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A Universal Coefficient Theorem for Actions of Finite Groups on C*-*Algebras*

By Ralf Meyer and George Nadareishvili

Abstract. The equivariant bootstrap class in the Kasparov category of actions of a finite group G consists of those actions that are equivalent to one on a Type I C*-algebra. Using a result by Arano and Kubota, we show that this bootstrap class is already generated by the continuous functions on G/H for all cyclic subgroups H of G. Then we prove a Universal Coefficient Theorem for the localisation of this bootstrap class at the group order $|G|$. This allows us to classify certain G-actions on stable Kirchberg algebras up to cocycle conjugacy.

1. Introduction

The Kirchberg–Phillips classification of nuclear, simple, purely infinite, separable, stable C*-algebras (see [19]) may be be split into two parts. The first, analytic part shows that two such C*-algebras are isomorphic once they are KK-equivalent. The second, topological part shows that they are KK-equivalent once they belong to the bootstrap class and have isomorphic K-theory. In addition, any pair of $\mathbb{Z}/2$-graded Abelian groups is the K-theory of some such C*-algebra. Both topological statements follow from the Universal Coefficient Theorem, which computes $\mathrm{KK}(A, B)$ in terms of $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$ provided A is in the bootstrap class.

There has been some recent progress on dynamical analogues of this classification, where the aim is to classify certain group actions. On the analytic side, the dynamical Kirchberg–Phillips theorem by Gabe and Szabó in [10] says that two pointwise outer actions of a discrete, countable, amenable group on stable Kirchberg algebras are cocycle conjugate if and only if they

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are equivalent in the equivariant Kasparov category KK^G. On the topological side, Manuel Köhler [12] showed for a finite cyclic group of prime order that isomorphism classes of objects in the equivariant bootstrap class are in bijection with isomorphism classes of "exact" modules over a certain ring. The relevant ring is, however, rather complicated. In [14], the classification is worked out in detail in the special case when the K-theory is a cyclic group. In addition, the case when G is torsion-free amenable is also treated in [14]: actions of such groups are equivalent to locally trivial bundles over the classifying space of G.

The classification in [14] for the cyclic group of prime order p is remarkably subtle in general, but it becomes rather simple if p is invertible in the ring $\mathrm{KK}^G(A, A)$. Here we generalise this easier part of the classification to an arbitrary finite group G, assuming that the group order $|G|$ is invertible in $\mathrm{KK}^G(A, A)$. Our result is related to recent independent work by Bouc, Dell'Ambrogio and Martos [6]. They prove that the localisation of the bootstrap class in KK^G at $\mathbb{Q}$ is semisimple and compute $\mathrm{KK}^G(A, A)$ when it is a $\mathbb{Q}$-vector space. Our approach is more elementary, using explicit polynomials for some key computations. This allows us to prove the main result after inverting only the group order $|G|$.

Another difference is that [6] treats only the Kasparov category of G-cell algebras. Using an important theorem by Arano and Kubota [1], we show that this subcategory is the same as the equivariant bootstrap class in KK^G, that is, the subcategory of all objects that are KK^G-equivalent to a G-action on a separable C*-algebra of Type I. As a result, our main result classifies pointwise outer actions of finite groups on A up to cocycle conjugacy provided A is a stable Kirchberg algebra, the action belongs to the G-equivariant bootstrap class, and $|G|$ is invertible in the ring $\mathrm{KK}^G(A, A)$. Incidentally, the results of Arano and Kubota [1] also imply that a G-action belongs to the G-equivariant bootstrap class if and only if the restrictions of the action belong to the H-equivariant bootstrap class for all cyclic subgroups $H \subseteq G$. This implies that A and $A \rtimes H$ for cyclic subgroups $H \subseteq G$ are in the bootstrap class in KK. We do not know whether this necessary condition is sufficient as well.

We end the introduction by formulating our main theorem. Writing down the classifying invariant needs some preparation. Let $H \subseteq G$ be a cyclic subgroup and $n := |H|$. The representation ring of H is isomorphic to $\mathbb{Z}[z]/(z^n-1)$. Let $\Phi_n \in \mathbb{Z}[z]$ be the nth cyclotomic polynomial, whose zeros are exactly the primitive nth roots of unity. This divides z^n-1, so that $\mathbb{Z}[z]/(\Phi_n)$ is a quotient of the representation ring of H. The representation ring is isomorphic to the ring $\mathrm{KK}_0^H(\mathbb{C},\mathbb{C})$, and the induction functor induces a map from this to $\mathrm{KK}_0^G(\mathrm{C}(G/H),\mathrm{C}(G/H))$. Let $N_H := \{g \in G \mid gHg^{-1} = H\}$. This acts on G/H and thus on $\mathrm{C}(G/H)$ by right translations, with the subgroup H of N_H acting trivially. Thus we get a homomorphism from the quotient group $W_H := N_H/H$ into the ring $\mathrm{KK}_0^G(\mathrm{C}(G/H),\mathrm{C}(G/H))$. The group W_H also acts on the representation ring of H because N_H acts on H by automorphisms and inner automorphisms act trivially on the representation ring. The homomorphisms from the representation ring $\mathbb{Z}[z]/(z^n-1)$ and from the group W_H to $\mathrm{KK}_0^G(\mathrm{C}(G/H),\mathrm{C}(G/H))$ are covariant and so combine to a homomorphism on $\mathbb{Z}[z]/(z^n-1)\rtimes W_H$. Hence $\mathbb{Z}[z]/(z^n-1)\rtimes W_H$ acts on the $\mathbb{Z}/2$-graded Abelian group $\mathrm{KK}_*^G(\mathrm{C}(G/H),B) \cong \mathrm{K}_0(B\rtimes H)$ for any G-C*-algebra B. Let

$$F_*^H(B) := \{x \in \mathrm{KK}_*^G(\mathrm{C}(G/H),B) \mid \Phi_n(z)\cdot x = 0\}.$$

The graded subgroup $F_*^H(B)$ in $\mathrm{KK}_*^G(\mathrm{C}(G/H),B)$ is even a $\mathbb{Z}/2$-graded module over the ring $\mathbb{Z}[z]/(\Phi_n(z))\rtimes W_H$.

Theorem 1.1. *Let G be a finite group. Let A and B be G-C*-algebras. Suppose that A is in the G-equivariant bootstrap class, that is, it is KK^G-equivalent to an action on a Type I C*-algebra. Suppose that B is $|G|$-divisible in the sense that multiplication by $|G|$ on $\mathrm{KK}^G(B,B)$ is invertible. Then there is a Universal Coefficient Theorem short exact sequence*

$$\prod_{H\subseteq \text{up to conjugacy}} \mathrm{Ext}^1_{\mathbb{Z}[z]/(\Phi_n(z))\rtimes W_H}\bigl(F^H_{*-1}(A),F^H_*(B)\bigr) \rightarrowtail \mathrm{KK}^G(A,B) \twoheadrightarrow \prod_{H\subseteq G \text{ cyclic}} \mathrm{Hom}_{\mathbb{Z}[z]/(\Phi_n(z))\rtimes W_H}\bigl(F^H_*(A),F^H_*(B)\bigr).$$

Here the products run over conjugacy classes of cyclic subgroups $H\subseteq G$. If M_H are countable $\mathbb{Z}/2$-graded modules over $\mathbb{Z}[z,1/|G|]/(\Phi_n(z))\rtimes W_H$ for all cyclic subgroups $H\subseteq G$, then there is a $|G|$-divisible object B in the bootstrap class in KK^G with $F_*^H(B)\cong M_H$ for all cyclic subgroups $H\subseteq G$, and this is unique up to KK^G-equivalence.

Together with the dynamical Kirchberg–Phillips theorem by Gabe and Szabó, this theorem implies a classification of certain outer group actions on Kirchberg algebras up to cocycle conjugacy. The proof of Theorem 1.1 is based on ideas of Manuel Köhler [12].

2. Equivariant KK-Theory

Let G be a second countable locally compact group. Let KK^G denote the Kasparov category of separable G-C*-algebras. The spatial tensor product of C*-algebras with diagonal G-action induces a symmetric monoidal structure on KK^G, which we denote by $\otimes$. By $\oplus$ we denote the C_0-direct sum, which exists for countable collections of G-C*-algebras; it makes KK^G an additive category with countable coproducts. The category KK^G is triangulated (see [15], and see [18] for an introduction to triangulated categories in general). The suspension functor is $\Sigma := \mathrm{C}_0(\mathbb{R}) \otimes -$. It is an involutive equivalence by Bott periodicity. The exact triangles come either from mapping cones of equivariant *-homomorphisms or from extensions of G-C*-algebras with a G-equivariant, completely positive contractive section (for details see the Appendix of [15]).

2.1. Functors on KK^G

Let $H \subseteq G$ be a closed subgroup. The *restriction* of G-actions defines a functor $\mathrm{Res}_H^G \colon \mathrm{KK}^G \to \mathrm{KK}^H$. It preserves coproducts, is triangulated, and symmetric monoidal (see, for instance, [15]). The *induction functor* $\mathrm{Ind}_H^G \colon \mathrm{KK}^H \to \mathrm{KK}^G$ is defined by

$$\mathrm{Ind}_H^G(A) := \{G \xrightarrow{\phi} A \mid \phi \text{ continuous, } h\phi(gh) = \phi(g) \text{ for all } g \in G, h \in H, \\ (gH \mapsto \|\phi(g)\|) \in \mathrm{C}_0(G/H)\},$$

equipped with the G-action $(g \cdot \phi)(s) = \phi(g^{-1}s)$ for $g, s \in G$. This is a triangulated functor preserving coproducts. If G/H is compact, then Ind_H^G is right adjoint to Res_H^G (see [15, Equation (19)]). If G/H is discrete, then Ind_H^G is left adjoint to Res_H^G (see [15, Equation (20)]). Thus, if G is finite, then the restriction and induction functors are adjoint both ways for any subgroup $H \subseteq G$.

For $g \in G$ and a subgroup $H \subseteq G$, the *conjugation functor*

$$^g(-)\colon \mathrm{KK}^H \to \mathrm{KK}^{gH}$$

sends a C*-algebra A with an H-action to the same C*-algebra A with an action of ${}^gH := gHg^{-1}$ defined by $ghg^{-1} \cdot a := ha$ for $h \in H$ and $a \in A$. The functor of conjugation by g is a triangulated, monoidal equivalence: the inverse is conjugation by g^{-1}. As such, it preserves coproducts.

This article is based on the following remarkable result of Arano and Kubota:

Theorem 2.1 ([1, Corollary 3.13.(1)]). *Let G be a finite group. If $A \in \mathrm{KK}^G$ is such that $\mathrm{Res}_G^H(A) \cong 0$ for all cyclic subgroups $H \subseteq G$, then already $A \cong 0$ in KK^G.*

In fact, the statement in [1] is more general. First, it allows G to be a compact Lie group. Secondly, it allows A and B to be σ-C*-algebras instead of C*-algebras.

2.2. Homological algebra in $\mathbf{KK}^G$

The main result of this paper is based on a Universal Coefficient Theorem for KK^G, and this fits in the context of relative homological algebra in a triangulated category $\mathfrak{T}$ (see [4, 7, 8, 13, 16]). Accordingly, we recall some facts from the general theory. As in every non-abelian category, doing homological algebra in a triangulated category requires extra structure. We usually specify this through a stable homological functor.

A *stable additive category* is an additive category $\mathfrak{A}$ with an auto-equivalence functor $\Sigma\colon \mathfrak{A} \to \mathfrak{A}$, which is called the *suspension* in $\mathfrak{A}$. A *stable homological functor* from a triangulated category $\mathfrak{T}$ to a stable abelian category $\mathfrak{A}$ is a functor $F\colon \mathfrak{T} \to \mathfrak{A}$ that maps exact triangles in $\mathfrak{T}$ to exact sequences in $\mathfrak{A}$ and that commutes with the suspension up to a natural isomorphism. The *kernel on morphisms* of F is the family of subgroups $\ker F(A,B) := \{\phi \in \mathfrak{T}(A,B) \mid F(\phi) = 0\}$. This is an ideal in $\mathfrak{T}$, and an ideal of this form for a stable homological functor F is called a *stable homological ideal.* If $F\colon \mathfrak{T} \to \mathfrak{D}$ is a triangulated functor to another triangulated category, then the kernel on morphisms is a stable homological ideal as well (see [16]). Such homological ideals play an important role in the localisation approach to the Baum–Connes assembly map developed in [15] and will also be crucial below.

Another homological functor H is called $\mathfrak{I}$-*exact* if $H(\phi) = 0$ for all $\phi \in \mathfrak{I}$. An $\mathfrak{I}$-*exact* stable homological functor $U\colon \mathfrak{T} \to \mathfrak{A}_{\mathfrak{I}}$ is called *universal* if any $\mathfrak{I}$-exact stable homological functor $H\colon \mathfrak{T} \to \mathfrak{A}$ factors uniquely as $\bar{H} \circ U$ for a stable exact functor $\bar{H}\colon \mathfrak{A}_{\mathfrak{I}} \to \mathfrak{A}$. Such a universal functor often exists, and then the homological algebra in $\mathfrak{T}$ is very closely related to homological algebra in the abelian category $\mathfrak{A}_{\mathfrak{I}}$. In particular, the derived functors in $\mathfrak{T}$ are those in $\mathfrak{A}_{\mathfrak{I}}$ composed with U.

Assume $\mathfrak{T}$ to have countable coproducts. An object $C \in \mathfrak{T}$ is called $\aleph_1$-*compact* if the functor $\mathfrak{T}(C, -)\colon \mathfrak{T} \to \mathfrak{Ab}$ commutes with countable coproducts. Let $\mathfrak{C}$ be an at most countable set of $\aleph_1$-compact objects in $\mathfrak{T}$, such that $\mathfrak{T}_n(C, A) := \mathfrak{T}(\Sigma^n C, A)$ is countable for all $A \in \mathfrak{T}$, $n \in \mathbb{Z}$. Let $\mathfrak{Ab}^{\mathbb{Z}}$ denote the abelian category of $\mathbb{Z}$-graded abelian groups with the suspension homomorphism shifting degrees. Define the functor

$$F_{\mathfrak{C}}\colon \mathfrak{T} \to \prod_{C\in\mathfrak{C}} \mathfrak{Ab}^{\mathbb{Z}}, \qquad A \mapsto \bigl(\mathfrak{T}_n(C, A)\bigr)_{C\in\mathfrak{C}, n\in\mathbb{Z}}.$$

Let $\mathfrak{I}_{\mathfrak{C}}$ be the kernel on morphisms of $F_{\mathfrak{C}}$. Let $\langle\mathfrak{C}\rangle \subseteq \mathfrak{T}$ be the smallest triangulated subcategory of $\mathfrak{T}$ containing $\mathfrak{C}$ and closed under countable coproducts. We are going to describe the universal $\mathfrak{I}_{\mathfrak{C}}$-exact stable homological functor. Let $\mathfrak{C}$ also denote the $\mathbb{Z}$-graded pre-additive category with $\mathfrak{C}$ as its object space and groups of arrows $\bigoplus_{n\in\mathbb{Z}} \mathfrak{T}_n(A, B)$ for $A, B \in \mathfrak{C}$. A *right* $\mathfrak{C}$-*module* is defined as a contravariant stable additive functor $\mathfrak{C} \to \mathfrak{Ab}^{\mathbb{Z}}$. These modules form a stable abelian category with direct sums and enough projective objects, which we denote by $\mathfrak{Mod}(\mathfrak{C}^{\mathrm{op}})$. The subcategory of countable modules is denoted by $\mathfrak{Mod}(\mathfrak{C}^{\mathrm{op}})_{\aleph_1}$. Giving $\bigl(\mathfrak{T}_n(C, A)\bigr)_{n\in\mathbb{Z}}$ the right $\mathfrak{C}$-module structure coming from the composition in $\mathfrak{T}$, we enrich $F_{\mathfrak{C}}$ to a functor

$$U_{\mathfrak{C}}\colon \mathfrak{T} \to \mathfrak{Mod}(\mathfrak{C}^{\mathrm{op}})_{\aleph_1}.$$

Lemma 2.2. *The universal $\mathfrak{I}_{\mathfrak{C}}$-exact stable homological functor is $U_{\mathfrak{C}}$.*

Proof. This is shown during the proof of [17, Theorem 4.4]. □

Theorem 2.3. *Let $\mathfrak{T}$ be a triangulated category with countable coproducts and let $\mathfrak{C} \subseteq \mathfrak{T}$ be a set of $\aleph_1$-compact objects. Let $A \in \langle\mathfrak{C}\rangle$ and $B \in \mathfrak{T}$. Then there is a natural, cohomologically indexed, right half-plane, conditionally convergent spectral sequence of the form*

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathfrak{Mod}(\mathfrak{C}^{\mathrm{op}})}\bigl(U_{\mathfrak{C}}(A), U_{\mathfrak{C}}(B)\bigr)_{-q} \Rightarrow \mathfrak{T}_{p+q}(A, B).$$

If the object $U_{\mathfrak{C}}(A)$ *has a projective resolution of length* 1*, then there is a natural short exact sequence*

$$\mathrm{Ext}^1_{\mathfrak{C}}(U_{\mathfrak{C}}(\Sigma A), U_{\mathfrak{C}}(B)) \rightarrowtail \mathfrak{T}(A,B) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{C}}(U_{\mathfrak{C}}(A), U_{\mathfrak{C}}(B)).$$

Proof. These statements are contained in [8, Theorem 5.12] and [16, Theorem 4.4]. □

Example 2.4. Let $\mathfrak{T} = \mathrm{KK}$ and $\mathfrak{C} = \{\mathbb{C}\}$. Then $\langle \mathfrak{C} \rangle$ is the well known bootstrap class, and the universal functor $U_{\mathfrak{C}}$ is K-theory, viewed as a functor to the stable abelian category $\mathfrak{Ab}^{\mathbb{Z}/2}_{\aleph_1}$ of countable $\mathbb{Z}/2$-graded abelian groups. This has global homological dimension 1. So the second part of Theorem 2.3 gives the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet [20].

Remark 2.5. The suspension functor $\Sigma = \mathrm{C}_0(\mathbb{R}) \otimes -$ in KK^G squares to the identity. In this situation, the $\mathbb{Z}$-graded modules in the above discussion become $\mathbb{Z}/2$-graded.

The following example is crucial for us. Fix a finite group G and a conjugation-invariant family $\mathcal{F}$ of subgroups of G. Let $\mathfrak{T} = \mathrm{KK}^G$ and let $\mathfrak{C} \subseteq \mathrm{KK}^G$ consist of $\mathrm{C}(G/H)$ with the G-action by translation, for all subgroups $H \in \mathcal{F}$. Since $\mathrm{C}(G/H) = \mathrm{Ind}_H^G \mathbb{C}$ and Ind_H^G is left adjoint to Res_G^H, we compute

$$\mathrm{KK}^G_*(\mathrm{C}(G/H), B) \cong \mathrm{KK}^H_*(\mathbb{C}, B) \cong \mathrm{K}_*(B \rtimes H).$$

Consequently, $\mathfrak{C}$ consists of $\aleph_1$-compact objects. So Lemma 2.2 applies. To describe the universal exact functor in this case, it mostly remains to understand the arrows in KK^G between the generators $\mathrm{C}(G/H)$ for $H \in \mathcal{F}$. This was done by Dell'Ambrogio in [8]. He shows that the family of $\mathbb{Z}/2$-graded countable Abelian groups $\mathrm{KK}^G_*(\mathrm{C}(G/H), B)$ carries the extra structure of a Mackey module over the representation Green ring R^G of G. We denote this Mackey module by $\mathrm{k}^G_*(B)$ (see [21] for a general, brief introduction to Mackey and Green functors). We will do some computations with Mackey modules in the proofs below and give more details when they are needed. It is shown in [8] that the functor k_*^G to the category $R^G\text{-Mac}_{\mathbb{Z}/2,\aleph_1}$ of countable $\mathbb{Z}/2$-graded Mackey modules over the representation Green ring R^G of G is the universal homological invariant for the homological ideal $\mathfrak{I}_{\mathfrak{C}}$. In particular, the following theorem holds:

Theorem 2.6 (Dell'Ambrogio [8, Theorem 4.9]). *The restriction of* $\mathrm{k}^G\colon \mathrm{KK}^G \to R^G\text{-Mac}$ *to the full subcategory* $\{\mathrm{C}(G/H) \mid H \subseteq G\}$ *of* KK^G *is fully faithful, that is, for all pairs of subgroups* $H, L \subseteq G$ *there are canonical isomorphisms*

$$\mathrm{KK}^G(\mathrm{C}(G/H), \mathrm{C}(G/L)) \xrightarrow{\mathrm{k}^G} R^G\text{-Mac}(\mathrm{k}^G\mathrm{C}(G/H), \mathrm{k}^G\mathrm{C}(G/L)).$$

3. Generators for the Equivariant Bootstrap Class

One way to define the bootstrap class in ordinary KK-theory is as the class of all separable C*-algebras that are KK-equivalent to a commutative C*-algebra. Since all C*-algebras of Type I belong to the bootstrap class, we may also say that it is the class of all separable C*-algebras that are KK-equivalent to a Type I C*-algebra. We choose this definition in the equivariant case. For any compact group G, it is shown in [9, Theorem 3.10] that a separable G-C*-algebras is KK^G-equivalent to a G-action on a Type I C*-algebra if and only if it belongs to the localising subcategory of KK^G that is generated by the G-actions on "elementary" C*-algebras. Here a G-action on a C*-algebra is called *elementary* if it is isomorphic to $\mathrm{Ind}_H^G \mathbb{M}_n(\mathbb{C})$ for some closed subgroup $H \subseteq G$ and some group action of H on the matrix algebra $\mathbb{M}_n(\mathbb{C})$ (by automorphisms). It is shown in the proof that any G-action on a C*-algebra of the form $\bigoplus A_n$ where each A_n is isomorphic to $\mathbb{K}(\mathcal{H})$ for a finite-dimensional or separable Hilbert space $\mathcal{H}$ is equivariantly Morita equivalent to a direct sum of elementary G-actions. We also call a C*-algebra of this form $\bigoplus A_n$ elementary. The Arano–Kubota Theorem 2.1 shows that many of the above generators are redundant:

Theorem 3.1. *Let* G *be a finite group. Then* $A \in \mathrm{KK}^G$ *belongs to the localising subcategory of* KK^G *that is generated by* $\mathrm{C}(G/H) \otimes A$ *for cyclic subgroups* $H \subseteq G$.

Proof. Let $\mathfrak{I} := \bigcap_H \ker(\mathrm{Res}_G^H)$, where the intersection runs over all cyclic subgroups $H \subseteq G$. Since Res_G^H has Ind_H^G as a left adjoint functor, objects of the form $\mathrm{Ind}_H^G(A)$ for $A \in \mathrm{KK}^H$ are $\mathfrak{I}$-projective and there are enough $\mathfrak{I}$-projective objects in KK^G (see [16, Proposition 55]). Since both restriction and induction functors commute with direct sums, the localising subcategory generated by the induced objects and the localising subcategory of $\mathfrak{I}$-contractible objects are a complementary pair by [13, Theorem 3.16]. Theorem 2.1 says that any $\mathfrak{I}$-contractible object is already 0. This means that the induced objects generate all of KK^G.

Next, we build a specific $\mathfrak{I}$-projective resolution of A. First, $\mathfrak{I}$ is the kernel on morphisms of the triangulated functor

$$(\mathrm{Res}_G^H)_{H \text{ cyclic}} \colon \mathrm{KK}^G \to \prod_{H \text{ cyclic}} \mathrm{KK}^H.$$

This functor has a left adjoint, namely, the functor $\prod_H \mathrm{KK}^H \to \mathrm{KK}^G$, $(A_H) \mapsto \bigoplus_H \mathrm{Ind}_H^G(A_H)$. Then the functor

$$T\colon \mathrm{KK}^G \to \mathrm{KK}^G, \qquad A \mapsto \bigoplus_{H \text{ cyclic}} \mathrm{Ind}_H^G \mathrm{Res}_G^H(A)$$

with the counit of the adjunction $\varepsilon\colon T \Rightarrow \mathrm{id}_{\mathrm{KK}^G}$ and the comultiplication $T \Rightarrow T^2$ induced by the unit of the adjunction is a comonad in KK^G. Now we can build the bar resolution of A with the objects $T^{n+1}(A)$ and the boundary map $\sum_{j=1}^{n+1}(-1)^j \varepsilon_j \colon T^n(A) \to T^{n-1}(A)$, where ε_j is the whiskering of $\varepsilon\colon T \Rightarrow \mathrm{id}_{\mathrm{KK}^G}$ by T^{j-1} on the left and T^{n-j} on the right (see [2] for the construction and properties of the bar resolution in this generality).

Since the objects of the form $T(A)$ are all $\mathfrak{I}$-projective, the bar resolution above is an $\mathfrak{I}$-projective resolution of A. Next, we build a "phantom castle" from this $\mathfrak{I}$-projective resolution as in [13, Section 3]. This contains $\mathfrak{I}$-cellular approximations of A, and their homotopy colimit is isomorphic to A by [13, Proposition 3.18] because all $\mathfrak{I}$-contractible objects are 0. It follows that A belongs to the localising subcategory of KK^G that is generated by $T^k(A)$ for $k \geq 1$.

By construction, $T^k(A)$ is the direct sum of the tensor products

$$\begin{aligned} &\mathrm{C}(G/H_1) \otimes \mathrm{C}(G/H_2) \otimes \cdots \otimes \mathrm{C}(G/H_k) \otimes A \\ &\qquad\qquad \cong \mathrm{C}(G/H_1 \times G/H_2 \times \cdots \times G/H_k, A) \end{aligned}$$

for cyclic subgroups $H_1, \ldots, H_k \subseteq G$. Decomposing $G/H_1 \times \cdots \times G/H_k$ into orbits, we further decompose this as a direct sum of $\mathrm{C}(G/H, A)$ where

$H \subseteq G$ is the stabiliser of an orbit representative. Each such stabiliser will be contained in a group that is conjugate to H_1, making it cyclic as well. Therefore, $T^k(A)$ is isomorphic to a direct sum of $\mathrm{C}(G/H, A)$ for cyclic subgroups $H \subseteq G$. □

Corollary 3.2. *An object A in KK^G belongs to the equivariant bootstrap class if and only if $\mathrm{Res}_G^H(A)$ belongs to the equivariant bootstrap class in KK^H for each cyclic subgroup $H \subseteq G$.*

Proof. Both restriction and induction functors map actions on Type I C*-algebras again to actions on Type I C*-algebras. Therefore, they map the equivariant bootstrap classes to each other. If $\mathrm{Res}_G^H A$ is in the H-equivariant bootstrap class, so is $\mathrm{C}(G/H, A) \cong \mathrm{Ind}_H^G \mathrm{Res}_G^H A$. Now Theorem 3.1 implies the result. □

Corollary 3.3. *The objects $\mathrm{C}(G/H)$ for cyclic subgroups $H \subseteq G$ generate the equivariant bootstrap class in KK^G.*

Proof. It suffices to prove that the localising subcategory generated by $\mathrm{C}(G/H)$ for cyclic subgroups $H \subseteq G$ contains all the generators of the equivariant bootstrap class. We therefore pick one or, a bit more generally, a direct sum of these generators. So let A be a G-action on an elementary C*-algebra. By Theorem 3.1, A belongs to the localising subcategory generated by $\mathrm{C}(G/H, A) \cong \mathrm{Ind}_H^G \mathrm{Res}_G^H A$ for cyclic subgroups $H \subseteq G$. Thus G-C*-algebras of the form $\mathrm{Ind}_H^G B$ for cyclic subgroups $H \subseteq G$ and an action of H on an elementary C*-algebra B also generate the equivariant bootstrap class; they cannot generate a larger subcategory because $\mathrm{Ind}_H^G B$ is an elementary C*-algebra if B is. Now for a cyclic group H, any 2-cocycle is trivial, so any elementary H-C*-algebra is Morita equivalent to $\mathrm{C}(H/K)$ for a subgroup $K \subseteq H$, which is again cyclic. Thus A belongs to the localising subcategory generated by $\mathrm{Ind}_H^G \mathrm{C}(H/K) \cong \mathrm{C}(G/K)$ for cyclic subgroups $K \subseteq G$. □

The next corollary removes the finite generation assumption from [1, Corollary 3.23.(1)].

Corollary 3.4. *Let A and B be objects of KK^G. If $\mathrm{KK}_*^H(A, B) = 0$ for all cyclic subgroups $H \subseteq G$, then $\mathrm{KK}_*^G(A, B) = 0$.*

Proof. Since induction is left adjoint to restriction, the assumption is equivalent to $\mathrm{KK}^G_*(\mathrm{C}(G/H) \otimes A, B) = 0$ for all cyclic subgroups $H \subseteq G$. The class of objects D with $\mathrm{KK}^G_*(D, B) = 0$ is localising. So the claim follows from Theorem 3.1. □

The following corollary relates certain conditions that are clearly necessary for $A \in \mathrm{KK}^G$ to belong to the equivariant bootstrap class. We do not know whether they are also sufficient.

Corollary 3.5. *Let A be an object of KK^G. If $A \rtimes H$ belongs to the bootstrap class in KK for all cyclic subgroups $H \subseteq G$, then $A \rtimes K$ is in the bootstrap class in KK for all subgroups, cyclic or not. In addition, $(A \otimes B) \rtimes G$ is in the bootstrap class in KK if B is in the bootstrap class in KK^G.*

Proof. Since $\big(A \otimes \mathrm{C}(G/H)\big) \rtimes G$ is Morita–Rieffel equivalent to $A \rtimes H$, the assumption means that $\big(A \otimes \mathrm{C}(G/H)\big) \rtimes G$ belongs to the bootstrap class. Since tensoring with A and the crossed product with G are triangulated functors that commute with countable direct sums, this implies that $\big(A \otimes B\big) \rtimes G$ belongs to the bootstrap class for all B in the localising subcategory generated by $\mathrm{C}(G/H)$ for the cyclic subgroups $H \subseteq G$. This is the equivariant bootstrap class by Corollary 3.3. Since it contains $\mathrm{C}(G/K)$ for any subgroup $K \subseteq G$, we also get the claim about $A \rtimes K$, which is Morita–Rieffel equivalent to $\big(A \otimes \mathrm{C}(G/K)\big) \rtimes G$. □

Let B be a separable G-C^*-algebra in the equivariant bootstrap class. Using the Ind-Res adjunction, for all $H \subseteq G$,

$$\mathrm{KK}^G_*(\mathrm{C}(G/H), B) \cong \mathrm{KK}^G_*(\mathrm{Ind}^G_H \mathbb{C}, B) \cong \mathrm{KK}^H_*(\mathbb{C}, \mathrm{Res}^G_H B) \cong \mathrm{K}^H_*(B)$$

is a countable $\mathbb{Z}/2$-graded module over the representation ring of H. The representation rings of all subgroups of G form a Green functor R^G (see [8]). The set of cyclic subgroups of G is closed under taking subgroups and conjugation. This allows to consider the representation Green functor only on cyclic subgroups of G. We denote it by R^G_{cy}. Then the countable $\mathbb{Z}/2$-graded $R(H)$-modules $\mathrm{K}^H_*(B)$ for the cyclic $H \subseteq G$ form a Mackey functor on cyclic subgroups of G over R^G_{cy}.

Proposition 3.6 ([8]). *The representable functor*

$$\mathrm{ck}^G_*\colon \mathrm{KK}^G \to \mathfrak{Mod}^{\mathbb{Z}/2}(R^G_{\mathrm{cy}})_{\aleph_1}, \qquad B \mapsto \{\mathrm{K}^H_*(B)\}^{\text{cyclic } H\subseteq G},$$

into the abelian category of $\mathbb{Z}/2$-graded countable right Mackey modules on cyclic subgroups of G over R^G_{cy} is the universal stable $\ker \mathrm{ck}^G_*$*-exact functor.*

Corollary 3.7. *Let G be a finite group. For every $A, B \in \mathrm{KK}^G$ with A in the G-equivariant boostrap class, there is a cohomologically indexed, right half plane, conditionally convergent spectral sequence*

$$E^{p,q}_2 = \mathrm{Ext}^p_{R^G_{\mathrm{cy}}}\big(\mathrm{ck}^G_*(A), \mathrm{ck}^G_*(B)\big)_{-q} \xRightarrow{n=p+q} \mathrm{KK}^G_n(A,B)$$

that depends functorially on A and B.

Proof. This follows from Corollary 3.3, Theorem 2.3, and Proposition 3.6. □

4. Localisation

4.1. Localisation at a set of primes

We recall how to localise KK^G at a set of primes S; this works also if G is an arbitrary locally compact group, or for other types of equivariant KK-theory. Let $\mathbb{Z}[S^{-1}] := \mathbb{Z}[1/p, p \in S]$. There are two useful ways to localise the category KK^G by $\mathbb{Z}[S^{-1}]$. We may either take the arrows between A and B to be $\mathrm{KK}^G(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{-1}]$ as in [11] or $\mathrm{KK}^G(A, B \otimes \mathbb{M}_{S^\infty})$ as in [5, Exercise 23.15.6]; here $\mathbb{M}_{S^\infty}$ denotes the UHF algebra of type $\prod_{p\in S} p^\infty$ with the trivial action of G. The first localisation yields again a triangulated category, but the canonical functor from KK^G to it does not preserve coproducts. Therefore, our machinery of relative homological algebra applies only partially. This is why we prefer the second approach to localisation here.

Definition 4.1. A separable G-C*-algebra A is *S-divisible* if $p\cdot \mathrm{id}_A \in \mathrm{KK}^G(A,A)$ is invertible for all $p \in S$.

Remark 4.2. If A is S-divisible, then for each $p \in S$ there is $h \in \mathrm{KK}^G(A,A)$ with $p\cdot h = \mathrm{id}_A$. The converse is also true because the Kasparov product with $p\cdot \mathrm{id}_A \in \mathrm{KK}^G(A,A)$ on either side simply multiplies by p.

PROPOSITION 4.3. *A separable G-C*-algebra A is S-divisible if and only if the canonical map $A \to A \otimes \mathbb{M}_{S^\infty}$ is a KK^G-equivalence, if and only if A is isomorphic to $B \otimes \mathbb{M}_{S^\infty}$ for some separable G-C*-algebra B. If B is S-divisible, then*

$$\mathrm{KK}^G(A, B) \cong \mathrm{KK}^G(A \otimes \mathbb{M}_{S^\infty}, B). \tag{4.1}$$

PROOF. Let B be S-divisible. We are going to prove that the canonical inclusion $A \hookrightarrow A \otimes \mathbb{M}_{S^\infty}$ induces an isomorphism as in (4.1). The C*-algebra $\mathbb{M}_{S^\infty}$ is defined as the C*-algebraic inductive limit of an inductive system formed of maps $\mathbb{M}_{m_n}(\mathbb{C}) \to \mathbb{M}_{m_{n+1}}(\mathbb{C})$ for $m_n \in \mathbb{N}^\times$ with $m_0 = 1$ and $m_{n+1} = p_n \cdot m_n$, such that $p_n \in S$ for all $n \in \mathbb{N}$ and each element of S occurs infinitely many times among the p_n. Since $\mathbb{M}_{S^\infty}$ is nuclear, the inductive system considered is "admissible" (see [15]). So $\mathbb{M}_{S^\infty}$ is a homotopy colimit as well, that is, there is an exact triangle

$$\bigoplus \mathbb{M}_{m_n}(\mathbb{C}) \xrightarrow{\mathrm{id}-\sigma} \bigoplus \mathbb{M}_{m_n}(\mathbb{C}) \to \mathbb{M}_{S^\infty} \to \Sigma \bigoplus \mathbb{M}_{m_n}(\mathbb{C}),$$

where σ is the map induced by the inclusions $\mathbb{M}_{m_n}(\mathbb{C}) \to \mathbb{M}_{m_{n+1}}(\mathbb{C})$. Since the tensor product with A is a triangulated functor, $A \otimes \mathbb{M}_{S^\infty}$ is the homotopy colimit of the induced inductive system

$$A = \mathbb{M}_{m_0}(A) \to \mathbb{M}_{m_1}(A) \to \cdots \to \mathbb{M}_{m_n}(A) \to \mathbb{M}_{m_{n+1}}(A) \to \cdots . \tag{4.2}$$

When we compose the induced map $\mathrm{KK}^G(\mathbb{M}_{m_{n+1}}(A), B) \to \mathrm{KK}^G(\mathbb{M}_{m_n}(A), B)$ with the canonical Morita equivalences between A and $\mathbb{M}_{p_n}(A)$ it becomes multiplication by p_n on $\mathrm{KK}^G(A, B)$. Since B is assumed S-divisible, this is invertible for all n. Therefore, the long exact sequence for a homotopy colimit simplifies to show that the inclusions $\mathbb{M}_{m_n}(A) \hookrightarrow A \otimes \mathbb{M}_{S^\infty}$ induce isomorphisms on $\mathrm{KK}^G(-, B)$ for all $n \in \mathbb{N}$. For $n = 0$, this becomes the isomorphism in (4.1).

Next, assume that A is S-divisible. Then the maps in (4.2) are KK^G-equivalences. It follows that the homotopy colimit of this inductive system is KK^G-equivalent to $\mathbb{M}_{m_n}(A)$ for all $n \in \mathbb{N}$. For $n = 0$, this gives the desired KK^G-equivalence between A and $A \otimes \mathbb{M}_{S^\infty}$. Conversely, since $p \cdot \mathrm{id}_{\mathbb{M}_{S^\infty}}$ is invertible in $\mathrm{KK}(\mathbb{M}_{S^\infty}, \mathbb{M}_{S^\infty})$ for all $p \in S$, anything KK^G-equivalent to $A \otimes \mathbb{M}_{S^\infty}$ for some A is S-divisible. □

Lemma 4.4. *The S-divisible objects in KK^G form a localising subcategory, that is, it is thick and closed under countable coproducts. We denote it by KK^G_S.*

Proof. The functoriality of suspensions and coproducts shows that a countable coproduct of suspensions of S-divisible objects is again S-divisible. Let $A \to B \to C \to \Sigma A$ be an exact triangle in KK^G. Multiplication by p is a KK^G-equivalence if and only if its mapping cone is KK^G-equivalent to 0. Since multiplication by p is natural, the cones of multiplication by p on A, B and C also form an exact triangle by [3, Proposition 1.1.11]. Therefore, if two of A, B and C are p-divisible for a prime p, so is the third. Thus the class of S-divisible objects in KK^G is triangulated.

It is known that a triangulated category with at least countable direct sums is Karoubian, that is, any idempotent has an image object; in particular, a triangulated subcategory closed under direct sums – such as that of S-divisible objects – is closed under direct summands, making it thick (see [18, Remark 3.2.7]). We recall how this is shown. A direct summand of an object A is the image of an idempotent endomorphism $p\colon A \to A$. The homotopy colimit of the constant inductive system

$$A \xrightarrow{p} A \xrightarrow{p} A \xrightarrow{p} A \to \cdots \tag{4.3}$$

exists and has the universal property of an image object for p. □

Proposition 4.5. *The localising subcategory $\mathrm{KK}^G_S \subseteq \mathrm{KK}^G$ is equivalent to the category with the same objects as KK^G and $\mathrm{KK}^G(A, B \otimes \mathbb{M}_{S^\infty})$ as the arrows from A to B, and with the composition induced by the Kasparov product in KK^G followed by the canonical KK^G-equivalence $\mathbb{M}_{S^\infty} \otimes \mathbb{M}_{S^\infty} \cong \mathbb{M}_{S^\infty}$.*

Proof. It is straightforward to check that there is a category with the same objects as KK^G, with the arrows $\mathrm{KK}^G(A, B \otimes \mathbb{M}_{S^\infty})$, and with the multiplication specified in the statement. Proposition 4.3 shows first that, in this category, every object A is isomorphic to $A \otimes \mathbb{M}_{S^\infty}$, secondly, that the latter is S-divisible, and, thirdly, that among S-divisible objects, arrows in this category simplify to $\mathrm{KK}^G(A, B)$ with the usual Kasparov product as composite. □

We now apply the machinery of relative homological algebra to the class of objects $\mathrm{C}(G/H)\otimes\mathbb{M}_{S^\infty}$ for cyclic $H\subseteq G$ in the category KK^G_S. Whereas these objects are not $\aleph_1$-compact in KK^G, they are $\aleph_1$-compact in KK^G_S because of (4.1).

Proposition 4.6. *The localising subcategory of KK^G_S generated by the objects $\mathrm{C}(G/H)\otimes\mathbb{M}_{S^\infty}$ for cyclic $H\subseteq G$ consists precisely of the S-divisible objects in the equivariant bootstrap class in KK^G. An object B in the equivariant bootstrap class in KK^G is S-divisible if and only if multiplication by p is an isomorphism on $\mathrm{KK}^G(\mathrm{C}(G/H),B)$ for all cyclic $H\subseteq G$ and all $p\in S$.*

Proof. As a homotopy colimit of objects of the form $\mathbb{M}_m(\mathrm{C}(G/H))$, the generators $\mathrm{C}(G/H)\otimes\mathbb{M}_{S^\infty}$ belong to the equivariant bootstrap class. Hence the localising subcategory generated by them is contained in the latter. It consists of S-divisible objects by Lemma 4.4.

Conversely, let B be S-divisible and in the equivariant bootstrap class. We do homological algebra in KK^G using the objects $\{\mathrm{C}(G/H)\mid H\subseteq G \text{ cyclic}\}$; these are the generators of the bootstrap class by Corollary 3.3. There are enough relative projective objects and we may build a cellular approximation tower for B from a projective resolution (P_n) as in [13]. This is a sequence of exact triangles

$$\tilde{B}_n\to\tilde{B}_{n+1}\to P_n\to\Sigma\tilde{B}_n.$$

with certain properties. It follows that $\tilde{B}_n$ and P_n are in the equivariant bootstrap class. The homotopy colimit of $(\tilde{B}_n)$ is B by [13, Proposition 3.18]. Now, the tensor product of our cellular approximation tower with $\mathbb{M}_{S^\infty}$,

$$\tilde{B}_n\otimes\mathbb{M}_{S^\infty}\to\tilde{B}_{n+1}\otimes\mathbb{M}_{S^\infty}\to P_n\otimes\mathbb{M}_{S^\infty}\to\Sigma\tilde{B}_n\otimes\mathbb{M}_{S^\infty},$$

gives a cellular approximation tower in KK^G_S. Its homotopy colimit is $B\otimes\mathbb{M}_{S^\infty}$. Therefore, the latter belongs to the localising subcategory generated by the objects $\mathrm{C}(G/H)\otimes\mathbb{M}_{S^\infty}$ with cyclic H. Since B is S-divisible, $B\cong B\otimes\mathbb{M}_{S^\infty}$.

Finally, we prove the criterion for S-divisibility. Since each $\mathrm{C}(G/H)$ is $\aleph_1$-compact,

$$\mathrm{KK}^G_*(\mathrm{C}(G/H),B\otimes\mathbb{M}_{S^\infty})\cong\mathrm{KK}^G_*(\mathrm{C}(G/H),B)\otimes\mathbb{Z}[S^{-1}]$$

holds for all B. (We remark that this isomorphism relates the approach to localisation that we follow here to that by tensoring the arrow spaces with $\mathbb{Z}[S^{-1}]$.) Therefore, $\mathrm{KK}^G_*(\mathrm{C}(G/H), B) \cong \mathrm{KK}^G_*(\mathrm{C}(G/H), B) \otimes \mathbb{Z}[S^{-1}]$ holds if B is S-divisible, and multiplication by p for $p \in S$ is invertible on this group. Conversely, assume multiplication by p for $p \in S$ is invertible on $\mathrm{KK}^G_*(\mathrm{C}(G/H), B)$ for all cyclic $H \subseteq G$. We must show that $p \cdot \mathrm{id}_B$ is invertible in $\mathrm{KK}^G_0(B, B)$. Equivalently, its mapping cone C is 0. Since this mapping cone still belongs to the equivariant bootstrap class, $C \cong 0$ if and only $\mathrm{KK}^G_*(\mathrm{C}(G/H), C) \cong 0$ for all cyclic $H \subseteq G$. By the Puppe sequence, this happens if and only if multiplication by p is an isomorphism on $\mathrm{KK}^G_*(\mathrm{C}(G/H), B)$. □

5. Localisation at the Group Order

While the modular representation theory of groups may be very complicated, it becomes relatively easy over a field in which the group order is invertible. In this section, we simplify the equivariant bootstrap class after localising at the group order. For finite cyclic groups, this is already shown by Manuel Köhler (see [12, Theorem 13.1]).

Let S be the (finite) set of primes that divide the order $|G|$ of G. We are going to work in the localising subcategory of S-divisible objects in the equivariant bootstrap class in KK^G. This is described in Proposition 4.6 as the localising subcategory generated by the objects $\mathrm{C}(G/H) \otimes \mathbb{M}_{S^\infty}$ for cyclic subgroups $H \subseteq G$.

Remark 5.1. We could also localise at a larger set of primes or even tensor with $\mathbb{Q}$ as in [6]. We do not discuss this because such a localisation may be obtained by first localising at the primes dividing $|G|$ and then localising once again at the remaining primes. So the statements we are going to prove imply the more general statements.

The main result in this section is a Universal Coefficient Theorem in this setting. We first formulate this theorem, which requires some notation. For a cyclic subgroup $H \subseteq G$, let $n := |H|$, let ϑ_n be a primitive nth root of unity, and let

$$N_H := \{y \in G \mid yHy^{-1} = H\}, \qquad W_H := N_H/H.$$

The representation ring of H is isomorphic to $\mathbb{Z}[z]/(z^n-1)$, and $\mathbb{Z}[\vartheta_n] \subseteq \mathbb{C}$ is a quotient of that by evaluation at ϑ_n. The nth cyclotomic polynomial Φ_n is the minimal polynomial of ϑ_n, that is, $\mathbb{Z}[\vartheta_n] \cong \mathbb{Z}[z]/(\Phi_n)$. The quotient $\mathbb{Z}[\vartheta_n]$ of the representation ring of H is invariant under the induced action of group automorphisms of H. Conjugation by elements of N_H defines automorphisms of H, so that N_H acts on $\mathbb{Z}[\vartheta_n]$ in a canonical way. Since elements of H act trivially, this induces an action of W_H on the representation ring and then on the quotient $\mathbb{Z}[\vartheta_n]$. Let $\mathbb{Z}[\vartheta_n] \rtimes W_H$ be the resulting semidirect product.

Let B be an object of KK^G_S, that is, B is a G-action on a separable C*-algebra and B is KK^G-equivalent to $\mathbb{M}_{S^\infty} \otimes B$. The elements corresponding to H-representations and conjugations span a subring in the endomorphism ring $\mathrm{KK}^G(\mathrm{C}(G/H), \mathrm{C}(G/H))$ that is isomorphic to $\mathbb{Z}[z]/(z^n-1) \rtimes W_H$. We quickly explained this in the introduction, and it also follows from Theorem 2.6. So $\mathrm{KK}^G_*(\mathrm{C}(G/H), B)$ becomes a $\mathbb{Z}/2$-graded module over the latter ring. Let

$$F^H_*(B) := \{x \in \mathrm{KK}^G_*(\mathrm{C}(G/H), B) \mid \Phi_n(z) \cdot x = 0\}.$$

This subgroup is a $\mathbb{Z}/2$-graded module over the quotient ring $\mathbb{Z}[\vartheta_n] \rtimes W_H$. Since B is S-divisible, multiplication by $|G|$ is invertible on $F^H_*(B)$, so that it becomes a $\mathbb{Z}/2$-graded module over $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$.

Theorem 5.2. *Let G be a finite group. Let $\mathfrak{A}_G$ be the product of the categories of $\mathbb{Z}/2$-graded, countable modules over the rings $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$, where H runs through a set of representatives for the conjugacy classes of cyclic subgroups in G. This stable Abelian category is hereditary, that is, any object has a projective resolution of length 1. The functors F^H_* for these H combine to a stable homological functor $F\colon \mathrm{KK}^G_S \to \mathfrak{A}_G$. If $A, B \in \mathrm{KK}^G_S$ and A belongs to the equivariant bootstrap class in KK^G, then there is a Universal Coefficient Theorem*

$$\mathrm{Ext}_{\mathfrak{A}_G}\bigl(F(A), F(\Sigma B)\bigr) \rightarrowtail \mathrm{KK}^G_S(A, B) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{A}_G}\bigl(F(A), F(B)\bigr).$$

The functor F induces a bijection between isomorphism classes of S-divisible objects in the G-equivariant bootstrap class and isomorphism classes of objects in $\mathfrak{A}_G$.

We will prove this theorem in the remainder of this section.

We begin by defining an idempotent in $\mathrm{C}(G/H)$ that produces the invariant F_*^H. It involves fractions with $|G|$ in the denominator, so that it only exists after inverting the primes in S. The construction uses some facts about representation rings and cyclotomic polynomials which are already used by Köhler in [12] to prove the special case of our main result when the whole group G is cyclic.

Let $\Phi_k(z) \in \mathbb{Z}[z]$ be the kth cyclotomic polynomial, whose roots are exactly the primitive kth roots of unity. Then

$$z^n - 1 = \prod_{k|n} \Phi_k(z). \tag{5.1}$$

For $k \mid n$, let

$$\psi_{n,k}(z) := \frac{z}{n}\frac{\mathrm{d}\Phi_k(z)}{\mathrm{d}z} \cdot \prod_{k'|n,k'\neq k} \Phi_{k'}(z) = \frac{z(z^n-1)}{n\Phi_k(z)}\frac{\mathrm{d}\Phi_k(z)}{\mathrm{d}z}.$$

By definition, $n \cdot \psi_{n,k} \in \mathbb{Z}[z]$. So $\psi_{n,k}$ only becomes available after localisation at n.

Lemma 5.3 ([12, Lemmas 22.5 and 22.6]). *The polynomials $\psi_{n,k}$ form a complementary set of idempotent elements in the ring $\mathbb{Z}[z,1/n]/(z^n-1)$, that is,*

$$\psi_{n,k} \cdot \psi_{n,l} \equiv \delta_{k,l}\psi_{n,k} \bmod (z^n-1), \qquad \sum_{k|n} \psi_{n,k} \equiv 1 \bmod (z^n-1).$$

Proof. The relation $\sum_{k|n} \psi_{n,k} \equiv 1 \bmod (z^n-1)$ follows by differentiating $z^n - 1 = \prod_{k|n} \Phi_k(z)$ and multiplying by z/n. The relation (5.1) implies that z^n-1 divides $\psi_{n,k}\psi_{n,l}$ if $k \neq l$, giving $\psi_{n,k} \cdot \psi_{n,l} \equiv 0 \bmod (z^n-1)$ in this case. Together with $\sum_{k|n} \psi_{n,k} \equiv 1 \bmod (z^n-1)$, this implies $\psi_{n,k} \cdot \psi_{n,k} \equiv \psi_{n,k} \bmod (z^n-1)$. □

The lemma implies an isomorphism of rings

$$\mathbb{Z}[z,1/n]/(z^n-1) \cong \bigoplus_{k|n} \mathbb{Z}[z,1/n]/(\Phi_k) \cong \bigoplus_{k|n} \mathbb{Z}[\vartheta_k,1/n], \tag{5.2}$$

where $\vartheta_k \in \mathbb{C}$ is a primitive kth root of unity (see [12, Proposition 22.8]).

Next we are going to compute the character of $\psi_{n,k}$. Mapping a representation to its character defines an injective map from the representation ring to the ring of class functions with pointwise multiplication. This remains injective after inverting some rational numbers. Since our cyclic group H is Abelian, all functions are class functions. The character homomorphism maps the generator $z \in \mathbb{Z}[z]/(z^n - 1)$ of the representation ring to the function $\mathbb{Z}/n \to \mathbb{C}$, $j \mapsto \vartheta_n^j$. Thus the image of $p \in \mathbb{Z}[z, 1/n]/(z^n - 1)$ is the function that maps $j \in \mathbb{Z}/n$ to $p(\vartheta_n^j)$.

Lemma 5.4. *The character of $\psi_{n,k}$ is the characteristic function of the subset of elements of $\mathbb{Z}/n$ of order equal to k.*

Proof. By construction, $\psi_{n,k}$ vanishes at the primitive lth roots of unity for all divisors $l \mid n$ with $l \neq k$. Then Lemma 5.3 shows that the character of $\psi_{n,k}$ is the characteristic function of the set of all $j \in \mathbb{Z}/n$ for which ϑ_n^j is a primitive kth root of unity. This is equivalent to j having order equal to k in $\mathbb{Z}/n$. □

The endomorphism ring $\mathrm{KK}^G(\mathrm{C}(G/H), \mathrm{C}(G/H))$ in Theorem 2.6 was computed by Köhler [12] for cyclic groups and in general by Ivo Dell'Ambrogio [8]. We have already explained in the introduction how to map the representation ring of H into it. When H is a cyclic subgroup of order n, we thus get an embedding

$$\mathbb{Z}[z]/(z^n - 1) \hookrightarrow \mathrm{KK}^G(\mathrm{C}(G/H), \mathrm{C}(G/H)).$$

In particular, the integer polynomials $n\psi_{n,k}$ give elements in this ring. After localising at $|G|$, we may divide these elements by n and get idempotent elements

$$p_{n,k} \in \mathrm{KK}^G_S(\mathrm{C}(G/H) \otimes \mathbb{M}_{S^\infty}, \mathrm{C}(G/H) \otimes \mathbb{M}_{S^\infty}).$$

These satisfy the relations in Lemma 5.3. In particular, they are complementary idempotents. Actually, we only need the idempotent element $p_n := p_{n,n}$. The proof of Lemma 4.4 shows that it has an image object in KK^G_S, which we denote by A^0_H.

Proposition 5.5. *Let $H \subseteq G$ be a cyclic subgroup. Assume that S contains all prime divisors of $|H|$. In the localisation of KK^G at S, the* *object* $\mathrm{C}(G/H)$ *in* KK^G *becomes isomorphic to a direct sum of* A^0_H *and certain direct summands of* A^0_K *for subgroups* $K \subseteq H$.

PROOF. Recall that any idempotent in KK^G has an image object. Therefore, we may write A_H as the direct sum of the image objects of the complementary orthogonal idempotents $p_{n,k}$ for $k \mid n$. By definition, A^0_H is an image object for $p_{n,n}$. We finish the proof of the proposition by showing that the image object of $p_{n,k}$ for a proper divisor k of n is isomorphic to a direct summand of A^0_K, where $K \subseteq H$ is the cyclic subgroup with k elements. Here we use the Frobenius relation for the induction and restriction generators for the subgroup $K \subseteq H$ and the idempotent element $p_{k,k}$ that projects A_K onto A^0_K (see [8, Section 3.1]); this applies here because the groups $\mathrm{KK}^G_*(\mathrm{C}(G/H), -)$ for $H \subseteq G$ form a Mackey module over the Green functor of representation rings, tensored by $\mathbb{Z}[S^{-1}]$. The Frobenius formula says that $\mathrm{ind}^H_K(\mathrm{res}^H_K(y) \cdot x) = y \cdot \mathrm{ind}^H_K(x)$ for all $x \in R(K) \otimes \mathbb{Z}[S^{-1}]$, $y \in R(H) \otimes \mathbb{Z}[S^{-1}]$; this is a relation in $\mathrm{KK}^G_S(\mathrm{C}(G/H), \mathrm{C}(G/H))$. We are interested in $x = p_{k,k}$, $y = p_{n,k}$. The induction of representations is computed most easily on the level of characters: there we simply map a function $\chi\colon K \to \mathbb{C}$ to the function $\chi'\colon H \to \mathbb{C}$ given by $\chi'(h) = 0$ for $h \notin K$ and $\chi'(h) = |H : K| \cdot \chi(h)$ for $h \in H$ because H is Abelian. Using Lemma 5.4, we see that the induced character of $p_{k,k}$ is $|H : K| \cdot p_{n,k}$. Therefore, $|H : K|^{-1}$ times the product of the restriction generator, $p_{k,k}$ and the induction generator gives the idempotent $p_{n,k}$ in $\mathrm{KK}^G_S(\mathrm{C}(G/H), \mathrm{C}(G/H))$. Thus, the restriction and induction generators provide a Murray–von Neumann equivalence between $p_{n,k}$ and a certain subprojection of $p_{k,k}$ in $\mathrm{KK}^G_S(\mathrm{C}(G/K), \mathrm{C}(G/K))$, as needed. □

COROLLARY 5.6. *The homological functors that combine* $\mathrm{KK}^G_S(\mathrm{C}(G/H), -)$ *and* $\mathrm{KK}^G_S(A^0_H, -)$, *respectively, for all cyclic subgroups* $H \subseteq G$, *have the same kernels on morphisms. As a consequence, they generate the same relative homological algebra.*

PROOF. The kernel on morphisms does not change if we add a direct summand of a homological functor to a list of homological functors or if we leave out several objects that are direct sums of others in the list. Using this repeatedly, the claim follows from Proposition 5.5. □

The objects A^0_H for two conjugate cyclic subgroups are isomorphic in

KK^G_S through conjugation. Therefore, our homological ideal does not change if we only take one cyclic subgroup $H \subseteq G$ in each conjugacy class. The resulting generators have the nice extra property that they are orthogonal, that is, any element $\mathrm{KK}^G(A^0_H, A^0_{H'})$ for H not conjugate to H' vanishes:

Lemma 5.7. *If there is a nonzero element in* $\mathrm{KK}^G_S(A^0_H, A^0_{H'})$ *for two cyclic subgroups* $H, H' \subseteq G$*, then* H *is conjugate to* H'*.*

Proof. In the proof of Theorem 2.6 in [8], it is shown that any element of

$$\mathrm{KK}^G(\mathrm{C}(G/H), \mathrm{C}(G/H'))$$

may be written as a sum of products

$$\mathrm{C}(G/H) \to \mathrm{C}\big(G/(H \cap (H')^{g^{-1}})\big) \xrightarrow[\cong]{c_g} \mathrm{C}\big(G/(H^g \cap H')\big) \xrightarrow{m_E} \mathrm{C}\big(G/(H^g \cap H')\big) \to \mathrm{C}(G/H'),$$

where the first and last arrow are the generators that act by induction and restriction on $\mathrm{K}^H_*(B)$, c_g is induced by right multiplication by g, and m_E is obtained by induction from $E \in \mathrm{KK}^{H^g \cap H'}(\mathbb{C}, \mathbb{C}) \cong R(H^g \cap H')$. When we replace $\mathrm{C}(G/H)$ and $\mathrm{C}(G/H')$ by A^0_H and $A^0_{H'}$, respectively, then we multiply these composites on both sides by the idempotents $p_{n,n}$ and $p_{n',n'}$, where $n := |H|$ and $n' := |H'|$. These idempotents restrict to 0 in any proper subgroup. Hence, by the Frobenius formula for Mackey modules in [8], these products kill the restriction and induction generators unless $H \cap (H')^{g^{-1}} = H$ and $H^g \cap H' = H'$ or, equivalently, $H^g = H'$. Therefore, $\mathrm{KK}^G_S(A^0_H, A^0_{H'})$ vanishes unless H and H' are conjugate. □

Lemma 5.8. *Let* $H \subseteq G$ *be a cyclic subgroup. Let* $N_H := \{g \in G \mid gHg^{-1} = H\}$ *and* $n := |H|$*. The canonical action of* N_H *on* H *induces an action on its representation ring, which further induces an action on* $\mathbb{Z}[\vartheta_n, 1/|G|]$*. This action is trivial on* $H \subseteq N_H$ *and therefore descends to an action of the quotient group* $W_H := N_H/H$*. The endomorphism ring* $\mathrm{KK}^G_S(A^0_H, A^0_H)$ *is isomorphic to the crossed product ring* $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$*.*

Proof. The direct summand in the localised representation ring of H that is the image of p_n is isomorphic to $\mathbb{Z}[\vartheta_n, 1/|G|]$ by (5.2). This summand is invariant under group automorphisms, so that W_H acts naturally on it. The span of the elements corresponding to representations of H in the endomorphism ring of A^0_H in KK^G_S is isomorphic to $\mathbb{Z}[\vartheta_n, 1/|G|]$. Conjugation by any one of $g \in N_H$ leaves this subspace invariant and acts by the canonical N_H-action on $\mathbb{Z}[\vartheta_n, 1/|G|]$ mentioned above. An argument as in the proof of Lemma 5.7 shows that products of elements corresponding to group representations and conjugations for $g \in N_H/H$ span the endomorphism ring of A^0_H in KK^G_S and satisfy the relations in the crossed product $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$. Theorem 2.6 shows that the canonical map from $\mathbb{Z}[z]/(z^n - 1) \rtimes W_H$ to the endomorphism ring of $\mathrm{C}(G/H)$ in KK^G is injective. This remains so after inverting $|G|$ because the groups involved are torsion-free. And then it follows that the map from $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$ to the endomorphism ring of A^0_H in KK^G_S is injective. □

Remark 5.9. Let X be a set of representatives for the conjugacy classes of cyclic subgroups of G. Recall that $A^0_H \cong A^0_{H'}$ if H and H' are conjugate. This and Proposition 5.5 imply that the endomorphism rings of $\bigoplus_{H\in X} A^0_H$ and $\bigoplus_{\text{cyclic } H\subseteq G} \mathrm{C}(G/H)$ in the localisation of KK^G at S are Morita equivalent: each ring is isomorphic to a corner in a matrix algebra over the other ring. These rings usually fail, however, to be isomorphic. The module category over the first ring is $\mathfrak{A}_G$, and the second ring is the localisation of R^G_{cy} at S. Therefore, the categories $\mathfrak{Mod}^{\mathbb{Z}/2}(R^G_{\mathrm{cy}})_{\aleph_1}[S^{-1}]$ and $\mathfrak{A}_G$ are equivalent, but not isomorphic.

At this point, the general machinery of homological algebra in triangulated categories shows that the universal Abelian approximation for KK^G_S with respect to the homological ideal that we are looking at is the functor to the category of countable $\mathbb{Z}/2$-graded modules over $\bigoplus_H \mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$ which maps an object B to the family $\mathrm{KK}^G_{S,*}(A^0_H, B)$; here H runs through a set of representatives for the conjugacy classes of cyclic subgroups in G. Inspection shows that this functor is naturally isomorphic to the functor F in Theorem 5.2. To get the Universal Coefficient Theorem in that theorem, it remains to prove that the target category is hereditary.

Lemma 5.10. *Let $H \subseteq G$ be a cyclic subgroup, let W_H and n be as in* *the previous lemma. The crossed product ring* $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$ *is hereditary, that is, any module over it has a projective resolution of length* 1.

Proof. We claim that a module over $\mathbb{Z}[\vartheta_n]$ is projective if and only if it is free as an Abelian group. This is shown in the proof of [12, Theorem 12.14]. That theorem is only stated if n is prime because that is the case that is needed by Köhler at the time. The proof, however, only uses that $\mathbb{Z}[\vartheta_p]$ for a prime p is a Dedekind domain, and this remains true for all n. As a consequence, any submodule of a free module over $\mathbb{Z}[\vartheta_n]$ is projective. Then it follows that any module M over $\mathbb{Z}[\vartheta_n]$ has a projective resolution $0 \to P_1 \to P_0 \to M$ of length 1. This remains a resolution if we tensor by $\mathbb{Z}[|G|^{-1}]$ because the latter is flat, and $P_j \otimes_Z \mathbb{Z}[|G|^{-1}]$ is projective as a module over $\mathbb{Z}[\vartheta_n, 1/|G|]$ because

$$\mathrm{Hom}(P_j \otimes_Z \mathbb{Z}[|G|^{-1}], N) \cong \mathrm{Hom}(P_j, N)$$

if N is a module over $\mathbb{Z}[\vartheta_n, 1/|G|]$. As a consequence, any submodule of a projective module over $\mathbb{Z}[\vartheta_n, 1/|G|]$ is itself projective.

Since $|W_H|$ divides $|G|$, averaging over the group W_H is possible after inverting $|G|$. Therefore, an extension of modules over $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$ that splits by a $\mathbb{Z}[\vartheta_n, 1/|G|]$-module map also splits by a $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$-module map. Thus, a module is projective over $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$ once it is projective over the subring $\mathbb{Z}[\vartheta_n, 1/|G|]$. Now it follows that any submodule of a projective module over $\mathbb{Z}[\vartheta_n, 1/|G|] \rtimes W_H$ is itself projective. This is equivalent to our statement. □

Any countable module over a countable ring is a quotient of a countable free module. Therefore, Lemma 5.10 implies that the category $\mathfrak{A}_G$ is hereditary. Now the Universal Coefficient in Theorem 5.2 follows from Theorem 2.3. This implies Theorem 1.1 by well known arguments, as with the usual Universal Coefficient Theorem (see, for example, [5, Section 23.10]).

By Proposition 4.6, an object B in the G-equivariant bootstrap class is S-divisible if and only if multiplication by $|G|$ is invertible on $\mathrm{KK}^G(\mathrm{C}(G/H), B)$ for all cyclic subgroups H. This makes it easier to check this hypothesis if we already know that B is in the equivariant bootstrap class. We do not know a checkable necessary and sufficient criterion for the latter, however.

Now we specialise to *Kirchberg algebras*, that is, nonzero, simple, purely infinite, nuclear C*-algebras. In that case, we may lift the classification of actions up to KK^G-equivalence to a classification up to cocycle conjugacy:

Theorem 5.11 (Gabe and Szabó [10]). *Let G be a finite group. Any G-action on a separable, nuclear* C*-*algebra is* KK^G*-equivalent to a pointwise outer action on a stable Kirchberg algebra. Two pointwise outer G-actions on stable Kirchberg algebras are* KK^G*-equivalent if and only if they are cocycle conjugate.*

Proof. The first claim is a special case of [14, Theorem 2.1]. The second claim is a special case of [10, Theorem A]. □

Corollary 5.12. *Let G be a finite group. There is a bijection between the set of isomorphism classes of objects of $\mathfrak{A}_G$ and the set of cocycle conjugacy classes of pointwise outer G-actions on stable Kirchberg algebras that belong to the G-equivariant bootstrap class and are S-divisible in* KK^G.

Example 5.13. Let $G = \mathbb{Z}/p$ be a cyclic group whose order is a prime number p. Then G only has the cyclic subgroups $\{1\}$ and G. So our invariant F takes values in the product of the categories of $\mathbb{Z}/2$-graded countable $\mathbb{Z} \rtimes G$-modules and of $\mathbb{Z}/2$-graded countable $\mathbb{Z}[\vartheta_p]$-modules. The crossed product $\mathbb{Z} \rtimes G$ is the group ring of G, which is naturally isomorphic to $\mathbb{Z}[z]/(z^p - 1)$. After inverting p, this splits as $\mathbb{Z}[1/p] \oplus \mathbb{Z}[\vartheta_p, 1/p]$ by (5.2). Thus our classification theorem implies that isomorphism classes of objects in KK^G are in bijection with triples (X, Y, Z) where X is a $\mathbb{Z}/2$-graded Abelian group and Y, Z are $\mathbb{Z}/2$-graded $\mathbb{Z}[\vartheta_p, 1/p]$-modules. This is equivalent to the classification that follows from Köhler's UCT and [14, Theorem 7.2].

Example 5.14. Let $V = \mathbb{Z}/2 \times \mathbb{Z}/2$ be the Klein four-group with generators a, b subject to the relations $a^2 = b^2 = (ab)^2 = 1$. Besides the trivial subgroup, this has exactly three cyclic subgroups, namely, those generated by a, b, ab, and they are of order 2. For these three subgroups, we find $\mathbb{Z}[\vartheta_2] = \mathbb{Z}$ and $N_H/H \cong \mathbb{Z}/2$, acting trivially. So the crossed product $\mathbb{Z}[\vartheta_2, 1/2] \rtimes \mathbb{Z}/2$ is isomorphic to $\mathbb{Z}[1/2]^{\oplus 2}$ because of the two characters of $\mathbb{Z}/2$. For the trivial subgroup, the relevant ring is $\mathbb{Z}[\vartheta_1, 1/2] \rtimes V$, the group ring of V with coefficients in $\mathbb{Z}[1/2]$. The evaluation at the four characters splits this group ring as $\mathbb{Z}[1/2]^{\oplus 4}$. So altogether, we get $4+6=10$ summands $\mathbb{Z}[1/2]$. Thus isomorphism classes of 2-divisible objects in the V-equivariant bootstrap class in KK^V are in bijection with 10-tuples of 2-divisible $\mathbb{Z}/2$-graded Abelian groups.

REMARK 5.15. Let B be any object in the G-equivariant bootstrap class. The inclusion map $B \to B \otimes \mathbb{M}_{S^\infty}$ is part of an exact triangle $B \to B \otimes \mathbb{M}_{S^\infty} \to B \otimes C \to \Sigma B$ with $\mathrm{K}_1(C) = 0$ and

$$\mathrm{K}_0(C) = \mathbb{Z}[S^{-1}]/\mathbb{Z} \cong \bigoplus_{p \in S} \mathbb{Z}[p^{-1}]/\mathbb{Z},$$

where the last isomorphism follows from the Chinese Remainder Theorem. Thus we may decompose $C = \bigoplus_{p \in S} C_p$ with $\mathrm{K}_1(C_p) = 0$, $\mathrm{K}_0(C_p) = \mathbb{Z}[1/p]/\mathbb{Z}$. We may write B as the desuspended mapping cone of the map $B \otimes \mathbb{M}_{S^\infty} \to \bigoplus_{p \in S} B \otimes C_p$ in the above exact triangle. If we know this arrow in KK^G, we know B up to isomorphism. We may further write $C_p = \varinjlim C_{p,n}$ where $C_{p,n}$ is the object in the bootstrap class with $\mathrm{K}_1(C_{p,n}) = 0$ and $\mathrm{K}_0(C_{p,n}) = \mathbb{Z}/p^n$. Thus $B \otimes C_{p,n}$ has the extra property that multiplication by p^n vanishes in its KK^G-endomorphism ring. Thus, one may try to classify general objects of the equivariant bootstrap class by first classifying those objects with the property that multiplication by p^n vanishes in their KK^G-endomorphism ring for some $p \mid |G|$, $n \in \mathbb{N}$. Even when this can be done, it still remains to classify the maps in KK^G from an S-divisible object to one of the form $B \otimes C_p$.

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Ralf Meyer
Mathematisches Institut
Universität Göttingen
Bunsenstraße 3–5
37073 Göttingen, Germany
E-mail: rmeyer2@uni-goettingen.de

George Nadareishvili
School of Mathematics
Kutaisi International University
Akhalgazrdoba Ave. Lane 5/7
4600 Kutaisi, Georgia
E-mail: giorgi.nadareishvili@kiu.edu.ge