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Combinatorial Construction of the Absolute Galois Group of the Field of Rational Numbers

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Abstract. In this paper, we give a *purely combinatorial/group-theoretic construction* of the conjugacy class of subgroups of the Grothendieck-Teichmüller group GT determined by the *absolute Galois group* $G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [where $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers] of the field of rational numbers $\mathbb{Q}$. In fact, this construction also yields, as a by-product, a *purely combinatorial/group-theoretic characterization* of the GT-conjugates of *closed subgroups of* $G_{\mathbb{Q}}$ that are "*sufficiently large*" in a certain sense. We then introduce the notions of *TKND-fields* [i.e., "torally Kummer-nondegenerate fields"] and *AVKF-fields* [i.e., "abelian variety Kummer-faithful fields"], which generalize, respectively, the notions of "torally Kummer-faithful fields" and "Kummer-faithful fields" [notions that appear in previous work of Mochizuki]. For instance, if we write $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ for the maximal abelian extension field of $\mathbb{Q}$, then every finite extension of $\mathbb{Q}^{\mathrm{ab}}$ is a *TKND-AVKF-field* [i.e., both TKND and AVKF]. We then apply the purely combinatorial/group-theoretic characterization referred to above to prove that, if a subfield $K \subseteq \overline{\mathbb{Q}}$ is TKND-AVKF, then the commensurator in GT of the subgroup $G_K \subseteq G_{\mathbb{Q}}$ determined by K is contained in $G_{\mathbb{Q}}$. Finally, we combine this computation of the commensurator with a result of Hoshi-Minamide-Mochizuki concerning GT to prove a *semi-absolute version of the Grothendieck Conjecture* for higher dimensional [i.e., of dimension ≥ 2] configuration spaces associated to hyperbolic curves of genus zero over TKND-AVKF-fields.

Contents

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Introduction

The present paper builds on the theory of *combinatorial Belyi cuspidalization* developed in [Tsjm], §1. The theory of combinatorial Belyi cuspidalization may be understood as a certain combinatorial version of the theory of Belyi cuspidalization developed in [AbsTopII], §3.

In the present paper, we apply the theory of combinatorial Belyi cuspidalization to give a *purely combinatorial/group-theoretic definition* of a certain class of closed subgroups "BGT" [cf. Definition 3.3, (v)] of the *Grothendieck-Teichmüller group*

$$\mathrm{GT}\ (\subseteq \mathrm{Out}(\Pi_n^{\mathrm{tpd}})),$$

where, for $n \geq 1$, Π_n^{tpd} denotes the étale fundamental group of the n-th configuration space associated to the projective line, minus the three points "0", "1", "∞", over the field of algebraic numbers $\overline{\mathbb{Q}}$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1; the first display of [CbGT], Corollary C]. In the following, we shall also write $\Pi^{\mathrm{tpd}} \overset{\mathrm{def}}{=} \Pi_1^{\mathrm{tpd}}$. This class of closed subgroups "BGT" is defined to be the class of closed subgroups of GT that satisfy certain properties, which may be summarized roughly as follows:

- the **COF-property**, i.e., "cofiltered property" [cf. Definition 3.3, (ii)]: for any pair of arithmetic Belyi diagrams [cf. [Tsjm], Definition 1.4], there exists an arithmetic Belyi diagram that *dominates* [cf. Definition 3.3, (i)] both of the given arithmetic Belyi diagrams;

- the **RGC-property**, i.e.,"Relative Grothendieck Conjecture property" [cf. Definition 3.3, (iii)]: if there exists a *geometric domination* between two arithmetic Belyi diagrams, then it is the *unique* geometric domination between the two arithmetic Belyi diagrams.

At a more conceptual level, these conditions may be understood as a *single condition* of **compatibility** with **Zariski localization** on the projective line minus three points.

Our first main result is the following [cf. Theorem 4.4]:

THEOREM A (Combinatorial construction of an algebraic closure of the field of rational numbers). *Let* BGT $\subseteq$ GT *be a closed subgroup that satisfies the* **COF-** *and* **RGC-properties** *[cf. Definition 3.3, (ii), (iii), (v)]. Then one may construct from* BGT *a set*

$$\overline{\mathbb{Q}}_{\mathrm{BGT}}$$

equipped with a natural action by the commensurator $C_{\mathrm{GT}}(\mathrm{BGT})$ *of* BGT *in* GT *that satisfies the following properties:*

(i) The set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ *is equipped with natural operations*

$$\boxplus_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}} \to \overline{\mathbb{Q}}_{\mathrm{BGT}},$$

$$\boxtimes_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}} \to \overline{\mathbb{Q}}_{\mathrm{BGT}},$$

as well as natural involutions [i.e., self-bijections which are their own inverses]

$$\square^{-1}_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\} \to \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\},$$

$$(1-\square)_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\} \to \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\},$$

all of which are equivariant with respect to the natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ *on* $\overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$*. These operations and involutions satisfy the following properties:*

$$\boxplus_{\mathrm{BGT}}(0,y) \stackrel{\mathrm{def}}{=} y, \quad \boxtimes_{\mathrm{BGT}}(0,y) \stackrel{\mathrm{def}}{=} 0, \quad \boxtimes_{\mathrm{BGT}}(1,y) \stackrel{\mathrm{def}}{=} y,$$

$$\square^{-1}_{\mathrm{BGT}}(0) \stackrel{\mathrm{def}}{=} \infty, \quad \square^{-1}_{\mathrm{BGT}}(1) \stackrel{\mathrm{def}}{=} 1, \quad \square^{-1}_{\mathrm{BGT}}(\infty) \stackrel{\mathrm{def}}{=} 0,$$

$$(1-\square)_{\mathrm{BGT}}(0) \stackrel{\mathrm{def}}{=} 1, \quad (1-\square)_{\mathrm{BGT}}(1) \stackrel{\mathrm{def}}{=} 0, \quad (1-\square)_{\mathrm{BGT}}(\infty) \stackrel{\mathrm{def}}{=} \infty.$$

(ii) *If the operations* $\boxplus_{\mathrm{BGT}}$ *and* $\boxtimes_{\mathrm{BGT}}$ *determine, on* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$*, the addition and multiplication operations of a structure, on* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$*, of field isomorphic to* $\overline{\mathbb{Q}}$*, then we shall say that* BGT *satisfies the* **ArBC-property** *[i.e., "arithmetic Belyi compatibility property"]. If* BGT *satisfies the ArBC-property, then there exists a field isomorphism* $\overline{\mathbb{Q}} \xrightarrow{\sim} \overline{\mathbb{Q}}_{\mathrm{BGT}}$*, as well as a* **natural homomorphism**

$$C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}_{\mathrm{BGT}}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$$

to the group of automorphisms of the field $\overline{\mathbb{Q}}_{\mathrm{BGT}}$*. [We refer to Theorem F below for a special case, which is of central interest in the present paper, of this sort of situation.] In particular, one may construct a natural outer homomorphism*

$$C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

to the absolute Galois group $G_{\mathbb{Q}}$ *of* $\mathbb{Q}$*.*

(iii) *Suppose that* BGT *admits a* **conducting field** K *that satisfies the* **ZISC-property** *[cf. Definition 3.3, (vi)]. Then* BGT *satisfies the* **ArBC-property***.*

At the time of writing, the authors do not know whether or not the outer homomorphism $C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}}$ of Theorem A, (ii), is injective in general. On the other hand, by imposing further purely combinatorial/group-theoretic conditions — i.e., the *QAA*- and *AA-properties* [cf. Definition 5.12; the brief description following Theorem C below] — on BGT, one may conclude that the following hold [cf. Theorems 5.15, (iii); 5.17, (i), (ii)]:

THEOREM B (Injectivity of the natural outer homomorphism $C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}}$). *Let* BGT $\subseteq$ GT *be a closed subgroup that satisfies the COF- and RGC-properties [cf. Definition 3.3, (ii), (iii), (v)]. Suppose further that* BGT *satisfies the* **QAA-property** *[cf. Definition 5.12]. Then the natural outer homomorphism*

$$C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}}$$

of Theorem A, (ii), is **injective***.*

THEOREM C (Combinatorial construction of $G_{\mathbb{Q}}$).

(i) Write $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\mathrm{tpd}}) \subseteq \mathrm{Out}(\Pi^{\mathrm{tpd}})$ *for the closed subgroup of outer automorphisms that induce the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of* Π^{tpd}*. Then the* **conjugacy class** *of subgroups of* $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\mathrm{tpd}})$ *determined by the* **absolute Galois group** *of* $\mathbb{Q}$ *may be constructed from the abstract topological group* Π_2^{tpd} *[cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/group-theoretic way, as the set of* **maximal** *elements [relative to the relation of inclusion] in the set of closed subgroups of* $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\mathrm{tpd}})$ *that arise as* $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\mathrm{tpd}})$*-conjugates of closed subgroups of* GT *that satisfy the* **QAA-property** *[cf. Definition 3.3, (v); Theorem 4.4, (ii); Definition 5.12].*

(ii) The **conjugacy class** *of subgroups of* GT *determined by the* **absolute Galois group** *of* $\mathbb{Q}$ *may be constructed from the abstract topological group* Π_2^{tpd} *[cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/group-theoretic way, as the set of* **maximal** *elements [relative to the relation of inclusion] in the set of closed subgroups of* GT *that arise as closed subgroups of* GT *that satisfy the* **AA-property** *[cf. Definition 3.3, (v); Theorem 4.4, (ii); Definition 5.12].*

The class of closed subgroups "BGT" satisfying the **QAA-property** [i.e., "quasi-algebraically ample property"] (respectively, the **AA-property** [i.e., "algebraically ample property"]) is defined to be the class of closed subgroups of GT that satisfy the COF- and RGC-properties, together with the *ArBC-property* [cf. Theorem A, (ii)], as well as certain further properties (i), (ii), (iii) (respectively, (i), (ii), (iii), (iv)), which may be summarized roughly as follows:

(i) The *Kummer theory* associated to BGT is sufficiently *nondegenerate.*

(ii) The *Kummer theory* associated to the various *arithmetic Belyi diagrams arising from* BGT is sufficiently *nondegenerate.*

(iii) There exists a *family of* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$*-valued set-theoretic functions* on a certain set of *cuspidal inertia subgroups* associated to the various *arithmetic Belyi diagrams arising from* BGT that satisfies properties satisfied by the function fields arising from these arithmetic Belyi diagrams.

(iv) The family of set-theoretic functions in (iii) determines a *Galois group* that satisfies a certain compatibility property involving Π_2^{tpd}.

Of course, it is by no means the case that the approach of Theorem C to constructing the conjugacy class of subgroups of GT determined by $G_{\mathbb{Q}}$ is, in any sense, *unique*. On the other hand, the approach of Theorem C is an attractive application of the technique of *combinatorial Belyi cuspidalization* developed in [Tsjm], §1. Moreover, the approach of Theorem C has interesting applications, i.e., Theorems F and G, given below.

The approach of Theorem C to constructing the conjugacy class of subgroups of GT determined by $G_{\mathbb{Q}}$ may be thought of as a sort of

> **conditional** [cf. the condition of *maximality* within a certain collection of closed subgroups] **surjectivity** counterpart of the well-known **injectivity** result of **Belyi**, i.e., to the effect that the natural outer homomorphism $G_{\mathbb{Q}} \to \mathrm{GT}$ is injective, or, alternatively, as **one** [of many possible] **natural answer(s)** to the problem posed by **Belyi** in the discussion following the Corollary to [Belyi], Theorem 4, of giving a *group-theoretic description of the image of this outer injection* $G_{\mathbb{Q}} \hookrightarrow \mathrm{GT}$.

The idea that there should exist such a *[conditional] surjectivity* counterpart of *Belyi injectivity* that could be proven by applying *Belyi maps* in some suitable fashion [i.e., just as in the case of *Belyi injectivity*!] was motivated in part by the proofs given in [CmbCsp], §2, §3, of the injectivity/bijectivity of the natural homomorphism

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$$

of [CmbCsp], Theorem A, (i). That is to say, these proofs given in [CmbCsp], §2, §3, are *remarkable* in the sense that

> the **conditional surjectivity** proven in [CmbCsp], §3, is proven by applying an argument that is *entirely similar* to the argument applied in the proof of the corresponding **injectivity** result in [CmbCsp], §2.

In this context, it is of interest to note that this fascinating general phenomenon — i.e., of obtaining *[conditional] surjectivity* results by means of essentially similar arguments to the arguments used to verify corresponding *injectivity* results — may also be observed in numerous well-known aspects of *algebraic topology*, such as the theory of *long exact sequences of (co)homology groups* and the *homotopy theory of CW-complexes*.

The proofs of Theorems B and C depend on the following *elementary field-theoretic results* proven in §1 [cf. Theorem 1.2, Corollary 1.3]:

Theorem D (Non-algebricity of field automorphisms of algebraically closed fields). *Let K be an algebraically closed field. Write* $\mathrm{Aut}(K)$ *for the group of field automorphisms of K. Let $\alpha \in \mathrm{Aut}(K)$. Write*

$$\alpha_\Gamma : K \hookrightarrow K \times K = \mathbb{A}^2(K)$$

for the graph of α, i.e., the map $K \ni x \mapsto (x, x^\alpha) \in K \times K$. If K is of characteristic 0 *(respectively, $p > 0$), then we shall write* $\mathrm{Fr} \in \mathrm{Aut}(K)$ *for the identity automorphism (respectively, the Frobenius automorphism [i.e., given by raising to the p-th power]) of K;* $\mathrm{Fr}^{\mathbb{Z}} \subseteq \mathrm{Aut}(K)$ *for the subgroup generated by* Fr. *Then the* **image** $\mathrm{Im}(\alpha_\Gamma) \subseteq \mathbb{A}^2(K)$ *of α_Γ is* **Zariski-dense** *if and only if $\alpha \notin \mathrm{Fr}^{\mathbb{Z}}$.*

Corollary E (A criterion for the algebricity of certain set-theoretic automorphisms). *In the notation of Theorem D, write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_K$ [i.e., the projective line over K]. Let $Y \to X$ be a finite [possibly] ramified Galois covering of smooth, proper, connected curves over K. Write $X(K)$ (respectively, $Y(K)$) for the set of K-valued points of X (respectively, Y);* $\mathrm{Aut}_{X(K)}(Y(K))$ *for the group of bijections $Y(K) \xrightarrow{\sim} Y(K)$ which preserve the fibers of the natural map $Y(K) \to X(K)$; $K(Y)$ for the rational function field of Y. For $\tau \in \mathrm{Aut}_{X(K)}(Y(K))$, $f \in \mathrm{Fn}(Y(K), K \cup \{\infty\})$ [where "$\mathrm{Fn}(-,-)$" denotes the set of maps from the first argument to the second argument], write*

$$f^\tau \stackrel{\mathrm{def}}{=} f \circ \tau \in \mathrm{Fn}(Y(K), K \cup \{\infty\}).$$

We shall regard $K(Y)$ as a subset of $\mathrm{Fn}(Y(K), K \cup \{\infty\})$ *by evaluating rational functions at closed points of Y and* $\mathrm{Gal}(Y/X)$ *as a subgroup of* $\mathrm{Aut}_{X(K)}(Y(K))$ *by means of the natural action of* $\mathrm{Gal}(Y/X)$ *on $Y(K)$. Let $k \subseteq K$ be a subfield such that the covering $Y \to X$ descends to a Galois covering $Y_k \to X_k$ defined over k, and*

$$(\mathrm{Aut}(K) \supseteq)\ \mathrm{Aut}(K/k) \not\subseteq \mathrm{Fr}^{\mathbb{Z}}\ (\subseteq \mathrm{Aut}(K)),$$

where we write $\mathrm{Aut}(K/k) \subseteq \mathrm{Aut}(K)$ *for the subgroup of automorphisms that restrict to the identity automorphism of* k*. Let* $\sigma \in \mathrm{Aut}_{X(K)}(Y(K))$ *that satisfies the following property: for each* $f \in K(Y)^{\times}$*, there exist*

$$\phi_f \in \mathrm{Fn}(Y(K), k^{\times}) \ (\subseteq \mathrm{Fn}(Y(K), K \cup \{\infty\})), \quad g_f \in K(Y)^{\times}$$

such that $f^{\sigma} = \phi_f \cdot g_f$*. Then* $\sigma \in \mathrm{Gal}(Y/X)$*.*

Next, let $K \subseteq \overline{\mathbb{Q}}$ be a subfield. Write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. If K is *stably* $\times\mu$*-indivisible* [cf. [Tsjm], Definition 3.3, (v)], then we recall from [Tsjm], Corollary E, that one may construct a *natural homomorphism*

$$C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$$

whose restriction to $C_{G_{\mathbb{Q}}}(G_K) \subseteq C_{\mathrm{GT}}(G_K)$ is the natural inclusion.

In the present paper, we shall say that the subfield $K \subseteq \overline{\mathbb{Q}}$ is an *AVKF-field* [i.e., "abelian variety Kummer-faithful field"] if the following property holds [cf. Definition 6.1, (iii)]:

> Let A be an abelian variety over a finite extension L of K. Write $A(L)^{\infty}$ for the group of divisible elements $\in A(L)$. Then $A(L)^{\infty} = \{1\}$.

Here, we recall in passing that any finite extension of the maximal abelian extension field $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}$ is a *stably* $\times\mu$*-indivisible AVKF-field* [cf. Proposition 6.3, (i)]. On the other hand, it is not clear to the authors at the time of writing

- whether or not there exist AVKF-fields that are *not* stably $\times\mu$-indivisible;
- whether or not there exist stably $\times\mu$-indivisible fields that are *not* AVKF.

If K is an *AVKF-field*, then G_K satisfies the *COF-*, *RGC-*, and *ArBC-properties* [cf. Corollary 6.5], hence may be taken to be the subgroup "BGT" of Theorem A. In particular, by applying Theorem A, (ii), (iii) [cf. also Proposition 6.4], one may also construct a *natural homomorphism*

$$C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$$

whose restriction to $C_{G_{\mathbb{Q}}}(G_K) \subseteq C_{\mathrm{GT}}(G_K)$ is the natural inclusion [cf. Corollary 6.5, (iii)].

At the time of writing, the authors do not know whether or not these natural homomorphisms [i.e., of Corollary 6.5, (iii), and [Tsjm], Corollary E] are injective in general. On the other hand, by imposing a further condition on K, one may conclude that the natural homomorphism $C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$ arising from Corollary 6.5, (iii), is *injective* [cf. Theorem F below]. We shall say that the subfield $K \subseteq \overline{\mathbb{Q}}$ is a *TKND-field* [i.e., "torally Kummer-nondegenerate field"] if the following property holds [cf. Definition 6.6, (ii)]:

> Write
> $$K_{\mathrm{div}} \stackrel{\mathrm{def}}{=} \bigcup_{L/K} L_{\times\infty} \subseteq \overline{\mathbb{Q}},$$
> where $L \subseteq \overline{\mathbb{Q}}$ ranges over the finite extensions of K, and we write
> $$L^{\times} \stackrel{\mathrm{def}}{=} L \setminus \{0\}, \quad L^{\times\infty} \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} (L^{\times})^m, \quad L_{\times\infty} \stackrel{\mathrm{def}}{=} \mathbb{Q}(L^{\times\infty}) \subseteq L.$$
> Then $\overline{\mathbb{Q}}$ is an infinite field extension of K_{div}.

We shall say that the subfield $K \subseteq \overline{\mathbb{Q}}$ is a *TKND-AVKF-field* if K is both TKND and AVKF. Our main result concerning TKND-AVKF-fields is the following [cf. Theorem 6.8]:

Theorem F (Injectivity of the natural homomorphism $C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$). *Suppose that $K \subseteq \overline{\mathbb{Q}}$ is a* **TKND-AVKF-field**. *Then its absolute Galois group G_K satisfies the AA-, hence also the COF-, RGC-, ArBC-, and QAA-properties. In particular, [cf. Theorem B] the natural homomorphism $C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$ of Theorem A, (ii), is* **injective** *and restricts to the natural inclusion $C_{G_{\mathbb{Q}}}(G_K) \hookrightarrow G_{\mathbb{Q}}$ on $C_{G_{\mathbb{Q}}}(G_K) \subseteq C_{\mathrm{GT}}(G_K)$.*

Theorem F is proved by applying the theory developed in §3, §4, §5 of the present paper, i.e., the theory that underlies the proof of Theorem C [cf. the discussion surrounding Theorem C].

Finally, by combining Theorem F with certain combinatorial anabelian results proven in §2 of the present paper and applying the theory of [CbGT] [cf. [CbGT], Theorem A; [CbGT], Corollary B; the first display of [CbGT], Corollary C], we obtain a semi-absolute version of the Grothendieck Conjecture for higher dimensional [i.e., of dimension ≥ 2] configuration spaces [cf. [MT], Definition 2.1, (i)] associated to hyperbolic curves of genus 0 over K [cf. Theorem 6.10]:

Theorem G (Semi-absolute Grothendieck Conjecture-type result over TKND-AVKF-fields). *Let (m, n) be a pair of positive integers; $K, L \subseteq \overline{\mathbb{Q}}$ TKND-AVKF-fields; X_K (respectively, Y_L) a hyperbolic curve over K (respectively, L). Write (g_X, r_X) (respectively, (g_Y, r_Y)) for the type [i.e., genus and degree of the divisor of marked points] of X_K (respectively, Y_L); $(X_K)_m$ (respectively, $(Y_L)_n$) for the m-th (respectively, n-th) configuration space associated to X_K (respectively, Y_L); $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ (respectively, $G_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/L)$);*

$$\mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L)$$

for the set of outer isomorphisms $\Pi_{(X_K)_m} \xrightarrow{\sim} \Pi_{(Y_L)_n}$ that induce outer isomorphisms between G_K and G_L. Then the following hold:

(i) Suppose that

- *$m \geq 4$ or $n \geq 4$ if $r_X = 0$ or $r_Y = 0$;*
- *$m \geq 3$ or $n \geq 3$ if $r_X \neq 0$ or $r_Y \neq 0$.*

Then the outer isomorphism

$$G_K \xrightarrow{\sim} G_L$$

induced by any outer isomorphism $\in \mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L)$ arises from a **field isomorphism** *$K \xrightarrow{\sim} L$.*

(ii) Suppose that

- *$m \geq 2$ or $n \geq 2$;*
- *$g_X = 0$ or $g_Y = 0$.*

Then the natural map

$$\mathrm{Isom}((X_K)_m, (Y_L)_n) \longrightarrow \mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L)$$

is **bijective***.*

In this context, we observe that any finite extension K of $\mathbb{Q}^{\mathrm{ab}}$ is a *TKND-AVKF-field* [cf. Proposition 6.3, (i); Remark 6.6.3]. Other interesting examples of *TKND-AVKF-fields* are given in Proposition 6.3, (ii) [cf. also Remarks 6.3.3, 6.3.4, 6.3.5, 6.6.3, 6.6.4]. In particular, we observe [cf. Remark 6.3.5] that

> *Theorem G constitutes an interesting example of* **[semi-absolute] anabelian geometry** *over fields that* **cannot** *be treated by means of well-known techniques of anabelian geometry that require the use of* **p-adic Hodge theory** *or* **Frobenius elements** *of absolute Galois groups of finite fields [cf. [Tama], Theorem 0.4; [LocAn], Theorem A; [AnabTop], Theorem 4.12].*

Next, suppose that K is a *sub-p-adic subfield* [cf. [LocAn], Definition 15.4, (i)] of $\overline{\mathbb{Q}}$, i.e., [as is easily verified] a subfield of $\overline{\mathbb{Q}}$ that is isomorphic to a subfield of a finite extension of the field of p-adic numbers $\mathbb{Q}_p$, for some prime number p. Then K is a *Kummer-faithful* field [cf. [AbsTopIII], Definition 1.5; [AbsTopIII], Remark 1.5.4, (i)], hence, in particular, a *TKND-AVKF-field.* Thus, Theorem G may be regarded as a sort of partial generalization of [AbsTopIII], Theorem 1.9. On the other hand, let us recall that the proof of [AbsTopIII], Theorem 1.9, depends, in an essential way, on [LocAn], Theorem A, hence, in particular, on Faltings' *p-adic Hodge theory.* By contrast, we observe [cf. Remark 3.3.2] that

> *the proof of Theorem G [say, in the case where K and L are assumed to be sub-p-adic subfields of $\overline{\mathbb{Q}}$] is based* **solely** *on results and techniques from* **combinatorial anabelian geometry** *and hence is, in particular,* **entirely independent** *of results concerning the* **Grothendieck Conjecture for hyperbolic curves** *over sub-p-adic fields* [i.e., [LocAn], Theorem A; [Tama], Theorem 0.4].

Moreover, unlike, for instance, [LocAn], Theorem A; [Tama], Theorem 0.4; [AbsCsp], Theorem 3.2,

> *the proof of Theorem G [say, in the case where K and L are assumed to be sub-p-adic local subfields of $\overline{\mathbb{Q}}$] does* **not** *involve*

the use of any arguments involving theories of **"weights"***, i.e., theories such as Faltings'* ***p*-adic Hodge theory** *or the* **Weil Conjectures**.

Here, we recall that a somewhat *weaker version* of Theorem G in the case where $m = n = 1$ and K and L are assumed to be *stably* p-$\times\mu/\times\mu$-*indivisible fields of characteristic* 0 [cf. [Tsjm], Definition 3.3, (v)], i.e., but *not necessarily* to be TKND-AVKF, is given in [Tsjm], Theorem F. Also, we recall that a version of Theorem G in the case where $m = n = 1$ and K and L are assumed to be *generalized sub-p-adic* may be found in [Hsh2], Corollary 5.6, (ii), (iii).

This paper is organized as follows. In §1, we prove Theorem D and Corollary E, which will be of use in §5. In §2, we give some preliminaries on combinatorial anabelian geometry which will be of use in later sections. In §3, we give a purely combinatorial/group-theoretic definition of a *certain class of closed subgroups* BGT of GT [cf. the discussion preceding Theorem A] and discuss the basic properties of this class of closed subgroups of GT. In §4, for each such closed subgroup BGT $\subseteq$ GT, we give a purely combinatorial/group-theoretic construction of a *set* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ that is equipped with *"field-like" operations*, as well as a *natural action by* $C_{\mathrm{GT}}(\mathrm{BGT})$. In particular, when these "field-like" operations determine a structure of field isomorphic to $\overline{\mathbb{Q}}$, we obtain a natural outer homomorphism $C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}}$ [cf. Theorem A, (ii)]. In §5, by imposing on BGT certain further combinatorial/group-theoretic conditions, we obtain a *certain class of closed subgroups* BGT [cf. the discussion following Theorem C] — whose definition is purely combinatorial/group-theoretic — for which the natural outer homomorphism $C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}}$ is *injective* [cf. Theorem B]. Moreover, we obtain Theorem C as a consequence of this injectivity. Finally, in §6, we study various types of fields and apply the theory of §1, §2, §3, §4, §5, to prove Theorems F and G.

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Notations and Conventions

Sets: Let A, B be sets. Then we shall write $\mathrm{Fn}(A, B)$ for the set of maps from A to B. If $\mathrm{Fn}(A, B) \ni f : A \to B$ is held *fixed* in a discussion, then we shall write $\mathrm{Aut}_B(A)$ for the group of bijections $A \xrightarrow{\sim} A$ which preserve the fibers of f over B.

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. The notation $\mathbb{N}$ will be used to denote the set or, by a slight abuse of notation, additive monoid of non-negative integers.

Fields: The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers of $\mathbb{Q}$; by a slight abuse of notation, the notation $\mathbb{Z}$ will also be used to denote the underlying additive group of this ring. The notation $\mathbb{C}$ will be used to denote the field of complex numbers. The notation $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ will be used to denote the set or field of algebraic numbers $\in \mathbb{C}$. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field*. If q is a power of a prime number, then we shall write $\mathbb{F}_q$ for the finite field consisting of q elements.

Let F be a field, p a prime number, n a positive integer. Then we shall write $\mathrm{Aut}(F)$ for the group of field automorphisms of F;

$$F^\times \stackrel{\mathrm{def}}{=} F \setminus \{0\}; \quad F^\pitchfork \stackrel{\mathrm{def}}{=} F \setminus \{0, 1\}; \quad \mu_n(F) \stackrel{\mathrm{def}}{=} \{x \in F^\times \mid x^n = 1\};$$

$$\mu(F) \stackrel{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_m(F); \quad F^{\times\infty} \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} (F^\times)^m;$$

$$\mu_{p^\infty}(F) \stackrel{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_{p^m}(F); \quad F^{\times p^\infty} \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} (F^\times)^{p^m},$$

where m ranges over the positive integers. If K is an extension field of F, then we shall write $\mathrm{Aut}(K/F) \subseteq \mathrm{Aut}(K)$ for the subgroup of automorphisms that restrict to the identity automorphism of F.

Topological groups: Let G be a topological group and $H \subseteq G$ a closed subgroup of G. Then we shall denote by $Z_G(H)$ (respectively, $N_G(H)$; $C_G(H)$) the *centralizer* (respectively, *normalizer*; *commensurator*) of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$$
$$\text{(respectively, } N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\};$$
$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1}\}),$$

and write

$$Z^{\text{loc}}(G) \stackrel{\text{def}}{=} \varinjlim_{U} Z_G(U),$$

where U ranges over the open subgroups of G, for the *local centralizer* of G. We shall say that the closed subgroup H is *normally terminal* in G if $H = N_G(H)$. We shall say that the closed subgroup H is *commensurably terminal* in G if $H = C_G(H)$. We shall say that G is *slim* if $Z_G(U) = \{1\}$ for any open subgroup U of G.

Let G be a topological group. Then we shall write G^{ab} for the quotient of G by the closure of the commutator subgroup $[G, G] \subseteq G$; $\text{Aut}(G)$ for the group of [continuous] automorphisms of G; $\text{Inn}(G) \subseteq \text{Aut}(G)$ for the group of inner automorphisms of G; $\text{Out}(G) \stackrel{\text{def}}{=} \text{Aut}(G)/\text{Inn}(G)$. Now suppose that G is *center-free* [i.e., $Z_G(G) = \{1\}$]. Then we have an exact sequence of groups

$$1 \longrightarrow G \ \ (\xrightarrow{\sim} \text{Inn}(G)) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1.$$

If J is a group, and $\rho : J \to \text{Out}(G)$ is a homomorphism, then we shall denote by

$$G \overset{\text{out}}{\rtimes} J$$

the group obtained by pulling back the above exact sequence of groups via ρ. Thus, we have a *natural exact sequence* of groups

$$1 \longrightarrow G \longrightarrow G \overset{\text{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

Suppose further that G is *profinite* and *topologically finitely generated*. Then one verifies immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\text{Aut}(G)$ and $\text{Out}(G)$ with respect to which the above exact sequence relating $\text{Aut}(G)$ and $\text{Out}(G)$ determines an exact sequence of *profinite groups*. In particular, one verifies easily that if, moreover, J is *profinite*, and $\rho : J \to \mathrm{Out}(G)$ is *continuous*, then the above exact sequence relating $G \overset{\mathrm{out}}{\rtimes} J$ to G and J determines an exact sequence of *profinite groups*.

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint.

Schemes: For a morphism of scheme $S \to T$, we shall write $\mathrm{Aut}_T(S)$ for the group of automorphisms of the T-scheme S. If $T = \mathrm{Spec}\ \mathbb{Z}$, then we shall write $\mathrm{Aut}(S)$ for $\mathrm{Aut}_T(S)$.

Log schemes: We shall, by a slight abuse of notation, regard schemes as log schemes equipped with the trivial log structure. If $S^{\log}$ is a log scheme, then we shall write S for the underlying scheme of $S^{\log}$ and $U_S \subseteq S$ for the *interior* of $S^{\log}$, i.e., the largest open subscheme of S over which the log structure of $S^{\log}$ is trivial.

Curves: We shall use the terms "*hyperbolic curve*", "*cusp*", "*stable log curve*", "*smooth log curve*", and "*tripod*" as they are defined in [CmbGC], §0; [CmbCsp], §0. We shall use the terms "*n-th configuration space*" and "*n-th log configuration space*" as they are defined in [MT], Definition 2.1, (i).

1. The Non-Algebricity of Field Automorphisms

In this section, we discuss an interesting elementary property of field automorphisms of algebraically closed fields, namely, that, with the exception of integral powers of the Frobenius automorphism, such field automorphisms *cannot be expressed algebraically* [cf. Theorem 1.2]. We then apply this property to give a *criterion for the algebricity* of certain *set-theoretic automorphisms* of sets of rational points of curves valued in algebraically closed fields [cf. Corollary 1.3]. This criterion will play an important role in the theory to be developed in the present paper.

Lemma 1.1 (The inversion map on the multiplicative group of a field). *Let k be a field. Write*

$$\sigma : k^{\times} \cup \{0\} \xrightarrow{\sim} k^{\times} \cup \{0\}$$

for the bijection such that

- $\sigma(x) = x^{-1}$ *for each* $x \in k^\times$,
- $\sigma(0) = 0$.

Then the following hold:

(i) The bijection σ *is a field automorphism if and only if* $k \xrightarrow{\sim} \mathbb{F}_2$, $\mathbb{F}_3$, *or* $\mathbb{F}_4$.

(ii) If $k \xrightarrow{\sim} \mathbb{F}_2$ *or* $\mathbb{F}_3$ *(respectively,* $k \xrightarrow{\sim} \mathbb{F}_4$*), then* σ *is the identity (respectively, the unique non-trivial) automorphism of* k.

Proof. First, we verify assertion (i). *Sufficiency* is immediate. Next, to verify *necessity*, we observe that if σ is a field automorphism, then, for $x \in k \setminus \{0, -1\}$,

$$1 + \tfrac{1}{x} \;=\; \sigma(1) + \sigma(x) \;=\; \sigma(1+x) \;=\; \tfrac{1}{1+x} \quad (\Longleftrightarrow x^2 + x + 1 = 0).$$

Since the equation $x^2 + x + 1 = 0$ has at most 2 solutions in k, we thus conclude that the cardinality of k is ≤ 4. Assertion (ii) follows immediately from the definitions. This completes the proof of Lemma 1.1. $\square$

Theorem 1.2 (Non-algebricity of field automorphisms of algebraically closed fields). *Let* K *be an algebraically closed field;* $\alpha \in \mathrm{Aut}(K)$. *Write*

$$\alpha_\Gamma : K \hookrightarrow K \times K = \mathbb{A}^2(K)$$

for the graph of α, *i.e., the map* $K \ni x \mapsto (x, x^\alpha) \in K \times K$. *If* K *is of characteristic* 0 *(respectively,* $p > 0$*), then we shall write* $\mathrm{Fr} \in \mathrm{Aut}(K)$ *for the identity automorphism (respectively, the Frobenius automorphism [i.e., given by raising to the p-th power]) of* K; $\mathrm{Fr}^{\mathbb{Z}} \subseteq \mathrm{Aut}(K)$ *for the subgroup generated by* Fr. *Then the image* $\mathrm{Im}(\alpha_\Gamma) \subseteq \mathbb{A}^2(K)$ *of* α_Γ *is Zariski-dense if and only if* $\alpha \notin \mathrm{Fr}^{\mathbb{Z}}$.

Proof. *Necessity* is immediate. Thus, it remains to verify *sufficiency.* If α_Γ is *not Zariski-dense*, then there exists a nonzero polynomial

$$0 \neq f = f(X,Y) = \sum a_{i,j} X^i Y^j \in K[X,Y]$$

such that

$$\mathrm{Im}(\alpha_\Gamma) \subseteq V(f) \subseteq \mathbb{A}^2(K),$$

where $V(f)$ denotes the zero set of f. In particular, for $x \in K$, we have

$$\sum a_{i,j} x^i (x^j)^\alpha = 0.$$

For $x \in K^\times$, write $\rho_{i,j}(x) \stackrel{\mathrm{def}}{=} x^i (x^j)^\alpha \in K^\times$. Then $\rho_{i,j} : K^\times \to K^\times$ is a character. Thus, it follows immediately from Artin's well-known result on the *linear independence of characters* that there exist pairs of integers $(i_1, j_1) \neq (i_2, j_2) \in \mathbb{N} \times \mathbb{N}$ such that $\rho_{i_1,j_1} = \rho_{i_2,j_2}$. In particular, there exists a pair of integers

$$(i, j) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$$

such that

$$x^i = (x^j)^\alpha$$

for every $x \in K^\times$. Since K is algebraically closed, it follows that $i \neq 0$, $j \neq 0$. Moreover, since $K^\times$ is divisible, we may assume without loss of generality that i and j are co-prime.

Now suppose that the *characteristic* of K is $p > 0$. Write $\phi_i : K^\times \to K^\times$ (respectively, $\phi_j : K^\times \to K^\times$) for the surjection determined by $x \mapsto x^i$ (respectively, $x \mapsto x^j$). Since $x^i = (x^j)^\alpha$ for $x \in K^\times$, it follows that $\mathrm{Ker}(\phi_i) = \mathrm{Ker}(\phi_j)$. Since i and j are co-prime, we thus conclude that $i, j \in \{\pm p^{\mathbb{Z}}\}$. Moreover, we may assume without loss of generality that $j = 1$. Thus, by applying Lemma 1.1, (i), we conclude that $\alpha \in \mathrm{Fr}^{\mathbb{Z}}$.

Next, we consider the case where the *characteristic* of K is 0. In this case, we have, for example, $2^i = 2^j$. This implies that $i = j$. Thus, since i and j are co-prime, we conclude that $\alpha \in \mathrm{Fr}^{\mathbb{Z}}$. This completes the proof of Theorem 1.2. $\square$

REMARK 1.2.1.

(i) Theorem 1.2 was in some sense *motivated* by the following *complex analytic analogue* of Theorem 1.2, i.e., the *non-holomorphicity* of the automorphism of $\mathbb{C}$ given by *complex conjugation*. Let n be a positive integer; $U \subseteq \mathbb{C}$ a nonempty relatively compact open subset; $\{f_j(z)\}_{1 \le j \le n}$ a *family of holomorphic functions* on U. Write μ for the

Lebesgue measure on $\mathbb{C}$; $\overline{z} \in \mathbb{C}$ for the complex conjugate of $z \in \mathbb{C}$. Then

$$\exists z \in U \text{ such that } \overline{z} \notin \{f_j(z)\}_{1 \leq j \leq n}.$$

Indeed, suppose that $\overline{z} \in \{f_j(z)\}_{1 \leq j \leq n}$ for every $z \in U$. By enlarging the family of holomorphic functions $\{f_j(z)\}_{1 \leq j \leq n}$ if necessary, we may assume without loss of generality that it is *stabilized by multiplication by* -1. Write

$$g_j(z) \stackrel{\text{def}}{=} f_j(z) + z, \quad E_j \stackrel{\text{def}}{=} \{z \in U \mid \pm \overline{z} = f_j(z)\}.$$

Then it follows immediately from the definitions that $E_j \subseteq U$ is a closed [hence, in particular, Lebesgue measurable] subset, and $U = \bigcup_{1 \leq j \leq n} E_j$. Thus, we conclude that

$$0 < \mu(U) \leq \sum_{1 \leq j \leq n} \mu(E_j) < \infty.$$

In particular, there exists an element $j \in \{1, \ldots, n\}$ such that $\mu(E_j) > 0$. Fix such an element j. Since the family of holomorphic functions $\{f_j(z)\}_{1 \leq j \leq n}$ is *stabilized by multiplication by* -1, by possibly replacing j by $j' \in \{1, \ldots, n\}$ such that $f_j(z) = -f_{j'}(z)$ for $z \in U$ [which implies that $E_j = E_{j'}$], we may assume without loss of generality that $g_j(z)$ is a *non-constant holomorphic function.* But then $g_j(E_j) \subseteq \mathbb{R} \cup \sqrt{-1} \cdot \mathbb{R}$, which implies that

$$0 < \mu(g_j(E_j)) \leq \mu(\mathbb{R} \cup \sqrt{-1} \cdot \mathbb{R}) = 0$$

— a contradiction!

(ii) Finally, we observe that Theorem 1.2 in the case where $K = \mathbb{C}$, and α is the automorphism given by complex conjugation follows immediately from the fact verified in Remark 1.2.1, (i). Indeed, if α_Γ is *not Zariski-dense*, then there exists a nonzero polynomial

$$0 \neq f = f(X, Y) = \sum a_{i,j} X^i Y^j \in \mathbb{C}[X, Y]$$

such that

$$\operatorname{Im}(\alpha_\Gamma) \subseteq V(f) \subseteq \mathbb{A}^2(\mathbb{C}),$$

where $V(f)$ denotes the zero set of f. Since the map $V(f) \to \mathbb{C}$ induced by the first projection $\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a *nonconstant algebraic* map [i.e., corresponds to a dominant morphism between one-dimensional schemes of finite type over $\mathbb{C}$], there exists a nonempty relatively compact open subset $U \subseteq \mathbb{C}$ such that the induced map

$$V(f)|_U \stackrel{\text{def}}{=} V(f) \cap (U \times \mathbb{C}) \;\to\; U$$

determines a *split finite étale morphism* of complex analytic spaces. The finite collection of *sections* of this induced map thus determines a *family of holomorphic functions* as in Remark 1.2.1, (i). This yields the desired contradiction.

Corollary 1.3 (A criterion for the algebricity of certain set-theoretic automorphisms). *In the notation of Theorem 1.2, write* $X \stackrel{\text{def}}{=} \mathbb{P}^1_K$ *[i.e., the projective line over* K*]. Let* $Y \to X$ *be a finite [possibly] ramified Galois covering of smooth, proper, connected curves over* K*. Write* $X(K)$ *(respectively,* $Y(K)$*) for the set of* K*-valued points of* X *(respectively,* Y*);* $\mathrm{Aut}_{X(K)}(Y(K))$ *for the group of bijections* $Y(K) \xrightarrow{\sim} Y(K)$ *which preserve the fibers of the natural map* $Y(K) \to X(K)$*;* $K(Y)$ *for the rational function field of* Y*. For* $\tau \in \mathrm{Aut}_{X(K)}(Y(K))$*,* $f \in \mathrm{Fn}(Y(K), K \cup \{\infty\})$*, write*

$$f^\tau \stackrel{\text{def}}{=} f \circ \tau \in \mathrm{Fn}(Y(K), K \cup \{\infty\}).$$

We shall regard $K(Y)$ *as a subset of* $\mathrm{Fn}(Y(K), K \cup \{\infty\})$ *by evaluating rational functions at closed points of* Y *and* $\mathrm{Gal}(Y/X)$ *as a subgroup of* $\mathrm{Aut}_{X(K)}(Y(K))$ *by means of the natural action of* $\mathrm{Gal}(Y/X)$ *on* $Y(K)$*. Let* $k \subseteq K$ *be a subfield such that the covering* $Y \to X$ *descends to a Galois covering* $Y_k \to X_k$ *defined over* k*, and*

$$(\mathrm{Aut}(K) \supseteq)\ \mathrm{Aut}(K/k) \not\subseteq \mathrm{Fr}^{\mathbb{Z}}\ (\subseteq \mathrm{Aut}(K)).$$

Let $\sigma \in \mathrm{Aut}_{X(K)}(Y(K))$ *that satisfies the following property: for each* $f \in K(Y)^\times$*, there exist*

$$\phi_f \in \mathrm{Fn}(Y(K), k^\times)\ (\subseteq \mathrm{Fn}(Y(K), K \cup \{\infty\})), \quad g_f \in K(Y)^\times$$

such that $f^\sigma = \phi_f \cdot g_f$*. Then* $\sigma \in \mathrm{Gal}(Y/X)$*.*

Proof. Write n for the degree of the covering $Y \to X$; $\sigma_1, \ldots, \sigma_n$ for the n distinct elements of $\mathrm{Gal}(Y/X)$. Let $\alpha \in \mathrm{Aut}(K/k) \setminus \mathrm{Fr}^{\mathbb{Z}}$. Write

$$\alpha_{\Gamma,X} : X(K) \to X(K) \times X(K),$$

$$\alpha_{\Gamma,Y} : Y(K) \to Y(K) \times Y(K)$$

for the respective graphs of α, i.e., the maps $X(K) \ni x \mapsto (x, x^{\alpha}) \in X(K) \times X(K)$ and $Y(K) \ni y \mapsto (y, y^{\alpha}) \in Y(K) \times Y(K)$. Then it follows immediately from Theorem 1.2 that the subset $\mathrm{Im}(\alpha_{\Gamma,X}) \subseteq X(K) \times X(K)$ is *Zariski-dense* in $X(K) \times X(K)$. Next, we *observe* that

- the covering $Y \to X$, hence also the morphism $Y \times Y \to X \times X$ [i.e., the product over K of two copies of the covering $Y \to X$] is *finite*;
- the map $\mathrm{Im}(\alpha_{\Gamma,Y}) \to \mathrm{Im}(\alpha_{\Gamma,X})$ induced by the finite morphism $Y \times Y \to X \times X$ is *surjective*.

Thus, since the Zariski closure of $\mathrm{Im}(\alpha_{\Gamma,Y})$ is an algebraic set in $Y(K) \times Y(K)$, it follows immediately from the above *observations* that $\mathrm{Im}(\alpha_{\Gamma,Y})$ is *Zariski-dense* in $Y(K) \times Y(K)$.

Next, we observe that the existence of the Galois covering $Y_k \to X_k$ [i.e., whose base-change over k to K is the covering $Y \to X$] implies that the natural action of $\mathrm{Aut}(K/k)$ on K induces a *natural action* of $\mathrm{Aut}(K/k)$ on $Y(K)$ that *commutes* with the natural action of $\mathrm{Gal}(Y/X)$ on $Y(K)$. If, moreover, $\beta \in \mathrm{Aut}(K/k)$, $h \in \mathrm{Fn}(Y(K), K \cup \{\infty\})$, then we shall write

$$h^{\beta} \stackrel{\mathrm{def}}{=} \beta^{-1} \circ h \circ \beta \in \mathrm{Fn}(Y(K), K \cup \{\infty\}).$$

For each pair of integers (i, j) such that $1 \leq i, j \leq n$, write

$$Y_{i,j} \stackrel{\mathrm{def}}{=} \{(y_1, y_2) \in Y(K) \times Y(K) \mid y_1^{\sigma\sigma_i} = y_1,\ y_2^{\sigma\alpha^{-1}\sigma_i} = y_2^{\alpha^{-1}\sigma_j}\}.$$

Since $\sigma \in \mathrm{Aut}_{X(K)}(Y(K))$, it follows immediately that

$$Y(K) \times Y(K) = \bigcup_{1 \leq i,j \leq n} Y_{i,j}.$$

Write

$$Z_{i,j}$$

for the Zariski closure of $\mathrm{Im}(\alpha_{\Gamma,Y}) \cap Y_{i,j}$ in $Y(K) \times Y(K)$. Since the subset $\mathrm{Im}(\alpha_{\Gamma,Y}) \subseteq Y(K) \times Y(K)$ is Zariski-dense, there exists a pair of integers (i,j) such that

$$Y(K) \times Y(K) = Z_{i,j}.$$

Fix such a pair of integers (i,j).

Next, we *observe* that, for each $f \in K(Y)^{\times}$, we have equalities

$$\begin{aligned}(\phi_f^{\sigma_i},\ (\phi_f^{\alpha^{-1}})^{\sigma_i}) &= (f^{\sigma\sigma_i} \cdot (g_f^{-1})^{\sigma_i},\ \{(f^{\sigma})^{\alpha^{-1}}\}^{\sigma_i} \cdot \{(g_f^{-1})^{\alpha^{-1}}\}^{\sigma_i}) \\ &= (f \cdot (g_f^{-1})^{\sigma_i},\ (f^{\alpha^{-1}})^{\sigma_j} \cdot \{(g_f^{-1})^{\alpha^{-1}}\}^{\sigma_i})\end{aligned}$$

[of ordered pairs of elements of $\mathrm{Fn}(Y(K), K \cup \{\infty\})$] on some subset $Y_{i,j}^{*} \subseteq Y_{i,j}$ [i.e., so that all of the values of functions that appear are *finite*] such that $Y_{i,j} \setminus Y_{i,j}^{*}$ is contained in an *algebraic set* $\subseteq Y(K) \times Y(K)$ *of dimension* 1 — which implies that the Zariski closure $Z_{i,j}^{*}$ of $\mathrm{Im}(\alpha_{\Gamma,Y}) \cap Y_{i,j}^{*}$ is equal to $Y(K) \times Y(K)$. Now consider the morphism

$$\psi \stackrel{\mathrm{def}}{=} (h_f^{\dagger}, h_f^{\ddagger}) : Y \times_K Y \to \mathbb{P}_K^1 \times_K \mathbb{P}_K^1.$$

determined by the rational functions $h_f^{\dagger} \stackrel{\mathrm{def}}{=} f \cdot (g_f^{-1})^{\sigma_i}$ and $h_f^{\ddagger} \stackrel{\mathrm{def}}{=} (f^{\alpha^{-1}})^{\sigma_j} \cdot \{(g_f^{-1})^{\alpha^{-1}}\}^{\sigma_i}$. Write Δ for the diagonal divisor of $\mathbb{P}_K^1 \times_K \mathbb{P}_K^1$. Then it follows immediately from the above *observation* [i.e., the *observation* discussed at the beginning of the present paragraph], together with the fact that the natural actions of α and σ_i on $Y(K)$ *commute*, that

$$\psi(\mathrm{Im}(\alpha_{\Gamma,Y}) \cap Y_{i,j}^{*}) \subseteq \Delta(k) \subseteq \Delta(K)\ (\subseteq \mathbb{P}_K^1(K) \times \mathbb{P}_K^1(K)).$$

Since $Y(K) \times Y(K) = Z_{i,j}^{*}$, we conclude that $\mathrm{Im}(\psi) \subseteq \Delta(K)$, hence, in particular, that the morphism ψ is *not dominant*. On the other hand, if *both* $h_f^{\dagger}$ and $h_f^{\ddagger}$ are *nonconstant* rational functions, then the morphism ψ is easily verified to be *dominant*. Thus, we conclude that *either* $h_f^{\dagger}$ or $h_f^{\ddagger}$ is *constant*, and hence, since $\mathrm{Im}(\psi) \subseteq \Delta(K)$, that *both* $h_f^{\dagger}$ and $h_f^{\ddagger}$ are *constant*. Write $c_f \in K$ for the unique constant value of $h_f^{\dagger}$. Thus,

$$f^{\sigma} = \phi_f \cdot g_f = c_f^{-1} \cdot \phi_f \cdot f^{\sigma_i^{-1}},$$

for every $f \in K(Y)^{\times}$. In particular, if we write $\tau \stackrel{\mathrm{def}}{=} \sigma\sigma_i$, $\phi_f^{\dagger} \stackrel{\mathrm{def}}{=} \phi_f^{\sigma_i}$, then

$$f^{\tau} = c_f^{-1} \cdot \phi_f^{\dagger} \cdot f,$$

for every $f \in K(Y)^{\times}$. For each $y \in Y(K)$, let $f_y \in K(Y)^{\times}$ be a rational function on Y such that f_y has a pole at y and no pole on $Y(K) \setminus \{y\}$. [The existence of such rational functions follows immediately from the Riemann-Roch theorem.] Thus, since $f_y^{\tau} = c_{f_y}^{-1} \cdot \phi_{f_y}^{\dagger} \cdot f_y$, we conclude that $y^{\tau} = y$ for each $y \in Y(K)$, hence that τ is the identity automorphism, i.e., $\sigma = \sigma_i^{-1} \in \mathrm{Gal}(Y/X)$. This completes the proof of Corollary 1.3. □

REMARK 1.3.1.

(i) Corollary 1.3 was in some sense *motivated* by the following *complex analytic analogue* of Corollary 1.3. Write $\mathbb{S}^1 \stackrel{\mathrm{def}}{=} \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}^{\times}$. In the notation of Corollary 1.3 in the case where $K \subseteq \mathbb{C}$, let $\zeta \in \mathrm{Aut}_{X(K)}(Y(K))$ that satisfies the following property: for each $f \in K(Y)^{\times}$, there exist

$$\omega_f \in \mathrm{Fn}(Y(K), \mathbb{S}^1), \quad q_f \in K(Y)^{\times}$$

such that $f^{\zeta} = \omega_f \cdot q_f$. Then

$$\zeta \in \mathrm{Gal}(Y/X).$$

Indeed, write μ for the Lebesgue measure on $\mathbb{C}$; μ_Y for the measure on $Y(\mathbb{C})$ induced by a [nowhere-vanishing] volume form on the Riemann surface associated to $Y \times_K \mathbb{C}$; n for the degree of the covering $Y \to X$; $\zeta_1, \ldots, \zeta_n$ for the n distinct elements of $\mathrm{Gal}(Y/X)$. For each $j = 1, \ldots, n$, write

$$E_j \stackrel{\mathrm{def}}{=} \{y \in Y(K) \mid y^{\zeta\zeta_j} = y\} \subseteq Y(\mathbb{C});$$

$$F_j \subseteq Y(\mathbb{C})$$

for the closure of $E_j \subseteq Y(\mathbb{C})$ in the complex topology [i.e., the topology induced by the topology of the topological field $\mathbb{C}$]. Thus, $F_j \subseteq Y(\mathbb{C})$ is measurable [i.e., with respect to the measure μ_Y]. Note that, since $\zeta \in \mathrm{Aut}_{X(K)}(Y(K))$,

$$\bigcup_{1 \leq j \leq n} E_j = Y(K).$$

Since the subset $Y(K) \subseteq Y(\mathbb{C})$ is easily verified to be dense in the complex topology, it follows immediately that

$$\bigcup_{1 \leq j \leq n} F_j = Y(\mathbb{C}).$$

Thus, we conclude that

$$0 < \mu_Y(Y(\mathbb{C})) \leq \sum_{1 \leq j \leq n} \mu_Y(F_j) < \infty.$$

In particular, there exists an element $j \in \{1, \ldots, n\}$ such that $\mu_Y(F_j) > 0$. Fix such an element j. Next, for each $f \in K(Y)^\times$, it follows immediately that

$$\omega_f^{\zeta_j} = f^{\zeta\zeta_j} \cdot (q_f^{\zeta_j})^{-1} = f \cdot (q_f^{\zeta_j})^{-1}$$

on some subset $E_j^* \subseteq E_j$ [i.e., so that all of the values of functions that appear are *finite*] such that $E_j \setminus E_j^*$ is a *finite set* — which implies that $\mu_Y(F_j^*) > 0$, where F_j^* denotes the closure of $E_j^* \subseteq Y(\mathbb{C})$ in the complex topology. Thus, we conclude that, for $y \in F_j^*$ $(\subseteq Y(\mathbb{C}))$,

$$|(f \cdot (q_f^{\zeta_j})^{-1})(y)| = 1 \ (\Longleftrightarrow (f \cdot (q_f^{\zeta_j})^{-1})(y) \in \mathbb{S}^1).$$

In particular, since $\mu(\mathbb{S}^1) = 0$ and $\mu_Y(F_j^*) > 0$, the *meromorphicity* of [the function $Y(\mathbb{C}) \to \mathbb{C} \cup \{\infty\}$ determined by] $f \cdot (q_f^{\zeta_j})^{-1}$ implies that $f \cdot (q_f^{\zeta_j})^{-1}$ is in fact a constant function. Thus, we conclude as in the final portion of the proof of Corollary 1.3 that $\zeta \in \mathrm{Gal}(Y/X)$.

(ii) Finally, we observe that Corollary 1.3 in the case where

- $K = \overline{\mathbb{Q}}$, $k = \mathbb{Q}^{\mathrm{ab}} \stackrel{\mathrm{def}}{=} \mathbb{Q}(\mu(\overline{\mathbb{Q}}))$ $(\subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C})$;
- for each $f \in K(Y)^\times$,

$$\phi_f \in \mathrm{Fn}(Y(\overline{\mathbb{Q}}), \mu(\mathbb{Q}^{\mathrm{ab}})) \ (\subseteq \mathrm{Fn}(Y(\overline{\mathbb{Q}}), (\mathbb{Q}^{\mathrm{ab}})^\times)),$$

follows immediately [since $\mu(\mathbb{Q}^{\mathrm{ab}}) \subseteq \mathbb{S}^1$] from the fact verified in Remark 1.3.1, (i).

2. Preliminaries on Combinatorial Anabelian Geometry

In this section, we give some preliminaries on combinatorial anabelian geometry which will be of use in the theory developed in the present paper.

THEOREM 2.1 (Outer automorphisms of configuration space groups induced by open immersions). *Let n be an integer such that $n \geq 2$; k an algebraically closed field of characteristic 0; X a hyperbolic curve over k of type (g, r_X); U an open subscheme of X which is a hyperbolic curve over k of type (g, r_U), where $r_U > r_X$ [which implies that $(g, r_U) \notin \{(0,3),(1,1)\}$; $r_U > 0$]. Write $\mathfrak{S}_n$ for the symmetric group on n letters; X_n (respectively, U_n) for the n-th configuration space associated to X (respectively, U). Let*

$$\alpha \in \mathrm{Out}(\Pi_{U_n}).$$

Recall that there exists a unique permutation $\sigma \in \mathfrak{S}_n \subseteq \mathrm{Out}(\Pi_{U_n})$ of the factors of U_n [cf. [CbTpII], Theorem B] such that

- *$\alpha \circ \sigma \in \mathrm{Out}^{\mathrm{F}}(\Pi_{U_n})$ [cf. [CbTpII], Theorem B, (i)];*
- *the outer automorphism $\alpha_1 \in \mathrm{Out}(\Pi_U)$ induced by $\alpha \circ \sigma$ [which does not depend on the choice of projection morphisms of co-length 1 — cf. [CbTpI], Theorem A, (i)] preserves the set of cuspidal inertia subgroups of Π_U [cf. [CbTpI], Theorem A, (ii)].*

Suppose that

(a) if $n = 2$, then either $r_X > 0$ or $\alpha \circ \sigma \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{U_n})$ [cf. [CmbCsp], Definition 1.1, (ii)];

(b) α_1 stabilizes the set of conjugacy classes of cuspidal inertia subgroups of Π_U associated to the cusps of U that arise from the cusps of X;

Then α determines an outer automorphism of Π_{X_n} via the natural outer surjection $\Pi_{U_n} \twoheadrightarrow \Pi_{X_n}$ induced by the natural open immersion $U_n \hookrightarrow X_n$.

PROOF. First, since $\mathfrak{S}_n$ acts compatibly on U_n and X_n, by replacing $\alpha \circ \sigma$ by α, we may assume without loss of generality that

$$\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_{U_n}).$$

Next, observe that it follows immediately from condition (b) that, by replacing α by the composite of α with a suitable element $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{U_n})$ that

- arises, via various specialization and generization isomorphisms, from [log] scheme theory, and, moreover,
- determines an outer automorphism of Π_{X_n} via the natural outer surjection $\Pi_{U_n} \twoheadrightarrow \Pi_{X_n}$

[cf. the proof of [CmbCsp], Lemma 2.4], we may also assume without loss of generality that

(c) α_1 induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of Π_U.

Let $V \subsetneq U$ be an open subscheme which is a hyperbolic curve over k of type $(g, r_U + 1)$; $\tilde{\alpha} \in \mathrm{Aut}^{\mathrm{F}}(\Pi_{U_n})$ a lifting of $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_{U_n})$. Write

$$\{x\} \stackrel{\mathrm{def}}{=} U \setminus V, \quad X_x \stackrel{\mathrm{def}}{=} X \setminus \{x\} \subseteq X.$$

Then, for suitable choices of basepoints, we obtain a commutative diagram of homomorphisms of profinite groups

$$\begin{tikzcd} 1 \arrow[r] & \Pi_{V_{n-1}} \arrow[r] \arrow[d] & \Pi_{U_n} \arrow[r] \arrow[d, "\phi_n"'] & \Pi_U \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \Pi_{(X_x)_{n-1}} \arrow[r] & \Pi_{X_n} \arrow[r] & \Pi_X \arrow[r] & 1, \end{tikzcd}$$

where V_{n-1} (respectively, $(X_x)_{n-1}$) denotes the $(n-1)$-th configuration space of V (respectively, X_x); the horizontal sequences denote the homotopy exact sequences induced by the first projections $U_n \to U$ and $X_n \to X$; the vertical arrows denote the homomorphisms induced by the natural open immersions $V_{n-1} \hookrightarrow (X_x)_{n-1}$, $U_n \hookrightarrow X_n$, and $U \hookrightarrow X$ [cf. [MT], Proposition 2.4, (i)].

Next, we verify the following assertion:

> Claim 2.1.A: Suppose that $n = 2$. Then the automorphism $\tilde{\alpha}|_{\Pi_V} \in \mathrm{Aut}(\Pi_V)$ [induced by $\tilde{\alpha} \in \mathrm{Aut}^{\mathrm{F}}(\Pi_{U_n})$ via the injection $\Pi_V \hookrightarrow \Pi_{U_2}$ in the above commutative diagram] preserves and fixes the conjugacy classes of cuspidal inertia subgroups of Π_V that are not associated to x.

In the case where $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{U_n})$, it follows immediately from condition (c) that $\tilde{\alpha}|_{\Pi_V}$ preserves and fixes the conjugacy classes of cuspidal inertia subgroups of Π_V [cf. [CmbCsp], Proposition 1.2, (iii); [CbTpII], Lemma 3.2, (iv)]. Thus, by condition (a), we may assume without loss of generality that $r_X > 0$. Then it follows from our assumption that $r_U > r_X$ that $r_U \geq 2$. Write

- $\mathrm{Cusp}(U)$ for the set of cusps of U;
- $\rho_U : \Pi_U \to \mathrm{Out}(\Pi_V)$ for the outer representation determined by the exact sequence in the above commutative diagram
$$1 \longrightarrow \Pi_V \longrightarrow \Pi_{U_2} \longrightarrow \Pi_U \longrightarrow 1;$$
- $Y^{\log}$ for the [uniquely determined, up to unique isomorphism] smooth log curve over $\mathrm{Spec}\, k$ such that $U_Y = U$;
- $Y_2^{\log}$ for the second log configuration space associated to $Y^{\log}$;
- for each $y \in \mathrm{Cusp}(U)$, $y^{\log} \stackrel{\mathrm{def}}{=} y \times_Y Y^{\log}$ [where the fiber product is determined by the natural morphism $Y^{\log} \to Y$ obtained by forgetting the log structure];
- $Y_y^{\log} \stackrel{\mathrm{def}}{=} Y_2^{\log} \times_{Y^{\log}} y^{\log}$ [where the fiber product is determined by the first projection $Y_2^{\log} \to Y^{\log}$ and the natural projection $y^{\log} \to Y^{\log}$];
- $\mathcal{G}_y$ for the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by the stable log curve $Y_y^{\log}$ over $y^{\log}$ [cf. [CmbGC], Definition 1.1, (i)];
- v_y^{new} (respectively, v_y) for the vertex of $\mathcal{G}_y$ associated to the irreducible component that contains (respectively, does not contain) the cusp that arises from the diagonal divisor of $Y_2^{\log}$;
- $\Pi_{\mathcal{G}_y}$ for the PSC-fundamental group of $\mathcal{G}_y$ [cf. [CmbGC], Definition 1.1, (ii)].

Thus, for each $y \in \mathrm{Cusp}(U)$, we have a *natural* $\mathrm{Im}(\rho_U)$ $(\subseteq \mathrm{Out}(\Pi_V))$-*torsor of outer isomorphisms*
$$\Pi_V \xrightarrow{\sim} \Pi_{\mathcal{G}_y}$$

that induces a bijection between the respective sets of cuspidal inertia subgroups. For each $y \in \mathrm{Cusp}(U)$, let us *fix* an outer isomorphism

$$\Pi_V \xrightarrow{\sim} \Pi_{\mathcal{G}_y}$$

that belongs to this collection. Then, by conjugating by this fixed outer isomorphism, we conclude that $\tilde{\alpha}|_{\Pi_V}$ determines an outer automorphism $\alpha_y \in \mathrm{Out}(\Pi_{\mathcal{G}_y})$ for each $y \in \mathrm{Cusp}(U)$.

Let $y, z \in \mathrm{Cusp}(U)$ such that $y \neq z$. [Recall that $r_U \geq 2$.] Then observe [by varying $y, z \in \mathrm{Cusp}(U)$] that it suffices to prove that α_y *preserves and fixes the conjugacy class of cuspidal inertia subgroups of* $\Pi_{\mathcal{G}_y}$ *associated to* z [where we identify naturally the set of cusps of V with the set of cusps of $\mathcal{G}_y$].

Next, we recall that $\alpha_1 \in \mathrm{Out}(\Pi_U)$ preserves and fixes the conjugacy class of cuspidal inertia subgroups of Π_U associated to y [cf. condition (c)]. Thus, it follows from [CbTpII], Theorem 1.9, (ii), that, by replacing $\tilde{\alpha}$ by the composite of $\tilde{\alpha}$ with an inner automorphism of Π_{U_2}, we may assume without loss of generality that α_y preserves the set of verticial subgroups of $\Pi_{\mathcal{G}_y}$. Since $(g, r_U) \notin \{(0,3), (1,1)\}$, it follows [cf. [MT], Remark 1.2.2] that α_y *preserves and fixes the conjugacy classes of verticial subgroups of* $\Pi_{\mathcal{G}_y}$. Let $\Pi_{v_y} \subseteq \Pi_{\mathcal{G}_y}$ be a verticial subgroup associated to v_y; $\tilde{\alpha}_y \in \mathrm{Aut}(\Pi_{\mathcal{G}_y})$ a lifting of α_y such that $\tilde{\alpha}_y(\Pi_{v_y}) = \Pi_{v_y}$. On the other hand, observe that the composite

$$\Pi_{v_y} \subseteq \Pi_{\mathcal{G}_y} \xleftarrow{\sim} \Pi_V \hookrightarrow \Pi_{U_2} \twoheadrightarrow \Pi_U$$

— where the final arrow denotes the natural outer surjection induced by the *second projection* $U_2 \to U$ — determines an outer isomorphism $\Pi_{v_y} \xrightarrow{\sim} \Pi_U$ that induces a bijection between the respective sets of cuspidal inertia subgroups and is compatible with the respective outer automorphisms α_y and α_1. Here, we recall that the cusp z abuts to the vertex v_y. Thus, by condition (c), we conclude that α_y preserves and fixes the conjugacy class of cuspidal inertia subgroups of $\Pi_{\mathcal{G}_y}$ associated to z. This completes the proof of Claim 2.1.A.

In the remainder of the proof of Theorem 2.1, we proceed by induction on $n \geq 2$. Next, we verify the following assertion:

Claim 2.1.B: Suppose that $n = 2$. Then Theorem 2.1 holds.

Indeed, let us note that, by condition (c), α_1 preserves the kernel of the natural surjection $\Pi_U \twoheadrightarrow \Pi_X$. On the other hand, it follows immediately from Claim 2.1.A that $\tilde{\alpha}|_{\Pi_V} \in \mathrm{Aut}(\Pi_V)$ preserves the kernel of the surjection $\Pi_V \twoheadrightarrow \Pi_{X_x}$. Thus, since Π_{X_x} is center-free, we conclude that $\tilde{\alpha}$ induces an automorphism of $\Pi_{X_2} = \Pi_{X_x} \overset{\mathrm{out}}{\rtimes} \Pi_X$. This completes the proof of Claim 2.1.B.

Next, we verify the following assertion [by a similar argument to the argument used to prove Claim 2.1.B]:

> Claim 2.1.C: Let m be an integer such that $m \geq 2$. Suppose that Theorem 2.1 holds in the case where $n = m$. Then Theorem 2.1 holds in the case where $n = m + 1$.

Indeed, let us note that, by condition (c), α_1 preserves the kernel of the natural surjection $\Pi_U \twoheadrightarrow \Pi_X$. Moreover, since $m \geq 2$, it follows from [CbTpI], Theorem A, (ii) [cf. also condition (c); [CbTpI], Theorem A, (i); [CbTpII], Lemma 3.2, (iv)], that the automorphism $\tilde{\alpha}|_{\Pi_{V_m}} \in \mathrm{Aut}(\Pi_{V_m})$ [induced by $\tilde{\alpha} \in \mathrm{Aut}^{\mathrm{F}}(\Pi_{U_{m+1}})$ via the injection $\Pi_{V_m} \hookrightarrow \Pi_{U_{m+1}}$ in the above commutative diagram] induces an automorphism of Π_V that induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of Π_V. On the other hand, since X_x is an affine hyperbolic curve, it follows from the induction hypothesis that the automorphism $\tilde{\alpha}|_{\Pi_{V_m}} \in \mathrm{Aut}(\Pi_{V_m})$ preserves the kernel of the surjection $\Pi_{V_m} \to \Pi_{(X_x)_m}$. Thus, since $\Pi_{(X_x)_m}$ is center-free [cf. [MT], Proposition 2.2, (ii)], we conclude that $\tilde{\alpha}$ induces an automorphism of $\Pi_{X_{m+1}} = \Pi_{(X_x)_m} \overset{\mathrm{out}}{\rtimes} \Pi_X$. This completes the proof of Claim 2.1.C, hence of Theorem 2.1. $\Box$

COROLLARY 2.2 (Group-theoreticity of cuspidal inertia subgroups in configuration space groups of genus 0). *In the notation of Theorem 2.1, suppose that $g = 0$ [so $r_U \geq 4$]. Then*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{U_n}) = \mathrm{Out}^{\mathrm{F}}(\Pi_{U_n})$$

[cf. [CmbCsp], Definition 1.1, (ii); [CbTpII], Theorem A, (ii), in the case where $n \geq 3$]. In particular,

$$\begin{aligned} \mathrm{Out}(\Pi_{U_n}) &= \mathrm{Out}^{\mathrm{gF}}(\Pi_{U_n}) \times \mathfrak{S}_n \\ &= \mathrm{Out}^{\mathrm{F}}(\Pi_{U_n}) \times \mathfrak{S}_n \\ &= \mathrm{Out}^{\mathrm{FC}}(\Pi_{U_n}) \times \mathfrak{S}_n \end{aligned}$$

[cf. [CbGT], Corollary B].

Proof. Write

$$p_{1,\ldots,n-1} : \Pi_{U_n} \twoheadrightarrow \Pi_{U_{n-1}}$$

for the surjection induced by the projection $U_n \to U_{n-1}$ obtained by forgetting the n-th factor. Let Z be a hyperbolic curve over k of genus 0 that arises as a fiber of the projection $U_{n-1} \to U_{n-2}$ obtained by forgetting the $(n-1)$-th factor. Write Z_2 for the second configuration space associated to Z; $p_Z : \Pi_{Z_2} \twoheadrightarrow \Pi_Z$ for the surjection induced by the first projection $Z_2 \to Z$. Then, for suitable choices of basepoints, we obtain a commutative diagram of homomorphisms of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Ker}(p_Z) & \longrightarrow & \Pi_{Z_2} & \xrightarrow{p_Z} & \Pi_Z & \longrightarrow & 1 \\
 & & \wr\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}(p_{1,\ldots,n-1}) & \longrightarrow & \Pi_{U_n} & \xrightarrow{p_{1,\ldots,n-1}} & \Pi_{U_{n-1}} & \longrightarrow & 1.
\end{array}$$

Thus, by replacing U by Z and applying [CbTpI], Theorem A, (ii), we may assume without loss of generality that $n = 2$.

Let $\beta \in \mathrm{Out}^{\mathrm{F}}(\Pi_{U_2})$. Write $\beta_1 \in \mathrm{Out}(\Pi_U)$ for the outer automorphism induced by β [cf. [CbTpI], Theorem A, (i)]. Observe that, by replacing β by the composite of β with a suitable element $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{U_2})$ [cf. [CmbCsp], Lemma 2.4], we may also assume without loss of generality that β_1 induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of Π_U.

In the remainder of the proof, we use the notation in the proof of Claim 2.1.A in the proof of Theorem 2.1 in the case where $(g, r_X) = (0, 3)$ and $\alpha \stackrel{\mathrm{def}}{=} \beta$. Observe that it follows from Claim 2.1.A that $\alpha \in \mathrm{Out}^{\mathrm{FwC}}(\Pi_{U_2})$ [cf. [CbTpII], Definition 2.1, (ii)].

Suppose that $y, z \in \mathrm{Cusp}(U)$, where $y \neq z$, arise from cusps of X. Then it suffices to prove that the outer automorphism $\alpha_y \in \mathrm{Out}(\Pi_{\mathcal{G}_y})$ [which preserves and fixes the conjugacy classes of verticial subgroups of $\Pi_{\mathcal{G}_y}$] preserves and fixes the conjugacy class of cuspidal inertia subgroups of $\Pi_{\mathcal{G}_y}$ associated to x, i.e., the cusp associated to the diagonal divisor of $Y_2^{\log}$. Let $\Pi_{v_y^{\mathrm{new}}} \subseteq \Pi_{\mathcal{G}_y}$ be a verticial subgroup associated to v_y^{new}; $\tilde{\alpha}_y^{\mathrm{new}} \in \mathrm{Aut}(\Pi_{\mathcal{G}_y})$ a lifting of α_y such that $\tilde{\alpha}_y^{\mathrm{new}}(\Pi_{v_y^{\mathrm{new}}}) = \Pi_{v_y^{\mathrm{new}}}$. Write

$$\tilde{\alpha}_X \in \mathrm{Aut}^{\mathrm{FwC}}(\Pi_{X_2})$$

for the automorphism induced by $\tilde{\alpha} \in \mathrm{Aut}^{\mathrm{FwC}}(\Pi_{U_2})$ and the natural surjection $\phi_2 : \Pi_{U_2} \twoheadrightarrow \Pi_{X_2}$ [cf. Theorem 2.1]. Write $T \supseteq X_x$ for the tripod over k obtained by eliminating the cusp z of X_x. Then it follows immediately from the various definitions involved that the composite

$$\Pi_{v_y^{\mathrm{new}}} \subseteq \Pi_{\mathcal{G}_y} \overset{\sim}{\leftarrow} \Pi_V \twoheadrightarrow \Pi_{X_x} \twoheadrightarrow \Pi_T$$

— where $\Pi_V \twoheadrightarrow \Pi_{X_x}$ (respectively, $\Pi_{X_x} \twoheadrightarrow \Pi_T$) denotes the natural outer surjection induced by the natural open immersion $V \hookrightarrow X_x$ (respectively, $X_x \hookrightarrow T$) — determines an outer isomorphism $\Pi_{v_y^{\mathrm{new}}} \overset{\sim}{\rightarrow} \Pi_T$ that induces a bijection between the respective sets of cuspidal inertia subgroups and is compatible with the outer automorphisms [of $\Pi_{v_y^{\mathrm{new}}}$, Π_T, respectively] induced by $\tilde{\alpha}_y^{\mathrm{new}}$ and the restriction $\tilde{\alpha}_X|_{\Pi_{X_x}}$ of $\tilde{\alpha}_X$ to Π_{X_x} [cf. Claim 2.1.A]. On the other hand, since $\tilde{\alpha}_X \in \mathrm{Aut}^{\mathrm{FwC}}(\Pi_{X_2}) = \mathrm{Aut}^{\mathrm{FC}}(\Pi_{X_2})$ [cf. [CbTpII], Theorem A, (ii)], it follows that $\tilde{\alpha}_X$ preserves and fixes the conjugacy classes of the cuspidal inertia subgroups of Π_{X_x} [cf. condition (c); [CmbCsp], Proposition 1.2, (iii); [CbTpII], Lemma 3.2, (iv)], hence of Π_T. Thus, we conclude that $\tilde{\alpha}_y^{\mathrm{new}}$ preserves and fixes the conjugacy classes of cuspidal inertia subgroups of $\Pi_{v_y^{\mathrm{new}}}$, hence that $\alpha_y \in \mathrm{Out}(\Pi_{\mathcal{G}_y})$ preserves and fixes the conjugacy class of cuspidal inertia subgroups of $\Pi_{\mathcal{G}_y}$ associated to x. This completes the proof of Corollary 2.2. □

Remark 2.2.1. One verifies immediately that Theorem 2.1 and Corollary 2.2, as well as their proofs, go through without change when the various "Π's" are replaced by their respective *maximal pro-l quotients*, for some prime number l. We leave the routine details to the reader. On the other hand, in the present paper, we shall not need these pro-l versions of Theorem 2.1 and Corollary 2.2.

3. Various Properties of Closed Subgroups of the Grothendieck-Teichmüller Group

In this section, we apply the technique developed in [Tsjm], §1, i.e., *combinatorial Belyi cuspidalization*, to give a purely combinatorial/group-theoretic definition of certain classes of closed subgroups of GT [cf. Definition 3.3]. Moreover, we prove a certain relationship between two of these classes [cf. Corollary 3.7] by applying Theorem 2.1.

Write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$; X_n for the n-th configuration space associated to X, where $n \geq 2$ denotes a positive integer; $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$ for the *Grothendieck-Teichmüller group* [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1]. Then recall from the first display of [CbGT], Corollary C, that we have a natural inclusion $\mathrm{GT} \hookrightarrow \mathrm{Out}(\Pi_{X_n})$. We shall write $\mathrm{GT}_n \subseteq \mathrm{Out}(\Pi_{X_n})$ for the image of this inclusion.

Corollary 3.1 (Purely combinatorial/group-theoretic reconstruction of the symmetric group). *For each positive integer m, write $\mathfrak{S}_m$ for the symmetric group on m letters; $\mathfrak{A}_m$ ($\subseteq \mathfrak{S}_m$) for the alternating group on m letters. Let us regard $\mathfrak{A}_{n+3} \subseteq \mathfrak{S}_{n+3}$ as subgroups of $\mathrm{Out}(\Pi_{X_n})$ via the natural injection $\mathfrak{S}_{n+3} \hookrightarrow \mathrm{Out}(\Pi_{X_n})$ induced by the natural action of $\mathfrak{S}_{n+3}$ on X_n [cf. [CbGT], Remark 2.1.1]. Let*

$$\psi_n : \mathrm{Out}(\Pi_{X_n}) \twoheadrightarrow \mathfrak{S}_{n+3}$$

be a representative of the outer surjection ξ_n induced by the natural action of $\mathrm{Out}(\Pi_{X_n})$ on the set of generalized fiber subgroups of length 1 [cf. [CbGT], Theorem A, (i), (ii)]. Then the following hold:

(i) Write

$$F \subseteq \Pi_{X_n}$$

for the generalized fiber subgroup of co-length 1 associated to the subset $\{5, \ldots, n+3\} \subseteq \{1, \ldots, n+3\}$ of labels of cardinality $n-1$ [cf. [CbGT], Theorem A, (i), (ii); [CbGT], Definition 2.1, (ii)]. Let

$$\alpha \in \mathrm{Out}(\Pi_{X_n})$$

be an outer automorphism of Π_{X_n} such that $\psi_n(\alpha) = (1\ 2)(3\ 4)$, and α induces the identity outer automorphism of Π_{X_n}/F ($\xrightarrow{\sim} \Pi_X$) via the natural surjection $\Pi_{X_n} \twoheadrightarrow \Pi_{X_n}/F$. Then

$$\alpha = (1\ 2)(3\ 4) \in \mathfrak{A}_{n+3} \subseteq \mathfrak{S}_{n+3} \subseteq \mathrm{Out}(\Pi_{X_n})$$

[cf. [CmbCsp], Corollary 4.2, (ii); the first display of [CbGT], Corollary C; [CbGT], Definition 2.7], and the subgroup $\mathfrak{A}_{n+3} \subseteq \mathrm{Out}(\Pi_{X_n})$ may be reconstructed, in a purely combinatorial/group-theoretic way, from Π_{X_n} as the subgroup of $\mathrm{Out}(\Pi_{X_n})$ generated by the $\mathrm{Out}(\Pi_{X_n})$-conjugacy class of α [which depends only on the outer surjection ξ_n].

(ii) Suppose that $n \geq 3$. Write

$$F \subseteq \Pi_{X_n}$$

for the generalized fiber subgroup of length 2 associated to the subset $\{1, 2\} \subseteq \{1, \ldots, n+3\}$ of labels of cardinality 2 [cf. [CbGT], Theorem A, (i), (ii); [CbGT], Definition 2.1, (ii)]. Let

$$\alpha \in \mathrm{Out}(\Pi_{X_n})$$

be an outer automorphism of Π_{X_n} such that $\psi_n(\alpha) = (1\ 2)$, and α induces the identity outer automorphism of Π_{X_n}/F ($\xrightarrow{\sim} \Pi_{X_{n-2}}$) via the natural surjection $\Pi_{X_n} \twoheadrightarrow \Pi_{X_n}/F$. Then

$$\alpha = (1\ 2) \in \mathfrak{S}_{n+3} \subseteq \mathrm{Out}(\Pi_{X_n})$$

[cf. [CmbCsp], Corollary 4.2, (ii); the first display of [CbGT], Corollary C; [CbGT], Definition 2.7], and the subgroup $\mathfrak{S}_{n+3} \subseteq \mathrm{Out}(\Pi_{X_n})$ may be reconstructed, in a purely combinatorial/group-theoretic way, from Π_{X_n} as the subgroup of $\mathrm{Out}(\Pi_{X_n})$ generated by the $\mathrm{Out}(\Pi_{X_n})$-conjugacy class of α [which depends only on the outer surjection ξ_n].

Proof. Write $\mathfrak{A} \subseteq \mathrm{Out}(\Pi_{X_n})$ (respectively, $\mathfrak{S} \subseteq \mathrm{Out}(\Pi_{X_n})$) for the subgroup constructed by the algorithm of assertion (i) (respectively, assertion (ii)). Then it follows immediately from the well-known structure of $\mathfrak{S}_{n+3}$ [where we recall that $n + 3 \geq 5$] that $\mathfrak{A}_{n+3} \subseteq \mathfrak{A}$ (respectively, $\mathfrak{S}_{n+3} \subseteq \mathfrak{S}$). [Here, we recall that the kernel of the unique outer surjection $\mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3$ [through which the natural outer action of $\mathfrak{S}_4$ on Π_X factors] is normally generated by $(1\ 2)(3\ 4)$.] On the other hand, by applying the first display of [CbGT], Corollary C, we conclude that $\mathfrak{A}_{n+3} = \mathfrak{A}$ (respectively, $\mathfrak{S}_{n+3} = \mathfrak{S}$). This completes the proof of Corollary 3.1. $\square$

Remark 3.1.1. In the second display of [CbGT], Corollary C, the subgroup $\mathfrak{S}_{n+3} \subseteq \mathrm{Out}(\Pi_{X_n})$ is reconstructed by forming the *local center* $Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_{X_n}))$ of $\mathrm{Out}(\Pi_{X_n})$. This local center is calculated by applying the *Grothendieck Conjecture for hyperbolic curves over number fields* [cf. [LocAn], Theorem A; [Tama], Theorem 0.4]. On the other hand, if $n \geq 3$, then, by applying the algorithm given in Corollary 3.1, (ii), the subgroup

$\mathfrak{S}_{n+3} \subseteq \mathrm{Out}(\Pi_{X_n})$ may be reconstructed, in a purely combinatorial/group-theoretic way, from Π_{X_n} *without applying the Grothendieck Conjecture for hyperbolic curves over number fields.* In fact, moreover, by regarding Π_{X_2} [cf. Corollary 3.1, (ii); [CbGT], Theorem A, (i), (ii); the first display of [CbGT], Corollary C] as an object reconstructed from Π_{X_3} in a purely combinatorial/group-theoretic way, one concludes that this assumption that $n \geq 3$ is *unnecessary* [cf. the discussion of Remark 4.5.1, (i), below]. Finally, we recall from the theory of [CbGT] that [unlike the *second display* of [CbGT], Corollary C!] the *first display* of [CbGT], Corollary C, is proved in [CbGT] *without applying the Grothendieck Conjecture for hyperbolic curves over number fields.*

Definition 3.2. Let n be an integer such that $n \geq 2$; k an algebraically closed field of characteristic 0; U a hyperbolic curve over k. Write U_n for the n-th configuration space associated to U. Recall the subgroup

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_{U_n}) \subseteq \mathrm{Out}(\Pi_{U_n})$$

[cf. [CbGT], Definition 2.1, (iv)]. Then we shall write

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_{U_n})^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{gF}}(\Pi_{U_n})$$

for the subgroup of elements that induce outer automorphisms of Π_U that preserve and fix the conjugacy classes of cuspidal inertia subgroups of Π_U [cf. [CbTpI], Theorem A, (i), (ii)].

Definition 3.3. Let $J \subseteq \mathrm{GT}$ be a closed subgroup of GT; N (respectively, $N^\dagger$) a normal open subgroup of J;

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \big\downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & & \end{array}$$

(respectively,

$$\begin{array}{ccc} \Pi_{U^\dagger} \overset{\mathrm{out}}{\rtimes} N^\dagger & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N^\dagger \\ \big\downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N^\dagger) & & \end{array}$$

an *arithmetic Belyi diagram* [cf. [Tsjm], Definition 1.4, where we take "M" to be N (respectively, $N^\dagger$), and we note that the "N" of *loc. cit.* does *not necessarily coincide* with the N of the present discussion; Remark 3.3.2 below], which we denote by $\mathbb{B}^{\rtimes}$ (respectively, ${}^\dagger\mathbb{B}^{\rtimes}$). Here, we recall that the notion of an *arithmetic Belyi diagram* may be understood as an abstract group-theoretic/combinatorial version of the notion of a scheme-theoretic diagram consisting of an open immersion [i.e., the horizontal arrow] of a finite étale covering of X [i.e., the vertical arrow] into X itself [where we think of the base field $\overline{\mathbb{Q}}$ as a direct limit of finite extensions of $\mathbb{Q}$].

(i) Write U_2 (respectively, $U_2^\dagger$) for the second configuration space associated to U (respectively, $U^\dagger$); $p : \Pi_{U_2} \to \Pi_U$ (respectively, $p^\dagger : \Pi_{U_2^\dagger} \to \Pi_{U^\dagger}$) for the outer surjection induced by the first projection. Note that it follows from Remark 3.3.4 below that there exists a(n) [unique — cf. the final portion of Remark 3.3.4] outer action $N \to \mathrm{Out}^{\mathrm{gF}}(\Pi_{U_2})$ (respectively, $N^\dagger \to \mathrm{Out}^{\mathrm{gF}}(\Pi_{U_2^\dagger})$) which induces the given outer action of N (respectively, $N^\dagger$) on Π_U (respectively, $\Pi_{U^\dagger}$) via the outer surjection p (respectively, $p^\dagger$). Then we shall say that ${}^\dagger\mathbb{B}^{\rtimes}$ *dominates* $\mathbb{B}^{\rtimes}$ if there exist a normal open subgroup

$$M \subseteq N \cap N^\dagger$$

of J and a Π_U-outer surjection

$$\phi : \Pi_{U^\dagger} \overset{\mathrm{out}}{\rtimes} M \twoheadrightarrow \Pi_U \overset{\mathrm{out}}{\rtimes} M$$

such that the following *purely combinatorial/group-theoretic* [cf. Corollary 2.2; the first display of [CbGT], Corollary C] conditions (a), (b) hold:

(a) There exists a [necessarily unique — cf. Proposition 3.4 below; the argument given in the proof of Claims 3.7.A, 3.7.B, 3.7.C in the proof of Corollary 3.7 below; [MT], Theorem 1.5, applied to the images via ϕ_2 of fiber subgroups of length 1; [MT], Proposition 2.4, (v), and its proof [applied in the case of $\Pi_{U_2^\dagger}$]; [CmbCsp], Proposition 1.7, (d) [applied in the case of Π_{U_2}]; [CmbCsp], Propositions 1.2, (iii), and 1.3, (v) [applied in the case of Π_{U_2},

$\Pi_{U_2^\dagger}$]; [CmbCsp], Theorem A, (i) [applied in the case of Π_{U_2}]] Π_{U_2}-outer surjection

$$\phi_2 : \Pi_{U_2^\dagger} \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_{U_2} \overset{\text{out}}{\rtimes} M$$

such that

- the diagram of $\Pi_{(-)}$-outer homomorphisms

$$\begin{array}{ccc} \Pi_{U_2^\dagger} \overset{\text{out}}{\rtimes} M & \xrightarrow{\phi_2} & \Pi_{U_2} \overset{\text{out}}{\rtimes} M \\ {\scriptstyle p^\dagger \overset{\text{out}}{\rtimes} \mathrm{id}_M} \big\downarrow & & {\scriptstyle p \overset{\text{out}}{\rtimes} \mathrm{id}_M} \big\downarrow \\ \Pi_{U^\dagger} \overset{\text{out}}{\rtimes} M & \xrightarrow{\phi} & \Pi_U \overset{\text{out}}{\rtimes} M \end{array}$$

  commutes;
- ϕ_2 maps the *fiber subgroups* of $\Pi_{U_2^\dagger}$ to the *fiber subgroups* of Π_{U_2};
- the kernel of ϕ_2 is topologically generated by [certain of the] *cuspidal inertia subgroups* of fiber subgroups of $\Pi_{U_2^\dagger}$ of length 1 [which implies, in particular, that the kernel of ϕ is topologically generated by [certain of the] cuspidal inertia subgroups of $\Pi_{U^\dagger}$];
- the image via ϕ_2 of any cuspidal inertia subgroup of a fiber subgroup of $\Pi_{U_2^\dagger}$ of length 1 is either trivial or a *cuspidal inertia subgroup* of a fiber subgroup of Π_{U_2} of length 1 [which implies, in particular, that the image via ϕ of any cuspidal inertia subgroup of $\Pi_{U^\dagger}$ is either trivial or a cuspidal inertia subgroup of Π_U].

(b) The composite of ϕ with the restriction to $\Pi_U \overset{\text{out}}{\rtimes} M$ of the Π_X-outer surjection

$$\Pi_U \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi_X \overset{\text{out}}{\rtimes} N$$

[i.e., the horizontal arrow in $\mathbb{B}^{\rtimes}$] coincides with the restriction to $\Pi_{U^\dagger} \overset{\text{out}}{\rtimes} M$ of the Π_X-outer surjection

$$\Pi_{U^\dagger} \overset{\text{out}}{\rtimes} N^\dagger \twoheadrightarrow \Pi_X \overset{\text{out}}{\rtimes} N^\dagger$$

[i.e., the horizontal arrow in ${}^\dagger\mathbb{B}^{\rtimes}$].

In this situation, we shall refer to $\phi : \Pi_{U^\dagger} \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_U \overset{\text{out}}{\rtimes} M$ as an *arithmetic domination* [of $\mathbb{B}^{\rtimes}$ by ${}^\dagger\mathbb{B}^{\rtimes}$] and to the Π_U-outer surjection $\phi_\Pi : \Pi_{U^\dagger} \twoheadrightarrow \Pi_U$ obtained by restricting ϕ to $\Pi_{U^\dagger}$ [a restriction whose image lies in Π_U, by either condition (a) or (b)] as a *geometric domination* [of $\mathbb{B}^{\rtimes}$ by ${}^\dagger\mathbb{B}^{\rtimes}$]. [Here, we observe in passing that it follows immediately from the definition of "$\overset{\text{out}}{\rtimes}$" that [up to possibly replacing M by an open subgroup of M that is normal in J] ϕ is *uniquely determined* by ϕ_Π, ${}^\dagger\mathbb{B}^{\rtimes}$, and $\mathbb{B}^{\rtimes}$.]

(ii) We shall say that the pair $(\mathbb{B}^{\rtimes}, {}^\dagger\mathbb{B}^{\rtimes})$ satisfies the *COF-property* [i.e., "cofiltered property"] if the pair $(\mathbb{B}^{\rtimes}, {}^\dagger\mathbb{B}^{\rtimes})$ satisfies the following condition:

- there exist a normal open subgroup $N^\ddagger$ of J and an *arithmetic Belyi diagram* ${}^\ddagger\mathbb{B}^{\rtimes}$

$$\begin{array}{ccc} \Pi_{U^\ddagger} \overset{\text{out}}{\rtimes} N^\ddagger & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N^\ddagger \\ \downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} N^\ddagger & & \end{array}$$

  such that ${}^\ddagger\mathbb{B}^{\rtimes}$ *dominates* $\mathbb{B}^{\rtimes}$ and ${}^\dagger\mathbb{B}^{\rtimes}$.

(iii) We shall say that the pair $(\mathbb{B}^{\rtimes}, {}^\dagger\mathbb{B}^{\rtimes})$ satisfies the *RGC-property* [i.e., "Relative Grothendieck Conjecture property"] if the pair $(\mathbb{B}^{\rtimes}, {}^\dagger\mathbb{B}^{\rtimes})$ satisfies the following condition:

- the cardinality of the set of *geometric dominations* [cf. (i)] of $\mathbb{B}^{\rtimes}$ by ${}^\dagger\mathbb{B}^{\rtimes}$ is ≤ 1.

(iv) Write $\mathrm{Cusp}(\Pi_U)$ (respectively, $\mathrm{Cusp}(\Pi_X)$) for the set of cusps of Π_U (respectively, Π_X) [cf. [Tsjm], Theorem 1.3, (i)]. Note that the horizontal arrow in $\mathbb{B}^{\rtimes}$ induces a natural injection $\mathrm{Cusp}(\Pi_X) = \{0, 1, \infty\} \hookrightarrow \mathrm{Cusp}(\Pi_U)$; we shall regard $\mathrm{Cusp}(\Pi_X)$ as a subset of $\mathrm{Cusp}(\Pi_U)$ via this injection. Let $T \subseteq \mathrm{Cusp}(\Pi_U) \setminus \mathrm{Cusp}(\Pi_X)$. Write $I(\Pi_U)$ for the set of cuspidal inertia subgroups of Π_U [cf. [Tsjm], Theorem 1.3, (i)]. Thus, $\mathrm{Cusp}(\Pi_U)$ may be identified with $I(\Pi_U)/\Pi_U$.

Write $\Pi_U \twoheadrightarrow \Pi_T$ for the quotient by the normal closed subgroup topologically generated by the cuspidal inertia subgroups of Π_U associated to the cusps $\in T$; $\Pi_U \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi_T \overset{\text{out}}{\rtimes} N$ for the natural quotient induced by the quotient $\Pi_U \twoheadrightarrow \Pi_T$. For $I_c \in I(\Pi_U)$, write $D_c \overset{\text{def}}{=} N_{\Pi_U \overset{\text{out}}{\rtimes} N}(I_c)$; $D_{T,c}$ for the image of D_c via the quotient $\Pi_U \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi_T \overset{\text{out}}{\rtimes} N$. Then we shall say that the arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$ satisfies the *CS-property* [i.e.,"cuspidal separatedness property"] if, for any T as above, $\mathbb{B}^{\rtimes}$ satisfies the following condition:

- for $I_c, I_{c'} \in I(\Pi_U)$, $D_{T,c}$ is commensurable to $D_{T,c'}$ if and only if there exists $\sigma \in \mathrm{Ker}(\Pi_U \twoheadrightarrow \Pi_T)$ such that $(I_c)^\sigma \overset{\text{def}}{=} \sigma I_c \sigma^{-1} = I_{c'}$.

One verifies immediately that this condition implies that $D_{T,c} \subseteq \Pi_T \overset{\text{out}}{\rtimes} N$ is *commensurably terminal*, hence *normally terminal*.

(v) We shall say that J satisfies the *COF-property* (respectively, the *RGC-property*) if every pair of arithmetic Belyi diagrams satisfies the *COF-property* (respectively, the *RGC-property*). We shall say that J satisfies the *CS-property* if every arithmetic Belyi diagram satisfies the *CS-property*. We shall say that J satisfies the *BC-property* [i.e., "Belyi compatibility property"] if J satisfies the *COF-* and the *RGC-properties*. By a slight abuse of notation, we shall use the notation BGT to denote a closed subgroup of GT that satisfies the *BC-property*. [We refer to Remark 4.4.1 below for some concrete examples.]

(vi) We shall refer to a field K of characteristic 0 as a *conducting field for J* if the image of [any representative of] the natural outer homomorphism $G_K \to G_{\mathbb{Q}}$ in $G_{\mathbb{Q}}$, where we think of $G_{\mathbb{Q}}$ as a subgroup of GT via the natural inclusion

$$G_{\mathbb{Q}} \overset{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$$

[cf. the discussion at the beginning of [Tsjm], Introduction], is contained in some GT-conjugate of J. We shall say that a field K of characteristic 0 satisfies the *ISC-property* if, for any two *distinct* points $y_1, y_2 \in Y(L)$ of a hyperbolic curve Y over a finite field extension

L of K, the Π_Y-conjugacy classes of the corresponding decomposition groups $D_{y_1}, D_{y_2} \subseteq \Pi_Y$ are *distinct.* We shall say that a field K of characteristic 0 satisfies the *ZISC-property* if, for any two *distinct* points $y_1, y_2 \in Y(L)$ of a hyperbolic curve Y of genus 0 over a finite field extension L of K, the Π_Y-conjugacy classes of the corresponding decomposition groups $D_{y_1}, D_{y_2} \subseteq \Pi_Y$ are *distinct.*

Remark 3.3.1. Note that it follows immediately from the various definitions involved that:

(a) each notion defined in Definition 3.3, (i), (ii), (iii) (respectively, Definition 3.3, (iv)), concerning $\mathbb{B}^{\rtimes}$, ${}^{\dagger}\mathbb{B}^{\rtimes}$ (respectively, concerning $\mathbb{B}^{\rtimes}$) is equivalent to the corresponding notion concerning the restrictions of $\mathbb{B}^{\rtimes}$, ${}^{\dagger}\mathbb{B}^{\rtimes}$ (respectively, the restriction of $\mathbb{B}^{\rtimes}$) to arbitrary open subgroups of N, $N^{\dagger}$ (respectively, N) that are normal in J;

(b) each notion defined in Definition 3.3, (v), concerning J is equivalent to the corresponding notion concerning an arbitrary open subgroup of J.

Remark 3.3.2. Let us recall that there are *precisely two* situations in [Tsjm] in which the *Grothendieck Conjecture for hyperbolic curves over number fields* [cf. [LocAn], Theorem A; [Tama], Theorem 0.4] is applied, namely:

(a) Claim 1.3.A in the proof of [Tsjm], Theorem 1.3, (ii) [which is applied in [Tsjm], Definition 1.4, to define the notion of an *arithmetic Belyi diagram*];

(b) the proof of [Tsjm], Theorem 1.3, (iii) [which must be applied in order to give a *purely combinatorial/group-theoretic* construction of the outer isomorphism that is used to identify the *two copies* of Π_X that appear in a *Belyi diagram*].

On the other hand, in Remark 3.3.3 below,

> we shall give a *purely combinatorial/group-theoretic* algorithm for constructing, via the algorithm of Corollary 3.1, (ii), the

> *identifying outer isomorphism* between the *two copies* of Π_X that appear in a Belyi diagram.

In particular, in the context of the theory of the present paper, instead of applying [Tsjm], Theorem 1.3, (iii), one may apply the *purely combinatorial/ group-theoretic* algorithm of Remark 3.3.3, which does not require any use of the Grothendieck Conjecture for hyperbolic curves over number fields [cf. Remark 3.1.1]. In addition, [Tsjm], Theorem 1.3, (ii) [i.e., the compatibility of the *identifying outer isomorphism* between the *two copies* of Π_X with the respective outer actions on the two copies] follows immediately from the *functoriality* of the *purely combinatorial/group-theoretic* algorithm given in Remark 3.3.3 below. Thus, in summary, in the theory of the present paper,

> one may in fact *avoid* any use of the *Grothendieck Conjecture for hyperbolic curves over number fields* when applying the theory/results of [Tsjm] in the present paper.

REMARK 3.3.3. In the following discussion, we use the notation that appears in the statement and proof of [Tsjm], Theorem 1.3.

(i) In the remainder of the present Remark 3.3.3, we shall reconstruct the *identifying outer isomorphism between the copies of* Π_X that appear in a *given Belyi diagram* $\mathbb{B}$ [cf. Remark 3.3.2] — by means of a *purely combinatorial/group-theoretic algorithm* — from [the underlying purely combinatorial/group-theoretic structure of] the collection of data

(a) the *profinite group* Π_{X_3};

(b) the outer surjections $\mathrm{pr}_{i,j} : \Pi_{X_3} \twoheadrightarrow \Pi_{X_2}$, where $(i,j) \in \{(1,2), (1,3),(2,3)\}$, determined by the natural projection $X_3 \twoheadrightarrow X_2$ to the i-th and j-th factors, i.e., to be precise, the *normal closed subgroups* $\mathrm{Ker}(\mathrm{pr}_{i,j}) \subseteq \Pi_{X_3}$, together with the *composite outer isomorphisms*

$$\Pi_{X_3}/\mathrm{Ker}(\mathrm{pr}_{i,j}) \xleftarrow{\sim} \Pi_{X_2} \xrightarrow{\sim} \Pi_{X_3}/\mathrm{Ker}(\mathrm{pr}_{i',j'}),$$

where $(i,j),(i',j') \in \{(1,2),(1,3),(2,3)\}$;

(c) the outer surjections $p_i : \Pi_{X_2} \twoheadrightarrow \Pi_X$ ($i \in \{1, 2\}$) determined by the natural projection $X_2 \twoheadrightarrow X$ to the i-th factor, i.e., to be precise, the *normal closed subgroups* $\mathrm{Ker}(p_1), \mathrm{Ker}(p_2) \subseteq \Pi_{X_2}$, together with the *composite outer isomorphism* $\Pi_{X_2}/\mathrm{Ker}(p_1) \overset{\sim}{\leftarrow} \Pi_X \overset{\sim}{\rightarrow} \Pi_{X_2}/\mathrm{Ker}(p_2)$;

(d) the profinite groups Π_{X_2} and Π_X, i.e., to be precise, the *quotients* of Π_{X_3} discussed in (b) and (c);

(e) *surjections*

$$\mathrm{pr}_1 : \Pi_{X_3} \twoheadrightarrow \Pi_X, \quad \mathrm{pr}_2 : \Pi_{X_3} \twoheadrightarrow \Pi_X, \quad \mathrm{pr}_3 : \Pi_{X_3} \twoheadrightarrow \Pi_X,$$

that represent the respective outer surjections $p_1 \circ \mathrm{pr}_{1,3}$, $p_1 \circ \mathrm{pr}_{2,3}$, $p_2 \circ \mathrm{pr}_{2,3}$.

(f) the *open subgroup* $\Pi_U \subseteq \Pi_X$;

(g) the *subset of labeled elements* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_U)$ [cf, [Tsjm], Theorem 1.3, (i)];

(h) the *subset of labeled elements* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_X)$ [cf, [Tsjm], Theorem 1.3, (i)]

— i.e., *without applying the Grothendieck Conjecture for hyperbolic curves over number fields*. Here, the data (f), (g), (h) correspond to the given Belyi diagram $\mathbb{B}$ [cf. the data "$C(\Pi_X)$" of [Tsjm], Theorem 1.3, (iii)]. Also, we note that any two collections of choices of surjections as in (e) are related to one another by composition with a single inner automorphism of Π_{X_3}. Moreover, by applying Corollary 3.1, (ii); [CbGT], Theorem A, (ii), one may regard the data of (b), (c), (d), (e) as data reconstructed [i.e., by using the action of the symmetric group $\mathfrak{S}_6 \subseteq \mathrm{Out}(\Pi_{X_3})$], up to unique isomorphism, from the data of (a).

(ii) Next, observe that the *identifying outer isomorphism between the copies of* Π_X in $\mathbb{B}$ coincides with the *composite*

$$\Pi_X \overset{\sim}{\leftarrow} \Pi^{\mathrm{ctpd}} \overset{\sim}{\rightarrow} \Pi_U^{\mathrm{ctpd}} \overset{\sim}{\rightarrow} \Pi_X^{\mathrm{ctpd}} \overset{\sim}{\rightarrow} \Pi_X,$$

where the first and the final arrows denote the outer isomorphisms arising from the [*scheme-theoretic!*] isomorphisms of tripods determined by the data of (i), (e), (h) [which may be used to rigidify the correspondences between *cusps*]; the second and the third arrows denote the natural isomorphisms induced, respectively, by the natural outer surjections $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ and $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$. Recall that the open subgroup $\Pi_{V_3} \subseteq \Pi_{X_3}$ is defined to be the inverse image of the open subgroup $\Pi_U^{\times 3} \subseteq \Pi_X^{\times 3}$ [determined by the open subgroup $\Pi_U \subseteq \Pi_X$] via the surjection $\Pi_{X_3} \twoheadrightarrow \Pi_X^{\times 3}$ determined by the surjection $\mathrm{pr}_i : \Pi_{X_3} \twoheadrightarrow \Pi_X$, where $i = 1, 2, 3$. Thus, to reconstruct the above *composite* in a purely combinatorial/group-theoretic way, it suffices to reconstruct the following data:

(a) the 3-*central tripods* $\subseteq \Pi_{X_3}$ [i.e., such as Π^{ctpd}];

(b) the *kernel of the natural outer surjection* $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ [which allows us to characterize Π^{ctpd} [cf. Claim 1.3.C in the proof of [Tsjm], Theorem 1.3, (ii)] and reconstruct Π_U^{ctpd}];

(c) the *outer isomorphism* $\Pi_X \overset{\sim}{\leftarrow} \Pi^{\mathrm{ctpd}}$;

(d) the *kernel of the natural outer surjection* $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$ [which allows us to reconstruct Π_X^{ctpd}];

(e) the *outer isomorphism* $\Pi_X^{\mathrm{ctpd}} \overset{\sim}{\rightarrow} \Pi_X$, where we regard both "$\Pi_X^{\mathrm{ctpd}}$" and "$\Pi_X$" as *subquotients* of

$$\Pi_3 \overset{\mathrm{def}}{=} \Pi_{U_3}/\mathrm{Ker}(\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}) \ (\overset{\sim}{\rightarrow} \Pi_{X_3}).$$

(iii) The data of (ii), (a), may be reconstructed by applying the algorithm implicit in the proof of [CbTpII], Theorem 3.16, (v) [cf. also [CbGT], Corollary B], where we allow the central tripod "T" of [CbTpII], Theorem 3.16, (v), to vary among all 3-central tripods. [Indeed, the proof of Claim 3.16.B in the proof of [CbTpII], Theorem 3.16, (v), consists precisely of a reconstruction algorithm for the 3-central tripods.] Once the data of (ii), (b) (respectively, (d)), has been reconstructed, the data of (ii), (c) (respectively, (e)), may be reconstructed by using the action of the symmetric group $\mathfrak{S}_6 \subseteq \mathrm{Out}(\Pi_{X_3})$ (respectively, $\mathfrak{S}_6 \subseteq \mathrm{Out}(\Pi_3)$) [cf. Corollary 3.1, (ii); the construction of the *geometric outer isomorphism* "$\Pi_{v^{\mathrm{new}}} \overset{\sim}{\rightarrow} \Pi_v$" in the proof of [CbTpII], Lemma 3.13, (iii)]. Thus, it suffices to reconstruct the data of (ii), (b), (d) [cf. (v), (vi), below].

(iv) Recall the set I_{X_3} of inertia subgroups $\subseteq \Pi_{X_3}$ of the discussion immediately following Claim 1.3.B in the proof of [Tsjm], Theorem 1.3, (ii). Write

$$I^{\mathrm{F}}_{X_3} \subseteq I_{X_3}$$

for the subset consisting of inertia subgroups $\subseteq \mathrm{Ker}(\mathrm{pr}_{i,j})$ for some $(i,j) \in \{(1,2),(1,3),(2,3)\}$. Let $(i,j) \in \{(1,2),(1,3),(2,3)\}$. Recall from [CbGT], Theorem A, (ii); the first display of [CbGT], Corollary C, that

(a) the image $\mathrm{GT}_3 \subseteq \mathrm{Out}(\Pi_{X_3})$

of the natural inclusion $\mathrm{GT} \hookrightarrow \mathrm{Out}(\Pi_{X_3})$ may be reconstructed from the data of (i), (a). Next, observe that the natural outer action of $\mathrm{GT}_3 = \mathrm{Out}^{\mathrm{gF}}(\Pi_{X_3})$ on Π_{X_3} stabilizes $\mathrm{Ker}(\mathrm{pr}_{i,j}) \subseteq \Pi_{X_3}$, as well as the set of *cuspidal inertia subgroups* of $\mathrm{Ker}(\mathrm{pr}_{i,j})$ [cf. [CbTpII], Theorem A, (ii)], hence determines

(b) an outer representation $\Pi_{X_3} \overset{\mathrm{out}}{\rtimes} \mathrm{GT}_3 \rightarrow \mathrm{Out}^{\mathrm{C}}(\mathrm{Ker}(\mathrm{pr}_{i,j}))$ [cf. [CbTpI], Definition 6.1],

which is l-cyclotomically full [cf. [CmbGC], Definition 2.3, (ii), where we regard $\mathrm{Ker}(\mathrm{pr}_{i,j})$ as the étale fundamental group of a geometric fiber of $\mathrm{pr}_{i,j}$, i.e., a smooth affine curve over an algebraically closed field of characteristic 0, which implies *formally*, from the definitions of the notation involved, that "$\mathrm{Out}^{\mathrm{C}}(\mathrm{Ker}(\mathrm{pr}_{i,j}))$" in the present discussion corresponds precisely to the notation "$\mathrm{Aut}(\mathcal{G})$" in [CmbGC], Definition 2.3] for any prime number l [where we apply the fact that $G_{\mathbb{Q}} \subseteq \mathrm{GT}$]. In particular, by applying the algorithm implicit in the proof of [CmbGC], Corollary 2.7, (i), we conclude that the cuspidal inertia subgroups of $\mathrm{Ker}(\mathrm{pr}_{i,j})$ may be reconstructed group-theoretically from the data of (b). Thus, by varying $(i,j) \in \{(1,2),(1,3),(2,3)\}$, we conclude that

(c) the inertia subgroups $\in I^{\mathrm{F}}_{X_3}$

may be reconstructed group-theoretically from the data of (i), (a), (b), (c), (d).

(v) Next, we reconstruct the data of (ii), (b). Let $I \in I^{\mathrm{F}}_{X_3}$ be such that, for each $h = 1, 2, 3$, $\mathrm{pr}_h(I) = \{1\}$. Then there exists a unique pair $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ such that $\mathrm{pr}_{i,j}(I) \neq \{1\}$. Write

- $\Pi_W \subseteq \Pi_X$ for the maximal normal open subgroup such that $\Pi_W \subseteq \Pi_U$;
- $\Pi_{Z_3} \stackrel{\mathrm{def}}{=} \Pi_{X_3} \times_{(\Pi_X \times \Pi_X \times \Pi_X)} (\Pi_W \times \Pi_W \times \Pi_W) \subseteq \Pi_{X_3}$, i.e., the inverse image via the surjection $\Pi_{X_3} \twoheadrightarrow \Pi_X \times \Pi_X \times \Pi_X$ induced by p_1, p_2, and p_3 of the open subgroup $\Pi_W \times \Pi_W \times \Pi_W \subseteq \Pi_X \times \Pi_X \times \Pi_X$ [determined by the inclusion $\Pi_W \subseteq \Pi_X$];

Note that $I \subseteq \Pi_{Z_3} \subseteq \Pi_{V_3} \subseteq \Pi_{X_3}$. Then it follows from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.2, (iii), that pr_i and pr_j induce natural isomorphisms

$$g_{i,I} : N_{\Pi_{Z_3}}(I)/I \cdot \big(\mathrm{Ker}(\mathrm{pr}_{i,j}) \cap N_{\Pi_{Z_3}}(I)\big) \xrightarrow{\sim} \Pi_W,$$

$$g_{j,I} : N_{\Pi_{Z_3}}(I)/I \cdot \big(\mathrm{Ker}(\mathrm{pr}_{i,j}) \cap N_{\Pi_{Z_3}}(I)\big) \xrightarrow{\sim} \Pi_W,$$

and that the outer automorphism of Π_W determined by $g_{j,I} \circ g_{i,I}^{-1}$ coincides with the outer automorphism determined by a(n) [unique] element $g \in \Pi_X/\Pi_W$. [That is to say, at a more conceptual level, one may think of the various groups that appear in the above display as decomposition groups of various *Galois [i.e.,* Π_X/Π_W*-] conjugates* of the (i, j)*-diagonal* of $W \times W \times W$.] Next, for each $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ and $g \in \Pi_X/\Pi_W$, we shall write

$$I_{i,j;g} \subseteq I^{\mathrm{F}}_{X_3}$$

for the subset consisting of the elements $I \in I^{\mathrm{F}}_{X_3}$ such that

- for each $h = 1, 2, 3$, $\mathrm{pr}_h(I) = \{1\}$;
- $\mathrm{pr}_{i,j}(I) \neq \{1\}$;
- $g_{j,I} \circ g_{i,I}^{-1}$ coincides with the outer automorphism of Π_W determined by $g \in \Pi_X/\Pi_W$.

Then we may reconstruct the kernel of the natural surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ as the normal closed subgroup of Π_{V_3} topologically normally generated by the elements of the subset

$$\bigcup_{i,j;\ g \notin \Pi_U/\Pi_W} I_{i,j;g} \subseteq I^{\mathrm{F}}_{X_3}.$$

(vi) Finally, we reconstruct the data of (ii), (d). Write

- $I^{\mathrm{F}}_{V_3} \stackrel{\mathrm{def}}{=} \{I \cap \Pi_{V_3} \ (\subseteq \Pi_{X_3}) \mid I \in I^{\mathrm{F}}_{X_3}\}$;
- $I^{\mathrm{F}}_{U_3}$ for the set of images of elements of $I^{\mathrm{F}}_{V_3}$ via the natural surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ [cf. (v)].

On the other hand, for each $i = 1, 2, 3$, pr_i naturally induces an outer surjection $q_i : \Pi_{U_3} \twoheadrightarrow \Pi_U$. Thus, we may reconstruct the kernel of the natural outer surjection $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$ as the normal closed subgroup topologically generated by the elements $I \in I^{\mathrm{F}}_{U_3}$ satisfying the following condition:

> there exists $i \in \{1, 2, 3\}$ such that $q_i(I) \subseteq \Pi_U$ is a cuspidal inertia subgroup that is not associated to 0, 1, ∞ [cf. (i), (g)].

Remark 3.3.4. We maintain the notation of Remark 3.3.3. Let $J \subseteq \mathrm{GT}$ be a closed subgroup; N a normal open subgroup of J;

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \big\downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & & \end{array}$$

an *arithmetic Belyi diagram*, which we denote by $\mathbb{B}^{\rtimes}$ [i.e., whose underlying *Belyi diagram* is the Belyi diagram $\mathbb{B}$ of Remark 3.3.3, (i)]. Recall the notation U_2 (respectively, X_2) for the second configuration space associated to U (respectively, X); write $p_U : \Pi_{U_2} \to \Pi_U$ (respectively, $p_X : \Pi_{X_2} \to \Pi_X$) for the outer surjection induced by the first projection. Let us recall from [Tsjm], Lemma 1.2, (b) [cf. also [Tsjm], Theorem 1.3, (ii); [Tsjm], Definition 1.4], that the outer action of N on Π_U extends uniquely [cf. the slimness of Π_X] to a Π_U-*outer action on* Π_X that is compatible, relative to the *vertical arrow* of the Belyi diagram $\mathbb{B}$, with the outer action of J ($\supseteq N$) on Π_X. Then observe that this Π_U-*outer action of* N *on* Π_X allows one to construct

- a natural outer action of N on Π_{X_3} that determines an injection $N \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$, together with

- a compatible natural Π_{V_3}-outer action of N on Π_{X_3} that stabilizes Π_{V_3}

[cf. the discussion preceding Claim 1.3.B in the proof of [Tsjm], Theorem 1.3, (ii)]. Next, recall from Remark 3.3.3, (ii), (b), (d) [cf. also Remark 3.3.3, (v), (vi)], that the resulting outer action of N on Π_{V_3} determines injections

$$N \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_{U_3}), \quad N \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$$

compatible with the outer surjections $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$. The F-admissibility of these outer actions implies that these natural outer actions of N on Π_{U_3} and Π_{X_3} determine injections

$$N \hookrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_{U_2})^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_{U_2}),$$

$$N \hookrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_{X_2})^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_2})$$

[cf. Corollary 2.2; Definition 3.2; [CbTpII], Theorem A, (ii)] and a commutative diagram

$$\begin{array}{ccc}
\Pi_{U_2} \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_{X_2} \overset{\mathrm{out}}{\rtimes} N \\
{\scriptstyle p_U \overset{\mathrm{out}}{\rtimes} \mathrm{id}_N} \downarrow & & {\scriptstyle p_X \overset{\mathrm{out}}{\rtimes} \mathrm{id}_N} \downarrow \\
\Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N,
\end{array}$$

where the lower horizontal arrow is the horizontal arrow of $\mathbb{B}^{\rtimes}$. Note that the outer action of N on Π_{U_2} (respectively, Π_{X_2}) just constructed is uniquely determined by the following two conditions [cf. Corollary 2.2; [CbTpII], Theorem A, (ii); [CmbCsp], Theorem A, (i)]:

- the outer action of N on Π_{U_2} (respectively, Π_{X_2}) determines an injection

$$N \hookrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_{U_2})^{\mathrm{cusp}} \quad (\text{respectively, } N \hookrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_{X_2})^{\mathrm{cusp}});$$

- the outer action of N on Π_{U_2} (respectively, Π_{X_2}) induces the given outer action of N on Π_U (respectively, Π_X) via the outer surjection p_U (respectively, p_X).

Proposition 3.4 (Functorial behavior of cuspidal inertia subgroups with respect to geometric dominations). *In the situation of Definition 3.3, (i), every conjugacy class of cuspidal inertia subgroups of* Π_U *arises as the image via* ϕ *of a unique conjugacy class of cuspidal inertia subgroups of* $\Pi_{U^\dagger}$.

Proof. We regard $U, U^\dagger$ as open subschemes of X via the respective natural open immersions $U \hookrightarrow X$, $U^\dagger \hookrightarrow X$. Write $\mathrm{Cusp}(U^\dagger)$ for the set of cusps of $U^\dagger$; $S \subseteq \mathrm{Cusp}(U^\dagger)$ for the subset of cusps $s \in \mathrm{Cusp}(U^\dagger)$ such that some [or equivalently, every] cuspidal inertia subgroup of $\Pi_{U^\dagger}$ associated to s is contained in $\mathrm{Ker}(\phi)$; $U^\dagger \subseteq U^\dagger_S$ $(\subseteq X)$ for the partial compactification of $U^\dagger$ such that $U^\dagger = U^\dagger_S \setminus S$. Thus, the natural outer surjection $\Pi_{U^\dagger} \twoheadrightarrow \Pi_{U^\dagger_S}$ induces a *bijection* between the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{U^\dagger}$ associated to cusps $\in \mathrm{Cusp}(U^\dagger) \setminus S$ and the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{U^\dagger_S}$. Next, observe that it follows immediately from Definition 3.3, (i), (a), (b), that ϕ induces an outer isomorphism

$$\phi_S : \Pi_{U^\dagger_S} \xrightarrow{\sim} \Pi_U$$

such that

(i) ϕ_S maps every cuspidal inertia subgroup of $\Pi_{U^\dagger_S}$ to a cuspidal inertia subgroup of Π_U;

(ii) ϕ_S maps every cuspidal inertia subgroup of $\Pi_{U^\dagger_S}$ associated to 0, 1, ∞ to a cuspidal inertia subgroup of Π_U associated to 0, 1, ∞, respectively.

Thus, to complete the proof of Proposition 3.4, it suffices to verify that ϕ_S induces [cf. (i)] a *bijection* between the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{U^\dagger_S}$ and the set of conjugacy classes of cuspidal inertia subgroups of Π_U. To this end, let us first observe that *injectivity* follows immediately from the fact that ϕ_S is an outer isomorphism. On the other hand, since ϕ_S is an outer isomorphism, *surjectivity* follows immediately, in light of (ii), from the fact that [since the hyperbolic curves $U^\dagger$ and U are of genus 0] $\Pi_{U^\dagger_S}$ and Π_U are topologically freely generated by their respective collections of cuspidal inertia subgroups associated to cusps $\neq \infty$. This completes the proof of Proposition 3.4. $\square$

Proposition 3.5 (Natural action of GT on the set of geometric dominations). *In the notation of Definition 3.3, (i), one may construct a natural action of* $C_{\mathrm{GT}}(J)$ ($\subseteq \mathrm{Out}(\Pi_X)$) *on the set of geometric dominations between arbitrary arithmetic Belyi diagrams.*

Proof. Let us consider the data of Remark 3.3.3, (i), (a), (b), (c), (d), (e), (f), (g), (h), associated to $\mathbb{B}^{\rtimes}$ and ${}^{\dagger}\mathbb{B}^{\rtimes}$. Then the data of

- "$\Pi_{U_2} \overset{\mathrm{out}}{\rtimes} M$", "$\Pi_{U_2^{\dagger}} \overset{\mathrm{out}}{\rtimes} M$", together with
- the respective fiber subgroups of length 1 and cuspidal inertia subgroups of such fiber subgroups

[cf. Definition 3.3, (i)] may be reconstructed from the data of Remark 3.3.3, (i), (a), (b); Remark 3.3.3, (ii), (b); Remark 3.3.3, (vi) [i.e., "$I^{\mathrm{F}}_{U_3}$"]. Thus, Proposition 3.5 follows immediately, in light of the various definitions involved, from the *functoriality* of the *purely combinatorial/group-theoretic* algorithm given in Remark 3.3.3. $\square$

Theorem 3.6 (Faithfulness via the CS-property for certain outer actions on configuration space groups induced by open immersions). *Let* $J \subseteq \mathrm{GT}$ *be a closed subgroup;* N *a normal open subgroup of* J*;*

$$\begin{array}{ccc}
\Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\
\Big\downarrow & & \\
\Pi_X \overset{\mathrm{out}}{\rtimes} N & &
\end{array}$$

an **arithmetic Belyi diagram***, which we denote by* $\mathbb{B}^{\rtimes}$*. Write* U_2 *(respectively,* X_2*) for the second configuration space associated to* U *(respectively,* X*);* $p_U : \Pi_{U_2} \to \Pi_U$ *(respectively,* $p_X : \Pi_{X_2} \to \Pi_X$*) for the outer surjection induced by the first projection. Thus, we have a commutative diagram*

$$\begin{array}{ccc}
\Pi_{U_2} \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_{X_2} \overset{\mathrm{out}}{\rtimes} N \\
{\scriptstyle p_U \overset{\mathrm{out}}{\rtimes} \mathrm{id}_N}\Big\downarrow & & {\scriptstyle p_X \overset{\mathrm{out}}{\rtimes} \mathrm{id}_N}\Big\downarrow \\
\Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N
\end{array}$$

as in Remark 3.3.4. We regard U *as an open subscheme of* X *via the natural open immersion* $U \hookrightarrow X$. *For each sequence*

$$U \subseteq V \subseteq W \subseteq X$$

of open subschemes of X, *write* V_2, W_2 *for the second configuration spaces associated to the hyperbolic curves* V, W, *respectively;*

$$h_{V,W} : \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{gF}}(\Pi_{W_2})^{\mathrm{cusp}}$$

for the homomorphism induced by the upper horizontal arrow of the above commutative diagram [cf. Theorem 2.1; [CbGT], Corollary B; the well-known elementary structure of the natural inclusion $\mathfrak{S}_3 \hookrightarrow \mathfrak{S}_5$*];* $N_{V_2} \subseteq \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}$ *for the image via the composite*

$$N \hookrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_{U_2})^{\mathrm{cusp}} \overset{h_{U,V}}{\to} \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}$$

[cf. Remark 3.3.4]. Suppose that $\mathbb{B}^{\rtimes}$ *satisfies the* **CS-property** *[cf. Definition 3.3, (iv)]. Then, for any* V, W *as above, the composite*

$$Z_{\mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}}(N_{V_2}) \subseteq \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}} \overset{h_{V,W}}{\to} \mathrm{Out}^{\mathrm{gF}}(\Pi_{W_2})^{\mathrm{cusp}}$$

is injective.

PROOF. Write $h \overset{\mathrm{def}}{=} h_{V,W}$; $\mathrm{Cusp}(V)$, $\mathrm{Cusp}(W)$ for the set of cusps of V, W, respectively. First, let us note that we may assume without loss of generality [i.e., by forming the composite of the $h_{V,W}$ for suitable V, W] that the cardinality of the set $\mathrm{Cusp}(V) \setminus \mathrm{Cusp}(W)$ is 1. Let

$$\beta \in Z_{\mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}}(N_{V_2}) \ (\subseteq \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}})$$

be such that $h(\beta) = 1$. Then it suffices to verify that

$$\beta = 1.$$

Note that the natural composites

$$N \overset{\sim}{\to} N_{V_2} \subseteq \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}, \quad N \overset{\sim}{\to} N_{W_2} \subseteq \mathrm{Out}^{\mathrm{gF}}(\Pi_{W_2})^{\mathrm{cusp}}$$

determine natural outer actions of N on Π_{V_2}, Π_{W_2}, hence also on Π_V, Π_W [by applying the natural outer surjections $\Pi_{V_2} \to \Pi_V$, $\Pi_{W_2} \to \Pi_W$ determined by the respective first projections].

Next, let us write

- y for the unique element $\in \mathrm{Cusp}(V) \setminus \mathrm{Cusp}(W)$;

- $\eta_j : \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2}) \to \mathrm{Out}^{\mathrm{gF}}(\Pi_V)$ for the natural homomorphism induced by the j-th projection, where $j \in \{1, 2\}$ [where we note that in fact, $\eta_1 = \eta_2$ — cf. Corollary 2.2; [CmbCsp], Proposition 1.2, (iii)];

- $Y^{\log}$ for the [uniquely determined, up to unique isomorphism] smooth log curve over $\mathrm{Spec}\ \overline{\mathbb{Q}}$ such that $U_Y = V$;

- $Y_2^{\log}$ for the second log configuration space associated to $Y^{\log}$;

- $y^{\log} \stackrel{\mathrm{def}}{=} y \times_Y Y^{\log}$ [where the fiber product is determined by the natural map $Y^{\log} \to Y$ obtained by forgetting the log structure];

- $Y_y^{\log} \stackrel{\mathrm{def}}{=} Y_2^{\log} \times_{Y^{\log}} y^{\log}$ [where the fiber product is determined by the first projection $Y_2^{\log} \to Y^{\log}$ and the natural map $y^{\log} \to Y^{\log}$];

- $\mathcal{G}_y$ for the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by the stable log curve $Y_y^{\log}$ [cf. [CmbGC], Definition 1.1, (i)];

- c_y, c_Δ for the cusps of $\mathcal{G}_y$ that arise from y, the diagonal divisor of $Y_2^{\log}$, respectively;

- v_y for the vertex of $\mathcal{G}_y$ associated to the irreducible component that does not contain c_Δ;

- $\Pi_{\mathcal{G}_y}$ for the PSC-fundamental group of $\mathcal{G}_y$ [cf. [CmbGC], Definition 1.1, (ii)].

Then we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{\mathcal{G}_y} & \longrightarrow & \Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_V \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & 1 \\
& & {\scriptstyle q_y}\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \Pi_V & \longrightarrow & \Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_W \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & 1,
\end{array}$$

where the middle and right-hand vertical arrows denote surjections that represent the outer surjection induced by the natural open immersion $V \hookrightarrow W$; $\Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N \to \Pi_V \overset{\mathrm{out}}{\rtimes} N$, $\Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N \to \Pi_W \overset{\mathrm{out}}{\rtimes} N$ denote surjections that represent the outer surjections induced by the respective first projections; q_y denotes the induced surjection. [Note that $\mathrm{Ker}(q_y)$ coincides with the normal closed subgroup topologically generated by the cuspidal inertia subgroups of $\Pi_{\mathcal{G}_y}$ associated to c_y.]

Since $\beta \in Z_{\mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}}(N_{V_2})$ ($\subseteq \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}$), and Π_{V_2} is center-free [cf. [MT], Proposition 2.2, (ii)], β determines a Π_{V_2}-outer automorphism γ_V of $\Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N$ that lies over the identity automorphism of N. Let I_y be a cuspidal inertia subgroup of Π_V associated to y; $\widetilde{\gamma}_V \in \mathrm{Aut}(\Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N)$ a lifting of γ_V. Write $(\widetilde{\gamma}_V)_1$ for the automorphism of $\Pi_V \overset{\mathrm{out}}{\rtimes} N$ induced by $\widetilde{\gamma}_V$ via the surjection $\Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi_V \overset{\mathrm{out}}{\rtimes} N$ in the above commutative diagram. Then since $\beta \in \mathrm{Out}^{\mathrm{gF}}(\Pi_{V_2})^{\mathrm{cusp}}$, by replacing $\widetilde{\gamma}_V$ by a suitable composite with an inner automorphism of $\Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N$ [determined by an element of Π_{V_2}] if necessary, we may assume without loss of generality that

$$(\widetilde{\gamma}_V)_1(I_y) = I_y.$$

Let $\Pi_{v_y} \subseteq \Pi_{\mathcal{G}_y}$ be a verticial subgroup associated to v_y. Note that since $V \subsetneq W$, v_y is not of type $(0,3)$. Thus, it follows immediately from [CbTpII], Theorem 1.9, (ii), that the restriction $\widetilde{\gamma}_V|_{\Pi_{\mathcal{G}_y}}$ of $\widetilde{\gamma}_V$ to $\Pi_{\mathcal{G}_y}$ preserves and fixes the conjugacy class of Π_{v_y}. Moreover, by replacing $\widetilde{\gamma}_V$ by a suitable composite with an inner automorphism of $\Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N$ [determined by an element of $\Pi_{\mathcal{G}_y}$] if necessary, we may assume without loss of generality that

$$\widetilde{\gamma}_V|_{\Pi_{\mathcal{G}_y}}(\Pi_{v_y}) = \Pi_{v_y}.$$

Write $\widetilde{\gamma}_W \in \mathrm{Aut}(\Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N)$ for the automorphism [that lies over N] induced by $\widetilde{\gamma}_V$ [cf. Theorem 2.1] via the surjection $\Pi_{V_2} \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N$ in the above commutative diagram.

Next, we verify the following assertion:

> Claim 3.6.A: The outer automorphism $\gamma \in \mathrm{Out}(\Pi_V)$ determined by the restriction $\widetilde{\gamma}_W|_{\Pi_V}$ of $\widetilde{\gamma}_W$ to Π_V ($\hookrightarrow \Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N$) coincides with $\eta_2(\beta) \in \mathrm{Out}(\Pi_V)$.

Recall that $\widetilde{\gamma}_V|_{\Pi_{\mathcal{G}_y}}$ preserves the cuspidal inertia subgroups of $\Pi_{\mathcal{G}_y}$ [cf. Corollary 2.2]. Write $q_\Delta : \Pi_{\mathcal{G}_y} \twoheadrightarrow \Pi_V$ for the natural outer surjection induced by the *second projection* $V_2 \to V$. Note that $\mathrm{Ker}(q_\Delta)$ coincides with the normal closed subgroup topologically generated by the cuspidal inertia subgroups of $\Pi_{\mathcal{G}_y}$ associated to c_Δ. On the other hand, it follows immediately from the various definitions involved that

- γ (respectively, $\eta_2(\beta)$) coincides with the outer automorphism induced by $\widetilde{\gamma}_V|_{\Pi_{\mathcal{G}_y}}$ via the surjection q_y (respectively, q_Δ);

- q_y and q_Δ determine the same outer isomorphism $(\Pi_{\mathcal{G}_y} \supseteq)\ \Pi_{v_y} \xrightarrow{\sim} \Pi_V$.

Thus, since $\widetilde{\gamma}_V|_{\Pi_{\mathcal{G}_y}}(\Pi_{v_y}) = \Pi_{v_y}$, we obtain the desired conclusion. This completes the proof of Claim 3.6.A.

Next, observe that since $h(\beta) = 1$, we have

$$\widetilde{\gamma}_W \in \mathrm{Inn}(\Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N) \subseteq \mathrm{Aut}(\Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N),$$

where the inner automorphism $\widetilde{\gamma}_W$ is determined by an element $\in \Pi_{W_2}$. Write

- $(\widetilde{\gamma}_W)_1$ for the inner automorphism of $\Pi_W \overset{\mathrm{out}}{\rtimes} N$ [determined by an element $\in \Pi_W$] induced by $\widetilde{\gamma}_W$ via the surjection $\Pi_{W_2} \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi_W \overset{\mathrm{out}}{\rtimes} N$ in the above commutative diagram;

- D_y $(\xrightarrow{\sim} N)$ [cf. [CmbGC], Proposition 1.2, (ii)] for the image of $N_{\Pi_V \overset{\mathrm{out}}{\rtimes} N}(I_y)$ via the surjection $\Pi_V \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi_W \overset{\mathrm{out}}{\rtimes} N$ in the above commutative diagram.

Then it follows from our assumption that $(\widetilde{\gamma}_V)_1(I_y) = I_y$ that $(\widetilde{\gamma}_W)_1(D_y) = D_y$. Recall that since $\mathbb{B}^{\rtimes}$ satisfies the *CS-property*, D_y is normally terminal in $\Pi_W \overset{\mathrm{out}}{\rtimes} N$ [cf. the final sentence of Definition 3.3, (iv); [CmbGC], Proposition 1.2, (ii)]. Thus, we conclude that the inner automorphism $(\widetilde{\gamma}_W)_1 \in \mathrm{Inn}(\Pi_W \overset{\mathrm{out}}{\rtimes} N)$ is determined by a(n) [unique] element $\in D_y \cap \Pi_W = \{1\}$, hence, in particular, that the inner automorphism $\widetilde{\gamma}_W$ is determined by an element $\in \Pi_V \subseteq \Pi_{W_2}$, i.e., that $\gamma = 1$. Finally, it follows immediately from the injectivity of η_2 [cf. Corollary 2.2; [CmbCsp], Theorem A, (i)], together with Claim 3.6.A, that $\beta = 1$. This completes the proof of Theorem 3.6. $\square$

COROLLARY 3.7 (The CS-property implies the RGC-property). *Let* $J \subseteq \mathrm{GT}$ *be a closed subgroup satisfying the* **CS-property** *[cf. Definition 3.3, (v)]. Then* J *satisfies the* **RGC-property** *[cf. Definition 3.3, (v)].*

PROOF. In the notation of Definition 3.3, (i), let ϕ, ϕ' be *arithmetic dominations* of $\mathbb{B}^{\rtimes}$ by ${}^{\dagger}\mathbb{B}^{\rtimes}$, defined over a normal open subgroup $M \subseteq J$. Then it suffices to prove that $\phi = \phi'$. Since $\mathrm{Ker}(\phi)$ and $\mathrm{Ker}(\phi')$ are topologically generated by [certain of the] cuspidal inertia subgroups of $\Pi_{U^{\dagger}}$ [cf. Definition 3.3, (i), (a)], it follows immediately from the *CS-property* [where we take the "T" of Definition 3.3, (iv), to be "$\mathrm{Cusp}(\Pi_{U^{\dagger}}) \setminus \mathrm{Cusp}(\Pi_X)$", "$\mathrm{Cusp}(\Pi_U) \setminus \mathrm{Cusp}(\Pi_X)$"], together with Definition 3.3, (i), (b) [cf. also Proposition 3.4], that

$$\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi').$$

Fix Π_{U_2}-outer surjections

$$\phi_2 : \Pi_{U_2^{\dagger}} \overset{\mathrm{out}}{\rtimes} M \twoheadrightarrow \Pi_{U_2} \overset{\mathrm{out}}{\rtimes} M, \quad \phi_2' : \Pi_{U_2^{\dagger}} \overset{\mathrm{out}}{\rtimes} M \twoheadrightarrow \Pi_{U_2} \overset{\mathrm{out}}{\rtimes} M$$

[that lie over ϕ, ϕ'] as in Definition 3.3, (i), (a), respectively.

Next, we make the following observation:

> Claim 3.7.A: ϕ_2 and ϕ_2' map the inertia subgroups of $\Pi_{U_2^{\dagger}}$ associated to the diagonal divisor of $U_2^{\dagger}$ isomorphically onto the inertia subgroups of Π_{U_2} associated to the diagonal divisor of U_2.

Indeed, Claim 3.7.A follows immediately from the discussion of Definition 3.3, (i), (a). That is to say, it suffices to show that the inertia subgroups of $\Pi_{U_2^{\dagger}}$ associated to the diagonal divisor of $U_2^{\dagger}$ do not lie in the kernel of ϕ_2 or ϕ_2'. On the other hand, the assumption that any such inertia group lies in the kernel of ϕ_2 or ϕ_2' leads immediately to a contradiction [cf. [MT], Theorem 1.5, applied to the images via ϕ_2 and ϕ_2' of fiber subgroups of length 1; [MT], Proposition, 2.4, (v), and its proof [applied in the case of $\Pi_{U_2^{\dagger}}$]; [CmbCsp], Proposition 1.7, (d) [applied in the case of Π_{U_2}]].

Next, we verify the following assertion:

> Claim 3.7.B: $\mathrm{Ker}(\phi_2) = \mathrm{Ker}(\phi_2')$.

Indeed, write

$$\phi_* : \Pi_{U^\dagger} \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_U \overset{\text{out}}{\rtimes} M, \quad \phi'_* : \Pi_{U^\dagger} \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_U \overset{\text{out}}{\rtimes} M$$

for the Π_U-outer surjections determined by ϕ_2, ϕ'_2, respectively, via the outer surjections $\Pi_{U_2^\dagger} \twoheadrightarrow \Pi_{U^\dagger}$, $\Pi_{U_2} \twoheadrightarrow \Pi_U$ induced by the respective *second projections* [cf. the portion of Definition 3.3, (i), (a) concerning *fiber subgroups*]. Then it follows immediately from Claim 3.7.A, together with a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.2, (iii), that the following assertion holds:

> Claim 3.7.C: $\phi = \phi_*$, $\phi' = \phi'_*$. In particular, $\mathrm{Ker}(\phi_*) = \mathrm{Ker}(\phi) = \mathrm{Ker}(\phi') = \mathrm{Ker}(\phi'_*)$.

Thus, since $\mathrm{Ker}(\phi_2)$ and $\mathrm{Ker}(\phi'_2)$ are topologically generated by [certain of the] cuspidal inertia subgroups of fiber subgroups of $\Pi_{U_2^\dagger}$ of length 1 [cf. Definition 3.3, (i), (a)], we conclude, again from Claim 3.7.A [cf. also [CbTpII], Lemma 3.6, (i), (ii)], that $\mathrm{Ker}(\phi_2) = \mathrm{Ker}(\phi'_2)$. This completes the proof of Claim 3.7.B.

It follows immediately from Claim 3.7.B that there exists a unique Π_{U_2}-outer automorphism $\alpha : \Pi_{U_2} \overset{\text{out}}{\rtimes} M \xrightarrow{\sim} \Pi_{U_2} \overset{\text{out}}{\rtimes} M$ such that $\phi_2 = \alpha \circ \phi'_2$. On the other hand, it follows from the *CS-property*, together with Definition 3.3, (i), (b), that we may apply Theorem 3.6 to conclude that α is the identity, hence that $\phi_2 = \phi'_2$, $\phi = \phi'$. This completes the proof of Corollary 3.7. $\square$

4. Combinatorial Construction of the Field $\overline{\mathbb{Q}}_{\mathrm{BGT}}$

In §3, we defined a certain class of closed subgroups BGT of GT [cf. Definition 3.3, (v)]. In this section, for each such closed subgroup BGT, we give a purely combinatorial/group-theoretic construction of a set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ associated to BGT equipped with *"field-like operations"*, together with a natural action by $C_{\mathrm{GT}}(\mathrm{BGT})$ that is compatible with these operations [cf. Theorem 4.4, (i)]. In particular, when these operations determine a structure of field isomorphic to $\overline{\mathbb{Q}}$, we construct a natural outer homomorphism $C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}} \overset{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [cf. Theorem 4.4, (ii), (iii)].

Write $X \overset{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$.

Definition 4.1. Let $\mathrm{BGT} \subseteq \mathrm{GT}$ be a closed subgroup satisfying the *BC-property* [cf. Definition 3.3, (v)]. For any *arithmetic Belyi diagram* $\mathbb{B}^{\rtimes}$

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & & \end{array}$$

[where N is a normal open subgroup of BGT], write $\Pi_{\mathbb{B}^{\rtimes}} \overset{\mathrm{def}}{=} \Pi_U$;

$$\mathrm{Cusp}(\mathbb{B}^{\rtimes})$$

for the set of conjugacy classes of cuspidal inertia subgroups [cf. [Tsjm], Theorem 1.3, (i)] of $\Pi_{\mathbb{B}^{\rtimes}}$. Write

$$I_{\mathrm{BGT}}$$

for the set of the *arithmetic Belyi diagrams* over normal open subgroups of BGT. We shall regard I_{BGT} as a preordered set [i.e., a set equipped with a *reflexive* and *transitive* binary relation] by means of the relation determined by *domination*, i.e., the *existence of an arithmetic domination* [cf. Definition 3.3, (i); Proposition 3.4]. It follows immediately from the *functorial* nature of the algorithm of Remark 3.3.3 [cf. also Remark 3.3.2; Proposition 3.5; [Tsjm], Definition 1.4] that there is a *natural action* of $C_{\mathrm{GT}}(\mathrm{BGT})$ on the *preordered set* I_{BGT}. Since BGT satisfies the *COF-property* [cf. Definition 3.3, (ii)], it follows formally that the preordered set I_{BGT} is *directed*, i.e., any pair of elements of the set admits a(n) [not necessarily minimal!] *upper bound*. Since BGT also satisfies the *RGC-property* [cf. Definition 3.3, (iii)], if ${}^{\ddagger}\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}$ *dominates* ${}^{\dagger}\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}$, then the *unique geometric domination*

$$\Pi_{{}^{\ddagger}\mathbb{B}^{\rtimes}} \twoheadrightarrow \Pi_{{}^{\dagger}\mathbb{B}^{\rtimes}}$$

of ${}^{\dagger}\mathbb{B}^{\rtimes}$ by ${}^{\ddagger}\mathbb{B}^{\rtimes}$ determines [cf. Proposition 3.4] a *natural injection*

$$\kappa_{\dagger,\ddagger} : \mathrm{Cusp}({}^{\dagger}\mathbb{B}^{\rtimes}) \hookrightarrow \mathrm{Cusp}({}^{\ddagger}\mathbb{B}^{\rtimes})$$

[which we shall often use to regard $\mathrm{Cusp}({}^{\dagger}\mathbb{B}^{\rtimes})$ as a *subset* of $\mathrm{Cusp}({}^{\ddagger}\mathbb{B}^{\rtimes})$]. Thus, we obtain a direct system $(\mathrm{Cusp}({}^{\ddagger}\mathbb{B}^{\rtimes}), \kappa_{\dagger,\ddagger})$. We shall write

$$\overline{\mathbb{Q}}_{\mathrm{BGT}} \overset{\mathrm{def}}{=} \varinjlim_{\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}} \mathrm{Cusp}(\mathbb{B}^{\rtimes}) \setminus \{\infty\},$$

$$\overline{\mathbb{Q}}^{\times}_{\mathrm{BGT}} \stackrel{\mathrm{def}}{=} \overline{\mathbb{Q}}_{\mathrm{BGT}} \setminus \{0\}, \qquad \overline{\mathbb{Q}}^{\pitchfork}_{\mathrm{BGT}} \stackrel{\mathrm{def}}{=} \overline{\mathbb{Q}}_{\mathrm{BGT}} \setminus \{0, 1\},$$

where $0, 1, \infty \in \mathrm{Cusp}(\mathbb{B}^{\rtimes})$ denote the elements determined by the Π_X-outer surjection $\Pi_U \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} N$ [i.e., the horizontal arrow in $\mathbb{B}^{\rtimes}$] and the conjugacy classes of cuspidal inertia subgroups of Π_X associated to $0, 1, \infty$, respectively. We shall refer to $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ as the BGT-*realization of* $\overline{\mathbb{Q}}$.

REMARK 4.1.1. In the notation of Definition 4.1, it follows immediately from the various definitions involved that the kernel of the *unique geometric domination*

$$\Pi_{{}^{\ddagger}\mathbb{B}^{\rtimes}} \twoheadrightarrow \Pi_{{}^{\dagger}\mathbb{B}^{\rtimes}}$$

of ${}^{\dagger}\mathbb{B}^{\rtimes}$ by ${}^{\ddagger}\mathbb{B}^{\rtimes}$ is the normal closed subgroup of $\Pi_{{}^{\ddagger}\mathbb{B}^{\rtimes}}$ topologically generated by the cuspidal inertia subgroups associated to $\mathrm{Cusp}({}^{\ddagger}\mathbb{B}^{\rtimes}) \setminus \mathrm{Cusp}({}^{\dagger}\mathbb{B}^{\rtimes})$.

PROPOSITION 4.2 (Countability of I_{BGT}). *In the notation of Definition 4.1, I_{BGT} is countable.*

PROOF. Let us observe that since Π_X is topologically finitely generated,

- the set of open subgroups of Π_X is countable;
- there exists a countable open basis of $\mathrm{BGT} \subseteq \mathrm{Out}(\Pi_X)$.

Thus, since $\mathrm{Cusp}(\mathbb{B}^{\rtimes})$ is finite, it follows from the various definitions involved that I_{BGT} is countable. This completes the proof of Proposition 4.2. $\square$

PROPOSITION 4.3 (Natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on the set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$). *There is a natural continuous action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on the discrete set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ [cf. Definition 4.1].*

PROOF. In the notation of Definition 4.1, let $\sigma \in C_{\mathrm{GT}}(\mathrm{BGT})$; $x \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$; $\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}$ an *arithmetic Belyi diagram*

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \big\downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & & \end{array}$$

[where N is a normal open subgroup of BGT] such that $N^\sigma \stackrel{\text{def}}{=} \sigma N \sigma^{-1} \subseteq$ BGT and $x \in \mathrm{Cusp}(\mathbb{B}^{\rtimes})$. Recall that x is the conjugacy class of some cuspidal inertia subgroup I_x of Π_U.

Next, let us recall the right-hand square in the diagram of the final display of the proof of [Tsjm], Corollary 1.6, (i), in the case where we take "J" to be GT [cf. also Remark 3.3.2]. In the notation of the present discussion, this right-hand square determines a commutative diagram of profinite groups

$$\begin{array}{ccc} \Pi_U \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N \\ \sigma \big\downarrow \wr & & \sigma \big\downarrow \wr \\ \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N^\sigma & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N^\sigma, \end{array}$$

where the horizontal arrows are the Π_X-outer surjections induced by the natural open immersions $U \hookrightarrow X$, $U^\sigma \hookrightarrow X$ of hyperbolic curves; the left- (respectively, the right-) hand vertical arrow is a Π_{U^σ}-outer (respectively, Π_X-outer) isomorphism of profinite groups. Write $x^\sigma \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$ for the element determined by $\sigma(I_x)$. Thus, to obtain a well-defined action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on $\overline{\mathbb{Q}}_{\mathrm{BGT}}$, it suffices to show that x^σ does not depend on the choice of $\mathbb{B}^{\rtimes}$. But this follows formally from the *COF-property* of BGT, together with Proposition 3.5 and the construction of x^σ. To verify that the resulting action is continuous, it suffices to observe that there exists an open subgroup $H \subseteq C_{\mathrm{GT}}(\mathrm{BGT})$ [which may be obtained, for instance, by forming the intersection of $C_{\mathrm{GT}}(\mathrm{BGT})$ with the open subgroup "$N \subseteq \mathrm{GT}$" of [Tsjm], Definition 1.4] such that $x^\sigma = x$ for $\sigma \in H$. This completes the proof of Proposition 4.3. $\square$

THEOREM 4.4 (Natural "field-like" operations on $\overline{\mathbb{Q}}_{\mathrm{BGT}}$). *The set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$, equipped with its natural action by $C_{\mathrm{GT}}(\mathrm{BGT})$ [cf. Proposition 4.3], satisfies the following properties:*

(i) The set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ is equipped with natural operations

$$\boxplus_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}} \to \overline{\mathbb{Q}}_{\mathrm{BGT}},$$

$$\boxtimes_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}} \to \overline{\mathbb{Q}}_{\mathrm{BGT}},$$

as well as natural involutions [i.e., self-bijections which are their own inverses]

$$\square^{-1}_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\} \to \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\},$$

$$(1-\square)_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\} \to \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\},$$

all of which are equivariant with respect to the natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ *on* $\overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$. *These operations and involutions satisfy the following properties:*

$$\boxplus_{\mathrm{BGT}}(0,y) \stackrel{\mathrm{def}}{=} y, \quad \boxtimes_{\mathrm{BGT}}(0,y) \stackrel{\mathrm{def}}{=} 0, \quad \boxtimes_{\mathrm{BGT}}(1,y) \stackrel{\mathrm{def}}{=} y,$$

$$\square^{-1}_{\mathrm{BGT}}(0) \stackrel{\mathrm{def}}{=} \infty, \quad \square^{-1}_{\mathrm{BGT}}(1) \stackrel{\mathrm{def}}{=} 1, \quad \square^{-1}_{\mathrm{BGT}}(\infty) \stackrel{\mathrm{def}}{=} 0,$$

$$(1-\square)_{\mathrm{BGT}}(0) \stackrel{\mathrm{def}}{=} 1, \quad (1-\square)_{\mathrm{BGT}}(1) \stackrel{\mathrm{def}}{=} 0, \quad (1-\square)_{\mathrm{BGT}}(\infty) \stackrel{\mathrm{def}}{=} \infty.$$

(ii) If the operations $\boxplus_{\mathrm{BGT}}$ *and* $\boxtimes_{\mathrm{BGT}}$ *determine, on* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$, *the addition and multiplication operations of a structure, on* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$, *of field isomorphic to* $\overline{\mathbb{Q}}$, *then we shall say that* BGT *satisfies the* **ArBC-property** *[i.e., "arithmetic Belyi compatibility property"]. If* BGT *satisfies the ArBC-property, then there exists a field isomorphism* $\overline{\mathbb{Q}} \xrightarrow{\sim} \overline{\mathbb{Q}}_{\mathrm{BGT}}$, *as well as a natural outer homomorphism* $C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}}$.

(iii) Suppose that BGT *admits a* **conducting field** K *that satisfies the* **ZISC-property** *[cf. Definition 3.3, (vi)]. Then* BGT *satisfies the* **ArBC-property**.

Proof. First, we construct *natural "field-like" operations* on the set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$, as described in assertion (i). Write $0, 1 \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$ for the elements determined, respectively, by the conjugacy classes of cuspidal inertia subgroups of Π_X associated to the cusps "0", "1" of X. Let

$$y \in \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$$

(respectively,

$$y \in \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\};$$

$$x \in \overline{\mathbb{Q}}^{\pitchfork}_{\mathrm{BGT}}, \ y \in \overline{\mathbb{Q}}_{\mathrm{BGT}});$$

$\mathbb{B}^{\rtimes}$ an arithmetic Belyi diagram

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} N & \xrightarrow{f} & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \big\downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & & \end{array}$$

[where N is a normal open subgroup of BGT] such that $x, y \in \mathrm{Cusp}(\mathbb{B}^{\rtimes})$. Write $\iota : U \hookrightarrow X$ for the open immersion that gives rise to the horizontal arrow f of $\mathbb{B}^{\rtimes}$ [cf. [Tsjm], Definition 1.1, (i); [Tsjm], Definition 1.4]; t for the standard coordinate on $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$;

$$\iota^{t^{-1}} : U \hookrightarrow X$$

(respectively,

$$\iota^{1-t} : U \hookrightarrow X;$$

$$\iota^{t/x} : U \hookrightarrow X)$$

for the open immersion obtained from $\iota : U \hookrightarrow X$ by composing with the automorphism $t \mapsto t^{-1}$ of X [i.e., the automorphism of X that switches the cusps "0" and "∞"] (respectively, composing with the automorphism $t \mapsto 1-t$ of X [i.e., the automorphism of X that switches the cusps "0" and "1"]; compactifying at the cusps other than "0", "x", "∞" [i.e., instead of at the cusps other than "0", "1", "∞"] and then dividing by x). Then it follows immediately from [Tsjm], Theorem 1.3, (ii) [cf. Remark 3.3.2], that the open immersion $\iota^{t^{-1}} : U \hookrightarrow X$ (respectively, $\iota^{1-t} : U \hookrightarrow X$; $\iota^{t/x} : U \hookrightarrow X$) determines a Π_X-outer surjection

$$f^{t^{-1}} : \Pi_U \stackrel{\mathrm{out}}{\rtimes} N \to \Pi_X \stackrel{\mathrm{out}}{\rtimes} N$$

(respectively,

$$f^{1-t} : \Pi_U \stackrel{\mathrm{out}}{\rtimes} N \to \Pi_X \stackrel{\mathrm{out}}{\rtimes} N;$$

$$f^{t/x} : \Pi_U \stackrel{\mathrm{out}}{\rtimes} N \to \Pi_X \stackrel{\mathrm{out}}{\rtimes} N).$$

Thus, by considering y relative to $f^{t^{-1}}$ (respectively, f^{1-t}; $f^{t/x}$) [cf. Definition 4.1], we obtain a new element $y^{t^{-1}} \in \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$ (respectively, $y^{1-t} \in \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$; $y^{t/x} \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$). In particular, by *applying the COF-property* of BGT, one verifies immediately that we obtain natural bijections

- $\{t^{-1}\} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\} \stackrel{\sim}{\to} \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$;
- $\{1-t\} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\} \stackrel{\sim}{\to} \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$;
- $\{t/x\} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \stackrel{\sim}{\to} \overline{\mathbb{Q}}_{\mathrm{BGT}}$

such that $\{t^{-1}\}(y) = y^{t^{-1}}$, $\{1-t\}(y) = y^{1-t}$, and $\{t/x\}(y) = y^{t/x}$. Here, we observe that $\{t^{-1}\}$ and $\{1-t\}$ are *involutions*, while $\{t/x\}$ and $\{t/x^{-1}\}$, where we write $x^{-1} \stackrel{\mathrm{def}}{=} \{t/x\}(1) \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$, are *inverse* to one another. Write

$$\Box^{-1}_{\mathrm{BGT}} \stackrel{\mathrm{def}}{=} \{t^{-1}\}, \quad (1-\Box)_{\mathrm{BGT}} \stackrel{\mathrm{def}}{=} \{1-t\}.$$

Then it follows immediately from the various definitions involved that

$$\Box^{-1}_{\mathrm{BGT}}(0) \stackrel{\mathrm{def}}{=} \infty, \quad \Box^{-1}_{\mathrm{BGT}}(1) \stackrel{\mathrm{def}}{=} 1, \quad \Box^{-1}_{\mathrm{BGT}}(\infty) \stackrel{\mathrm{def}}{=} 0,$$

$$(1-\Box)_{\mathrm{BGT}}(0) \stackrel{\mathrm{def}}{=} 1, \quad (1-\Box)_{\mathrm{BGT}}(1) \stackrel{\mathrm{def}}{=} 0, \quad (1-\Box)_{\mathrm{BGT}}(\infty) \stackrel{\mathrm{def}}{=} \infty.$$

For each $(x,y) \in \overline{\mathbb{Q}}^{\pitchfork}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}}$, write

$$\boxtimes_{\mathrm{BGT}}(x,y) \stackrel{\mathrm{def}}{=} \{t/\{t^{-1}\}(x)\}(y),$$

$$\boxtimes_{\mathrm{BGT}}(0,y) \stackrel{\mathrm{def}}{=} 0, \qquad \boxtimes_{\mathrm{BGT}}(1,y) \stackrel{\mathrm{def}}{=} y.$$

Thus, we obtain a *multiplication map*

$$\boxtimes_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}} \to \overline{\mathbb{Q}}_{\mathrm{BGT}}.$$

Write

$$\mathbb{B}^{\rtimes}_{-1}$$

for the arithmetic Belyi diagram [over a suitable normal open subgroup of BGT — cf. the subgroup "M" of [Tsjm], Definition 1.4] determined by the unique [up to isomorphism] connected finite étale covering of X of degree 2 ramified over 0 and ∞;

$$-1_{\mathrm{BGT}} \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$$

for the element of $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ determined by the unique element of $\mathrm{Cusp}(\mathbb{B}^{\rtimes}_{-1}) \setminus \{0,1,\infty\}$. Then we obtain an *addition map*

$$\boxplus_{\mathrm{BGT}} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}} \to \overline{\mathbb{Q}}_{\mathrm{BGT}}$$

by taking

$$\boxplus_{\mathrm{BGT}}(x,y) \stackrel{\mathrm{def}}{=} \boxtimes_{\mathrm{BGT}}(x, \{1-t\}(\boxtimes_{\mathrm{BGT}}(-1_{\mathrm{BGT}}, \boxtimes_{\mathrm{BGT}}(\{t^{-1}\}(x), y)))),$$

$$\boxplus_{\mathrm{BGT}}(0,y) \stackrel{\mathrm{def}}{=} y,$$

where $(x, y) \in \overline{\mathbb{Q}}^{\times}_{\mathrm{BGT}} \times \overline{\mathbb{Q}}_{\mathrm{BGT}}$.

Next, we verify that the natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on the set $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ [cf. Proposition 4.3] is *compatible* with the *"field-like" operations* constructed above. Let $\sigma \in C_{\mathrm{GT}}(\mathrm{BGT})$. Recall that the maps $\boxtimes_{\mathrm{BGT}}$ and $\boxplus_{\mathrm{BGT}}$ are completely determined by $\Box^{-1}_{\mathrm{BGT}} = \{t^{-1}\}$, $(1 - \Box)_{\mathrm{BGT}} = \{1 - t\}$, $\{t/x\}$ (for $x \in \overline{\mathbb{Q}}^{\pitchfork}_{\mathrm{BGT}}$), and -1_{BGT}. Thus, since $0^\sigma = 0$ and $1^\sigma = 1$, it suffices to verify the following assertion:

Claim 4.4.A: Let $x \in \overline{\mathbb{Q}}^{\pitchfork}_{\mathrm{BGT}}$, $y \in \overline{\mathbb{Q}}^{\times}_{\mathrm{BGT}}$. Then

- $\{t^{-1}\}(y^\sigma) = \{t^{-1}\}(y)^\sigma$,
- $\{1 - t\}(y^\sigma) = (\{1 - t\}(y))^\sigma$,
- $\{t/x^\sigma\}(y^\sigma) = (\{t/x\}(y))^\sigma$,
- $(-1_{\mathrm{BGT}})^\sigma = -1_{\mathrm{BGT}}$.

First, it follows from the uniqueness of the connected finite étale covering of X of degree 2 ramified over 0 and ∞ that σ induces an automorphism of $\mathbb{B}^{\rtimes}_{-1}$. Then since $0^\sigma = 0$ and $1^\sigma = 1$, the equality $(-1_{\mathrm{BGT}})^\sigma = -1_{\mathrm{BGT}}$ follows immediately from the definition of -1_{BGT}. Next, let $\mathbb{B}^{\rtimes}$ be an arithmetic Belyi diagram

$$\begin{tikzcd} \Pi_U \overset{\mathrm{out}}{\rtimes} N \arrow[r, "f"] \arrow[d] & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & \end{tikzcd}$$

[where N is a normal open subgroup of BGT] such that $N^\sigma \overset{\mathrm{def}}{=} \sigma N \sigma^{-1} \subseteq \mathrm{BGT}$, and $x, y \in \mathrm{Cusp}(\mathbb{B}^{\rtimes})$. Then, by considering the [right-hand square in the final display of the] proof of [Tsjm], Corollary 1.6, (i) [cf. also Remark 3.3.2; the *functorial algorithm* of Remark 3.3.3], in the case where $J = \mathrm{GT}$, we obtain a *commutative diagram*

$$\begin{tikzcd} \Pi_U \overset{\mathrm{out}}{\rtimes} N \arrow[r, "f"'] \arrow[d, "\sigma"', "\wr"] & \Pi_X \overset{\mathrm{out}}{\rtimes} N \arrow[d, "\sigma"', "\wr"] \\ \Pi_{U^\sigma} \overset{\mathrm{out}}{\rtimes} N^\sigma \arrow[r, "f^\sigma"'] & \Pi_X \overset{\mathrm{out}}{\rtimes} N^\sigma, \end{tikzcd}$$

where the horizontal arrows are the Π_X-outer surjections induced by the natural open immersions $U \hookrightarrow X$, $U^\sigma \hookrightarrow X$ of hyperbolic curves; the left- (respectively, the right-) hand vertical arrow is a Π_{U^σ}-outer (respectively, Π_X-outer) isomorphism of profinite groups.

Note that $\{t^{-1}\}(y^\sigma)$ (respectively, $\{1-t\}(y^\sigma)$; $\{t/x^\sigma\}(y^\sigma)$) is completely determined by y^σ and the Π_X-outer surjection

$$(f^\sigma)^{t^{-1}} : \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N^\sigma \to \Pi_X \overset{\text{out}}{\rtimes} N^\sigma$$

(respectively,

$$(f^\sigma)^{1-t} : \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N^\sigma \to \Pi_X \overset{\text{out}}{\rtimes} N^\sigma;$$

$$(f^\sigma)^{t/x^\sigma} : \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N^\sigma \to \Pi_X \overset{\text{out}}{\rtimes} N^\sigma),$$

which sends $(\infty, 1, 0)$ (respectively, $(1, 0, \infty)$; $(0, x^\sigma, \infty)$) to $(0, 1, \infty)$.

On the other hand, $(\{t^{-1}\}(y))^\sigma$ (respectively, $(\{1-t\}(y))^\sigma$; $(\{t/x\}(y))^\sigma$) is completely determined by y^σ and the Π_X-outer surjection

$$\sigma \circ f^{t^{-1}} \circ \sigma^{-1} : \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N^\sigma \to \Pi_X \overset{\text{out}}{\rtimes} N^\sigma$$

(respectively,

$$\sigma \circ f^{1-t} \circ \sigma^{-1} : \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N^\sigma \to \Pi_X \overset{\text{out}}{\rtimes} N^\sigma;$$

$$\sigma \circ f^{t/x} \circ \sigma^{-1} : \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N^\sigma \to \Pi_X \overset{\text{out}}{\rtimes} N^\sigma),$$

which sends $(\infty, 1, 0)$ (respectively, $(1, 0, \infty)$; $(0, x^\sigma, \infty)$) to $(0, 1, \infty)$.

Note that the Π_X-outer surjections of the displays of the last two paragraphs exhibit analogous behavior on the cusps [i.e., more precisely, on the conjugacy classes of cuspidal inertia subgroups]. Thus, we conclude from the above *commutative diagram* [cf. also Remark 3.3.2; the *functorial algorithm* of Remark 3.3.3] that

- $(f^\sigma)^{t^{-1}} = \sigma \circ f^{t^{-1}} \circ \sigma^{-1}$,
- $(f^\sigma)^{1-t} = \sigma \circ f^{1-t} \circ \sigma^{-1}$,
- $(f^\sigma)^{t/x^\sigma} = \sigma \circ f^{t/x} \circ \sigma^{-1}$.

This completes the proof of Claim 4.4.A, hence of assertion (i). Assertion (ii) follows immediately from the various definitions involved.

Next, we verify assertion (iii). In the following discussion, we shall identify $X(\overline{\mathbb{Q}})$ with $\overline{\mathbb{Q}}^{\pitchfork}$. We begin by *observing* that, for any *pair* consisting of

- an *arithmetic Belyi diagram* $\mathbb{B}^{\rtimes}$

$$\begin{tikzcd} \Pi_U \overset{\mathrm{out}}{\rtimes} N \arrow[r] \arrow[d] & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & \end{tikzcd}$$

  [where N is a normal open subgroup of BGT] and

- a *finite subset* $F \subseteq \overline{\mathbb{Q}}^{\pitchfork}$,

there exist

- an *open immersion* $U^{\dagger} \hookrightarrow U$ $(\hookrightarrow X)$ over $\overline{\mathbb{Q}}$ such that

$$F \subseteq X(\overline{\mathbb{Q}}) \setminus U^{\dagger}(\overline{\mathbb{Q}}) \subseteq X(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}^{\pitchfork}$$

  [where we regard $U^{\dagger}(\overline{\mathbb{Q}})$ as a subset of $X(\overline{\mathbb{Q}})$ by means of the composite of the open immersion $U^{\dagger} \hookrightarrow U$ with the open immersion $U \hookrightarrow X$ that gives rise to the horizontal arrow of the given arithmetic Belyi diagram],

- a *normal open subgroup* $M^{\dagger} \subseteq N$ of BGT, and

- an *arithmetic Belyi diagram* ${}^{\dagger}\mathbb{B}^{\rtimes}$

$$\begin{tikzcd} \Pi_{U^{\dagger}} \overset{\mathrm{out}}{\rtimes} M^{\dagger} \arrow[r] \arrow[d] & \Pi_X \overset{\mathrm{out}}{\rtimes} M^{\dagger} \\ \Pi_X \overset{\mathrm{out}}{\rtimes} M^{\dagger} & \end{tikzcd}$$

  [where the restriction $\Pi_{U^{\dagger}} \to \Pi_X$ of the horizontal arrow to $\Pi_{U^{\dagger}}$ is the Π_X-outer surjection that arises from the above open immersion $U^{\dagger} \hookrightarrow U$ $(\hookrightarrow X)$ over $\overline{\mathbb{Q}}$]

such that the outer action of $M^\dagger$ on $\Pi_{U^\dagger}$ is compatible, relative to the outer surjection $\Pi_{U^\dagger} \twoheadrightarrow \Pi_U$ [induced by the open immersion $U^\dagger \hookrightarrow U$], with the restriction to $M^\dagger \subseteq N$ of the outer action of N on Π_U. Indeed, write $g : U \to X$ for the connected finite étale covering that gives rise to the vertical arrow of the given arithmetic Belyi diagram. Let ${}^*\mathbb{B}^{\rtimes}$ be an arithmetic Belyi diagram

$$\begin{tikzcd}
\Pi_{U^*} \overset{\mathrm{out}}{\rtimes} M^* \arrow[r] \arrow[d] & \Pi_X \overset{\mathrm{out}}{\rtimes} M^* \\
\Pi_X \overset{\mathrm{out}}{\rtimes} M^* &
\end{tikzcd}$$

[where M^* is a normal open subgroup of BGT] such that

$$U^*(\overline{\mathbb{Q}}) \subseteq X(\overline{\mathbb{Q}}) \setminus g(U(\overline{\mathbb{Q}}) \cap F) \subseteq X(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}^{\pitchfork}$$

[cf., e.g., [NCBel], Corollary 1.1], where we regard $U^*(\overline{\mathbb{Q}})$ as a subset of $X(\overline{\mathbb{Q}})$ by means of the open immersion $U^* \hookrightarrow X$ that gives rise to the horizontal arrow of ${}^*\mathbb{B}^{\rtimes}$. Write $U^\dagger \overset{\mathrm{def}}{=} g^{-1}(U^*)$. Thus, we conclude that there exist a normal open subgroup $M^\dagger \subseteq M^* \subseteq N$ of BGT and a diagram

$$\begin{tikzcd}
\Pi_{U^\dagger} \overset{\mathrm{out}}{\rtimes} M^\dagger \arrow[r] \arrow[d] & \Pi_U \overset{\mathrm{out}}{\rtimes} N|_{M^\dagger} \arrow[r] \arrow[d] & \Pi_X \overset{\mathrm{out}}{\rtimes} N|_{M^\dagger} \\
\Pi_{U^*} \overset{\mathrm{out}}{\rtimes} M^*|_{M^\dagger} \arrow[r] \arrow[d] & \Pi_X \overset{\mathrm{out}}{\rtimes} N|_{M^\dagger} & \\
\Pi_X \overset{\mathrm{out}}{\rtimes} M^*|_{M^\dagger} & &
\end{tikzcd}$$

— where the upper right-hand portion of the diagram is the diagram obtained by restricting $\mathbb{B}^{\rtimes}$ to $M^\dagger$; the lower left-hand portion of the diagram is the diagram obtained by restricting ${}^*\mathbb{B}^{\rtimes}$ to $M^\dagger$; the upper left-hand square of the diagram is cartesian — such that the composite of the upper horizontal arrows and the composite of the left-hand vertical arrows determine an arithmetic Belyi diagram ${}^{\dagger}\mathbb{B}^{\rtimes}$

$$\begin{array}{ccc} \Pi_{U^{\dagger}} \overset{\mathrm{out}}{\rtimes} M^{\dagger} & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} M^{\dagger} \\ \big\downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} M^{\dagger} & & \end{array}$$

satisfying the desired property. This completes the proof of the above *observation.*

Next, let us fix an element $\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}$. Then by applying the above *observation* in a *recursive* fashion [i.e., by applying the *observation* to $\mathbb{B}^{\rtimes}$ and some finite subset F to obtain ${}^{\dagger}\mathbb{B}^{\rtimes}$, then applying the *observation* to ${}^{\dagger}\mathbb{B}^{\rtimes}$ and some other finite subset ${}^{\dagger}F$ to obtain ${}^{\ddagger}\mathbb{B}^{\rtimes}$, etc.], we conclude [cf. the definition of $\overline{\mathbb{Q}}_{\mathrm{BGT}}$] that one may construct a *family of injections*

$$\left\{\phi_{\mathbb{B}^{\rtimes},F} : F \cup \{0,1\} \hookrightarrow \overline{\mathbb{Q}}_{\mathrm{BGT}}\right\}_{\{F \subseteq \overline{\mathbb{Q}}^{\pitchfork}\}}$$

[indexed by the finite subsets $F \subseteq \overline{\mathbb{Q}}^{\pitchfork}$] such that the following conditions are satisfied:

- $\mathrm{Cusp}(\mathbb{B}^{\rtimes}) \setminus \{\infty\} \subseteq \bigcup_{F \subseteq \overline{\mathbb{Q}}^{\pitchfork}} \mathrm{Im}(\phi_{\mathbb{B}^{\rtimes},F})$.
- If $F_1 \subseteq F_2 \subseteq \overline{\mathbb{Q}}^{\pitchfork}$, then $(\phi_{\mathbb{B}^{\rtimes},F_2})|_{F_1} = \phi_{\mathbb{B}^{\rtimes},F_1}$.

Thus, the various injections $\phi_{\mathbb{B}^{\rtimes},F}$, indexed by the finite subsets $F \subseteq \overline{\mathbb{Q}}^{\pitchfork}$, determine an *injection*

$$\phi_{\mathbb{B}^{\rtimes}} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_{\mathrm{BGT}}$$

associated to $\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}$ such that $\mathrm{Cusp}(\mathbb{B}^{\rtimes}) \setminus \{\infty\} \subseteq \mathrm{Im}(\phi_{\mathbb{B}^{\rtimes}})$.

Next, let K be a *conducting field* for BGT that satisfies the *ZISC-property.* Then one verifies immediately that, to verify assertion (iii), by replacing K and BGT, respectively, by $K \cap \overline{\mathbb{Q}}$ [where we think of K as being embedded in some algebraic closure $\overline{K}$ of K that contains $\overline{\mathbb{Q}}$] and a suitable GT-conjugate of BGT, we may assume without loss of generality that

$$G_K \subseteq \mathrm{BGT}\ (\subseteq \mathrm{GT}).$$

Then, to verify assertion (iii), it suffices to verify the following assertion:

Claim 4.4.B: The injection $\phi_{\mathbb{B}^{\rtimes}}$ is, in fact, a *bijection*. Moreover, the "field-like" operations $\boxplus_{\mathrm{BGT}}$ and $\boxtimes_{\mathrm{BGT}}$ on $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ induce the usual operations of *addition* and *multiplication* on $\overline{\mathbb{Q}}$ via $\phi_{\mathbb{B}^{\rtimes}}$.

Indeed, let $x \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$; ${}^{\dagger}\mathbb{B}^{\rtimes}$ an arithmetic Belyi diagram

$$\begin{array}{ccc} \Pi_{U^{\dagger}} \overset{\mathrm{out}}{\rtimes} N^{\dagger} & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N^{\dagger} \\ \downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N^{\dagger} & & \end{array}$$

[where $N^{\dagger}$ is a normal open subgroup of BGT] such that $x \in \mathrm{Cusp}({}^{\dagger}\mathbb{B}^{\rtimes})$. Then observe that, by restricting ${}^{\dagger}\mathbb{B}^{\rtimes}$ to $N^{\dagger} \cap G_K$, we obtain an element $x_{\overline{\mathbb{Q}}} \in \overline{\mathbb{Q}}$ associated to $x \in \mathrm{Cusp}({}^{\dagger}\mathbb{B}^{\rtimes})$ that, in light of the *ZISC-property* of K and the *COF-property* of BGT, is *independent* of the choice of ${}^{\dagger}\mathbb{B}^{\rtimes}$. Therefore, it follows immediately from the definition of $\phi_{\mathbb{B}^{\rtimes}}$, together with the *ZISC-property* of K and the *COF-property* of BGT, that $\phi_{\mathbb{B}^{\rtimes}}(x_{\overline{\mathbb{Q}}}) = x$. In particular, we conclude that $\phi_{\mathbb{B}^{\rtimes}}$ is *bijective*. Next, we recall that $\{t^{-1}\}$, $\{1-t\}$, and $\{t/x\}$ (for $x \in \overline{\mathbb{Q}}^{\pitchfork}_{\mathrm{BGT}}$) are defined by using the scheme-theoretic morphisms $\iota^{t^{-1}}$, ι^{1-t}, and $\iota^{t/x}$. In particular, by restricting via $G_K \subseteq \mathrm{BGT}$ [cf. the *functorial algorithm* of Remark 3.3.3] and applying the *ZISC-property* of K, we conclude that the operations $\{t^{-1}\}$, $\{1-t\}$, and $\{t/x\}$ (for $x \in \overline{\mathbb{Q}}^{\pitchfork}_{\mathrm{BGT}}$) induce, via $\phi_{\mathbb{B}^{\rtimes}}$, the usual involutions and operation of multiplication by $x_{\overline{\mathbb{Q}}}^{-1}$ on $\overline{\mathbb{Q}}$. In a similar vein, it follows immediately from the definition of -1_{BGT}, together with the *ZISC-property* of K, that $\phi_{\mathbb{B}^{\rtimes}}(-1) = -1_{\mathrm{BGT}}$. Thus, it follows immediately from the various definitions involved [cf. also the *bijectivity* of $\phi_{\mathbb{B}^{\rtimes}}$] that the "field-like" operations $\boxplus_{\mathrm{BGT}}$ and $\boxtimes_{\mathrm{BGT}}$ on $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ induce the usual operations of *addition* and *multiplication* on $\overline{\mathbb{Q}}$ via $\phi_{\mathbb{B}^{\rtimes}}$. This completes the proof of Claim 4.4.B, hence of assertion (iii) [and indeed of Theorem 4.4]. $\square$

REMARK 4.4.1. Let p be a prime number, F a field which is a finite extension of the *field of rational numbers* $\mathbb{Q}$ or the *field of p-adic numbers* $\mathbb{Q}_p$. Thus, we have a natural inclusion $\mathbb{Q} \subseteq F$. Let $\overline{F}$ be an algebraic closure of F. By abuse of notation, we shall identify $\overline{\mathbb{Q}}$ with the algebraic closure of $\mathbb{Q}$ in $\overline{F}$. Write $G_F \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$. Thus, we obtain *natural injections*

$$G_F \hookrightarrow G_{\mathbb{Q}} \hookrightarrow \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$$

[cf. the discussion at the beginning of [Tsjm], Introduction], which we use to identify G_F with its image in GT. Then it follows immediately from the fact that F is *Kummer-faithful* [cf. [AbsTopIII], Definition 1.5; [AbsTopIII], Remark 1.5.4, (i)], together with a similar argument to the argument applied in the proof of [Tsjm], Corollary 3.2, that F satisfies the *ISC-property*, and that G_F satisfies the *CS-property*. Thus, we conclude from Corollary 3.7 that G_F satisfies the *RGC-property*. Since, in this situation, the *COF-property* is immediate, we thus conclude [cf. Theorem 4.4, (iii)] that G_F satisfies the *ArBC-property*, i.e., that we may take "BGT" to be G_F, and, moreover, that the additional condition of Theorem 4.4, (ii), holds. Finally, we observe that the evident scheme-theoretic interpretation of the various arithmetic Belyi diagrams that appear determines a natural isomorphism of fields $\overline{\mathbb{Q}}_{G_F} \xrightarrow{\sim} \overline{\mathbb{Q}}$ that is compatible, relative to the natural injection $G_F \hookrightarrow G_{\mathbb{Q}}$, with the respective natural actions, i.e., we obtain a diagram as follows:

$$\begin{array}{ccc} G_F & \hookrightarrow & G_{\mathbb{Q}} \\ \curvearrowright & & \curvearrowright \\ \overline{\mathbb{Q}}_{G_F} & \xrightarrow{\sim} & \overline{\mathbb{Q}}. \end{array}$$

REMARK 4.4.2. It is not clear to the authors at the time of writing whether or not GT satisfies the *BC-property*, i.e., whether or not "GT = BGT".

COROLLARY 4.5 (Group-theoretic nature of BGT). *Let n be an integer such that $n \geq 2$. Write X_n for the n-th configuration space of $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$; $\mathrm{GT}_n \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{gF}}(\Pi_{X_n}) \subseteq \mathrm{Out}(\Pi_{X_n})$. Recall that we have a natural isomorphism $\mathrm{GT}_n \xrightarrow{\sim} \mathrm{GT}$ [cf. the first display of [CbGT], Corollary C]. Then one may reconstruct from Π_{X_n}, in a purely combinatorial/group-theoretic way, i.e., in a way that only involves the structure of Π_{X_n} as a topological group [cf. also Remark 4.5.1 below],*

- *the subgroups $\mathrm{GT}_n \subseteq \mathrm{Out}(\Pi_{X_n})$, $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$, where we regard Π_X as the quotient of Π_{X_n} by a generalized fiber subgroup, and we recall [cf. the first display of [CbGT], Corollary C] that $\mathrm{Out}(\Pi_{X_n})$ normalizes GT_n and acts, by conjugation, on GT_n via inner automorphisms of GT_n;*
- *the natural isomorphism $\mathrm{GT}_n \xrightarrow{\sim} \mathrm{GT}$;*

- *the collection of closed subgroups* $J \subseteq \mathrm{GT}_n$ *such that* J *satisfies [i.e., the image of* J*, via the natural isomorphism, in* GT *satisfies] the BC-property [cf. Definition 3.3, (v)];*

- *the collection of closed subgroups* $J \subseteq \mathrm{GT}_n$ *such that* J *satisfies [i.e., the image of* J*, via the natural isomorphism, in* GT *satisfies] the ArBC-property [cf. Theorem 4.4, (ii)].*

If, moreover, a closed subgroup $J = \mathrm{BGT} \subseteq \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$ *satisfies the BC-property, then the construction from* Π_{X_n} *[cf. also Remark 4.5.1 below] of*

- *the preordered set of arithmetic Belyi diagrams* I_{BGT} *[cf. Definition 4.1],*

- *the natural action of* $C_{\mathrm{GT}}(\mathrm{BGT})$ *on the preordered set* I_{BGT} *[cf. Definition 4.1],*

- *the set* $\mathrm{Cusp}(-)$ *associated to any element of* I_{BGT} *[cf. Definition 4.1],*

- *the direct limit* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ *[cf. Definition 4.1],*

- *the natural continuous action of* $C_{\mathrm{GT}}(\mathrm{BGT})$ *on* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ *[cf. Proposition 4.3], and*

- *the field structure on* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$*, whenever* J *satisfies the ArBC-property [cf. Theorem 4.4, (ii)],*

may be phrased in purely combinatorial/group-theoretic terms, i.e., in terms that only involve the structure of Π_{X_n} *as a topological group.*

Proof. The various assertions of Corollary 4.5 follow immediately from Definitions 3.3, 4.1; Remarks 3.3.2, 3.3.3 [cf. also Remark 4.5.1 below]; Proposition 4.3 [and its proof]; Theorem 4.4 [and its proof]; [CbGT], Theorem A, (ii); the first display of [CbGT], Corollary C; [Tsjm], Theorem 1.3, (i); [Tsjm], Definition 1.4. □

Remark 4.5.1.

(i) Here, in the context of Remark 3.3.3, (i), we observe that the natural isomorphism $\mathrm{GT}_n \xrightarrow{\sim} \mathrm{GT}$ [cf. the first display of [CbGT], Corollary C], together with the algorithm of Corollary 3.1, (ii), implies that there is in fact *no substantive difference* between

- *constructions starting from* Π_{X_n} [where we recall that $n \geq 2$] and
- *constructions starting from* Π_{X_3}.

(ii) In the situation discussed in (i) [cf. also Remark 3.3.3, (i)], suppose that we apply the constructions discussed in Corollary 4.5 to Π_{X_3}, regarded as an *abstract topological group.* Then the algorithm of Corollary 3.1, (ii), determines a subgroup

$$\mathfrak{S}_3 \subseteq \mathrm{Out}(\Pi_X)$$

[i.e., where, by a slight abuse of notation, we use the notation "$\mathfrak{S}_3$" to denote this subgroup which is isomorphic to the symmetric group on 3 letters] of the group of outer automorphisms $\mathrm{Out}(\Pi_X)$ of the quotient Π_X of the given abstract topological group Π_{X_3} discussed in Remark 3.3.3, (i), (d).

(iii) We maintain the notation of (ii). Then observe that since the quotient Π_X of the given *abstract topological group* Π_{X_3} is *not* equipped with a natural bijection between its set of cusps and the set of symbols $\{0, 1, \infty\}$, it follows that this quotient Π_X is only related to any of the "Π_X's" that appear in the arithmetic Belyi diagrams discussed in the statement of Corollary 4.5 [*not* by a single outer isomorphism, but rather] by an $\mathfrak{S}_3$-*torsor of outer isomorphisms.*

5. Combinatorial Construction of the Conjugacy Class of Subgroups of GT Determined by $G_{\mathbb{Q}}$

Write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$; X_n for the n-th configuration space associated to X, where $n \geq 2$. In this section, we reconstruct from the topological group Π_{X_n}, in a *purely combinatorial/group-theoretic* way, the conjugacy class of subgroups of the Grothendieck-Teichmüller group $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$

determined by the absolute Galois group of $\mathbb{Q}$ as the set of maximal closed subgroups BGT of GT satisfying a certain purely combinatorial/group-theoretic condition that we refer to as the **AA-property** [cf. Definition 5.12; Theorem 5.17, (ii)].

Write $\Pi_{X_{0\infty}}$ for the quotient of Π_X by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to the cusp "1" [so $\Pi_{X_{0\infty}}$ is isomorphic to $\widehat{\mathbb{Z}}$ as an abstract topological group]. Let J be a closed subgroup of $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$. Then we shall write [by a slight abuse of notation]

$$\Pi_X \overset{\mathrm{out}}{\rtimes} J \twoheadrightarrow \Pi_{X_{0\infty}} \rtimes J$$

for the quotient by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to the cusp "1".

Definition 5.1. In the notation of Definition 4.1:

(i) Write

$$\Pi \stackrel{\mathrm{def}}{=} \varprojlim_{\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}} \Pi_{\mathbb{B}^{\rtimes}},$$

where the transition morphisms are the *unique geometric dominations*

$$\Pi_{\ddagger\mathbb{B}^{\rtimes}} \twoheadrightarrow \Pi_{\dagger\mathbb{B}^{\rtimes}}.$$

Here, we observe that even though these transition morphisms are, strictly speaking, *outer [surjective] homomorphisms*, it follows immediately from Proposition 4.2 that one may choose a *coherent system of homomorphism representatives* of the given system of outer homomorphisms; in particular, Π is well-defined as a *profinite group*, up to inner automorphisms. It follows immediately from Proposition 3.5, together with the various definitions involved, that the natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on I_{BGT} [cf. Definition 4.1] induces a *natural outer action* of $C_{\mathrm{GT}}(\mathrm{BGT})$ on the group Π.

(ii) In the context of the inverse limit of Definition 5.1, (i), we shall refer to an inverse limit of cuspidal inertia subgroups of some cofinal collection of $\Pi_{\mathbb{B}^{\rtimes}}$'s [where the induced transition morphisms are necessarily isomorphisms] as a *cuspidal inertia subgroup* of Π. For each open subgroup Π^* of Π, we shall refer to the intersection of Π^* with a cuspidal inertia subgroup of Π as a *cuspidal inertia subgroup* of Π^* and write

$$\mathrm{Cusp}(\Pi^*)$$

for the set of Π^*-conjugacy classes of cuspidal inertia subgroups of Π^*. Thus, it follows immediately from the definitions that we obtain a *natural surjection*

$$\mathrm{Cusp}(\Pi^*) \twoheadrightarrow \mathrm{Cusp}(\Pi)$$

with *finite fibers.* For each finite subset $E^* \subseteq \mathrm{Cusp}(\Pi^*)$, write

$$\Pi^* \twoheadrightarrow \Pi^*_{E^*}$$

for the *topologically finitely generated* [cf. Remark 5.1.1 below] quotient profinite group of Π^* obtained by forming the quotient of Π^* by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to $\mathrm{Cusp}(\Pi^*) \setminus E^*$. Observe that the natural outer action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on Π [cf. Definition 5.1, (i)] induces a *natural action* of $C_{\mathrm{GT}}(\mathrm{BGT})$ on $\mathrm{Cusp}(\Pi)$. Finally, we observe that it follows immediately from the various definitions involved [cf., especially, Definition 4.1] that we have a *natural* $C_{\mathrm{GT}}(\mathrm{BGT})$-*equivariant bijection*

$$\mathrm{Cusp}(\Pi) \ \stackrel{\sim}{\to}\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}.$$

(iii) Write

$$C_{\mathrm{BGT}}$$

for the set of finite subsets of $\mathrm{Cusp}(\Pi)$ that contain $\{0, 1, \infty\}$. Observe that the natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on $\mathrm{Cusp}(\Pi)$ [cf. Definition 5.1, (ii)] induces a *natural action* of $C_{\mathrm{GT}}(\mathrm{BGT})$ on C_{BGT}. We shall write

$$C^{\mathrm{st}}_{\mathrm{BGT}} \subseteq\ C_{\mathrm{BGT}}$$

for the subset of $C_{\mathrm{GT}}(\mathrm{BGT})$-*stable* elements, i.e., elements fixed by the action of $C_{\mathrm{GT}}(\mathrm{BGT})$. Finally, we observe that the assignment $I_{\mathrm{BGT}} \ni \mathbb{B}^{\rtimes} \mapsto \mathrm{Cusp}(\mathbb{B}^{\rtimes}) \in C_{\mathrm{BGT}}$ induces a *natural* $C_{\mathrm{GT}}(\mathrm{BGT})$-*equivariant map*

$$I_{\mathrm{BGT}} \ \to\ C_{\mathrm{BGT}}.$$

REMARK 5.1.1. In the notation of Definition 5.1, it follows immediately from Remark 4.1.1 that the kernel of the natural outer surjection

$$\Pi \twoheadrightarrow \Pi_{\mathbb{B}^{\rtimes}}$$

is the normal closed subgroup of Π topologically generated by the cuspidal inertia subgroups associated to $\mathrm{Cusp}(\Pi) \setminus \mathrm{Cusp}(\mathbb{B}^{\rtimes})$. In particular, the quotient $\Pi \twoheadrightarrow \Pi_{\mathbb{B}^{\rtimes}}$ may be *naturally identified* with the quotient

$$\Pi \twoheadrightarrow \Pi_{\mathrm{Cusp}(\mathbb{B}^{\rtimes})}$$

of the third display of Definition 5.1, (ii) [i.e., where we take "Π^*" to be Π and "E^*" to be $\mathrm{Cusp}(\mathbb{B}^{\rtimes})$].

REMARK 5.1.2. Let $E \in C^{\mathrm{st}}_{\mathrm{BGT}}$ [cf. Definition 5.1, (iii)]. Then it follows immediately from the various definitions involved that the natural outer action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on Π [cf. Definition 5.1, (i)] induces, via the natural outer surjection $\Pi \twoheadrightarrow \Pi_E$, a *natural continuous outer action* of $C_{\mathrm{GT}}(\mathrm{BGT})$ on the *topologically finitely generated profinite group* Π_E [cf. the discussion entitled "Topological groups" in Notations and Conventions; Definition 5.1, (ii); [Tsjm], Lemma 1.2, (b); [Tsjm], Theorem 1.3, (ii); [Tsjm], Definition 1.4].

REMARK 5.1.3. Observe that it follows immediately from the *continuity* [cf. Proposition 4.3] of the natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ on $\overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$ ($\xrightarrow{\sim} \mathrm{Cusp}(\Pi)$) [cf. Definition 5.1, (ii)], together with the *COF-property* of BGT, that

> for any $E \in C_{\mathrm{BGT}}$, there exists an element $E^{\mathrm{st}} \in C^{\mathrm{st}}_{\mathrm{BGT}}$ (respectively, $\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}$) such that $E \subseteq E^{\mathrm{st}}$ (respectively, $E \subseteq \mathrm{Cusp}(\mathbb{B}^{\rtimes})$).

In particular, we conclude [cf. Remarks 5.1.1, 5.1.2; Proposition 5.2, (ii), below] that we may write

$$\Pi = \varprojlim_{E \in C_{\mathrm{BGT}}} \Pi_E = \varprojlim_{E^{\mathrm{st}} \in C^{\mathrm{st}}_{\mathrm{BGT}}} \Pi_{E^{\mathrm{st}}},$$

$$\Pi \overset{\mathrm{out}}{\rtimes} \mathrm{BGT} = \varprojlim_{E^{\mathrm{st}} \in C^{\mathrm{st}}_{\mathrm{BGT}}} \Pi_{E^{\mathrm{st}}} \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}$$

— where, in the inverse limits, we regard C_{BGT} and $C^{\mathrm{st}}_{\mathrm{BGT}}$ as directed preordered sets by means of the relation of inclusion of subsets of $\mathrm{Cusp}(\Pi)$.

PROPOSITION 5.2 (Basic properties of Π). *In the notation of Definition 5.1, the following hold:*

(i) For each $E \in C_{\mathrm{BGT}}$ of cardinality r, there exists an isomorphism of profinite groups between Π_E, on the one hand, and the étale fundamental group of an open subscheme of X obtained by removing $r-3$ distinct points from X, on the other, that induces a bijection between the respective sets of cuspidal inertia subgroups.

(ii) For each $E \in C_{\mathrm{BGT}}$, Π_E is slim. In particular, Π is slim.

(iii) The group $\Pi \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}$ admits a natural structure of profinite group.

(iv) Let Π^* be a normal open subgroup of Π. Then, for any sufficiently small normal open subgroup $M \subseteq \mathrm{BGT}$, there exist an outer action of M on Π^* and an open injection $\Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}$ such that

(a) the outer action of M on Π^* preserves the set of cuspidal inertia subgroups of Π^*;

(b) the outer action of M on Π^* extends uniquely [cf. the slimness of Π] to a Π^*-outer action on Π that is compatible with the outer action of BGT $(\supseteq M)$ on Π; the injection $\Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}$ is the injection determined by the inclusions $\Pi^* \subseteq \Pi$ and $M \subseteq \mathrm{BGT}$, together with the Π^*-outer actions of M on Π^* and Π.

(v) In the notation of (iv), the homomorphism $\Pi^* \overset{\mathrm{out}}{\rtimes} M \to \mathrm{Aut}(\Pi^*)$ determined by conjugation is injective.

PROOF. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from [MT], Proposition 1.4. Assertion (iii) follows immediately, in light of the second line of the final display of Remark 5.1.3, from Remark 5.1.2. Next, since, in the notation of Definition 5.1, (i), Π^* arises as the inverse image in Π of some normal open subgroup of some $\Pi_{\mathbb{B}^{\rtimes}}$, assertion (iv) follows immediately from a similar argument to the argument applied in the proof of [Tsjm], Lemma 1.2.

Finally, we verify assertion (v). First, we note that since Π, hence also Π^*, is slim [cf. Proposition 5.2, (ii)], the restriction of the homomorphism $\Pi^* \overset{\text{out}}{\rtimes} M \to \mathrm{Aut}(\Pi^*)$ to Π^* is injective. Note also that since the natural surjection $\Pi \twoheadrightarrow \Pi_X$ is compatible with the respective outer actions of M, and $M \subseteq \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$, the natural homomorphism $M \to \mathrm{Out}(\Pi)$ is injective. In particular, since Π is slim, it follows immediately from condition (b) of Proposition 5.2, (iv), that the natural homomorphism $M \to \mathrm{Out}(\Pi^*)$ is injective. Thus, we conclude that the homomorphism $\Pi^* \overset{\text{out}}{\rtimes} M \to \mathrm{Aut}(\Pi^*)$ is injective. This completes the proof of assertion (v), hence of Proposition 5.2. $\square$

Definition 5.3. In the following, we consider the analogues of [Tsjm], Definition 1.5, (i), (ii); [Tsjm], Corollary 1.6, (ii), obtained by replacing "Π_X" by $\Pi_{X_{0\infty}}$. Let J be a closed subgroup of $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$.

(i) Fix an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$

$$\begin{tikzcd} \Pi_U \overset{\text{out}}{\rtimes} M \arrow[r] \arrow[d] & \Pi_X \overset{\text{out}}{\rtimes} M \\ \Pi_X \overset{\text{out}}{\rtimes} M & \end{tikzcd}$$

[cf. [Tsjm], Definition 1.4]. Write

$$\mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}, M, J)$$

for the set consisting of the images via the natural composite $\Pi_{X_{0\infty}}$-outer homomorphism $\Pi_U \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_X \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} J \twoheadrightarrow \Pi_{X_{0\infty}} \rtimes J$ of the normalizers in $\Pi_U \overset{\text{out}}{\rtimes} M$ of the cuspidal inertia subgroups of Π_U that are not associated to 0 and ∞;

$$\mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}, J)$$

for the quotient set $\big(\sqcup_{M^\dagger \subseteq J} \mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}|_{M^\dagger}, M^\dagger, J)\big)/\sim$, where $M^\dagger$ ranges over the normal open subgroups of J contained in M, and we write

$\mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}|_{M^\dagger}, M^\dagger, J) \ni G_{M^\dagger} \sim G_{M^\ddagger} \in \mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}|_{M^\ddagger}, M^\ddagger, J)$ if $G_{M^\dagger} \cap G_{M^\ddagger}$ is open in both $G_{M^\dagger}$ and $G_{M^\ddagger}$. Finally, we observe that $\Pi_{X_{0\infty}}$ acts naturally on $\mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}, M, J)$ and $\mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}, J)$.

(ii) Write

$$\mathbb{D}_{0\infty}(J)$$

for the quotient set $\left(\sqcup_{\mathbb{B}^{\rtimes}} \mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}, J)\right)/\sim$, where $\mathbb{B}^{\rtimes}$ ranges over all arithmetic Belyi diagrams, and we write $\mathbb{D}_{0\infty}({}^\dagger\mathbb{B}^{\rtimes}, J) \ni G_{{}^\dagger\mathbb{B}^{\rtimes}} \sim G_{{}^\ddagger\mathbb{B}^{\rtimes}} \in \mathbb{D}_{0\infty}({}^\ddagger\mathbb{B}^{\rtimes}, J)$ if $G_{M^\dagger} \cap G_{M^\ddagger}$ is open in both $G_{M^\dagger}$ and $G_{M^\ddagger}$ for some representative $G_{M^\dagger}$ (respectively, $G_{M^\ddagger}$) of $G_{{}^\dagger\mathbb{B}^{\rtimes}}$ (respectively, $G_{{}^\ddagger\mathbb{B}^{\rtimes}}$). Observe that $\Pi_{X_{0\infty}}$ acts naturally on $\mathbb{D}_{0\infty}(J)$.

(iii) Write

$$D_{0\infty}(J)$$

for the quotient set $\mathbb{D}_{0\infty}(J)/\Pi_{X_{0\infty}}$.

Remark 5.3.1. In the following, we consider certain slightly generalized analogues of [Tsjm], Corollary 1.6, (ii), (iii), obtained by replacing "Π_X" by $\Pi_{X_{0\infty}}$. Let J be a closed subgroup of $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$. Then it follows immediately from a similar argument [cf. also Remarks 3.3.2, 3.3.3] to the argument applied in the proof of [Tsjm], Corollary 1.6, together with the various definitions involved, that:

- $D_{0\infty}(J)$ admits a natural action by $C_{\mathrm{GT}}(J)$, hence, in particular, by J.
- Let J_1 and J_2 be closed subgroups of GT. If $J_1 \subseteq J_2 \subseteq \mathrm{GT}$, then the inclusion $J_1 \subseteq J_2$ induces, by considering the intersection of subgroups of $\Pi_{X_{0\infty}} \rtimes J_2$ with $\Pi_{X_{0\infty}} \rtimes J_1$, a natural surjection

$$D_{0\infty}(J_2) \twoheadrightarrow D_{0\infty}(J_1)$$

that is equivariant with respect to the natural actions of J_1 ($\subseteq J_2$) on the domain and codomain.

Lemma 5.4 (Kummer classes of group-theoretic constant functions). *We maintain the notation of Definitions 4.1, 5.3. Then the following hold:*

(i) There exists a natural injection

$$\iota_{\mathrm{BGT}} : D_{0\infty}(\mathrm{BGT}) \hookrightarrow \varinjlim_{M\subseteq \mathrm{BGT}} H^1(M, \Pi_{X_{0\infty}}),$$

where M ranges over the normal open subgroups of BGT.

(ii) There exists a natural surjection

$$\psi_{\mathrm{BGT}} : \overline{\mathbb{Q}}^{\times}_{\mathrm{BGT}} \twoheadrightarrow D_{0\infty}(\mathrm{BGT}).$$

(iii) The above maps ι_{BGT} and ψ_{BGT} are compatible with the respective natural actions of $C_{\mathrm{GT}}(\mathrm{BGT})$ [cf. Proposition 4.3, Remark 5.3.1].

(iv) The composite

$$\iota_{\mathrm{BGT}} \circ \psi_{\mathrm{BGT}} : \overline{\mathbb{Q}}^{\times}_{\mathrm{BGT}} \to \varinjlim_{M\subseteq \mathrm{BGT}} H^1(M, \Pi_{X_{0\infty}})$$

is compatible with the operations "$\boxtimes_{\mathrm{BGT}}$" and "$\square^{-1}_{\mathrm{BGT}}$" on the domain [cf. Theorem 4.4, (i)] and the corresponding operations arising from the natural group structure on the codomain. In particular, the image of this composite is a subgroup of the codomain.

Proof. First, we verify assertion (i). Let I_1 be a cuspidal inertia subgroup of Π_X associated to the cusp "1". Then the image of the normalizer

$$N_{\Pi_X \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}}(I_1) \subseteq \Pi_X \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}$$

via the natural surjection $\Pi_X \overset{\mathrm{out}}{\rtimes} \mathrm{BGT} \twoheadrightarrow \Pi_{X_{0\infty}} \rtimes \mathrm{BGT}$ determines a section s_1 [cf. [CmbGC], Proposition 1.2, (ii)] of the second to last arrow of the natural exact sequence

$$1 \longrightarrow \Pi_{X_{0\infty}} \longrightarrow \Pi_{X_{0\infty}} \rtimes \mathrm{BGT} \longrightarrow \mathrm{BGT} \longrightarrow 1.$$

On the other hand, note that an element $x \in \mathbb{D}_{0\infty}(\mathbb{B}^{\rtimes}, M, \mathrm{BGT})$, where $\mathbb{B}^{\rtimes}$ denotes an arithmetic Belyi diagram as in Definition 5.3, (i) [i.e., where we take "J" to be BGT], determines a section s_x [cf. [CmbGC], Proposition 1.2, (ii)] of the restriction to M of the second to last arrow of the above exact sequence. Thus, by forming the difference κ_x between s_x and the restriction to M of s_1, one verifies immediately that the assignment $s_x \mapsto \kappa_x$ determines, by allowing $\mathbb{B}^{\rtimes} \in I_{\mathrm{BGT}}$ [hence also "M"] to *vary*, a natural map

$$\iota_{\mathrm{BGT}} : D_{0\infty}(\mathrm{BGT}) \to \varinjlim_{M \subseteq \mathrm{BGT}} H^1(M, \Pi_{X_{0\infty}}),$$

where M ranges over the normal open subgroups of BGT. Finally, the injectivity of ι_{BGT} follows immediately from the definitions of $D_{0\infty}(-)$ and $H^1(-,-)$. This completes the proof of assertion (i). Assertion (ii) follows immediately from the definitions of $\overline{\mathbb{Q}}^{\times}_{\mathrm{BGT}}$ and $D_{0\infty}(\mathrm{BGT})$. Assertion (iii) follows immediately from the definitions of the natural actions of $C_{\mathrm{GT}}(\mathrm{BGT})$ [cf., especially, the proof of Proposition 4.3]. Assertion (iv) follows immediately from the construction of the multiplication operation on the field $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ [i.e., the construction of "$\boxtimes_{\mathrm{BGT}}$" in the proof of Theorem 4.4, (i)] by means of the well-known *natural group structure* on $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, \infty\}$, i.e., "$(\mathbb{G}_m)_{\overline{\mathbb{Q}}}$". This completes the proof of Lemma 5.4. □

In the remainder of the present paper, we shall identify $D_{0\infty}(\mathrm{BGT})$ with $\mathrm{Im}(\iota_{\mathrm{BGT}})$ via the natural injection ι_{BGT}.

PROPOSITION 5.5 (Synchronizations of cuspidal inertia subgroups). *We maintain the notation of Definition 5.1. Then the following hold:*

(i) For each cuspidal inertia subgroup I_x of Π associated to $x \in \mathrm{Cusp}(\Pi)$, the natural scheme-theoretic isomorphism

$$I_x \xrightarrow{\sim} \Pi_{X_{0\infty}}$$

may be reconstructed, in a purely combinatorial/group-theoretic way, from the collection of data

$$(\Pi;\ \mathrm{Cusp}(\Pi);\ \{0, \infty\} \subseteq \mathrm{Cusp}(\Pi))$$

consisting of

- *a profinite group Π;*

- *a set* $\mathrm{Cusp}(\Pi)$ *of conjugacy classes of subgroups of* Π*;*
- *a subset* $\{0, \infty\} \subseteq \mathrm{Cusp}(\Pi)$ *of cardinality* 2 *[equipped with labels "0", "∞"] of the set* $\mathrm{Cusp}(\Pi)$.

(ii) Let $\Pi^* \subseteq \Pi$ *be an open subgroup;* $x \in \mathrm{Cusp}(\Pi^*)$*;* I_x^* *a cuspidal inertia subgroup of* Π^* *associated to* x*. Then one may construct a natural isomorphism*

$$I_x^* \xrightarrow{\sim} \Pi_{X_{0\infty}}$$

as follows: Write $I_x \stackrel{\mathrm{def}}{=} N_\Pi(I_x^*)$*. Note that* $I_x = N_\Pi(I_x) = C_\Pi(I_x) = C_\Pi(I_x^*)$ *is the unique cuspidal inertia subgroup of* Π *containing* I_x^* *[cf. Proposition 5.2, (i); [CmbGC], Proposition 1.2, (i), (ii)], and that the subgroup* $I_x^* \subseteq I_x$ *is of finite index* m*. Then since cuspidal inertia subgroups are abstractly isomorphic to* $\widehat{\mathbb{Z}}$ *[cf. [CmbGC], Remark 1.1.3], division by* m *determines an isomorphism* $I_x^* \xrightarrow{\sim} I_x$*. Thus, by composing with the isomorphism of (i), we obtain a natural isomorphism* $I_x^* \xrightarrow{\sim} \Pi_{X_{0\infty}}$*.*

Proof. First, we verify assertion (i). Let I_0 be a cuspidal inertia subgroup of Π associated to the cusp "$0 \in \mathrm{Cusp}(\Pi)$". Write

$$\Pi \twoheadrightarrow \Pi_{\{0,x\}}$$

for the quotient profinite group of Π obtained by forming the quotient of Π by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to $\mathrm{Cusp}(\Pi) \setminus \{0, x\}$. Then the surjection $\Pi \twoheadrightarrow \Pi_{\{0,x\}}$ induces isomorphisms

$$\alpha_0 : I_0 \xrightarrow{\sim} \Pi_{\{0,x\}}, \quad \alpha_x : I_x \xrightarrow{\sim} \Pi_{\{0,x\}}.$$

Write $\alpha : I_x \xrightarrow{\sim} I_0$ for the composite of $\alpha_0^{-1} \circ \alpha_x$ with the inversion map $I_0 \xrightarrow{\sim} I_0$. Thus, by composing α with the restriction to I_0 of the natural surjection $\Pi \twoheadrightarrow \Pi_{X_{0\infty}}$, we obtain an isomorphism $I_x \xrightarrow{\sim} \Pi_{X_{0\infty}}$. The desired functoriality follows immediately from the construction. This completes the proof of assertion (i).

Assertion (ii) follows immediately from the various definitions involved. This completes the proof of Proposition 5.5. □

Definition 5.6. In the notation of Definition 5.1, let $\Pi^* \subseteq \Pi$ be an open subgroup. Fix

- a *normal open subgroup* $M \subseteq \mathrm{BGT}$,
- an *outer action* of M on Π^*, and
- an *open injection* $f_M : \Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}$

such that

(a) the outer action of M on Π^* preserves the set of cuspidal inertia subgroups of Π^*;

(b) the outer action of M on Π^* extends uniquely [cf. the slimness of Π] to a Π^*-outer action on Π that is compatible with the outer action of BGT ($\supseteq M$) on Π; the injection $\Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} \mathrm{BGT}$ is the injection determined by the inclusions $\Pi^* \subseteq \Pi$ and $M \subseteq \mathrm{BGT}$, together with the Π^*-outer actions of M on Π^* and Π

[cf. Proposition 5.2, (iv)]. Write

$$I(\Pi^*, \Pi)$$

for the *set of open injections* $f_{\Pi^*} : \Pi^* \hookrightarrow \Pi$ satisfying the following properties:

(1) For each cuspidal inertia subgroup I^* of Π^*, the commensurator $C_\Pi(f_{\Pi^*}(I^*))$ of $f_{\Pi^*}(I^*)$ in Π is a cuspidal inertia subgroup of Π [which implies, by Proposition 5.5, (ii), that $C_\Pi(f_{\Pi^*}(I^*)) = N_\Pi(f_{\Pi^*}(I^*))$].

(2) For each cuspidal inertia subgroup I of Π, the inverse image $f_{\Pi^*}^{-1}(I) \subseteq \Pi^*$ is a cuspidal inertia subgroup of Π^*.

(3) Let I^* be a cuspidal inertia subgroup of Π^*; I a cuspidal inertia subgroup of Π such that $I^* = f_{\Pi^*}^{-1}(I)$. Then the composite

$$\Pi_{X_{0\infty}} \overset{\sim}{\leftarrow} I^* \hookrightarrow I \overset{\sim}{\rightarrow} \Pi_{X_{0\infty}}$$

— where the first and final arrows are the isomorphisms of Proposition 5.5, (i), (ii) — coincides with the homomorphism determined by multiplication by some positive integer.

(4) For any sufficiently small normal open subgroup $N^* \subseteq M$ of BGT, there exists a(n) [necessarily *unique* — cf. Remark 5.6.1 below] open injection

$$\Pi^* \overset{\text{out}}{\rtimes} N^* \hookrightarrow \Pi \overset{\text{out}}{\rtimes} N^*$$

that is compatible with the open injection between respective subgroups $f_{\Pi^*} : \Pi^* \hookrightarrow \Pi$ and the surjections to N^* ($\subseteq$ BGT).

Remark 5.6.1. Note that any open injection $\Pi^* \overset{\text{out}}{\rtimes} N^* \hookrightarrow \Pi \overset{\text{out}}{\rtimes} N^*$ as in Definition 5.6, (4), is *unique*. Indeed, let $f : \Pi^* \overset{\text{out}}{\rtimes} N^* \hookrightarrow \Pi \overset{\text{out}}{\rtimes} N^*$ be an *open injection* as in Definition 5.6, (4); $\Pi^{**} \subseteq \Pi$ an open subgroup such that $\Pi^{**} \subseteq f_{\Pi^*}(\Pi^*)$, and $\Pi^{**} \subseteq \Pi \overset{\text{out}}{\rtimes} N^*$ is a normal closed subgroup. Write $\mathrm{Aut}_{\Pi^{**}}(\Pi) \subseteq \mathrm{Aut}(\Pi)$ for the subgroup of automorphisms that preserve the normal open subgroup $\Pi^{**} \subseteq \Pi$. Then we have a commutative diagram

$$\begin{array}{ccc}
\Pi^* \overset{\text{out}}{\rtimes} N^* & \xrightarrow{f} & \Pi \overset{\text{out}}{\rtimes} N^* \\
\downarrow & & \downarrow \\
\mathrm{Aut}(\Pi^{**}) & \longleftarrow & \mathrm{Aut}_{\Pi^{**}}(\Pi),
\end{array}$$

where the vertical arrows denote the injections determined by the respective actions by conjugation; the lower horizontal arrow denotes the natural injection [cf. Proposition 5.2, (ii)]. Thus, we conclude that the open injection $f : \Pi^* \overset{\text{out}}{\rtimes} N^* \hookrightarrow \Pi \overset{\text{out}}{\rtimes} N^*$ is uniquely determined by the open injection f_{Π^*} and the respective outer actions of N^* on Π^* and Π, hence that any open injection as in Definition 5.6, (4), is unique.

Remark 5.6.2. In the notation of Definition 5.6, let $\Pi^{**} \subseteq \Pi$ be an open subgroup contained in Π^*. Then the inclusion $\Pi^{**} \subseteq \Pi^*$ determines a natural map $I(\Pi^*, \Pi) \to I(\Pi^{**}, \Pi)$ [cf. Propositions 5.2, (iv); 5.5, (ii)].

Proposition 5.7 (Kummer classes of group-theoretic nonconstant functions). *In the notation of Definition 5.6, let $f_{\Pi^*} \in I(\Pi^*, \Pi)$. Then f_{Π^*} naturally determines an element of*

$$\varinjlim_{N^* \subseteq \mathrm{BGT}} H^1(\Pi^* \overset{\text{out}}{\rtimes} N^*, \Pi_{X_{0\infty}}),$$

where N^* *ranges over the normal open subgroups of* BGT. *In particular, we obtain a natural map*

$$\kappa_{\Pi^*} : I(\Pi^*, \Pi) \to \varinjlim_{N^* \subseteq \mathrm{BGT}} H^1(\Pi^* \overset{\mathrm{out}}{\rtimes} N^*, \Pi_{X_{0\infty}}).$$

Proof. Let $\Pi^* \overset{\mathrm{out}}{\rtimes} N^* \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N^*$ be an *open injection* as in Definition 5.6, (4). Write

$$s_{f_{\Pi^*}} : \Pi^* \overset{\mathrm{out}}{\rtimes} N^* \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N^* \twoheadrightarrow \Pi_{X_{0\infty}} \rtimes N^*$$

for the composite of this *open injection* $\Pi^* \overset{\mathrm{out}}{\rtimes} N^* \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N^*$ with the natural $\Pi_{X_{0\infty}}$-outer surjection $\Pi \overset{\mathrm{out}}{\rtimes} N^* \twoheadrightarrow \Pi_{X_{0\infty}} \rtimes N^*$. Let I_1 be a cuspidal inertia subgroup of Π_X associated to the cusp "1". Then I_1 determines a section $s_1|_{N^*}$ of the surjection $\Pi_{X_{0\infty}} \rtimes N^* \twoheadrightarrow N^*$ [cf. the proof of Lemma 5.4, (i)]. In particular, by composing $s_1|_{N^*}$ with the natural surjection $\Pi^* \overset{\mathrm{out}}{\rtimes} N^* \twoheadrightarrow N^*$, we obtain a homomorphism

$$s_1|_{\Pi^* \overset{\mathrm{out}}{\rtimes} N^*} : \Pi^* \overset{\mathrm{out}}{\rtimes} N^* \to \Pi_{X_{0\infty}} \rtimes N^*.$$

Thus, by forming the difference between $s_{f_{\Pi^*}}$ and $s_1|_{\Pi^* \overset{\mathrm{out}}{\rtimes} N^*}$, we obtain an element $\in H^1(\Pi^* \overset{\mathrm{out}}{\rtimes} N^*, \Pi_{X_{0\infty}})$, hence an element

$$f^{\kappa}_{\Pi^*} \in \varinjlim_{N^* \subseteq \mathrm{BGT}} H^1(\Pi^* \overset{\mathrm{out}}{\rtimes} N^*, \Pi_{X_{0\infty}}).$$

Finally, it follows immediately from the various definitions involved that $f^{\kappa}_{\Pi^*}$ is *independent* of the choice of I_1 [within its Π_X-conjugacy class]. This completes the proof of Proposition 5.7. □

Definition 5.8. We maintain the notation of Definition 5.6. Let $f_{\Pi^*} \in I(\Pi^*, \Pi)$; $x \in \mathrm{Cusp}(\Pi^*)$; I_x a cuspidal inertia subgroup of Π^* associated to x. Then we define the *value*

$$f_{\Pi^*}(x) \in \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$$

of f_{Π^*} at x to be the image of the element $\in \mathrm{Cusp}(\Pi)$ determined by the cuspidal inertia subgroup $N_{\Pi}(f_{\Pi^*}(I_x)) \subseteq \Pi$ via the *natural* $C_{\mathrm{GT}}(\mathrm{BGT})$-*equivariant bijection* $\mathrm{Cusp}(\Pi) \xrightarrow{\sim} \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$ [cf. Definition 5.1, (ii)]. It follows immediately from the various definitions involved that $f_{\Pi^*}(x) \in \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}$ does not depend on the choice of I_x within its Π^*-conjugacy class.

Definition 5.9. We maintain the notation of Definition 5.8.

(i) Write

$$F_{\Pi^*} : I(\Pi^*, \Pi) \to \mathrm{Fn}(\mathrm{Cusp}(\Pi^*),\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\})$$

(respectively,

$$B_{\Pi^*} : \overline{\mathbb{Q}}_{\mathrm{BGT}} \to \mathrm{Fn}(\mathrm{Cusp}(\Pi^*),\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}))$$

for the natural map determined by considering the *value* (respectively, the *constant value*) at each of the elements $\in \mathrm{Cusp}(\Pi^*)$. Then we shall write

$$L_{\Pi^*} \overset{\mathrm{def}}{=} \mathrm{Im}\ F_{\Pi^*} \bigcup \mathrm{Im}\ B_{\Pi^*} \subseteq \mathrm{Fn}(\mathrm{Cusp}(\Pi^*),\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\})).$$

(ii) For each finite subset $S \subseteq \mathrm{Cusp}(\Pi^*)$, we shall write

$$\Pi^*_S$$

for the quotient of Π^* by the normal closed subgroup topologically generated by the cuspidal inertia subgroups associated to $\mathrm{Cusp}(\Pi^*) \setminus S$ [cf. Definition 5.1, (ii)]. Suppose that the open subgroup $N \subseteq \mathrm{BGT}$ [cf. Definition 4.1] is contained in the open subgroup $M \subseteq \mathrm{BGT}$ [cf. Definition 5.6], and, moreover, that

$$N \subseteq \mathrm{BGT}\ \textit{induces the identity automorphism on}\ S.$$

Then we shall write

$$\Pi^*_{\rtimes N} \overset{\mathrm{def}}{=} \Pi^* \overset{\mathrm{out}}{\rtimes} N, \quad \Pi^*_{S \rtimes N} \overset{\mathrm{def}}{=} \Pi^*_S \overset{\mathrm{out}}{\rtimes} N.$$

[cf. Proposition 5.2, (ii)]. Write

$$I_S(\Pi^*, \Pi)$$

for the inverse image of

$$\mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S,\ \overline{\mathbb{Q}}_{\mathrm{BGT}}^{\times})\ (\subseteq \mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S,\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}))$$

by the composite of F_{Π^*} with the restriction map

$$\mathrm{Fn}(\mathrm{Cusp}(\Pi^*),\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}) \twoheadrightarrow \mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S,\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\});$$

$$F_{\Pi^*,S} : I_S(\Pi^*, \Pi) \to \mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S,\ \overline{\mathbb{Q}}_{\mathrm{BGT}}^{\times})$$

for the natural map induced by F_{Π^*};

$$\kappa_{\Pi^*,S} : I_S(\Pi^*, \Pi) \to \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{\rtimes N^*}, \Pi_{X_{0\infty}})$$

— where N^* ranges over the normal open subgroups of BGT contained in N — for the restriction of κ_{Π^*} to $I_S(\Pi^*, \Pi)$ [cf. Proposition 5.7]. Here, we note that it follows immediately from the various definitions involved [cf. the proof of Proposition 5.7] that $\kappa_{\Pi^*,S}$ factors as the composite of a natural map

$$\kappa_{\Pi^*_S} : I_S(\Pi^*, \Pi) \to \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}})$$

with the injection given by the inflation map

$$\varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}}) \hookrightarrow \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{\rtimes N^*}, \Pi_{X_{0\infty}}).$$

(iii) In the notation of (ii), let $x \in \mathrm{Cusp}(\Pi^*) \setminus S$; N_x a normal open subgroup of BGT contained in N that stabilizes x; $I_x \subseteq \Pi^*$ a cuspidal inertia subgroup associated to x. Then the image of $N_{\Pi^*_{\rtimes N_x}}(I_x)$ via the natural surjection $\Pi^*_{\rtimes N_x} \twoheadrightarrow \Pi^*_{S \rtimes N_x}$ determines a section $N_x \hookrightarrow \Pi^*_{S \rtimes N_x}$ of the natural surjection $\Pi^*_{S \rtimes N_x} \twoheadrightarrow N_x$ [cf. the proof of Lemma 5.4, (i)]. Thus, in particular, by allowing "N_x" to vary, we obtain a natural map

$$D_{\Pi^*_S} : \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}}) \longrightarrow \mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S,\ \vec{H}^1(N, \Pi_{X_{0\infty}})),$$

where $\vec{H}^1(N, \Pi_{X_{0\infty}}) \stackrel{\mathrm{def}}{=} \varinjlim_{N^* \subseteq N} H^1(N^*, \Pi_{X_{0\infty}})$.

REMARK 5.9.1. In the remainder of the present paper, we shall use the injection given by the inflation map in the final display of Definition 5.9, (ii), to regard the group $\varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}})$ as a subgroup of $\varinjlim_{N^* \subseteq N} H^1(\Pi^*_{\rtimes N^*}, \Pi_{X_{0\infty}})$.

REMARK 5.9.2. We maintain the notation of Definition 5.9. Note that, for each element $f_{\Pi^*} \in I(\Pi^*, \Pi)$, the set of Π^*-conjugacy classes of cuspidal inertia subgroups I^* of Π^* such that $f_{\Pi^*}(I^*)$ is contained in a *fixed* Π-conjugacy class of cuspidal inertia subgroups of Π is *finite*. Indeed, this follows immediately from the fact that f_{Π^*} is an *open injection* that induces a *bijection* between the cuspidal inertia subgroups of Π^* and Π — cf. Definition 5.6, (1), (2). Thus, it follows immediately from the various definitions involved that

$$I(\Pi^*, \Pi) = \bigcup_{S \subseteq \mathrm{Cusp}(\Pi^*)} I_S(\Pi^*, \Pi),$$

where S ranges over the finite subsets of $\mathrm{Cusp}(\Pi^*)$.

DEFINITION 5.10. We maintain the notation of Definition 5.9, (ii). Suppose that $S \neq \emptyset$, and that, for each normal open subgroup N^* of BGT contained in N,

$$H^1(\Pi^*_{\emptyset}, \Pi_{X_{0\infty}})^{N^*} = \{0\}.$$

Then we shall construct a subgroup

$$K^{\kappa}_{\Pi^*_S} \subseteq \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}})$$

as follows: First, we observe that the natural exact sequence

$$1 \longrightarrow \Pi^*_S \longrightarrow \Pi^*_{S \rtimes N^*} \longrightarrow N^* \longrightarrow 1$$

determines an exact sequence

$$0 \longrightarrow H^1(N^*, \Pi_{X_{0\infty}}) \longrightarrow H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}}) \xrightarrow{r} H^1(\Pi^*_S, \Pi_{X_{0\infty}})^{N^*}.$$

Thus, by allowing the normal open subgroup N^* to vary, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \varinjlim_{N^* \subseteq N} H^1(N^*, \Pi_{X_{0\infty}}) &\longrightarrow \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}}) \\ &\longrightarrow \varinjlim_{N^* \subseteq N} H^1(\Pi^*_S, \Pi_{X_{0\infty}})^{N^*}. \end{aligned}$$

Here, we observe that

$$H^1(\Pi^*_S, \Pi_{X_{0\infty}})^{N^*} = H^1((\Pi^*_S)^{\mathrm{ab}}, \Pi_{X_{0\infty}})^{N^*}.$$

Next, for each $x \in S$, let I_x be a cuspidal inertia subgroup of Π^* associated to x. Then we have an exact sequence of N^*-modules

$$\bigoplus_{x \in S} I_x \longrightarrow (\Pi^*_S)^{\mathrm{ab}} \longrightarrow (\Pi^*_\emptyset)^{\mathrm{ab}} \longrightarrow 0,$$

which determines an exact sequence of modules

$$\begin{aligned} 0 \longrightarrow H^1((\Pi^*_\emptyset)^{\mathrm{ab}}, \Pi_{X_{0\infty}})^{N^*} &\longrightarrow H^1((\Pi^*_S)^{\mathrm{ab}}, \Pi_{X_{0\infty}})^{N^*} \\ \longrightarrow \bigoplus_{x \in S} H^1(I_x, \Pi_{X_{0\infty}}). \end{aligned}$$

Thus, by applying our assumption that $H^1((\Pi^*_\emptyset)^{\mathrm{ab}}, \Pi_{X_{0\infty}})^{N^*} = \{0\}$, we obtain a natural injection

$$i : H^1((\Pi^*_S)^{\mathrm{ab}}, \Pi_{X_{0\infty}})^{N^*} \hookrightarrow \bigoplus_{x \in S} H^1(I_x, \Pi_{X_{0\infty}}).$$

Write

$$1_x \in H^1(I_x, \Pi_{X_{0\infty}}) = \mathrm{Hom}(I_x, \Pi_{X_{0\infty}})$$

for the isomorphism $I_x \xrightarrow{\sim} \Pi_{X_{0\infty}}$ of Proposition 5.5, (ii);

$$\mathbb{Z}_x \subseteq H^1(I_x, \Pi_{X_{0\infty}})$$

for the subgroup generated by 1_x;

$$i_x : N^* \hookrightarrow \Pi^*_{\emptyset \rtimes N^*}$$

for the section of the natural surjection $\Pi^*_{\emptyset \rtimes N^*} \twoheadrightarrow N^*$ determined by the image of $N_{\Pi^*_{S \rtimes N^*}}(I_x)$ via the natural surjection $\Pi^*_{S \rtimes N^*} \twoheadrightarrow \Pi^*_{\emptyset \rtimes N^*}$ [cf. the proof of Lemma 5.4, (i)]. Next, we fix $x_0 \in S$. Write

$$D_x \in H^1(N^*, (\Pi^*_\emptyset)^{\mathrm{ab}})$$

for the element obtained by forming the difference between i_{x_0} and i_x;

$$\mathcal{P}_S \ \subseteq \ \bigoplus_{x \in S} \mathbb{Z}_x \ (\subseteq \ \bigoplus_{x \in S} H^1(I_x, \Pi_{X_{0\infty}}))$$

for the subgroup consisting of $(n_x)_{x \in S} \in \bigoplus_{x \in S} \mathbb{Z}_x$ such that

$$\sum_{x \in S} n_x \ = \ 0, \qquad \sum_{x \in S} n_x \cdot D_x \ = \ 0 \ (\in H^1(N^*, (\Pi^*_\emptyset)^{\mathrm{ab}}))$$

[where we note that one verifies immediately that these conditions on $(n_x)_{x \in S}$ are *independent* of the choice of $x_0 \in S$];

$$\mathcal{P}^\kappa_S$$

for the image of $(i \circ r)^{-1}(\mathcal{P}_S)$ via the natural homomorphism $H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}}) \to \varinjlim_{M^* \subseteq N} H^1(\Pi^*_{S \rtimes M^*}, \Pi_{X_{0\infty}})$, where M^* ranges over the normal open subgroups of BGT contained in N. Then we define

$$\begin{aligned} K^\kappa_{\Pi^*_S} &\stackrel{\mathrm{def}}{=} \mathcal{P}^\kappa_S \bigcap D^{-1}_{\Pi^*_S}(\mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S, \ D_{0\infty}(\mathrm{BGT}))) \\ &\subseteq \varinjlim_{M^* \subseteq N} H^1(\Pi^*_{S \rtimes M^*}, \Pi_{X_{0\infty}}) \end{aligned}$$

[cf. Lemma 5.4, (i); Definition 5.9, (iii)] and

$$K^\kappa_{\Pi^*} \stackrel{\mathrm{def}}{=} \bigcup_T K^\kappa_{\Pi^*_T} \subseteq \varinjlim_{M^* \subseteq N} H^1(\Pi^*_{\rtimes M^*}, \Pi_{X_{0\infty}}),$$

where T ranges over the finite subsets of $\mathrm{Cusp}(\Pi^*)$.

REMARK 5.10.1. In the notation of Definition 5.10, suppose that $\mathrm{BGT} = G_{\mathbb{Q}}$ [cf. Remark 4.4.1]. Then the above construction of $K^\kappa_{\Pi^*}$ may be understood as a sort of reconstruction of the set of Kummer classes of rational functions associated to Π^*, i.e., in the spirit of [AbsTopIII], Proposition 1.8.

Lemma 5.11 (Kummer classes of abstract functions). *We maintain the notation of Definitions 5.9, 5.10. Suppose that the restriction $D_{\Pi^*_S}|_{K^\kappa_{\Pi^*_S}}$ of $D_{\Pi^*_S}$ to $K^\kappa_{\Pi^*_S}$ [cf. Definition 5.9, (iii); Definition 5.10] is injective for arbitrary choices of "S" and "N" as in Definition 5.10. Then there exists a unique map*

$$\mathrm{Im}(F_{\Pi^*,S}) \to \mathrm{Im}(\kappa_{\Pi^*_S})$$

[cf. Definition 5.9, (ii)] whose composite with the natural surjection $I_S(\Pi^*,\Pi) \twoheadrightarrow \mathrm{Im}(F_{\Pi^*,S})$ determined by $F_{\Pi^*,S}$ coincides with the natural surjection $I_S(\Pi^*,\Pi) \twoheadrightarrow \mathrm{Im}(\kappa_{\Pi^*_S})$ determined by $\kappa_{\Pi^*_S}$, and whose image lies in $K^\kappa_{\Pi^*_S}$. Moreover, by allowing S to vary, one obtains a natural map

$$L_{\Pi^*} \setminus \{0\} \to \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{\rtimes N^*}, \Pi_{X_{0\infty}})$$

[cf. Remarks 5.9.1, 5.9.2] whose image lies in $K^\kappa_{\Pi^*}$.

Proof. First, we observe that it follows from the various definitions involved that there exists a commutative diagram

$$\begin{array}{ccc}
I_S(\Pi^*,\Pi) & \xrightarrow{F_{\Pi^*,S}} & \mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S,\ \overline{\mathbb{Q}}^\times_{\mathrm{BGT}}) \\
{\scriptstyle \kappa_{\Pi^*_S}}\downarrow & & \downarrow \\
\varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}}) & \xrightarrow[D_{\Pi^*_S}]{} & \mathrm{Fn}(\mathrm{Cusp}(\Pi^*) \setminus S,\ \varinjlim_{N^* \subseteq N} H^1(N^*, \Pi_{X_{0\infty}})),
\end{array}$$

where the right-hand vertical arrow is the natural map induced by the homomorphism

$$\iota_{\mathrm{BGT}} \circ \psi_{\mathrm{BGT}} : \overline{\mathbb{Q}}^\times_{\mathrm{BGT}} \to \varinjlim_{N^* \subseteq N} H^1(N^*, \Pi_{X_{0\infty}})$$

[cf. Lemma 5.4, (iv)].

Next, we observe that $\kappa_{\Pi^*_S}$ factors as the composite of a map

$$I_S(\Pi^*,\Pi) \longrightarrow K^\kappa_{\Pi^*_S}$$

with the inclusion $K^{\kappa}_{\Pi^*_S} \subseteq \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{S \rtimes N^*}, \Pi_{X_{0\infty}})$ [cf. Definition 5.10]. Indeed, since $(\Pi_\emptyset)^{\mathrm{ab}} = \{0\}$ [hence, in particular, $H^1(N^*, (\Pi_\emptyset)^{\mathrm{ab}}) = \{0\}$], it follows immediately from the various definitions involved that $\kappa_{\Pi_{\{0,\infty\}}}$ maps $\mathrm{id} \in I_{\{0,\infty\}}(\Pi, \Pi)$ [cf. Proposition 5.5, (ii)] to an element of $K^{\kappa}_{\Pi_{\{0,\infty\}}}$. Thus, since any element $f_{\Pi^*} \in I_S(\Pi^*, \Pi)$ may be thought of as the pull-back "via f_{Π^*}" of $\mathrm{id} \in I_{\{0,\infty\}}(\Pi, \Pi)$, by applying the functoriality of the constructions involved [cf. also Definition 5.6, (3)], we obtain the desired conclusion.

Next, we apply our assumption that $D_{\Pi^*_S}|_{K^{\kappa}_{\Pi^*_S}}$ is *injective*. Thus, since the above diagram is commutative, there exists a unique map $\mathrm{Im}(F_{\Pi^*,S}) \to \mathrm{Im}(\kappa_{\Pi^*_S})$ that is compatible with the maps $F_{\Pi^*,S}$ and $\kappa_{\Pi^*_S}$ in the desired sense. In particular, since all of the constructions involved are *functorially compatible* with enlargement of the finite subset $S \subseteq \mathrm{Cusp}(\Pi^*)$, by allowing $S \subseteq \mathrm{Cusp}(\Pi^*)$ to vary, we obtain a natural map

$$\mathrm{Im}(F_{\Pi^*}) \to \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{\rtimes N^*}, \Pi_{X_{0\infty}})$$

[cf. Remarks 5.9.1, 5.9.2]. On the other hand, by considering the composite of $\iota_{\mathrm{BGT}} \circ \psi_{\mathrm{BGT}}$ with the inflation map

$$\varinjlim_{N^* \subseteq N} H^1(N^*, \Pi_{X_{0\infty}}) \hookrightarrow \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{\rtimes N^*}, \Pi_{X_{0\infty}}),$$

we obtain a natural map

$$\mathrm{Im}(B_{\Pi^*}) \setminus \{0\} \to \varinjlim_{N^* \subseteq N} H^1(\Pi^*_{\rtimes N^*}, \Pi_{X_{0\infty}}).$$

Thus, since $L_{\Pi^*} = \mathrm{Im}\ F_{\Pi^*} \cup \mathrm{Im}\ B_{\Pi^*}$ [where we note that $\mathrm{Im}\ F_{\Pi^*} \cap \mathrm{Im}\ B_{\Pi^*} = \emptyset$ — cf. Remark 5.9.2], we obtain the desired conclusion. This completes the proof of Lemma 5.11. □

Definition 5.12. Let $\mathrm{BGT} \subseteq \mathrm{GT}$ be a closed subgroup that satisfies the *ArBC-property* [cf. Theorem 4.4, (ii)]. In the following discussion, we apply the notation of Definitions 4.1, 5.1, 5.6, 5.8, 5.9. Write $t \in L_\Pi$ for the element determined by $\mathrm{id} \in I(\Pi, \Pi)$ [cf. Proposition 5.5, (ii)]. Then, if BGT satisfies the following *purely combinatorial/group-theoretic* [cf. Corollary 4.5, together with the various definitions involved] conditions (i), (ii), (iii) (respectively, (i), (ii), (iii), (iv)), then we shall say that BGT satisfies the *QAA-property* [i.e., "quasi-algebraically ample property"] (respectively, *AA-property* [i.e., "algebraically ample property"]):

(i) Write $(\overline{\mathbb{Q}}_{\mathrm{BGT}})_{\mathrm{div}} \subseteq \overline{\mathbb{Q}}_{\mathrm{BGT}}$ for the subfield generated over $\mathbb{Q}$ by $\mathrm{Ker}(\iota_{\mathrm{BGT}} \circ \psi_{\mathrm{BGT}})$ [cf. Lemma 5.4, (iv)]. Then $(\overline{\mathbb{Q}}_{\mathrm{BGT}})_{\mathrm{div}} \subseteq \overline{\mathbb{Q}}_{\mathrm{BGT}}$ is an *infinite extension of fields.*

(ii) For

- each normal open subgroup $\Pi^\dagger \subseteq \Pi$,
- each nonempty finite subset $S \subseteq \mathrm{Cusp}(\Pi^\dagger)$, and
- any sufficiently small normal open subgroup $N^\dagger$ of BGT,

it holds that $H^1(\Pi^\dagger_\emptyset, \Pi_{X_{0\infty}})^{N^\dagger} = \{0\}$ [cf. Definition 5.10], and the restriction $D_{\Pi^\dagger_S}|_{K^\kappa_{\Pi^\dagger_S}}$ of $D_{\Pi^\dagger_S}$ to $K^\kappa_{\Pi^\dagger_S}$ [cf. Definition 5.9, (iii); Definition 5.10] is *injective* [cf. Lemma 5.11].

(iii) Assume that condition (ii) holds. There exists a *family of subsets*

$$\{K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$$

— where $\Pi^\dagger$ ranges over the normal open subgroups of Π — satisfying the following conditions:

(a) Let $\Pi^\ddagger \subseteq \Pi^\dagger$ be normal open subgroups of Π. Then the natural injection $L_{\Pi^\dagger} \hookrightarrow L_{\Pi^\ddagger}$ [determined by the natural surjection $\mathrm{Cusp}(\Pi^\ddagger) \twoheadrightarrow \mathrm{Cusp}(\Pi^\dagger)$ — cf. Proposition 5.5, (ii); Remarks 5.6.2, 5.9.2] induces an *injection*

$$K_{\Pi^\dagger} \hookrightarrow K_{\Pi^\ddagger}.$$

In the remainder of the present paper, we regard $K_{\Pi^\dagger}$ as a subset of $K_{\Pi^\ddagger}$ via this injection.

(b) For each normal open subgroup $\Pi^\dagger \subseteq \Pi$, and each finite subset $R \subseteq \mathrm{Cusp}(\Pi^\dagger)$, the restriction to $K_{\Pi^\dagger}$ of the natural restriction map

$$\mathrm{Fn}(\mathrm{Cusp}(\Pi^\dagger),\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}) \twoheadrightarrow \mathrm{Fn}(\mathrm{Cusp}(\Pi^\dagger) \setminus R,\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\})$$

is *injective.*

(c) For each normal open subgroup $\Pi^\dagger \subseteq \Pi$, $K_{\Pi^\dagger}$ admits a [necessarily *unique* — cf. (b)] *field structure* compatible with the ring structure of $\mathrm{Fn}(\mathrm{Cusp}(\Pi^\dagger),\ \overline{\mathbb{Q}}_{\mathrm{BGT}})$ in the following sense: Let $f, g \in K_{\Pi^\dagger}$, $T \subseteq \mathrm{Cusp}(\Pi^\dagger)$ a finite subset such that $f(x), g(x) \in \overline{\mathbb{Q}}_{\mathrm{BGT}}$ for any $x \in \mathrm{Cusp}(\Pi^\dagger) \setminus T$. [For given elements $f, g \in K_{\Pi^\dagger}$, the existence of such a finite set T follows immediately from Remark 5.9.2.] Then the images of $f+g$ and fg via the restriction map

$$\mathrm{Fn}(\mathrm{Cusp}(\Pi^\dagger),\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\}) \twoheadrightarrow \mathrm{Fn}(\mathrm{Cusp}(\Pi^\dagger) \setminus T,\ \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\})$$

coincide, respectively, with the functions

$$\mathrm{Cusp}(\Pi^\dagger) \setminus T \ni x \mapsto f(x) + g(x) \in \overline{\mathbb{Q}}_{\mathrm{BGT}},$$

$$\mathrm{Cusp}(\Pi^\dagger) \setminus T \ni x \mapsto f(x)g(x) \in \overline{\mathbb{Q}}_{\mathrm{BGT}}.$$

Moreover, relative to these unique field structures, $K_\Pi \subseteq K_{\Pi^\dagger}$ is a *finite Galois extension.*

(d) $\overline{\mathbb{Q}}_{\mathrm{BGT}} = \mathrm{Im}\ B_\Pi \subseteq K_\Pi$, and $t \in K_\Pi$. Moreover, if we write $\overline{\mathbb{Q}}_{\mathrm{BGT}}(t) \subseteq K_\Pi$ for the *subfield generated by* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ *and* t, then $K_\Pi = \overline{\mathbb{Q}}_{\mathrm{BGT}}(t)$.

(e) For each normal open subgroup $\Pi^\dagger \subseteq \Pi$, the natural action of Π on $L_{\Pi^\dagger}$ [cf. Proposition 5.2, (iv)] preserves $K_{\Pi^\dagger}$. Moreover, the natural homomorphism

$$\Pi/\Pi^\dagger \to \mathrm{Gal}(K_{\Pi^\dagger}/K_\Pi)$$

is an *isomorphism.*

(f) For each normal open subgroup $\Pi^\dagger \subseteq \Pi$, the restriction to $K^\times_{\Pi^\dagger}$ ($\subseteq L_{\Pi^\dagger}$) of the natural map

$$L_{\Pi^\dagger} \setminus \{0\} \to K^\kappa_{\Pi^\dagger} \quad \left(\subseteq \varinjlim_{N^\dagger \subseteq \mathrm{BGT}} H^1(\Pi^\dagger_{\rtimes N^\dagger}, \Pi_{X_{0\infty}}) \right)$$

[cf. condition (ii); Definition 5.10; Lemma 5.11] is *surjective.*

(iv) Assume that conditions (ii), (iii) hold. In the notation of condition (iii), write $\overline{\mathbb{Q}}_{\mathrm{BGT}}[t, \frac{1}{t}, \frac{1}{t-1}] \subseteq L_\Pi$ for the $\overline{\mathbb{Q}}_{\mathrm{BGT}}$-subalgebra generated by t, $\frac{1}{t}$, and $\frac{1}{t-1}$; $X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}\ \overline{\mathbb{Q}}_{\mathrm{BGT}}[t, \frac{1}{t}, \frac{1}{t-1}]$. [Thus, it follows immediately from Lemma 5.13, (i), (ii), below that the natural outer surjection $\Pi \twoheadrightarrow \Pi_X$ determines a natural outer isomorphism $\Pi_{X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}} \xrightarrow{\sim} \Pi_X$.] Then the natural outer isomorphism $\Pi_{X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}} \xrightarrow{\sim} \Pi_X$ is induced by a(n) [*uniquely determined, up to composition with an element of* $\mathfrak{S}_5 \subseteq \mathrm{Out}(\Pi_{X_2})$ that fixes the element $5 \in \{1,2,3,4,5\}$ — cf. Corollary 3.1, (ii); Remark 4.5.1; [CbTpII], Theorem A, (i); the first display of [CbGT], Corollary C] *outer isomorphism*

$$\Pi_{X_2} \xrightarrow{\sim} \Pi_{(X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}})_2}$$

via the natural outer surjections $\Pi_{X_2} \twoheadrightarrow \Pi_X$ and $\Pi_{(X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}})_2} \twoheadrightarrow \Pi_{X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}}$ determined by the respective first projections [cf. Remark 5.12.2 below].

REMARK 5.12.1. In the notation of Remark 4.4.1, it follows immediately from Remark 4.4.1, together with the various definitions involved and the fact that F is *Kummer-faithful* [cf. [AbsTopIII], Definition 1.5; [AbsTopIII], Remark 1.5.4, (i)], that G_F satisfies the *AA-property* [cf. Theorem 6.8, (i) [and its proof, as well as Remark 6.6.2], below, for more details].

REMARK 5.12.2. In condition (iv), we regard Π_{X_2} as an abstract topological group and Π_X as a quotient of Π_{X_2}, i.e., as in Corollary 4.5 [cf. also Remark 4.5.1].

LEMMA 5.13 (Geometric interpretation of the set of cuspidal inertia subgroups of Π). *Suppose that* BGT *satisfies conditions (ii), (iii) of Definition 5.12. Let*

$$\{K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$$

be a family of subsets as in Definition 5.12, (iii). Write

$$\widetilde{K}_\Pi \stackrel{\mathrm{def}}{=} \varinjlim_{\Pi^\dagger \subseteq \Pi} K_{\Pi^\dagger},$$

where $\Pi^\dagger$ *ranges over the normal open subgroups of* Π. *Then the following hold:*

(i) Let $\Pi^\dagger \subseteq \Pi$ *be a normal open subgroup. Write* $Y_{\Pi^\dagger} \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$ *for the finite ramified Galois covering of smooth, proper, connected curves over* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ *corresponding to the extension of function fields* $\overline{\mathbb{Q}}_{\mathrm{BGT}}(t) = K_\Pi \subseteq K_{\Pi^\dagger}$ *[cf. Definition 5.12, (iii), (a), (c), (d), (e)];* $Y_{\Pi^\dagger}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$ *for the set of* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$*-valued points of* $Y_{\Pi^\dagger}$. *Then the natural map*

$$\mathrm{ev}_{\Pi^\dagger} : \mathrm{Cusp}(\Pi^\dagger) \to Y_{\Pi^\dagger}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$$

induced by evaluating elements of $K_{\Pi^\dagger}$ *at elements of* $\mathrm{Cusp}(\Pi^\dagger)$ *is* **bijective**.

(ii) $\widetilde{K}_\Pi$ *is an* **algebraic closure** *of* $\overline{\mathbb{Q}}_{\mathrm{BGT}}(t) = K_\Pi$. *Moreover, the natural action of* Π *on* $\widetilde{K}_\Pi$ *determines an* **isomorphism**

$$\Pi \xrightarrow{\sim} G_{\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{K}_\Pi/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))$$

that induces a **bijection** *between the respective sets of cuspidal inertia subgroups of* Π *and* $G_{\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)}$.

Proof. Let K_Π^{alg} be an algebraic closure of $\widetilde{K}_\Pi$. First, we verify assertion (i). Note that it follows immediately from the various definitions involved [cf. especially, Definitions 5.1, (ii); 5.12, (iii), (d)] that ev_Π is *bijective*. Note, moreover, that the natural map $\mathrm{ev}_{\Pi^\dagger} : \mathrm{Cusp}(\Pi^\dagger) \to Y_{\Pi^\dagger}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$ is compatible with the isomorphism $\Pi/\Pi^\dagger \xrightarrow{\sim} \mathrm{Gal}(K_{\Pi^\dagger}/K_\Pi)$ [cf. Definition 5.12, (iii), (e)] and the respective natural actions of $\Pi/\Pi^\dagger$ and $\mathrm{Gal}(K_{\Pi^\dagger}/K_\Pi)$. Thus, it follows immediately from the *transitivity* of the natural action of $\mathrm{Gal}(K_{\Pi^\dagger}/K_\Pi)$ on the fibers of the finite ramified Galois covering $Y_{\Pi^\dagger} \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$ that $\mathrm{ev}_{\Pi^\dagger}$ is *surjective*.

Write

$$\widetilde{\mathrm{Cusp}}(\Pi) \stackrel{\mathrm{def}}{=} \varprojlim_{\Pi^\ddagger \subseteq \Pi} \mathrm{Cusp}(\Pi^\ddagger), \quad \widetilde{Y}(\overline{\mathbb{Q}}_{\mathrm{BGT}}) \stackrel{\mathrm{def}}{=} \varprojlim_{\Pi^\ddagger \subseteq \Pi} Y_{\Pi^\ddagger}(\overline{\mathbb{Q}}_{\mathrm{BGT}}),$$

where $\Pi^\ddagger$ ranges over the normal open subgroups of Π. Observe that the natural maps $\{\mathrm{ev}_{\Pi^\ddagger}\}_{\Pi^\ddagger \subseteq \Pi}$ induce a natural map $\widetilde{\mathrm{ev}} : \widetilde{\mathrm{Cusp}}(\Pi) \to \widetilde{Y}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$

that, for each normal open subgroup $\Pi^{\ddagger}$ of Π, fits into a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Cusp}}(\Pi) & \xrightarrow{\widetilde{\mathrm{ev}}} & \widetilde{Y}(\overline{\mathbb{Q}}_{\mathrm{BGT}}) \\ \downarrow & & \downarrow \\ \mathrm{Cusp}(\Pi^{\ddagger}) & \xrightarrow{\mathrm{ev}_{\Pi^{\ddagger}}} & Y_{\Pi^{\ddagger}}(\overline{\mathbb{Q}}_{\mathrm{BGT}}). \end{array}$$

One verifies easily that this commutative diagram is compatible with the natural isomorphism $\Pi \overset{\sim}{\to} \mathrm{Gal}(\widetilde{K}_{\Pi}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))$ [cf. Definition 5.12, (iii), (e)] and the respective natural actions of Π and $\mathrm{Gal}(\widetilde{K}_{\Pi}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))$.

Suppose that $\mathrm{ev}_{\Pi^{\dagger}}(c_1) = \mathrm{ev}_{\Pi^{\dagger}}(c_2)$, where $c_1, c_2 \in \mathrm{Cusp}(\Pi^{\dagger})$. Let $I_1 \subseteq \Pi^{\dagger}$, $I_2 \subseteq \Pi^{\dagger}$, $J \subseteq \mathrm{Gal}(\widetilde{K}_{\Pi}/K_{\Pi^{\dagger}})$ be cuspidal inertia subgroups associated to c_1, c_2, $\mathrm{ev}_{\Pi^{\dagger}}(c_1)$, respectively. Thus, since $\widetilde{\mathrm{ev}}$ is compatible with the isomorphism $\Pi^{\dagger} \overset{\sim}{\to} \mathrm{Gal}(\widetilde{K}_{\Pi}/K_{\Pi^{\dagger}})$ and the respective natural actions, one verifies immediately that by choosing suitable conjugates of I_1, I_2, and J, we may assume without loss of generality that the natural isomorphism $\Pi^{\dagger} \overset{\sim}{\to} \mathrm{Gal}(\widetilde{K}_{\Pi}/K_{\Pi^{\dagger}})$ induces inclusions $\iota_1 : I_1 \hookrightarrow J$, $\iota_2 : I_2 \hookrightarrow J$. Next, observe that any cuspidal inertia subgroup of $\mathrm{Gal}(\widetilde{K}_{\Pi}/K_{\Pi^{\dagger}})$ is a *quotient* of some cuspidal inertia subgroup of $\mathrm{Gal}(K_{\Pi}^{\mathrm{alg}}/K_{\Pi^{\dagger}})$ via the natural surjection $\mathrm{Gal}(K_{\Pi}^{\mathrm{alg}}/K_{\Pi^{\dagger}}) \twoheadrightarrow \mathrm{Gal}(\widetilde{K}_{\Pi}/K_{\Pi^{\dagger}})$, and that every cuspidal inertia subgroup of $\mathrm{Gal}(K_{\Pi}^{\mathrm{alg}}/K_{\Pi^{\dagger}})$ is isomorphic to $\widehat{\mathbb{Z}}$. Thus, we conclude that J is *abelian*, and hence, by applying the inclusions ι_1, ι_2, that $I_1 \subseteq N_{\Pi^{\dagger}}(I_2)$, $I_2 \subseteq N_{\Pi^{\dagger}}(I_1)$, which [cf. Proposition 5.5, (ii)] implies that $I_1 = I_2$, as desired. This completes the proof of the *injectivity* of $\mathrm{ev}_{\Pi^{\dagger}}$ and hence of assertion (i).

Next, we verify assertion (ii). For each $E \in C_{\mathrm{BGT}}$ [cf. Definition 5.1, (iii)], write

$$\mathrm{Gal}(K_{\Pi}^{\mathrm{alg}}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)) \twoheadrightarrow \mathrm{Gal}(K_{\Pi}^{\mathrm{alg}}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))_E,$$

$$\mathrm{Gal}(\widetilde{K}_{\Pi}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)) \twoheadrightarrow \mathrm{Gal}(\widetilde{K}_{\Pi}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))_E$$

for the respective quotients determined by the field extensions of $\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)$ that are *unramified* outside of E. Recall from Proposition 5.2, (i), that there exists an isomorphism $\xi_E : \Pi_E \overset{\sim}{\to} \mathrm{Gal}(K_{\Pi}^{\mathrm{alg}}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))_E$ of profinite groups. In particular, since the natural isomorphism $\Pi \overset{\sim}{\to} \mathrm{Gal}(\widetilde{K}_{\Pi}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))$ [cf. Definition 5.12, (iii), (e)] induces a bijection between the respective sets of cuspidal inertia subgroups of Π and $\mathrm{Gal}(\widetilde{K}_{\Pi}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))$ [cf. Lemma 5.13, (i)], hence, in particular, a natural isomorphism $\Pi_E \xrightarrow{\sim} \mathrm{Gal}(\widetilde{K}_\Pi/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))_E$, it follows that the *composite morphism*

$$\begin{aligned}\mathrm{Gal}(K_\Pi^{\mathrm{alg}}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))_E &\twoheadrightarrow \mathrm{Gal}(\widetilde{K}_\Pi/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))_E \\ &\xleftarrow{\sim} \Pi_E \underset{\xi_E}{\xrightarrow{\sim}} \mathrm{Gal}(K_\Pi^{\mathrm{alg}}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t))_E\end{aligned}$$

is a *surjective endomorphism* of a *topologically finitely generated* profinite group [i.e., which, as is well-known, satisfies the "*Hopfian property*"], hence is an *isomorphism*. Thus, by allowing $E \in C_{\mathrm{BGT}}$ to vary, we conclude that $K_\Pi^{\mathrm{alg}} = \widetilde{K}_\Pi$. This completes the proof of assertion (ii), hence of Lemma 5.13. $\square$

THEOREM 5.14 (Uniqueness of function fields). *Suppose that* BGT *satisfies the QAA-property [cf. Definition 3.3, (v); Theorem 4.4, (ii); Definition 5.12]. Then any family*

$$\{K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$$

of subsets as in Definition 5.12, (iii), is unique.

PROOF. Let $\{K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$, $\{{}^\bullet K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$ be families of subsets as in Definition 5.12, (iii). Recall that, if $\Pi^\ddagger \subseteq \Pi^\dagger$ are normal open subgroups of Π, then $K_{\Pi^\dagger} \subseteq K_{\Pi^\ddagger}$ and ${}^\bullet K_{\Pi^\dagger} \subseteq {}^\bullet K_{\Pi^\ddagger}$ [cf. Definition 5.12, (iii), (a)]. Write

$$\widetilde{K}_\Pi \overset{\mathrm{def}}{=} \varinjlim_{\Pi^\dagger \subseteq \Pi} K_{\Pi^\dagger}, \quad {}^\bullet\widetilde{K}_\Pi \overset{\mathrm{def}}{=} \varinjlim_{\Pi^\dagger \subseteq \Pi} {}^\bullet K_{\Pi^\dagger},$$

where $\Pi^\dagger$ ranges over the normal open subgroups of Π. Then since $\widetilde{K}_\Pi$ and ${}^\bullet\widetilde{K}_\Pi$ are *algebraic closures* of K_Π [cf. Lemma 5.13, (ii)], there exists an abstract field isomorphism $\beta : \widetilde{K}_\Pi \xrightarrow{\sim} {}^\bullet\widetilde{K}_\Pi$ over K_Π, which determines an isomorphism of profinite groups $\alpha : \mathrm{Gal}({}^\bullet\widetilde{K}_\Pi/K_\Pi) \xrightarrow{\sim} \mathrm{Gal}(\widetilde{K}_\Pi/K_\Pi)$. Fix a normal open subgroup $\Pi^\dagger \subseteq \Pi$.

Write

- ${}^\circ K_{\Pi^\dagger} \overset{\mathrm{def}}{=} \beta^{-1}({}^\bullet K_{\Pi^\dagger}) \subseteq \widetilde{K}_\Pi$;
- $Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$ (respectively, ${}^\bullet Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$, ${}^\circ Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$) for the finite ramified Galois covering of smooth, proper, connected curves over

$\overline{\mathbb{Q}}_{\mathrm{BGT}}$ corresponding to the extension of function fields $\overline{\mathbb{Q}}_{\mathrm{BGT}}(t) = K_\Pi \subseteq K_{\Pi^\dagger}$ (respectively, $\overline{\mathbb{Q}}_{\mathrm{BGT}}(t) = K_\Pi \subseteq {}^\bullet K_{\Pi^\dagger}$, $\overline{\mathbb{Q}}_{\mathrm{BGT}}(t) = K_\Pi \subseteq {}^\circ K_{\Pi^\dagger}$) [cf. Definition 5.12, (iii), (a), (c), (d), (e)];

- $\mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$, $Y(\overline{\mathbb{Q}}_{\mathrm{BGT}})$, ${}^\bullet Y(\overline{\mathbb{Q}}_{\mathrm{BGT}})$, ${}^\circ Y(\overline{\mathbb{Q}}_{\mathrm{BGT}})$ for the respective sets of $\overline{\mathbb{Q}}_{\mathrm{BGT}}$-valued points of $\mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$, Y, ${}^\bullet Y$, ${}^\circ Y$.

Observe that there exist *natural bijections*

$$\mathrm{Cusp}(\Pi) \underset{\mathrm{ev}_\Pi}{\xrightarrow{\sim}} \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}(\overline{\mathbb{Q}}_{\mathrm{BGT}}), \quad \mathrm{Cusp}(\Pi^\dagger) \underset{\mathrm{ev}_{\Pi^\dagger}}{\xrightarrow{\sim}} Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}),$$
$$\mathrm{Cusp}(\Pi^\dagger) \underset{{}^\bullet\mathrm{ev}_{\Pi^\dagger}}{\xrightarrow{\sim}} {}^\bullet Y(\overline{\mathbb{Q}}_{\mathrm{BGT}})$$

[cf. Lemma 5.13, (i)] that fit into a *commutative diagram*

$$\begin{array}{ccccccc}
\mathrm{Gal}(\widetilde{K}_\Pi/K_\Pi) & \xleftarrow{\sim} & \Pi & \xrightarrow{\sim} & \mathrm{Gal}({}^\bullet\widetilde{K}_\Pi/K_\Pi) & \underset{\alpha}{\xrightarrow{\sim}} & \mathrm{Gal}(\widetilde{K}_\Pi/K_\Pi) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Gal}(K_{\Pi^\dagger}/K_\Pi) & \xleftarrow{\sim} & \Pi/\Pi^\dagger & \xrightarrow{\sim} & \mathrm{Gal}({}^\bullet K_{\Pi^\dagger}/K_\Pi) & \underset{\alpha}{\xrightarrow{\sim}} & \mathrm{Gal}({}^\circ K_{\Pi^\dagger}/K_\Pi) \\
\curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\
Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}) & \underset{\mathrm{ev}_{\Pi^\dagger}}{\xleftarrow{\sim}} & \mathrm{Cusp}(\Pi^\dagger) & \underset{{}^\bullet\mathrm{ev}_{\Pi^\dagger}}{\xrightarrow{\sim}} & {}^\bullet Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}) & \underset{\beta}{\xrightarrow{\sim}} & {}^\circ Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}(\overline{\mathbb{Q}}_{\mathrm{BGT}}) & \underset{\mathrm{ev}_\Pi}{\xleftarrow{\sim}} & \mathrm{Cusp}(\Pi) & \underset{\mathrm{ev}_\Pi}{\xrightarrow{\sim}} & \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}(\overline{\mathbb{Q}}_{\mathrm{BGT}}) & = & \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}(\overline{\mathbb{Q}}_{\mathrm{BGT}}),
\end{array}$$

where the vertical arrows denote the natural surjections; the horizontal arrows $\mathrm{Gal}({}^\bullet K_{\Pi^\dagger}/K_\Pi) \underset{\alpha}{\xrightarrow{\sim}} \mathrm{Gal}({}^\circ K_{\Pi^\dagger}/K_\Pi)$ and ${}^\bullet Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}) \underset{\beta}{\xrightarrow{\sim}} {}^\circ Y(\overline{\mathbb{Q}}_{\mathrm{BGT}})$ denote the bijections induced, respectively, by α and β.

Note that it follows immediately from the above *commutative diagram* that the sets $\subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$ of branch points of the finite ramified Galois coverings $Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$ and ${}^\circ Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$ *coincide*. Write $T \subseteq \mathrm{Cusp}(\Pi)$ for the image of the set of branch points of the finite ramified Galois covering $Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}$ via the bijection ev_Π^{-1}. Then, by replacing the normal open subgroup $\Pi^\dagger \subseteq \Pi$ by the pull-back of a suitable characteristic open subgroup of Π_T [cf. Definition 5.1, (ii)] via the natural surjection $\Pi \twoheadrightarrow \Pi_T$, we may assume without loss of generality that $K_{\Pi^\dagger} = {}^\circ K_{\Pi^\dagger}$, $Y = {}^\circ Y$.

Write

$$\sigma : Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}) \xrightarrow{\sim} {}^\circ Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}) = Y(\overline{\mathbb{Q}}_{\mathrm{BGT}})$$

for the composite of the horizontal arrows in the third row of the above *commutative diagram*. Recall that the images of $K^{\times}_{\Pi^\dagger}, {}^{\bullet}K^{\times}_{\Pi^\dagger}$ $(\subseteq L_{\Pi^\dagger})$ via the natural map

$$L_{\Pi^\dagger} \setminus \{0\} \to \varinjlim_{N^\dagger \subseteq \mathrm{BGT}} H^1(\Pi^\dagger_{\rtimes N^\dagger}, \Pi_{X_{0\infty}})$$

coincide with $K^{\kappa}_{\Pi^\dagger}$ [cf. Definition 5.12, (iii), (f)]. In particular, for each $f \in K^{\times}_{\Pi^\dagger}$, there exist

$$\phi_f \in \mathrm{Fn}(Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}), (\overline{\mathbb{Q}}_{\mathrm{BGT}})^{\times}_{\mathrm{div}}) \; (\subseteq \mathrm{Fn}(Y(\overline{\mathbb{Q}}_{\mathrm{BGT}}), \overline{\mathbb{Q}}_{\mathrm{BGT}} \cup \{\infty\})), \quad g_f \in K^{\times}_{\Pi^\dagger}$$

such that $f^{\sigma} \stackrel{\mathrm{def}}{=} f \circ \sigma = \phi_f \cdot g_f$ [cf. Definitions 5.9; 5.10; 5.12, (i)]. Note that it follows immediately from the above *commutative diagram* that σ lies over the identity automorphism of $\mathbb{P}^1_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$. Thus, we conclude from Corollary 1.3 [cf. also Definition 5.12, (i)] that, relative to the notational conventions of *loc. cit.*, $\sigma \in \mathrm{Gal}(K_{\Pi^\dagger}/K_{\Pi})$ and hence that $K_{\Pi^\dagger} = {}^{\bullet}K_{\Pi^\dagger}$. This completes the proof of Theorem 5.14. $\square$

THEOREM 5.15 (Injectivity of $C_{\mathrm{GT}}(\mathrm{BGT}) \to \mathrm{Aut}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$). *Suppose that* BGT *satisfies the QAA-property [cf. Definition 3.3, (v); Theorem 4.4, (ii); Definition 5.12]. Write*

$$\{K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$$

for the unique family of subsets as in Definition 5.12, (iii) [cf. Theorem 5.14];

$$\widetilde{K}_{\Pi} \stackrel{\mathrm{def}}{=} \varinjlim_{\Pi^\dagger \subseteq \Pi} K_{\Pi^\dagger},$$

where $\Pi^\dagger$ *ranges over the normal open subgroups of* Π*;*

$$G_{\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{K}_{\Pi}/K_{\Pi}) \; (= \mathrm{Gal}(\widetilde{K}_{\Pi}/\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)))$$

[cf. Definition 5.12, (iii), (d)];

$$\rho : C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}_{\mathrm{BGT}}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\overline{\mathbb{Q}}_{\mathrm{BGT}})$$

for the homomorphism induced by the natural action of $C_{\mathrm{GT}}(\mathrm{BGT})$ *on the field* $\overline{\mathbb{Q}}_{\mathrm{BGT}}$ *[cf. Theorem 4.4, (ii)]. Then the following hold:*

(i) $\Pi \overset{\text{out}}{\rtimes} C_{\mathrm{GT}}(\mathrm{BGT})$ *acts naturally on the algebraically closed field* $\widetilde{K}_{\Pi}$ *[cf. Lemma 5.13, (ii)]. Moreover, this action induces a commutative diagram*

$$\begin{array}{ccc} C_{\mathrm{GT}}(\mathrm{BGT}) & \xrightarrow{\rho} & G_{\mathbb{Q}_{\mathrm{BGT}}} \\ \downarrow & & \downarrow \\ \mathrm{Out}(\Pi) & \xrightarrow{\sim} & \mathrm{Out}(G_{\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)}), \end{array}$$

where the left-hand vertical arrow denotes the homomorphism induced by the natural outer action of $C_{\mathrm{GT}}(\mathrm{BGT})$ *on* Π *[cf. Definition 5.1, (i)]; the right-hand vertical arrow denotes the natural outer representation; the lower horizontal arrow denotes the isomorphism induced by the isomorphism* $\Pi \xrightarrow{\sim} G_{\overline{\mathbb{Q}}_{\mathrm{BGT}}(t)}$ *[cf. Lemma 5.13, (ii)].*

(ii) The commutative diagram of (i) induces a commutative diagram

$$\begin{array}{ccc} C_{\mathrm{GT}}(\mathrm{BGT}) & \xrightarrow{\rho} & G_{\mathbb{Q}_{\mathrm{BGT}}} \\ \downarrow & & \downarrow \\ \mathrm{Out}(\Pi_X) & \xrightarrow{\sim} & \mathrm{Out}(\Pi_{X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}}), \end{array}$$

where the left-hand vertical arrow denotes the homomorphism induced by the natural faithful outer action of $C_{\mathrm{GT}}(\mathrm{BGT}) \subseteq \mathrm{GT}$ *on* Π_X*; the right-hand vertical arrow denotes the natural outer representation; the lower horizontal arrow denotes the isomorphism induced by the isomorphism* $\Pi_X \xrightarrow{\sim} \Pi_{X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}}}$ *[cf. Lemma 5.13, (i), (ii)].*

(iii) The homomorphism ρ *is* **injective**. *In particular, the restriction* $\rho|_{\mathrm{BGT}}$ *of* ρ *to* BGT *is injective.*

(iv) Suppose, moreover, that BGT *satisfies the AA-property. Write* $\mathrm{GT}_{\mathrm{BGT}} \subseteq \mathrm{Out}(\Pi_{(X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}})_2})$ *for the Grothendieck-Teichmüller group associated [cf. Corollary 4.5] to* $\Pi_{(X_{\overline{\mathbb{Q}}_{\mathrm{BGT}}})_2}$*. Then the commutative diagram of (ii) induces a commutative diagram*

$$\begin{array}{ccc} C_{\mathrm{GT}}(\mathrm{BGT}) & \xrightarrow{\rho} & G_{\mathbb{Q}_{\mathrm{BGT}}} \\ \downarrow & & \downarrow \\ \mathrm{GT} & \xrightarrow{\sim} & \mathrm{GT}_{\mathrm{BGT}}, \end{array}$$

where the vertical arrows denote the natural injections; the lower horizontal arrow denotes the isomorphism induced by an outer isomorphism $\Pi_{X_2} \xrightarrow{\sim} \Pi_{(X_{\mathbb{Q}_{\mathrm{BGT}}})_2}$ *as in Definition 5.12, (iv).*

Proof. First, we verify assertion (i). Note that it follows immediately from the various definitions involved that $\Pi \overset{\mathrm{out}}{\rtimes} C_{\mathrm{GT}}(\mathrm{BGT})$ acts naturally on the family of sets $\{L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$, where $\Pi^\dagger$ ranges over the normal open subgroups of Π [cf. Definition 5.8]. Thus, we conclude from the *uniqueness* of the family of subsets $\{K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$ [cf. Theorem 5.14] that $\Pi \overset{\mathrm{out}}{\rtimes} C_{\mathrm{GT}}(\mathrm{BGT})$ acts naturally on the algebraically closed field $\widetilde{K}_\Pi$. Moreover, it follows immediately from the various definitions involved that this natural action induces the desired commutative diagram. This completes the proof of assertion (i). Next, since the natural surjection $\Pi \twoheadrightarrow \Pi_X$ is compatible with the respective outer actions of $C_{\mathrm{GT}}(\mathrm{BGT})$ [cf. Definition 5.1, (i)], assertion (ii) follows immediately from Theorem 5.15, (i). Assertion (iii) follows immediately from Theorem 5.15, (ii). Assertion (iv) follows immediately from the various definitions involved. This completes the proof of Theorem 5.15. □

Lemma 5.16 (Elementary property of profinite groups). *Let* G *be a profinite group,* $H \subseteq G$ *a closed subgroup,* $g \in G$ *an element such that* $H \subseteq H^g := g \cdot H \cdot g^{-1}$*. Then* $H = H^g$*.*

Proof. By considering quotients of G by normal open subgroups, one reduces immediately to the case where G is *finite*. Then the equality $H = H^g$ follows immediately from the fact that H and H^g have the same cardinality. This completes the proof of Lemma 5.16. □

Theorem 5.17 (Combinatorial construction of $G_{\mathbb{Q}}$).

(i) *Write* $\mathrm{Out}^{|\mathrm{C}|}(\Pi_X) \subseteq \mathrm{Out}(\Pi_X)$ *for the closed subgroup of outer automorphisms that induce the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of* Π_X*. Then the* **conjugacy class** *of subgroups of* $\mathrm{Out}^{|\mathrm{C}|}(\Pi_X)$ *determined by the* **absolute Galois group** *of* $\mathbb{Q}$ *may be constructed from the abstract topological group* Π_{X_2} *[cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/ group-theoretic way, as the set of* **maximal** *elements [relative to the* *relation of inclusion] in the set of closed subgroups of* $\mathrm{Out}^{|\mathrm{C}|}(\Pi_X)$ *that arise as* $\mathrm{Out}^{|\mathrm{C}|}(\Pi_X)$*-conjugates of closed subgroups of* GT *that satisfy the* **QAA-property** *[cf. Definition 3.3, (v); Theorem 4.4, (ii); Definition 5.12].*

(ii) *The* **conjugacy class** *of subgroups of* GT *determined by the* **absolute Galois group** *of* $\mathbb{Q}$ *may be constructed from the abstract topological group* Π_{X_2} *[cf. Corollary 4.5, Remark 4.5.1], in a purely combinatorial/group-theoretic way, as the set of* **maximal** *elements [relative to the relation of inclusion] in the set of closed subgroups of* GT *that arise as closed subgroups of* GT *that satisfy the* **AA-property** *[cf. Definition 3.3, (v); Theorem 4.4, (ii); Definition 5.12].*

Proof. Recall from Remark 5.12.1 that $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ may be regarded as a closed subgroup of GT that satisfies the *AA-property*, hence may itself be taken to be "BGT". Thus, it follows formally from Theorem 5.15, (ii) [cf. also Lemma 5.13, (ii)] (respectively, Theorem 5.15, (iv)), that *any* $\mathrm{Out}^{|\mathrm{C}|}(\Pi_X)$*-conjugate* (respectively, GT-*conjugate*) of a *closed subgroup* of GT that satisfies the *QAA-property* (respectively, *AA-property*) is *contained in* — hence *equal to*, whenever it is *maximal* with respect to the relation of inclusion among such conjugates of closed subgroups — some $\mathrm{Out}^{|\mathrm{C}|}(\Pi_X)$-conjugate (respectively, GT-conjugate) of $G_{\mathbb{Q}}$. In particular, the maximality of any $\mathrm{Out}^{|\mathrm{C}|}(\Pi_X)$-conjugate (respectively, GT-conjugate) of $G_{\mathbb{Q}}$ follows formally from Lemma 5.16. This completes the proof of Theorem 5.17. □

6. Application to Semi-Absolute Anabelian Geometry over TKND-AVKF-Fields

In this section, we introduce the notion of a *TKND-AVKF-field* [cf. Definition 6.6, (iii)] and show that the absolute Galois group of a TKND-AVKF subfield of $\overline{\mathbb{Q}}$ satisfies the *AA-property* [cf. Theorem 6.8, (i)]. We then apply the theory developed in the present paper to prove a *semi-absolute version of the Grothendieck Conjecture* for higher dimensional configuration spaces [of dimension ≥ 2] associated to hyperbolic curves of genus 0 over TKND-AVKF-fields [cf. Theorem 6.10, (ii)].

Write $\mathbb{Q}^{\mathrm{ab}}$ ($\subseteq \overline{\mathbb{Q}}$) for the maximal abelian extension of $\mathbb{Q}$.

DEFINITION 6.1. Let $p \in \mathfrak{Primes}$; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset.

(i) Let M be an abelian group. Then we shall say that M is *p^∞-tor-finite* if the subgroup of p-power torsion elements of M is finite. We shall say that M is *Σ^∞-tor-finite* if, for each $l \in \Sigma$, M is l^∞-tor-finite.

(ii) Let G be a profinite group. Then we shall say that G is *p-subfree* if there exists a closed subgroup of G isomorphic to $\mathbb{Z}_p$. We shall say that G is *Σ-subfree* if, for each $l \in \Sigma$, G is l-subfree. We shall say that G is *p-sparse* if the maximal pro-p quotient of every open subgroup of G is finite. We shall say that G is *Σ-sparse* if, for each $l \in \Sigma$, G is l-sparse.

(iii) Let K be a field. If K satisfies the following condition, then we shall say that K is an *AVKF-field* [i.e., "abelian variety Kummer-faithful field"]:

> Let A be an abelian variety over a finite extension L of K. Write $A(L)^\infty$ for the group of divisible elements $\in A(L)$. Then $A(L)^\infty = \{1\}$.

If K is an AVKF-field, then we shall also say that K is *AVKF*.

(iv) Let K be a field. If K satisfies the following condition, then we shall say that K is *p-AV-tor-indivisible* (respectively, *p^∞-AV-tor-finite*):

> Let A be an abelian variety over a finite extension L of K. Write
> - $A(L)^{p^\infty}$ for the *group of p-divisible elements* $\in A(L)$;
> - $A(L)_\infty$ for the *group of torsion elements* $\in A(L)$;
> - $A(L)_{p^\infty}$ for the *group of p-power torsion elements* $\in A(L)$.
>
> Then $A(L)^{p^\infty} \subseteq A(L)_\infty$ (respectively, $A(L)_{p^\infty}$ is finite).

We shall say that K is *Σ-AV-tor-indivisible* (respectively, *Σ^∞-AV-tor-finite*) if, for each $l \in \Sigma$, K is l^∞-AV-tor-indivisible (respectively, l^∞-AV-tor-finite).

(v) Let K be a field. Then we shall say that K is *stably* Σ-$\times\mu$-*indivisible* (respectively, *stably* μ_{Σ^∞}-*finite*) if, for each $l \in \Sigma$, K is *stably* l-$\times\mu$-*indivisible* (respectively, *stably* μ_{l^∞}-*finite*) [cf. the final portion of Remark 6.1.2; [Tsjm], Definition 3.3, (v), (vii)].

Remark 6.1.1. If a profinite group G is Σ-*subfree* (respectively, Σ-*sparse*), then so is any open subgroup of G.

Remark 6.1.2. Let $\square$ be one of the following properties:

- *AVKF*,
- Σ-*AV-tor-indivisible*,
- Σ^∞-*AV-tor-finite*,
- *stably* Σ-$\times\mu$-*indivisible*,
- *stably* μ_{Σ^∞}-*finite*.

Then one verifies immediately that if L is an *extension field* of a field K, then the following implication holds:

$$L \text{ is } \square \;\Rightarrow\; K \text{ is } \square.$$

Also, we observe that the *second* and *third* properties are the respective analogues for *abelian varieties* of the *fourth* and *fifth* properties, which may be regarded as properties concerning rational points of the *"torus"* $\mathbb{G}_{\mathrm{m}}$.

Remark 6.1.3. In the notation of Definition 6.1, (iii), suppose further that K is of *characteristic* 0. Then it follows immediately from [AbsTopIII], Definition 1.5, that the following implication concerning K holds:

$$\Bigl(\textit{torally Kummer-faithful} \text{ and } \textit{AVKF}\Bigr) \iff \textit{Kummer-faithful}.$$

Lemma 6.2 (Stably p-$\times\mu$-indivisible and p-AV-tor-indivisible fields). *Let $p \in \mathfrak{Primes}$, K a field of characteristic $\neq p$. Then:*

(i) Let L be a [not necessarily finite!] **Galois extension** *of K such that* $\mathrm{Gal}(L/K)$ *is* **p-sparse**. *Let $\square$ be one of the following properties:*

- **stably p-$\times\mu$-indivisible**,

- **stably $\mu_{\boldsymbol{p}^\infty}$-finite**,
- **$\boldsymbol{p}$-AV-tor-indivisible**,
- **$\boldsymbol{p}^\infty$-AV-tor-finite**.

Then if K is $\square$, then so is L.

(ii) Let L be a [not necessarily finite!] **Galois extension** *of K.*

(ii$^\times$) Suppose that L is **stably $\mu_{\boldsymbol{p}^\infty}$-finite**. *Then if K is* **stably $\boldsymbol{p}$-$\times\mu$-indivisible**, *then so is L.*

(ii^{AV}) Suppose that L is **$\boldsymbol{p}^\infty$-AV-tor-finite**. *Then if K is* **$\boldsymbol{p}$-AV-tor-indivisible**, *then so is L.*

(iii) The following properties hold:

(iii$^\times$) Suppose that K is **stably $\boldsymbol{p}$-$\times\mu$-indivisible**, **stably $\mu_{\mathfrak{Primes}^\infty}$-finite**, *and of characteristic* 0. *Then K is* **torally Kummer-faithful**. *If, moreover, K is* **AVKF**, *then K is* **Kummer-faithful** *[cf. Remark 6.1.3].*

(iii^{AV}) Suppose that K is **$\boldsymbol{p}$-AV-tor-indivisible** *and* **$\mathfrak{Primes}^\infty$-AV-tor-finite**. *Then K is* **AVKF**.

(iv) The following properties hold:

(iv$^\times$) If K is **torally Kummer-faithful**, *then K is* **stably $\mu_{\mathfrak{Primes}^\infty}$-finite**.

(iv^{AV}) If K is **AVKF**, *then K is* **$\mathfrak{Primes}^\infty$-AV-tor-finite**.

(v) Suppose that K is a **sub-$\boldsymbol{p}$-adic field** *[cf. [LocAn], Definition 15.4, (i)]. Then K is*

- **stably $\boldsymbol{p}$-$\times\mu$-indivisible**,
- **stably $\mu_{\mathfrak{Primes}^\infty}$-finite**,
- **$\boldsymbol{p}$-AV-tor-indivisible**,
- **$\mathfrak{Primes}^\infty$-AV-tor-finite**.

PROOF. First, we consider assertion (i). We begin by *observing* that any finite extension field $L^\dagger$ of L arises as a Galois extension of some finite extension field $K^\dagger$ of K such that $\mathrm{Gal}(L^\dagger/K^\dagger)$ is *p-sparse*. Next, we *observe* that the Galois group $\mathrm{Gal}(M/K)$ of any [*not necessarily finite*!] *Galois extension M* of K that arises by

- adjoining compatible systems of *p-power roots* of elements of K or by
- adjoining infinitely many *p-power roots of unity*,

admits an open subgroup which is a *pro-p group*. Assertion (i) in the case where $\square$ is taken to be one of the *first two properties* then follows immediately from the above *observations*, together with our assumption that $\mathrm{Gal}(L/K)$ is *p-sparse*. Assertion (i) in the case where $\square$ is taken to be one of the *latter two properties* follows by a similar argument [cf. the final portion of Remark 6.1.2]. This completes the proof of assertion (i).

Assertion (ii$^{\times}$) follows immediately from [Tsjm], Lemma D, (v). Next, we verify assertion (ii^{AV}). Let $L^{\dagger}$ be a finite extension field of L; $A^{\dagger}$ an abelian variety over $L^{\dagger}$. To verify assertion (ii^{AV}), it suffices to prove that $A^{\dagger}(L^{\dagger})^{p^{\infty}} \subseteq A^{\dagger}(L^{\dagger})_{\infty}$. Let $x \in A^{\dagger}(L^{\dagger})^{p^{\infty}}$. By replacing K by a finite extension field of K, we may assume without loss of generality that

- $L^{\dagger} = L$;
- $A^{\dagger} = A \times_K L$, where A is an abelian variety over K;
- $x \in A(K)$.

Thus, since K is *p-AV-tor-indivisible*, it suffices to verify the following assertion:

Claim 6.2.A: $x \in A(K)^{p^{\infty}}$.

Indeed, let n be a positive integer. Since L is *p^{∞}-AV-tor-finite*, $A(L)_{p^{\infty}}$ is finite. Write p^m for the cardinality of $A(L)_{p^{\infty}}$. Then since $x \in A(L)^{p^{\infty}}$, there exists an element $x_{m+n} \in A(L)$ such that $p^{m+n} \cdot x_{m+n} = x$. Write $x_n \stackrel{\mathrm{def}}{=} p^m \cdot x_{m+n}$. Thus, since $p^n \cdot x_n = x$, it suffices to prove that $x_n \in A(K)$. Let $\sigma \in \mathrm{Gal}(L/K)$. Observe that

$$p^{m+n} \cdot ((x_{m+n})^{\sigma} - x_{m+n}) = x^{\sigma} - x = 0,$$

hence, in particular, that $(x_{m+n})^{\sigma} - x_{m+n} \in A(L)_{p^{\infty}}$. Thus, we conclude that

$$x_n^{\sigma} - x_n = p^m \cdot ((x_{m+n})^{\sigma} - x_{m+n}) = 0,$$

hence that $x_n \in A(K)$. This completes the proof of Claim 6.2.A, hence of assertion (ii^{AV}).

Assertions (iii$^{\times}$), (iii^{AV}) follow immediately from the fact that, for any $l \in \mathfrak{Primes}$, the divisible group $\mathbb{Q}_l/\mathbb{Z}_l$ has *no nontrivial finite quotient.*

Next, we verify assertion (iv). Recall that, for any $l \in \mathfrak{Primes}$, the group of l-torsion points of an abelian variety over an algebraically closed field is *finite* [cf. e.g., [Mumf], p. 64]. Thus, assertion (iv) follows immediately from the fact that, for any $l \in \mathfrak{Primes}$, every infinite subgroup of $\mathbb{Q}_l/\mathbb{Z}_l$ is *divisible.*

Finally, we consider assertion (v). One verifies immediately that we may assume without loss of generality that K is a finitely generated field extension of $\mathbb{Q}_p$. Moreover, by applying the *"relative Mordell-Weil Theorem"* [cf. [Lang], Chapter 6, Theorem 2], together with well-known elementary facts concerning the multiplicative group of a function field, one concludes that we may assume without loss of generality that K is a finite field extension of $\mathbb{Q}_p$. Then assertion (v) follows immediately from a similar argument to the argument applied in [AbsTopIII], Remark 1.5.4, (i). This completes the proof of Lemma 6.2. $\square$

Remark 6.2.1. The argument applied in the proof of Claim 6.2.A [in the proof of Lemma 6.2, (ii^{AV})] is similar to the argument applied in the proof of [Moon], Proposition 7.

Proposition 6.3 (Examples of AVKF-fields). *Let $F \subseteq \overline{\mathbb{Q}}$ be a number field.*

(i) Let L be a [not necessarily finite!] Galois extension of $F \cdot \mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ such that $\mathrm{Gal}(L/F \cdot \mathbb{Q}^{\mathrm{ab}})$ *is* **$\mathfrak{Primes}$-sparse**. *Then L is*

- **stably $\mathfrak{Primes}$-$\times\mu$-indivisible**,
- **$\mathfrak{Primes}$-AV-tor-indivisible**,
- **$\mathfrak{Primes}^{\infty}$-AV-tor-finite**.

In particular, L is a **stably $\times\mu$-indivisible AVKF-field** *[cf. Lemma 6.2, (iii^{AV}); [Tsjm], Lemma D, (i)].*

(ii) Let $\{v_1, v_2, \ldots\}$ be an infinite set of non-archimedean primes of F. [Here, we assume, for simplicity, that the indices of the "v_j" are chosen in such a way that $v_j \neq v_{j'}$ for $j \neq j'$.] Let $\{\Sigma_j \subseteq \mathfrak{Primes}\}_{j \geq 1}$ be

a family of subsets such that, for any positive integer j,

$$\bigcup_{i \geq j} \Sigma_i = \mathfrak{Primes},$$

where i ranges over the positive integers $\geq j$; $M \subseteq \overline{\mathbb{Q}}$ a [not necessarily finite!] **Galois extension** *of F; L a [not necessarily finite!]* **Galois extension** *of $M \subseteq \overline{\mathbb{Q}}$ such that* $\mathrm{Gal}(L/M)$ *is* **$\mathfrak{Primes}$-sparse**. *Suppose that for each positive integer j, the absolute Galois group of the residue field of the ring of integers of M at [every prime that divides] v_j is* **Σ_j-subfree**. *Then L is*

- **stably $\mathfrak{Primes}$-$\times\mu$-indivisible**,
- **stably $\mu_{\mathfrak{Primes}^\infty}$-finite**,
- **$\mathfrak{Primes}$-AV-tor-indivisible**,
- **$\mathfrak{Primes}^\infty$-AV-tor-finite**.

In particular, L is a **Kummer-faithful** *field [cf. Lemma 6.2, (iii$^\times$), (iii^{AV})].*

Proof. First, we verify assertion (i). Note that it follows immediately from Lemma 6.2, (i), that we may assume without loss of generality that $L = F \cdot \mathbb{Q}^{\mathrm{ab}}$. Then since L is an abelian extension of a number field, it follows immediately from [Tsjm], Lemma D, (iii), (iv), that L is *stably $\mathfrak{Primes}$-$\times\mu$-indivisible.* On the other hand, it follows immediately from [KLR], Appendix, Theorem 1, that L is *$\mathfrak{Primes}^\infty$-AV-tor-finite.* Next, observe that F is *$\mathfrak{Primes}$-AV-tor-indivisible* [cf. Lemma 6.2, (v)]. Thus, since L is a *$\mathfrak{Primes}^\infty$-AV-tor-finite* Galois extension of F, we conclude from Lemma 6.2, (ii^{AV}), that L is *$\mathfrak{Primes}$-AV-tor-indivisible.* This completes the proof of assertion (i).

Next, we verify assertion (ii). Note that it follows immediately from Lemma 6.2, (i), that we may assume without loss of generality that $L = M$. For each positive integer j, write p_j for the residue characteristic of v_j. Then it follows immediately from our assumption on various unions of the subsets $\Sigma_i \subseteq \mathfrak{Primes}$ that, for any positive integer j,

$$\bigcup_{i \geq j} \bigl(\Sigma_i \setminus \{p_i\}\bigr) = \mathfrak{Primes},$$

where i ranges over the positive integers $\geq j$. Let $p \in \mathfrak{Primes}$; $L^\dagger$ a finite extension of L; $A^\dagger$ an abelian variety over $L^\dagger$; j a positive integer such that $p \in \Sigma_j \setminus \{p_j\}$, and $A^\dagger$ has good reduction at some prime $\tilde{v}_j$ of $L^\dagger$ that divides v_j [cf. the above display!]. Write

- $\mathcal{O}^\dagger_{\tilde{v}_j} \subseteq L^\dagger$ for the ring of integers at $\tilde{v}_j$;
- $k^\dagger_{\tilde{v}_j}$ for the residue field of $\mathcal{O}^\dagger_{\tilde{v}_j}$;
- $\mathcal{A}^\dagger_j$ for the abelian scheme over $\mathcal{O}^\dagger_{\tilde{v}_j}$ whose generic fiber is $A^\dagger$;
- $\mathcal{A}^\dagger_{\tilde{v}_j} \stackrel{\mathrm{def}}{=} \mathcal{A}^\dagger_j \times_{\mathcal{O}^\dagger_{\tilde{v}_j}} k^\dagger_{\tilde{v}_j}$.

Then since the morphism $\mathcal{A}^\dagger_j \to \mathcal{A}^\dagger_j$ given by multiplication by a power of p is *finite étale*, it follows immediately that there exists a natural injection

$$A^\dagger(L^\dagger)_{p^\infty} \hookrightarrow \mathcal{A}^\dagger_{\tilde{v}_j}(k^\dagger_{\tilde{v}_j}).$$

Thus, it follows immediately from

- our assumption [cf. Remark 6.1.1] that the *absolute Galois group* of $k^\dagger_{\tilde{v}_j}$ is Σ_j*-subfree*,
- the well-known fact that the *absolute Galois group* of a *finite field* is isomorphic to $\widehat{\mathbb{Z}}$, and
- the well-known fact that, for any positive integer n, $GL_n(\mathbb{Z}_p)$ contains an open subgroup which is a *pro-p-group*

that $A^\dagger(L^\dagger)_{p^\infty}$ is *finite*. Thus, by allowing p to vary, we conclude that L is $\mathfrak{Primes}^\infty$*-AV-tor-finite*. A similar argument applied to the multiplicative group $\mathbb{G}_m$ implies that L is *stably* $\mu_{\mathfrak{Primes}^\infty}$*-finite*. Next, observe that L is a $\mathfrak{Primes}^\infty$*-AV-tor-finite* Galois extension of the $\mathfrak{Primes}$*-AV-tor-indivisible* field F [cf. Lemma 6.2, (v)]. Thus, we conclude from Lemma 6.2, (ii^{AV}), that L is $\mathfrak{Primes}$*-AV-tor-indivisible*. A similar argument implies that L is *stably* $\mathfrak{Primes}$-$\times\mu$*-indivisible*. This completes the proof of assertion (ii), hence of Proposition 6.3. □

Remark 6.3.1. The following example was suggested to the authors of the present paper by *A. Tamagawa*. Let $\{G_i\}_{i \in I}$ be a family of *nonabelian* *finite simple groups* [i.e., such as the alternating group on n letters $\mathfrak{A}_n$, where $n \geq 5$]. Then the direct product group

$$G \stackrel{\mathrm{def}}{=} \prod_{i \in I} G_i$$

endowed with the product topology is $\mathfrak{Primes}$-*sparse*. Indeed, this follows immediately from the definition of the product topology, together with the elementary fact that, for each $p \in \mathfrak{Primes}$, $i \in I$, the maximal pro-p quotient of G_i is trivial. If I is *countable*, and we assume that there exists a positive integer $n \geq 5$ such that G_i is *isomorphic* to $\mathfrak{A}_n$ for all $i \in I$, then it follows immediately from well-known facts concerning the *Hilbertian* nature of *number fields* and the *regularity* of $\mathfrak{A}_n$ [cf. [FJ], §6.2; [FJ], Theorem 13.4.2; [FJ], Proposition 16.2.8, (b); [FJ], Proposition 16.7.6] that G may be realized as the *Galois group* of a Galois extension E of a number field F. Here, we note that such a Galois extension E of F is necessarily *linearly disjoint* from any *abelian field extension* of F.

Remark 6.3.2. Later [cf. Remark 6.6.3 below], we shall see that the fields "L" of Proposition 6.3, (i), (ii), are in fact *"TKND-AVKF-fields"*.

Remark 6.3.3. Let $F \subseteq \overline{\mathbb{Q}}$ be a number field such that $\sqrt{-1} \in F$; $\{v_1, v_2, \dots\}$ an *infinite* set of non-archimedean primes of F. [Here, we assume, for simplicity, that the indices of the "v_j" are chosen in such a way that $v_j \neq v_{j'}$ for $j \neq j'$.] Let $\{\Sigma_j \subseteq \mathfrak{Primes}\}_{j \geq 1}$ be a family of *finite* subsets such that, for any positive integer j,

$$\bigcup_{i \geq j} \Sigma_i = \mathfrak{Primes},$$

where i ranges over the positive integers $\geq j$. For each positive integer j, write $\mathfrak{Primes} \setminus \Sigma_j = \{p_{j,m}\}_{m \geq 1}$; F_{v_j} for the completion of F at v_j. For each pair of positive integers i, j such that $j \leq i$, write $F^{\dagger}_{v_j}[i]$ for the *finite unramified [abelian] extension* of F_{v_j} of degree

$$\prod_{1 \leq m \leq i} p^i_{j,m}.$$

For each positive integer j, let $F^{\ddagger}_{v_j}$ be an *abelian totally wildly ramified infinite extension* of F_{v_j}. For each pair of positive integers i, j such that $j \leq i$, let $F^{\ddagger}_{v_j}[i] \subseteq F^{\ddagger}_{v_j}$ be a *finite subextension* of F_{v_j} such that

$$F^{\ddagger}_{v_j}[i] \subseteq F^{\ddagger}_{v_j}[i+1], \quad \bigcup_{j \leq m} F^{\ddagger}_{v_j}[m] = F^{\ddagger}_{v_j},$$

where m ranges over the positive integers $\geq j$. [Here, we observe that the *existence* of such extensions of F_{v_j} follows immediately from [NSW], Theorem 7.2.11.] Next, let i be a positive integer; M_i an *abelian extension* of F such that, for each pair of positive integers i, j such that $j \leq i$, the local extensions of M_i/F at v_j coincide with the extension $F^{\dagger}_{v_j}[i] \cdot F^{\ddagger}_{v_j}[i]/F_{v_j}$. [Here, we observe that, in light of our assumption that $\sqrt{-1} \in F$, the *existence* of such an abelian extension M_i of F follows immediately from [NSW], Definitions 9.1.5, 9.1.7; [NSW], Theorem 9.2.8.] Write

$$M \subseteq \overline{\mathbb{Q}}$$

for the field generated by $\{M_i\}_{i \geq 1}$ over F. Then we make the following *observations*, each of which follows immediately from the construction of M:

(a) M is an *abelian extension* of F;

(b) for each positive integer j, the *absolute Galois group* of the *residue field* of the ring of integers of M at [every prime that divides] v_j is Σ_j*-subfree*;

(c) for each positive integer j, the *ramification index* of the extension M/F at v_j is *infinite* [so if $\{v_1, v_2, \dots\}$ coincides with the set of all non-archimedean primes of F, then M is *not* a *generalized sub-p-adic field* for any prime number p — cf. [AnabTop], Definition 4.11];

(d) for each positive integer j, the *residue field* of the ring of integers of M at [every prime that divides] v_j is *infinite*.

Thus, in particular, any Galois extension L of M whose Galois group is $\mathfrak{Primes}$*-sparse* — such as, for instance, a composite field $L \stackrel{\text{def}}{=} M \cdot E$, where E is as in Remark 6.3.1 — satisfies the assumptions of Proposition 6.3, (ii), as well as the properties discussed in (c), (d).

Remark 6.3.4. Note that it follows immediately from the various definitions involved that the field "L" of Proposition 6.3, (i), satisfies properties analogous to the properties (c), (d) of Remark 6.3.3. That is to say, in the notation of Proposition 6.3, (i),

- the *ramification index* of the extension L/F at every non-archimedean prime of L is *infinite* [so L is *not* a *generalized sub-p-adic field* for any prime number p — cf. [AnabTop], Definition 4.11];
- the *residue field* of the ring of integers of L at every non-archimedean prime of L is *algebraically closed*, hence *infinite*.

Remark 6.3.5. The properties (c), (d) of Remark 6.3.3 [cf. also Remark 6.3.4] are of interest in that they show that

> anabelian geometry over fields such as the fields L of Proposition 6.3, (i), (ii) [cf. Theorem 6.10 below] *cannot* be treated by means of well-known techniques of anabelian geometry that require the use of *p-adic Hodge theory* or *Frobenius elements* of absolute Galois groups of finite fields [cf. [Tama], Theorem 0.4; [LocAn], Theorem A; [AnabTop], Theorem 4.12].

Proposition 6.4 (AVKF-fields satisfy the ISC-, CS-properties). *Let $K \subseteq \overline{\mathbb{Q}}$ be an AVKF-field [cf. Definition 6.1, (iii)]. Write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K) \subseteq G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus, we obtain natural injections*

$$G_K \subseteq G_{\mathbb{Q}} \hookrightarrow \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$$

[cf. the discussion at the beginning of [Tsjm], Introduction], which we use to identify G_K with its image in GT*. Then K satisfies the* **ISC-property***, and the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies the* **CS-property***.*

Proof. Indeed, it follows immediately from a similar argument to the argument applied in the proof of [Tsjm], Theorem 3.1, and [Tsjm], Corollary 3.2, that K satisfies the *ISC-property.* The *CS-property* for the closed subgroup $G_K \subseteq \mathrm{GT}$ then follows formally. This completes the proof of Proposition 6.4. □

COROLLARY 6.5 (AVKF-fields satisfy the ArBC-property). *In the notation of Proposition 6.4, the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies the* **ArBC-property** *[cf. Theorem 4.4, (ii), (iii)]. Moreover, if one takes "*BGT*" to be G_K [cf. Definition 3.3, (v); Theorem 4.4, (ii)], then the following hold:*

(i) In the notation of Theorem 4.4, (ii), there exists a **natural isomorphism of fields**

$$\overline{\mathbb{Q}}_{G_K} \xrightarrow{\sim} \overline{\mathbb{Q}}$$

that is compatible, relative to the respective natural actions, with the inclusion $G_K \subseteq G_{\mathbb{Q}}$. In the remainder of the present §6, we shall use this natural isomorphism to identify $\overline{\mathbb{Q}}_{G_K}$ with $\overline{\mathbb{Q}}$.

(ii) In the notation of Definition 5.1, there exists a **natural outer isomorphism**

$$\Pi \xrightarrow{\sim} G_{K_X}$$

between the profinite group Π and the **absolute Galois group** *G_{K_X} of the* **function field** *K_X of $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$. This natural outer isomorphism is compatible with the respective natural outer actions of $\mathrm{BGT} = G_K$ on Π and G_{K_X}.*

(iii) There exists a **natural homomorphism**

$$C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$$

[cf. Theorem 4.4, (ii)] whose restriction to $C_{G_{\mathbb{Q}}}(G_K)$ is the natural inclusion $C_{G_{\mathbb{Q}}}(G_K) \subseteq G_{\mathbb{Q}}$.

PROOF. First, we observe that it follows immediately — from the evident *scheme-theoretic interpretation* of the various *arithmetic Belyi diagrams* that arise — that the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies the *COF-property.* Thus, it follows immediately from Corollary 3.7; Theorem 4.4, (iii); Proposition 6.4, together with the various definitions involved, that the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies the *ArBC-property.* Next, we observe that it follows immediately — from the evident *scheme-theoretic interpretation* of the various *arithmetic Belyi diagrams* that arise — that these arithmetic Belyi diagrams determine

- a *natural isomorphism of fields* $\overline{\mathbb{Q}}_{G_K} \xrightarrow{\sim} \overline{\mathbb{Q}}$ that is compatible with the inclusion $G_K \subseteq G_{\mathbb{Q}}$, and
- a *natural outer isomorphism* $\Pi \xrightarrow{\sim} G_{K_X}$ that is compatible with the respective natural outer actions of $\mathrm{BGT} = G_K$ on Π and G_{K_X}

[cf. the proof of Theorem 4.4, (ii), (iii)]. Thus, we conclude [cf. the proof of Theorem 4.4, (ii), (iii)] that there exists a natural homomorphism

$$C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$$

whose restriction to $C_{G_{\mathbb{Q}}}(G_K)$ is the natural inclusion $C_{G_{\mathbb{Q}}}(G_K) \subseteq G_{\mathbb{Q}}$. This completes the proof of Corollary 6.5. $\square$

DEFINITION 6.6. Let K be a field, $\overline{K}$ an algebraic closure of K. Write $K_{\mathrm{prm}} \subseteq K$ for the prime field of K.

(i) Write

$$K_{\mathrm{div}} \stackrel{\mathrm{def}}{=} \bigcup_{L/K} L_{\times\infty} \subseteq \overline{K},$$

where L ($\subseteq \overline{K}$) ranges over the finite extensions of K, and we write

$$L_{\times\infty} \stackrel{\mathrm{def}}{=} K_{\mathrm{prm}}(L^{\times\infty}) \subseteq L$$

[cf. the discussion entitled "Fields" in Notations and Conventions].

(ii) If $K_{\mathrm{div}} \subseteq \overline{K}$ is an infinite field extension, then we shall say that K is a *TKND-field* [i.e., "torally Kummer-nondegenerate field"]. If K is a TKND-field, then we shall say that K is *TKND*.

(iii) If $K \subseteq \overline{K}$ is both TKND and AVKF, then we shall say that K is a *TKND-AVKF-field*. If K is a TKND-AVKF-field, then we shall say that K is *TKND-AVKF*.

REMARK 6.6.1. One verifies immediately that if L is an *algebraic extension* of a field K [which implies that K and L admit a *common algebraic closure*], then the following implication holds:

$$L \text{ is } \textit{TKND} \;\Rightarrow\; K \text{ is } \textit{TKND}.$$

Remark 6.6.2. In the notation of Definition 6.6, suppose further that K is of *characteristic* 0. Then the following implications concerning K hold [cf. Definition 6.1, (iii); [AbsTopIII], Definition 1.5; [Tsjm], Definition 3.3, (v); the well-known fact that $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ is an infinite field extension [cf., e.g., [Tsjm], Lemma D, (iii), (iv)]]:

$$\textit{torally Kummer-faithful} \;\Rightarrow\; \textit{stably } \times\mu\textit{-indivisible} \;\Rightarrow\; \textit{TKND};$$

$$\textit{Kummer-faithful} \;\Rightarrow\; \textit{stably } \times\mu\textit{-indivisible} \text{ and } \textit{AVKF} \;\Rightarrow\; \textit{TKND-AVKF}.$$

Remark 6.6.3. It follows immediately from Remark 6.6.2 that the fields "L" of Proposition 6.3, (i), (ii), are *TKND-AVKF-fields*.

Remark 6.6.4. Recall that

- the *TKND-field* "L" of Proposition 6.3, (i) [cf. Remark 6.6.3], contains the *entire subset* $\mu(\overline{\mathbb{Q}})$, while
- the *TKND-field* "L" of Proposition 6.3, (ii) [cf. Remark 6.6.3], is *stably* $\mu_{\mathfrak{Primes}^{\infty}}$*-finite*.

That is to say, the TKND-fields of Proposition 6.3, (i), (ii), may be thought of as two *"extremal cases"*, i.e., with regard to the property of containing roots of unity. On the other hand, a detailed analysis of the various *"intermediate cases"* that, in some sense, lie in between these two "extremal cases" is beyond the scope of the present paper.

Lemma 6.7 (Generalities on rational functions). *Let K be a field of characteristic 0; $\overline{K}$ an algebraic closure of K; Y a smooth curve over K. For each algebraic extension M ($\subseteq \overline{K}$) of K, write $G_M \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/M)$; $Y_M \overset{\mathrm{def}}{=} Y \times_K M$; $Y(M)$ for the set of M-rational points of Y; $O^{\times}_{Y_M}$ for the group of invertible regular functions on Y_M;*

$$\kappa_Y : O^{\times}_{Y_{\overline{K}}} = \varinjlim_{K \subseteq K^{\dagger}} O^{\times}_{Y_{K^{\dagger}}} \longrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(\Pi_{Y_{K^{\dagger}}}, \mu_{\widehat{\mathbb{Z}}}(\overline{K}))$$

for the **Kummer map**, *where $\mu_{\widehat{\mathbb{Z}}}(\overline{K}) \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu(\overline{K}))$; $K^{\dagger}$ ($\subseteq \overline{K}$) ranges over the finite extensions of K. Let $y \in Y(K^{\dagger})$, where $K^{\dagger}$ ($\subseteq \overline{K}$) is a finite extension of K. Thus, $y \in Y(K^{\dagger})$ determines a section $G_{K^{\dagger}} \hookrightarrow \Pi_{Y_{K^{\dagger}}}$*

[i.e., strictly speaking, an outer homomorphism] of the natural surjection $\Pi_{Y_{K^\dagger}} \twoheadrightarrow G_{K^\dagger}$. *In particular, by allowing* $K^\dagger$ *and* $y \in Y(K^\dagger)$ *to vary, we obtain a natural homomorphism*

$$D_Y : \varinjlim_{K \subseteq K^\dagger} H^1(\Pi_{Y_{K^\dagger}}, \mu_{\widehat{\mathbb{Z}}}(\overline{K})) \longrightarrow \mathrm{Fn}(Y(\overline{K}), \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \mu_{\widehat{\mathbb{Z}}}(\overline{K}))).$$

Then the following hold:

(i) *Suppose that* K *is AVKF, and that* Y *is proper over* K. *Then*

$$H^1(\Pi_{Y_{\overline{K}}}, \mu_{\widehat{\mathbb{Z}}}(\overline{K}))^{G_K} = \{0\}.$$

(ii) *Suppose that*

- $K \subseteq \overline{K} = \overline{\mathbb{Q}}$, *and* K *is AVKF;*
- Y *is affine, and the function field of* $Y_{\overline{\mathbb{Q}}}$ *is equipped with the structure of a finite Galois extension of* K_X *[cf. Corollary 6.5, (ii)].*

We apply the notation of Definition 5.9, (ii), where we take "BGT*" to be* G_K *[cf. Corollary 6.5], "*$\Pi^* \subseteq \Pi$*" to be the normal open subgroup determined by* $Y_{\overline{\mathbb{Q}}}$ *[cf. Corollary 6.5, (ii)], and "*$S \subseteq \mathrm{Cusp}(\Pi^*)$*" to be the subset corresponding to the set of cusps of the hyperbolic curve* $Y_{\overline{\mathbb{Q}}}$. *Then the natural outer isomorphism* $\Pi^*_S \xrightarrow{\sim} \Pi_{Y_{\overline{\mathbb{Q}}}}$ *[which is compatible with the respective outer actions of* N $(\subseteq \mathrm{BGT} = G_K)$ *— cf. Corollary 6.5, (ii)] and the natural scheme-theoretic isomorphism* $\Pi_{X_{0\infty}} \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(\overline{K})$ *induce an isomorphism* $\mathrm{Im}(\kappa_Y) \xrightarrow{\sim} K^{\kappa}_{\Pi^*_S}$ *[cf. (i); Definition 5.10].*

(iii) *Suppose that* K *is TKND-AVKF. Then the restriction* $D_Y|_{\mathrm{Im}(\kappa_Y)}$ *of* D_Y *to* $\mathrm{Im}(\kappa_Y)$ *is injective.*

Proof. First, we verify assertion (i). Recall that since Y is a smooth, proper curve over K, $\Pi^{\mathrm{ab}}_{Y_{\overline{K}}}$ is naturally isomorphic to the Tate module of the Jacobian J of Y. In particular, if $(\Pi^{\mathrm{ab}}_{Y_{\overline{K}}})^{G_K} \neq \{1\}$, then there exists a nontrivial divisible element of $J(K)$. Thus, since K is AVKF, we conclude that $(\Pi^{\mathrm{ab}}_{Y_{\overline{K}}})^{G_K} = \{1\}$. On the other hand, Poincaré duality yields a G_K-equivariant isomorphism of topological modules

$$H^1(\Pi_{Y_{\overline{K}}}, \mu_{\widehat{\mathbb{Z}}}(\overline{K})) \ = \ \mathrm{Hom}(\Pi^{\mathrm{ab}}_{Y_{\overline{K}}}, \mu_{\widehat{\mathbb{Z}}}(\overline{K})) \ \stackrel{\sim}{\to} \ \Pi^{\mathrm{ab}}_{Y_{\overline{K}}}.$$

Thus, we conclude that $H^1(\Pi_{Y_{\overline{K}}}, \mu_{\widehat{\mathbb{Z}}}(\overline{K}))^{G_K} = \{0\}$. This completes the proof of assertion (i). Assertion (ii) follows immediately from the various definitions involved [cf. Remark 5.10.1; the argument applied in the proof of [Tsjm], Theorem 3.1].

Finally, we verify assertion (iii). First, we observe that it follows from the various definitions involved that there exists a commutative diagram

$$\begin{array}{ccc}
O^{\times}_{Y_{\overline{K}}} & \xrightarrow{\mathrm{ev}_Y} & \mathrm{Fn}(Y(\overline{K}),\ \overline{K}^{\times}) \\
\kappa_Y \big\downarrow & & \big\downarrow \\
\varinjlim_{K \subseteq K^\dagger} H^1(\Pi_{Y_{K^\dagger}}, \mu_{\widehat{\mathbb{Z}}}(\overline{K})) & \xrightarrow{D_Y} & \mathrm{Fn}(Y(\overline{K}),\ \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \mu_{\widehat{\mathbb{Z}}}(\overline{K}))),
\end{array}$$

where ev_Y denotes the homomorphism induced by evaluating elements of $O^{\times}_{Y_{\overline{K}}}$ at elements of $Y(\overline{K})$; the right-hand vertical arrow denotes the natural homomorphism induced by the Kummer map

$$\overline{K}^{\times} = \varinjlim_{K \subseteq K^\dagger} (K^\dagger)^{\times} \ \longrightarrow \ \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \mu_{\widehat{\mathbb{Z}}}(\overline{K})).$$

Let $f \in \mathrm{Ker}(D_Y \circ \kappa_Y)$. Then the commutativity of the above diagram implies that $\mathrm{Im}(\mathrm{ev}_Y(f)) \subseteq K^{\times}_{\mathrm{div}} \subseteq \overline{K}^{\times}$. On the other hand, we note that, for any *nonconstant* rational function $g \in O^{\times}_{Y_{\overline{K}}}$, the complement $\overline{K}^{\times} \setminus \mathrm{Im}(\mathrm{ev}_Y(g))$ is a *finite set.* In particular, it follows immediately from our assumption that K is *TKND* [i.e., the fact that $K_{\mathrm{div}} \subseteq \overline{K}$ is an *infinite field extension*] that f is a *constant* function such that $\kappa_Y(f) = 0$. Thus, we conclude that $D_Y|_{\mathrm{Im}(\kappa_Y)}$ is injective. This completes the proof of assertion (iii), hence of Lemma 6.7. $\square$

Theorem 6.8 (TKND-AVKF-fields satisfy the AA-property). *Let $K \subseteq \overline{\mathbb{Q}}$ be a TKND-AVKF-field. Then the following hold:*

(i) The closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies the **AA-property** *[cf. Definition 5.12].*

(ii) The natural homomorphism

$$C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$$

[cf. Corollary 6.5, (iii)] is **injective** *and compatible with the respective natural injections* $C_{\mathrm{GT}}(G_K) \hookrightarrow \mathrm{GT}$ *and* $G_{\mathbb{Q}} \hookrightarrow \mathrm{GT}$ *into* GT *[cf. Corollary 5.15, (iv)].*

Proof. First, we verify assertion (i). Since K is AVKF, it follows from Corollary 6.5 that the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies the *ArBC-property.* Next, since K is TKND, it follows immediately from the various definitions involved that the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies condition (i) of Definition 5.12. Moreover, since K is TKND-AVKF, it follows immediately from Lemma 6.7, (i), (ii), (iii), together with the various definitions involved, that the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies condition (ii) of Definition 5.12. On the other hand, since K is AVKF, it follows immediately from Lemma 6.7, (ii), together with the various definitions involved, that the function fields of finite ramified Galois coverings of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ [i.e., the projective line over $\overline{\mathbb{Q}}$] determine a family

$$\{K_{\Pi^\dagger} \subseteq L_{\Pi^\dagger}\}_{\Pi^\dagger \subseteq \Pi}$$

of subsets as in Definition 5.12, (iii). Finally, it follows immediately from the various definitions involved that condition (iv) of Definition 5.12 holds. Thus, we conclude that the closed subgroup $G_K \subseteq \mathrm{GT}$ satisfies the *AA-property.* This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with Theorem 5.15, (iii), (iv). This completes the proof of Theorem 6.8. □

Remark 6.8.1. Theorem 6.8, (i), may be regarded as a generalization of Remark 5.12.1 [cf. Remark 6.6.2]. In this context, we observe that the proof of Theorem 6.8, (i), (ii), can be *simplified considerably* in the case where K is assumed to be *Kummer-faithful*, in which case one may combine the techniques of [AbsTopIII], Theorem 1.11, or [Hsh1], Theorem A, with the *combinatorial approach* to *Belyi cuspidalizations* developed in §3 of the present paper.

Corollary 6.9 (Semi-absolute Grothendieck Conjecture-type result over TKND-AVKF-fields for tripods). *Let n be an integer ≥ 2; $K, L \subseteq \overline{\mathbb{Q}}$*

TKND-AVKF-fields. Write $X_K \stackrel{\text{def}}{=} \mathbb{P}^1_K \backslash \{0, 1, \infty\}$*;* $X_L \stackrel{\text{def}}{=} \mathbb{P}^1_L \backslash \{0, 1, \infty\}$*;* $(X_K)_n$ *(respectively,* $(X_L)_n$*) for the* n*-th configuration space associated to* X_K *(respectively,* X_L*);* $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ *(respectively,* $G_L \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/L)$*);*

$$\mathrm{Out}(\Pi_{(X_K)_n}/G_K, \Pi_{(X_L)_n}/G_L)$$

for the set of outer isomorphisms $\Pi_{(X_K)_n} \xrightarrow{\sim} \Pi_{(X_L)_n}$ *that induce outer isomorphisms* $G_K \xrightarrow{\sim} G_L$*. Then the natural map*

$$\mathrm{Isom}((X_K)_n, (X_L)_n) \longrightarrow \mathrm{Out}(\Pi_{(X_K)_n}/G_K, \Pi_{(X_L)_n}/G_L)$$

is **bijective***.*

Proof. Write $X \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$; X_n for the n-th configuration space associated to X. Let $\sigma \in \mathrm{Out}(\Pi_{(X_K)_n}/G_K, \Pi_{(X_L)_n}/G_L)$;

$$\tilde{\sigma} : \Pi_{(X_K)_n} \xrightarrow{\sim} \Pi_{(X_L)_n}$$

an isomorphism that lifts σ. Write $\sigma_{\overline{\mathbb{Q}}} \in \mathrm{Out}(\Pi_{X_n})$ for the outer automorphism determined by the restriction of $\tilde{\sigma}$ to Π_{X_n}; $\tilde{\sigma}_{\mathrm{Gal}} : G_K \xrightarrow{\sim} G_L$ for the isomorphism induced by the isomorphism $\tilde{\sigma}$. Thus, it follows immediately from the various definitions involved that there exists a *commutative diagram*

$$\begin{array}{ccc} G_K & \longrightarrow & \mathrm{Out}(\Pi_{X_n}) \\ {\scriptstyle \tilde{\sigma}_{\mathrm{Gal}}} \downarrow \wr & & {\scriptstyle \iota_{\sigma_{\overline{\mathbb{Q}}}}} \downarrow \wr \\ G_L & \longrightarrow & \mathrm{Out}(\Pi_{X_n}), \end{array}$$

where the horizontal arrows denote the natural outer representations; the right-hand vertical arrow denotes the automorphism $\iota_{\sigma_{\overline{\mathbb{Q}}}}$ obtained by conjugating by $\sigma_{\overline{\mathbb{Q}}}$. Next, we verify the following assertion:

> Claim 6.9.A: The isomorphism $\tilde{\sigma}_{\mathrm{Gal}}$ arises from an isomorphism $\overline{\mathbb{Q}} \xrightarrow{\sim} \overline{\mathbb{Q}}$ that maps $K \subseteq \overline{\mathbb{Q}}$ onto $L \subseteq \overline{\mathbb{Q}}$.

Indeed, [cf. the above *commutative diagram*] since the closed subgroups $G_K \subseteq \mathrm{GT}$ and $G_L \subseteq \mathrm{GT}$ satisfy the *ArBC-property* [cf. Corollary 6.5], the *functorial constructions* of Corollary 4.5, together with the isomorphism of Corollary 6.5, (i) [applied to G_K and G_L], determine a commutative diagram

$$\begin{array}{ccccccc} G_K & = & G_K & \overset{\tilde{\sigma}_{\mathrm{Gal}}}{\overset{\sim}{\to}} & G_L & = & G_L \\ \curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\ \overline{\mathbb{Q}} & \overset{\sim}{\leftarrow} & \overline{\mathbb{Q}}_{G_K} & \overset{\sim}{\to} & \overline{\mathbb{Q}}_{G_L} & \overset{\sim}{\to} & \overline{\mathbb{Q}}, \end{array}$$

where the lower horizontal arrows are isomorphisms of fields. Thus, we obtain the desired conclusion. This completes the proof of Claim 6.9.A.

Now it follows from Claim 6.9.A that we may assume without loss of generality that $K = L \subseteq \overline{\mathbb{Q}}$. Next, it follows from Theorem 6.8, (ii), together with the various definitions involved, that

$$N_{\mathrm{GT}}(G_K) \subseteq C_{\mathrm{GT}}(G_K) \subseteq G_{\mathbb{Q}}.$$

In particular, we conclude that $N_{\mathrm{GT}}(G_K)/G_K = N_{G_{\mathbb{Q}}}(G_K)/G_K$. Note that since Π_{X_n} is center-free [cf. [MT], Proposition 2.2, (ii)], there exists [cf. the above *commutative diagram* involving $\iota_{\sigma_{\overline{\mathbb{Q}}}}$] a natural isomorphism

$$\mathrm{Out}(\Pi_{(X_K)_n}/G_K) \overset{\sim}{\to} N_{\mathrm{Out}(\Pi_{X_n})}(G_K)/G_K,$$

where $\mathrm{Out}(\Pi_{(X_K)_n}/G_K)$ denotes the set of outer automorphisms of $\Pi_{(X_K)_n}$ that induce outer automorphisms of G_K. In particular, $\sigma \in \mathrm{Out}(\Pi_{(X_K)_n}/G_K)$ determines an element of

$$\begin{aligned} N_{\mathrm{Out}(\Pi_{X_n})}(G_K)/G_K &= N_{\mathrm{GT}\times S_{n+3}}(G_K)/G_K \\ &= (N_{\mathrm{GT}}(G_K)/G_K) \times S_{n+3} \end{aligned}$$

[cf. the first display of [CbGT], Corollary C]. Thus, in light of the *natural isomorphism*

$$\mathrm{Aut}(K) \overset{\sim}{\to} N_{G_{\mathbb{Q}}}(G_K)/G_K = N_{\mathrm{GT}}(G_K)/G_K,$$

we conclude that the *natural group homomorphism*

$$\mathrm{Aut}((X_K)_n) \longrightarrow \mathrm{Out}(\Pi_{(X_K)_n}/G_K)$$

is *surjective*, and [by considering the various *fiber subgroups* of Π_{X_n} and *cuspidal inertia subgroups* of Π_X] that any element $\alpha \in \mathrm{Aut}((X_K)_n)$ in the kernel of this group homomorphism is *K-linear* and *compatible* with the *identity automorphism* of X_K relative to any of the $n+3$ *generalized projection morphisms* $(X_K)_n \to X_K$ [cf. [CbGT], Definition 2.1, (i)]. But this implies that any such α is equal to the *identity automorphism* of $(X_K)_n$. This completes the proof of Corollary 6.9. $\square$

REMARK 6.9.1. Note that Corollary 6.9 [cf. the final portion of the proof of Corollary 6.9], together with the well-known *commensurable terminality* of $G_{\mathbb{Q}_p}$ in $G_{\mathbb{Q}}$ [cf., e.g., [AbsAnab], Theorem 1.1.1, (i)] gives a *new proof* of the equalities

$$C_{\mathrm{GT}}(G_{\mathbb{Q}}) = G_{\mathbb{Q}}, \quad C_{\mathrm{GT}}(G_{\mathbb{Q}_p}) = G_{\mathbb{Q}_p},$$

hence also, by applying the well-known *slimness* of $G_{\mathbb{Q}}$ [cf., e.g., [AbsAnab], Theorem 1.1.1, (ii)], the equality

$$Z^{\mathrm{loc}}(\mathrm{GT}) = \{1\}.$$

In particular, Corollary 6.9 yields a *purely combinatorial/group-theoretic* proof of the portion of [CbGT], Corollary C, concerning "$Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$" [in the notation of *loc. cit.*, where we take"Σ" to be $\mathfrak{Primes}$] that does *not* depend on the proofs of the *Grothendieck Conjecture for hyperbolic curves over number fields* given in [LocAn], Theorem A; [Tama], Theorem 0.4 [cf. the discussion of Remark 3.1.1].

THEOREM 6.10 (Semi-absolute Grothendieck Conjecture-type result over TKND-AVKF-fields for arbitrary hyperbolic curves). *Let (m, n) be a pair of positive integers; $K, L \subseteq \overline{\mathbb{Q}}$ TKND-AVKF-fields; X_K (respectively, Y_L) a hyperbolic curve over K (respectively, L). Write (g_X, r_X) (respectively, (g_Y, r_Y)) for the type [i.e., genus and degree of the divisor of marked points] of X_K (respectively, Y_L); $(X_K)_m$ (respectively, $(Y_L)_n$) for the m-th (respectively, n-th) configuration space associated to X_K (respectively, Y_L); $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ (respectively, $G_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/L)$);*

$$\mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L)$$

for the set of outer isomorphisms $\Pi_{(X_K)_m} \xrightarrow{\sim} \Pi_{(Y_L)_n}$ that induce outer isomorphisms between G_K and G_L. Then the following hold:

(i) Suppose that

- $m \geq 4$ *or* $n \geq 4$ *if* $r_X = 0$ *or* $r_Y = 0$*;*
- $m \geq 3$ *or* $n \geq 3$ *if* $r_X \neq 0$ *or* $r_Y \neq 0$.

Then the outer isomorphism

$$G_K \xrightarrow{\sim} G_L$$

induced by any outer isomorphism $\in \mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L)$ *arises from a* **field isomorphism** $K \xrightarrow{\sim} L$.

(ii) Suppose that

- $m \geq 2$ *or* $n \geq 2$*;*
- $g_X = 0$ *or* $g_Y = 0$.

Then the natural map

$$\mathrm{Isom}((X_K)_m, (Y_L)_n) \longrightarrow \mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L)$$

is **bijective***.*

PROOF. Write

- $Z_K \stackrel{\mathrm{def}}{=} \mathbb{P}^1_K \backslash \{0, 1, \infty\}$; $Z_L \stackrel{\mathrm{def}}{=} \mathbb{P}^1_L \backslash \{0, 1, \infty\}$;
- $X \stackrel{\mathrm{def}}{=} X_K \times_K \overline{\mathbb{Q}}$; $Y \stackrel{\mathrm{def}}{=} Y_L \times_L \overline{\mathbb{Q}}$; $Z \stackrel{\mathrm{def}}{=} Z_K \times_K \overline{\mathbb{Q}} = Z_L \times_L \overline{\mathbb{Q}}$;

For each positive integer i, write

- X_i (respectively, Y_i, Z_i) for the i-th configuration space associated to X (respectively, Y, Z).

Note that, to verify assertions (i), (ii), it follows immediately from the various definitions involved that we may assume without loss of generality that

$$\mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L) \neq \emptyset.$$

Thus, we conclude from [CbGT], Theorem A, (i), that

$$m = n \geq 2, \quad g_X = g_Y, \quad r_X = r_Y.$$

Let $\sigma \in \mathrm{Out}(\Pi_{(X_K)_n}/G_K, \Pi_{(Y_L)_n}/G_L)$;

$$\tilde{\sigma} : \Pi_{(X_K)_n} \xrightarrow{\sim} \Pi_{(Y_L)_n}$$

an isomorphism that lifts σ. Write $\sigma_{\overline{\mathbb{Q}}} : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n}$ for the outer isomorphism determined by the restriction of $\tilde{\sigma}$ to Π_{X_n}; $\tilde{\sigma}_{\mathrm{Gal}} : G_K \xrightarrow{\sim} G_L$ for the isomorphism induced by the isomorphism $\tilde{\sigma}$.

Next, we verify assertion (i). Note that $m = n \geq 3$. Let $\Pi_X^{\mathrm{ctpd}} \subseteq \Pi_{X_3}$ (respectively, $\Pi_Y^{\mathrm{ctpd}} \subseteq \Pi_{Y_3}$) be a *3-central* $\{$*1, 2, 3*$\}$*-tripod* of Π_{X_n} (respectively, Π_{Y_n}) [cf. [CbTpII], Definition 3.7, (ii)]. Then since $m = n$, $g_X = g_Y$, and $r_X = r_Y$, it follows immediately from [CbGT], Corollary B; [CbTpII], Theorem A, (ii); [CbTpII], Theorem C, (ii); the discussion of [CbTpII], Remark 4.14.1, that, after possibly replacing σ by the composite of σ with an element $\in \mathrm{Out}(\Pi_{(X_K)_n}/G_K, \Pi_{(X_K)_n}/G_K)$ that arises from an element of the symmetric group "$\mathfrak{S}_{n^*}$" of [CbGT], Corollary B, we may assume without loss of generality that

- $\sigma_{\overline{\mathbb{Q}}}$ induces *bijections* between the respective sets of *fiber subgroups* and *inertia subgroups*;

- the outer isomorphism $\Pi_{X_3} \xrightarrow{\sim} \Pi_{Y_3}$ induced by $\sigma_{\overline{\mathbb{Q}}}$ determines an *outer isomorphism* $\sigma_{\mathrm{ctpd}} : \Pi_X^{\mathrm{ctpd}} \xrightarrow{\sim} \Pi_Y^{\mathrm{ctpd}}$;

- there exists a *commutative diagram* of profinite groups

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_n}) & \xrightarrow[T_X]{} & \mathrm{Out}(\Pi_X^{\mathrm{ctpd}}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Out}^{\mathrm{FC}}(\Pi_{Y_n}) & \xrightarrow[T_Y]{} & \mathrm{Out}(\Pi_Y^{\mathrm{ctpd}}), \end{array}$$

  where the vertical arrows denote the isomorphisms induced by the outer isomorphisms $\sigma_{\overline{\mathbb{Q}}}$ and σ_{ctpd}, and T_X and T_Y denote the respective *tripod homomorphisms.*

Here, we *identify* Π_Z with Π_X^{ctpd}, Π_Y^{ctpd}, via *outer isomorphisms* $\Pi_Z \xrightarrow{\sim} \Pi_X^{\mathrm{ctpd}}$, $\Pi_Z \xrightarrow{\sim} \Pi_Y^{\mathrm{ctpd}}$ that arise from the respective $\mathfrak{S}_3$-torsors of scheme-theoretic isomorphisms of tripods over $\overline{\mathbb{Q}}$ in such a way that

- the outer automorphism $\sigma_Z : \Pi_Z \xrightarrow{\sim} \Pi_X^{\text{ctpd}} \xrightarrow{\sim} \Pi_Y^{\text{ctpd}} \xleftarrow{\sim} \Pi_Z$ obtained by conjugating σ_{ctpd} by these *identifying outer isomorphisms* determines an element $\in \text{GT} \subseteq \text{Out}(\Pi_Z)$

[cf. [CbTpII], Theorem C, (iv), together with our assumptions on $m = n$]. Moreover, it follows immediately [again from [CbTpII], Theorem C, (iv), together with our assumptions on $m = n$] that

- the images of T_X and T_Y are contained in $\text{GT} \subseteq \text{Out}(\Pi_Z)$.

In particular, the above *commutative diagram*, together with the natural outer representations $G_K \to \text{Out}^{\text{FC}}(\Pi_{X_n})$, $G_L \to \text{Out}^{\text{FC}}(\Pi_{Y_n})$, determines a commutative diagram of profinite groups

$$\begin{array}{ccc} G_K & \longrightarrow & \text{GT} \\ {\scriptstyle \tilde{\sigma}_{\text{Gal}}}\downarrow\wr & & {\scriptstyle \sigma_Z}\downarrow\wr \\ G_L & \longrightarrow & \text{GT}, \end{array}$$

where the right-hand vertical arrow denotes the inner automorphism obtained by conjugating by σ_Z; the horizontal arrows denote the natural injections. Observe that since Π_{Z_2} is center-free [cf. [MT], Proposition 2.2, (ii)], this last commutative diagram determines an outer isomorphism $\Pi_{(Z_K)_2} \xrightarrow{\sim} \Pi_{(Z_L)_2}$ that lies over $\tilde{\sigma}_{\text{Gal}}$ between the second configuration spaces $(Z_K)_2$, $(Z_L)_2$ associated to Z_K, Z_L, respectively. Thus, we conclude from Corollary 6.9 that the outer isomorphism determined by $\tilde{\sigma}_{\text{Gal}} : G_K \xrightarrow{\sim} G_L$ arises from a field isomorphism $K \xrightarrow{\sim} L$. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, it follows from a similar argument to the argument applied in the final portion of the proof of Corollary 6.9 [after possibly passing to suitable *finite Galois extensions* of K and L and, if $r_X = r_Y \geq 4$, applying Corollary 2.2] that the natural map

$$\text{Isom}((X_K)_n, (Y_L)_n) \longrightarrow \text{Out}(\Pi_{(X_K)_n}/G_K, \Pi_{(Y_L)_n}/G_L)$$

is *injective*. Thus, it suffices to prove that this map is *surjective*. We begin by observing that, by applying the *injectivity* that has already been verified, we may pass to suitable *finite Galois extensions* of K and L and apply *Galois descent*. In particular, we may assume without loss of generality that every cusp of X (respectively, Y) is *K-rational* (respectively, *L-rational*). On the other hand, since $g_X = g_Y = 0$, it suffices to consider the case where $r_X = r_Y \geq 4$ [cf. Corollary 6.9].

Next, we verify the following assertion:

> Claim 6.10.A: There exists an *isomorphism of schemes* $X_K \xrightarrow{\sim} Y_L$.

Indeed, observe that it follows from Theorem 2.1 [cf. our assumption that $r_X = r_Y \geq 4$] that there exist open immersions $X_K \hookrightarrow Z_K$, $Y_L \hookrightarrow Z_L$ over K, L, respectively, which, together with $\tilde{\sigma}$, determine a Π_{Z_n}-outer isomorphism $\sigma_{Z_n} : \Pi_{(Z_K)_n} \xrightarrow{\sim} \Pi_{(Z_L)_n}$ that lies over the isomorphism $\tilde{\sigma}_{\mathrm{Gal}}$ and *fixes the cusps* of Z. Thus, by applying Corollary 6.9 [cf. also the final portion of the proof of Corollary 6.9], we may assume without loss of generality that

- $K = L$;
- $\tilde{\sigma}_{\mathrm{Gal}}$ is the identity automorphism;
- σ_{Z_n} is the identity Π_{Z_n}-outer automorphism.

In particular, since $\sigma_{\overline{\mathbb{Q}}}$ induces a bijection between the respective sets of *fiber subgroups* and *inertia subgroups* [cf. Corollary 2.2; the discussion of [CbTpII], Remark 4.14.1], $\tilde{\sigma}$ determines a Π_Y-outer isomorphism $\sigma_1 : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_K}$ [cf. [CbTpI], Theorem A, (i)] such that

- σ_1 lies over G_K;
- σ_1 induces a bijection between the respective sets of cuspidal inertia subgroups.

Thus, we conclude from the fact that K satisfies the *ISC-property* [cf. Proposition 6.4], applied to Z_K [cf. the proof of Lemma 5.4, (i)], that there exists an isomorphism $X_K \xrightarrow{\sim} Y_K$ over K. This completes the proof of Claim 6.10.A.

In summary, it follows formally from Claim 6.10.A, together with the above discussion, that we may assume without loss of generality that

- $K = L$, $X_K = Y_K$;

- $\tilde{\sigma}$ is an automorphism of $\Pi_{(X_K)_n}$ that lies over the *identity automorphism* of G_K;
- the Π_{Z_n}-outer automorphism $\sigma_{Z_n} : \Pi_{(Z_K)_n} \xrightarrow{\sim} \Pi_{(Z_K)_n}$ [induced by $\tilde{\sigma}$ and the open immersion $X_K \hookrightarrow Z_K$ over K] is the *identity* Π_{Z_n}*-outer automorphism*;
- the outer automorphism $\sigma_{\overline{\mathbb{Q}}} : \Pi_{X_n} \xrightarrow{\sim} \Pi_{X_n}$ [determined by $\tilde{\sigma}$] induces the *identity automorphism* on the set of *fiber subgroups*;
- the Π_X-outer automorphism $\sigma_1 : \Pi_{X_K} \xrightarrow{\sim} \Pi_{X_K}$ [determined by $\tilde{\sigma}$] induces the *identity automorphism* on the set of conjugacy classes of *cuspidal inertia subgroups* of Π_X [cf. the discussion above of the ISC-property applied to Z_K].

Thus, if we regard G_K as a subgroup of $\mathrm{Out}^{\mathrm{gF}}(\Pi_{X_n})^{\mathrm{cusp}}$ via the natural injection $G_K \hookrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_{X_n})^{\mathrm{cusp}}$ [cf. [CbTpI], Theorem A, (i), (ii)], then $\sigma_{\overline{\mathbb{Q}}} \in Z_{\mathrm{Out}^{\mathrm{gF}}(\Pi_{X_n})^{\mathrm{cusp}}}(G_K)$. Write

$$\beta \in Z_{\mathrm{Out}^{\mathrm{gF}}(\Pi_{X_2})^{\mathrm{cusp}}}(G_K)$$

for the element determined by $\sigma_{\overline{\mathbb{Q}}}$ via the natural injection $\mathrm{Out}^{\mathrm{gF}}(\Pi_{X_n})^{\mathrm{cusp}} \hookrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_{X_2})^{\mathrm{cusp}}$ [cf. [CbTpII], Theorem A, (i)];

$$h : \mathrm{Out}^{\mathrm{gF}}(\Pi_{X_2})^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{gF}}(\Pi_{Z_2})^{\mathrm{cusp}}$$

for the natural homomorphism induced by the natural open immersion $X_2 \hookrightarrow Z_2$ [cf. Theorem 2.1]. Then it follows immediately from our assumption that $\sigma_{Z_n} : \Pi_{(Z_K)_n} \xrightarrow{\sim} \Pi_{(Z_K)_n}$ is the identity Π_{Z_n}-outer automorphism that $h(\beta) = 1$. Thus, we conclude from Theorem 3.6 [where we apply [NCBel], Corollary 1.1, and we take "$V \subseteq W$" to be the open immersion $X \hookrightarrow Z$ in the above discussion], together with Proposition 6.4, that $\beta = 1$, hence that $\sigma_{\overline{\mathbb{Q}}} = 1$. Finally, since Π_{X_n} is center-free [cf. [MT], Proposition 2.2, (ii)], it holds that $\tilde{\sigma}$ is an inner automorphism, hence that $\sigma = 1$. Thus, we obtain the desired *surjectivity*. This completes the proof of assertion (ii), hence of Theorem 6.10. □

Remark 6.10.1. In the notation of Theorem 6.10, write

$$\mathrm{Out}(\Pi_{(X_K)_m}, \Pi_{(Y_L)_n})$$

for the set of outer isomorphisms $\Pi_{(X_K)_m} \xrightarrow{\sim} \Pi_{(Y_L)_n}$. Suppose that G_K and G_L are *very elastic* [cf. [AbsTopI], Definition 1.1, (ii)]. Then since Π_{X_m} and Π_{Y_n} are topologically finitely generated [cf. [MT], Proposition 2.2, (ii)], it follows formally that

$$\mathrm{Out}(\Pi_{(X_K)_m}, \Pi_{(Y_L)_n}) = \mathrm{Out}(\Pi_{(X_K)_m}/G_K, \Pi_{(Y_L)_n}/G_L),$$

i.e., that the *"absolute version"* of Theorem 6.10 holds.

Remark 6.10.2. In the notation of Theorem 6.10, suppose that K and L arise as fields "L" of the sort discussed in Proposition 6.3, (i), (ii) [cf. Remark 6.6.3]. Suppose, further, that K and L are *abelian extensions* of *number fields* [cf., e.g., the field "$F \cdot \mathbb{Q}^{\mathrm{ab}}$" of Proposition 6.3, (i); the field "M" of Remark 6.3.3]. Then G_K and G_L are *very elastic* [cf. [FJ], §6.2; [FJ], Theorem 13.4.2; [FJ], Theorem 16.11.3; [Mi], Theorem 2.1]. In particular, it follows immediately from Remark 6.10.1 that the *absolute version* of Theorem 6.10 holds.

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