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Elementary Proof of Representation of Submodular Function as Supremum of Measures on σ-Algebra with Totally Ordered Generating Class

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Abstract. We give an alternative proof of a fact that a finite continuous non-decreasing submodular set function on a measurable space can be expressed as a supremum of measures dominated by the function, if there exists a chain (class of sets which is totally ordered with respect to inclusion) which generates the sigma-algebra of the space. The proof is elementary in the sense that the measure attaining the supremum in the claim is constructed by a standard extension theorem of measures. As a consequence, unique existence of the supremum attaining measure follows. A Polish space is an example of a measurable space which has a chain that generates the Borel sigma-algebra.

1. Introduction

Let $(\Omega, \mathcal{F})$ be a measurable space, namely, a σ-algebra $\mathcal{F}$ is a class of subsets of Ω and is closed under complements and countable unions. For a measurable set $A \in \mathcal{F}$ denote by $\mathcal{F}|_A$, the class of measurable sets restricted to A, and denote the set of finite measures on the measurable space $(A, \mathcal{F}|_A)$ by $\mathcal{M}(A)$.

For a set function $v: \ \mathcal{F} \to \mathbb{R}$ and a measurable set $A \in \mathcal{F}$, let $\mathcal{C}_{-,v}(A)$ be a class of measures defined by

$$\mathcal{C}_{-,v}(A) = \{\mu \in \mathcal{M}(A) \mid \mu(A) = v(A),\ \mu(B) \leqq v(B),\ B \in \mathcal{F}|_A\}. \tag{1.1}$$

If

$$v(B) = \sup_{\mu \in \mathcal{C}_{-,v}(A)} \mu(B), \tag{1.2}$$

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holds for all $A, B \in \mathcal{F}$ satisfying $B \subset A$, then it is easy to see that

$$v(A) + v(B) \geqq v(A \cup B) + v(A \cap B), \quad A, B \in \mathcal{F}, \tag{1.3}$$

holds (see Proposition 1 in §2). A set function which satisfies (1.3) is called a submodular function.

The converse that a submodular function satisfies (1.2) is also known to hold under mild and natural assumptions. In fact, a proof in [5] proves existence of a measure μ which satisfies $v(B) = \mu(B)$, so that the supremum in (1.2) is attained.

The formula (1.2) has significance in the related fields of study, such as coherent risk measures in mathematical finance and cores of convex games in cooperative game theory. In view of wide interest in this formula, it may be worthwhile to find an alternative elementary proof.

In contrast to a proof in [5] which seeks for wide applicability even beyond measurable spaces, we keep ourselves as close as possible to measures, except for (1.3) which characterizes the submodular property. See the definitions in §2 for detail. Our proof in §3 is elementary in the sense that we prove the existence of μ satisfying $v(B) = \mu(B)$ by the extension theorem of a measure on a finite algebra to the σ-algebra generated by the finite algebra, in contrast to the proof in [5] which uses the Hahn–Banach Theorem. As a consequence, uniqueness of μ does not follow in general in the latter proof, while we have certain uniqueness result for μ (see Theorem 3 in §2). In this uniqueness result, it is essential that there exists a chain $\mathcal{I} \subset \mathcal{F}$ (a class of sets which is totally ordered with respect to inclusion) which generates the σ-algebra $\mathcal{F}$. Examples of measurable spaces $(\Omega, \mathcal{F})$ which have such chains are given in §4. Polish spaces are in the examples, hence our main result holds for spaces which are extensively used in the theory of stochastic processes.

We also note that $L^\infty([0,1))$ is not separable, hence Proposition 7 in §4.3 is not applicable. Whether our main result is applicable to a non-separable metric space such as $L^\infty([0,1))$, is left as an open problem.

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2. Definition and Main Result

Throughout this paper, we assume that a set function $v: \mathcal{F} \to \mathbb{R}$ satisfies the following conditions (2.1), (2.2) and (2.3):

$$v(A) \leqq v(B),\ A \subset B,\ A, B \in \mathcal{F}, \tag{2.1}$$

$$\lim_{n\to\infty} v(A_n) = v(\bigcup_{n\in\mathbb{N}} A_n), \quad A_1 \subset A_2 \subset \cdots, \quad A_n \in \mathcal{F},\ n = 1, 2, 3, \ldots, \tag{2.2}$$

$$\lim_{n\to\infty} v(A_n) = v(\bigcap_{n\in\mathbb{N}} A_n) \quad A_1 \supset A_2 \supset \cdots, \quad A_n \in \mathcal{F},\ n = 1, 2, 3, \ldots. \tag{2.3}$$

In measure theory, a finite measure is defined to be a non-negative real valued σ-additive set function defined on a σ-algebra $\mathcal{F}$. Equivalently, we can say that a real valued set function $\mu: \mathcal{F} \to \mathbb{R}$ is a finite measure if μ satisfies $\mu(\emptyset) = 0$ and (2.1), (2.2), (2.3), and

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B), \quad A, B \in \mathcal{F}. \tag{2.4}$$

In analogy to the terminology of measure theory, in this paper we call a set function v non-decreasing if it satisfies (2.1), and continuous if it satisfies both (2.2) and (2.3).

Submodular and supermodular functions are defined by replacing the equality (2.4) with inequalities. In this paper, we say that a set function $v: \mathcal{F} \to \mathbb{R}$ is submodular, if v is non-decreasing, continuous, $v(\emptyset) = 0$, and satisfies (1.3), and $v: \mathcal{F} \to \mathbb{R}$ is supermodular, if v is non-decreasing, continuous, $v(\emptyset) = 0$, and satisfies

$$v(A) + v(B) \leqq v(A \cup B) + v(A \cap B), \quad A, B \in \mathcal{F}. \tag{2.5}$$

(Incidentally, while $\mu(\emptyset) = 0$ is crucial for a measure μ to be additive, submodular and supermodular functions generally lack this property. Imposing $v(\emptyset) = 0$ is thus only for notational simplicity, and the formula in this paper can easily be generalized to the case $v(\emptyset) \neq 0$ by the replacements $v(A) \mapsto v(A) - v(\emptyset)$.)

To keep the definitions close to that of measures, we assume the non-decreasing property (2.1) and continuity (2.2) and (2.3) in the definitions of submodular and supermodular functions. We remark that the set of conditions (2.2) and (2.3) is strong. It is known that with this continuity condition, the core only consists of sigma-additive measures and is weakly compact [8, 2], and for submodular functions satisfying (2.2) and not necessarily the stronger (2.3) sigma-additive measures are dense in the core consisting of finitely additive measures [7, 3]. See [4, Chapter 7] for details and more results.

We now move on to the relation between (1.2) and submodularity.

PROPOSITION 1. *Let $(\Omega, \mathcal{F})$ be a measurable space, and $v : \mathcal{F} \to \mathbb{R}$ a set function. If (1.2) holds for all $A, B \in \mathcal{F}$ satisfying $B \subset A$, then v is a submodular function.*

PROOF. Assume (1.2). Since a measure μ is non-decreasing and continuous, $v(B) = \sup\limits_{\mu \in \mathcal{C}_{-,v}(\Omega)} \mu(B)$ obtained by $A = \Omega$ in (1.2) implies that v is also non-decreasing and continuous.

To prove (1.3), let $A, B \in \mathcal{F}$ and substitute A in (1.2) by $A \cup B$ to obtain $v(B) = \sup\limits_{\mu \in \mathcal{C}_{-,v}(A \cup B)} \mu(B)$ and $v(A) = \sup\limits_{\mu \in \mathcal{C}_{-,v}(A \cup B)} \mu(A)$, which further imply $v(B) \geqq \mu(B)$ and $v(A) \geqq \mu(A)$ for all $\mu \in \mathcal{C}_{-,v}(A \cup B)$. Also (1.1) implies $v(A \cup B) = \mu(A \cup B)$ for all $\mu \in \mathcal{C}_{-,v}(A \cup B)$. Finally, $v(A \cap B) = \sup\limits_{\mu \in \mathcal{C}_{-,v}(A \cup B)} \mu(A \cap B)$ implies that for any $\epsilon > 0$ there exists $\mu \in \mathcal{C}_{-,v}(A \cup B)$ such that $v(A \cap B) \leqq \mu(A \cap B) + \epsilon$. These equality and inequalities imply, with (2.4),

$$\begin{aligned} &v(A \cup B) + v(A \cap B) - v(A) - v(B) \\ &\leqq \mu(A \cup B) + \mu(A \cap B) + \epsilon - \mu(A) - \mu(B) = \epsilon. \end{aligned}$$

ϵ can be any positive constant, hence (1.3) follows. □

Concerning the converse of Proposition 1, note first that the definition (1.1) of $\mathcal{C}_{-,v}(A)$ implies $v(B) \geqq \sup\limits_{\mu \in \mathcal{C}_{-,v}(A)} \mu(B)$ for any $A, B \in \mathcal{F}$ satisfying $B \subset A$. Hence if there exists $\mu \in \mathcal{C}_{-,v}(A)$ such that $\mu(B) = v(B)$ then (1.2) holds for this pair (A, B). The main result of this paper is on the construction of such μ. To state the main theorem, we consider the following set of conditions on a class of measurable sets $\mathcal{I} \subset \mathcal{F}$;

$$
\left\{
\begin{array}{l}
\emptyset \in \mathcal{I},\ \ \Omega \in \mathcal{I}, \\
\mathcal{I} \text{ is a chain, i.e., for all } I_1, I_2 \in \mathcal{I} \text{ either } I_1 \subset I_2 \text{ or } I_2 \subset I_1 \,, \\
\sigma[\mathcal{I}] = \mathcal{F},\ \text{ where } \sigma[\mathcal{I}] \text{ denotes the smallest} \\
\sigma\text{-algebra containing } \mathcal{I}.
\end{array}
\right. \tag{2.6}
$$

For a class $\mathcal{I} \subset \mathcal{F}|_A$ denote by $\sigma_A[\mathcal{I}]$ the smallest σ-algebra in the measurable space $(A, \mathcal{F}|_A)$ containing $\mathcal{I}$. With this notation we have $\sigma[\mathcal{I}] = \sigma_\Omega[\mathcal{I}]$ for $\mathcal{I} \subset \mathcal{F}$. The following elementary property will be crucial in the proof of the main result to come. For a class of measurable sets $\mathcal{I} \subset \mathcal{F}$ and a pair of measurable sets $A, B \in \mathcal{F}$ satisfying $B \subset A$, denote the restriction of $\mathcal{I}$ on A by $\mathcal{I}|_A := \{I \cap A \mid I \in \mathcal{I}\}$ and the insertion of B into $\mathcal{I}|_A$ by

$$
\mathcal{I}|_{A,B} = \{B \cap I \mid I \in \mathcal{I}|_A\} \cup \{B \cup I \mid I \in \mathcal{I}|_A\} \tag{2.7}
$$

In particular, $\mathcal{I}|_{A,A} = \mathcal{I}|_A$ if $\Omega \in \mathcal{I}$.

LEMMA 2. *Let $(\Omega, \mathcal{F})$ be a measurable space and assume that a class of measurable sets $\mathcal{I} \subset \mathcal{F}$ satisfies (2.6). Then for any pair of measurable sets A and B satisfying $B \subset A$, $\mathcal{I}|_{A,B}$ of (2.7) satisfies (2.6) with the total space $(\Omega, \mathcal{F})$ replaced by $(A, \mathcal{F}|_A)$.*

PROOF. Since $\mathcal{I}$ satisfies (2.6), it suffices to prove $\sigma_A[\mathcal{I}|_{A,B}] \supset \mathcal{F}|_A$, all the other properties in (2.6) with the substitution $(\Omega, \mathcal{F}) = (A, \mathcal{F}|_A)$ being direct consequences of the assumptions.

Put $\mathcal{G} = \{F \in \mathcal{F} \mid F \cap A \in \sigma_A[\mathcal{I}|_{A,B}]\}$. Then as in a standard elementary argument in measure theory. $\sigma_A[\mathcal{I}|_{A,B}] \supset \mathcal{F}|_A$ is equivalent to $\mathcal{G} \supset \mathcal{F}$. Since by assumption $\mathcal{F} = \sigma[\mathcal{I}]$, it suffices to prove $\mathcal{G} \supset \mathcal{I}$ and that it is a σ-algebra in Ω. The latter is a straightforward consequence of the definition of $\mathcal{G}$ and that $\sigma_A[\mathcal{I}|_{A,B}]$ is a σ-algebra in A.

Finally, to prove $\mathcal{G} \supset \mathcal{I}$, let $I \in \mathcal{I}$. Then $B = B \cap A \in \mathcal{I}|_{A,B}$ and $B \cap I = B \cap (I \cap A) \in \mathcal{I}|_{A,B}$ imply $B \cap I^c = B \cap (A \cap (B \cap I)^c) \in \sigma_A[\mathcal{I}|_{A,B}]$, hence, with $B \cup (I \cap A) \in \mathcal{I}|_{A,B}$, it follows that

$$
A \cap I = (B \cup (I \cap A)) \cap (A \cap (B \cap I^c)^c) \in \sigma_A[\mathcal{I}|_{A,B}],
$$

which implies $I \in \mathcal{G}$. $\square$

We are ready to state the main theorem. Note that as stated at the beginning of §2, we assume that v is non-decreasing, continuous, and $v(\emptyset) = 0$, when we say that v is a submodular function.

THEOREM 3. *Let $(\Omega, \mathcal{F})$ be a measurable space, and assume that there exists $\mathcal{I} \subset \mathcal{F}$ satisfying (2.6). Then for any submodular function $v: \mathcal{F} \to \mathbb{R}$ and for any $A, B \in \mathcal{F}$ satisfying $B \subset A$, there exists unique $\mu \in \mathcal{C}_{-,v}(A)$ such that*

$$v(I) = \mu(I), \ I \in \mathcal{I}|_{A,B}\,, \tag{2.8}$$

where $\mathcal{C}_{-,v}(A)$ is as in (1.1). In particular, $v(B) = \mu(B)$ holds.

Consequently, (1.2) holds for all $A, B \in \mathcal{F}$ satisfying $B \subset A$.

We will prove Theorem 3 in §3.

The corresponding results in this paper for supermodular functions hold through a well-known correspondence

$$\tilde{v}(A) = v(\Omega) - v(A^c) + v(\emptyset), \ A \in \mathcal{F}, \tag{2.9}$$

which gives a non-decreasing continuous supermodular (resp., submodular) function $\tilde{v}$ from a non-decreasing continuous submodular (resp., supermodular) function v satisfying $\tilde{v}(\Omega) = v(\Omega$ and $\tilde{v}(\emptyset) = v(\emptyset)$. In analogy to $\mathcal{C}_{-,v}(A)$ in (1.1), let

$$\mathcal{F}|_A = \{B \in \mathcal{F} \mid B \subset A\} = \{B \cap A \mid B \in \mathcal{F}\} \tag{2.10}$$

and let $\mathcal{C}_{+,v}(A)$ be a class of measures defined by

$$\mathcal{C}_{+,v}(A) = \{\mu \in \mathcal{M}(A) \mid \mu(A) = v(A), \ \mu(B) \geqq v(B), \ B \in \mathcal{F}|_A\}, \tag{2.11}$$

and consider as an analog of (1.2)

$$v(B) = \inf_{\mu \in \mathcal{C}_{+,v}(A)} \mu(B). \tag{2.12}$$

COROLLARY 4. *Let $(\Omega, \mathcal{F})$ be a measurable space and $v: \mathcal{F} \to \mathbb{R}$ a set function.*

If (2.12) holds for all $A, B \in \mathcal{F}$ satisfying $B \subset A$, then v is a supermodular function.

Conversely, assume further that there exists $\mathcal{I} \subset \mathcal{F}$ satisfying (2.6). Then for any supermodular function $v: \mathcal{F} \to \mathbb{R}$ and for any $A, B \in \mathcal{F}$ satisfying $B \subset A$, there exists unique $\mu \in \mathcal{C}_{+,v}(A)$ such that (2.8) holds, where $\mathcal{C}_{+,v}(A)$ is as in (2.11). In particular, $v(B) = \mu(B)$ holds. Consequently, (2.12) holds for all $A, B \in \mathcal{F}$ satisfying $B \subset A$.

Theorem 3 and Corollary 4 imply corresponding results on Choquet integrable functions. For a non-decreasing, continuous, and finite (real-valued) set function $v: \mathcal{F} \to \mathbb{R}$ on a measurable space $(\Omega, \mathcal{F})$ and a measurable function $f: \Omega \to \mathbb{R}$, we define

$$v(f) = \lim_{y \to -\infty} \left(y\, v(\Omega) + \int_y^{\infty} v(\{\omega \in \Omega \mid f(\omega) > z\})\, dz \right) \tag{2.13}$$

whenever the right-hand side is a real value and we say that f is v-integrable. If either the Lebesgue integration or the limit diverges in the right-hand side of (2.13) we do not define $v(f)$. If $v(f)$ of (2.13) is defined it is equal to the Choquet integration, or the asymmetric integral in terms of [5, Chap. 5]. If in addition v is submodular, it is known [5, Prop. 10.3] that

$$v(f) = \sup_{\mu \in \mathcal{C}_{-,v}(\Omega)} \int_\Omega f\, d\mu \tag{2.14}$$

holds. (In the reference, the statements are for finitely additive measures and algebras, but the corresponding results hold for (σ-additive) measures when working on measurable space with σ-algebra as we do here.) The functional ρ defined by $\rho(f) = v(-f)$ is called the coherent risk measure in mathematical finance and (2.14) is known to be a basic formula [1, 3, 6].

The definition of $\mathcal{C}_{-,v}(\Omega)$ and monotonicity of Choquet integration is known to imply $v(f) \leqq \sup_{\mu \in \mathcal{C}_{-,v}(\Omega)} \int_\Omega f\, d\mu$. Hence, in a similar spirit as in Theorem 3, to prove (2.14) it is sufficient to prove the existence of a measure $\mu \in \mathcal{C}_{-,v}(\Omega)$ such that $v(f) = \int_\Omega f\, d\mu$ holds. A proof in [5] for the existence of such μ starts with considering a maximal set of commonotonic v-integrable functions $\mathcal{X}$. With aid of the Hahn–Banach Theorem, the domain of the functional v is then extended to to the linear space $\mathcal{V}$ generated by $\mathcal{X}$, as a linear non-negative functional $\tilde{v}$. Then a set function μ defined by $\mu(A) = \tilde{v}(\mathbf{1}_A)$, where $\mathbf{1}_A$ denotes the indicator function of the set A, is proved to be a measure. Since the indicator functions of the level sets $I_{f,z} = \{\omega \in \Omega \mid f(\omega) > z\}$, $z \in \mathbb{R}$, are commonotonic with f, $\mu(I_{f,z}) = v(I_{f.z})$ holds, so that the definition of Choquet integration implies $v(f) = \int_\Omega f\, d\mu$. This μ gives equality $v(B) = \mu(B)$ in Theorem 3, by choosing $f = \mathbf{1}_A$.

We note that since the above-outlined proof in the reference uses the Hahn–Banach Theorem, uniqueness of measure μ which attains the equality in (1.2) does not follow in general. The unique existence claims in Theorem 3 and Corollary 4 are consequences of our proof in the next section which uses the extension theorem of measures from measures on finitely additive algebra.

As a simple and direct application of Theorem 3, we can state a following result for the Choquet integration $v(f)$ in (2.14). Denote the class of level sets by $\mathcal{I}_f = \{I_{f,z} \mid z \in \mathbb{R}\}$. Then Theorem 3 implies the following.

COROLLARY 5. *Let $(\Omega, \mathcal{F})$ be a measurable space, $v : \mathcal{F} \to \mathbb{R}$ be a submodular function, and $f : \Omega \to \mathbb{R}$ be a v-integrable function. If the class of level sets $\mathcal{I}_f$ satisfies (2.6) then there exists a unique $\mu \in \mathcal{C}_{-,v}(\Omega)$ such that $v(I_{f,z}) = \mu(I_{f,z})$, $z \in \mathbb{R}$. Moreover, $v(f) = \int_\Omega f\, d\mu$ holds for this μ, hence (2.14) also holds.*

3. Proof of Main Theorem

PROOF OF THEOREM 3. Assume $A, B \in \mathcal{F}$ satisfy $B \subset A$. We fix A and B throughout the proof.

Let $\mathcal{F}|_A$ be the restriction (2.10) to A of $\mathcal{F}$, and $\mathcal{I}|_{A,B}$ be the insertion (2.7) of B to $\mathcal{I}|_A$. Lemma 2 then implies that $\emptyset, B, A \in \mathcal{I}|_{A,B}$ and that $\mathcal{I}|_{A,B}$ is totally ordered with respect to inclusion with $\sigma[\mathcal{I}|_{A,B}] = \mathcal{F}|_A$.

Denote by $\mathcal{J}|_{A,B}$ the finitely additive class generated by $\mathcal{I}|_{A,B}$. Namely, $\mathcal{J}|_{A,B}$ is an algebra of sets satisfying $\mathcal{I}|_{A,B} \subset \mathcal{J}|_{A,B} \subset \mathcal{F}|_A$, is closed under complement and union, and is the smallest class with these properties. Since $\mathcal{I}|_{A,B}$ is totally ordered with respect to inclusion, we have an explicit representation

$$\mathcal{J}|_{A,B} = \{\bigcup_{i=1}^{n}(C_i \cap D_i^c) \mid C_1 \supset D_1 \supset C_2 \supset \cdots \supset D_n\,, \quad C_i,\ D_i \in \mathcal{I}|_{A,B}\,,\ i = 1, 2, \ldots, n,\ \ n = 1, 2, 3, \ldots\}. \tag{3.1}$$

Using the notation in the right-hand side of (3.1), define a set function $\mu_{A,B} :\ \mathcal{J}|_{A,B} \to \mathbb{R}$ by

$$\mu_{A,B}(\bigcup_{i=1}^{n}(C_i \cap D_i^c)) = \sum_{i=1}^{n}(v(C_i) - v(D_i)). \tag{3.2}$$

The definition implies that $\mu_{A,B}$ is finitely additive. Note that $\mu_{A,B}$ is well-defined as a set function on $\mathcal{J}|_{A,B}$ because the right-hand side of (3.2) has the same value for different expressions of a set $J \in \mathcal{J}|_{A,B}$. The reason is just as in the case of the Lebesgue measure on a line. For example, let $n = 2$ and $D_1 = C_2$ in (3.1). Then we see that $J = (C_1 \cap D_1^c) \cup (C_2 \cap D_2^c)$ has another expression $J = C_1 \cap D_2^c$, but since $D_1 = C_2$, we have

$$v(C_1) - v(D_1) + v(C_2) - v(D_2) = v(C_1) - v(D_2),$$

and the right-hand sides of (3.2) for the two expressions of this $J \in \mathcal{J}|_{A,B}$ give the same value. In general, if $J \in \mathcal{J}|_{A,B}$ has 2 different expressions of the form (3.1), say $J = \bigcup_{i=1}^{n}(C_i \cap D_i^c) = \bigcup_{i=1}^{n'}(C_i' \cap D'^c_i)$, one finds $D_i = C_{i+1}$ or $D_i' = C'_{i+1}$ for some i, hence by induction in $\max\{n, n'\}$, we can conclude that $\sum_{i=1}^{n}(v(C_i) - v(D_i)) = \sum_{i=1}^{n'}(v(C_i') - v(D'^c_i))$. Therefore (3.2) defines $\mu_{A,B} :\ \mathcal{J}|_{A,B} \to \mathbb{R}$.

Lemma 6. *The finitely additive measure $\mu_{A,B} :\ \mathcal{J}|_{A,B} \to \mathbb{R}$ defined on the finitely additive class $\mathcal{J}|_{A,B}$ satisfies $\mu_{A,B}(J) \leqq v(J),\ J \in \mathcal{J}|_{A,B}$, and, in particular, $\mu_{A,B}(I) = v(I),\ I \in \mathcal{I}|_{A,B}$,*

Proof. If $I \in \mathcal{I}|_{A,B}$ we can put $n = 1,\ C_1 = I,\ D_1 = \emptyset$ in (3.2) to obtain $\mu_{A,B}(I) = v(I)$.

Let $J \in \mathcal{J}|_{A,B}$. We can use the expression (3.1) and write

$$J = \bigcup_{i=1}^{n} (C_i \cap (D_i)^c); \quad C_1 \supset D_1 \supset C_2 \supset \cdots \supset D_n\,, \\ C_i,\ D_i \in \mathcal{I}|_{A,B}\,,\ i = 1, 2, \ldots, n, \tag{3.3}$$

where $C_i,\ D_i \in \mathcal{I}|_{A,B}$ implies

$$\mu_{A,B}(C_i) = v(C_i),\ \mu_{A,B}(D_i) = v(D_i),\ i = 1, 2, \ldots .n. \tag{3.4}$$

Therefore,

$$\begin{aligned} \mu_{A,B}(J) &= \sum_{i=1}^{n} \mu_{A,B}(C_i \cap (D_i)^c) \\ &= \sum_{i=1}^{n} (\mu_{A,B}(C_i) - \mu_{A,B}(D_i)) \\ &= \sum_{i=1}^{n} (v(C_i) - v(D_i)) \end{aligned} \tag{3.5}$$

Put

$$A_i = \bigcup_{j=i}^{n} (C_j \cap (D_j)^c), \quad i = 1, 2, \ldots, n, \tag{3.6}$$

and $A_{n+1} = \emptyset$. Then (3.3) and (3.6) imply $A_1 = J$, and

$$A_i \cup D_i = C_i\,,\ A_i \cap D_i = A_{i+1}\,, \quad i = 1, 2, \ldots, n,$$

Since v is submodular, we can apply (1.3) with $A = A_i$ and $B = D_i$ to find

$$v(C_i) + v(A_{i+1}) \leqq v(A_i) + v(D_i),\ i = 1, 2, \ldots, n.$$

Summing this up with i and using (3.5) leads to $\mu_{A,B}(J) \leqq v(J)$. □

We assume non-decreasing property and continuity in the definitions of submodular functions. Therefore $\mu_{A,B}$ is non-negative valued and σ-additive on the finitely additive class $\mathcal{J}|_{A,B}$. Here, to prove that $\mu_{A,B}$ is σ-additive on $\mathcal{J}|_{A,B}$, it suffices to prove $\lim_{n\to\infty} \mu_{A,B}(E_n) = 0$ for any sequence

$E_n \in \mathcal{J}|_{A,B}$, $n \in \mathbb{N}$, satisfying $E_1 \supset E_2 \supset \cdots$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. Since we assume $v(\emptyset) = 0$ and continuity (2.3) for v, we have $\lim_{n\to\infty} v(E_n) = 0$ for $E_n \in \mathcal{J}|_{A,B}$, $n \in \mathbb{N}$, satysfing the conditions. This and Lemma 6 and non-negativity imply $\lim_{n\to\infty} \mu_{A,B}(E_n) = 0$.

The extension theorem of measures now implies that $\mu_{A,B}$ is uniquely extended to a measure on $\sigma[\mathcal{J}|_{A,B}] = \sigma[\mathcal{I}|_{A,B}] = \mathcal{F}|_A$. We denote this measure by the same symbol so that $\mu_{A,B} \in \mathcal{M}(A)$. We complete a proof of the theorem by proving that $\mu_{A,B} \in \mathcal{C}_{-,v}(A)$ and $v(B) = \mu_{A,B}(B)$. Since $A, B \in \mathcal{I}|_{A,B}$, Lemma 6 implies $\mu_{A,B}(A) = v(A)$ and $\mu_{A,B}(B) = v(B)$, hence it remains to prove $\mu_{A,B}(E) \leqq v(E)$ for $E \in \mathcal{F}|_A$.

Let ϵ be an arbitrary positive real. Since $E \in \mathcal{F}|_A$ implies $A \cap E^c \in \mathcal{F}|_A$, the extension theorem of measures implies that there exists a countable union of sets in $\mathcal{J}|_{A,B}$, which we denote by $K \in \mathcal{F}|_A$ such that

$$A \cap E^c \subset K \subset A \quad \text{and} \quad \mu_{A,B}(A \cap E^c) + \epsilon \geqq \mu_{A,B}(K). \tag{3.7}$$

We can use the expression in (3.1) for sets in $\mathcal{J}|_{A,B}$ to express K as

$$K = \bigcup_{i\in\mathbb{N}} (C_i \cap D_i^c),\ C_i,\ D_i \in \mathcal{I}|_{A,B};\ C_i \supset D_i,\ i = 1, 2, \dots . \tag{3.8}$$

For each $n \in \mathbb{N}$ put $K_n = \bigcup_{i=1}^{n} (C_i \cap D_i^c)$. Then

$$K_n \in \mathcal{J}|_{A,B},\ n \in \mathbb{N},\quad K_1 \subset K_2 \subset \cdots,\quad \bigcup_{n\in\mathbb{N}} K_n = K. \tag{3.9}$$

$\mu_{A,B}(A \cap E^c) + \epsilon \geqq \mu_{A,B}(K)$ in (3.7) and (3.9) then imply

$$\mu_{A,B}(E) - \epsilon \leqq \mu_{A,B}(A \cap K_n^c),\ n \in \mathbb{N}, \tag{3.10}$$

while $A \cap E^c \subset K \subset A$ in (3.7) and (3.9) and monotonicity and continuity of v imply that there exists $n_0 \in \mathbb{N}$ such that

$$v(E) + \epsilon \geqq v(A \cap K_n^c),\ n = n_0, n_0 + 1. \dots . \tag{3.11}$$

Since $K_n \in \mathcal{J}|_{A,B}$ and $\mathcal{J}|_{A,B}$ is a finite algebra, $A \cap K_n^c \in \mathcal{J}|_{A,B}$. Lemma 6 therefore impiles $\mu_{A,B}(A \cap K_n^c) \leqq v(A \cap K_n^c)$. This with (3.10) and (3.11) implies $\mu_{A,B}(E) - \epsilon \leqq v(E) + \epsilon$. Since ϵ is an arbitrary positive real this implies $\mu_{A,B}(E) \leqq v(E)$, which completes the proof. □

4. Example

In this section we give examples of measure spaces $(\Omega, \mathcal{F})$ and $\mathcal{I}$ satisfying (2.6) so that Theorem 3 is applicable.

4.1. Countable set

Consider the case $(\Omega, \mathcal{F}) = (\mathbb{N}, 2^{\mathbb{N}})$, and put $\mathcal{I} = \{\{1, 2, \dots, i\} \mid i \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$. Then $\mathcal{I}$ satisfies (2.6), hence Theorem 3 is applicable.

For example, assume further that $m, n \in \mathbb{N}$ satisfy $m < n$, and $b_i \in \mathbb{N}$, $i = 1, 2, \dots, m$, and $c_j \in \mathbb{N}$, $j = 1, 2, \dots, n-m$, satisfy

$$\begin{aligned}&b_1 < b_2 < \cdots < b_m\,, \;\; c_1 < c_2 < \cdots < c_{n-m}\,, \text{ and}\\ &c_j \ne b_i\,, \; i = 1, \dots, m, \; j = 1, \dots, n-m.\end{aligned}$$

Put $B = \{b_1, \dots, b_m\}$ and $A = B \cup \{c_1, \dots, c_{n-m}\}$. Then (2.7) implies

$$\begin{aligned}\mathcal{I}|_{A,B} = \{\emptyset\} &\cup \{\{b_1, \dots, b_i\} \mid i = 1, 2, \dots, m\}\\ &\cup \{B \cup \{c_1, \dots, c_j\} \mid j = 1, 2, \dots, n-m\}.\end{aligned}$$

Define a measure μ defined on $\mathcal{F}|_A$ by

$$\begin{aligned}&\mu(\{b_1\}) = v(\{b_1\}),\\ &\mu(\{b_i\}) = v(\{b_1, \dots, b_i\}) - v(\{b_1, \dots, b_{i-1}\}), \; i = 2, \dots, m,\\ &\mu(\{c_1\}) = v(A \cup \{c_1\}) - v(A),\\ &\mu(\{c_j\}) = v(A \cup \{c_1, \dots, c_j\}) - v(A \cup \{c_1, \dots, c_{j-1}\}), \; j = 2, \dots, n-m.\end{aligned}$$

Then (2.8) holds and Theorem 3 also implies $\mu \in \mathcal{C}_{-,v}(A)$.

This reproduces, with the correspondence (2.9) between submodular function and supermodular function (convex game), a classical theory of cores of convex games by Shapley [9].

4.2. 1-dimensional Borel σ-algebra

Consider the case $(\Omega, \mathcal{F}) = ([0, 1), \mathcal{B}([0, 1)))$, where $[0, 1)$ is a unit interval and $\mathcal{B}([0, 1))$ denotes the σ-algebra generated by the open sets in $[0, 1)$.

Put $\mathcal{I} = \{[0, x) \mid 0 \leqq x \leqq 1\}$. Then $\mathcal{I}$ satisfies (2.6), hence Theorem 3 is applicable.

For $A, B \in \mathcal{B}([0,1)))$ satisfying $B \subset A$, we have, from (2.7),

$$\mathcal{I}|_{A,B} = \{B \cap [0,x) \mid 0 \leqq x \leqq 1\} \cup \{B \cup (A \cap [0,x)) \mid 0 \leqq x \leqq 1\}.$$

For each $I \in \mathcal{I}|_{A,B}$ put $\mu(I) = v(I)$. Then Lemma 2 and the proof of Theorem 3 in §3 imply that μ is uniquely extended to a measure on $\mathcal{B}([0,1))|_A$ and $\mu \in \mathcal{C}_{-,v}(A)$.

4.3. Borel algebra on Polish space

Let Ω be a separable, complete, metric space (a Polish space) and $\mathcal{B}(\Omega)$ be the Borel σ-algebra, the σ-algebra generated by the open sets. The collection $\mathcal{J}$ of all the open balls, each with a radius of positive rational and the center chosen from a fixed dense subset of Ω, generates the class of open sets, i.e., $\mathcal{J}$ is a countable open basis, hence $\sigma[\mathcal{J}] = \mathcal{B}(\Omega)$ holds. It turns out that we can construct a class $\mathcal{I}$ which satisfies (2.6) on $(\Omega, \mathcal{F}) = (\Omega, \mathcal{B}(\Omega))$ from $\mathcal{J}$, so that Theorem 3 is applicable.

More generally, we have the following.

PROPOSITION 7. *Let $(\Omega, \mathcal{F})$ be a measurable space and assume that there exists a countable class of measurable sets $\mathcal{J} = \{J_1, J_2, \dots\} \subset \mathcal{F}$ such that $\sigma[\mathcal{J}] = \mathcal{F}$.*

Define a $\mathcal{F}/\mathcal{B}([0,1))$ measurable function $f: \Omega \to [0,1)$ by

$$f = \sum_{n=1}^{\infty} 3^{-n}\, \mathbf{1}_{J_n}\,, \tag{4.1}$$

and define a class of sets $\mathcal{I} \subset \mathcal{F}$ by $\mathcal{I} = \{f^{-1}([0,a)) \mid 0 \leqq a \leqq 1\}$, where $f^{-1}([0,a)) = \{\omega \in \Omega \mid f(\omega) < a\}$. Then $\mathcal{I}$ satisfies (2.6).

PROOF. Since f is measurable, $\sigma[\mathcal{I}] \subset \mathcal{F}$. Since $\mathcal{F} = \sigma[\mathcal{J}]$, to prove $\mathcal{F} \subset \sigma[\mathcal{I}]$ It suffices to prove $J_N \in \sigma[\mathcal{I}]$ for each $N \in \mathbb{N}$. Note that the right-hand side of (4.1) has a form of ternary expansion, because the indicator function $\mathbf{1}_{J_n}$ takes values in $\{0,1\}$. In particular, f is an injection. We can therefore write, for each $N \in \mathbb{N}$,

$$J_N = f^{-1}\Big(\bigcup_{(a_1,\cdots,a_{N-1})\in\{0,1\}^{N-1}} [\sum_{n=1}^{N-1} a_n 3^{-n} + 3^{-N},\ \sum_{n=1}^{N-1} a_n 3^{-n} + 2\cdot 3^{-N})\Big)$$
$$= \bigcup_{(a_1,\cdots,a_{N-1})\in\{0,1\}^{N-1}} \Big(f^{-1}(\ [0,\ \sum_{n=1}^{N-1} a_n 3^{-n} + 2\cdot 3^{-N})\ ) \cap f^{-1}(\ [0,\ \sum_{n=1}^{N-1} a_n 3^{-n} + 3^{-N})\ )^c \Big)$$
$$\in \sigma[\mathcal{I}].$$

This proves $\sigma[\mathcal{I}] = \mathcal{F}$. The remaining properties stated in (2.6) follows from $f^{-1}([0,1)) = \Omega$, $f^{-1}(\emptyset) = \emptyset$, and $0 \leqq a < a' \leqq 1 \Rightarrow f^{-1}([0,a)) \subset f^{-1}([0,a'))$. □

References

[1] Artzner, P., Delbaen, F., Eber, J.-M. and D. Heath, Coherent measures of risk, Mathematical Finance **9–3** (1999), 203–228.

[2] Delbaen, F., Convex games and extreme points, Journal of Mathematical Analysis and Applications **45** (1974), 210–233.

[3] Delbaen, F., Coherent risk measures on general probability spaces, *Advances in Finance and Stochastics*, Springer, Berlin, 2002, 1–37.

[4] Delbaen, F., Monetary utility functions, Lecture notes: University of Osaka, Osaka 2008.

[5] Denneberg, D., *Non-additive measure and integral*, Kluwer, 1994.

[6] Kusuoka, S., On law invariant coherent resk measres, RIMS KOKYUROKU **1215** (2001), 158–168.

[7] Parker, J., The sigma–core of a cooperative game, Manuscripta Mathematica **70** (1991), 247–253.

[8] Schmeidler, D., Cores of convex games, Journal of Mathematical Analysis and Applications **40** (1972), 214–225.

[9] Shapley, L. S., Cores of convex games, International Journal of Game Theory **1** (1971), 11–26.

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