.1}$$

Proof. First, assume that E_G admits an equivariant holomorphic connection. Then, from the above criterion of [AB] it follows immediately that (4.1) holds.

To prove converse, assume that (4.1) holds for every triple $(P, L(P), \chi)$ as above, and every Γ–equivariant holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$. This implies that the principal G–bundle E_G admits a holomorphic connection (see [Bi, p. 274, Lemma 4.2]). Let

$$\delta \; : \; TX \; \longrightarrow \; \mathrm{At}(E_G)$$

be a holomorphic connection on E_G; so we have $\varpi \circ \delta = \mathrm{Id}_{TX}$, where ϖ is the homomorphism in (3.2) (see [At]). For any $\gamma \in \Gamma$, let

$$\delta_\gamma \; : \; TX \; \longrightarrow \; \mathrm{At}(E_G)$$

be the homomorphism given by the following composition of maps:

$$TX \xrightarrow{\gamma\cdot} TX \xrightarrow{\delta} \mathrm{At}(E_G) \xrightarrow{\gamma^{-1}\cdot} \mathrm{At}(E_G),$$

where $\gamma\cdot$ (respectively, $\gamma^{-1}\cdot$) is the action of γ (respectively, γ^{-1}) on TX (respectively, $\mathrm{At}(E_G)$); recall that Γ acts on both TX and $\mathrm{At}(E_G)$.

Now consider the homomorphism

$$\widehat{\delta} \; := \; \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} \delta_\gamma \; : \; TX \; \longrightarrow \; \mathrm{At}(E_G).$$

Since $\varpi\circ\delta = \mathrm{Id}_{TX}$, it follows immediately that $\varpi\circ\widehat{\delta} = \mathrm{Id}_{TX}$. It is also evident that $\widehat{\delta}$ is Γ–equivariant. Consequently, $\widehat{\delta}$ is an equivariant holomorphic connection on the equivariant principal G–bundle E_G. $\square$

The second part of the proof of Lemma 4.1 gives the following:

Corollary 4.2. *An equivariant principal G–bundle E_G admits a holomorphic connection if and only if E_G admits an equivariant holomorphic connection.*

4.2. Split equivariant Lie algebroid connections

Let (V, ϕ) be a split equivariant Lie algebroid (see Definition 2.2). As before, G is a connected reductive affine algebraic group defined over $\mathbb{C}$.

PROPOSITION 4.3. *An equivariant principal G–bundle E_G over X admits an equivariant holomorphic Lie algebroid connection (see Definition 3.2) if and only if for every triple $(P, L(P), \chi)$ as in Lemma 4.1, and every Γ–equivariant holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$,*

$$\text{degree}(\mathcal{L}(E_{L(P)}, \chi)) \;=\; 0. \tag{4.2}$$

PROOF. We will show that E_G admits an equivariant holomorphic Lie algebroid connection if and only if E_G admits an equivariant holomorphic connection. To prove this, first assume that E_G admits an equivariant holomorphic connection. Take an equivariant holomorphic connection

$$\delta_0 \;:\; TX \;\longrightarrow\; \text{At}(E_G)$$

on E_G. Since $\varpi \circ \delta_0 \;=\; \text{Id}_{TX}$, where ϖ is the homomorphism in (3.2), there is a unique holomorphic homomorphism

$$\delta_0' \;:\; \text{At}(E_G) \;\longrightarrow\; \text{ad}(E_G)$$

such that $\text{kernel}(\delta_0') \;=\; \delta_0(TX)$ and $\delta_0' \circ \iota \;=\; \text{Id}_{\text{ad}(E_G)}$, where ι is the homomorphism in (3.2). Now, consider the homomorphism

$$\delta_0' \circ \varphi \;:\; \mathcal{A}(E_{\mathbb{G}}) \;\longrightarrow\; \text{ad}(E_G),$$

where φ is the homomorphism in (3.5). There is a unique holomorphic homomorphism

$$\delta \;:\; V \;\longrightarrow\; \mathcal{A}(E_{\mathbb{G}})$$

such that $\delta(V) \;=\; \text{kernel}(\delta_0' \circ \varphi)$ and $\rho \circ \delta \;=\; \text{Id}_V$, where ρ is the homomorphism in (3.4). Since δ is also Γ–equivariant, it defines an equivariant holomorphic Lie algebroid connection on E_G.

To prove the converse, assume that E_G has an equivariant holomorphic Lie algebroid connection

$$\delta \;:\; V \;\longrightarrow\; \mathcal{A}(E_{\mathbb{G}}).$$

Fix a Γ–equivariant holomorphic homomorphism

$$\eta \,:\, TX \,\longrightarrow\, V$$

such that $\phi \circ \eta = \mathrm{Id}_{TX}$; see Definition 2.2 (recall that (V, ϕ) is a split equivariant Lie algebroid). Now it is straightforward to check that the composition of homomorphisms

$$\varphi \circ \delta \circ \eta \,:\, TX \,\longrightarrow\, \mathrm{At}(E_G),$$

where φ is the homomorphism in (3.5), is an equivariant holomorphic connection on E_G.

Since E_G admits an equivariant holomorphic Lie algebroid connection if and only if E_G admits an equivariant holomorphic connection, Lemma 4.1 completes the proof of the proposition. □

Proposition 4.3, Corollary 4.2 and Lemma 4.1 together give the following:

COROLLARY 4.4. *Let (V, ϕ) be a split equivariant Lie algebroid and G a reductive affine algebraic group over $\mathbb{C}$. Let E_G be an equivariant principal G–bundle over X. The following four statements are equivalent:*

(1) *E_G over X admits an equivariant holomorphic Lie algebroid connection.*

(2) *E_G admits an equivariant holomorphic connection.*

(3) *E_G admits a holomorphic connection.*

(4) *For every triple $(P, L(P), \chi)$ as in Lemma 4.1, and every Γ–equivariant holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$,*

$$\mathrm{degree}(\mathcal{L}(E_{L(P)}, \chi)) \;=\; 0.$$

5. Nonsplit Equivariant Lie Algebroid Connections

Let (V, ϕ) be a nonsplit equivariant Lie algebroid (see Definition 2.2). As before, G is a connected reductive affine algebraic group defined over $\mathbb{C}$.

We will show that any equivariant principal G–bundle over X admits an equivariant holomorphic Lie algebroid connection.

REMARK 5.1. Take an equivariant principal G–bundle E_G over X. There is a Levi subgroup $L(P)$ of a parabolic subgroup $P \subset G$, and a holomorphic reduction of structure group $E_{L(P)} \subset E_G$ of E_G to $L(P) \subset G$, satisfying the following conditions:

(1) The action of Γ on E_G preserves $E_{L(P)} \subset E_G$, and

(2) the maximal torus of the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$ coincides with the center of $L(P)$. (Note that any element z of the center of $L(P)$ gives a Γ–equivariant holomorphic automorphisms of $E_{L(P)}$ defined by $x \longmapsto xz$.)

Moreover if $P' \subset G$ is another parabolic subgroup, $L(P')$ is a Levi subgroup of P', and $E_{L(P')} \subset E_G$ is a holomorphic reduction of structure group of E_G to $L(P')$ satisfying the above two conditions, then there is an element $x \in G$ such that $L(P') = x^{-1}L(P)x$ and $E_{L(P')} = E_{L(P)}x$. (See [BP, p. 63, Theorem 4.1].)

LEMMA 5.2. *Assume that the equivariant principal $L(P)$–bundle $E_{L(P)}$ in Remark 5.1 admits an equivariant holomorphic Lie algebroid connection. Then the equivariant principal G–bundle E_G admits an equivariant holomorphic Lie algebroid connection.*

PROOF. There are natural homomorphisms $a : \mathrm{ad}(E_{L(P)}) \hookrightarrow \mathrm{ad}(E_G)$ and $b : \mathrm{At}(E_{L(P)}) \hookrightarrow \mathrm{At}(E_G)$ because $E_{L(P)}$ is a holomorphic reduction of structure group of E_G to $L(P)$, and they fit in the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_{L(P)}) & \longrightarrow & \mathrm{At}(E_{L(P)}) & \longrightarrow & TX & \longrightarrow & 0 \\
 & & \Big\downarrow a & & \Big\downarrow b & & \Big\| & & \\
0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G) & \longrightarrow & TX & \longrightarrow & 0
\end{array}$$

where the rows are the Atiyah exact sequences (for $E_{L(P)}$ and E_G); see (3.2). This commutative diagram produces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_{L(P)}) & \xrightarrow{\iota'} & \mathcal{A}(E_{L(P)}) & \xrightarrow{\rho'} & V & \longrightarrow & 0 \\ & & \big\downarrow a' & & \big\downarrow b' & & \| & & \\ 0 & \longrightarrow & \mathrm{ad}(E_G) & \xrightarrow{\iota} & \mathcal{A}(E_G) & \longrightarrow & V & \longrightarrow & 0 \end{array} \tag{5.1}$$

(see (3.6)).

Since the principal $L(P)$–bundle $E_{L(P)}$ admits an equivariant holomorphic Lie algebroid connection, we have a Γ–equivariant holomorphic homomorphism

$$\delta' \ : \ V \ \longrightarrow \ \mathcal{A}(E_{L(P)})$$

such that $\rho' \circ \delta' = \mathrm{Id}_V$, where ρ' is the homomorphism in (5.1). Now, the homomorphism

$$b' \circ \delta' \ : \ V \ \longrightarrow \ \mathcal{A}(E_G),$$

where b' is the homomorphism in (5.1), is an equivariant holomorphic Lie algebroid connection on E_G. $\square$

As before, $L(P)$ and $E_{L(P)}$ are as in Remark 5.1. Consider the Atiyah exact sequence

$$0 \ \longrightarrow \ \mathrm{ad}(E_{L(P)}) \ \longrightarrow \ \mathrm{At}(E_{L(P)}) \ \longrightarrow \ TX \ \longrightarrow \ 0 \tag{5.2}$$

for the equivariant principal $L(P)$–bundle $E_{L(P)}$. Let

$$\beta \ \in \ H^1(X,\, \mathrm{ad}(E_{L(P)}) \otimes K_X)$$

be the extension class for the short exact sequence in (5.2). Since (5.2) is an exact sequence of Γ–equivariant vector bundles, we have

$$\beta \ \in \ H^1(X,\, \mathrm{ad}(E_{L(P)}) \otimes K_X)^\Gamma \ \subset \ H^1(X,\, \mathrm{ad}(E_{L(P)}) \otimes K_X). \tag{5.3}$$

Consider the dual homomorphism $\phi^* \ : \ K_X \ \longrightarrow \ V^*$ for the anchor map. Tensoring it with the identity map of $\mathrm{ad}(E_{L(P)})$, we have the homomorphism

$$\Psi \ := \ \mathrm{Id}_{\mathrm{ad}(E_{L(P)})} \otimes \phi^* \ : \ \mathrm{ad}(E_{L(P)}) \otimes K_X \ \longrightarrow \ \mathrm{ad}(E_{L(P)}) \otimes V^*.$$

Let

$$(5.4)\qquad \Psi_* \;:\; H^1(X,\, \mathrm{ad}(E_{L(P)})\otimes K_X) \;\longrightarrow\; H^1(X,\, \mathrm{ad}(E_{L(P)})\otimes V^*)$$

be the homomorphism of cohomologies induced by the above homomorphism Ψ.

LEMMA 5.3. *The equivariant principal $L(P)$–bundle $E_{L(P)}$ admits an equivariant holomorphic Lie algebroid connection if and only if*

$$\Psi_*(\beta) \;=\; 0,$$

where β and Ψ_* are constructed in (5.3) *and* (5.4) *respectively.*

PROOF. Consider the short exact sequence

$$(5.5)\qquad 0 \longrightarrow \mathrm{ad}(E_{L(P)}) \xrightarrow{\iota'} \mathcal{A}(E_{L(P)}) \xrightarrow{\rho'} V \longrightarrow 0$$

in (5.1). Let

$$\beta_V \;\in\; H^1(X,\, \mathrm{ad}(E_{L(P)})\otimes V^*)$$

be the extension class for it. Since (5.5) is an exact sequence of Γ–equivariant vector bundles, we have

$$(5.6)\qquad \beta_V \;\in\; H^1(X,\, \mathrm{ad}(E_{L(P)})\otimes V^*)^{\Gamma} \;\subset\; H^1(X,\, \mathrm{ad}(E_{L(P)})\otimes V^*).$$

Note that the equivariant principal $L(P)$–bundle $E_{L(P)}$ admits an equivariant holomorphic Lie algebroid connection if and only if we have $\beta_V = 0$; indeed, $\beta_V = 0$ if and only if $E_{L(P)}$ admits a holomorphic Lie algebroid connection, and, exactly as shown in Corollary 4.2, $E_{L(P)}$ admits a holomorphic Lie algebroid connection if and only if if admits an equivariant holomorphic Lie algebroid connection.

Now consider the commutative diagram

$$(5.7)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_{L(P)}) & \xrightarrow{\iota'} & \mathcal{A}(E_{L(P)}) & \xrightarrow{\rho'} & V & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow\phi & & \\ 0 & \longrightarrow & \mathrm{ad}(E_{L(P)}) & \longrightarrow & \mathrm{At}(E_{L(P)}) & \longrightarrow & TX & \longrightarrow & 0 \end{array}$$

(see (3.6)). From (5.7) it follows immediately that

$$(5.8)\qquad \Psi_*(\beta) \;=\; \beta_V,$$

where β and β_V are the extension classes in (5.3) and (5.6) respectively while Ψ_* is the homomorphism in (5.4). Since $E_{L(P)}$ admits an equivariant holomorphic Lie algebroid connection if and only if we have $\beta_V = 0$, it follows from (5.8) that $E_{L(P)}$ admits an equivariant holomorphic Lie algebroid connection if and only if $\Psi_*(\beta) = 0$. $\square$

As before, $L(P)$ and $E_{L(P)}$ are as in Remark 5.1. Denote the Lie algebra of $L(P)$ by $\ell(\mathfrak{p})$. Since $L(P)$ is reductive, there is an $L(P)$–invariant nondegenerate symmetric bilinear form on $\ell(\mathfrak{p})$. Fix a $L(P)$–invariant nondegenerate symmetric bilinear form

$$\mathcal{B} \in \mathrm{Sym}^2(\ell(\mathfrak{p})^*)^{L(P)}. \tag{5.9}$$

The form $\mathcal{B}$ in (5.9) produces a holomorphic isomorphism

$$\mathrm{ad}(E_{L(P)}) \xrightarrow{\sim} \mathrm{ad}(E_{L(P)})^*. \tag{5.10}$$

By Serre duality,

$$H^1(X,\, \mathrm{ad}(E_{L(P)}) \otimes K_X)^\Gamma = (H^0(X,\, \mathrm{ad}(E_{L(P)})^*)^*)^\Gamma \tag{5.11}$$

$$= (H^0(X,\, \mathrm{ad}(E_{L(P)}))^*)^\Gamma = (H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma)^*;$$

see (5.10).

Let

$$\widehat{\beta} \in (H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma)^* = \mathrm{Hom}(H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma,\, \mathbb{C}) \tag{5.12}$$

be the element corresponding to β in (5.3) for the isomorphism in (5.11).

Let

$$\mathcal{Z}(\ell(\mathfrak{p})) \subset \ell(\mathfrak{p}) \tag{5.13}$$

be the center of $\ell(\mathfrak{p})$. Since the adjoint action of $L(P)$ on its Lie algebra $\ell(\mathfrak{p})$ fixes $\mathcal{Z}(\ell(\mathfrak{p}))$ pointwise, we have an injective homomorphism

$$\Phi : \mathcal{Z}(\ell(\mathfrak{p})) \longrightarrow H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma. \tag{5.14}$$

Proposition 5.4.

(1) *Take any* $\xi_n \in H^0(X, \mathrm{ad}(E_{L(P)}))^\Gamma$ *which is nilpotent over some point of* X*. Then*

$$\widehat{\beta}(\xi_n) \;=\; 0,$$

where $\widehat{\beta}$ *is the homomorphism in* (5.12).

(2) *Take any* $\xi_s \in H^0(X, \mathrm{ad}(E_{L(P)}))^\Gamma$ *which is semisimple over every point of* X*. Then there is an element* $w \in \mathcal{Z}(\ell(\mathfrak{p}))$ *such that*

$$\Phi(w) \;=\; \xi_s,$$

where Φ *is the homomorphism in* (5.14).

Proof. The first statement follows immediately from [AB, p. 341, Proposition 3.9].

For the proof of second statement, first recall from Remark 5.1 that the center of $L(P)$ is the maximal torus of the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$. The automorphisms of $E_{L(P)}$ given by the center of $L(P)$ evidently commute with all the automorphisms of $E_{L(P)}$. On the other hand, the image of the connected component, containing the identity element, of the center of $L(P)$ under the natural map to the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$, is a maximal torus of the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$. Therefore, we conclude that a maximal torus of the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$ is contained in the center of the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$.

If the maximal torus of a connected complex reductive algebraic group $\mathcal{G}$ is contained in the center of $\mathcal{G}$, then $\mathcal{G}$ is abelian, which means that $\mathcal{G}$ is a torus. Therefore, the semisimple part, i.e., the Levi factor, of the Lie algebra of the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$ coincides with the center $\mathcal{Z}(\ell(\mathfrak{p}))$. Note that $H^0(X, \mathrm{ad}(E_{L(P)}))^\Gamma$ is the Lie algebra of the group of all Γ–equivariant holomorphic automorphisms of $E_{L(P)}$. From these, the second statement of the proposition follows immediately. □

Proposition 5.5. *The equivariant principal* $L(P)$*–bundle* $E_{L(P)}$ *admits an equivariant holomorphic Lie algebroid connection.*

Proof. In view of (5.8) and Lemma 5.3, it suffices to show that

$$\beta_V \;=\; \Psi_*(\beta) \;=\; 0, \tag{5.15}$$

where β and Ψ_* are constructed in (5.3) and (5.4) respectively.

By Serre duality,

$$H^1(X,\, \mathrm{ad}(E_{L(P)}) \otimes V^*)^\Gamma \;=\; (H^0(X,\, \mathrm{ad}(E_{L(P)})^* \otimes V \otimes K_X)^*)^\Gamma \tag{5.16}$$

$$=\; (H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^*)^\Gamma \;=\; (H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma)^*;$$

see (5.10). Let

$$\widehat{\beta}_V \;\in\; (H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma)^* \tag{5.17}$$

$$=\; \mathrm{Hom}(H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma,\, \mathbb{C})$$

be the element corresponding to β_V in (5.6) for the isomorphism in (5.16).

Consider the anchor map $\phi \in H^0(X,\, V^* \otimes TX)^\Gamma$. We have the homomorphism

$$\Phi_1 \;:\; H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma \tag{5.18}$$

$$\longrightarrow\; H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma \otimes H^0(X,\, V^* \otimes TX)^\Gamma$$

that sends any $s \in H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma$ to $s \otimes \phi$. There is a natural homomorphism

$$\Phi_2 \;:\; H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma \otimes H^0(X,\, V^* \otimes TX)^\Gamma \tag{5.19}$$

$$\longrightarrow\; H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X \otimes V^* \otimes TX)^\Gamma$$

$$=\; H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes \mathrm{End}(V) \otimes \mathrm{End}(TX))^\Gamma.$$

Using the trace maps

$$\mathrm{End}(V) \longrightarrow \mathcal{O}_X \quad \text{and} \quad \mathrm{End}(TX) \longrightarrow \mathcal{O}_X, \tag{5.20}$$

we have the map

$$\Phi_3 \;:\; H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes \mathrm{End}(V) \otimes \mathrm{End}(TX))^\Gamma \tag{5.21}$$

$$\longrightarrow\; H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma.$$

Now consider the homomorphism

$$\widetilde{\Phi} \;:=\; \Phi_3 \circ \Phi_2 \circ \Phi_1 \;:\; H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma \longrightarrow H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma, \tag{5.22}$$

where Φ_1, Φ_2 and Φ_3 are constructed in (5.18), (5.19) and (5.21) respectively. From (5.8) we know that the following diagram is commutative:

$$\begin{array}{ccc} H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma & \xrightarrow{\widetilde{\Phi}} & H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma \\ \Big\downarrow \widehat{\beta}_V & & \Big\downarrow \widehat{\beta} \\ \mathbb{C} & \xrightarrow{\mathrm{Id}} & \mathbb{C} \end{array}$$

where $\widehat{\beta}$ and $\widehat{\beta}_V$ are the homomorphisms constructed in (5.12) and (5.17) respectively, and $\widetilde{\Phi}$ is defined in (5.22). In other words, we have

$$\widehat{\beta} \circ \widetilde{\Phi} \;=\; \widehat{\beta}_V \tag{5.23}$$

as elements of $\mathrm{Hom}(H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma,\, \mathbb{C})$.

To prove (5.15) by contradiction, assume that

$$\beta_V \;=\; \Psi_*(\beta) \;\neq\; 0. \tag{5.24}$$

From (5.24) it follows that there is a section $s \in H^0(X,\, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma$ such that

$$\widehat{\beta}_V(s) \;\neq\; 0, \tag{5.25}$$

where $\widehat{\beta}_V$ is constructed in (5.17). Consider the section

$$\widehat{s} \;:=\; \widetilde{\Phi}(s) \;\in\; H^0(X,\, \mathrm{ad}(E_{L(P)}))^\Gamma, \tag{5.26}$$

where s is the section in (5.25) and $\widetilde{\Phi}$ is constructed in (5.22). From (5.24), (5.23) and (5.26) we know that

$$\widehat{\beta}(\widehat{s}) \;\neq\; 0. \tag{5.27}$$

In view of (5.27), from Proposition 5.4(1) we know that for each point $x \in X$, the element $\widehat{s}(x) \in \mathrm{ad}(E_{L(P)})_x$ is *not* nilpotent.

So for each point $x \in X$, the semisimple component of $\widehat{s}(x) \in \mathrm{ad}(E_{L(P)})_x$, for the Jordan decomposition, is nonzero. Moreover, the conjugacy class of the semisimple component of $\widehat{s}(x)$ is actually independent of the point $x \in X$. To see this, take any $L(P)$–invariant holomorphic function I on $\ell(\mathfrak{p})$. Then $x \longmapsto I(\widehat{s}(x))$ is a holomorphic function on X. This function is a constant one because X is compact and connected. From this it follows that the conjugacy class of the semisimple component of $\widehat{s}(x)$ is independent of $x \in X$.

Take a holomorphic character $\chi : L(P) \longrightarrow \mathbb{G}_m = \mathbb{C}^*$. Let

$$d\chi : \ell(\mathfrak{p}) \longrightarrow \mathbb{C}$$

be the homomorphism of Lie algebras given by χ. This homomorphism $d\chi$ produces a homomorphism

$$\widetilde{\chi} : \mathrm{ad}(E_{L(P)}) \longrightarrow \mathcal{O}_X. \tag{5.28}$$

Let

$$\widetilde{\chi}_* : H^0(X, \mathrm{ad}(E_{L(P)}))^\Gamma \longrightarrow H^0(X, \mathcal{O}_X) = \mathbb{C} \tag{5.29}$$

be the homomorphism of global sections given by $\widetilde{\chi}$ in (5.28).

From Proposition 5.4(2) it follows that there is a holomorphic character $\chi : L(P) \longrightarrow \mathbb{C}^*$ such that

$$\widetilde{\chi}_*(\widehat{s}) \neq 0, \tag{5.30}$$

where $\widehat{s}$ and $\widetilde{\chi}_*$ are constructed in (5.26) and (5.29) respectively.

The homomorphism $\widetilde{\chi}$ in (5.28) produces a homomorphism

$$\widetilde{\chi}' : H^0(X, \mathrm{ad}(E_{L(P)}) \otimes V \otimes K_X)^\Gamma \longrightarrow H^0(X, V \otimes K_X)^\Gamma. \tag{5.31}$$

Define the map

$$\Psi_1 : H^0(X, V \otimes K_X)^\Gamma \tag{5.32}$$

$$\longrightarrow H^0(X, V \otimes K_X) \otimes H^0(X, V^* \otimes TX)^\Gamma, \quad v \longmapsto v \otimes \phi,$$

where ϕ is the anchor map. We have the natural map

$$\Psi_2 : H^0(X, V \otimes K_X)^\Gamma \otimes H^0(X, V^* \otimes TX)^\Gamma \tag{5.33}$$

$$\longrightarrow\ H^0(X,\, V\otimes K_X\otimes V^*\otimes TX)^\Gamma$$
$$=\ H^0(X,\, \mathrm{End}(V)\otimes \mathrm{End}(TX))^\Gamma.$$

Using the trace maps in (5.20) we have the homomorphism

$$\Psi_3\ :\ H^0(X,\, \mathrm{End}(V)\otimes\mathrm{End}(TX))^\Gamma\ \longrightarrow\ H^0(X,\, \mathcal{O}_X)\ =\ \mathbb{C}. \tag{5.34}$$

Now define

$$\widetilde{\Psi}\ :=\ \Psi_3\circ\Psi_2\circ\Psi_1\ :\ H^0(X,\, V\otimes K_X)\ \longrightarrow\ \mathbb{C}, \tag{5.35}$$

where Ψ_1, Ψ_2 and Ψ_3 are constructed in (5.32), (5.33) and (5.34) respectively. We note that the following diagram is commutative:

$$\begin{array}{ccc} H^0(X,\, \mathrm{ad}(E_{L(P)})\otimes V\otimes K_X)^\Gamma & \xrightarrow{\widetilde{\chi}'} & H^0(X,\, V\otimes K_X)^\Gamma \\ \Big\downarrow \widetilde{\Phi} & & \Big\downarrow \widetilde{\Psi} \\ \mathbb{C} & \xrightarrow{\mathrm{Id}} & \mathbb{C} \end{array}$$

where $\widetilde{\chi}'$, $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are constructed in (5.31), (5.22) and (5.35) respectively. Consequently, using (5.26) we have

$$\widetilde{\Psi}\circ\widetilde{\chi}'(s)\ =\ \widetilde{\chi}_*(\widehat{s}) \tag{5.36}$$

as elements of $\mathbb{C}$. From (5.30) and (5.36) we conclude that

$$\widetilde{\chi}'(s)\ \neq\ 0. \tag{5.37}$$

In view of the construction of $\widetilde{\chi}'$ done in (5.31), from (5.37) it is deduced that the following composition of maps

$$TX\ \xrightarrow{\widetilde{\chi}'(s)}\ V\ \xrightarrow{\phi}\ TX$$

coincides with multiplication by the nonzero constant $\widetilde{\chi}_*(\widehat{s})\ \in\ \mathbb{C}\setminus\{0\}$ in (5.30). Consequently, the homomorphism

$$\frac{1}{\widetilde{\chi}_*(\widehat{s})}\cdot\widetilde{\chi}'(s)\ :\ TX\ \longrightarrow\ V$$

gives a splitting of the equivariant Lie algebroid $(V,\, \phi)$. But $(V,\, \phi)$ does not split. In view of this contradiction, it follows that (5.24) does not hold. This completes the proof. □

Corollary 5.6. *The equivariant principal G–bundle E_G admits an equivariant holomorphic Lie algebroid connection.*

Proof. This follows from the combination of Lemma 5.2 and Proposition 5.5. □

6. Criterion for Lie Algebroid Connection

As before, G is a complex reductive affine algebraic group. The combination of Corollary 4.4 and Corollary 5.6 gives the following:

Theorem 6.1.

- *Let $(V,\, \phi)$ be a nonsplit equivariant Lie algebroid. Then any equivariant principal G–bundle over X admits an equivariant holomorphic Lie algebroid connection.*
- *Let $(V,\, \phi)$ be a split equivariant Lie algebroid. Let E_G be an equivariant principal G–bundle over X. The following four statements are equivalent:*
  1. *E_G over X admits an equivariant holomorphic Lie algebroid connection.*
  2. *E_G admits an equivariant holomorphic connection.*
  3. *E_G admits a holomorphic connection.*
  4. *For every triple $(P,\, L(P),\, \chi)$ as in Lemma 4.1, and every Γ–equivariant holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$,*
  $$\text{degree}(\mathcal{L}(E_{L(P)}, \chi)) \;=\; 0.$$

Remark 6.2. Take a complex Lie group $\mathbb{G}$ and a holomorphic principal $\mathbb{G}$–bundle $E_{\mathbb{G}}$ over X. Consider the corresponding Lie algebroid $(\text{At}(E_{\mathbb{G}}), \varpi)$ as in (3.2). Assume that $E_{\mathbb{G}}$ does not admit any holomorphic connection. It was shown in Example 3.1 that the Lie algebroid $(\text{At}(E_{\mathbb{G}}),\, \varpi)$ is nonsplit. Assume that $E_{\mathbb{G}}$ is equivariant. Therefore, Theorem 6.1 says that any equivariant principal G–bundle E_G over X admits an equivariant holomorphic Lie algebroid connection for the Lie algebroid $(\text{At}(E_{\mathbb{G}}),\, \varpi)$.

Next consider the case where the equivariant holomorphic principal $\mathbb{G}$–bundle $E_{\mathbb{G}}$ does not admit any holomorphic connection. Now Theorem 6.1 says that the following four statements are equivalent:

(1) E_G over X admits an equivariant holomorphic Lie algebroid connection.

(2) E_G admits an equivariant holomorphic connection.

(3) E_G admits a holomorphic connection.

(4) For every triple $(P,\, L(P),\, \chi)$ as in Lemma 4.1, and every Γ–equivariant holomorphic reduction of structure group $E_{L(P)}$ of E_G to $L(P)$,

$$\text{degree}(\mathcal{L}(E_{L(P)}, \chi)) \;=\; 0.$$

We will reformulate Theorem 6.1 in the set-up of parabolic bundles.

Fix n ordered distinct points

$$S \;=\; \{s_1,\, \cdots,\, s_n\} \;\subset\; X. \tag{6.1}$$

For each $1 \,\leq\, i \,\leq n$, fix an integer $N_i \,\geq\, 2$. We assume the following:

(1) If genus$(X) \,=\, 0$, then $n \,\neq\, 1$.

(2) If genus$(X) \,=\, 0$, and $n \,=\, 2$, then $N_1 \,=\, N_2$.

A parabolic G–bundle consists of a complex manifold $\mathcal{E}_G$, a surjective holomorphic map $p \,:\, \mathcal{E}_G \,\longrightarrow\, X$ and a holomorphic right action of G

$$\Psi \;:\; \mathcal{E}_G \times G \;\longrightarrow\; \mathcal{E}_G$$

such that the following conditions hold:

(1) $\Psi(y,\, h) \,=\, \Psi(x)$ for all $x \,\in\, \mathcal{E}_G$ and $h \,\in\, G$.

(2) For every $x \,\in\, X$, the action of G on $p^{-1}(x)$ is transitive.

(3) The restriction $p\big|_{p^{-1}(X\setminus S)} \;:\; p^{-1}(X \setminus S) \;\longrightarrow\; X \setminus S$ (see (6.1)) is a holomorphic principal G–bundle on $X \setminus S$.

(4) The isotropy subgroup for any $y \in p^{-1}(s_i)$ is a finite cyclic subgroup of G whose order divides N_i.

(See [BBN], [Bi], [BS].) For $1 \leq i \leq n$, the parabolic weights, at s_i, of a parabolic vector bundle will be integral multiples of $\frac{1}{N_i}$.

The parabolic tangent bundle, denoted by $(TX)_*$, is $TX \otimes \mathcal{O}_X(-\sum_{i=1}^n s_i)$ equipped with parabolic weight $\frac{1}{N_i}$ at s_i, $1 \leq i \leq n$.

A parabolic Lie algebroid is a pair (V_*, ϕ), where V_* is a parabolic vector bundle, and $\phi : V_* \longrightarrow (TX)_*$ is a parabolic homomorphism, such that

(1) V_* is equipped with a $\mathbb{C}$–Lie algebra structure

$$[-, -] : V_* \otimes_{\mathbb{C}} V_* \longrightarrow V_*$$

which is compatible with the parabolic structure, and

(2) $[s, f \cdot t] = f \cdot [s, t] + \phi(s)(f) \cdot t$ for locally defined holomorphic sections s, t of V_* and all locally defined holomorphic functions f on X.

A parabolic Lie algebra (V_*, ϕ) is called split if there is a parabolic homomorphism $\beta : (TX)_* \longrightarrow V_*$ such that $\phi \circ \beta = \mathrm{Id}_{(TX)_*}$. A parabolic Lie algebra (V_*, ϕ) is called nonsplit if it is not split.

Take a parabolic G–bundle $p : \mathcal{E}_G \longrightarrow X$. The invariant direct image, on X, of the holomorphic tangent bundle $T\mathcal{E}_G$

$$(p_* T\mathcal{E}_G)^G \subset p_* T\mathcal{E}_G$$

has a natural parabolic structure. The resulting parabolic vector bundle is called the Atiyah bundle for $\mathcal{E}_G$, and it is denoted by $\mathrm{At}(\mathcal{E}_G)_*$. Let $T_p \subset T\mathcal{E}_G$ be the relative tangent bundle for the projection p. The parabolic adjoint bundle $\mathrm{ad}(\mathcal{E}_G)_*$ is defined to be the invariant direct image.$(p_* T_p)^G$. The parabolic vector bundle $\mathrm{At}(\mathcal{E}_G)_*$ fits in the following short exact sequence of parabolic vector bundles over X:

$$0 \longrightarrow \mathrm{ad}(\mathcal{E}_G)_* \longrightarrow \mathrm{At}(\mathcal{E}_G)_* \xrightarrow{\psi} (TX)_* \longrightarrow 0. \tag{6.2}$$

A holomorphic connection on $\mathcal{E}_G$ is a holomorphic splitting of (6.2) (see [Bi]).

Define $\mathcal{A}(\mathcal{E}_{\mathbb{G}})_*$ to be the parabolic vector bundle given by the kernel of the parabolic homomorphism

$$V^* \oplus \mathrm{At}(\mathcal{E}_G)_* \longrightarrow (TX)_*, \quad (a,\, b) \longmapsto \phi(a) - \psi(b),$$

where ψ is the homomorphism in (6.2). The parabolic vector bundle $\mathcal{A}(\mathcal{E}_G)_*$ fits in the following short exact sequence of parabolic vector bundles over X:

$$0 \longrightarrow \mathrm{ad}(\mathcal{E}_G)_* \longrightarrow \mathcal{A}(\mathcal{E}_G)_* \longrightarrow V \longrightarrow 0. \tag{6.3}$$

A holomorphic Lie algebroid connection on $\mathcal{E}_G$ is a holomorphic splitting of (6.3).

There is a ramified Galois covering $\varpi : Y \longrightarrow X$ such that

(1) the branch locus of ϖ is $S = \{s_1, \cdots, s_n\} \subset X$ (see (6.1)), and

(2) for every $1 \leq i \leq n$, the multiplicity of ϖ at any $y \in \varpi^{-1}(s_i)$ is N_i.

(See [Na, p. 26, Proposition 1.2.12] for the existence of such a covering ϖ.)

Let $\Gamma = \mathrm{Gal}(\varpi)$ be the Galois group for ϖ.

The parabolic G–bundles over X correspond to Γ–equivariant principal G–bundles on Y [BBN], [BS]. In particular, the parabolic vector bundles over X correspond to Γ–equivariant vector bundles on Y. The parabolic Lie algebroids over X correspond to the Γ–equivariant Lie algebroids on Y. Parabolic G–bundles over X equipped with a parabolic connection correspond to the Γ–equivariant principal G–bundles on Y equipped with a Γ–equivariant connection.

Consequently, Theorem 6.1 gives the following:

Theorem 6.3.

- *Let (V_*, ϕ) be a nonsplit parabolic Lie algebroid. Then any parabolic G–bundle over X admits a parabolic Lie algebroid connection.*
- *Let (V_*, ϕ) be a split parabolic Lie algebroid. Let $\mathcal{E}_G$ be a parabolic G–bundle over X. The following three statements are equivalent:*
  (1) *$\mathcal{E}_G$ over X admits a parabolic Lie algebroid connection.*
  (2) *$\mathcal{E}_G$ admits a parabolic holomorphic connection.*

(3) *For every triple $(P, L(P), \chi)$ as in Lemma 4.1, and every holomorphic reduction of structure group $\mathcal{E}_{L(P)}$ of $\mathcal{E}_G$ to $L(P)$, the parabolic line bundle over X associated to $\mathcal{E}_{L(P)}$ for χ has parabolic degree zero.*

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Department of Mathematics
Shiv Nadar University
NH91, Tehsil Dadri, Greater Noida
Uttar Pradesh 201314, India
E-mail: indranil.biswas@snu.edu.in
indranil29@gmail.com