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Category-Theoretic Reconstruction of Log Schemes from Categories of Reduced fs Log Schemes

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Abstract. Let $S^{\log}$ be a locally Noetherian fs log scheme and $\blacklozenge/S^{\log}$ a set of properties of fs log schemes over $S^{\log}$. In the present paper, we shall mainly be concerned with the properties "reduced", "quasi-compact over $S^{\log}$", "quasi-separated over $S^{\log}$", "separated over $S^{\log}$", and "of finite type over $S^{\log}$". We shall write $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ for the full subcategory of the category of fs log schemes over $S^{\log}$ determined by the fs log schemes over $S^{\log}$ that satisfy every property contained in $\blacklozenge/S^{\log}$. In the present paper, we discuss a purely category-theoretic reconstruction of the log scheme $S^{\log}$ from the intrinsic structure of the abstract category $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

Contents

Introduction

Throughout the present paper, we **fix** a Grothendieck universe **U**. Let

$$S^{\log}$$

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be a (**U**-small) fs log scheme. Write S for the underlying scheme of $S^{\log}$. Let

$$\blacklozenge$$

be a set of properties of morphisms of (**U**-small) schemes (where we *identify* properties of morphisms of schemes with certain full subcategories of the category of morphisms of schemes, cf. Notations and Conventions — Properties of Schemes and Log Schemes). We shall write $\mathsf{Sch}_{/S}$ for the category of (**U**-small) S-schemes, $\mathsf{Sch}_{\blacklozenge/S} \subset \mathsf{Sch}_{/S}$ for the full subcategory of objects of $\mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property contained in $\blacklozenge/S$ (cf. Notations and Conventions — Properties of Schemes and Log Schemes), $\mathsf{Sch}^{\log}_{/S^{\log}}$ for the category of (**U**-small) fs log schemes over $S^{\log}$, and

$$\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$$

for the full subcategory determined by the fs log schemes over $S^{\log}$ whose underlying S-scheme is contained in $\mathsf{Sch}_{\blacklozenge/S}$. In the present paper, we shall mainly be concerned with the situation where $\blacklozenge/S^{\log}$ (cf. Notations and Conventions — Properties of Schemes and Log Schemes) is contained in the following set of properties of log schemes over $S^{\log}$:

“red”, “qcpt”, “qsep”, “sep”, “ft”,

i.e., (the source scheme is) “reduced”, “quasi-compact over $S^{\log}$”, “quasi-separated over $S^{\log}$”, “separated over $S^{\log}$”, and “of finite type over $S^{\log}$”.

In the present paper, we consider the problem of reconstructing the log scheme $S^{\log}$ from the intrinsic structure of the abstract category $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. The problem of reconstructing the scheme S from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$ in the case where the elements of $\blacklozenge/S$ amount essentially to the property of being “finite étale over S” is closely related to Grothendieck’s anabelian conjectures and has been investigated by many mathematicians. By contrast, the case where the elements of $\blacklozenge/S$ differ substantially from the property of being “finite étale over S” has only been investigated to a limited degree. In this case, there are some known results, mainly as follows: In [Mzk04, Section 1], Mochizuki gave a solution to this reconstruction problem in the case where S is locally Noetherian, and $\blacklozenge = \{\mathrm{ft}\}$. In [vDdB19], van Dobben de Bruyn gave a solution to this reconstruction problem in the case where S is an arbitrary scheme, and $\blacklozenge = \varnothing$. The arguments in [Mzk04, Section 1] and [vDdB19] make essential use of the existence of non-reduced schemes. On the other hand, in [YJ], the author gave a solution to this reconstruction problem in the case where S is a locally Noetherian normal scheme, and one allows an arbitrary subset $\blacklozenge \subset \{\text{red}, \text{qcpt}, \text{qsep}, \text{sep}\}$.

There are even fewer known results concerning the problem of reconstructing a log scheme $S^{\log}$ from the intrinsic structure of the abstract category $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. In [Mzk15] (and [Mzk04, Section 2]), S. Mochizuki proved that if $\blacklozenge = \{\text{ft}\}$, then a locally Noetherian fs log scheme $S^{\log}$ may be reconstructed category-theoretically from the intrinsic structure of the abstract category $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. In [HoNa], Y. Hoshi and C. Nakayama gave a category-theoretic characterization of strict morphisms in the case where $S^{\log}$ is locally Noetherian, and $\blacklozenge = \{\text{ft}\}$. As discussed in [HoNa, Introduction], the arguments of [Mzk15] can be applied in more general situations where $\blacklozenge = \{\text{ft}\}$. For instance, the condition assumed in [Mzk15] that $\blacklozenge = \{\text{ft}\}$ may be replaced by the assumption that $\blacklozenge = \{\text{ft}, \text{sep}\}$ (for a detailed discussion, cf. [HoNa, Introduction]). On the other hand, the proof given in [Mzk15] is based on somewhat complicated combinatorial properties of monoids. By contrast, while the arguments in [HoNa] are somewhat more straightforward than the arguments of [Mzk15], the result of Hoshi and Nakayama depends essentially on the existence of non-separated log schemes in $\mathsf{Sch}^{\log}_{\{\text{ft}\}/S^{\log}}$ (for a detailed discussion, cf. [HoNa, Introduction]). In particular, the arguments in [HoNa] cannot be applied in the situation, for instance, where $\blacklozenge = \{\text{ft}, \text{sep}\}$. Here we note that the arguments in [Mzk15] also make essential use of the existence of non-reduced schemes to give a characterization of "SLEM" morphisms (cf. [Mzk15, Definition 2.1, Proposition 2.2]). Hence, the arguments in [Mzk15] cannot be applied in the case, for instance, where $\blacklozenge = \{\text{ft}, \text{sep}, \text{red}\}$.

In the present paper, we give a **relatively simple** solution to this problem of reconstructing log structures in a situation that **generalizes** the situations discussed in [Mzk04], [Mzk15], and [HoNa] to include the log scheme version of the situation discussed in [YJ]. Our main result is the following:

THEOREM A (cf. Theorem 5.10). *Let $S^{\log}, T^{\log}$ be locally Noetherian* *fs log schemes,*

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$$

[possibly empty] subsets, and $F : \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ *an equivalence of categories. Assume that one of the following conditions (A), (B) holds:*

(A) $\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$*, and the underlying schemes of* $S^{\log}$ *and* $T^{\log}$ *are normal.*

(B) $\blacklozenge = \lozenge = \{\mathrm{ft}\}$*.*

Then the following assertions hold:

(i) Let $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *be an object. Then there exists an isomorphism of log schemes* $X^{\log} \xrightarrow{\sim} F(X^{\log})$ *that is functorial with respect to* $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$*.*

(ii) Assume that $\blacklozenge = \lozenge$*. Then there exists a unique isomorphism of log schemes* $S^{\log} \xrightarrow{\sim} T^{\log}$ *such that* F *is isomorphic to the equivalence of categories* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\blacklozenge/T^{\log}}$ *induced by composing with this isomorphism of log schemes* $S^{\log} \xrightarrow{\sim} T^{\log}$*.*

By combining the theory of [YJ] with the above Theorem A (ii), we conclude the following corollary (cf. Corollary 5.11):

Corollary B (cf. Corollary 5.11). *Let* $S^{\log}, T^{\log}$ *be locally Noetherian normal fs log schemes and*

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$$

subsets such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$*, and* $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \lozenge$*. If the categories* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *and* $\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ *are equivalent, then* $\blacklozenge = \lozenge$*, and* $S^{\log} \cong T^{\log}$*.*

Our proof of Theorem A proceeds by giving category-theoretic characterizations of various properties of log schemes and morphisms of log schemes as follows:

- In Section 1, we introduce some notions related to monoids and discuss various generalities that will play an important role in the theory of the present paper. In particular, we construct certain *non-quasi-integral* push-out monoids (Corollary 1.8). Here, we recall that the notion of quasi-integral monoids was introduced by C. Nakayama (cf. [Nak, Definition 2.2.4]) to study the surjectivity of base-changes of morphisms of log schemes whose underlying morphism of sets is surjective.

- In Section 2, we introduce some notions related to log schemes and discuss various properties of push-outs in the category of fs log schemes. In particular, we prove that certain push-outs exist in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ (cf. Corollary 2.11).

- In Section 3, we give a category-theoretic characterization of the objects of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ whose underlying log scheme is an fs log point (cf. Proposition 3.3). This characterization also yields a category-theoretic characterization of the morphisms of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ whose underlying morphism of log schemes is isomorphic to a *log residue field* (cf. Definition 2.7), i.e., the natural strict morphism that arises from the spectrum of the residue field at a point of the target log scheme (cf. Corollary 3.5).

- In Section 4, we give a category-theoretic characterization of the morphisms in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ whose underlying morphism of log schemes is strict (cf. Corollary 4.6). We then use this characterization and apply [YJ, Corollary 4.11] to obtain the first equality of Corollary B (cf. Corollary 4.7).

- In Section 5, we give a category-theoretic characterization of the morphisms of monoid objects in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ that represent the functor

$$\begin{aligned} \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} &\to \mathsf{Mor}(\mathsf{Mon}) \\ X^{\log} &\mapsto [\alpha_X(X) : \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X)], \end{aligned}$$

  which arises from the log structures of the objects of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ (cf. Proposition 5.7). We then use this characterization to prove the main theorem of the present paper (cf. Theorem 5.10).

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Notations and Conventions

We shall use the notation $\mathbb{N}$ to denote the additive monoid of non-negative rational integers $n \geq 0$. We shall use the notation $\mathbb{Z}$ to denote the ring of rational integers. We shall use the notation $\mathbb{Q}$ to denote the field of fractions of $\mathbb{Z}$. Throughout the present paper, we **fix** a Grothendieck universe $\mathbf{U}$.

Categories. Let $\mathcal{C}$ be a category. We shall write $\mathcal{C}^{\mathrm{op}}$ for the opposite category associated to $\mathcal{C}$. By a slight abuse of notation, we shall use the notation $X \in \mathcal{C}$ to denote that X is an object of $\mathcal{C}$. We shall write $\mathsf{Mor}(\mathcal{C})$ for the category of morphisms of $\mathcal{C}$, i.e., the category consisting of the following data:

- An object of $\mathsf{Mor}(\mathcal{C})$ is a morphism in $\mathcal{C}$.
- A morphism $[f : X \to Y] \to [g : X' \to Y']$ in $\mathsf{Mor}(\mathcal{C})$ is a pair $(h_X : X \to X', h_Y : Y \to Y')$ of morphisms in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{h_X} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{h_Y} & Y'. \end{array}$$

- The composite of morphisms is given on each component by composing morphisms in $\mathcal{C}$.

Let $X \in \mathcal{C}$ be an object. We shall write $\mathcal{C}_{/X}$ for the slice category of objects and morphisms equipped with a structure morphism to X. We shall write $\mathcal{C}_{X/}$ for the over category of objects and morphisms equipped with a structure morphism from X.

Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory. We shall say that $\mathcal{D}$ is a **strictly full subcategory** of $\mathcal{C}$ if $\mathcal{D}$ is closed under isomorphism, i.e., for any object

$X \in \mathcal{D}$ and any isomorphism $f : X \to Y$ in $\mathcal{C}$, Y (hence also f) is contained in $\mathcal{D}$.

Rings and Schemes. We shall use the notation Sch to denote the category of (**U**-small) schemes.

Let $f : Y \to X$ be a morphism of schemes. We shall write $\mathcal{O}_X$ for the structure sheaf of X. We shall write $|X|$ for the underlying topological space of X. We shall write $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ for the morphism of sheaves of rings on $|X|$ which defines the morphism of schemes $f : Y \to X$. If $|F| \subset |X|$ is a closed subset, then we shall write F_{red} for the reduced closed subscheme of X determined by $|F| \subset |X|$. We shall write $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ for the morphism induced by f. Let A and B be (commutative) rings (with unity). If $Y = \mathrm{Spec}(B)$, and $X = \mathrm{Spec}(A)$, then we shall write $f^{\#} : A \to B$ for the ring homomorphism induced by f. By a slight abuse of notation, if $f^{\#} : A \to B$ is a ring homomorphism, then we shall use the notation f to denote the corresponding morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. For any point $x \in X$, we shall write $k(x)$ for the residue field at $x \in X$.

Properties of Schemes and Log Schemes. Let $S^{\log}$ be a (**U**-small) fs log scheme. We shall write S for the underlying scheme of $S^{\log}$. We shall use the notation

$$\mathsf{Sch}^{\log}$$

to denote the category of (**U**-small) fs log schemes. We shall write $\mathsf{Sch}_{/S}$ for the category of (**U**-small) S-schemes and $\mathsf{Sch}^{\log}_{/S^{\log}}$ for the category of (**U**-small) fs log schemes over $S^{\log}$.

We shall refer to a strictly full subcategory (cf. Notations and Conventions — Categories) of Sch, $\mathsf{Mor}(\mathsf{Sch})$, $\mathsf{Sch}^{\log}$, $\mathsf{Mor}(\mathsf{Sch}^{\log})$, $\mathsf{Sch}_{/S}$, and $\mathsf{Sch}^{\log}_{/S^{\log}}$ as a **property** of (**U**-small) schemes, morphisms of (**U**-small) schemes, (**U**-small) fs log schemes, morphisms of (**U**-small) fs log schemes, (**U**-small) S-schemes, and (**U**-small) fs log schemes over $S^{\log}$, respectively.

Let

$$\blacklozenge$$

be a (not necessarily **U**-small) set of properties of morphisms of (**U**-small) schemes. For any property $\mathtt{P} \in \blacklozenge$, write $\mathtt{P}/S \subset \mathsf{Sch}_{/S}$ for the full subcategory consisting of S-schemes whose structure morphism is contained in the full subcategory $\mathtt{P} \subset \mathsf{Mor}(\mathsf{Sch})$ and $\mathtt{P}/S^{\log} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$ for the full subcategory consisting of fs log schemes over $S^{\log}$ whose underlying S-scheme is contained in $\mathtt{P}/S$. By a slight abuse of notation, we shall write $\blacklozenge/S \stackrel{\mathrm{def}}{:=} \{\mathtt{P}/S \mid \mathtt{P} \in \blacklozenge\}$ and $\blacklozenge/S^{\log} \stackrel{\mathrm{def}}{:=} \{\mathtt{P}/S^{\log} \mid \mathtt{P} \in \blacklozenge\}$. In the present paper, we shall mainly be concerned with the situation where

$$\blacklozenge \subset \{ \text{ red, qcpt, qsep, sep, ft } \},$$

and

- "red" denotes the strictly full subcategory of $\mathsf{Mor}(\mathsf{Sch})$ consisting of morphisms whose source scheme is reduced,
- "qcpt" denotes the strictly full subcategory of $\mathsf{Mor}(\mathsf{Sch})$ consisting of quasi-compact morphisms,
- "qsep" denotes the strictly full subcategory of $\mathsf{Mor}(\mathsf{Sch})$ consisting of quasi-separated morphisms,
- "sep" denotes the strictly full subcategory of $\mathsf{Mor}(\mathsf{Sch})$ consisting of separated morphisms, and
- "ft" denotes the strictly full subcategory of $\mathsf{Mor}(\mathsf{Sch})$ consisting of morphisms of finite type.

Let $f : X \to Y$ be a morphism of schemes. Then we shall say that f **satisfies** every property contained in $\blacklozenge$ if for any $\mathtt{P} \in \blacklozenge$, $f \in \mathtt{P} \subset \mathsf{Mor}(\mathsf{Sch})$. Hence, in particular, if $\blacklozenge = \varnothing$, then every morphism of schemes satisfies every property contained in $\blacklozenge$. If $\blacklozenge = \{\mathtt{P}\}$, and f satisfies every property contained in $\blacklozenge$, then we shall say that f **satisfies** the property $\mathtt{P}$.

We shall write $\mathsf{Sch}_{\blacklozenge/S} \subset \mathsf{Sch}_{/S}$ for the full subcategory consisting of S-schemes whose structure morphisms satisfy every property contained in $\blacklozenge$ and

$$\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$$

for the full subcategory consisting of fs log schemes over $S^{\log}$ whose underlying structure morphism of schemes is contained in $\mathsf{Sch}_{\blacklozenge/S}$. Thus, if $\blacklozenge = \varnothing$, then $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} = \mathsf{Sch}^{\log}_{/S^{\log}}$.

We shall write $\lim^{\blacklozenge}, \operatorname{colim}^{\blacklozenge}, \times^{\blacklozenge}, \sqcup^{\blacklozenge}$ for the (inverse) limit, colimit, fiber product, and push-out in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ (if these exist in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$). We shall write $\lim, \operatorname{colim}, \times, \sqcup$ for the (inverse) limit, colimit, fiber product, and push-out in $\mathsf{Sch}^{\log}_{/S^{\log}}$ (if these exist in $\mathsf{Sch}^{\log}_{/S^{\log}}$). By a slight abuse of notation, we shall use the notation $\varnothing$ to denote the empty log scheme.

Let

- $S^{\log}$ be a ($\mathbf{U}$-small) locally Noetherian fs log scheme,
- $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$ a subset,
- $\mathtt{P}$ a property of ($\mathbf{U}$-small) fs log schemes over $S^{\log}$,
- $\mathtt{Q}$ a property of morphisms of ($\mathbf{U}$-small) fs log schemes over $S^{\log}$.

Then we shall say that

> **the property that an object $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ satisfies $\mathtt{P}$ may be characterized category-theoretically** from the data $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, X^{\log})$

if for any object $Y^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, any ($\mathbf{U}$-small) locally Noetherian fs log scheme $T^{\log}$, any subset $\lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$, and any equivalence $F : \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$, it holds that

$$Y^{\log} \text{ satisfies } \mathtt{P} \iff F(Y^{\log}) \text{ satisfies } \mathtt{P}.$$

We shall say that

> **the property that a morphism $f^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ satisfies $\mathtt{Q}$ may be characterized category-theoretically** from the data $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, f^{\log})$

if for any morphism $g^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, any ($\mathbf{U}$-small) locally Noetherian fs log scheme $T^{\log}$, any subset $\lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$, and any equivalence $F : \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$, it holds that

$$g^{\log} \text{ satisfies } \mathtt{Q} \iff F(g^{\log}) \text{ satisfies } \mathtt{Q}.$$

1. Quasi-integral Monoids

In this section, we discuss various generalities concerning monoids that will play an important role in the theory of the present paper. Our main result (Corollary 1.8) concerns the construction of an extension of sharp fs monoids that satisfies a certain non-quasi-integrality property.

Definition 1.1. Let M be a (commutative) monoid.

(i) We shall write Mon for the category of ($\mathbf{U}$-small) monoids and Ab for the category of ($\mathbf{U}$-small) abelian groups.

(ii) We shall write M^{gp} for the groupification of M.

(iii) We shall write $M^{\times}$ for the unit group of M.

(iv) We shall write M^{int} for the image of the natural morphism $M \to M^{\mathrm{gp}}$.

(v) We shall write

$$M^{\mathrm{sat}} \overset{\mathrm{def}}{:=} \left\{ m \in M^{\mathrm{gp}} \;\middle|\; \exists a \in \mathbb{N} \setminus \{0\}, am \in M^{\mathrm{int}} \right\}.$$

(vi) We shall say that M is **sharp** if $M^{\times} = 0$.

(vii) We shall say that M is **integral** if the natural morphism $M \to M^{\mathrm{gp}}$ is injective. If M is an integral monoid, then we regard M as a submonoid of M^{gp} via the natural injection $M \hookrightarrow M^{\mathrm{gp}}$. We shall write $\mathsf{Mon}^{\mathrm{int}} \subset \mathsf{Mon}$ for the full subcategory of Mon consisting of integral monoids.

(viii) We shall say that M is **saturated** if M is integral, and, moreover, for any element $m \in M^{\mathrm{gp}}$, if there exists a positive integer $a \in \mathbb{N} \setminus \{0\}$ such that $am \in M$, then $m \in M$. We shall write $\mathsf{Mon}^{\mathrm{sat}} \subset \mathsf{Mon}$ for the full subcategory of Mon consisting of saturated monoids.

(ix) We shall say that M is **finitely generated** if there exist an integer $a \in \mathbb{N}$ and a surjection $\mathbb{N}^{\oplus a} \to M$ of monoids.

(x) Let $S \subset M$ be a subset, $k \in \mathbb{N} \setminus \{0\}$ an integer, and $m_1, ..., m_k \in M$ elements. We shall write $\langle S \rangle \subset M$ for the submonoid of M generated by S, i.e., the smallest submonoid of M that contains S. We shall write $\langle S, m_1, \cdots, m_k \rangle \overset{\mathrm{def}}{:=} \langle S \cup \{m_1, \cdots, m_k\} \rangle$.

(xi) We shall say that M is **fine** if M is integral and finitely generated.

(xii) We shall say that M is **fs** if M is saturated and finitely generated.

REMARK 1.2.

(i) The functor $(-)^{\mathrm{int}} : \mathsf{Mon} \to \mathsf{Mon}^{\mathrm{int}}$ is a left adjoint functor to the natural inclusion $\mathsf{Mon}^{\mathrm{int}} \subset \mathsf{Mon}$. In particular, the limit of any diagram of integral monoids in Mon is integral.

(ii) The functor $(-)^{\mathrm{sat}} : \mathsf{Mon} \to \mathsf{Mon}^{\mathrm{sat}}$ is a left adjoint functor to the natural inclusion $\mathsf{Mon}^{\mathrm{sat}} \subset \mathsf{Mon}$. In particular, the limit of any diagram of saturated monoids in Mon is saturated.

(iii) Let M be a fine monoid and K a field. Then observe that the natural inclusion of K-subalgebras $K[M] \hookrightarrow K[M^{\mathrm{sat}}]$ of $K[M^{\mathrm{gp}}]$ is integral and induces an isomorphism on quotient fields, hence, by a well-known result in commutative algebra, is *finite*. This *finiteness* of $K[M^{\mathrm{sat}}]$ as a $K[M]$-module implies that M^{sat} is a *finitely generated monoid*.

DEFINITION 1.3. Let $f : M \to N$ be a morphism of monoids.

(i) We shall write $f^{\mathrm{gp}} : M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ for the morphism induced on groupifications by f.

(ii) We shall say that f is **local** if $f^{-1}(N^\times) = M^\times$.

LEMMA 1.4. *Let M be a sharp saturated monoid and $n \in M^{\mathrm{gp}} \setminus M$. Then $\langle M, -n\rangle^{\mathrm{sat}}$ is sharp.*

PROOF. Write $N \overset{\mathrm{def}}{:=} \langle M, -n\rangle^{\mathrm{sat}}$. Let $\tilde{n} \in N^\times$ be an element. Then $\tilde{n}, -\tilde{n} \in N^\times \subset N = \langle M, -n\rangle^{\mathrm{sat}}$. Hence, by Definition 1.1 (v), there exist **positive** integers $a_1, a_2 \geq 1$ such that $a_1\tilde{n}, -a_2\tilde{n} \in \langle M, -n\rangle$. Moreover, by Definition 1.1 (x), there exist non-negative integers $b_1, b_2 \in \mathbb{N}$, and elements $m_1, m_2 \in M$ such that

$$a_1\tilde{n} = m_1 - b_1 n, \quad \text{and} \quad -a_2\tilde{n} = m_2 - b_2 n.$$

Thus it holds that $a_2(m_1 - b_1 n) + a_1(m_2 - b_2 n) = a_2 a_1 \tilde{n} - a_1 a_2 \tilde{n} = 0$, hence

$$(a_2 b_1 + a_1 b_2) n = a_2 m_1 + a_1 m_2 \in M.$$

Since $n \in M^{\mathrm{gp}} \setminus M$, and M is saturated, it holds that $a_2b_1 + a_1b_2 = 0$. Since $a_1, a_2 \geq 1$, and $b_1, b_2 \in \mathbb{N}$, it holds that $b_1 = b_2 = 0$. Thus $a_1\tilde{n} = m_1 \in M$, and $-a_2\tilde{n} = m_2 \in M$. Since $a_1, a_2 \geq 1$, and M is saturated, it holds that $\tilde{n}, -\tilde{n} \in M$. Since M is sharp, $\tilde{n} = 0$. This completes the proof of Lemma 1.4. $\square$

Corollary 1.5. *Let M be a monoid, N a sharp fs monoid, and $i_1, i_2 : N \to M$ morphisms of monoids. Assume that neither i_1 nor i_2 is injective. Then there exist elements $n_1, n_2 \in N^{\mathrm{gp}} \setminus N$ such that $i_1^{\mathrm{gp}}(n_1) = i_2^{\mathrm{gp}}(n_2) = 0$, and $\langle N, -n_1, -n_2\rangle^{\mathrm{sat}}$ is a sharp fs monoid.*

Proof. Since i_1 is not injective, it holds that $\ker(i_1^{\mathrm{gp}}) \neq 0$. Since N is sharp, it holds that $\ker(i_1^{\mathrm{gp}}) \not\subset N$. Let $n_1 \in \ker(i_1^{\mathrm{gp}}) \setminus N$ be an element. Then, by Remark 1.2 (iii) and Lemma 1.4, $L_1 \stackrel{\mathrm{def}}{:=} \langle N, -n_1\rangle^{\mathrm{sat}}$ is a sharp fs monoid. Since $n_1 \in \ker(i_1^{\mathrm{gp}})$, $i_1^{\mathrm{gp}}(n_1) = 0$.

Since i_2 is not injective, $\ker(i_2^{\mathrm{gp}}) \neq 0$. Since L_1 is sharp, $\ker(i_2^{\mathrm{gp}}) \not\subset L_1$. Let $n_2 \in \ker(i_2^{\mathrm{gp}}) \setminus L_1$ be an element. Then, by Remark 1.2 (iii) and Lemma 1.4, $\langle L_1, -n_2\rangle^{\mathrm{sat}} = \langle N, -n_1, -n_2\rangle^{\mathrm{sat}}$ is a sharp fs monoid. Since $n_2 \in \ker(i_2^{\mathrm{gp}})$, $i_2^{\mathrm{gp}}(n_2) = 0$. This completes the proof of Corollary 1.5. $\square$

Definition 1.6. We shall say that a monoid M is **quasi-integral** if for any $m, n \in M$, the equality $m + n = m$ implies that $n = 0$, or, equivalently, any element of M that becomes trivial in M^{gp} is trivial in M.

In Appendix A, we prove an extension of the following lemma to the case where M, L are quasi-integral, and N is an arbitrary monoid (cf. Corollary A.5).

Lemma 1.7 (cf. [Nak, Lemma 2.2.6 (i)], Corollary A.5). *Let L, M, N be sharp fs monoids and $f : N \to M, g : N \to L$ local morphisms of monoids. Write $P \stackrel{\mathrm{def}}{:=} M \sqcup_N L$. Then the following assertions are equivalent:*

(i) P is quasi-integral.

(ii) For any element $n \in N^{\mathrm{gp}}$, if $f^{\mathrm{gp}}(n) \in M$, and $-g^{\mathrm{gp}}(n) \in L$, then $f^{\mathrm{gp}}(n) = 0$, and $g^{\mathrm{gp}}(n) = 0$.

Proof. Lemma 1.7 follows immediately from [Nak, Lemma 2.2.6 (i)]. $\square$

Corollary 1.8. *Let M, N be sharp fs monoids and $i_1, i_2 : N \to M$ local morphisms of monoids. Assume that neither i_1 nor i_2 is injective. Then there exists a sharp fs monoid L such that $N \subset L \subset N^{\mathrm{gp}}$, and neither $M \sqcup_{i_1,N} L$ nor $M \sqcup_{i_2,N} L$ is quasi-integral.*

Proof. Since neither i_1 nor i_2 is injective, it follows from Corollary 1.5 that there exist elements $n_1, n_2 \in N^{\mathrm{gp}} \setminus N$ such that $\langle N, -n_1, -n_2 \rangle^{\mathrm{sat}}$ is a sharp fs monoid, and $i_1^{\mathrm{gp}}(n_1) = i_2^{\mathrm{gp}}(n_2) = 0$. Write

$$L \stackrel{\mathrm{def}}{:=} \langle N, -n_1, -n_2 \rangle^{\mathrm{sat}}.$$

Then, for each $k \in \{1, 2\}$, since M, N, L are sharp fs monoids, $i_k : N \to M$ is local, the inclusion morphism $N \hookrightarrow L$ is local, $i_k^{\mathrm{gp}}(n_k) = 0 \in M$, $-n_k \in L$, and $n_k \neq 0$, it follows from Lemma 1.7 that $M \sqcup_{i_k,N} L$ is not quasi-integral. This completes the proof of Corollary 1.8. $\square$

2. Some Remarks on Log Schemes

In this section, we introduce some notions related to log schemes and give proofs of several elementary results on log schemes. In particular, we prove that certain push-outs exist in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ (cf. Corollary 2.11).

Definition 2.1. Let $X^{\log}, Y^{\log}$ be log schemes and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes (cf. [KK, Section 1]).

(i) We shall write X for the underlying scheme of the log scheme $X^{\log}$. We shall write f for the underlying morphism of schemes $f : X \to Y$ of the morphism of log schemes $f^{\log}$.

(ii) We shall write $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ for the morphism of sheaves of monoids on the étale site of X which defines the log structure of $X^{\log}$. We shall write $\overline{\mathcal{M}}_X \stackrel{\mathrm{def}}{:=} \mathcal{M}_X/(\alpha_X^{-1}\mathcal{O}_X^{\times})$.

(iii) We shall write $f^{\flat} : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ for the morphism of sheaves of monoids on the étale site of X which defines the morphism of log schemes $f^{\log} : X^{\log} \to Y^{\log}$. We shall write $\bar{f}^{\flat} : f^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ for the morphism of sheaves of monoids on the étale site of X induced by $f^{\flat} : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$.

(iv) We shall write $f^*\alpha_Y : f^*\mathcal{M}_Y \to \mathcal{O}_X$ for the log structure on X determined by the pull-back of the log structure $\alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y$ on Y via f.

(v) Let $|Z| \subset |X|$ be a closed subspace. Then we shall write $Z^{\log}_{\mathrm{red}}$ for the log scheme whose underlying scheme is Z_{red}, and whose log structure is induced from $X^{\log}$ via the natural closed immersion $Z_{\mathrm{red}} \hookrightarrow X$. We shall write $(X^{\log})_{\mathrm{red}} \stackrel{\mathrm{def}}{:=} X^{\log}_{\mathrm{red}}$.

Definition 2.2. Let $X^{\log}, Y^{\log}$ be log schemes, $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes, $\mathtt{P}$ a property of schemes [such as "quasi-compact", "separated", "quasi-separated", "reduced", "connected"], and $\mathtt{Q}$ a property of morphisms of schemes [such as "quasi-compact", "separated", "quasi-separated", "of finite type", "étale"].

(i) We shall say that $X^{\log}$ satisfies $\mathtt{P}$ if the underlying scheme X satisfies $\mathtt{P}$.

(ii) We shall say that $f^{\log}$ satisfies $\mathtt{Q}$ if the underlying morphism of schemes f satisfies $\mathtt{Q}$.

(iii) We shall say that $f^{\log}$ is **strict** (or, in the terminology of [Mzk04], **scheme-like**), if the morphism of sheaves of monoids $f^*\mathcal{M}_Y \to \mathcal{M}_X$ on the étale site of X induced by $f^{\flat}$ is an isomorphism.

(iv) We shall say that $f^{\log}$ is a **strict closed immersion** if $f^{\log}$ is strict and a closed immersion.

(v) We shall say that $f^{\log}$ is a **strict open immersion** if $f^{\log}$ is strict and an open immersion.

(vi) Let $\bar{x} \to X$ be a geometric point. We shall say that a morphism of log schemes $i^{\log} : U^{\log} \to X^{\log}$ is a **strict étale neighborhood** of $\bar{x}$ if $i : U \to X$ is an étale neighborhood of $\bar{x}$, and $i^{\log}$ is strict.

Definition 2.3. Let $X^{\log}$ be a log scheme.

(i) We shall say that the log structure of $X^{\log}$ is **integral** if for any geometric point $\bar{x} \to X$, the stalk $\mathcal{M}_{X,\bar{x}}$ of $\mathcal{M}_X$ at $\bar{x} \to X$ is an integral monoid.

(ii) We shall say that the log structure of $X^{\log}$ is **saturated** if for any geometric point $\bar{x} \to X$, the stalk $\mathcal{M}_{X,\bar{x}}$ of $\mathcal{M}_X$ at $\bar{x} \to X$ is a saturated monoid.

(iii) We shall say that $X^{\log}$ **has a global chart** if there exist a monoid M and a morphism of monoids $M \to \Gamma(X, \mathcal{O}_X)$ such that the log structure of $X^{\log}$ is isomorphic to the log structure associated to the adjoint morphism of sheaves of monoids $\tilde{M} \to \mathcal{O}_X$ (cf. [KK, (1.1), (1.3)]), where $\tilde{M}$ is the constant sheaf on the étale site of X associated to M. In this situation, we shall refer to the morphism of monoids $M \to \Gamma(X, \mathcal{O}_X)$, or, equivalently, the morphism of sheaves of monoids $\tilde{M} \to \mathcal{O}_X$ on X, as a **global chart** of $X^{\log}$.

(iv) Let $\bar{x} \to X$ be a geometric point. Then we shall say that $X^{\log}$ has a **chart** at $\bar{x}$ if there exists a strict étale neighborhood (cf. Definition 2.2 (vi)) $U^{\log} \to X^{\log}$ of $\bar{x}$ such that the log scheme $U^{\log}$ has a global chart $M \to \Gamma(U, \mathcal{O}_U)$. In this situation, we shall say that $(U^{\log} \to X^{\log}, M \to \Gamma(U, \mathcal{O}_U))$ is a **chart** at $\bar{x}$.

(v) Assume that the log structure of $X^{\log}$ is integral. Then we shall say that $X^{\log}$ is a **fine** log scheme if for any geometric point $\bar{x} \to X$, $X^{\log}$ has a chart $(U^{\log} \to X^{\log}, M \to \Gamma(U, \mathcal{O}_U))$ at $\bar{x}$ such that M is finitely generated.

(vi) Assume that the log structure of $X^{\log}$ is saturated. Then we shall say that $X^{\log}$ is an **fs** log scheme if for any geometric point $\bar{x} \to X$, $X^{\log}$ has a chart $(U^{\log} \to X^{\log}, M \to \Gamma(U, \mathcal{O}_U))$ at $\bar{x}$ such that M is finitely generated.

Remark 2.4. If $f^{\log} : Y^{\log} \to X^{\log}$ is a strict morphism of log schemes and $g^{\log} : Z^{\log} \to X^{\log}$ is a morphism of log schemes, then one verifies immediately that the natural morphism $Y^{\log} \times_{X^{\log}} Z^{\log} \to Z^{\log}$ is strict.

Remark 2.5. Assume that the log structure of a log scheme $X^{\log}$ is integral. Then one verifies immediately that the log structure of $X^{\log}$ is saturated if and only if for any geometric point $\bar{x} \to X$, $\overline{\mathcal{M}}_{X,\bar{x}}$ is a saturated monoid.

Definition 2.6. Let $X^{\log}$ be an fs log scheme.

(i) We shall say that $X^{\log}$ is an **fs log point** if the underlying scheme X of $X^{\log}$ is isomorphic to the spectrum of a field.

(ii) Assume that $X^{\log}$ is an fs log point. We shall say that $X^{\log}$ is a **split log point** if $\overline{\mathcal{M}}_X$ is a constant sheaf on the étale site of X.

Definition 2.7. Let $f^{\log} : Y^{\log} \to X^{\log}$ be a morphism of fs log schemes. Assume that $Y^{\log}$ is an fs log point. Write $y \in Y$ for the unique point of Y. Then we shall say that $f^{\log}$ is a **log residue field** of $X^{\log}$ if $f^{\log}$ is strict, and $f : Y \to X$ is isomorphic as an X-scheme to the natural morphism $\mathrm{Spec}(k(f(y))) \to X$ that arises from the spectrum of the residue field at $f(y) \in X$.

Lemma 2.8. *Let $X^{\log}$ be an fs log point and $\bar{x} \to X$ a geometric point. Write $M \stackrel{\mathrm{def}}{:=} \overline{\mathcal{M}}_{X,\bar{x}}$. Let $\varphi : M \to N$ be a local morphism between sharp fs monoids. Then there exists a morphism between fs log points $f^{\log} : Z^{\log} \to X^{\log}$ such that the following conditions hold:*

(i) $f^{\#} : \Gamma(X, \mathcal{O}_X) \to \Gamma(Z, \mathcal{O}_Z)$ is a finite separable field extension in $k(\bar{x})$.

(ii) $Z^{\log}$ is a split log point.

(iii) For any geometric point $\bar{z} \to Z$, there exists an isomorphism $\psi : N \xrightarrow{\sim} \overline{\mathcal{M}}_{Z,\bar{z}}$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \| & & \wr\downarrow\psi \\ M & \xrightarrow{\bar{f}^{\flat}_{\bar{z}}} & \overline{\mathcal{M}}_{Z,\bar{z}}. \end{array}$$

Proof. Write $k \stackrel{\mathrm{def}}{:=} \Gamma(X, \mathcal{O}_X)$, $k_s \subset k(\bar{x})$ for the subfield consisting of the elements which are separable algebraic over k, G for the automorphism group of the k-algebra k_s, and

$$H \stackrel{\text{def}}{:=} \{\sigma \in G \mid \sigma \text{ acts trivially on } M\}.$$

Then, since M is finitely generated, the subgroup $H \subset G$ is of finite index. Write $K \subset k_s$ for the fixed field of H and $f : Z \to X$ for the morphism of schemes determined by the field extension K/k. Then, since $H \subset G$ is of finite index, $f^{\#} : \Gamma(X, \mathcal{O}_X) \to \Gamma(Z, \mathcal{O}_Z)$ satisfies condition (i).

Write

- $\varphi_Z : M_Z \to N_Z$ for the morphism between constant étale sheaves of monoids on the étale site of Z determined by the morphism of monoids $\varphi : M \to N$;

- $\Gamma(Z, f^{-1}\overline{\mathcal{M}}_X)_Z$ for the constant étale sheaf of monoids on the étale site of Z determined by the monoid $\Gamma(Z, f^{-1}\overline{\mathcal{M}}_X)$ (which is naturally isomorphic to M);

- $Z^{\log}$ for the split log point such that $Z = \mathrm{Spec}(K)$, and the log structure of $Z^{\log}$ is the morphism of sheaves of monoids

$$\alpha_Z : \mathcal{M}_Z \stackrel{\text{def}}{:=} \mathcal{O}_Z^{\times} \times N_Z \to \mathcal{O}_Z$$

  on the étale site of Z determined by the natural inclusion $\mathcal{O}_Z^{\times} \hookrightarrow \mathcal{O}_Z$ and the unique local morphism of monoids $N \to K$, where we regard K as a commutative monoid by the multiplication operation, and we recall that N is assumed to be sharp.

Then, since the log structure of $Z^{\log}$ arises from the unique local morphism of monoids $N \to K$, $Z^{\log}$ satisfies condition (ii).

Since H acts trivially on M, the natural morphisms $f^{-1}\overline{\mathcal{M}}_X \xleftarrow{\sim} \Gamma(Z, f^{-1}\overline{\mathcal{M}}_X)_Z \xrightarrow{\sim} M_Z$ of sheaves of monoids on the étale site of Z are isomorphisms. Hence there exists an isomorphism $f^{-1}\mathcal{O}_X^{\times} \times M_Z \xrightarrow{\sim} f^{-1}\mathcal{M}_X$. Since $\varphi : M \to N$ is local, the morphism of sheaves of monoids $f^{\flat} : f^{-1}\mathcal{M}_X \to \mathcal{M}_Z$ determined by $f^{-1}\mathcal{O}_X^{\times} \times M_Z \xrightarrow{\sim} f^{-1}\mathcal{M}_X$ and $f^{\#,\times} \times \varphi_Z : f^{-1}\mathcal{O}_X^{\times} \times M_Z \to \mathcal{O}_Z^{\times} \times N_Z$ satisfies $\alpha_Z \circ f^{\flat} = f^{\#} \circ f^{-1}(\alpha_X)$. Thus the pair $(f, f^{\flat})$ is a morphism of log schemes $f^{\log} : Z^{\log} \to X^{\log}$. Then $f^{\log}$ satisfies condition (iii). This completes the proof of Lemma 2.8. $\square$

Next, we discuss some basic properties of push-outs in the category of fs log schemes.

Lemma 2.9. *Let $i : Y \hookrightarrow X$ be a closed immersion of schemes, $\bar{y} \to Y$ a geometric point of Y, and $g : V \to Y$ an étale neighborhood of $\bar{y} \to Y$. Then there exist an étale neighborhood $f : U \to X$ of the geometric point $\bar{y} \to Y \to X$ of X and a morphism $h : U \times_X Y \to V$ of Y-schemes such that the following diagram commutes and indeed is cartesian:*

$$\begin{tikzcd} U \times_X Y \arrow[r, "h"] \arrow[d, hook] & V \arrow[r, "g"] & Y \arrow[d, hook, "i"] \\ U \arrow[rr, "f"] & & X. \end{tikzcd}$$

Proof. Write $\bar{x} \to X$ for the composite $\bar{y} \to Y \to X$ and $\mathcal{I} \stackrel{\mathrm{def}}{:=} \ker(i^{\#} : \mathcal{O}_X \to i_* \mathcal{O}_Y)$. Then, by [Stacks, Tag 05WS], the morphism $\mathcal{O}_{X,\bar{x}}/\mathcal{I}_{\bar{x}} \xrightarrow{\sim} \mathcal{O}_{Y,\bar{y}}$ induced by $i^{\#}$ is an isomorphism. Thus Lemma 2.9 follows immediately from [Stacks, Tag 04GW]. $\square$

Lemma 2.10. *Let $X^{\log} \xleftarrow{s^{\log}} Z^{\log} \xrightarrow{t^{\log}} Y^{\log}$ be morphisms of fs log schemes. Assume that the following conditions hold:*

- *s and t are closed immersions.*
- *Either $s^{\log}$ or $t^{\log}$ is strict.*

Write $W \stackrel{\mathrm{def}}{:=} X \sqcup_Z Y$ for the push-out of schemes (where we observe that the existence of W follows from [Stacks, Tag 0E25]*), $p : X \to W$ and $q : Y \to W$ for the natural morphisms, $r \stackrel{\mathrm{def}}{:=} p \circ s = q \circ t$, $\mathcal{M}_W \stackrel{\mathrm{def}}{:=} p_* \mathcal{M}_X \times_{r_* \mathcal{M}_Z} q_* \mathcal{M}_Y$ (an étale sheaf of monoids on W), and $\alpha_W : \mathcal{M}_W \to p_* \mathcal{O}_X \times_{r_* \mathcal{O}_Z} q_* \mathcal{O}_Y \cong \mathcal{O}_W$ (cf.* [Stacks, Tag 0E25]*):*

$$\begin{tikzcd} Z \arrow[r, hook, "s"] \arrow[d, hook, "t"] \arrow[dr, "r"] & X \arrow[d, hook, "p"] \\ Y \arrow[r, hook, "q"] & W. \end{tikzcd}$$

Then the following assertions hold:

(i) The triple $W^{\log} \overset{\mathrm{def}}{:=} (W, \mathcal{M}_W, \alpha_W)$ *is an fs log scheme.*

(ii) $W^{\log}$ *represents the push-out of the diagram* $X^{\log} \overset{s^{\log}}{\hookleftarrow} Z^{\log} \overset{t^{\log}}{\hookrightarrow} Y^{\log}$ *in the category of fs log schemes.*

Proof. It follows immediately from the definition of $W^{\log}$ that assertion (ii) follows from assertion (i). In the remainder of the proof of Lemma 2.10, we prove assertion (i). Since the functor $(-)^{\times} : \mathsf{Mon} \to \mathsf{Ab}$ preserves limits, it follows immediately that $W^{\log}$ is a log scheme. Moreover, it follows immediately from Remark 1.2 (ii) and the definition of the notion of a saturated log structure that the log structure of $W^{\log}$ is saturated. Hence, to prove Lemma 2.10, it suffices to prove that for any geometric point $\bar{w} \to W$, $W^{\log}$ has a chart $(U_0^{\log} \to W^{\log}, M_0 \to \Gamma(U_0, \mathcal{O}_{U_0}))$ at $\bar{w}$ such that M_0 is an fs monoid. Let $\bar{w} \to W$ be a geometric point.

Write $U^{\log}$ for the fs log scheme determined by the open subset $|X| \setminus \mathrm{Im}(s)$, $V^{\log}$ for the fs log scheme determined by the open subset $|Y| \setminus \mathrm{Im}(t)$, and $f^{\log} : (U \sqcup V)^{\log} \to W^{\log}$ for the natural morphism of log schemes determined by $p^{\log}$ and $q^{\log}$. Then $f^{\log}$ is a strict open immersion, and $\mathrm{Im}(f) = (\mathrm{Im}(p) \setminus \mathrm{Im}(q)) \cup (\mathrm{Im}(q) \setminus \mathrm{Im}(p))$. Hence if $\mathrm{Im}(\bar{w} \to W) \subset (\mathrm{Im}(p) \setminus \mathrm{Im}(q)) \cup (\mathrm{Im}(q) \setminus \mathrm{Im}(p))$, then $W^{\log}$ has a chart at $\bar{w}$ of the desired type.

Assume that $\mathrm{Im}(\bar{w} \to W) \subset \mathrm{Im}(r)$. Then the geometric point $\bar{w} \to W$ arises from a geometric point $\bar{w} \to Z$. Since $s^{\log}$ and $t^{\log}$ are morphisms of fs log schemes, it follows from Lemma 2.9 and [KK, Definition 2.9 (2) and Lemma 2.10] that there exist a chart $(U_X^{\log} \to X^{\log}, M_X \to \Gamma(U_X, \mathcal{O}_{U_X}))$ at $\bar{w} \to X$, a chart $(U_Y^{\log} \to Y^{\log}, M_Y \to \Gamma(U_Y, \mathcal{O}_{U_Y}))$ at $\bar{w} \to Y$, a chart $(U_Z^{\log} \to Z^{\log}, M_Z \to \Gamma(U_Z, \mathcal{O}_{U_Z}))$ at $\bar{w} \to Z$, a morphism of monoids $\tilde{s}^{\flat} : M_X \to M_Z$, and a morphism of monoids $\tilde{t}^{\flat} : M_Y \to M_Z$ such that M_X, M_Y, M_Z are fs monoids, and, moreover, for each $u \in \{s, t\}$, if $u^{\log}$ is strict, then $\tilde{u}^{\flat}$ is an isomorphism. By Lemma 2.9, there exists an étale neighborhood $U_W \to W$ of $\bar{w} \to W$ such that for each $(*, ?) \in \{(X, p), (Y, q), (Z, r)\}$, there exists a morphism of $*$-schemes $U_W \times_W * \to U_*$ such that the following diagram commutes and indeed is cartesian:

$$\begin{tikzcd}
U_W \times_W * \arrow[r] \arrow[d, hook] & U_* \arrow[r, "\text{ét}"] & * \arrow[d, hook, "?"] \\
U_W \arrow[rr, "\text{ét}"] & & W.
\end{tikzcd}$$

Write $M_W \stackrel{\mathrm{def}}{:=} M_X \times_{M_Z} M_Y$. Since either $\tilde{s}^\flat : M_X \to M_Z$ or $\tilde{t}^\flat : M_Y \to M_Z$ is an isomorphism, M_W is an fs monoid. Thus $U_W \to W$ and the natural morphism $M_W \to \Gamma(U_W, \mathcal{O}_{U_W})$ determine a chart of $W^{\log}$ at $\bar{w}$ of the desired type. This completes the proof of Lemma 2.10. □

Corollary 2.11. *Let $S^{\log}$ be an fs log scheme,*

$$\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$$

a subset, and $X^{\log} \stackrel{s^{\log}}{\hookleftarrow} Z^{\log} \stackrel{t^{\log}}{\hookrightarrow} Y^{\log}$ strict closed immersions in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Write $W^{\log} \stackrel{\mathrm{def}}{:=} X^{\log} \sqcup_{Z^{\log}} Y^{\log}$ (cf. Lemma 2.10). Then the following assertions hold.

(i) If $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$, then $W^{\log}$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$, i.e., the push-out of $s^{\log}, t^{\log}$ exists in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

(ii) If $S^{\log}$ is locally Noetherian, then $W^{\log}$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$, i.e., the push-out of $s^{\log}, t^{\log}$ exists in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

Proof. First, we note that by Lemma 2.10 (i) (ii), it holds that $W \cong X \sqcup_Z Y$. Hence assertion (i) follows immediately from [Stacks, Tag 0E26], [Stacks, Tag 04ZD], and the fact that the fiber product of reduced rings is reduced. Assertion (ii) follows immediately from assertion (i) and [Stacks, Tag 0E27]. This completes the proof of Corollary 2.11. □

3. Fs Log Points

In this section, we assume that

$$\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}.$$

In the present Section 3, we give a category-theoretic characterization of the objects of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ whose underlying log scheme is an fs log point for locally Notherian fs log schemes $S^{\log}$ (cf. Proposition 3.3).

First, we note the following lemma:

Lemma 3.1. *Let $S^{\log}$ be an fs log scheme and $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ an object. Then the following assertions hold:*

(i) $X \neq \varnothing$ *if and only if* $X^{\log}$ *is not an initial object of* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. *In particular, the property that* $X \neq \varnothing$ *may be characterized category-theoretically from the data* $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, X^{\log})$.

(ii) $|X|$ *is connected if and only if* $X \neq \varnothing$, *and, moreover,* $X^{\log}$ *does not admit a representation as a coproduct of two non-initial objects of* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. *In particular, the property that* $|X|$ *is connected may be characterized category-theoretically from the data* $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, X^{\log})$.

Proof. Assertions (i) and (ii) follow immediately from elementary log scheme theory. □

Next, we give a category-theoretic characterization of fs log points (cf. Proposition 3.3).

Lemma 3.2. *Let* $S^{\log}$ *be a locally Noetherian fs log scheme;* $i_1^{\log}, i_2^{\log} : X^{\log} \to Y^{\log}$ *morphisms in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$; $p^{\log} : Y^{\log} \to X^{\log}$ *a morphism in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. *Assume that the following conditions hold:*

(i) $X^{\log}$ *is an fs log point.*

(ii) $p^{\log} \circ i_1^{\log} = p^{\log} \circ i_2^{\log} = \mathrm{id}_{X^{\log}}$.

(iii) The morphism $i^{\log} : X^{\log} \sqcup X^{\log} \to Y^{\log}$ *determined by* $i_1^{\log}$ *and* $i_2^{\log}$ *is an epimorphism in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

(iv) $|Y|$ *is connected.*

Then i_1, i_2, p *are isomorphisms, and* $i_1 = i_2 = p^{-1}$.

Proof. Write $Z^{\log} \stackrel{\mathrm{def}}{:=} \overline{\mathrm{Im}(i)}^{\log}_{\mathrm{red}}$ (cf. Definition 2.1 (v)), where we note that $Z^{\log}$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$. By condition (i), $X \sqcup X$ is reduced. Hence, by [Stacks, Tag 056B], the scheme-theoretic image of i is equal to Z (cf. [Stacks, Tag 01R7], [Stacks, Tag 01R6]). Hence the morphism $i^{\log} : X^{\log} \sqcup X^{\log} \to Y^{\log}$ factors uniquely through the strict closed immersion $Z^{\log} \hookrightarrow Y^{\log}$. By condition (iii), $Z^{\log} \hookrightarrow Y^{\log}$ is an epimorphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Hence the two natural inclusions $Y^{\log} \hookrightarrow$

$Y^{\log} \sqcup_{Z^{\log}} Y^{\log}$ coincide (where we note that by Corollary 2.11 (ii), the push-out $Y^{\log} \sqcup_{Z^{\log}} Y^{\log}$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$). Thus, by the construction of $Y^{\log} \hookrightarrow Y^{\log} \sqcup_{Z^{\log}} Y^{\log}$ (cf. Lemma 2.10 (i) and the proof of [Stacks, Tag 0E25]) and Corollary 2.11 (ii), the strict closed immersion $Z^{\log} \hookrightarrow Y^{\log}$ is an isomorphism. In particular, it holds that $Y^{\log} = \overline{\mathrm{Im}(i)}^{\log}_{\mathrm{red}}$.

Write $x \in X$ for the unique point (cf. condition (i)). By condition (ii), the morphisms of fields $k(x) \to k(i_1(x))$ and $k(x) \to k(i_2(x))$ induced by $p : Y \to X$ are isomorphisms. Hence, by [Stacks, Tag 01TE], $i_1(x)$ and $i_2(x)$ are closed points of Y. Thus it holds that

$$|Y| = \overline{\mathrm{Im}(i)} = \overline{\{i_1(x), i_2(x)\}} = \overline{\{i_1(x)\}} \cup \overline{\{i_2(x)\}} = \{i_1(x), i_2(x)\}.$$

By condition (iv), $|Y|$ is of cardinality 1. Moreover, since $Y = \overline{\mathrm{Im}(i)}_{\mathrm{red}}$ is reduced, Y is isomorphic to the spectrum of a field. Thus, since $p \circ i_1 = p \circ i_2 = \mathrm{id}_X$, the morphisms i_1, i_2, p are isomorphisms, and $i_1 = i_2 = p^{-1}$. This completes the proof of Lemma 3.2. $\square$

PROPOSITION 3.3. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $X^{\log}$ an object of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Assume that $|X|$ is connected. Then $X^{\log}$ is* ***not*** *an fs log point if and only if there exist an object $Y^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, morphisms $i_1^{\log}, i_2^{\log} : X^{\log} \to Y^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, and a morphism $p^{\log} : Y^{\log} \to X^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that the following conditions hold:*

(i) $p^{\log} \circ i_1^{\log} = p^{\log} \circ i_2^{\log} = \mathrm{id}_{X^{\log}}$.

(ii) The morphism $i^{\log} : X^{\log} \sqcup X^{\log} \to Y^{\log}$ determined by $i_1^{\log}$ and $i_2^{\log}$ is an epimorphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

(iii) $|Y|$ is connected.

(iv) Neither $i_1^{\log}$ nor $i_2^{\log}$ is an isomorphism.

(v) For any morphism $f^{\log} : Z^{\log} \to Y^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $Z \neq \varnothing$, there exists a commutative diagram

$$\begin{array}{ccc} W^{\log} & \longrightarrow & Z^{\log} \\ \big\downarrow & & \big\downarrow{\scriptstyle f^{\log}} \\ X^{\log} \sqcup X^{\log} & \xrightarrow{i^{\log}} & Y^{\log} \end{array}$$

in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *such that* $W \neq \varnothing$.

In particular, the property that $X^{\log}$ *is an fs log point may be characterized category-theoretically (cf. Lemma 3.1 (i) (ii)) from the data* $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, X^{\log})$.

Proof. First, we prove necessity. Assume that $X^{\log}$ is not an fs log point. Then X is not isomorphic to the spectrum of a field. Hence there exists a closed immersion $j : X_1 \hookrightarrow X$ such that X_1 is reduced, $X_1 \neq \varnothing$, and j is not an isomorphism. The pull-back of the log structure of $X^{\log}$ to X_1 determines a log structure on X_1, together with a strict closed immersion $j^{\log} : X_1^{\log} \hookrightarrow X^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Write $Y^{\log} :\stackrel{\mathrm{def}}{=} X^{\log} \sqcup_{X_1^{\log}} X^{\log}$ (cf. Corollary 2.11 (ii)); $i_1^{\log}, i_2^{\log} : X^{\log} \to Y^{\log}$ for the natural inclusions; $p^{\log} : Y^{\log} \to X^{\log}$ for the unique morphism such that $p^{\log} \circ i_1^{\log} = p^{\log} \circ i_2^{\log} = \mathrm{id}_{X^{\log}}$. Then, by Corollary 2.11 (ii), $Y^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, and $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ satisfies conditions (i) and (ii). Since $|X|$ is connected, $X_1 \neq \varnothing$, and $j : X_1 \hookrightarrow X$ is not an isomorphism, it follows from the construction of $X^{\log} \sqcup_{X_1^{\log}} X^{\log}$ (cf. Lemma 2.10 (i) and the proof of [Stacks, Tag 0E25]) that $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ satisfies conditions (iii) and (iv). Since $j^{\log} : X_1^{\log} \hookrightarrow X^{\log}$ is a strict closed immersion, it follows from the construction of the log structure of $X^{\log} \sqcup_{X_1^{\log}} X^{\log}$ (cf. Lemma 2.10 (i)) that the morphism $i^{\log} : X^{\log} \sqcup X^{\log} \to Y^{\log}$ determined by $i_1^{\log}$ and $i_2^{\log}$ is strict. Thus, since i is surjective, $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ satisfies condition (v). This completes the proof of necessity.

Next, we prove sufficiency. Assume that $X^{\log}$ is an fs log point. Let $Y^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ be an object; $i_1^{\log}, i_2^{\log} : X^{\log} \to Y^{\log}$ morphisms in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$; $p^{\log} : Y^{\log} \to X^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ satisfies conditions (i) (ii) (iii) (iv). To prove sufficiency, it suffices to prove that $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ does not satisfy condition (v). Since $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ satisfies conditions (i) (ii) (iii), it follows from Lemma 3.2 that i_1, i_2, p are isomorphisms, and $i_1 = i_2 = p^{-1}$. Hence, to prove that $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ does not satisfy condition (v), we may assume without loss of generality that $X = Y$, and $i_1 = i_2 = p = \mathrm{id}_X$.

Let $\bar{x} \to X$ be a geometric point. Write $M :\stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{X,\bar{x}}$, $N :\stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{Y,\bar{x}}$, $\bar{i}_1^{\flat} :\stackrel{\mathrm{def}}{=} \bar{i}_{1,\bar{x}}^{\flat}$, $\bar{i}_2^{\flat} :\stackrel{\mathrm{def}}{=} \bar{i}_{2,\bar{x}}^{\flat}$, and $\bar{p}^{\flat} :\stackrel{\mathrm{def}}{=} \bar{p}_{\bar{x}}^{\flat}$. Then M, N are sharp fs monoids, and $\bar{i}_1^{\flat}$,

$\bar{i}_2^\flat$, $\bar{p}^\flat$ are local morphisms of monoids. Moreover, since $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ satisfies condition (i), it holds that $\bar{i}_1^\flat \circ \bar{p}^\flat = \bar{i}_2^\flat \circ \bar{p}^\flat = \mathrm{id}_M$:

$$\begin{tikzcd} M \arrow[r, "\bar{p}^\flat"'] \arrow[rr, bend left, "\mathrm{id}_M"] & N \arrow[r, shift left] \arrow[r, shift right, "\bar{i}_1^\flat, \bar{i}_2^\flat"'] & M. \end{tikzcd}$$

Since $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ satisfies condition (iv), neither $\bar{i}_1^\flat$ nor $\bar{i}_2^\flat$ is injective. Hence, by Corollary 1.8, there exists a sharp fs monoid $N \subset L \subset N^{\mathrm{gp}}$ such that neither $M \sqcup_{\bar{i}_1^\flat, N} L$ nor $M \sqcup_{\bar{i}_2^\flat, N} L$ is quasi-integral. Since $N \hookrightarrow L$ is local, it follows from Lemma 2.8 that there exists a morphism $f^{\log} : Z^{\log} \to Y^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $Z^{\log} \neq \varnothing$, and for any geometric point $\bar{z} \to Z$, $\bar{f}_{\bar{z}}^\flat : (f^{-1}\overline{\mathcal{M}}_Y)_{\bar{z}} \to \overline{\mathcal{M}}_{Z,\bar{z}}$ is isomorphic as an object of $\mathsf{Mon}_{N/}$ to $N \subset L$. Since neither $M \sqcup_{\bar{i}_1^\flat, N} L$ nor $M \sqcup_{\bar{i}_2^\flat, N} L$ is quasi-integral, it follows from [Nak, Lemma 2.2.5] that for each $k \in \{1, 2\}$, $X^{\log} \times_{i_k^{\log}, Y^{\log}, f^{\log}} Z^{\log} = \varnothing$. Thus $(Y^{\log}, i_1^{\log}, i_2^{\log}, p^{\log})$ does not satisfy condition (v). This completes the proof of Proposition 3.3. □

Next, we prove the following property concerning fiber products in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

Lemma 3.4. *Let $S^{\log}$ be a locally Noetherian fs log scheme, $f^{\log} : X^{\log} \to Y^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, and $g^{\log} : Y_0^{\log} \to Y^{\log}$ a quasi-compact morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Then the fiber product $X^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log}$ exists in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Moreover, the natural morphism*

$$(X^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log})_{\mathrm{red}} \xrightarrow{\sim} (X^{\log} \times_{Y^{\log}} Y_0^{\log})_{\mathrm{red}}$$

induced by the morphism of log schemes $X^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log} \to X^{\log} \times_{Y^{\log}} Y_0^{\log}$ is an isomorphism of log schemes. In particular, if $f^{\log}$ is strict, then the natural projection $X^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log} \to Y_0^{\log}$ is also strict.

Proof. Write $\lozenge \stackrel{\mathrm{def}}{:=} \blacklozenge \setminus \{\mathrm{red}\}$. By Remark 1.2 (iii), the underlying scheme of $X^{\log} \times_{Y^{\log}} Y_0^{\log}$ is finite over $X \times_Y Y_0$. Since $g^{\log} : Y_0^{\log} \to Y^{\log}$ is quasi-compact, the natural projection $X \times_Y Y_0 \to X$ is also quasi-compact. Thus $X^{\log} \times_{Y^{\log}} Y_0^{\log}$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\lozenge/S^{\log}} \subset$

$\mathsf{Sch}^{\log}_{/S^{\log}}$. If, moreover, $\{\text{red}\} \subset \blacklozenge$, then $(X^{\log} \times_{Y^{\log}} Y_0^{\log})_{\text{red}}$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$ and indeed may be interpreted as the fiber product $X^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. In particular, for arbitrary $\blacklozenge$, the natural morphism

$$(X^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log})_{\text{red}} \xrightarrow{\sim} (X^{\log} \times_{Y^{\log}} Y_0^{\log})_{\text{red}}$$

is an isomorphism. The final assertion follows immediately from Remark 2.4. This completes the proof of Lemma 3.4. □

Finally, we give a category-theoretic characterization of log residue fields in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ (cf. Definition 2.7).

COROLLARY 3.5. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $f^{\log} : Y^{\log} \to X^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Assume that $Y^{\log}$ is an fs log point. Then $f^{\log}$ is a log residue field of $X^{\log}$ if and only if $f^{\log}$ satisfies the following condition:*

(†) For any morphism $g^{\log} : Z^{\log} \to X^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $Z^{\log}$ is an fs log point, if $Z^{\log} \times^{\blacklozenge}_{X^{\log}} Y^{\log} \neq \varnothing$, then the natural projection $Z^{\log} \times^{\blacklozenge}_{X^{\log}} Y^{\log} \xrightarrow{\sim} Z^{\log}$ is an isomorphism (where we note that since $Y^{\log}$ is an fs log point, the fiber product $Z^{\log} \times^{\blacklozenge}_{X^{\log}} Y^{\log}$ exists in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, cf. Lemma 3.4):

$$\begin{tikzcd} Z^{\log} \times^{\blacklozenge}_{X^{\log}} Y^{\log} \neq \varnothing \arrow[r] \arrow[d, "\wr"'] & Y^{\log} \arrow[d, "f^{\log}"] \\ Z^{\log} \arrow[r, "g^{\log}"] & X^{\log} \end{tikzcd}$$

In particular, the property that $f^{\log} : Y^{\log} \to X^{\log}$ is a log residue field of $X^{\log}$ may be characterized category-theoretically (cf. Lemma 3.1 (i), Proposition 3.3) from the data $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, f^{\log})$.

PROOF. Necessity follows immediately from Lemma 3.4. In the remainder of the proof of Corollary 3.5, we prove sufficiency. Assume that $f^{\log} : Y^{\log} \to X^{\log}$ satisfies condition (†). Write $x \in \mathrm{Im}(f)$ for the unique point contained in $\mathrm{Im}(f) \subset X$ and $g^{\log} : Z^{\log} \to X^{\log}$ for the log residue field determined by $x \in X$. Then, by the necessity portion of Corollary 3.5, the natural projection $Z^{\log} \times^{\blacklozenge}_{X^{\log}} Y^{\log} \xrightarrow{\sim} Y^{\log} \neq \emptyset$ is an isomorphism. Hence, by condition (†), the natural projection $Z^{\log} \times^{\blacklozenge}_{X^{\log}} Y^{\log} \xrightarrow{\sim} Z^{\log}$ is an isomorphism. Thus $f^{\log} : Y^{\log} \to X^{\log}$ is isomorphic as a log scheme over $X^{\log}$ to the log residue field $g^{\log} : Z^{\log} \to X^{\log}$. This completes the proof of Corollary 3.5. □

4. Strict Morphisms

In this section, we assume that

$$\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}.$$

In the present section, we give a category-theoretic characterization of strict morphisms in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ (cf. Corollary 4.6).

First, we prove the following property of the *strict locus* of a morphism of fs log schemes.

Lemma 4.1. *Let $f^{\log} : X^{\log} \to Y^{\log}$ be a morphism of fs log schemes. Write*

$$\mathrm{Str}(f^{\log}) \overset{\mathrm{def}}{:=} \left\{ x \in X \;\middle|\; \begin{array}{l} \textit{the composite of } f^{\log} : X^{\log} \to Y^{\log} \textit{ with} \\ \textit{the log residue field } \mathrm{Spec}(k(x))^{\log} \to X^{\log} \\ \textit{determined by } x \in X \textit{ is strict} \end{array} \right\}.$$

Then the following assertions hold:

(i) $\mathrm{Str}(f^{\log}) \subset |X|$ is an open subset. Thus, $\mathrm{Str}(f^{\log}) \subset X$ may be regarded as an open subscheme.

(ii) Let

$$\begin{array}{ccc} X^{\log} & \xrightarrow[\text{strict}]{p^{\log}} & X_0^{\log} \\ {\scriptstyle f^{\log}}\big\downarrow & & \big\downarrow{\scriptstyle f_0^{\log}} \\ Y^{\log} & \xrightarrow[\text{strict}]{q^{\log}} & Y_0^{\log} \end{array}$$

be a commutative diagram of fs log schemes such that $p^{\log}$ and $q^{\log}$ are strict. Then it holds that $\mathrm{Str}(f^{\log}) = p^{-1}(\mathrm{Str}(f_0^{\log}))$.

(iii) The open immersion $\mathrm{Str}(f^{\log}) \hookrightarrow X$ *(cf. (i)) is quasi-compact.*

Proof. First, we prove assertion (i). Let $\bar{x}_0 \to X$ be a geometric point such that $\mathrm{Im}(\bar{x}_0 \to X) \subset \mathrm{Str}(f^{\log})$. Then $\bar{f}^{\flat}_{\bar{x}_0} : (f^{-1}\overline{\mathcal{M}}_Y)_{\bar{x}_0} \xrightarrow{\sim} \overline{\mathcal{M}}_{X,\bar{x}_0}$ is an isomorphism. Hence, there exists an étale neighborhood $i_0 : U_0 \to X$ of $\bar{x}_0$ such that $i_0^{-1} f^{-1}\overline{\mathcal{M}}_Y \xrightarrow{\sim} i_0^{-1}\overline{\mathcal{M}}_X$ is an isomorphism. Thus $\mathrm{Im}(\bar{x}_0 \to X) \subset |\mathrm{Im}(i_0)| \subset \mathrm{Str}(f^{\log})$. Since i_0 is étale, $|\mathrm{Im}(i_0)| \subset |X|$ is an open subset. This implies that $\mathrm{Str}(f^{\log}) \subset |X|$ is an open subset. This completes the proof of assertion (i).

Next, we prove assertion (ii). Let $h^{\log} : Z^{\log} \to X^{\log}$ be a strict morphism of fs log schemes. Since $p^{\log}$ is strict, $p^{\log} \circ h^{\log}$ is also strict. Since $q^{\log} \circ f^{\log} \circ h^{\log} = f_0^{\log} \circ p^{\log} \circ h^{\log}$, and $q^{\log}$ is strict, it holds that

$$f^{\log} \circ h^{\log} \text{ is strict if and only if } f_0^{\log} \circ p^{\log} \circ h^{\log} \text{ is strict.}$$

This implies that $\mathrm{Str}(f^{\log}) = p^{-1}(\mathrm{Str}(f_0^{\log}))$. This completes the proof of assertion (ii).

Next, we prove assertion (iii). By [Stacks, Tag 02KQ] and [Stacks, Tag 022C], to prove assertion (iii), it suffices to prove that there exists an étale covering $\{f_i : U_i \to X\}_{i \in I}$ of X such that for any $i \in I$, the open immersion $f_i^{-1}(\mathrm{Str}(f^{\log})) \hookrightarrow U_i$ is a quasi-compact morphism. Hence, by assertion (ii) and [KK, Definition 2.9 (2), Lemma 2.10], to prove assertion (iii), we may assume without loss of generality that there exist a morphism of fs monoids $\psi : N \to M$ and a commutative diagram of fs log schemes

$$\begin{tikzcd} X^{\log} \arrow[r, "p^{\log}", "\text{strict}"'] \arrow[d, "f^{\log}"'] & \mathrm{Spec}(\mathbb{Z}[M])^{\log} \arrow[d, "\psi^{\log}"] \\ Y^{\log} \arrow[r, "q^{\log}", "\text{strict}"'] & \mathrm{Spec}(\mathbb{Z}[N])^{\log} \end{tikzcd}$$

— where we write $\mathrm{Spec}(\mathbb{Z}[-])^{\log}$ for the log scheme determined by the monoid ring $\mathbb{Z}[-]$ and the morphism of monoids $(-) \to \mathbb{Z}[-]$, and we write $\psi^{\log} : \mathrm{Spec}(\mathbb{Z}[M])^{\log} \to \mathrm{Spec}(\mathbb{Z}[N])^{\log}$ for the morphism of log schemes determined by $\psi : N \to M$ — such that $p^{\log}$ and $q^{\log}$ are strict. Then, by assertion (ii), it holds that $\mathrm{Str}(f^{\log}) = p^{-1}(\mathrm{Str}(\psi^{\log}))$. Moreover, since $\mathrm{Spec}(\mathbb{Z}[M])$ is Noetherian, the open immersion $\mathrm{Str}(\psi^{\log}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[M])$ is a quasi-compact morphism (cf. [Stacks, Tag 01OX]). Thus $\mathrm{Str}(f^{\log}) \hookrightarrow X$

is also a quasi-compact morphism (cf. [Stacks, Tag 01K5]). This completes the proof of Lemma 4.1. □

Next, we give a category-theoretic characterization of strict morphisms (cf. Corollary 4.6).

Lemma 4.2. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $f^{\log}$: $X^{\log} \to Y^{\log}$ a morphism between fs log points in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Assume that the diagonal morphism $X^{\log} \to X^{\log} \times^{\blacklozenge}_{Y^{\log}} X^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ is strict (where we note that since $X^{\log}$ and $Y^{\log}$ are fs log points, the fiber product $X^{\log} \times^{\blacklozenge}_{Y^{\log}} X^{\log}$ exists in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, by Lemma 3.4). Then, for any geometric point $\bar{x} \to X$, it holds that*

$$\operatorname{coker}(\bar{f}^{\flat,\mathrm{gp}}_{\bar{x}} : (f^{-1}\overline{\mathcal{M}}_Y)^{\mathrm{gp}}_{\bar{x}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Proof. Let $\bar{x} \to X$ be a geometric point. Write

$$M \stackrel{\mathrm{def}}{:=} \overline{\mathcal{M}}_{X,\bar{x}} \sqcup_{\bar{f}^{\flat}_{\bar{x}},(f^{-1}\overline{\mathcal{M}}_Y)_{\bar{x}},\bar{f}^{\flat}_{\bar{x}}} \overline{\mathcal{M}}_{X,\bar{x}}.$$

Since the groupification functor $(-)^{\mathrm{gp}} : \mathsf{Mon} \to \mathsf{Ab}$ and the functor $(-)\otimes_{\mathbb{Z}}\mathbb{Q}$ preserve push-outs, it holds that

$$\dim_{\mathbb{Q}}(M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}) = 2\dim_{\mathbb{Q}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(\mathrm{Im}(\bar{f}^{\flat,\mathrm{gp}}_{\bar{x}}) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Hence, to prove Lemma 4.2, it suffices to prove that

$$\dim_{\mathbb{Q}}(M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \otimes_{\mathbb{Z}} \mathbb{Q}).$$

By (the proof of) Lemma 3.4, the diagonal morphism $\Delta^{\log} : X^{\log} \to X^{\log} \times_{Y^{\log}} X^{\log}$ in $\mathsf{Sch}^{\log}_{/S^{\log}}$ is strict. Hence the composite of the natural morphisms

$$\begin{aligned}\left(\mathcal{M}_{X,\bar{x}} \sqcup_{f^{\flat}_{\bar{x}},(f^{-1}\mathcal{M}_Y)_{\bar{x}},f^{\flat}_{\bar{x}}} \mathcal{M}_{X,\bar{x}}\right)^{\mathrm{sat}} &\xrightarrow{\sim} \left(\mathcal{M}_X \sqcup_{f^{\flat},f^{-1}\mathcal{M}_Y,f^{\flat}} \mathcal{M}_X\right)^{\mathrm{sat}}_{\bar{x}} \\ &\to \left(\Delta^{-1}\mathcal{M}_{X^{\log}\times_{Y^{\log}}X^{\log}}\right)_{\bar{x}}\end{aligned}$$

induces an isomorphism $\overline{\Delta}^{\flat}_{\bar{x}} : M^{\mathrm{sat}}/(M^{\mathrm{sat}})^{\times} \xrightarrow{\sim} \overline{\mathcal{M}}_{X,\bar{x}}$. By Lemma 1.7, Lemma A.1, and Lemma A.2, M is sharp, quasi-integral, and finitely generated. Hence M^{int} is sharp and fine. This implies that M^{sat} is fs (cf. Remark 1.2 (iii)), and, moreover, $(M^{\mathrm{sat}})^{\times}$ is a finite abelian group. Hence the morphism $M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by $\overline{\Delta}^{\flat}_{\bar{x}}$ is an isomorphism. In particular, it holds that $\dim_{\mathbb{Q}}(M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \otimes_{\mathbb{Z}} \mathbb{Q})$. This completes the proof of Lemma 4.2. □

LEMMA 4.3. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $p^{\log} : X^{\log} \to Y^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Assume that the following conditions hold:*

(i) $|X|$ is irreducible, and $Y^{\log}$ is an fs log point. Write $\eta \in X$ for the unique generic point.

(ii) For any geometric point $\bar{x} \to X$ such that the log residue field determined by the image of $\bar{x} \to X$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$, it holds that $\mathrm{coker}(\bar{p}^{\flat,\mathrm{gp}}_{\bar{x}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

(iii) The composite of $p^{\log} : X^{\log} \to Y^{\log}$ and the log residue field $\mathrm{Spec}(k(\eta))^{\log} \to X^{\log}$ determined by $\eta \in X$ is strict.

Then $p^{\log}$ is strict.

PROOF. Let $\bar{x} \to X$ be a geometric point such that the log residue field $\mathrm{Spec}(k(x))^{\log} \to X^{\log}$ determined by $\bar{x} \to X$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$ and $\bar{\eta} \to X$ a geometric point such that $\eta \in \mathrm{Im}(\bar{\eta} \to X)$. Then, since $Y^{\log}$ is an fs log point, the localization morphism $(p^{-1}\overline{\mathcal{M}}_Y)_{\bar{x}} \xrightarrow{\sim} (p^{-1}\overline{\mathcal{M}}_Y)_{\bar{\eta}}$ is an isomorphism. Moreover, by condition (iii), $\bar{p}^{\flat}_{\bar{\eta}} : (p^{-1}\overline{\mathcal{M}}_Y)_{\bar{\eta}} \xrightarrow{\sim} \overline{\mathcal{M}}_{X,\bar{\eta}}$ is an isomorphism. Since the following diagram of monoids commutes, $\mathrm{coker}(\bar{p}^{\flat,\mathrm{gp}}_{\bar{x}})$ is a direct summand of $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}}$:

$$\begin{tikzcd}
(p^{-1}\overline{\mathcal{M}}_Y)_{\bar{x}} \arrow[r, "\sim"] \arrow[d, "\bar{p}^{\flat}_{\bar{x}}"'] & (p^{-1}\overline{\mathcal{M}}_Y)_{\bar{\eta}} \arrow[d, "\bar{p}^{\flat}_{\bar{\eta}}", "\wr"'] \\
\overline{\mathcal{M}}_{X,\bar{x}} \arrow[r, "\text{localization}"] & \overline{\mathcal{M}}_{X,\bar{\eta}},
\end{tikzcd}$$

where $\overline{\mathcal{M}}_{X,\bar{x}} \to \overline{\mathcal{M}}_{X,\bar{\eta}}$ is the localization morphism. Thus, since $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}}$ is torsion-free, $\mathrm{coker}(\bar{p}^{\flat,\mathrm{gp}}_{\bar{x}})$ is also torsion-free. Hence, by condition (ii), $\bar{p}^{\flat,\mathrm{gp}}_{\bar{x}}$ is surjective. Since the above diagram commutes, $\bar{p}^{\flat,\mathrm{gp}}_{\bar{x}}$ is an isomorphism, and, moreover, $\overline{\mathcal{M}}_{X,\bar{x}} \to \overline{\mathcal{M}}_{X,\bar{\eta}}$ is injective. Hence, $\bar{p}^{\flat}_{\bar{x}}$ is also an isomorphism. Since $\overline{\mathcal{M}}_{X,\bar{x}}$ and $(p^{-1}\overline{\mathcal{M}}_Y)_{\bar{x}}$ are fs monoids, this implies that there exists an open subscheme $U \subset X$ such that the following conditions hold:

- $\bar{p}^{\flat}|_U : (p^{-1}\overline{\mathcal{M}}_Y)|_U \to \overline{\mathcal{M}}_X|_U$ is an isomorphism.
- For any geometric morphism $\bar{x} \to X$, if the log residue field $\mathrm{Spec}(k(x))^{\log} \to X^{\log}$ determined by $\bar{x} \to X$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$, then $\mathrm{Im}(\bar{x} \to X) \subset U$.

Since S is locally Noetherian, the second condition implies that $U = X$. Thus $\bar{p}^{\flat}$ is an isomorphism. This completes the proof of Lemma 4.3. $\square$

Proposition 4.4. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $p^{\log} : G^{\log} \to Y^{\log}$ a group object in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ over $Y^{\log}$ such that $Y^{\log}$ is an fs log point. Write $\mathrm{e}^{\log} : Y^{\log} \to G^{\log}$ for the identity section. Assume that the following conditions hold:*

(a) $|G|$ is connected.

(b) The identity section $\mathrm{e}^{\log} : Y^{\log} \to G^{\log}$ is a log residue field.

Then the following assertions hold:

(i) For any geometric point $\bar{g} \to G$ such that the log residue field determined by the image of $\bar{g} \to G$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$, it holds that $\mathrm{coker}(\bar{p}^{\flat,\mathrm{gp}}_{\bar{g}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

(ii) $p^{\log}$ is strict.

Proof. First, we prove assertion (i). Let $\bar{g} \to G$ be a geometric point such that the log residue field determined by the image of $\bar{g} \to G$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$. Write

- $g^{\log} : Y_0^{\log} \to G^{\log}$ for the log residue field determined by the image of $\bar{g} \to G$;

- $G_0^{\log} \overset{\mathrm{def}}{:=} G^{\log} \times^{\blacklozenge}_{Y^{\log}, p^{\log} \circ g^{\log}} Y_0^{\log} \to Y_0^{\log}$ for the group object in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ obtained by base-changing $p^{\log}$ by $p^{\log} \circ g^{\log}$ (where we note that since $p^{\log} \circ g^{\log} : Y_0^{\log} \to Y^{\log}$ is a morphism between fs log points, the fiber product $G^{\log} \times^{\blacklozenge}_{Y^{\log}, p^{\log} \circ g^{\log}} Y_0^{\log}$ exists in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, cf. Lemma 3.4);
- $e_0^{\log} \overset{\mathrm{def}}{:=} e^{\log} \times^{\blacklozenge}_{Y^{\log}, p^{\log} \circ g^{\log}} \mathrm{id}_{Y_0^{\log}} : Y_0^{\log} \to G_0^{\log}$ for the identity section of $G_0^{\log}$;
- $g_0^{\log} : Y_0^{\log} \to G_0^{\log}$ for the morphism over $Y_0^{\log}$ determined by $g^{\log} : Y_0^{\log} \to G^{\log}$;
- $\varphi^{\log} : G_0^{\log} \xrightarrow{\sim} G_0^{\log}$ for the left translation isomorphism of $G_0^{\log}$ determined by the morphism $g_0^{\log} : Y_0^{\log} \to G_0^{\log}$ over $Y_0^{\log}$.

Then $g_0^{\log} = \varphi^{\log} \circ e_0^{\log}$. Moreover, by condition (b) and Lemma 3.4, $e_0^{\log}$ is strict. In particular, $g_0^{\log}$ is strict. Since $g^{\log}$ is strict, it follows from Lemma 3.4 that the natural projection $Y_0^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log} \to G_0^{\log}$ (cf. the commutative diagram below) is also strict. Thus the diagonal morphism $Y_0^{\log} \to Y_0^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log}$ is strict:

$$\begin{tikzcd}
Y_0^{\log} \arrow[r] \arrow[rr, bend left, "g_0^{\log}\ \text{:strict}"] & Y_0^{\log} \times^{\blacklozenge}_{Y^{\log}} Y_0^{\log} \arrow[r, "\text{strict}"'] \arrow[d] \arrow[dr, phantom, "\square"] & G_0^{\log} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_0^{\log} \arrow[d, "p^{\log} \circ g^{\log}"] \\
& Y_0^{\log} \arrow[r, "g^{\log}", "\text{log residue field}"'] & G^{\log} \arrow[r, "p^{\log}"] & Y^{\log}.
\end{tikzcd}$$

Hence it follows from Lemma 4.2 that $\mathrm{coker}(\bar{p}_{\bar{g}}^{\flat,\mathrm{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. This completes the proof of assertion (i).

Next, we prove assertion (ii). By condition (b), $\mathrm{Str}(p^{\log}) \neq \varnothing$ (cf. Lemma 4.1). Let $\eta \in \mathrm{Str}(p^{\log})$ be a point. Then, by assertion (i) and Lemma 4.3, the composite

$$\overline{\{\eta\}}^{\log}_{\mathrm{red}} \hookrightarrow G^{\log} \xrightarrow{p^{\log}} Y^{\log}$$

(cf. Definition 2.1 (v)) is strict (where we note that since $G^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, and $\overline{\{\eta\}}^{\log}_{\mathrm{red}} \hookrightarrow G^{\log}$ is a strict closed immersion, $\overline{\{\eta\}}^{\log}_{\mathrm{red}}$ belongs to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$). Hence $\overline{\{\eta\}} \subset \mathrm{Str}(p^{\log})$. In particular, we conclude that $|\mathrm{Str}(p^{\log})| \subset |G|$ is stable under specialization. Moreover, by Lemma 4.1 (i) (iii), $\mathrm{Str}(p^{\log}) \hookrightarrow G$ is a quasi-compact open immersion. Thus, by [Stacks, Tag 05JL], $|\mathrm{Str}(p^{\log})| \subset |G|$ is closed. Since $|G|$ is connected, we conclude that $\mathrm{Str}(p^{\log}) = G$. This completes the proof of Proposition 4.4. □

Corollary 4.5. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Assume that $X^{\log}$ and $Y^{\log}$ are fs log points. Then $f^{\log}$ is strict if and only if there exist a group object $G^{\log} \to Y^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ over $Y^{\log}$ and a morphism $g^{\log} : X^{\log} \to G^{\log}$ over $Y^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that the following conditions hold:*

(i) $|G|$ is connected.

(ii) The identity section $e^{\log} : Y^{\log} \to G^{\log}$ is a log residue field.

(iii) $g^{\log}$ is a log residue field.

In particular, the property that

$f^{\log}$ is a strict morphism between fs log points

may be characterized category-theoretically (cf. Lemma 3.1 (ii), Proposition 3.3, Corollary 3.5) from the data $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, f^{\log})$.

Proof. First, we prove necessity. Assume that $f^{\log} : X^{\log} \to Y^{\log}$ is a strict morphism between fs log points. Write

- $k \overset{\mathrm{def}}{:=} \Gamma(Y, \mathcal{O}_Y)$,
- A for the symmetric k-algebra determined by the underlying k-linear space of $\Gamma(X, \mathcal{O}_X)$,
- $G \overset{\mathrm{def}}{:=} \mathrm{Spec}(A) \to Y$ for the affine space over Y determined by the k-linear space $\Gamma(X, \mathcal{O}_X)$ equipped with the natural (additive) group scheme structure over Y, and
- $p^{\log} : G^{\log} \to Y^{\log}$ for the strict morphism obtained by pulling back the log structure of $Y^{\log}$ via $G \to Y$.

Note that since $X^{\log}$ and $Y^{\log}$ are fs log points, if $\{\mathrm{ft}\} \subset \blacklozenge$, then the underlying morphism of schemes $f : X \to Y$ is finite. Hence, regardless of whether or not $\{\mathrm{ft}\} \subset \blacklozenge$, it holds that $G^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Since $p^{\log}$ is strict, and G is a geometrically integral affine scheme over Y, $G^{\log} \times_{Y^{\log}} G^{\log}$ and $G^{\log} \times_{Y^{\log}} G^{\log} \times_{Y^{\log}} G^{\log}$ belong to the full subcategory $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \subset \mathsf{Sch}^{\log}_{/S^{\log}}$. Hence the evident group object structure of $G^{\log}$ in $\mathsf{Sch}^{\log}_{/S^{\log}}$ over $Y^{\log}$ may be regarded as a group object structure of $G^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ over $Y^{\log}$. Moreover, the group object $G^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ over $Y^{\log}$ clearly satisfies conditions (i) and (ii).

Write $g : X \to G$ for the closed immersion determined by the tautological surjection of k-algebras $A \to \Gamma(X, \mathcal{O}_X)$. Then $f = p \circ g$. Since $p^{\log} : G^{\log} \to Y^{\log}$ and $f^{\log} : X^{\log} \to Y^{\log}$ are strict, the pull-back of the log structure of $G^{\log}$ to X via g is isomorphic to the log structure of $X^{\log}$. Thus we obtain a strict closed immersion $g^{\log} : X^{\log} \to G^{\log}$ such that $f^{\log} = p^{\log} \circ g^{\log}$. In particular, $g^{\log}$ satisfies condition (iii). This completes the proof of necessity.

Next, we prove sufficiency. Assume that there exist a group object $G^{\log} \to Y^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ over $Y^{\log}$ and a morphism $g^{\log} : X^{\log} \to G^{\log}$ over $Y^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $G^{\log} \to Y^{\log}$ and $g^{\log}$ satisfy conditions (i), (ii), and (iii). Since $G^{\log} \to Y^{\log}$ satisfies conditions (i) and (ii), it follows from Proposition 4.4 that $G^{\log} \to Y^{\log}$ is strict. Thus, by condition (iii), $f^{\log} : X^{\log} \to Y^{\log}$ is also strict. This completes the proof of Corollary 4.5. □

Corollary 4.6. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Then $f^{\log}$ is strict if and only if for any commutative diagram*

$$\begin{array}{ccc} Z^{\log} & \xrightarrow{i^{\log}} & X^{\log} \\ {\scriptstyle p^{\log}}\downarrow & & \downarrow{\scriptstyle f^{\log}} \\ W^{\log} & \xrightarrow{j^{\log}} & Y^{\log} \end{array}$$

in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, if $i^{\log}$ is a log residue field of $X^{\log}$, and $j^{\log}$ is a log residue field of $Y^{\log}$, then $p^{\log}$ is strict. In particular, the property that $f^{\log}$ is strict *may be characterized category-theoretically (cf. Corollary 3.5, Corollary 4.5) from the data* $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, f^{\log})$.

Proof. Corollary 4.6 follows immediately from Corollary 4.5. □

Now we prove the first equality of Corollary B.

Corollary 4.7. *Let* $S^{\log}, T^{\log}$ *be locally Noetherian fs log schemes and* $\blacklozenge, \lozenge \subset \{\text{red}, \text{qcpt}, \text{qsep}, \text{sep}\}$ *subsets such that* $\{\text{qsep}, \text{sep}\} \not\subset \blacklozenge$, *and* $\{\text{qsep}, \text{sep}\} \not\subset \lozenge$. *If the categories* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *and* $\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ *are equivalent, then* $\blacklozenge = \lozenge$.

Proof. Corollary 4.7 follows immediately from Corollary 4.6, [YJ, Cor-ollary 4.11], and [Stacks, Tag 01OY], where we apply [YJ, Corollary 4.11] to the categories obtained by considering the full subcategory of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ or $\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ determined by the objects whose structure morphism is strict. □

5. Log-like Morphisms

In this section, we assume that

$$\blacklozenge \subset \{\text{red}, \text{qcpt}, \text{qsep}, \text{sep}, \text{ft}\}.$$

In the present section, we give a category-theoretic characterization of the morphisms of monoid objects in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ that represent the functor

$$\begin{aligned} \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} &\to \mathsf{Mor}(\mathsf{Mon}) \\ X^{\log} &\mapsto [\alpha_X(X) : \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X)], \end{aligned}$$

which arises from the log structures of objects of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ (cf. Definition 5.2, Proposition 5.7). We then use this characterization to complete the proof of the main theorem of the present paper (cf. Theorem 5.9, Theorem 5.10).

First, we introduce some notation used in this section.

DEFINITION 5.1.

- Let $\mathcal{C}$ be a category. Then we shall write $\mathsf{Mon}(\mathcal{C})$ for the category of monoid objects in $\mathcal{C}$.

- Let $\mathcal{C}$ be a category and A a ring object in $\mathcal{C}$. Then we shall write $(A, \times)$ for the underlying multiplicative monoid object of the ring object A in $\mathcal{C}$.

- We shall write $\mathbb{A}^{1,\log}_{\mathbb{Z}}$ for the fs log scheme over $\mathbb{Z}$ whose underlying scheme is $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[t])$, and whose log structure is the log structure determined by the morphism of monoids
$$\begin{aligned} \mathbb{N} &\to \mathbb{Z}[t], \\ n &\mapsto t^n. \end{aligned}$$

- We shall write $\mathbb{G}_{m,\mathbb{Z}} \subset \mathbb{A}^1_{\mathbb{Z}}$ for the unit group scheme of the ring scheme $\mathbb{A}^1_{\mathbb{Z}}$ over $\mathbb{Z}$.

- Let $X^{\log}$ be an fs log scheme and Y a scheme. Then we shall write $X^{\log} \times_{\mathbb{Z}} Y$ for the fs log scheme whose underlying scheme is $X \times_{\mathbb{Z}} Y$, and whose log structure is the log structure obtained by pulling back the log structure of $X^{\log}$ via the natural projection $X \times_{\mathbb{Z}} Y \to X$.

DEFINITION 5.2 ($\alpha^{\log}_{S^{\log},\mathbb{A}}$). Let $S^{\log}$ be a locally Noetherian fs log scheme and $f^{\log} : X^{\log} \to S^{\log}$ a morphism of fs log schemes. Then an element $m \in \Gamma(X, \mathcal{M}_X)$ determines a morphism of monoids $\bar{g}^{\flat} : \mathbb{N} \to \Gamma(X, \mathcal{M}_X)$ and a morphism of rings $g^{\#} : \mathbb{Z}[t] \to \Gamma(X, \mathcal{O}_X)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\bar{g}^{\flat}:1\mapsto m"] \arrow[d, "1\mapsto t"'] & \Gamma(X, \mathcal{M}_X) \arrow[d, "\alpha_X(X)"] \\ \mathbb{Z}[t] \arrow[r, "g^{\#}:t\mapsto \alpha_X(X)(m)"] & \Gamma(X, \mathcal{O}_X). \end{tikzcd}$$

Hence we obtain a morphism of fs log schemes $g^{\log} : X^{\log} \to \mathbb{A}^{1,\log}_{\mathbb{Z}}$. Conversely, each morphism of fs log schemes $g^{\log} : X^{\log} \to \mathbb{A}^{1,\log}_{\mathbb{Z}}$ determines an element $g^\flat(1) \in \Gamma(X, \mathcal{M}_X)$. One verifies easily that these assignments determine an isomorphism of functors

$$\Gamma(-, \mathcal{M}_{(-)}) \xrightarrow{\sim} \mathrm{Hom}_{S^{\log}}(-, S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}).$$

In particular, $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}$ represents the functor

$$\begin{aligned} \mathsf{Sch}^{\log}_{/S^{\log}} &\to \mathsf{Mon}, \\ X^{\log} &\mapsto \Gamma(X, \mathcal{M}_X), \end{aligned}$$

which implies that $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}$ has a monoid object structure in $\mathsf{Sch}^{\log}_{/S^{\log}}$, hence also in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Moreover, the family of morphisms of monoids

$$\{\alpha_X(X) : \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X)\}_{X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}}$$

determines a morphism of monoid objects $\alpha^{\log}_{S^{\log},\mathbb{A}} : S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to (S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \times)$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

The log structure of $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}$ may be identified with the log structure obtained by pushing forward the log structure of $S^{\log} \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}$ to $S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ via the open immersion $S \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}} \hookrightarrow S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ as follows (in the case where the log structure of $S^{\log}$ is trivial, cf. [KK, Example 1.5 (1) (2)]):

Lemma 5.3. *Let $S^{\log}$ be an fs log scheme. Then the log structure of $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}$ is isomorphic to the log structure determined by the push-forward of the log structure of $S^{\log} \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}$ to $S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ via the open immersion $S \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}} \hookrightarrow S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$.*

Proof. Write

- $i : S \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}} \hookrightarrow S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ for the inclusion morphism,
- $p : S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to S$ for the natural projection,
- $\alpha_1 : \mathcal{M}_1 \to \mathcal{O}_{S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}}$ for the log structure of $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}$, and
- $\alpha_2 : \mathcal{M}_2 \to \mathcal{O}_{S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}}$ for the log structure on $S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ obtained by pushing forward the log structure of $S^{\log} \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}$ to $S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ via the open immersion $i : S \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}} \hookrightarrow S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$.

Since the log structure of $S^{\log} \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}$ is isomorphic to the log structure $i^*(\mathcal{M}_1, \alpha_1)$, there exists a unique morphism of sheaves of monoids $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ on the étale site of $S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ such that the following diagram of sheaves of monoids on $S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ commutes:

$$\begin{tikzcd}
\mathcal{O}_{S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}} \arrow[d, equal] & \mathcal{M}_1 \arrow[l, "\alpha_1"'] \arrow[r, "\theta_1"] \arrow[d, "\exists ! \varphi"] & i_* i^{-1} \mathcal{M}_1 & i_* i^{-1} p^* \mathcal{M}_S \arrow[l, "i_* i^{-1} \tilde{p}_1^{\flat}"', "\sim"] \arrow[d, equal] \\
\mathcal{O}_{S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}} & \mathcal{M}_2 \arrow[l, "\alpha_2"'] \arrow[r, "\theta_2"] & i_* i^{-1} \mathcal{M}_2 & i_* i^{-1} p^* \mathcal{M}_S, \arrow[l, "i_* i^{-1} \tilde{p}_2^{\flat}"', "\sim"]
\end{tikzcd}$$

where, for each $k \in \{1, 2\}$, $\tilde{p}_k^{\flat} : p^* \mathcal{M}_S \to \mathcal{M}_k$ is the morphism of sheaves of monoids on $S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ that arises from the morphism of log schemes $\tilde{p}_k^{\log} : (S \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \mathcal{M}_k, \alpha_k) \to S^{\log}$, and $\theta_k : \mathcal{M}_k \to i_* i^{-1} \mathcal{M}_k$ is the natural morphism. Then it follows immediately from various definitions involved that for each $k \in \{1, 2\}$, the natural morphism $\theta_k : \mathcal{M}_k \to i_* i^{-1} \mathcal{M}_k$ is injective. Thus, to prove that the morphism of sheaves of monoids $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism, i.e., to prove that the morphism of sheaves of monoids $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ is surjective, we may assume without loss of generality that S is isomorphic to the spectrum of an algebraically closed field. But then Lemma 5.3 follows immediately. $\square$

Definition 5.4. We shall say that a morphism of log schemes $f^{\log}$ is **log-like** if the underlying morphism of schemes is an isomorphism (cf. the terminology of [Mzk04]).

Lemma 5.5. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Then the following assertions hold:*

(i) (cf. [Mzk15, Proposition 1.11 (i)]*). There exists a factorization*

$$X^{\log} \xrightarrow{g^{\log}} Z^{\log} \xrightarrow{h^{\log}} Y^{\log}$$

of $f^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $g^{\log}$ is log-like, and $h^{\log}$ is strict.

(ii) (cf. [Mzk15, Proposition 1.11 (ii)]*). The factorization $X^{\log} \xrightarrow{g^{\log}} Z^{\log} \xrightarrow{h^{\log}} Y^{\log}$ of (i) may be characterized up to a unique isomorphism, via the following universal property: The morphism $h^{\log}$ is*

strict, and moreover, if

$$X^{\log} \xrightarrow{g_0^{\log}} Z_0^{\log} \xrightarrow{h_0^{\log}} Y^{\log}$$

is a factorization of $f^{\log}$ *in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *such that* $h_0^{\log}$ *is strict, then there exists a unique morphism* $r^{\log} : Z^{\log} \to Z_0^{\log}$ *in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *such that* $g_0^{\log} = r^{\log} \circ g^{\log}$*, and* $h_0^{\log} \circ r^{\log} = h^{\log}$*:*

$$\begin{tikzcd} X^{\log} \arrow[r, "g^{\log}", "\text{log-like}"'] \arrow[d, equal] & Z^{\log} \arrow[r, "h^{\log}", "\text{strict}"'] \arrow[d, "\exists! r^{\log}"'] & Y^{\log} \arrow[d, equal] \\ X^{\log} \arrow[r, "g_0^{\log}"] & Z_0^{\log} \arrow[r, "h_0^{\log}", "\text{strict}"'] & Y^{\log}. \end{tikzcd}$$

(iii) Let

$$\begin{tikzcd} X^{\log} \arrow[r, "f^{\log}"] \arrow[d, "p^{\log}"'] & Y^{\log} \arrow[d, "q^{\log}"] \\ X_0^{\log} \arrow[r, "f_0^{\log}"] & Y_0^{\log} \end{tikzcd}$$

be a commutative diagram in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$*;*

$$X^{\log} \xrightarrow{g^{\log}} Z^{\log} \xrightarrow{h^{\log}} Y^{\log}$$

a factorization of $f^{\log}$ *such that* $g^{\log}$ *is log-like, and* $h^{\log}$ *is strict;*

$$X_0^{\log} \xrightarrow{g_0^{\log}} Z_0^{\log} \xrightarrow{h_0^{\log}} Y_0^{\log}$$

is a factorization of $f_0^{\log}$ *in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *such that* $h_0^{\log}$ *is strict. Then there exists a unique morphism* $r^{\log} : Z^{\log} \to Z_0^{\log}$ *in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *such that* $g_0^{\log} \circ p^{\log} = r^{\log} \circ g^{\log}$*, and* $h_0^{\log} \circ r^{\log} = q^{\log} \circ h^{\log}$*:*

$$\begin{tikzcd} X^{\log} \arrow[r, "g^{\log}", "\text{log-like}"'] \arrow[d, "p^{\log}"'] & Z^{\log} \arrow[r, "h^{\log}", "\text{strict}"'] \arrow[d, "\exists! r^{\log}"'] & Y^{\log} \arrow[d, "q^{\log}"] \\ X_0^{\log} \arrow[r, "g_0^{\log}"] & Z_0^{\log} \arrow[r, "h_0^{\log}", "\text{strict}"'] & Y_0^{\log}. \end{tikzcd}$$

Proof. Assertions (i) and (ii) follow immediately by considering the pull-back of the log structure of $Y^{\log}$ to X via f. Since

$$X^{\log} \xrightarrow{(g_0^{\log} \circ p^{\log}, f^{\log})} Z_0^{\log} \times_{Y_0^{\log}} Y^{\log} \xrightarrow{h_0^{\log} \times \mathrm{id}_{Y^{\log}}} Y^{\log},$$

(where we note that the fiber product is taken in $\mathsf{Sch}^{\log}_{/S^{\log}}$) is a factorization of $f^{\log}$, and $h_0^{\log} \times \mathrm{id}_{Y^{\log}}$ is strict, the existence and uniqueness portions of assertion (iii) follow, respectively, by purely formal considerations, from the existence and uniqueness portions of assertion (ii) (applied to $\mathsf{Sch}^{\log}_{/S^{\log}}$). $\square$

Corollary 5.6. *Let $S^{\log}$ be a locally Noetherian fs log scheme and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Then $f^{\log}$ is log-like if and only if the following condition holds:*

For any factorization

$$X^{\log} \xrightarrow{g^{\log}} Z_0^{\log} \xrightarrow{h^{\log}} Y^{\log}$$

of $f^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $h^{\log}$ is strict, there exists a unique morphism $r^{\log} : Y^{\log} \to Z_0^{\log}$ in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ such that $g^{\log} = r^{\log} \circ f^{\log}$, and $h^{\log} \circ r^{\log} = \mathrm{id}_{Y^{\log}}$:

$$\begin{array}{ccccc} X^{\log} & \xrightarrow{f^{\log}} & Y^{\log} & = & Y^{\log} \\ \| & & \Big\downarrow{\scriptstyle \exists! r^{\log}} & & \| \\ X^{\log} & \xrightarrow{g^{\log}} & Z_0^{\log} & \xrightarrow[\text{strict}]{h^{\log}} & Y^{\log}. \end{array}$$

In particular, the property that $f^{\log}$ is log-like may be characterized category-theoretically (cf. Corollary 4.6) from the data $(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}, f^{\log})$.

Proof. Corollary 5.6 follows immediately from Lemma 5.5 (i) (ii). $\square$

Next, we give a characterization of the morphisms of monoid objects in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ that are isomorphic as objects of $\mathsf{Mor}(\mathsf{Mon}(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}))$ to $\alpha^{\log}_{S^{\log},\mathbb{A}} : S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to (S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \times)$ (cf. Definition 5.1, Definition 5.2).

Proposition 5.7. *Let*

- $S^{\log}$ *be a locally Noetherian fs log scheme,*
- $A^{\log}$ *a ring object of* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *that is isomorphic as a ring object in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *to* $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$*, and*
- $\alpha^{\log} : M^{\log} \to (A^{\log}, \times)$ *a morphism of monoid objects in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

Write $A^{\times,\log} \hookrightarrow A^{\log}$ *for the strict open immersion from the group of units of the ring object* $A^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *(where we note that since* $A^{\log} \cong S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1$*, it holds that* $A^{\times,\log} \cong S^{\log} \times_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{Z}}$*). Then* $\alpha^{\log} : M^{\log} \to (A^{\log}, \times)$ *is isomorphic as an object of* $\mathsf{Mor}(\mathsf{Mon}(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}))$ *to*

$$\alpha^{\log}_{S^{\log},\mathbb{A}} : S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to (S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \times) \qquad \textit{(cf. Definition 5.2)}$$

if and only if the following conditions hold:

(i) $\alpha^{\log}$ *is log-like.*

(ii) The natural projection $A^{\times,\log} \times^{\blacklozenge}_{A^{\log}} M^{\log} \xrightarrow{\sim} A^{\times,\log}$ *is an isomorphism (where we note that by Lemma 3.4, the fiber product*

$$A^{\times,\log} \times^{\blacklozenge}_{A^{\log}} M^{\log}$$

exists in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$*). Write* $i^{\log}_M : A^{\times,\log} \hookrightarrow M^{\log}$ *for the strict open immersion over* $A^{\log}$ *determined by the natural projections* $A^{\times,\log} \xleftarrow{\sim} A^{\times,\log} \times^{\blacklozenge}_{A^{\log}} M^{\log} \hookrightarrow M^{\log}$.

(iii) For any log-like morphism $f^{\log} : X^{\log} \to A^{\log}$ *in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *such that the natural projection* $A^{\times,\log} \times^{\blacklozenge}_{A^{\log}} X^{\log} \xrightarrow{\sim} A^{\times,\log}$ *is an isomorphism (where we note that by Lemma 3.4, the fiber product* $A^{\times,\log} \times^{\blacklozenge}_{A^{\log}} X^{\log}$ *exists in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$*), there exists a unique morphism* $g^{\log} : M^{\log} \to X^{\log}$ *in* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *such that* $f^{\log} \circ g^{\log} = \alpha^{\log}$*, and* $i^{\log}_X = g^{\log} \circ i^{\log}_M$*:*

$$\begin{tikzcd} A^{\times,\log} \arrow[r, hook, "i^{\log}_X"] \arrow[d, hook, "i^{\log}_M"'] & X^{\log} \arrow[d, "f^{\log}"] \\ M^{\log} \arrow[ur, "\exists! g^{\log}"] \arrow[r, "\alpha^{\log}"'] & A^{\log}, \end{tikzcd}$$

where $i_X^{\log} : A^{\times,\log} \hookrightarrow X^{\log}$ *is the strict open immersion over* $A^{\log}$ *determined by the natural projections* $A^{\times,\log} \xleftarrow{\sim} A^{\times,\log} \times^{\blacklozenge}_{A^{\log}} X^{\log} \hookrightarrow X^{\log}$.

Proof. Necessity follows immediately from Lemma 3.4 and Lemma 5.3. In the remainder of the proof of Proposition 5.7, we prove sufficiency. Assume that conditions (i), (ii), and (iii) hold. Let

$$\alpha_0^{\log} : M_0^{\log} \to (A^{\log}, \times)$$

be a morphism of monoid objects in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ that is isomorphic as an object of $\mathsf{Mor}(\mathsf{Mon}(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}))$ to $\alpha^{\log}_{S,\mathbb{A}} : S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to (S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \times)$. Write $i_0^{\log} : A^{\times,\log} \hookrightarrow M_0^{\log}$ for the strict open immersion determined by the natural projections $A^{\times,\log} \xleftarrow{\sim} A^{\times,\log} \times^{\blacklozenge}_{A^{\log}} M_0^{\log} \hookrightarrow M_0^{\log}$. By condition (iii), we obtain a unique morphism $g^{\log} : M^{\log} \to M_0^{\log}$ such that $\alpha_0^{\log} \circ g^{\log} = \alpha^{\log}$, and $i_0^{\log} = g^{\log} \circ i_M^{\log}$. On the other hand, by conditions (i) (ii) and the necessity portion of Proposition 5.7, we obtain a unique morphism $g_0^{\log} : M_0^{\log} \to M^{\log}$ such that $\alpha^{\log} \circ g_0^{\log} = \alpha_0^{\log}$, and $i_M^{\log} = g_0^{\log} \circ i_0^{\log}$. Then it holds that

$$\begin{aligned}
\alpha^{\log} &= \alpha_0^{\log} \circ g^{\log} = \alpha^{\log} \circ g_0^{\log} \circ g^{\log}, \\
\alpha_0^{\log} &= \alpha^{\log} \circ g_0^{\log} = \alpha_0^{\log} \circ g^{\log} \circ g_0^{\log}, \\
i_M^{\log} &= g_0^{\log} \circ i_0^{\log} = g_0^{\log} \circ g^{\log} \circ i_M^{\log}, \\
i_0^{\log} &= g^{\log} \circ i_M^{\log} = g^{\log} \circ g_0^{\log} \circ i_0^{\log}.
\end{aligned}$$

By the uniqueness portion of condition (iii), $g_0^{\log} \circ g^{\log} = \mathrm{id}_{M^{\log}}$. In a similar vein, by condition (iii) and the necessity portion of Proposition 5.7, $g^{\log} \circ g_0^{\log} = \mathrm{id}_{M_0^{\log}}$. Thus the underlying morphism of log schemes $\alpha^{\log} : M^{\log} \to A^{\log}$ is isomorphic as an object of $\mathsf{Mor}(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}})$ to $\alpha^{\log}_{S^{\log},\mathbb{A}} : S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$. Finally, since $i_M^{\log} = g_0^{\log} \circ i_0^{\log}$, the isomorphism $g_0^{\log}$ is compatible with the monoid object structures on the objects $M^{\log}$ and $M_0^{\log}$ of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. This completes the proof of Proposition 5.7. □

Next, we summarize the properties of fs log schemes and morphisms of fs log schemes that were characterized category-theoretically in the present paper.

Corollary 5.8. *Let $S^{\log}, T^{\log}$ be locally Noetherian fs log schemes,*

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$$

[possibly empty] subsets, and $F : \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ an equivalence of categories. Then the following assertions hold:

(i) Let $f^{\log} : Y^{\log} \to X^{\log}$ be a morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Then the following assertions hold:

(i-a) $X^{\log}$ is an fs log point if and only if $F(X^{\log})$ is an fs log point.

(i-b) $f^{\log}$ is strict if and only if $F(f^{\log})$ is strict.

(i-c) $f^{\log}$ is log-like if and only if $F(f^{\log})$ is log-like.

(ii) Let $A^{\log}$ be a ring object in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ and $\alpha^{\log} : M^{\log} \to (A^{\log}, \times)$ a morphism of monoid objects in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Assume that

- *$A^{\log}$ is isomorphic as a ring object of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ to $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$, and*
- *$F(A^{\log})$ is isomorphic as a ring object of $\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ to $T^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$.*

Then $\alpha^{\log}$ is isomorphic as an object of $\mathsf{Mor}(\mathsf{Mon}(\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}))$ to

$$\alpha^{\log}_{S,\mathbb{A}} : S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to (S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \times)$$

(cf. Definition 5.2) if and only if $F(\alpha^{\log})$ is isomorphic as an object of $\mathsf{Mor}(\mathsf{Mon}(\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}))$ to

$$\alpha^{\log}_{T,\mathbb{A}} : T^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to (T^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \times).$$

Proof. Assertion (i-a) follows immediately from Proposition 3.3. Assertion (i-b) follows immediately from Corollary 4.6. Assertion (i-c) follows immediately from Corollary 5.6. Assertion (ii) follows immediately from assertion (i-c) and Proposition 5.7. □

Let us recall that $\blacklozenge/S^{\log}$ is a set of properties of (**U**-small) schemes over the underlying scheme of $S^{\log}$ (cf. Notations and Conventions — Properties of Schemes and Log Schemes). For any ($\mathbf{U}$-small) scheme S, we shall write $\mathsf{Sch}_{/S}$ for the category of ($\mathbf{U}$-small) S-schemes and $\mathsf{Sch}_{\blacklozenge/S} \subset \mathsf{Sch}_{/S}$ for the full subcategory of the objects of $\mathsf{Sch}_{/S}$ that satisfy every property contained in $\blacklozenge/S$.

Finally, we prove the main result of the present paper.

THEOREM 5.9. *Let $S^{\log}, T^{\log}$ be locally Noetherian fs log schemes,*

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$$

[possibly empty] subsets, $F : \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ an equivalence of categories, and $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ an object. Assume that the following condition holds:

(†) For any equivalence $\underline{F} : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ and object $X \in \mathsf{Sch}_{\blacklozenge/S}$, there exists an isomorphism of schemes $X \xrightarrow{\sim} \underline{F}(X)$ that is functorial with respect to $X \in \mathsf{Sch}_{\blacklozenge/S}$ (for fixed $\underline{F}$).

Then the following assertions hold:

(i) There exists an isomorphism of log schemes $X^{\log} \xrightarrow{\sim} F(X^{\log})$ that is functorial with respect to $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

(ii) Assume that $\blacklozenge = \lozenge$. Then there exists a unique isomorphism of log schemes $S^{\log} \xrightarrow{\sim} T^{\log}$ such that F is isomorphic to the equivalence of categories $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\blacklozenge/T^{\log}}$ induced by composing with this isomorphism of log schemes $S^{\log} \xrightarrow{\sim} T^{\log}$.

PROOF. First, we prove assertion (i). For each $*/Z \in \{\blacklozenge/S, \lozenge/T\}$, write $\mathsf{Sch}^{\log}_{*/Z^{\log}}|_{\mathrm{schlk}} \subset \mathsf{Sch}^{\log}_{*/Z^{\log}}$ for the full subcategory determined by the objects of $\mathsf{Sch}^{\log}_{*/Z^{\log}}|_{\mathrm{schlk}}$ whose structure morphism to $Z^{\log}$ is strict. Then, by Corollary 5.8 (i-b), F induces an equivalence of categories

$$\left(\mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim}\right) \ \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}|_{\mathrm{schlk}} \underset{\sim}{\xrightarrow{F}} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}|_{\mathrm{schlk}} \ \left(\xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}\right).$$

Thus, condition (†) implies that if we write $\underline{F(X^{\log})}$ for the underlying scheme of the log scheme $F(X^{\log})$, then it follows immediately from Lemma 5.5 (i) (iii) that we obtain an isomorphism of schemes

$$X \xrightarrow{\sim} \underline{F(X^{\log})}$$

that is functorial with respect to $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Hence, in particular, for any ring object $A^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, it holds that

> $A^{\log}$ is isomorphic as a ring object of $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ to $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ if and only if
>
> $F(A^{\log})$ is isomorphic as a ring object of $\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ to $T^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$.

Since the map between sets of $(-)$-valued points of $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}$ and $S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ induced by composing with $\alpha^{\log}_{S,\mathbb{A}} : S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}} \to (S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, \times)$ may be naturally identified (cf. Definition 5.2) with the morphism between sheaves of monoids that defines the log structure on "$(-)$", it follows from Corollary 5.8 (ii) that assertion (i) holds.

Next, we prove the existence portion of assertion (ii). Write

$$\left\{\varphi_{X^{\log}} : X^{\log} \xrightarrow{\sim} F(X^{\log})\right\}_{X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}}$$

for the family of functorial isomorphisms of log schemes discussed in assertion (i) and $p^{\log} : F(S^{\log}) \xrightarrow{\sim} T^{\log}$ for the structure morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/T^{\log}}$. Then, it follows immediately from the functoriality of the family $\{\varphi_{X^{\log}}\}_{X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}}$ that F is isomorphic to the equivalence obtained by composing with $\varphi_{S^{\log}} \circ p^{\log} : S^{\log} \xrightarrow{\sim} T^{\log}$.

Finally, we prove the uniqueness portion of assertion (ii). Let

$$f^{\log}, g^{\log} : S^{\log} \xrightarrow{\sim} T^{\log}$$

be isomorphisms. For any object $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$, write $p^{\log}_{X^{\log}} : X^{\log} \to S^{\log}$ for the structure morphism. Let

$$\psi = \{\psi^{\log}_{X^{\log}} : X^{\log} \xrightarrow{\sim} X^{\log}\}_{X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}}$$

be a family of isomorphisms of log schemes such that $f^{\log} \circ p^{\log}_{X^{\log}} = g^{\log} \circ p^{\log}_{X^{\log}} \circ \psi^{\log}_{X^{\log}}$, and $\psi^{\log}_{X^{\log}}$ is functorial with respect to $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$. Then $f^{\log} = g^{\log} \circ \psi^{\log}_{S^{\log}}$. By the functoriality of ψ, for any strict open immersion $p^{\log}_{U^{\log}} : U^{\log} \hookrightarrow S^{\log}$, it holds that $\psi^{\log}_{S^{\log}} \circ p^{\log}_{U^{\log}} = p^{\log}_{U^{\log}} \circ \psi^{\log}_{U^{\log}}$. Hence, in particular, the underlying morphism of topological spaces of $\psi^{\log}_{S^{\log}}$ is equal to $\mathrm{id}_{|S|}$ (cf. [John, Theorem 7.24]). By the functoriality of ψ, for each $i \in \{0, 1, \infty\}$, it holds that $(\psi^{\log}_{S^{\log} \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}}}) \circ (\mathrm{id}_{S^{\log}} \times i) = (\mathrm{id}_{S^{\log}} \times i) \circ \psi^{\log}_{S^{\log}}$ (where we regard i as the morphism of schemes $\mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$ corresponding to $i \in \mathbb{Z} \cup \{\infty\}$):

$$\begin{array}{ccccc}
S^{\log} & \xrightarrow{\mathrm{id}_{S^{\log}} \times i} & S^{\log} \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}} & \hookleftarrow & S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \\
{\scriptstyle \psi^{\log}_{S^{\log}}} \downarrow \wr & & \wr \downarrow {\scriptstyle \psi^{\log}_{S^{\log} \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}}}} & & \wr \downarrow {\scriptstyle \psi^{\log}_{S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}}} \\
S^{\log} & \xrightarrow{\mathrm{id}_{S^{\log}} \times i} & S^{\log} \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}} & \hookleftarrow & S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}.
\end{array}$$

Hence $\psi^{\log}_{S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}} = \psi^{\log}_{S^{\log}} \times \mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}}$. In particular, by the functoriality of ψ, for any object $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ and any element $s \in \Gamma(X, \mathcal{O}_X)$, if we write $\tilde{s}^{\log} : X^{\log} \to S^{\log} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ for the morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ corresponding to s, then $\tilde{s}^{\log} \circ \psi^{\log}_{X^{\log}} = (\psi^{\log}_{S^{\log}} \times \mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}}) \circ \tilde{s}^{\log}$. This implies that $\psi^{\#}_{X^{\log}}(X)(s) = s \in \Gamma(X, \mathcal{O}_X)$, and hence that the underlying morphism of schemes of $\psi^{\log}_{S^{\log}}$ is equal to id_S. Next, observe that by the functoriality of ψ, it follows from Lemma 5.3 that $\psi^{\log}_{S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}} = \psi^{\log}_{S^{\log}} \times \mathrm{id}_{\mathbb{A}^{1,\log}_{\mathbb{Z}}}$. In particular, by the functoriality of ψ, for any object $Y^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ and any element $t \in \Gamma(Y, \mathcal{M}_Y)$, if we write $\tilde{t}^{\log} : Y^{\log} \to S^{\log} \times_{\mathbb{Z}} \mathbb{A}^{1,\log}_{\mathbb{Z}}$ for the morphism in $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ corresponding to t, then $\tilde{t}^{\log} \circ \psi^{\log}_{Y^{\log}} = (\psi^{\log}_{S^{\log}} \times \mathrm{id}_{\mathbb{A}^{1,\log}_{\mathbb{Z}}}) \circ \tilde{t}^{\log}$. This implies that $\psi^{\flat}_{Y^{\log}}(Y)(t) = t \in \Gamma(Y, \mathcal{M}_Y)$, and hence that $\psi^{\log}_{S^{\log}} = \mathrm{id}_{S^{\log}}$. Thus $f^{\log} = g^{\log}$. This completes the proof of Theorem 5.9. $\square$

THEOREM 5.10. *Let*

$$S^{\log}, T^{\log}$$

be locally Noetherian fs log schemes,

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}, \mathrm{ft}\}$$

[possibly empty] subsets, and $F : \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ *an equivalence of categories. Assume that one of the following conditions (A), (B) holds:*

(A) $\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$*, and the underlying schemes of* $S^{\log}$ *and* $T^{\log}$ *are normal.*

(B) $\blacklozenge = \lozenge = \{\mathrm{ft}\}$.

Then the following assertions hold:

(i) *Let* $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *be an object. Then there exists an isomorphism of log schemes* $X^{\log} \xrightarrow{\sim} F(X^{\log})$ *that is functorial with respect to* $X^{\log} \in \mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$.

(ii) *Assume that* $\blacklozenge = \lozenge$. *Then there exists a unique isomorphism of log schemes* $S^{\log} \xrightarrow{\sim} T^{\log}$ *such that* F *is isomorphic to the equivalence of categories* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}} \xrightarrow{\sim} \mathsf{Sch}^{\log}_{\blacklozenge/T^{\log}}$ *induced by composing with this isomorphism of log schemes* $S^{\log} \xrightarrow{\sim} T^{\log}$.

Proof. If condition (A) holds, then it follows immediately from Theorem 5.9 (i) (ii) and [YJ, Corollary 6.24] that assertions (i) and (ii) hold. If condition (B) holds, then it follows immediately from Theorem 5.9 (i) (ii) and (the proof of) [Mzk04, Theorem 1.7 (ii)] that assertions (i) and (ii) hold. This completes the proof of Theorem 5.10. □

Corollary 5.11. *Let* $S^{\log}, T^{\log}$ *be locally Noetherian normal fs log sch-emes and*

$$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$$

subsets such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$, *and* $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \lozenge$. *If the categories* $\mathsf{Sch}^{\log}_{\blacklozenge/S^{\log}}$ *and* $\mathsf{Sch}^{\log}_{\lozenge/T^{\log}}$ *are equivalent, then* $\blacklozenge = \lozenge$, *and* $S^{\log} \cong T^{\log}$.

Proof. Corollary 5.11 follows immediately from Corollary 4.7 and Theorem 5.10 (ii). □

Appendix A. A Lemma of Nakayama

In this appendix, we prove an extension of [Nak, Lemma 2.2.6 (i)] (cf. Lemma 1.7) to the case where M, L are quasi-integral, and N is an arbitrary sharp monoid (cf. Corollary A.5).

LEMMA A.1. *Let $f : N \to M$ and $g : N \to L$ be morphisms of (arbitrary) monoids. Write $P \stackrel{\mathrm{def}}{:=} M \sqcup_N L$; $i_M : M \to P$ and $i_L : L \to P$ for the natural inclusions; $\pi : M \times L \to P$ for the morphism determined by i_M and i_L. Then π is surjective.*

PROOF. Write $P' \stackrel{\mathrm{def}}{:=} \mathrm{Im}(\pi)$, $i : P' \hookrightarrow P$ for the inclusion morphism, and for each $* \in \{M, L\}$, $i'_* : * \to P'$ for the morphism induced by i_*. Then, by the universality of push-outs, there exists a unique morphism of monoids $r : P \to P'$ such that $r \circ i_M = i'_M$, and $r \circ i_L = i'_L$. This implies that $i \circ r = \mathrm{id}_P$. Thus i is surjective, which implies that π is surjective. This completes the proof of Lemma A.1. □

LEMMA A.2. *Let L, M, N be sharp monoids and $f : N \to M$, $g : N \to L$ local morphisms of monoids. Write $P \stackrel{\mathrm{def}}{:=} M \sqcup_N L$; $i_M : M \to P$ and $i_L : L \to P$ for the natural inclusions; $\pi : M \times L \to P$ for the morphism determined by i_M and i_L. Then i_M, i_L, and π are local, and P is sharp.*

PROOF. Write $(\mathbb{F}_2, \times)$ for the underlying multiplicative monoid of the finite field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ (so $\mathbb{F}_2^\times = \{1\}$). For any monoid $*$, write $\varphi_* : * \to (\mathbb{F}_2, \times)$ for the unique local morphism of monoids. Since f and g are local, it holds that $\varphi_M \circ f = \varphi_N = \varphi_L \circ g$. Hence there exists a unique morphism $h : P \to (\mathbb{F}_2, \times)$ such that $\varphi_M = h \circ i_M$, and $\varphi_L = h \circ i_L$. Thus $\varphi_{M \times L} = h \circ \pi$. Since π is surjective (cf. Lemma A.1), $h^{-1}(1) = 0$. This implies that P is sharp, that $h = \varphi_P$, and that i_M, i_L, and π are local. This completes the proof of Lemma A.2. □

DEFINITION A.3. Let M be a monoid. Then we shall write $\eta_M : M \to M^{\mathrm{gp}}$ for the natural morphism to the groupification.

LEMMA A.4. *Let L, M, N be sharp monoids and $f : N \to M, g : N \to L$ local morphisms of monoids. Then the following assertions are equivalent:*

(i) *$M \sqcup_N L$ is quasi-integral.*

(ii) *For any elements $m \in M$, $l \in L$, and $n \in N^{\mathrm{gp}}$, if $f^{\mathrm{gp}}(n) = \eta_M(m)$, and $-g^{\mathrm{gp}}(n) = \eta_L(l)$, then $m = 0$, and $l = 0$.*

(iii) *M, L are quasi-integral, and, moreover, for any element $n \in N^{\mathrm{gp}}$, if $f^{\mathrm{gp}}(n) \in M^{\mathrm{int}}$, and $-g^{\mathrm{gp}}(n) \in L^{\mathrm{int}}$, then $f^{\mathrm{gp}}(n) = 0$, and $g^{\mathrm{gp}}(n) = 0$.*

Proof. Write $P \overset{\text{def}}{:=} M \sqcup_N L$; $i_M : M \to P$ and $i_L : L \to P$ for the natural inclusions; $\pi : M \times L \to P$ for the morphism determined by i_M and i_L.

To prove that (i) implies (ii), assume that P is quasi-integral. Let $m \in M$, $l \in L$, and $n \in N^{\mathrm{gp}}$ be elements such that $f^{\mathrm{gp}}(n) = \eta_M(m)$, and $-g^{\mathrm{gp}}(n) = \eta_L(l)$. Then

$$\eta_P(\pi(m,l)) = \pi^{\mathrm{gp}}(\eta_M(m), \eta_L(l)) = \pi^{\mathrm{gp}}(f^{\mathrm{gp}}(n), -g^{\mathrm{gp}}(n)) = 0.$$

Since P is quasi-integral, $\pi(m,l) = 0$. Moreover, since π is local (cf. Lemma A.2), we conclude that $m = 0$, and $l = 0$. This completes the proof of the implication (i) $\Rightarrow$ (ii).

Next, we prove that (ii) implies (i). Assume that assertion (ii) holds. Let $p \in P$ be an element such that $\eta_P(p) = 0$. Since π is surjective (cf. Lemma A.1), there exist elements $m \in M$ and $l \in L$ such that $p = \pi(m,l)$. Then

$$\pi^{\mathrm{gp}}(\eta_M(m), \eta_L(l)) = \eta_P(\pi(m,l)) = \eta_P(p) = 0.$$

Hence there exists an element $n \in N^{\mathrm{gp}}$ such that $\eta_M(m) = f^{\mathrm{gp}}(n)$, and $\eta_L(l) = -g^{\mathrm{gp}}(n)$. Thus, by assertion (ii), it holds that $m = 0$, and $l = 0$. This implies that $p = 0$, i.e., that P is quasi-integral. This completes the proof of the implication (ii) $\Rightarrow$ (i).

Finally, by the definition of the notion of a quasi-integral monoid, assertion (ii) is equivalent to assertion (iii). This completes the proof of Lemma A.4. $\square$

Corollary A.5 (cf. [Nak, Lemma 2.2.6 (i)], Lemma 1.7). *Let L, M, N be sharp monoids and $f : N \to M, g : N \to L$ local morphisms of monoids. Assume that M and L are quasi-integral. Then the following assertions are equivalent:*

(i) $M \sqcup_N L$ is quasi-integral.

(ii) For any element $n \in N^{\mathrm{gp}}$, if $f^{\mathrm{gp}}(n) \in M^{\mathrm{int}}$, and $-g^{\mathrm{gp}}(n) \in L^{\mathrm{int}}$, then $f^{\mathrm{gp}}(n) = 0$, and $g^{\mathrm{gp}}(n) = 0$.

Proof. Corollary A.5 follows immediately from Lemma A.4 (i) $\Leftrightarrow$ (iii). $\square$

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