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Properties of Minimal Charts and Their Applications X: Charts of Type (5, 2)

By Teruo Nagase and Akiko Shima*

Abstract. Charts are oriented labeled graphs in a disk. Any simple surface braid (2-dimensional braid) can be described by using a chart. Also, a chart represents an oriented closed surface embedded in 4-space. In this paper, we investigate embedded surfaces in 4-space by using charts. Let Γ be a chart, and we denote by Γ_m the union of all the edges of label m. A chart Γ is of type $(5, 2)$ if there exists a label m such that $w(\Gamma) = 7$, $w(\Gamma_m \cap \Gamma_{m+1}) = 5$, $w(\Gamma_{m+1} \cap \Gamma_{m+2}) = 2$ where $w(G)$ is the number of white vertices in G. In this paper, we investigate a minimal chart of type $(5, 2)$.

1. Introduction

Charts are oriented labeled graphs in a disk (see [1],[5], and see Section 2 for the precise definition of charts). Let D_1^2, D_2^2 be 2-dimensional disks. Any simple surface braid (2-dimensional braid) can be described by using a chart, here a simple surface braid is a properly embedded surface S in the 4-dimensional disk $D_1^2 \times D_2^2$ such that a natural map $\pi : S \subset D_1^2 \times D_2^2 \to D_2^2$ is a simple branched covering map of D_2^2 and the boundary ∂S is a trivial closed braid in the solid torus $D_1^2 \times \partial D_2^2$ (see [4], [5, Chapter 14 and Chapter 18]). Also, from a chart, we can construct a simple closed surface braid in 4-space $\mathbb{R}^4$. This surface is an oriented closed surface embedded in $\mathbb{R}^4$. On the other hand, any oriented embedded closed surface in $\mathbb{R}^4$ is ambient isotopic to a simple closed surface braid (see [4],[5, Chapter 23]). A C-move is a local modification between two charts in a disk (see Section 2 for C-moves). A C-move between two charts induces an ambient isotopy between oriented closed surfaces corresponding to the two charts. In this paper, we investigate oriented closed surfaces in 4-space by using charts.

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We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat. In [18], we showed that there is no minimal chart with exactly five white vertices (see Section 2 for the precise definition of minimal charts). Hasegawa proved that there exists a minimal chart with exactly six white vertices [2]. This chart represents a 2-twist spun trefoil. In [3] and [17], we investigated minimal charts with exactly four white vertices. In this paper, we investigate properties of minimal charts which support a conjecture that there is no minimal chart with exactly seven white vertices (see [6],[7],[8],[9],[10],[11],[12],[13],[14],[15]).

Let Γ be a chart. For each label m, we denote by Γ_m the union of all the edges of label m.

Now we define a type of a chart: Let Γ be a chart with at least one white vertex, and $n_1, n_2, \ldots, n_k$ integers. The chart Γ is of *type* $(n_1, n_2, \ldots, n_k)$ if there exists a label m of Γ satisfying the following three conditions:

(i) For each $i = 1, 2, \ldots, k$, the chart Γ contains exactly n_i white vertices in $\Gamma_{m+i-1} \cap \Gamma_{m+i}$.

(ii) If $i < 0$ or $i > k$, then Γ_{m+i} does not contain any white vertices.

(iii) Both of the two subgraphs Γ_m and Γ_{m+k} contain at least one white vertex.

If we want to emphasize the label m, then we say that Γ is of *type* $(m; n_1, n_2, \ldots, n_k)$. Note that $n_1 \geq 1$ and $n_k \geq 1$ by Condition (iii).

We proved in [7, Theorem 1.1] that if there exists a minimal n-chart Γ with exactly seven white vertices, then Γ is a chart of type $(7), (5, 2), (4, 3)$, $(3, 2, 2)$ or $(2, 3, 2)$ (if necessary we change the label i by $n - i$ for all label i). In [10], we showed that there is no minimal chart of type $(3, 2, 2)$. In [11] and [12], there is no minimal chart of type $(2, 3, 2)$. In [13], there is no minimal chart of type (7). In [14], there is no minimal chart of type $(4, 3)$. In this paper, we investigate a minimal chart of type $(5, 2)$.

An edge in a chart is called a *terminal edge* if it has a white vertex and a black vertex.

In our argument we often construct a chart Γ. On the construction of a chart Γ, for a white vertex $w \in \Gamma_m$ for some label m, among the three edges of Γ_m containing w, if one of the three edges is a terminal edge (see Fig. 1(a) and (b)), then we remove the terminal edge and put a black dot

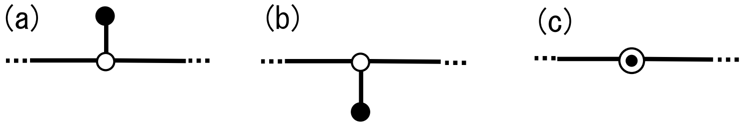

Fig. 1. (a),(b) White vertices in terminal edges. (c) BW-vertex.

at the center of the white vertex as shown in Fig. 1(c). Namely Fig. 1(c) means Fig. 1(a) or Fig. 1(b). We call the vertex in Fig. 1(c) a *BW-vertex* with respect to Γ_m.

In this paper we shall show the following:

THEOREM 1.1. *Let* Γ *be a minimal chart of type* $(m; 5, 2)$. *Suppose that there exists a connected component of* Γ_m *with exactly five white vertices. Then* Γ_m *contains one of the two graphs as shown in Fig. 2.*

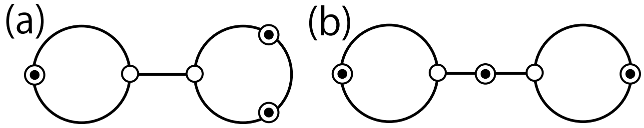

Fig. 2. Graphs with three black vertices.

In the last paper [15] in this series, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then there exists a connected component of Γ_m with exactly five white vertices. Moreover, by using the above theorem, we shall show that there is no minimal chart of type $(5, 2)$, and there is no minimal chart with exactly seven white vertices.

The paper is organized as follows. In Section 2, we define charts and minimal charts. Let Γ be a minimal chart, and m a label of Γ. In Section 3, we review a useful lemma for a disk called a lens. In Section 4, we investigate a disk called a k-angled disk of Γ_m with at most one white vertex in its interior, where a k-angled disk is a disk whose boundary contains exactly k white vertices and consists of edges of label m. In Section 5, we investigate a 5-angled disk of Γ_m. In Section 6, we investigate a 4-angled disk of Γ_m. In Section 7, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m contains neither graphs as shown in Fig. 13(a),(c). In Section 8, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(b). In Section 9, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(d). In Section 10, we review IO-Calculation(a property of numbers of inward arcs of label k and outward arcs of label k in a closed domain F with $\partial F \subset \Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}$ for some label k). In Section 11, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(e). In Section 12, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(f). In Section 13, we review Triangle Lemma. These lemmas will be used in Section 14. In Section 14, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(g). Moreover, we shall prove Theorem 1.1.

2. Preliminaries

In this section, we introduce the definition of charts and its related words.

Let n be a positive integer. An *n-chart* (a braid chart of degree n [1] or a surface braid chart of degree n [5]) is an oriented labeled graph in the interior of a disk, which may be empty or have closed edges without vertices satisfying the following four conditions (see Fig. 3):

(i) Every vertex has degree 1, 4, or 6.

(ii) The labels of edges are in $\{1, 2, \dots, n-1\}$.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i - j| > 1$.

We call a vertex of degree 1 a *black vertex*, a vertex of degree 4 a *crossing*, and a vertex of degree 6 a *white vertex* respectively.

Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a *middle arc* at the white vertex (see Fig. 3(c)). For each white vertex v, there are two middle arcs at v in a small neighborhood of v. An edge is said to be *middle at* a white vertex v if it contains a middle arc at v.

Let e be an edge connecting v_1 and v_2. If e is oriented from v_1 to v_2, then we say that e is oriented *outward at* v_1 and *inward at* v_2.

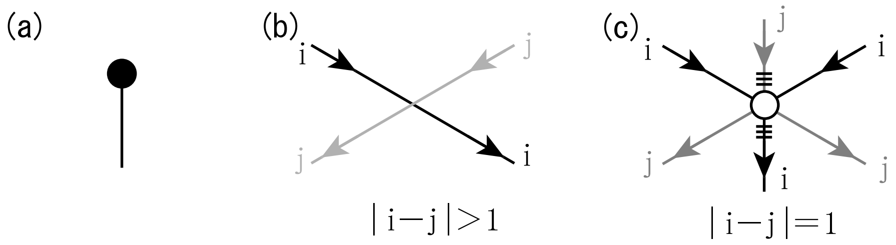

Fig. 3. (a) A black vertex. (b) A crossing. (c) A white vertex. Each arc with three transversal short arcs is a middle arc at the white vertex.

Now *C-moves* are local modifications of charts as shown in Fig. 4 (cf. [1], [5] and [19]). Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

An edge in a chart is called a *free edge* if it has two black vertices.

For each chart Γ, let $w(\Gamma)$ and $f(\Gamma)$ be the number of white vertices, and the number of free edges respectively. The pair $(w(\Gamma), -f(\Gamma))$ is called a *complexity* of the chart (see [4]). A chart Γ is called a *minimal chart* if its complexity is minimal among the charts C-move equivalent to the chart Γ with respect to the lexicographic order of pairs of integers.

We showed the difference of a chart in a disk and in a 2-sphere (see [6, Lemma 2.1]). This lemma follows from that there exists a natural one-to-one correspondence between {charts in S^2}/C-moves and {charts in D^2}/C-moves, conjugations ([5, Chapter 23 and Chapter 25]). To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk.

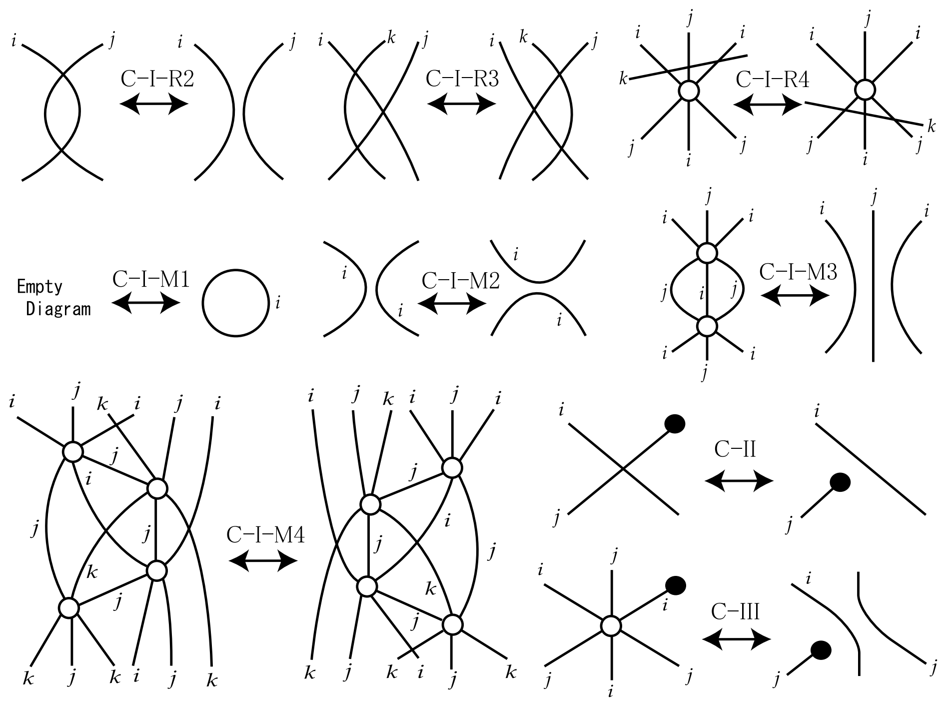

Fig. 4. For the C-III move, the edge with the black vertex is not middle at the white vertex in the left figure.

ASSUMPTION 1. *In this paper, all charts are contained in the* 2*-sphere* S^2.

We have the special point in the 2-sphere S^2, called the point at infinity, denoted by ∞. In this paper, all charts are contained in a disk such that the disk does not contain the point at infinity ∞.

Let Γ be a chart, and m a label of Γ. A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). A *ring* is a simple closed curve in Γ_m containing at least one crossing but not containing any white vertices. A hoop is said to be *simple* if one of the two complementary domains of the hoop does not contain any white vertices.

We can assume that all minimal charts Γ satisfy the following four conditions (see [6],[7],[8],[16]):

Assumption 2. *If an edge of* Γ *contains a black vertex, then the edge is a free edge or a terminal edge. Moreover any terminal edge contains a middle arc.*

Assumption 3. *All free edges and simple hoops in* Γ *are moved into a small neighborhood* U_∞ *of the point at infinity* ∞. *Hence we assume that* Γ *does not contain free edges nor simple hoops, unless otherwise mentioned.*

Assumption 4. *Each complementary domain of any ring and hoop must contain at least one white vertex.*

Assumption 5. *The point at infinity* ∞ *is moved into any complementary domain of* Γ.

In this paper for a subset X in a space we denote the interior of X, the boundary of X and the closure of X by $\text{Int}X$, ∂X and $Cl(X)$ respectively.

Let α be a simple arc or an edge, and p, q the endpoints of α. We denote $\partial\alpha = \{p, q\}$ and $\text{Int}\alpha = \alpha - \{p, q\}$.

3. Lenses

In this section, we review a useful lemma for a disk called a lens.

Let Γ be a chart, and m a label of Γ. Let L be the closure of a connected component of the set obtained by taking out all the white vertices from Γ_m. If L contains at least one white vertex but does not contain any black vertex, then L is called an *internal edge of label* m. Note that an internal edge may contain a crossing of Γ.

Let Γ be a chart. Let D be a disk such that

(1) the boundary ∂D consists of an internal edge e_1 of label m and an internal edge e_2 of label $m+1$, and

(2) any edge containing a white vertex in e_1 does not intersect the open disk $\text{Int}D$.

Note that ∂D may contain crossings. Let w_1 and w_2 be the white vertices in e_1. If the disk D satisfies one of the following conditions, then D is called *a lens of type* $(m, m+1)$ (see Fig. 5):

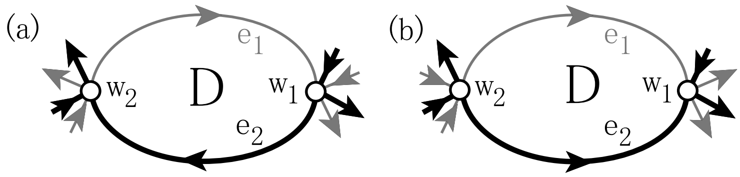

Fig. 5. Lenses.

(i) Neither e_1 nor e_2 contains a middle arc.

(ii) One of the two edges e_1 and e_2 contains middle arcs at both white vertices w_1 and w_2 simultaneously.

LEMMA 3.1 ([6, Theorem 1.1]). *There exist at least three white vertices in the interior of the lens for any minimal chart.*

LEMMA 3.2 ([7, Corollary 1.3]). *There is no lens in any minimal chart with at most seven white vertices.*

Let Γ be a chart, and m a label of Γ. A *loop* is a simple closed curve in Γ_m with exactly one white vertex (possibly with crossings).

In our argument, we often need a name for an unnamed edge by using a given edge and a given white vertex. For the convenience, we use the following naming: Let e', e_i, e'' be three consecutive edges containing a white vertex w_j. Here, the two edges e' and e'' are unnamed edges. There are six arcs in a neighborhood U of the white vertex w_j. If the three arcs $e' \cap U$, $e_i \cap U$, $e'' \cap U$ lie anticlockwise around the white vertex w_j in this order, then e' and e'' are denoted by a_{ij} and b_{ij} respectively (see Fig. 6). There is a possibility $a_{ij} = b_{ij}$ if they are contained in a loop.

LEMMA 3.3. *Let Γ be a chart, and k a label of Γ. Let e_1 be an internal edge of label k with two white vertices w_1 and w_2 (see Fig. 7). Suppose that $w_1, w_2 \in \Gamma_{k+\delta}$ for some $\delta \in \{+1, -1\}$, and suppose that one of the two edges a_{11}, b_{12} is a terminal edge. If $\Gamma_{k+2\delta} \cap e_1 = \emptyset$, and if Γ satisfies one of the following four conditions, then Γ is not a minimal chart.*

(a) *The two edges a_{11}, b_{12} are oriented inward (or outward) at w_1, w_2, respectively.*

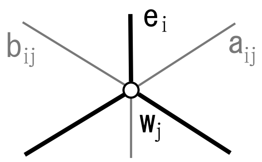

Fig. 6. The three edges a_{ij}, e_i, b_{ij} are consecutive edges around the white vertex w_j.

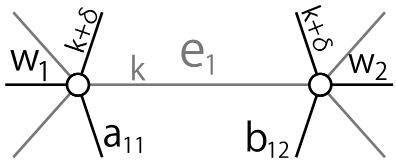

Fig. 7. The edge e_1 is of label k, and $\delta \in \{+1, -1\}$.

(b) *The edge* a_{11} (*resp.* b_{12}) *is a terminal edge, and* b_{12} (*resp.* a_{11}) *is not middle at the white vertex different from* w_2 (*resp.* w_1).

(c) *The two edges* a_{11}, b_{12} *are middle at* w_1, w_2, *respectively.*

(d) *Both of* a_{11}, b_{12} *are terminal edges.*

PROOF. Suppose that Γ is a minimal chart. Without loss of generality we can assume that

(1) a_{11} is a terminal edge and oriented inward at w_1.

Then by Assumption 2, the terminal edge a_{11} is middle at w_1. Thus

(2) the edge e_1 is oriented inward at w_1 (i.e. the edge e_1 is oriented from w_2 to w_1).

Since e_1 is an edge of label k, we have

(3) $\Gamma_{k+\delta} \cap \mathrm{Int} e_1 = \emptyset$.

Now by the condition of this lemma, we have

(4) $\Gamma_{k+2\delta} \cap e_1 = \emptyset$.

First, we shall show that if Γ satisfies Condition (a), then we have a contradiction. Since a_{11} is oriented inward at w_1 by (1), the edge b_{12} is also oriented inward at w_2 by Condition (a) (see Fig. 8(a)). Hence by (2)

(5) the edge b_{12} is not middle at w_2.

By (3),(4) and applying C-II moves along the edge e_1, we can move the black vertex in a_{11} near the white vertex w_2 (see Fig. 8(b)). Apply a C-I-M2 move between a_{11} and b_{12}, we obtain a new terminal edge of label $k + \delta$ at w_2 (see Fig. 8(c)). However by (5), the terminal edge is not middle at w_2. This contradicts Assumption 2.

Now by (1) and Lemma 3.3(a), we can assume that

(6) the edge b_{12} is oriented outward at w_2 (see Fig. 8(d)).

Next, we shall show that if Γ satisfies Condition (b), then we have a contradiction. Let w_3 be the white vertex in b_{12} different from w_2. Then by Condition (b),

(7) the edge b_{12} is not middle at w_3.

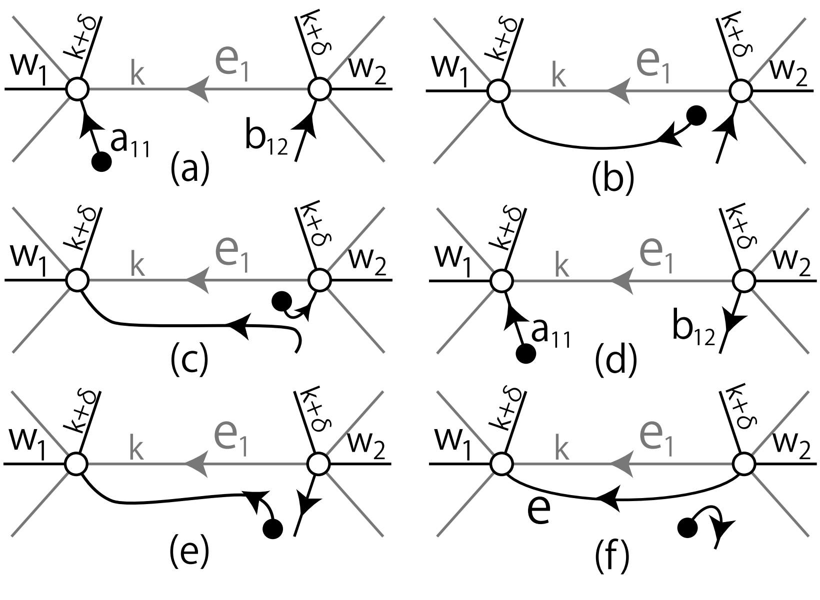

Fig. 8. The edge e_1 is of label k, and $\delta \in \{+1, -1\}$.

By the similar way of the proof of Lemma 3.3(a), we can move the black vertex in a_{11} near the white vertex w_2 by C-II moves (see Fig. 8(e)). Apply a C-I-M2 move between a_{11} and b_{12}, we obtain an internal edge of label $k+\delta$ with w_1 and w_2, and a terminal edge of label $k+\delta$ at w_3 (see Fig. 8(f)). However we have the same contradiction by (7).

Next, we shall show that if Γ satisfies Condition (c), then we have a contradiction. By the similar way of the proof of Lemma 3.3(b), we obtain an internal edge of label $k+\delta$ with w_1 and w_2, say e (see Fig. 8(f)). Thus by Condition (c), the edge e is middle at both white vertices w_1, w_2. Hence $e_1 \cup e$ bounds a lens whose interior does not contain any white vertices. This contradicts Lemma 3.1.

Finally, if Γ satisfies Condition (d), then by Assumption 2 the two terminal edges a_{11} and b_{12} are middle at w_1, w_2, respectively. Thus Γ satisfies Condition (c). Hence we have the same contradiction.

Therefore we have a contradiction for all cases. Thus Γ is not a minimal chart. □

4. Special k-Angled Disks

In this section, we investigate a disk called a k-angled disk.

Let X be a set in a chart Γ. Let

$$w(X) = \text{the number of white vertices in } X.$$

Lemma 4.1 ([10, Lemma 3.2(1)]). *Let Γ be a minimal chart, and m a label of Γ. Let G be a connected component of Γ_m. If $1 \leq w(G)$, then $2 \leq w(G)$.*

Lemma 4.2 ([8, Lemma 6.1]). *Let Γ be a minimal chart. Let C be a ring or a non simple hoop, and D a disk with $\partial D = C$. If $w(\Gamma \cap D) = 1$, then Γ is C-move equivalent to the minimal chart $Cl(\Gamma - C)$.*

Let Γ be a chart, m a label of Γ, D a disk with $\partial D \subset \Gamma_m$, and k a positive integer. If ∂D contains exactly k white vertices, then D is called *a k-angled disk of Γ_m*. Note that the boundary ∂D may contain crossings.

LEMMA 4.3. *Let* Γ *be a minimal chart, and* m *a label of* Γ*. Let* D *be a* k*-angled disk of* Γ_m*. Suppose that all the white vertices on* ∂D *are contained in* $\Gamma_{m+\varepsilon}$ *for some* $\varepsilon \in \{+1, -1\}$*. If* $w(\Gamma \cap \text{Int}D) \leqq 1$*, then* Γ *can be modified to a minimal chart* Γ' *by C-moves in* $\text{Int}D$ *such that* $\Gamma'_{m-\varepsilon} \cap D = \emptyset$*.*

PROOF. We shall show that $\text{Int}D$ does not contain any white vertex in $\Gamma_{m-\varepsilon}$. Suppose that there exists a white vertex w in $\Gamma_{m-\varepsilon}$. Let G be the connected component of $\Gamma_{m-\varepsilon}$ with $G \ni w$. Then by the condition of this lemma, we have $G \subset \text{Int}D$. Thus the condition $w(\Gamma \cap \text{Int}D) \leqq 1$ implies $w(G) = 1$. This contradicts Lemma 4.1. Hence $\text{Int}D$ does not contain any white vertex in $\Gamma_{m-\varepsilon}$.

Suppose that $\Gamma_{m-\varepsilon} \cap D \neq \emptyset$. Then we shall show that the set $\Gamma_{m-\varepsilon} \cap D$ consists of rings and non simple hoops. Let e be an internal edge (possibly a ring or a hoop) of label $m - \varepsilon$ intersecting the disk D. Since all the white vertices on ∂D are contained in $\Gamma_{m+\varepsilon}$, the condition $\partial D \subset \Gamma_m$ implies $e \subset \text{Int}D$. Since $\text{Int}D$ does not contain any white vertex in $\Gamma_{m-\varepsilon}$, the edge e is a ring or a hoop. Thus by Assumption 4, the curve e is a ring or a non simple hoop.

Since $w(\Gamma \cap \text{Int}D) \leqq 1$, by Lemma 4.2 we can eliminate each ring and non simple hoop in $\text{Int}D$. Hence the chart Γ can be modified to a minimal chart Γ' by C-moves in $\text{Int}D$ such that $\Gamma'_{m-\varepsilon} \cap D = \emptyset$. □

Let Γ be a chart, and m a label of Γ. An edge of label m is called a *feeler* of a k-angled disk D of Γ_m if the edge intersects $N - \partial D$ where N is a regular neighborhood of ∂D in D.

Let Γ be a chart, and D a k-angled disk of Γ_m. If any feeler of D of label m is a terminal edge, then D is called a *special* k-angled disk.

LEMMA 4.4. *Let* Γ *be a minimal chart, and* m *a label of* Γ*. Let* D *be a special* k*-angled disk of* Γ_m *with at least one feeler* e_1*. Let* w_1 *be the white vertex in* e_1*, and* $w_2, w_3, \cdots, w_k$ *the other white vertices on* ∂D *lying anticlockwise in this order. Suppose that all of* $w_1, w_2, \cdots, w_k$ *are contained in* $\Gamma_{m+\varepsilon}$ *for some* $\varepsilon \in \{+1, -1\}$*. Let* a_{11}, b_{11} *be internal edges* (*possibly terminal edges*) *of label* $m + \varepsilon$ *at* w_1 *in* D *such that* a_{11}, e_1, b_{11} *lie anticlockwise in this order* (*see Fig. 9*(*a*))*. If* $w(\Gamma \cap \text{Int}D) \leqq 1$*, then the following five conditions hold:*

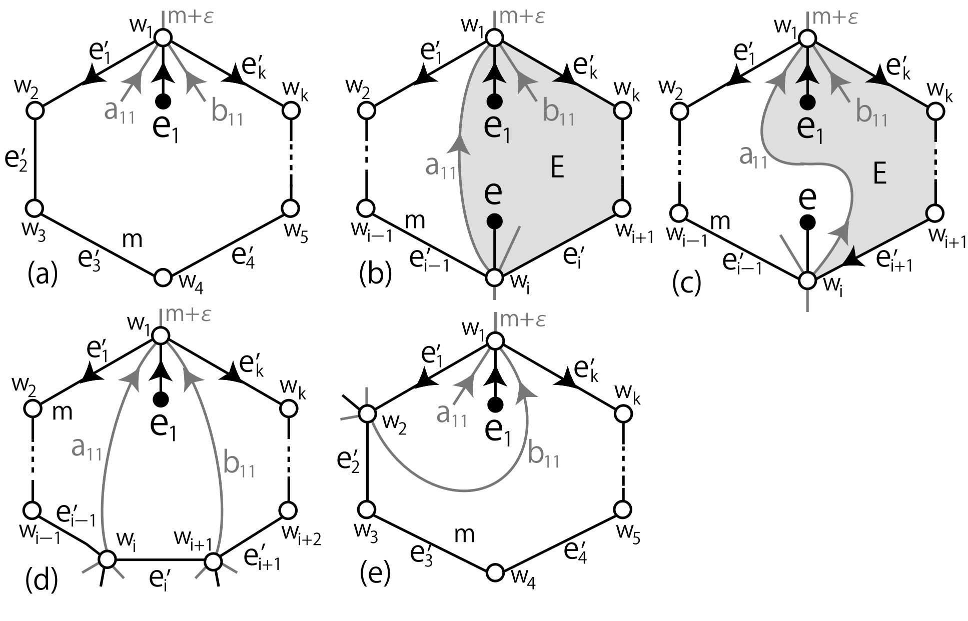

Fig. 9. The edge e_1 is a terminal edge of label m, and a_{11}, b_{11} are internal edges of label $m + \varepsilon$ for some $\{+1, -1\}$. The gray regions are disks E.

(a) *If* $a_{11} \ni w_i$ *or* $b_{11} \ni w_i$ *for some* $i \in \{2, 3, \cdots, k\}$, *then* D *does not contain a feeler at* w_i.

(b) $a_{11} \not\ni w_2$ *and* $b_{11} \not\ni w_k$.

(c) *If* $a_{11} \ni w_i$ *for some* $i \in \{3, 4, \cdots, k-1\}$, *then* $b_{11} \not\ni w_{i+1}$.

(d) *If* $b_{11} \ni w_i$ *for some* $i \in \{3, 4, \cdots, k-1\}$, *then* $a_{11} \not\ni w_{i-1}$.

(e) $b_{11} \not\ni w_2$ *and* $a_{11} \not\ni w_k$.

Proof. Let $e'_1, e'_2, \cdots, e'_k$ be the internal edges of label m in ∂D such that $\partial e'_i = \{w_i, w_{i+1}\}$ for $i = 1, 2, \cdots, k-1$ and $\partial e'_k = \{w_k, w_1\}$. Without loss of generality, we can assume that

(1) the terminal edge e_1 is oriented inward at w_1.

Then by Assumption 2,

(2) both of a_{11}, b_{11} are oriented inward at w_1 and

(3) both of e_1', e_k' are oriented outward at w_1 (see Fig. 9(a)).

Since $w(\Gamma \cap \mathrm{Int}D) \leqq 1$, by Lemma 4.3 we can assume that

(4) $\Gamma_{m-\varepsilon} \cap D = \emptyset$ ($\Gamma_{m-\varepsilon} \cap a_{11} = \emptyset$ and $\Gamma_{m-\varepsilon} \cap b_{11} = \emptyset$).

First, we shall show Statement (a). Now, suppose that $a_{11} \ni w_i$ for some $i \in \{2, 3, \cdots, k\}$. Then the edge a_{11} separates the disk D into two disks. One of the two disks contains the terminal edge e_1, say E. Suppose that D contains a feeler e at w_i. Then $e \subset E$ or $e \not\subset E$.

If $e \subset E$ (see Fig. 9(b)), then by (4) and Lemma 3.3(d) the chart Γ is not minimal. This contradicts the fact that Γ is minimal. Thus $e \not\subset E$ (see Fig. 9(c)).

By (2), the edge a_{11} is oriented outward at w_i. Thus by Assumption 2, the edge e_i' is oriented inward at w_i. Hence by (1),(4) and Lemma 3.3(a) the chart Γ is not minimal. This contradicts the fact that Γ is minimal. Therefore D does not contain a feeler at w_i.

Similarly if $b_{11} \ni w_i$, then we can show that D does not contain a feeler at w_i. Thus Statement (a) holds.

Next, we shall show Statement (b). If $a_{11} \ni w_2$, then by Lemma 4.4(a) the disk D does not contain a feeler at w_2. Thus the curve $e_1' \cup a_{11}$ bounds a lens whose interior contains at most one white vertex. This contradicts Lemma 3.1. Thus $a_{11} \not\ni w_2$.

Similarly we can show $b_{11} \not\ni w_k$. Thus Statement (b) holds.

Next, we shall show Statement (c). Now, suppose that $a_{11} \ni w_i$ for some $i \in \{3, 4, \cdots, k-1\}$. We shall show $b_{11} \not\ni w_{i+1}$. If $b_{11} \ni w_{i+1}$, then by Lemma 4.4(a), the disk D contains neither a feeler at w_i nor a feeler at w_{i+1} (see Fig. 9(d)).

Now, by (1), the terminal edge e_1 is oriented inward at w_1. Since the edge e_i' is oriented inward at w_i or w_{i+1}, the chart Γ is not minimal by (4) and Lemma 3.3(a). This contradicts the fact that Γ is minimal. Therefore $b_{11} \not\ni w_{i+1}$. Thus Statement (c) holds.

Similarly we can show Statement (d).

Finally, we shall show Statement (e). Suppose $b_{11} \ni w_2$. Then by Lemma 4.4(a), the disk D does not contain a feeler at w_2 (see Fig. 9(e)). Since the terminal edge e_1 is oriented inward at w_1 by (1), and since the edge e'_1 is oriented inward at w_2 by (3), the chart Γ is not minimal by (4) and Lemma 3.3(a). This contradicts the fact that Γ is minimal. Therefore $b_{11} \not\ni w_2$.

Similarly we can show $a_{11} \not\ni w_k$. Thus Statement (e) holds. □

5. Special 5-Angled Disks

In this section, we investigate a special 5-angled disk whose interior contains at most one white vertex.

Let Γ be a chart. Suppose that an object consists of some edges of Γ, arcs in edges of Γ and arcs around white vertices. Then the object is called a *pseudo chart.*

LEMMA 5.1 ([7, Corollary 5.8]). *Let* Γ *be a minimal chart. Let* D *be a* 2*-angled disk of* Γ_m *with at most one feeler. If* $w(\Gamma \cap \text{Int}D) = 0$*, then a regular neighborhood of* D *contains one of two pseudo charts as shown in Fig. 10.*

LEMMA 5.2. *Let* Γ *be a minimal chart, and* m *a label of* Γ. *Let* D *be a special* 5*-angled disk of* Γ_m *with at least one feeler* e_1. *Suppose that all the white vertices on* ∂D *are contained in* $\Gamma_{m+\varepsilon}$ *for some* $\varepsilon \in \{+1, -1\}$. *Let* w_1 *be the white vertex in* e_1. *Let* a_{11}, b_{11} *be internal edges* (*possibly terminal edges*) *of label* $m+\varepsilon$ *at* w_1 *in* D. *If* $w(\Gamma \cap \text{Int}D) \leqq 1$*, then one of* a_{11}, b_{11} *contains a white vertex in* $\text{Int}D$.

PROOF. Let $w_2, \cdots, w_5$ be the four white vertices on ∂D different from w_1 such that $w_1, w_2, \cdots, w_5$ lie anticlockwise in this order. Without loss of

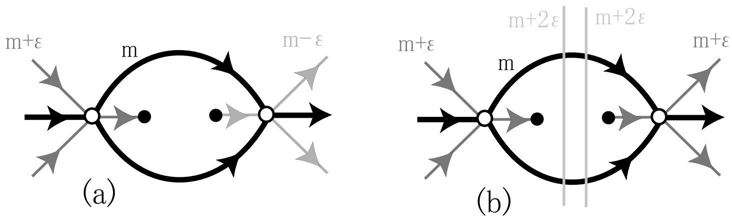

Fig. 10. The 2-angled disks have no feelers where $\varepsilon \in \{+1, -1\}$.

generality we can assume that a_{11}, e_1, b_{11} lie anticlockwise in this order. By Assumption 2,

(1) neither a_{11} nor b_{11} is a terminal edge.

Suppose that neither a_{11} nor b_{11} contains a white vertex in Int$D$. Then by applying Lemma 4.4(b),(e) for the edge a_{11}, we have $a_{11} \ni w_3$ or $a_{11} \ni w_4$.

If $a_{11} \ni w_3$, then by (1) we have $b_{11} \ni w_4$ or $b_{11} \ni w_5$. This contradicts Lemma 4.4(b),(c).

If $a_{11} \ni w_4$, then by (1) we have $b_{11} \ni w_5$. This contradicts Lemma 4.4(b).

Therefore we have a contradiction for both cases. Hence one of a_{11}, b_{11} contains a white vertex in Int$D$. □

Let Γ be a chart, D a disk, and G a pseudo chart with $G \subset D$. Let $r : D \to D$ be a reflection of D, and G^* the pseudo chart obtained from G by changing the orientations of all of the edges. Then the set $\{G, G^*, r(G), r(G^*)\}$ is called the *RO-family of the pseudo chart G*.

LEMMA 5.3. *Let Γ be a minimal chart, and m a label of Γ. Let D be a special 5-angled disk of Γ_m with at least one feeler e_1. Suppose that all the white vertices on ∂D are contained in $\Gamma_{m+\varepsilon}$ for some $\varepsilon \in \{+1, -1\}$. Let w_1 be the white vertex in e_1. Let a_{11}, b_{11} be internal edges (possibly terminal edges) of label $m + \varepsilon$ at w_1 in D. Suppose that one of a_{11}, b_{11} contains a white vertex in* Int$D$*, but the other contains a white vertex in ∂D different from w_1. If $w(\Gamma \cap \mathrm{Int}D) \leqq 1$, then the disk D contains exactly one feeler e_1. Moreover, the disk D contains one of the RO-family of the pseudo chart as shown in Fig. 11(a).*

PROOF. Let $w_2, \cdots, w_5$ be the four white vertices on ∂D different from w_1 such that $w_1, w_2, \cdots, w_5$ lie anticlockwise in this order. Let $e'_1, e'_2, \cdots, e'_5$ be the five internal edges of label m in ∂D with $\partial e'_i = \{w_i, w_{i+1}\}$ ($i = 1, 2, 3, 4$) and $\partial e'_5 = \{w_5, w_1\}$. Without loss of generality we can assume that a_{11}, e_1, b_{11} lie anticlockwise in this order (see Fig. 11(b)).

Since $w(\Gamma \cap \mathrm{Int}D) \leqq 1$, by Lemma 4.3 we can assume that

(1) $\Gamma_{m-\varepsilon} \cap D = \emptyset$ ($\Gamma_{m-\varepsilon} \cap a_{11} = \emptyset$).

Without loss of generality, we can assume that the terminal edge e_1 is oriented inward at w_1. Then by Assumption 2,

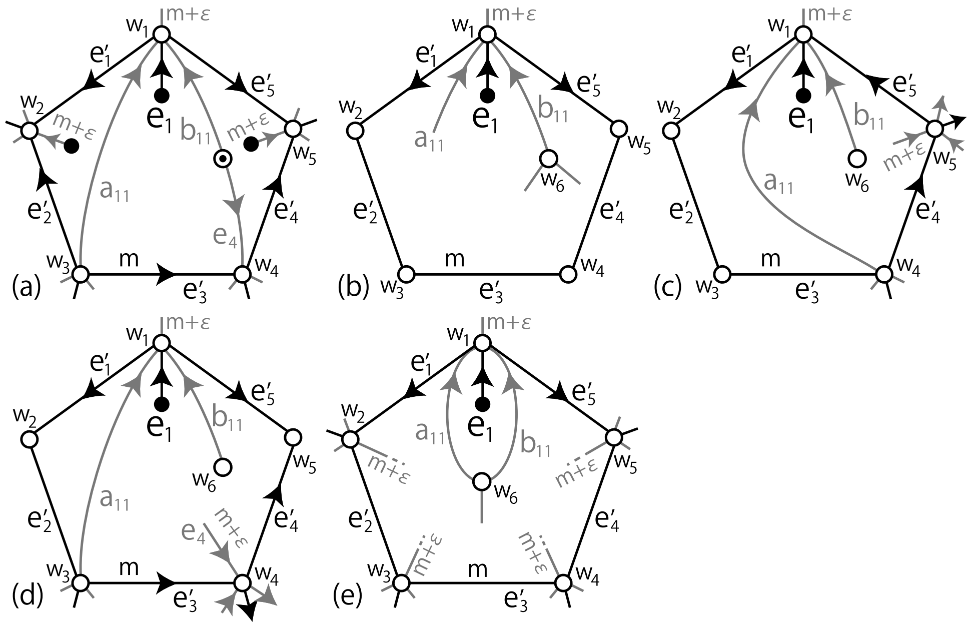

Fig. 11. Special 5-angled disks with one feeler e_1.

(2) both of a_{11}, b_{11} are oriented inward at w_1,

(3) both of e'_1, e'_5 are oriented outward at w_1.

Without loss of generality, we can assume that the edge b_{11} contains a white vertex in Int$D$, say w_6. Thus by the condition of this lemma, the edge a_{11} contains a white vertex in ∂D different from w_1. Hence by Lemma 4.4(b),(e), we have $a_{11} \ni w_3$ or $a_{11} \ni w_4$.

We shall show that $a_{11} \ni w_3$. If $a_{11} \ni w_4$, then by Lemma 4.4(a) the disk D does not contain a feeler at w_4. Hence by (1) and Lemma 3.3(a) the internal edge e'_4 is oriented from w_4 to w_5 (because if not, then the two edges e_1, e'_4 are oriented inward at w_1, w_4, respectively. Thus by (1) and Lemma 3.3(a), the chart Γ is not minimal. This is a contradiction.). Moreover, by (1) and Lemma 3.3(b), the edge e'_4 is middle at w_5. Hence by Assumption 2, the disk D does not contain a feeler at w_5. Thus by the definition of the chart, the internal edge e'_5 is oriented from w_5 to w_1. Hence the edge e'_5 must be oriented inward at w_1 (see Fig. 11(c)). This contradicts

(3). Thus $a_{11} \ni w_3$.

Similarly, by (1) and Lemma 3.3(a),(b), we can show that the disk D contains neither feeler at w_3 nor feeler at w_4, and

(4) the internal edge e'_3 is oriented from w_3 to w_4, and is middle at w_4,

(5) the internal edge e'_4 is oriented from w_4 to w_5 (see Fig. 11(d)).

Let e_4 be an internal edge (possibly a terminal edge) of label $m+\varepsilon$ at w_4 in D. Then by (4),

(6) the edge e_4 is oriented inward at w_4, but not middle at w_4.

Thus by Assumption 2, we have $e_4 \ni w_5$ or $e_4 \ni w_6$.

We shall show that $e_4 \ni w_6$. If $e_4 \ni w_5$, then the disk D does not contain a feeler at w_5 by Lemma 4.4(b),(e). Thus there are three consecitive edges e'_4, e_4, e'_5 at w_5 such that e'_4, e'_5 are oriented inward at w_5 by (3) and (5), but e_4 is oriented outward at w_5 by (6). This contradicts the definition of the chart. Hence $e_4 \ni w_6$.

Let e_6 be an internal edge (possibly a terminal edge) of label $m+\varepsilon$ at w_6 different from b_{11}, e_4. Then by (2) and (6),

(7) the edge e_6 is oriented inward at w_6.

We shall show that the disk D does not contain a feeler at w_5. If D contains a feeler at w_5, then two internal edges e, e' of label $m+\varepsilon$ at w_5 in D contain white vertices in Int$D$ by Assumption 2. Hence the condition $w(\text{Int}D) \leqq 1$ implies that the two edges e, e' contain the same white vertex w_6. Thus there exist four internal edges b_{11}, e_4, e, e' of label $m+\varepsilon$ at w_6. This contradicts the definition of the chart. Hence D does not contain a feeler at w_5.

Let e_5 be an internal edge (possibly a terminal edge) of label $m+\varepsilon$ at w_5 in D. Since both of e'_4, e'_5 are oriented inward at w_5 by (3) and (5), the edge e_5 is oriented inward at w_5. Thus by (7), we have $e_5 \neq e_6$. Hence both of e_5, e_6 are terminal edges.

Moreover by Assumption 2, we can show that the disk D does not contain a feeler at w_2 and there exists a terminal edge of label $m+\varepsilon$ at w_2 in D. Therefore the disk D contains the pseudo chart as shown in Fig. 11(a). □

LEMMA 5.4. *Let* Γ *be a minimal chart, and* m *a label of* Γ. *Let* D *be a special* 5*-angled disk of* Γ_m *with at least one feeler. Suppose that all* *the white vertices on ∂D are contained in $\Gamma_{m+\varepsilon}$ for some $\varepsilon \in \{+1, -1\}$. If $w(\Gamma \cap \mathrm{Int}D) \leqq 1$, then the disk D contains exactly one feeler. Moreover, the disk D contains one of RO-families of the two pseudo charts as shown in Fig. 11(a),(e).*

Proof. Let e_1 be a feeler of D, and w_1 the white vertex in e_1. Let $w_2, \cdots, w_5$ be the four white vertices on ∂D different from w_1 such that $w_1, w_2, \cdots, w_5$ lie anticlockwise in this order. Let a_{11}, b_{11} be internal edges (possibly terminal edges) of label $m+\varepsilon$ at w_1 such that a_{11}, e_1, b_{11} lie anticlockwise in this order (see Fig. 11(b)). By Lemma 5.2, one of a_{11}, b_{11} contains a white vertex in Int$D$. Without loss of generality, we can assume that the edge b_{11} contains a white vertex in Int$D$, say w_6. Moreover, by Assumption 2, the edge a_{11} is not a terminal edge. Hence either a_{11} contains a white vertex in Int$D$, or a_{11} contains a white vertex in ∂D different from w_1.

If a_{11} contains a white vertex in ∂D different from w_1, then by Lemma 5.3 the disk D contains one of the RO-family of the pseudo chart as shown in Fig. 11(a).

Suppose that a_{11} contains a white vertex in Int$D$. Since $w(\Gamma \cap \mathrm{Int}D) \leqq 1$, both of a_{11}, b_{11} contain the same white vertex w_6 in Int$D$. Thus $a_{11} \cup b_{11}$ bounds a 2-angled disk in D. Hence by Lemma 5.1, the 2-angled disk has no feeler.

Finally we shall show that D has exactly one feeler. If D has another feeler e_2 at some w_i $(i = 2, 3, 4, 5)$, then D has at least two feelers. Hence by Lemma 5.2 and Lemma 5.3, the two internal edges e, e' of label $m+\varepsilon$ at w_i in D contain white vertices in Int$D$. However, the condition $w(\Gamma \cap \mathrm{Int}D) \leqq 1$ implies that there are four edges a_{11}, b_{11}, e, e' of label $m+\varepsilon$ at w_6. This contradicts the definition of the chart. Thus D has exactly one feeler e_1. Hence D contains the pseudo chart as shown in Fig. 11(e). □

From the above three lemmas, we have the following corollary:

Corollary 5.5. *Let Γ be a minimal chart, and m a label of Γ. Let D be a special 5-angled disk of Γ_m. Suppose that all the white vertices on ∂D are contained in $\Gamma_{m+\varepsilon}$ for some $\varepsilon \in \{+1, -1\}$. Then we have the following:*

(a) *If the disk D contains at least one feeler, then $w(\Gamma \cap \mathrm{Int}D) \geqq 1$.*

(b) *If the disk D contains at least two feelers, then $w(\Gamma \cap \mathrm{Int}D) \geqq 2$.*

6. Special 4-Angled Disks

In this section, we investigate a 4-angled disk whose interior contains at most one white vertex. By the similar way of the proof of Lemma 5.2, we can show the following lemma:

LEMMA 6.1. *Let* Γ *be a minimal chart, and* m *a label of* Γ. *Let* D *be a special* 4*-angled disk of* Γ_m *with at least one feeler* e_1. *Suppose that all the white vertices on* ∂D *are contained in* $\Gamma_{m+\varepsilon}$ *for some* $\varepsilon \in \{+1, -1\}$. *Let* w_1 *be the white vertex in* e_1. *Let* a_{11}, b_{11} *be internal edges* (*possibly terminal edges*) *of label* $m + \varepsilon$ *at* w_1 *in* D. *If* $w(\Gamma \cap \text{Int}D) \leqq 1$, *then one of* a_{11}, b_{11} *contains a white vertex in* Int$D$.

By the similar way of the proof of Lemma 5.4, we can show the following lemma:

LEMMA 6.2. *Let* Γ *be a minimal chart, and* m *a label of* Γ. *Let* D *be a special* 4*-angled disk of* Γ_m *with at least one feeler. Suppose that all the white vertices on* ∂D *are contained in* $\Gamma_{m+\varepsilon}$ *for some* $\varepsilon \in \{+1, -1\}$. *If* $w(\Gamma \cap \text{Int}D) \leqq 1$, *then the disk* D *contains exactly one feeler. Moreover, the disk* D *contains one of RO-families of the two pseudo charts as shown in Fig. 12.*

From the above two lemmas, we have the following corollary:

COROLLARY 6.3. *Let* Γ *be a minimal chart, and* m *a label of* Γ. *Let* D

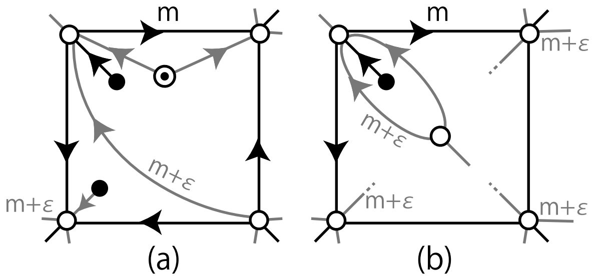

Fig. 12. Special 4-angled disks with one feeler.

be a special 4*-angled disk of* Γ_m*. Suppose that all the white vertices on* ∂D *are contained in* $\Gamma_{m+\varepsilon}$ *for some* $\varepsilon \in \{+1, -1\}$*. Then we have the following:*

(a) *If the disk* D *contains at least one feeler, then* $w(\Gamma \cap \text{Int}D) \geqq 1$*.*

(b) *If the disk* D *contains at least two feelers, then* $w(\Gamma \cap \text{Int}D) \geqq 2$*.*

7. Cases of the Graphs as Shown in Fig. 13(a),(c)

In this section, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m contains neither graphs as shown in Fig. 13(a),(c).

Lemma 7.1 ([11, Lemma 3.4]). *Let* Γ *be a minimal chart, and* m *a label of* Γ*. Let* G *be a connected component of* Γ_m*. If* G *contains exactly five white vertices, and if* G *has no loop, then* G *is one of nine graphs as shown in Fig. 2 and Fig. 13.*

Lemma 7.2 ([13, Lemma 7.2(a),(c)]). *Let* Γ *be a minimal chart, and* m *a label of* Γ*. Let* G *be a connected component of* Γ_m *with* $w(G) = 5$*. Then we have the following:*

(a) *If* G *is the graph as shown in Fig. 13*(*a*) (*resp. Fig. 13*(*b*))*, then* G *is one of the RO-family of the graph as shown in Fig. 14*(*a*) (*resp. Fig. 14*(*b*))*.*

(b) *If* G *is the graph as shown in Fig. 13*(*d*) (*resp. Fig. 13*(*e*))*, then* G *is one of the RO-family of the graph as shown in Fig. 14*(*c*) (*resp. Fig. 14*(*d*))*.*

Lemma 7.3 ([13, Lemma 3.6(a)]). *Let* Γ *be a minimal chart, and* m *a label of* Γ*. Let* D *be a* 2*-angled disk of* Γ_m *without feelers, and* w_1, w_2 *the white vertices in* ∂D*. Let* e_1, e_2 *be the internal edges* (*possibly terminal edges*) *of label* m *at* w_1, w_2*, respectively, such that* $e_1 \not\subset D$ *and* $e_2 \not\subset D$*. Suppose that the two edges* e_1, e_2 *are oriented inward* (*resp. outward*) *at* w_1, w_2*, respectively* (*see Fig. 15*(*a*) *and* (*b*))*. Then we have* $w(\Gamma \cap \text{Int}D) \geq 1$*.*

Lemma 7.4. *Let* Γ *be a minimal chart of type* $(m; 5, 2)$*. Then* Γ_m *does not contain the graph as shown in Fig. 13*(*a*)*.*

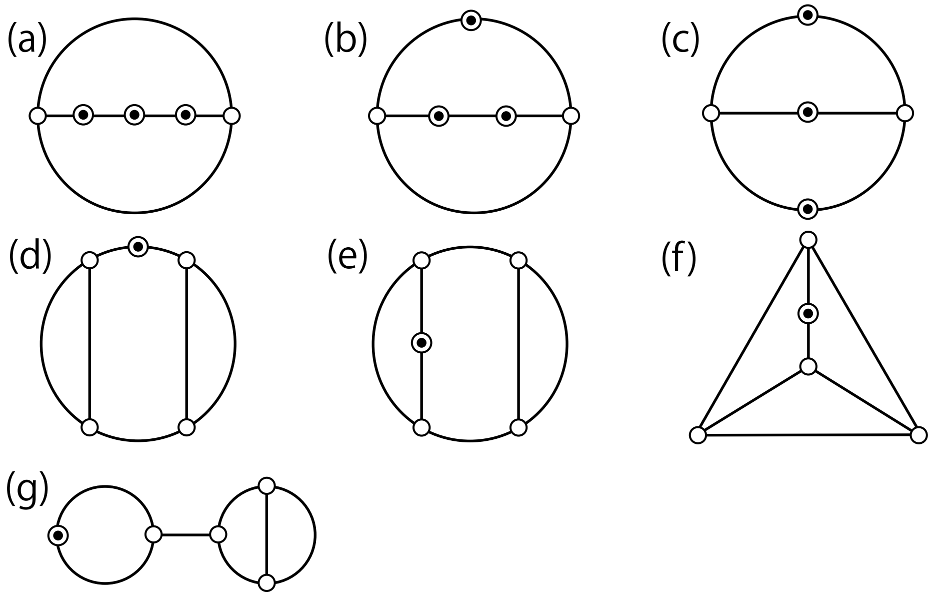

Fig. 13. (a),(b),(c) Graphs with three black vertices. (d),(e),(f),(g) Graphs with one black vertex.

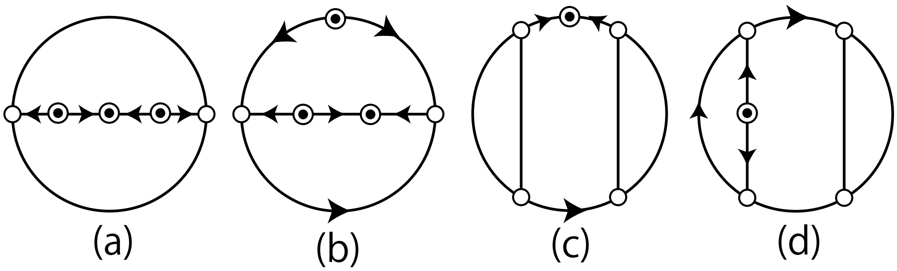

Fig. 14. Connected components of Γ_m with five white vertices.

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 13(a), say G. Then G separates the 2-sphere S^2 into three disks. One of the three disks is a 2-angled disk, say D_1. Let D_2, D_3 be the other disks. Then D_2 and D_3 are special 5-angled disks. Since one of D_2, D_3 has at least two feelers, by Corollary 5.5(b) we have

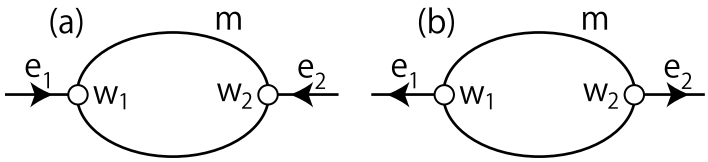

Fig. 15. 2-angled disks without feelers.

(1) $w(\Gamma \cap \text{Int}D_2) \geqq 2$ or $w(\Gamma \cap \text{Int}D_3) \geqq 2$.

By Lemma 7.2(a), the graph G is one of the RO-family of the graph as shown in Fig. 14(a). Without loss of generality, we can assume that the graph G is the graph as shown in Fig. 14(a). Let w_1, w_2 be the two white vertices in ∂D_1. Let e_1, e_2 be internal edges of label m at w_1, w_2, respectively, with $e_1 \not\subset \partial D_1$ and $e_2 \not\subset \partial D_1$. Then the two edges e_1, e_2 are oriented inward at w_1, w_2, respectively. Thus by Lemma 7.3, we have $w(\Gamma \cap \text{Int}D_1) \geqq 1$. Hence by (1), we have

$$\begin{aligned} 7 = w(\Gamma) &= w(G) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) + w(\Gamma \cap \text{Int}D_3) \\ &\geqq 5 + 1 + 2 = 8. \end{aligned}$$

This is a contradiction. Therefore Γ_m does not contain the graph as shown in Fig. 13(a). □

LEMMA 7.5. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 13(c).*

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 13(c), say G. Then G separates the 2-sphere S^2 into three disks. Let D_1, D_2, D_3 be the three disks. Then D_1, D_2, D_3 are special 4-angled disks. Without loss of generality, we can assume that D_1 has at least one feeler. Then D_1 has one feeler or two feelers.

If D_1 has two feelers, then one of D_2, D_3 has one feeler. Thus by Colorally 6.3, we have $w(\Gamma \cap \text{Int}D_1) \geqq 2$ and ($w(\Gamma \cap \text{Int}D_2) \geqq 1$ or $w(\Gamma \cap \text{Int}D_3) \geqq 1$). By the similar way of the proof of Lemma 7.4, we have a contradiction. Thus D_1 has exactly one feeler.

Since D_1 has exactly one feeler, one of D_2, D_3 has at least one feeler. If one of D_2, D_3 has exactly two feelers, then we have the same contradiction as above. Hence both of D_2, D_3 have exactly one feeler. Thus by Colorally 6.3(a), we have $w(\Gamma \cap \mathrm{Int} D_i) \geqq 1$ for $i = 1, 2, 3$. Hence

$$\begin{aligned} 7 = w(\Gamma) &= w(G) + w(\Gamma \cap \mathrm{Int} D_1) + w(\Gamma \cap \mathrm{Int} D_2) + w(\Gamma \cap \mathrm{Int} D_3) \\ &\geqq 5 + 1 + 1 + 1 = 8. \end{aligned}$$

This is a contradiction. Therefore Γ_m does not contain the graph as shown in Fig. 13(c). □

8. Case of the Graph as Shown in Fig. 13(b)

In this section, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(b).

LEMMA 8.1 ([14, Lemma 4.2(a)]). *Let Γ be a minimal chart, and m a label of Γ. Let D be a special 3-angled disk of Γ_m with at most two feelers. If $w(\Gamma \cap \mathrm{Int} D) = 0$, then a regular neighborhood of D contains one of the RO-families of the two pseudo charts as shown in Fig. 16.*

LEMMA 8.2. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 13(b).*

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 13(b), say G. Then G separates the 2-sphere S^2 into three disks. One of the three disks is a 3-angled disk, say D_1. One of the three disks is a 4-angled disk,

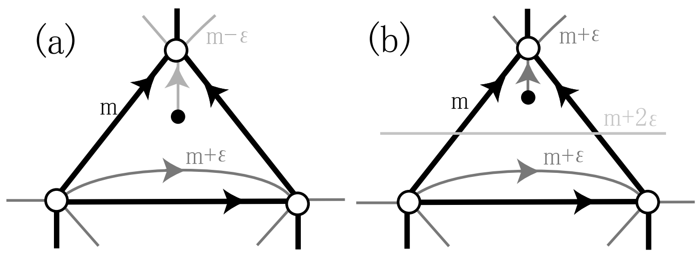

Fig. 16. The 3-angled disks have no feelers where m is a label, $\varepsilon \in \{+1, -1\}$.

say D_2. The last one is a 5-angled disk, say D_3. For the disk D_3, there are four cases: (i) D_3 has no feeler, (ii) D_3 has exactly one feeler, (iii) D_3 has exactly two feelers, (iv) D_3 has exactly three feelers.

Case (i). Since D_3 has no feeler, the 3-angled disk D_1 has exactly one feeler and the 4-angled disk D_2 has exactly two feelers. Thus by Corollary 6.3(b) and Lemma 8.1, we have $w(\Gamma\cap\text{Int}D_1) \geqq 1$ and $w(\Gamma\cap\text{Int}D_2) \geqq 2$. Hence, we have

$$\begin{aligned} 7 = w(\Gamma) &\geqq w(G) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) \\ &\geqq 5 + 1 + 2 = 8. \end{aligned}$$

This is a contradiction. Thus Case (i) does not occur.

Case (ii). Since the 5-angled disk D_3 has exactly one feeler, by Corollary 5.5(a) we have $w(\Gamma \cap \text{Int}D_3) \geqq 1$. Moreover, the 4-angled disk D_2 has one feeler or two feelers.

If D_2 has exactly two feelers, then by Corollary 6.3(b) we have $w(\Gamma \cap \text{Int}D_2) \geqq 2$. Thus we have the same contradiction of Case (i).

If D_2 has exactly one feeler, then the 3-angled disk D_1 has exactly one feeler. Thus by Corollary 6.3(a) and Lemma 8.1, we have $w(\Gamma \cap \text{Int}D_2) \geqq 1$ and $w(\Gamma \cap \text{Int}D_3) \geqq 1$. By the similar way of the proof of Lemma 7.5, we have a contradiction.

Therefore both cases do not occur. Thus Case (ii) does not occur.

Case (iii). Since the 5-angled disk D_3 has exactly two feelers, one of the disks D_1, D_2 has exactly one feeler. Thus by Corollary 5.5(b), Corollary 6.3(a) and Lemma 8.1, we have $w(\Gamma\cap\text{Int}D_3) \geqq 2$ and $(w(\Gamma\cap\text{Int}D_1) \geqq 1$ or $w(\Gamma \cap \text{Int}D_2) \geqq 1)$. Hence we have the same contradiction of Case (i). Thus Case (iii) does not occur.

Case (iv). Since the 5-angled disk D_3 has exactly three feelers, by Corollary 5.5(b) we have $w(\Gamma \cap \text{Int}D_3) \geqq 2$. Since $w(\Gamma) = 7$, we have $w(\Gamma \cap \text{Int}D_3) = 2$ and $w(\Gamma \cap \text{Int}D_2) = 0$.

By Lemma 7.2(a), the graph G is one of the RO-family of the graph as shown in Fig. 14(b). Without loss of generality, we can assume that the graph G is the graph as shown in Fig. 14(b). Thus the chart Γ contains the pseudo chart as shown in Fig. 17. We use the notations as shown in Fig. 17, where e_1, e_2, e_3 are internal edges (possibly terminal edges) of label $m+1$ oriented outward at w_1, w_2, w_3 in D_2, respectively. Hence the condition $w(\Gamma\cap\text{Int}D_2) = 0$ implies one of e_1 or e_2 is a terminal edge. However neither

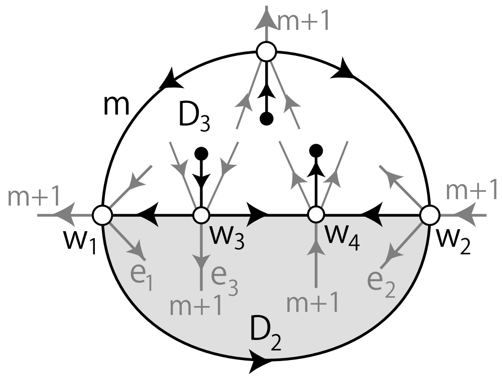

Fig. 17. The graph as shown in Fig. 13(b). The gray region is the 4-angled disk D_2.

e_1 nor e_2 is middle at w_1 or w_2 (see Fig. 17). This contradicts Assumption 2. Thus Case (iv) does not occur.

Therefore all four cases do not occur. Hence Γ_m does not contain the graph as shown in Fig. 13(b). □

9. Case of the Graph as Shown in Fig. 13(d)

In this section, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(d).

LEMMA 9.1. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 13(d).*

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 13(d), say G. By Lemma 7.2(b), the graph G is one of the RO-family of the graph as shown in Fig. 14(c). Without loss of generality, we can assume that the graph G is the graph as shown in Fig. 14(c). We use the notations as shown in Fig. 18(a), where $w_1, w_2, \cdots, w_5$ are five white vertices, and

(1) e_1, e_2 are internal edges of label m oriented outward at w_1, w_2, respectively.

Let D_1, D_2 be special 5-angled disks of Γ_m with $\mathrm{Int}D_1 \cap \mathrm{Int}D_2 = \emptyset$ such that the disk D_2 contains the point at infinity, ∞. If necessary, we move

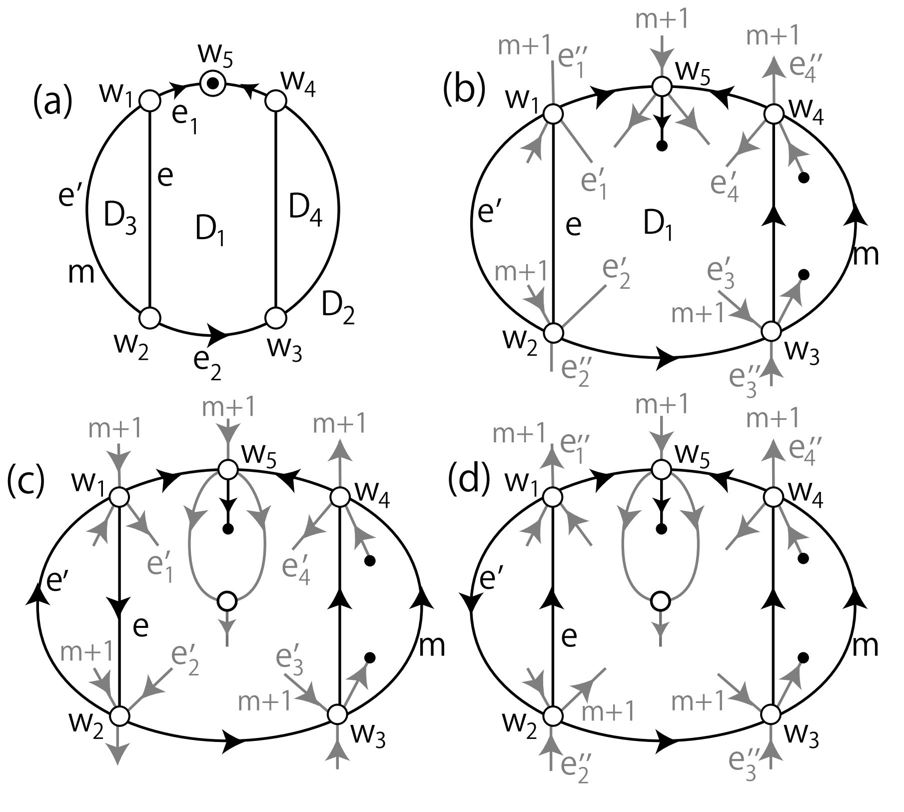

Fig. 18. The graphs as shown in Fig. 13(d). (c) The edge e is oriented from w_1 to w_2. (d) The edge e is oriented from w_2 to w_1.

the point ∞ by Assumption 5, and if necessary, we reflect the chart Γ, we can assume that the disk D_1 has one feeler. Thus by Corollary 5.5(a), we have

(2) $w(\Gamma \cap \mathrm{Int}D_1) \geqq 1$.

Let D_3, D_4 be special 2-angled disks of Γ_m with $\partial D_3 \ni w_1$ and $\partial D_4 \ni w_4$. Then by (1) and Lemma 7.3, we have $w(\Gamma \cap \mathrm{Int}D_3) \geqq 1$. Hence by (2), the condition $w(\Gamma) = 7$ implies that

(3) $w(\Gamma \cap \mathrm{Int}D_1) = 1$, $w(\Gamma \cap \mathrm{Int}D_2) = 0$, $w(\Gamma \cap \mathrm{Int}D_4) = 0$.

Thus by Lemma 5.1, a regular neighborhood of D_4 contains the pseudo chart as shown in Fig. 10(b). Therefore, the chart Γ contains the pseudo chart as shown in Fig. 18(b). We use the notations as shown in Fig. 18(b), where e'_i, e''_i $(i = 1, 2, 3, 4)$ are internal edges (possibly terminal edges) of label $m + 1$ at w_i with $e'_i \subset D_1$ and $e''_i \subset D_2$,

(4) e'_3, e''_3 are oriented inward at w_3, but not middle at w_3, and

neither e'_4 nor e''_4 is middle at w_4. Thus by Assumption 2,

(5) none of e'_3, e''_3, e'_4, e''_4 are terminal edges.

Since $w(\Gamma \cap \mathrm{Int} D_1) = 1$ by (3) and since e'_4 is not a terminal edge by (5), by Lemma 5.4 the disk D_1 contains the pseudo chart as shown in Fig. 11(e) (see Fig. 18(c),(d)). Let e, e' be internal edges of label m in ∂D_3 with $e \subset D_1$ and $e' \subset D_2$. Then there are two cases: (i) e is oriented from w_1 to w_2 (see Fig. 18(c)), (ii) e is oriented from w_2 to w_1 (see Fig. 18(d)).

Case (i). By looking around the white vertex w_1, we have that e' is oriented from w_2 to w_1, and

(6) e'_2 is oriented inward at w_2, but not middle at w_2.

Thus by (5) and Assumption 2, none of e'_2, e'_3, e'_4 are terminal edges. Moreover, by (4) and (6), the two edges e'_2 and e'_3 are oriented inward at w_2, w_3, respectively. Hence, for the edge e'_3, we must have $e'_3 = e'_4$. However, there exists a lens. This contradicts Lemma 3.2. Thus Case (i) does not occur.

Case (ii). Looking around the white vertex w_2, we have that e' is oriented from w_1 to w_2, and

(7) e''_2 is oriented inward at w_2, but not middle at w_2.

Now $w(\Gamma \cap \mathrm{Int} D_2) = 0$ by (3). By the similar way of the proof of Case (i) in this lemma, for the edge e''_3, we must have $e''_3 = e''_4$. However, there exists a lens. This contradicts Lemma 3.2. Thus Case (ii) does not occur.

Therefore both two cases do not occur. Hence Γ_m does not contain the graph as shown in Fig. 13(d). □

10. IO-Calculation

In this section, we review IO-Calculation.

Let Γ be a chart, and v a vertex. Let α be a short arc of Γ in a small neighborhood of v such that v is an endpoint of α. If the arc α is oriented to v, then α is called *an inward arc*, and otherwise α is called *an outward arc*.

Let Γ be an n-chart. Let F be a closed domain with $\partial F \subset \Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}$ for some label k of Γ, where $\Gamma_0 = \emptyset$ and $\Gamma_n = \emptyset$. By Condition (iii) for charts, in a small neighborhood of each white vertex, there are three inward arcs and three outward arcs. Also in a small neighborhood of each black vertex, there exists only one inward arc or one outward arc. We often use the following fact, when we fix (inward or outward) arcs near white vertices and black vertices:

(∗) *The number of inward arcs contained in $F \cap \Gamma_k$ is equal to the number of outward arcs in $F \cap \Gamma_k$.*

When we use this fact, we say that we use *IO-Calculation with respect to Γ_k in F*. For example, in a minimal chart Γ, consider the pseudo chart as shown in Fig. 19 where

(1) F is a 4-angled disk of $\Gamma_{k+\delta}$ without feelers for some $\delta \in \{+1, -1\}$,

(2) e_1, e_2, e_4 are internal edges (possibly terminal edges) of label k oriented outward at w_1, w_2, w_4, respectively,

(3) e_3 is an internal edge (possibly a terminal edge) of label k oriented inward at w_3,

(4) neither e_2 nor e_4 is middle at w_2 or w_4.

Then we can show that $w(\Gamma \cap \text{Int}F) \geq 1$. Suppose $w(\Gamma \cap \text{Int}F) = 0$. By (4) and Assumption 2,

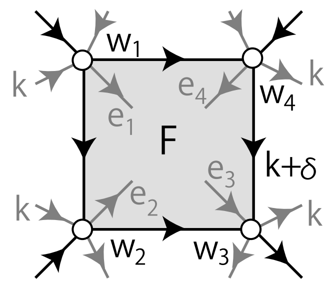

Fig. 19. The gray region is the 4-angled disk F where k is a label, $\delta \in \{+1, -1\}$.

(5) neither e_2 nor e_4 is a terminal edge.

If both two edges e_1, e_3 are a terminal edge, then by (2) and (3) the number of inward arcs in $F \cap \Gamma_k$ is two, but the number of outward arcs in $F \cap \Gamma_k$ is four. This contradicts the fact $(*)$. If e_1 is a terminal edge, but e_3 is not a terminal edge, then by (2) and (3) the number of inward arcs in $F \cap \Gamma_k$ is two, but the number of outward arcs in $F \cap \Gamma_k$ is three. This contradicts the fact $(*)$. Similarly for the other cases we have the same contradiction. Thus $w(\Gamma \cap \mathrm{Int}F) \geq 1$. Instead of the above argument, we say that

we have $w(\Gamma \cap \mathrm{Int}F) \geq 1$ *by IO-Calculation with respect to* Γ_k *in* F.

11. Case of the Graph as Shown in Fig. 13(e)

In this section, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(e).

Let Γ and Γ' be C-move equivalent charts. Suppose that a pseudo chart X of Γ is also a pseudo chart of Γ'. Then we say that Γ is modified to Γ' by *C-moves keeping X fixed.* In Fig. 20, we give examples of C-moves keeping pseudo charts fixed.

Let Γ be a chart, and X a subset of Γ. Let

$$c(X) = \text{the number of crossings in } X.$$

Let D be a k-angled disk of Γ_m for a minimal chart Γ. The pair of integers $(w(\Gamma \cap \mathrm{Int}D), c(\partial D))$ is called the *local complexity with respect to D*, denoted by $\ell c(D; \Gamma)$. Let $\mathbb{S}$ be the set of all minimal charts each of which

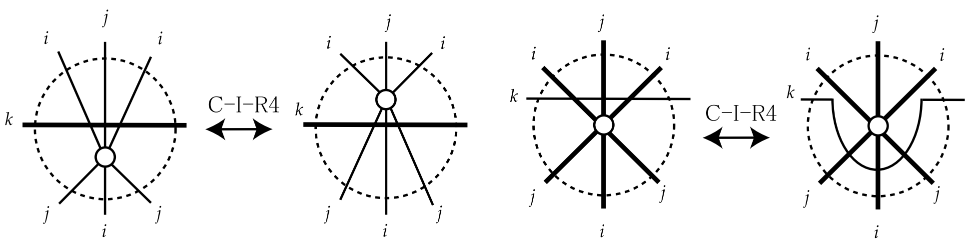

Fig. 20. C-moves keep thicken figures fixed.

can be moved from Γ by C-moves in a regular neighborhood of D keeping ∂D fixed. The chart Γ is said to be *locally minimal with respect to D* if its local complexity with respect to D is minimal among the charts in $\mathbb{S}$ with respect to the lexicographic order.

LEMMA 11.1 ([8, Theorem 1.1]). *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m with at most one feeler such that Γ is locally minimal with respect to D. If $w(\Gamma \cap \mathrm{Int}D) \leqq 1$, then a regular neighborhood of D contains an element in the RO-families of the five pseudo charts as shown in Fig. 10 and Fig. 21.*

LEMMA 11.2. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 13(e).*

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 13(e), say G. Then G separates the 2-sphere S^2 into four disks. One of the four disks is a 5-angled disk, say D_1. One of the four disks is a 4-angled disk, say D_2. One of the four disks is a 3-angled disk, say D_3. Let D_4 be the last disk.

Since one of D_1, D_3 has one feeler, by Corollary 5.5(a) and Lemma 8.1 we have

(1) $w(\Gamma \cap \mathrm{Int}D_1) \geqq 1$ or $w(\Gamma \cap \mathrm{Int}D_3) \geqq 1$.

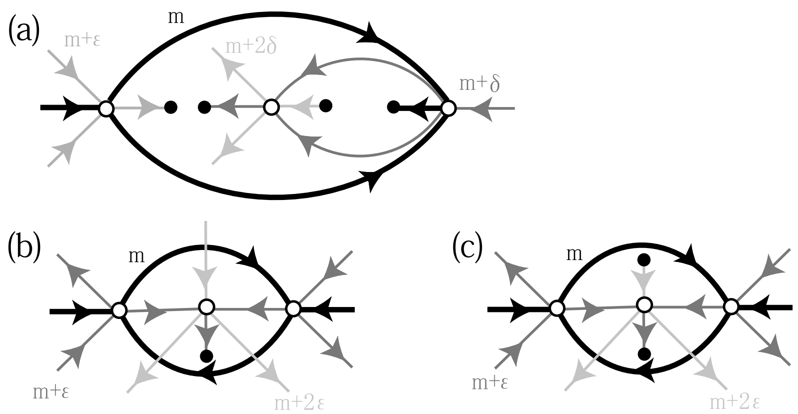

Fig. 21. The 2-angled disk (a) has one feeler, the others do not have any feelers.

Thus the condition $w(\Gamma) = 7$ implies that $w(\Gamma \cap \mathrm{Int}D_4) \leqq 1$. We can assume that Γ is locally minimal with respect to D_4. Hence by Lemma 11.1 a regular neighborhood of D_4 contains one of the RO-families of the three pseudo charts as shown in Fig. 10(b) and Fig. 21(b),(c). Moreover, by Lemma 7.2(b), the graph G is the graph as shown in Fig. 14(d). Thus the chart Γ contains one of the RO-families of the three pseudo charts as shown in Fig. 22, where the pseudo charts as shown in Fig. 22(b),(c) are contained in one of the pseudo charts as shown in Fig. 21(b),(c). Without loss of generality, we can assume that the chart Γ contains one of the three pseudo charts as shown in Fig. 22.

Suppose that the chart Γ contains the pseudo chart as shown in Fig. 22(b). Then we have $w(\Gamma \cap \mathrm{Int}D_4) \geqq 1$. Moreover, we have $w(\Gamma \cap \mathrm{Int}D_2) \geqq 1$ by considering as $F = D_2$, $k = m+1$ and $\delta = -1$ in the example of IO-Calculation in Section 10. Hence by (1)

$$\begin{aligned}7 &= w(\Gamma)\\ &= w(G) + w(\Gamma \cap \mathrm{Int}D_1) + w(\Gamma \cap \mathrm{Int}D_2) + w(\Gamma \cap \mathrm{Int}D_3) + w(\Gamma \cap \mathrm{Int}D_4)\\ &\geqq 5 + 1 + 1 + 1 = 8.\end{aligned}$$

This is a contradiction. Thus Γ does not contain the pseudo chart as shown in Fig. 22(b).

Similarly, we can show that Γ does not contain the pseudo chart as shown in Fig. 22(c).

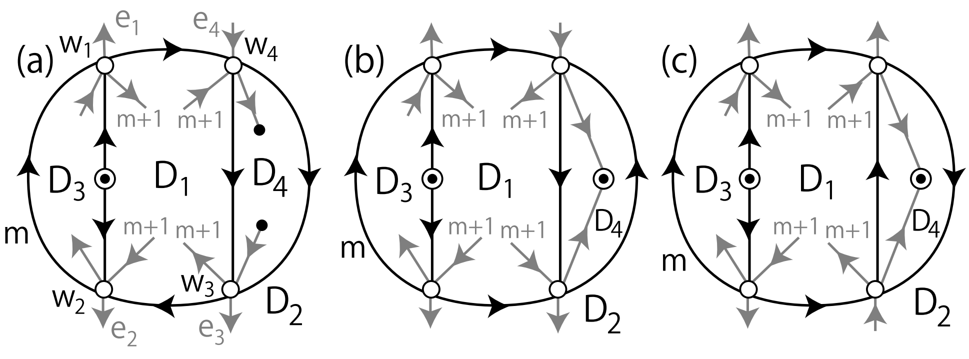

Fig. 22. The graphs as shown in Fig. 13(e). (a) The two internal edges in ∂D_4 are oriented from w_4 to w_3. (b) ∂D_4 is oriented anticlockwise. (c) ∂D_4 is oriented clockwise.

Now, suppose that the chart Γ contain the pseudo chart as shown in Fig. 22(a). We use the notations as shown in Fig. 22(a), where e_i ($i = 1, 2, 3, 4$) is an internal edge (possibly a terminal edge) of label $m + 1$ at w_i in the 4-angled disk D_2, and

(2) the three edges e_1, e_2, e_3 are oriented outward at w_1, w_2, w_3, respectively,

none of e_1, e_2, e_3, e_4 are middle at w_1, w_2, w_3 or w_4. Thus by Assumption 2,

(3) none of e_1, e_2, e_3, e_4 are terminal edges.

Hence by IO-Calculation with respect to Γ_{m+1} in D_2, we have $w(\Gamma \cap \text{Int}D_2) \geqq 1$. Thus by (1), the condition $w(\Gamma) = 7$ implies that

(4) $w(\Gamma \cap \text{Int}D_2) = 1$.

Let w_5 be the white vertex in $\text{Int}D_2$. Then for the edge e_4, there are four cases: (i) $e_4 = e_1$, (ii) $e_4 = e_2$, (iii) $e_4 = e_3$, (iv) $e_4 \ni w_5$.

Case (i) and Case (iii). There exists a lens. This contradicts Lemma 3.2. Hence neither Case (i) nor Case (iii) occurs.

Case (ii). By (2) and (3), both of e_1, e_3 contain the white vertex w_5. Thus one of the edges e_1, e_3 of label $m + 1$ intersects the edge e_4 of label $m + 1$. This contradicts the definition of the chart. Hence Case (ii) does not occur.

Case (iv). Since e_4 contains the white vertex w_5, one of the three edges e_1, e_2, e_3 does not contain white vertex w_5. Thus by (4), one of the three edges e_1, e_2, e_3 is a terminal edge. This contradicts (3). Hence Case (iv) does not occur.

Therefore all the four cases do not occur. Thus Γ does not contain the pseudo chart as shown in Fig. 22(a). Hence Γ_m does not contain the graph as shown in Fig. 13(e). □

12. Case of the Graph as Shown in Fig. 13(f)

In this section, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(f).

LEMMA 12.1. *Let Γ be a minimal chart of type $(m; 5, 2)$. If Γ_m contains the graph as shown in Fig. 13(f), then Γ contains one of RO-families of the three pseudo charts as shown in Fig 23(a),(b),(c).*

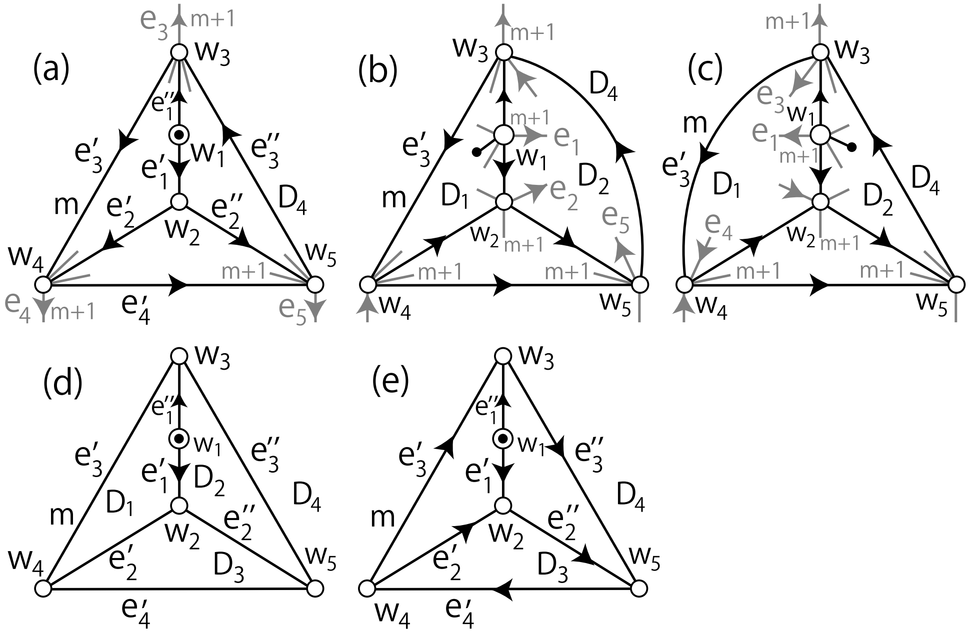

Fig. 23. The graphs as shown in Fig. 13(f).

Proof. Let G be the graph in Γ_m as shown in Fig. 13(f). We use the notations as shown in Fig. 23(d), where w_1 is the BW-vertex, and $e_1', e_1'', e_2', e_2'', e_3', e_3'', e_4'$ are seven internal edges of label m with $e_1' \cap e_1'' \ni w_1$, $e_1' \cap e_2' \cap e_2'' \ni w_2$, $e_1'' \cap e_3' \cap e_3'' \ni w_3$, and $\partial e_4' = \{w_4, w_5\}$.

Since the graph G separates the 2-sphere S^2 into four disks. Two of the four disks are 4-angled disks, say D_1, D_2. Two of the four disks are 3-angled disks, say D_3, D_4. Without loss of generality we can assume that $\partial D_1 \ni w_4$, $\partial D_2 \ni w_5$ and D_4 contains the point at infinity, ∞ (see Fig. 23(d)).

Without loss of generality we can assume that the terminal edge of label m at w_1 is oriented inward at w_1. Then by Assumption 2,

(1) both of e_1', e_1'' are oriented outward at w_1 (see Fig. 23(d)).

There are two cases: (i) one of e_1', e_1'' is middle at w_2 or w_3, (ii) neither e_1' nor e_1'' is middle at w_2 or w_3.

Case (i). If necessary we move the point ∞ in D_3, we can assume that e_1' is middle at w_2. By Condition (iii) of the definition of a chart, both of e_2', e_2'' are oriented outward at w_2. If necessary we reflect the chart Γ, we can assume that the edge e_4' is oriented from w_4 to w_5. Since both of e_2'', e_4' are oriented inward at w_5, the edge e_3'' is oriented from w_5 to w_3. Moreover, since both of e_1'', e_3'' are oriented inward at w_3 by (1), the edge e_3' is oriented from w_3 to w_4. Therefore Γ contains the pseudo chart as shown in Fig. 23(a).

Case (ii). One of e_2', e_2'' is oriented inward at w_2, and the other is oriented outward at w_2. If necessary we reflect the chart Γ, we can assume that the edge e_2' is oriented inward at w_2, and the edge e_2'' is oriented outward at w_2.

Next, we shall show that e_3' is oriented outward at w_3. If e_3' is oriented inward at w_3, then e_3'' is oriented outward at w_3 (because, e_1'' is oriented inward at w_3 by (1)). Thus both of e_2'' and e_3'' are oriented inward at w_5. Hence the edge e_4' is oriented from w_5 to w_4 (see Fig. 23(e)). Thus, both of ∂D_3 and ∂D_4 are oriented clockwise or anticlockwise. Hence, by Lemma 8.1, we have $w(\Gamma \cap \text{Int}D_3) \geqq 1$ and $w(\Gamma \cap \text{Int}D_4) \geqq 1$. Moreover, since one of D_1 and D_2 is a 4-angled disk with one feeler, by Corollary 6.3(a) we have $w(\Gamma \cap \text{Int}D_1) \geqq 1$ or $w(\Gamma \cap \text{Int}D_2) \geqq 1$. Thus

$$\begin{aligned} 7 &= w(\Gamma) \\ &= w(G) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) + w(\Gamma \cap \text{Int}D_3) + w(\Gamma \cap \text{Int}D_4) \\ &\geqq 5 + 1 + 1 + 1 = 8. \end{aligned}$$

This is a contradiction. Hence e_3' is oriented outward at w_3.

Since e_1'' is not middle at w_3, the edge e_3'' is oriented inward at w_3. If necessary we move the point ∞ in D_3, we can assume that the edge e_4' is oriented from w_4 to w_5. Therefore, if D_1 (resp. D_2) has one feeler, then Γ contains the pseudo chart as shown in Fig. 23(b) (resp. Fig. 23(c)). □

PROPOSITION 12.2. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 13(f).*

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 13(f), say G. Since the graph G separates the 2-sphere S^2 into four disks. Two of the four disks are 4-angled disks, say D_1, D_2. Two of the four disks are 3-angled disks, say D_3, D_4. Without loss of generality we can assume that D_4 contains the point at infinity, ∞ (see Fig. 23(d)). By Lemma 12.1, we can assume that Γ contains one of the three pseudo charts as shown in Fig. 23(a),(b),(c).

Suppose that Γ contains the pseudo chart as shown in Fig. 23(a). We use the notations as shown in Fig. 23(a), where

(1) e_3, e_4, e_5 are internal edges (possibly terminal edges) of label $m+1$ oriented outward at w_3, w_4, w_5 in D_4, respectively,

but none of e_3, e_4, e_5 are middle at w_3, w_4 or w_5. Thus by Assumption 2,

(2) none of e_3, e_4, e_5 are terminal edges.

Hence by IO-Calculation with respect to Γ_{m+1} in D_4, we have $w(\Gamma \cap \text{Int}D_4) \geqq 2$. Moreover, since one of D_1, D_2 contains one feeler, by Corollary 6.3(a) we have $w(\Gamma \cap \text{Int}D_1) \geqq 1$ or $w(\Gamma \cap \text{Int}D_2) \geqq 1$. Thus

$$\begin{aligned} 7 &= w(\Gamma) \\ &\geqq w(G) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) + w(\Gamma \cap \text{Int}D_4) \\ &\geqq 5 + 1 + 2 = 8. \end{aligned}$$

This is a contradiction. Hence Γ does not contain the pseudo chart as shown in Fig. 23(a).

Suppose that Γ contains the pseudo chart as shown in Fig. 23(b). Without loss of generality, we can assume that D_1 has one feeler. Thus by Corollary 6.3(a) we have $w(\Gamma \cap \text{Int}D_1) \geqq 1$.

We use the notations as shown in Fig. 23(b), where

(3) e_1, e_2, e_5 are internal edges (possibly terminal edges) of label $m+1$ oriented outward at w_1, w_2, w_5 in D_2, respectively,

but neither e_2 nor e_5 is middle at w_2 or w_5. Thus by Assumption 2,

(4) neither e_2 nor e_5 is a terminal edge.

Hence by IO-Caclulation with respect to Γ_{m+1} in D_2, we have $w(\Gamma \cap \text{Int}D_2) \geqq 1$.

Since the boundary ∂D_4 is oriented anticlockwise, by Lemma 8.1 we have $w(\Gamma \cap \text{Int}D_4) \geqq 1$. Thus

$$\begin{aligned} 7 &= w(\Gamma) \\ &\geqq w(G) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) + w(\Gamma \cap \text{Int}D_4) \\ &\geqq 5 + 1 + 1 + 1 = 8. \end{aligned}$$

This is a contradiction. Hence Γ does not contain the pseudo chart as shown in Fig. 23(b).

Suppose that Γ contains the pseudo chart as shown in Fig. 23(c). Without loss of generality, we can assume that D_2 has one feeler. Thus by Corollary 6.3(a) we have $w(\Gamma \cap \text{Int}D_2) \geqq 1$.

Since the boundary ∂D_4 is oriented anticlockwise, by Lemma 8.1 we have $w(\Gamma \cap \text{Int}D_4) \geqq 1$. Hence the condition $w(\Gamma) = 7$ implies that

(5) $w(\Gamma \cap \text{Int}D_1) = 0$.

We use the notations as shown in Fig. 23(c), where e_1, e_3, e_4 are internal edges (possibly terminal edges) of label $m+1$ at w_1, w_3, w_4 in D_1, respectively,

(6) e_1, e_3 are oriented outward at w_1, w_3, respectively,

but neither e_3 nor e_4 is middle at w_3 or w_4. Thus by Assumption 2, neither e_3 nor e_4 is a terminal edge. Hence by (5) and (6), we have $e_3 = e_4$. However there exists a lens. This contradicts Lemma 3.2. Thus Γ does not contain the pseudo chart as shown in Fig. 23(c).

Therefore we have a contradiction for all cases. Hence Γ_m does not contain the graph as shown in Fig. 13(f). □

13. Triangle Lemma

In this section, we review Triangle Lemma. These lemmas will be used in the next section.

LEMMA 13.1 ([6, Lemma 5.4]). *If a minimal chart Γ contains the pseudo chart as shown in Fig. 24, then the interior of the disk D contains at least one white vertex, where D is the disk with the boundary $e_3 \cup e_4 \cup e$.*

LEMMA 13.2 ([14, Lemma 4.2(b)]). *Let Γ be a minimal chart, and m a label of Γ. Let D be a special 3-angled disk of Γ_m with at most two feelers. If $w(\Gamma \cap \text{Int}D) = w(\Gamma_{m+\varepsilon} \cap \text{Int}D) = 1$ for some $\varepsilon \in \{+1, -1\}$, then a regular neighborhood of D contains one of the RO-families of the six pseudo charts as shown in Fig. 25.*

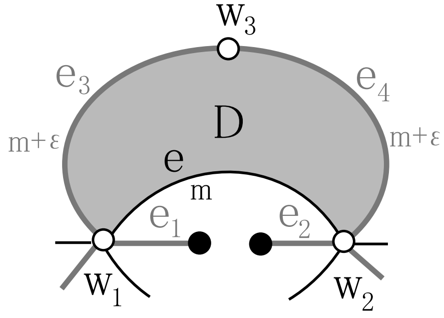

Fig. 24. The white vertices w_1 and w_2 are in $\Gamma_m \cap \Gamma_{m+\varepsilon}$ where $\varepsilon \in \{+1, -1\}$.

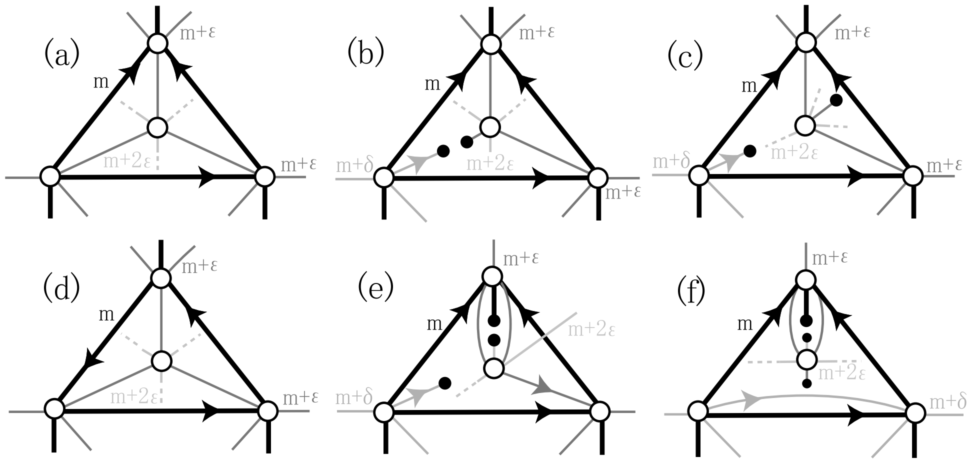

Fig. 25. (a),(b),(c),(d) 3-angled disks without feelers. (e),(f) 3-angled disks with one feeler. Here, $\varepsilon, \delta \in \{+1, -1\}$.

LEMMA 13.3 (Triangle Lemma) ([9, Lemma 8.3]).

(a) *For a chart* Γ*, if there exists a* 3*-angled disk* D_1 *of* Γ_m *without feelers in a disk* D *as shown in Fig. 26*(*a*) *and if* $w(\Gamma \cap \text{Int} D_1) = 0$*, then there exists a chart obtained from* Γ *by C-moves in* D *which contains the pseudo chart in* D *as shown in Fig. 26*(*b*)*.*

(b) *For a minimal chart* Γ*, if there exists a* 3*-angled disk* D_1 *of* Γ_m *without feelers in a disk* D *as shown in Fig. 26*(*c*)*, then* $w(\Gamma \cap \text{Int} D_1) \geq 1$*.*

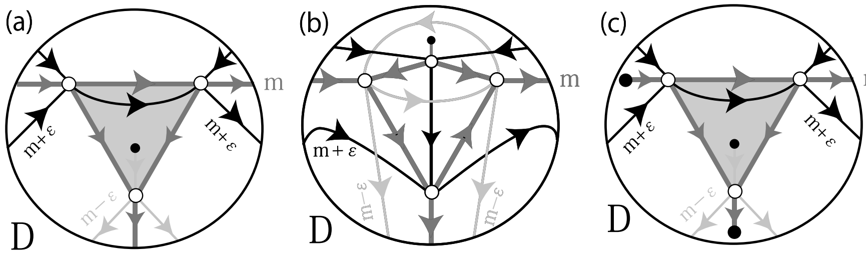

Fig. 26. The gray region is the 3-angled disk D_1. The thick lines are edges of label m, and $\varepsilon \in \{+1, -1\}$.

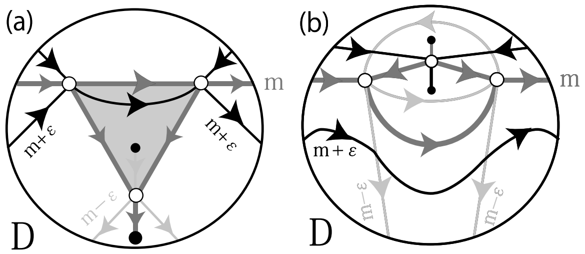

Fig. 27. The gray region is the 3-angled disk D_1. The thick lines are edges of label m, and $\varepsilon \in \{+1, -1\}$.

By the above lemma, we can show the following corollary by using C-II moves and a C-III move:

Corollary 13.4. *For a chart* Γ*, if there exists a* 3*-angled disk* D_1 *of* Γ_m *without feelers in a disk* D *as shown in Fig. 27*(*a*) *and if* $w(\Gamma \cap \text{Int} D_1) = 0$*, then there exists a chart obtained from* Γ *by C-moves in* D *which contains the pseudo chart in* D *as shown in Fig. 27*(*b*).

Lemma 13.5 ([14, Theorem 1.1]). *There is no minimal chart of type* $(4, 3)$.

14. Case of the Graph as Shown in Fig. 13(g)

In this section, we shall show that if Γ is a minimal chart of type $(m; 5, 2)$, then the graph Γ_m does not contain the graph as shown in Fig. 13(g).

Moreover, we shall show the main theorem.

Suppose that the graph Γ_m contains the graph as shown in Fig. 13(g). Form now on throughout this section, we use the notations as shown in Fig. 28, where

(a) $w_1, w_2, \cdots, w_5$ are five white vertices, and

(b) $e_1, e_2, \cdots, e_7$ are seven internal edges of label m with $\partial e_1 = \partial e_2 = \{w_1, w_2\}$, $\partial e_3 = \{w_2, w_3\}$, $\partial e_4 = \{w_3, w_4\}$, $\partial e_5 = \{w_3, w_5\}$, $\partial e_6 = \partial e_7 = \{w_4, w_5\}$,

(c) D_1, D_2 are special 2-angled disks with $\partial D_1 = e_1 \cup e_2$ and $\partial D_2 = e_6 \cup e_7$,

(d) D_3 is the special 3-angled disk with $\partial D_3 = e_4 \cup e_5 \cup e_6$.

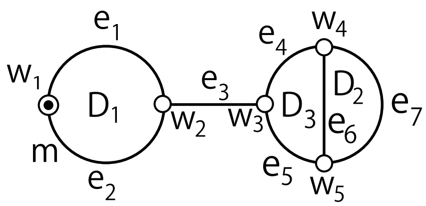

Fig. 28. The graph as shown in Fig. 13(g).

LEMMA 14.1. *Let Γ be a minimal chart of type $(m; 5, 2)$. If Γ_m contains the graph as shown in Fig. 13(g), then Γ_m contains one of RO-families of the four graphs as shown in Fig. 29.*

PROOF. We use the notations as shown in Fig. 28. Without loss of generality, we can assume that the terminal edge of label m at w_1 is oriented inward at w_1. Then by Assumption 2, both of e_1, e_2 are oriented from w_1 to w_2. Thus the edge e_3 is oriented form w_2 to w_3. There are two cases: (i) e_3 is middle at w_3, (ii) e_3 is not middle at w_3.

Case (i). Since e_3 is middle at w_3, both of e_4, e_5 are oriented outward at w_3. Hence

(1) e_5 is oriented inward at w_5.

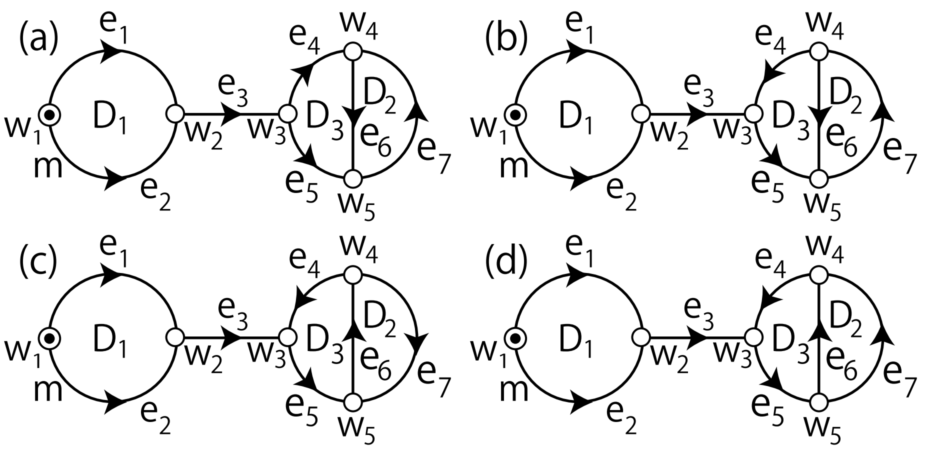

Fig. 29. The graphs as shown in Fig. 13(g).

If necessary we reflect the chart Γ, we can assume that e_6 is oriented from w_4 to w_5. Thus by (1), the edge e_7 is oriented from w_5 to w_4. Hence Γ_m contains the graph as shown in Fig. 29(a).

Case (ii). Since e_3 is not middle at w_3, one of e_4, e_5 is oriented inward at w_3 and the other is oriented outward at w_3. If necessary we reflect the chart Γ, we can assume that e_4 is oriented inward at w_3 and e_5 is oriented outward at w_3. Thus

(2) e_5 is oriented inward at w_5.

If e_6 is oriented from w_4 to w_5, then by (2) the edge e_7 is oriented from w_5 to w_4. Hence Γ_m contains the graph as shown in Fig. 29(b).

If e_6 is oriented from w_5 to w_4, then Γ_m contains one of the two graphs as shown in Fig. 29(c),(d). □

LEMMA 14.2. *Let* Γ *be a minimal chart of type* $(m; 5, 2)$. *Then* Γ_m *does not contain the graph as shown in Fig. 29*(*c*).

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 29(c), say G. We use the notations as shown in Fig. 28 and Fig. 29(c).

Since ∂D_2 is oriented clockwise, since e_4 is oriented outward at w_4 and since e_5 is oriented inward at w_5, by Lemma 11.1 we have $w(\Gamma \cap \text{Int} D_2) \geqq 2$.

Since ∂D_3 is oriented anticlockwise, by Lemma 8.1 we have $w(\Gamma \cap \mathrm{Int}D_3) \geqq 1$. Thus we have

$$7 = w(\Gamma) \geqq w(G) + w(\Gamma \cap \mathrm{Int}D_2) + w(\Gamma \cap \mathrm{Int}D_3) \geqq 5 + 2 + 1 = 8.$$

This is a contradiction. Therefore Γ_m does not contain the graph as shown in Fig. 29(c). □

Lemma 14.3. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 29(b).*

Proof. Suppose that Γ_m contains the graph as shown in Fig. 29(b). We use the notations as shown in Fig. 28 and Fig. 29(b).

By the similar way of the proof of Lemma 14.2, we have $w(\Gamma \cap \mathrm{Int}D_2) \geqq 2$. Thus the condition $w(\Gamma) = 7$ implies that

(1) $w(\Gamma \cap \mathrm{Int}D_1) = 0$ and $w(\Gamma \cap (S^2 - (D_1 \cup D_2 \cup D_3))) = 0$.

Hence by Lemma 5.1, a regular neighborhood of D_1 contains the pseudo chart as shown in Fig. 10(b) (see Fig. 30(a)).

We use the notations as shown in Fig. 30(a), where

(2) e_1', e_1'', e_3', e_4' are internal edges (possibly terminal edges) of label $m+1$ oriented inward at w_1, w_1, w_3, w_4, respectively,

(3) e_2', e_2'', e_3'', e_5' are internal edges (possibly terminal edges) of label $m+1$ oriented outward at w_2, w_2, w_3, w_5, respectively.

Moreover, none of e_2', e_2'', e_3'', e_5' are middle at w_2, w_3 or w_5. Thus by Assumption 2,

(4) none of e_2', e_2'', e_3'', e_5' are terminal edges.

Hence by (1),(2),(3), none of e_1', e_1'', e_3', e_4' are terminal edges. Thus for the edge e_2'', we have $e_2'' = e_1''$. However $e_2 \cup e_2''$ bounds a lens. This contradicts Lemma 3.2. Therefore Γ_m does not contain the graph as shown in Fig. 29(b). □

Lemma 14.4. *Let Γ be a minimal chart of type $(m; 5, 2)$. If Γ_m contains the graph as shown in Fig. 29(d), then $w(\Gamma \cap \mathrm{Int}D_2) = 0$ and $w(\Gamma \cap \mathrm{Int}D_3) = 1$.*

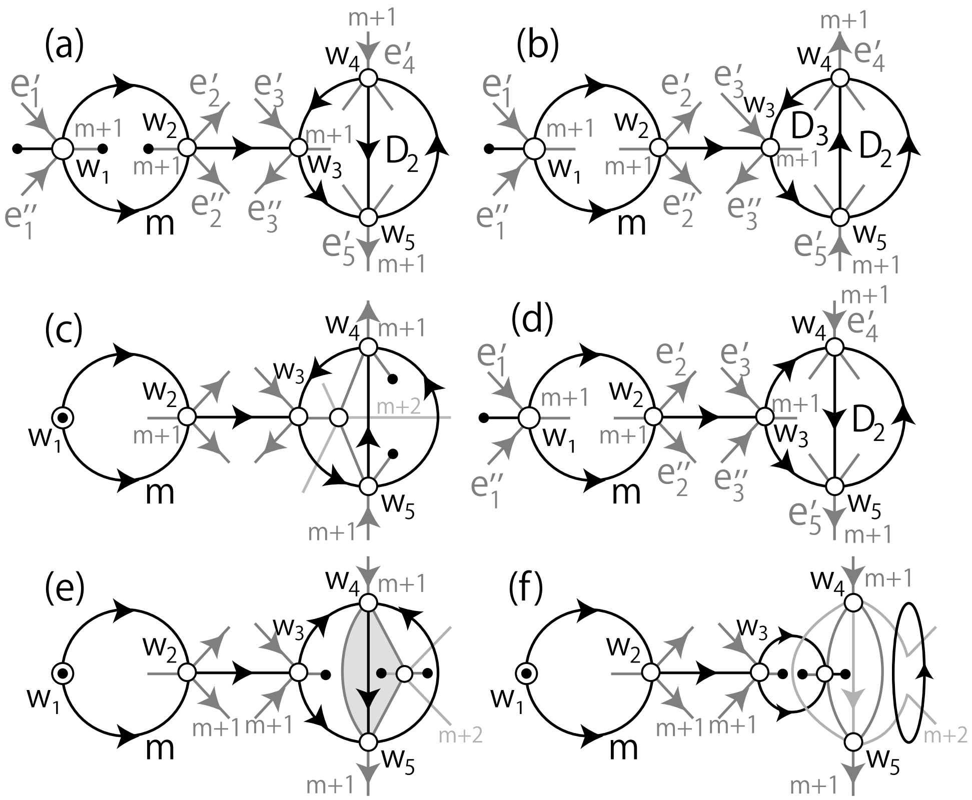

Fig. 30. The graphs as shown in Fig. 13(g). The gray region is the 3-angled disk D.

Proof. We use the notations as shown in Fig. 28 and Fig. 29(d). Since ∂D_3 is oriented anticlockwise, by Lemma 8.1 we have

(1) $w(\Gamma \cap \mathrm{Int}D_3) \geqq 1$.

Let e be the terminal edge of label m at w_1. If $e \subset D_1$, then by Lemma 5.1 we have $w(\Gamma \cap \mathrm{Int}D_1) \geqq 1$. Thus by (1) and $w(\Gamma) = 7$, we have $w(\Gamma \cap \mathrm{Int}D_2) = 0$ and $w(\Gamma \cap \mathrm{Int}D_3) = 1$.

Now, suppose $e \not\subset D_1$. We use the notations as shown in Fig. 30(b), where

(2) e_1', e_1'', e_3', e_5' are internal edges (possibly terminal edges) of label $m+1$ oriented inward at w_1, w_1, w_3, w_5, respectively, and

(3) e_2', e_2'', e_3'', e_4' are internal edges (possibly terminal edges) of label $m+1$ oriented outward at w_2, w_2, w_3, w_4, respectively.

Moreover, none of e_2', e_2'', e_3'', e_4' are middle at w_2, w_3 or w_4. Thus by Assumption 2, none of the four edges e_2', e_2'', e_3'', e_4' are terminal edges.

If $w(\Gamma \cap (S^2 - (D_1 \cup D_2 \cup D_3))) = 0$, then by (2) and (3) none of the four edges e_1', e_1'', e_3', e_5' are terminal edges. Hence for the edge e_2'', we have $e_2'' = e_1''$. However, there exists a lens. This contradicts Lemma 3.2. Thus $w(\Gamma \cap (S^2 - (D_1 \cup D_2 \cup D_3))) \geqq 1$.

Hence by (1) and $w(\Gamma) = 7$, we have $w(\Gamma \cap \mathrm{Int} D_2) = 0$ and $w(\Gamma \cap \mathrm{Int} D_3) = 1$. □

LEMMA 14.5. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 29(d).*

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 29(d). We use the notations as shown in Fig. 28 and Fig. 29(d).

Since ∂D_3 is oriented anticlockwise, by Lemma 13.2 and Lemma 14.4 a regular neighborhood of D_3 contains the pseudo chart as shown in Fig. 25(d). Moreover, by Lemma 5.1 and Lemma 14.4 a regular neighborhood of D_2 contains the pseudo chart as shown in Fig. 10(b) (see Fig. 30(c)). Hence the chart Γ contains the pseudo chart as shown in Fig. 24. Thus by Lemma 13.1, we have $w(\Gamma \cap \mathrm{Int} D_3) \geq 2$. This contradicts Lemma 14.4. Thus we complete the proof of Lemma 14.5. □

LEMMA 14.6. *Let Γ be a minimal chart of type $(m; 5, 2)$. If Γ_m contains the graph as shown in Fig. 29(a), then $w(\Gamma \cap \mathrm{Int} D_2) = 1$ and $w(\Gamma \cap \mathrm{Int} D_3) = 0$.*

PROOF. We use the notations as shown in Fig. 28 and Fig. 29(a). Since ∂D_2 is oriented anticlockwise, by Lemma 5.1 we have

(1) $w(\Gamma \cap \mathrm{Int} D_2) \geqq 1$.

Let e be the terminal edge of label m at w_1. If $e \subset D_1$, then by Lemma 5.1 we have $w(\Gamma \cap \mathrm{Int} D_1) \geqq 1$. Thus by (1) and $w(\Gamma) = 7$, we have $w(\Gamma \cap \mathrm{Int} D_2) = 1$ and $w(\Gamma \cap \mathrm{Int} D_3) = 0$.

Now, suppose $e \not\subset D_1$. We use the notations as shown in Fig. 30(d), where

(2) $e'_1, e''_1, e'_3, e''_3, e'_4$ are internal edges (possibly terminal edges) of label $m+1$ oriented inward at w_1, w_1, w_3, w_3, w_4, respectively.

Moreover, none of e'_1, e''_1, e'_3, e''_3 are middle at w_1 or w_3. Thus by Assumption 2, none of the four edges e'_1, e''_1, e'_3, e''_3 are terminal edges. Thus by (2) and by IO-Calculation with respect to Γ_{m+1} in $Cl(S^2 - (D_1 \cup D_2 \cup D_3))$, we have $w(\Gamma \cap (S^2 - (D_1 \cup D_2 \cup D_3))) \geqq 1$. Hence by (1) and $w(\Gamma) = 7$, we have $w(\Gamma \cap \text{Int}D_2) = 1$ and $w(\Gamma \cap \text{Int}D_3) = 0$. □

PROPOSITION 14.7. *Let Γ be a minimal chart of type $(m; 5, 2)$. Then Γ_m does not contain the graph as shown in Fig. 13(g).*

PROOF. Suppose that Γ_m contains the graph as shown in Fig. 13(g). Then by Lemma 14.1, the graph Γ_m contains one of RO-families of the four graphs as shown in Fig. 29. Hence by Lemma 14.2, Lemma 14.3 and Lemma 14.5, the graph Γ_m contains one of the RO-family of the graph as shown in Fig. 29(a). Without loss of generality, we can assume that the graph Γ_m contains of the graph as shown in Fig. 29(a).

By Lemma 8.1 and Lemma 14.6, a regular neighborhood of D_3 contains the pseudo chart as shown in Fig. 16(b). Moreover, by Lemma 11.1 and Lemma 14.6, a regular neighborhood of D_2 contains one of the two pseudo charts as shown in Fig. 21(b),(c). Hence there exists a 3-angled disk D of Γ_{m+1} in $D_2 \cup D_3$.

Let w_6 be the white vertex in $\text{Int}D_2$, and e' the terminal edge of label $m+1$ at w_6. Since $w(\Gamma \cap \text{Int}D) = 0$ by Lemma 14.6, a regular neighborhood of D contains the pseudo chart as shown in Fig. 16(a). Hence $e' \not\subset D$ (see Fig. 30(e)). Thus Γ contains the pseudo chart as shown in Fig. 27(a). Thus by Corollary 13.4, there exists a minimal chart Γ' obtained from Γ by C-moves which contains the pseudo chart as shown in Fig. 27(b) (see Fig. 30(f)). Hence Γ is C-move equivalnet to the minimal chart Γ' of type $(m; 4, 3)$. This contradicts Lemma 13.5. Hence Γ_m does not contain the graph as shown in Fig. 29(a).

Hence Γ_m does not contains the graph as shown in Fig. 13(g). Therefore we complete the proof of Proposition 14.7. □

LEMMA 14.8 ([9, Theorem 1.1]). *There is no loop in any minimal chart with exactly seven white vertices.*

Now, we shall show the main theorem.

PROOF OF THEOREM 1.1. Let Γ be a minimal chart of type $(m; 5, 2)$. Suppose that there exists a connected component G of Γ_m with $w(G) = 5$. Then by Lemma 14.8, the graph G does not contain any loop. Thus by Lemma 7.1, the graph G is one of nine graphs as shown in Fig. 2 and Fig. 13. Hence the main theorem follows from the seven propositions (Lemma 7.4, Lemma 7.5, Lemma 8.2, Lemma 9.1, Lemma 11.2, Proposition 12.2 and Proposition 14.7). Therefore we complete the proof of the main theorem. □

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Teruo NAGASE
Tokai University
4-1-1 Kitakaname, Hiratuka
Kanagawa, 259-1292 Japan
E-mail: nagase.teruo.t@tokai.ac.jp

Akiko SHIMA
Department of Mathematics
Tokai University
4-1-1 Kitakaname, Hiratuka
Kanagawa, 259-1292 Japan
E-mail: shima-a@tokai.ac.jp

List of terminologies

List of notations