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Real Regulators for Products of Elliptic Curves

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Abstract. Assuming the Künneth decomposition of the Chow groups of products of very general Kummer surfaces, we prove that the Hodge-$\mathscr{D}$-conjecture fails for the real regulator $r_{k,1}$ on a product of n very general elliptic curves for $2n \geq 3k-1 \geq 8$.

1. Introduction

Let X be a projective variety and let (k, m) be a pair of integers. The higher Chow groups $\mathrm{CH}^k(X, m)$ were introduced by S. Bloch [Blo86]. For the purpose of this paper, let us give a quick definition of $\mathrm{CH}^k(X, 1)$ as follows

$$Z^k(X,1) = \Big\{ \sum_j (f_j, Z_j) \ : \mathrm{cd}_X Z_j = k-1,\ f_j \in \mathbb{C}(Z_j)^\times \Big\}$$

$$\mathrm{CH}^k(X,1) = \frac{\{\xi = \sum (f_j, Z_j) \in Z^k(X,1) : \mathrm{div}(\xi) = \sum \mathrm{div}(f_j) = 0\}}{\mathrm{Image(Tame\ symbol)}}.$$

where Z_j are irreducible subvarieties of X of codimension $k-1$, $\mathbb{C}(Z_j)$ is the space of rational functions on Z_j and $\mathrm{div}(f_j)$ is the divisor on Z_j defined by f_j. We will not explain the Tame symbol since it is not needed in this paper.

If X is smooth, then similar to the cycle maps $\mathrm{CH}^k(X) \to H^{2k}(X)$ on Chow groups, there are maps, called *regulators*, from the higher Chow groups of X to its Deligne cohomologies (see, for example, [KLMS06,

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KL07]). Again, for our purpose, we just need the *real regulator* map

$$\begin{array}{ccc} \mathrm{CH}^k_{\mathbb{R}}(X,1) & \xrightarrow{r_{k,1}} & H^{k-1,k-1}(X,\mathbb{R}) \\ \| & & \| \\ \mathrm{CH}^k(X,1)\otimes\mathbb{R} & & (H^{n-k+1,n-k+1}(X,\mathbb{R}))^\vee \end{array}$$

defined on $\mathrm{CH}^k(X,1)$ for a smooth projective variety X of dimension n, which is explicitly given by

$$r_{k,1}(\xi)(\omega) = \sum_j \int_{Z_j} \log|f_j|\omega$$

for $\xi \in \mathrm{CH}^k(X,1)$ represented by $\xi = \sum(f_j, Z_i) \in Z^k(X,1)$ satisfying $\mathrm{div}(\xi) = 0$.

The Hodge-$\mathscr{D}$-conjecture states that this map is surjective. It is expected to be true for varieties over $\overline{\mathbb{Q}}$. For surfaces over $\mathbb{C}$, it is known to be true for rational surfaces and general Abelian and K3 surfaces [CL05]. It fails for very general surfaces in $\mathbb{P}^3$ of degree ≥ 5 [MS97].

Let us consider the real regulator for a product of elliptic curves.

Conjecture 1.1. *For n very general complex elliptic curves E_1, $E_2, ..., E_n$ and $X = E_1 \times E_2 \times ... \times E_n$, the real regulator map $r_{k,1}$ on X is surjective for $k = 2$ and "trivial" (explained below) for all $2n \geq 3k - 1 \geq 8$. The triviality of $r_{k,1}$ is measured by whether its image is orthogonal to one of the subspaces*

$$T_m(H^{2n-2k+2}(X)) = \sum_{\substack{|I|=m \\ I\subset\{1,2,...,n\}}} \bigotimes_{i\in I} H^1(E_i) \otimes H^{2n-2k+2-m}(\prod_{j\notin I} E_j) \tag{1.1}$$

of $H^{2n-2k+2}(X)$ for some $1 \leq m \leq n$. We expect that

$$r_{k,1}(\mathrm{CH}^k_{\mathbb{R}}(X,1)) \subset T_{2r+2}(H^{2n-2k+2}(X))^\perp \tag{1.2}$$

for all $3 \leq k \leq 2r+1$. For example, when $(k,r,n) = (3,1,4)$, we expect that

$$r_{3,1}(\mathrm{CH}^3_{\mathbb{R}}(X,1)) \subset T_4(H^4(X))^\perp = (H^1(E_1)\otimes H^1(E_2)\otimes H^1(E_3)\otimes H^1(E_4))^\perp.$$

It is easy to show that $r_{k,1}$ is surjective if and only if

$$\tag{1.3} \begin{aligned} &\bigotimes_{i=1}^{2l} H^1(E_{a_i}) \cap H^{l,l}(E_{a_1} \times E_{a_2} \times \dots \times E_{a_{2l}}, \mathbb{R}) \\ &\subset r_{l+1,1}(\mathrm{CH}^{l+1}_{\mathbb{R}}(E_{a_1} \times E_{a_2} \times \dots \times E_{a_{2l}}, 1)) \end{aligned}$$

for all $1 \leq l \leq k-1$ and $1 \leq a_1 < a_2 < \dots < a_{2l} \leq n$. This holds for $k = 2$ [CL05]. So the question here is whether $r_{k,1}$ is trivial for $3 \leq k \leq n-1$.

In this paper, we will reduce the question regarding regulators on products of elliptic curves to those on products of Kummer surfaces. Here a *Kummer surface* is the minimal resolution of $E_1 \times E_2 / \pm 1$ for a product of elliptic curves E_1 and E_2. Roughly, if we assume a Künneth decomposition of some Chow group of products of very general Kummer surfaces, then we have the triviality of $r_{k,1}$.

Theorem 1.2. *Let k, r and m be integers satisfying $3 \leq k \leq 2r+1 \leq 2m-1$. Suppose that the natural map*

$$\tag{1.4} \bigoplus_{\sum d_i = k-1} \bigotimes_{j=1}^{r} \mathrm{CH}^{d_j}_{\mathbb{Q}}(Y_j) \otimes \mathrm{CH}^{d_{r+1}}_{\mathbb{Q}}(\prod_{j=r+1}^{m} Y_j) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{CH}^{k-1}_{\mathbb{Q}}(\prod_{j=1}^{m} Y_j)$$

is surjective for very general Kummer surfaces $Y_1, \dots, Y_r, Y_{r+2}, \dots, Y_m$ and all Kummer surfaces Y_{r+1} with $\mathrm{rank}_{\mathbb{Z}} \mathrm{Pic}(Y_{r+1}) \leq 19$. Then (1.2) holds on a product X of $n \leq 2m$ very general elliptic curves.

In the above theorem, for example, if $n = 2m = 4$, $r = 1$ and $k = 3$, then

$$\sum_{i=1}^{2} d_i = 2.$$

And (1.4) becomes

$$\sum_{d_1+d_2=2} \mathrm{CH}^{d_1}_{\mathbb{Q}}(Y_1) \otimes \mathrm{CH}^{d_2}_{\mathbb{Q}}(Y_2) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{CH}^2_{\mathbb{Q}}(Y_1 \times Y_2),$$

which is equivalent to the Künneth decomposition of $\mathrm{CH}^2_{\mathbb{Q}}(Y_1 \times Y_2)$.

Keep in mind the difference between Y_{r+1} and the rest of Y_j.

Note that if (1.2) holds for $n = n_0$, then we see that it holds for all $k+1 \le n \le n_0$ by projecting $E_1 \times E_2 \times \ldots \times E_{n_0}$ to $E_1 \times E_2 \times \ldots \times E_n$. So we just have to prove the above theorem for $n = 2m$.

We will prove Theorem 1.2 in sections 2 and 3. In section 4, we will show that the Künneth decomposition of $\mathrm{CH}^2_{\mathbb{Q}}(Y_1 \times Y_2)$ is a consequence of the Bloch-Beilinson conjecture on Abel-Jacobi maps. Hence (1.2) holds for $(k, r, n) = (3, 1, 4)$ if we assume the Bloch-Beilinson conjecture. Consequently, either the Hodge-$\mathscr{D}$-conjecture fails for $r_{3,1}$ on a product of four very general elliptic curves or the Bloch-Beilinson conjecture fails.

We work exclusively over $\mathbb{C}$ unless otherwise stated.

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2. Completion of Higher Chow Cycles

Roughly speaking, we will follow the same argument in [CL06]. That is, we will construct a family of products of Kummer surfaces, extend a higher Chow cycle to the whole family and use a standard monodromy argument to show that it has trivial regulator on a general fiber. First we need a generalization of [CL06, Theorem 0.1], which allows us to extend a higher Chow cycle over the family after some modification.

Theorem 2.1. *Let $f : W \to \Gamma$ be a dominant morphism with connected fibers from a smooth projective variety W to a smooth projective curve Γ, let $Y_1, Y_2, \ldots, Y_r$ be smooth projective surfaces with $H^1(Y_j) = 0$ and let $k \le 2r+1$ be a positive integer such that the natural map*

$$(2.1)\qquad \bigoplus_{\sum d_i = k-1} \bigotimes_{j=1}^{r} \mathrm{CH}^{d_j}_{\mathbb{Q}}(Y_j) \otimes \mathrm{CH}^{d_{r+1}}_{\mathbb{Q}}(V) \twoheadrightarrow \mathrm{CH}^{k-1}_{\mathbb{Q}}\Big(\prod_{j=1}^{r} Y_j \times V\Big)$$

is surjective for all irreducible components V of W_t and all $t \in \Gamma$, where W_t is the fiber of W/Γ over t. Let $\xi \in \mathrm{CH}^k_{\mathbb{Q}}(\prod Y_j \times W_U, 1)$ be a higher Chow cycle defined on $\prod Y_j \times W_U = \prod Y_j \times (W \times_\Gamma U)$ for an open set $U \subset \Gamma$. Then there exist $\eta \in \mathrm{CH}^k_{\mathbb{Q}}(\prod Y_j \times W, 1)$ and pre-higher Chow cycles $\alpha_0, \alpha_1, \ldots, \alpha_r$ on $\prod Y_j \times W$ such that

$$(2.2)\qquad \alpha_0 \in f^* Z^k_{\mathbb{Q}}\Big(\prod Y_j \times \Gamma, 1\Big),$$

$$\alpha_i \in Z^0_{\mathbb{Q}}(Y_i) \otimes Z^k_{\mathbb{Q}}(\prod_{j\neq i} Y_j \times W, 1) \oplus Z^1_{\mathbb{Q}}(Y_i) \otimes Z^{k-1}_{\mathbb{Q}}(\prod_{j\neq i} Y_j \times W, 1) \tag{2.3}$$

for $i = 1, 2, ..., r$, and

$$\eta + \sum_{i=0}^{r} \alpha_i = \xi$$

on $\prod Y_j \times W_U$, where $Z^m(X)$ is the free abelian group of Chow cycles of codimension m on X and $Z^m(X, 1)$ is the free abelian group of pre-higher Chow cycles defined at the beginning.

Proof. We can extend ξ to a pre-higher Chow cycle $\overline{\xi}$ on $\prod Y_j \times W$ with $\operatorname{div}(\overline{\xi})$ supported on $\prod Y_j \times W_B$ for $B = \Gamma \backslash U$. By the surjection (2.1), we may choose the completion $\overline{\xi}$ of ξ, after some modification by a cycle in $Z^{k-1}_{\mathbb{Q}}(\prod Y_j \times W_B, 1)$, such that

$$\operatorname{div}(\overline{\xi}) = \sum_{i=1}^{r} R_i + R_0$$

where

$$R_i \in \bigotimes_{a=1}^{i-1} Z^2_{\mathbb{Q}}(Y_a) \otimes \Big(Z^0_{\mathbb{Q}}(Y_i) \otimes Z^{k-2i+1}_{\mathbb{Q}}(\prod_{j=i+1}^{r} Y_j \times W_B) \oplus Z^1_{\mathbb{Q}}(Y_i) \otimes Z^{k-2i}_{\mathbb{Q}}(\prod_{j=i+1}^{r} Y_j \times W_B) \Big)$$

for $i = 1, 2, ..., r$ and

$$R_0 \in \bigotimes_{a=1}^{r} Z^2_{\mathbb{Q}}(Y_a) \otimes Z^{k-2r-1}_{\mathbb{Q}}(W_B).$$

Note that

$$\sum_{i=0}^{r} R_i \sim_{\text{rat}} 0$$

in $\mathrm{CH}^k_{\mathbb{Q}}(\prod Y_j \times W)$.

Let us prove by induction that $R_i = \operatorname{div}(\alpha_i)$ for $i = 1, 2, ..., r$ and some α_i in the space (2.3).

Starting with R_1, we write

$$R_1 = R_{1,0} + R_{1,1} \qquad \text{for } R_{1,0} \in Z^0_{\mathbb{Q}}(Y_1) \otimes Z^{k-1}_{\mathbb{Q}}(\prod_{j=2}^{r} Y_j \times W_B)$$

$$R_{1,1} \in Z^1_{\mathbb{Q}}(Y_1) \otimes Z^{k-2}_{\mathbb{Q}}(\prod_{j=2}^{r} Y_j \times W_B)$$

Clearly, $R_{1,0}$ can be written as

$$R_{1,0} = Y_1 \otimes S \qquad \text{for some } S \in Z^{k-1}_{\mathbb{Q}}(\prod_{j=2}^{r} Y_j \times W_B)$$

By intersecting $\sum R_i$ with $p \times Y_2 \times ... \times Y_r \times W$ for a point $p \in Y_1$, we see that $S \sim_{\text{rat}} 0$ in $\mathrm{CH}^k_{\mathbb{Q}}(\prod_{j=2}^r Y_j \times W)$. Therefore,

$$R_{1,0} = \mathrm{div}(\alpha_{1,0}) \qquad \text{for some } \alpha_{1,0} \in Z^0_{\mathbb{Q}}(Y_1) \otimes Z^k_{\mathbb{Q}}(\prod_{j=2}^{r} Y_j \times W, 1)$$

Hence $R_{1,0} \sim_{\text{rat}} 0$ and

$$R_{1,1} + \sum_{i=2}^{r} R_i + R_0 \sim_{\text{rat}} 0$$

in $\mathrm{CH}^k_{\mathbb{Q}}(\prod Y_j \times W)$.

We can write

$$R_{1,1} = \sum L_a \otimes S_a \quad \text{for some } L_a \in Z^1_{\mathbb{Q}}(Y_1) \text{ and } S_a \in Z^{k-2}_{\mathbb{Q}}(\prod_{j=2}^{r} Y_j \times W_B)$$

We may assume that L_a are linearly independent in $\mathrm{CH}^1_{\mathbb{Q}}(Y_1)$, after further modifying $\overline{\xi}$ by some cycle supported on $\prod Y_j \times W_B$.

Since $H^1(Y_1) = 0$, the intersection pairing

$$\mathrm{CH}^1_{\mathbb{Q}}(Y_1) \otimes \mathrm{CH}^1_{\mathbb{Q}}(Y_1) \longrightarrow H^2(Y_1, \mathbb{Q}) \cong \mathbb{Q}$$

is nondegenerate. Therefore, by intersecting $R_{1,1} + R_2 + ... + R_r + R_0$ with cycles in

$$Z^1_{\mathbb{Q}}(Y_1) \otimes Z^0_{\mathbb{Q}}(\prod_{j=2}^{r} Y_j \times W),$$

we obtain $S_a \sim_{\text{rat}} 0$ in $\mathrm{CH}^{k-1}_{\mathbb{Q}}(\prod_{j=2}^r Y_j \times W)$ for all a. Consequently,

$$R_{1,1} = \operatorname{div}(\alpha_{1,1}) \qquad \text{for some } \alpha_{1,1} \in Z^1_{\mathbb{Q}}(Y_1) \otimes Z^{k-1}_{\mathbb{Q}}(\prod_{j=2}^{r} Y_j \times W, 1)$$

Then

$$R_1 = \operatorname{div}(\alpha_{1,0} + \alpha_{1,1}) = \operatorname{div}(\alpha_1)$$

and hence

$$\sum_{i=2}^{r} R_i + R_0 \sim_{\text{rat}} 0$$

in $\mathrm{CH}^k_{\mathbb{Q}}(\prod Y_j \times W)$.

In this way, we can inductively show that $R_i = \operatorname{div}(\alpha_i)$ for $i = 1, 2, ..., r$ by intersecting $R_i + ... + R_r + R_0$ with cycles in

$$Z^2_{\mathbb{Q}}(Y_i) \otimes Z^0_{\mathbb{Q}}(\prod_{j \neq i} Y_j \times W) \quad \text{and} \quad Z^1_{\mathbb{Q}}(Y_i) \otimes Z^0_{\mathbb{Q}}(\prod_{j \neq i} Y_j \times W).$$

It follows that

$$R_0 \sim_{\text{rat}} 0$$

in $\mathrm{CH}^k_{\mathbb{Q}}(\prod Y_j \times W)$. It remains to find α_0 in the space (2.2) such that $R_0 = \operatorname{div}(\alpha_0)$.

If $k < 2r + 1$, then $R_0 = 0$ and there is nothing to prove. Suppose that $k = 2r + 1$. In this case,

$$R_0 \in Z^{2r}_{\mathbb{Q}}(Y) \otimes Z^0_{\mathbb{Q}}(W_B) \qquad \text{for } Y = \prod_{j=1}^{r} Y_j.$$

Let us write

$$R_0 = \sum L_a \otimes S_a$$

where S_a are irreducible components of W_B and $L_a \in Z^{2r}_{\mathbb{Q}}(Y)$. Let μ_a be the multiplicity of S_a in W_B. We claim that for every pair S_a and S_b with $f(S_a) = f(S_b)$, i.e., for any two components S_a and S_b of W_p and all $p \in B$,

$$\mu_b L_a \sim_{\text{rat}} \mu_a L_b$$

over $\mathbb{Q}$ on Y.

Since W is smooth, the components of W_B are Cartier divisors of W. We take a sufficiently ample divisor A on W and cut W by $n-2$ general members $A_1, A_2, ..., A_{n-2} \in |A|$ for $n = \dim W$. The resulting $D = A_1 \cap A_2 \cap ... \cap A_{n-2}$ is a smooth projective surface and a flat family of curves over Γ. The basic intersection theory on surfaces tells us that for every $p \in B$, the intersection matrix of any $m-1$ irreducible components of D_p is negative definite, where m is the number of irreducible components of W_p. Therefore, for any two components S_a and S_b of W_p, there exists $\Lambda \in Z^1(W)$, supported on W_p, such that

$$\begin{gathered}\Lambda.D.S \equiv 0 \text{ for all components } S \neq S_a, S_b \text{ of } W_p,\\ \Lambda.D.S_a \not\equiv 0, \text{ and } \Lambda.D.(\mu_a S_a + \mu_b S_b) = 0\end{gathered}$$

where "$\equiv$" is numerical equivalence. And since Λ is supported on W_p, we actually have $\Lambda.D.S_i \equiv 0$ for all $i \neq a, b$. For simplicity, by choosing the cycle $\Lambda \in Z^1_{\mathbb{Q}}(W)$ over $\mathbb{Q}$, we may assume that $\Lambda.D.S_a \equiv \mu_b$. In summary, by letting $C = \Lambda.D$, we conclude that for every $p \in B$ and any two components S_a and S_b of W_p, we can find a 1-cycle $C \in Z^{n-1}_{\mathbb{Q}}(W)$ such that

$$C.S_i \equiv 0 \text{ for } i \neq a, b, \ C.S_a \equiv \mu_b, \text{ and } C.S_b \equiv -\mu_a.$$

Then

$$\begin{aligned} f_*((Y \otimes C).R_0) &= (C.S_a)L_a \otimes p + (C.S_b)L_b \otimes p \\ &= ((C.S_a)L_a + (C.S_b)L_b) \otimes p \end{aligned}$$

for $f : Y \times W \to Y \times \Gamma$. Thus

$$(C.S_a)L_a + (C.S_b)L_b \sim_{\text{rat}} 0 \Rightarrow \mu_b L_a - \mu_a L_b \sim_{\text{rat}} 0.$$

Therefore, $\mu_b L_a \sim_{\text{rat}} \mu_a L_b$ for all pairs of components S_a and S_b of W_p. This implies that after replacing $\overline{\xi}$ by $\overline{\xi} + \beta$ for some $\beta \in Z^0_{\mathbb{Q}}(W_B) \otimes Z^{k-1}_{\mathbb{Q}}(Y, 1)$, we may write R_0 as

$$R_0 = \sum_{p \in B} M_p \otimes W_p$$

where $M_p = (1/\mu_a)L_a$ for a component S_a of W_p. Namely, $R_0 = f^*G$ for some $G \in Z^{k-1}_{\mathbb{Q}}(Y) \otimes Z^1_{\mathbb{Q}}(\Gamma)$. Since $R_0 \sim_{\text{rat}} 0$ on $Y \times W$, $G \sim_{\text{rat}} 0$ on $Y \times \Gamma$. So there exists $\alpha_0 \in f^* Z^k_{\mathbb{Q}}(Y \times \Gamma, 1)$ such that $R_0 = \text{div}(\alpha_0)$.

In conclusion,

$$\eta = \overline{\xi} - \sum_{i=0}^{r} \alpha_i$$

is a higher Chow cycle in $\mathrm{CH}^k_{\mathbb{Q}}(Y \times W, 1)$ with the required property. □

3. Products of Kummer Surfaces

We will reduce the triviality of $r_{k,1}$ on products of elliptic curves to that on products of Kummer surfaces.

For a product $E_1 \times E_2$ of two elliptic curves, we fix two involutions σ_1 and σ_2 on E_i and let $E_1 \times E_2/\sigma_1 \times \sigma_2$ be the quotient of $E_1 \times E_2$ by the action $\sigma_1 \times \sigma_2$. Usually, we simply write it as $E_1 \times E_2/\pm 1$. Note that the action of $\sigma_1 \times \sigma_2$ is invariant on $H^2(E_1 \times E_2)$, i.e.,

$$(\sigma_1 \times \sigma_2)\omega = \omega \quad \text{for all } \omega \in H^2(E_1 \times E_2) \tag{3.1}$$

The resulting surface $E_1 \times E_2/\pm 1$ has 16 ordinary double points, corresponding to 16 fixed points of $\sigma_1 \times \sigma_2$. Blowing up at the 16 double points, we obtain a Kummer K3 surface Y. Indeed, we have a diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \longrightarrow & X/\pm 1 \end{array} \tag{3.2}$$

where $X = E_1 \times E_2$, Z is the blowup of X at the 16 fixed points of $\sigma_1 \times \sigma_2$ and f is a finite morphism of degree 2 ramified at the 16 exceptional divisors of $Z \to X$. The action $\sigma_1 \times \sigma_2$ on X extends to the Galois action σ on Z associated to f. Clearly, σ preserves the 16 exceptional divisors of g. Combining this with (3.1), we see that

$$\sigma(\omega) = \omega \quad \text{for all } \omega \in H^2(Z).$$

Thus, we have

$$f^* f_* \omega = \omega + \sigma(\omega) = 2\omega \quad \text{for all } \omega \in H^2(Z). \tag{3.3}$$

Combined with the projection formula $f_* f^* \omega = 2\omega$, (3.3) implies that f^* and f_* are isomorphims between $H^2(Y)$ and $H^2(Z)$ satisfying (with $\deg f = 2$)

$$\begin{tikzcd} H^2(Y) \arrow[r, "f^*", "\sim"'] \arrow[rr, bend right, "(\deg f)I"'] & H^2(Z) \arrow[r, "f_*", "\sim"'] \arrow[rr, bend left, "(\deg f)I"] & H^2(Y) \arrow[r, "f^*", "\sim"'] & H^2(Z) \end{tikzcd} \tag{3.4}$$

It follows that $(1/\deg f)f_* = (f^*)^{-1}$ preserves the intersection pairing and hence

$$\langle f_*\alpha, f_*\beta\rangle = (\deg f)f_*\langle\alpha,\beta\rangle \tag{3.5}$$

for all $\alpha, \beta \in H^2(Z)$.

Furthermore, f^* and f_* induce isomorphisms between the $\mathbb{Q}$-Hodge structures on $H^2(Y,\mathbb{Q})$ and $H^2(Z,\mathbb{Q})$. Thus they induce isomorphims between the algebraic/transcendental parts of $H^2(Y)$ and $H^2(Z)$.

We define

$$H^2_{\mathrm{tr}}(Y) = f_*g^*(H^1(E_1)\otimes H^1(E_2)). \tag{3.6}$$

Strictly speaking, this is not exactly the transcendental part of $H^2(Y)$. It is the subspace orthogonal to the 18 algebraic classes of $H^2(Y)$ corresponding to the two fibers of $E_1\times E_2$ over E_i and 16 exceptional divisors of g. For very general E_1 and E_2, this is the transcendental part of $H^2(Y)$. For arbitrary E_1 and E_2, it contains the transcendental part of $H^2(Y)$ as a subspace.

Based on the above observations, we have

PROPOSITION 3.1. *Let $E_1, E_2, ..., E_{2m}$ be $n = 2m$ elliptic curves and let Y_{ij} be the Kummer surface birational to $E_i\times E_j/\pm 1$. Then* (1.2) *holds if the real regulator $r_{k,1}$ on*

$$Y = Y_{a_1a_2}\times Y_{a_3a_4}\times ...\times Y_{a_{2m-1}a_{2m}}$$

satisfies

$$r_{k,1}(\mathrm{CH}^k_{\mathbb{R}}(Y,1)) \subset \Big(\bigotimes_{i=1}^{r+1} H^2_{\mathrm{tr}}(Y_{a_{2i-1}a_{2i}})\otimes H^{4m-2k-2r}(\prod_{i=r+2}^{m} Y_{a_{2i-1}a_{2i}})\Big)^{\perp}$$

for all $\{a_1, a_2, ..., a_{2m}\} = \{1, 2, ..., 2m\}$, where $H^2_{\mathrm{tr}}(Y_{ij})$ is the subspace of $H^2(Y_{ij})$ defined by (3.6).

PROOF. Clearly, $T_{2r+2}(H^{4m-2k+2}(X))$ is spanned by the forms

$$\underbrace{\omega_1\otimes\omega_2\otimes ...\otimes\omega_{2r+2}}_{\omega}\otimes\eta$$

for some $\omega_i \in H^1(E_{a_i})$ and

$$\eta \in H^{4m-2k-2r}(\prod_{i\neq a_1,\ldots,a_{2r+2}} E_i).$$

It suffices to prove that

$$r_{k,1}(\mathrm{CH}^k_{\mathbb{R}}(X,1)) \subset (\omega \otimes \eta)^{\perp} \tag{3.7}$$

For simplicity, we may assume that $(a_1, a_2, \ldots, a_{2r+2}) = (1, 2, \ldots, 2r+2)$.

Let Z_{ij} be the blowup of $E_i \times E_j$ at the 16 fixed points and let

$$Y = Y_{12} \times Y_{34} \times \ldots \times Y_{2m-1,2m}$$

and

$$Z = Z_{12} \times Z_{34} \times \ldots \times Z_{2m-1,2m}.$$

We have the commutative diagram (3.2).

By (3.4) and (3.5),

$$\begin{aligned} &\langle r_{k,1}(\xi), \omega \otimes \eta\rangle = g^*\langle r_{k,1}(\xi), \omega \otimes \eta\rangle = \langle g^*(r_{k,1}(\xi)), g^*(\omega \otimes \eta)\rangle \\ &= f_*\langle r_{k,1}(g^*\xi), g^*(\omega \otimes \eta)\rangle = \frac{1}{\deg f}\langle f_*(r_{k,1}(g^*\xi)), f_*g^*(\omega \otimes \eta)\rangle \\ &= \frac{1}{\deg f}\langle r_{k,1}(f_*g^*\xi), f_*g^*(\omega \otimes \eta)\rangle = 0 \end{aligned}$$

for all $\xi \in \mathrm{CH}^k(X,1)$ and (3.7) follows. $\square$

By the above proposition, to prove the triviality of $r_{k,1}$ on a product of $2m$ very general elliptic curves in our main Theorem 1.2, it suffices to prove

$$r_{k,1}(\mathrm{CH}^k_{\mathbb{R}}(\prod_{i=1}^{m} Y_i, 1)) \subset \Big(\bigotimes_{i=1}^{r+1} H^2_{\mathrm{tr}}(Y_i) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^{m} Y_i)\Big)^{\perp} \tag{3.8}$$

for a product of m very general Kummer surfaces $Y_1, Y_2, \ldots, Y_m$.

Now let us try to use the argument in [CL06] to prove (3.8) and thus Theorem 1.2. As in [CL06], we first construct a one-parameter family of Kummer surfaces with "nice" singularities.

We start with the construction of two flat projective families S/B and T/B of curves over a smooth projective curve B satisfying

- S and T are smooth,
- there is a nonempty finite set $\Sigma \subset B$ such that S_b and T_b are rational curves with a node for $b \in \Sigma$ and they are smooth elliptic curves for $b \notin \Sigma$,
- $S_b \times T_b$ is a general product of two elliptic curves for $b \in B$ general,

$$h^{1,1}(S_b \times T_b, \mathbb{Q}) := \dim H^{1,1}(S_b \times T_b, \mathbb{Q}) \leq 3 \qquad \text{for all } b \in B \backslash \Sigma, \tag{3.9}$$

- and both S/B and T/B have sections.

By (3.9), the one-parameter family of Kummer surfaces constructed from $S \times_B T$ is generically of Picard rank 18 and of Picard rank 19 (but never 20) at finitely many points of $B \backslash \Sigma$.

It is not hard to construct such S/B and T/B individually. The difficulty is that we have to make sure that S/B and T/B are singular over the same points $b \in B$. Here is one construction.

We let $G \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a general pencil of cubic curves. It is well known that $G/\mathbb{P}^1$ has exactly 12 nodal fibers over $p_1, p_2, ..., p_{12} \in \mathbb{P}^1$. We choose two different morphisms $g_i : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 12 that map all $p_1, p_2, ..., p_{12}$ to the same point $q \in \mathbb{P}^1$, i.e.,

$$g_i^*(q) = p_1 + p_2 + ... + p_{12} \qquad \text{for } i = 1, 2.$$

Let B be the normalization of the fiber product of $g_1 : \mathbb{P}^1 \to \mathbb{P}^1$ and $g_2 : \mathbb{P}^1 \to \mathbb{P}^1$ with the diagram

$$\begin{tikzcd}
 & B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\pi"] & \\
\mathbb{P}^1 \arrow[dr, "g_1"'] & & \mathbb{P}^1 \arrow[dl, "g_2"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

Then

$$\Sigma = \pi^{-1}(q) = \pi_1^{-1}\{p_1, p_2, ..., p_{12}\} = \pi_2^{-1}\{p_1, p_2, ..., p_{12}\}.$$

Indeed, for general choices of g_1 and g_2, B is irreducible and $\pi : B \to \mathbb{P}^1$ has degree 144.

Let $S = G \times_{\mathbb{P}^1} B$ be the fiber product of $G \to \mathbb{P}^1$ and $\pi_1 : B \to \mathbb{P}^1$ and let $T = G \times_{\mathbb{P}^1} B$ be the fiber product of $G \to \mathbb{P}^1$ and $\pi_2 : B \to \mathbb{P}^1$. It is not hard to see that S/B and T/B have the required properties for very general choices of g_1 and g_2: (3.9) holds since there are only countably many products of elliptic curves $E \times F$ with $h^{1,1}(E \times F, \mathbb{Q}) = 4$; it is easy to choose g_i such that $S_b \times T_b$ is not one of them for all $b \in B \backslash \Sigma$; the pencil $G/\mathbb{P}^1$ has infinitely many sections so the same holds for both S/B and T/B. This shows the existence of such S/B and T/B.

Since S/B has a section, we have an involution $\sigma_S : S/B \dashrightarrow S/B$ defined on smooth fibers of S/B. This involution extends to singular fibers of S/B as well: for a nodal fiber S_b, it extends to an automorphism $S_b \to S_b$ fixing three points including the node. Indeed, this is the Galois action induced by a degree 2 map $S_b \to \mathbb{P}^1$. So we have an automorphism $\sigma_S : S \to S$ preserving the base B of order 2. The fixed locus of σ_S consists of a multisection of S/B which meets each smooth fiber transversely at 4 points and each singular fiber at 3 points including the node. Of course, the same holds for T/B and we have an involution $\sigma_T : T/B \to T/B$.

Let $R = S \times_B T/\sigma$ for $\sigma = \sigma_S \times \sigma_T$. After resolving the singularities of R, we obtain a family of Kummer surfaces over B. Let $Y_1, ..., Y_r, Y_{r+2}, ..., Y_m$ be $m-1$ very general Kummer surfaces and let us try to prove (3.8) where Y_{r+1} is the Kummer surface birational to R_t for $t \in B$ general. If (3.8) fails, then there exist a finite base change $\phi : \Gamma \to B$, a desingularization $Z \to R \times_B \Gamma$ and a higher Chow cycle

$$\xi \in \mathrm{CH}^k_{\mathbb{Q}}(\prod_{i=1}^r Y_i \times Z_U \times \prod_{i=r+2}^m Y_i, 1)$$

over a nonempty open set $U \subset \Gamma$ such that

- Z_t is a Kummer surface birational to $R_{\phi(t)}$ for $t \notin \phi^{-1}(\Sigma)$, and
- for every $t \in U$,

$$r_{k,1}(\xi_t) \notin \big(\bigotimes_{i=1}^r H^2_{\mathrm{tr}}(Y_i) \otimes H^2_{\mathrm{tr}}(Z_t) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i)\big)^{\perp}.$$

We claim that we can choose Z, after a further finite base change $\Gamma' \to \Gamma$, such that every irreducible component of Z_t is a smooth rational surface for all $t \in \phi^{-1}(\Sigma)$. Namely, we claim

Proposition 3.2. *Let $R = S \times_B T/\sigma$ be constructed as above and let $\phi : \Gamma \to B$ be a finite morphism from a smooth projective curve Γ to B such that the ramification index of ϕ at each point of $\phi^{-1}(\Sigma)$ is even. Then there exists a desingularization $Z \to R \times_B \Gamma$ such that every irreducible component of Z_t is a smooth rational surface for all $t \in \phi^{-1}(\Sigma)$.*

The hypothesis on the ramification index in the above proposition can be easily met by a further finite base change $\Gamma' \to \Gamma$. Assuming Proposition 3.2, let us finish the proof of (3.8).

For $t \notin \phi^{-1}(\Sigma)$, by (3.9), we have $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(Z_t) \le 19$. Therefore, by our hypothesis (1.4), the map

(3.10)

$$\bigotimes_{i=1}^{r} \mathrm{CH}^{\bullet}_{\mathbb{Q}}(Y_i) \otimes \mathrm{CH}^{\bullet}_{\mathbb{Q}}(Z_t \times \prod_{i=r+2}^{m} Y_i) \twoheadrightarrow \mathrm{CH}^{k-1}_{\mathbb{Q}}(\prod_{i=1}^{r} Y_i \times Z_t \times \prod_{i=r+2}^{m} Y_i)$$

is surjective for all $t \notin \phi^{-1}(\Sigma)$.

For $t \in \phi^{-1}(\Sigma)$, every irreducible component $P \subset Z_t$ is a smooth rational surface by Proposition 3.2. The Chow groups of $P \times X$ have Künneth decomposition

$$\mathrm{CH}^{\bullet}(P \times X) = \mathrm{CH}^{\bullet}(P) \otimes \mathrm{CH}^{\bullet}(X) \tag{3.11}$$

for every smooth rational projective surface P and every smooth projective variety X.

Let Y_{r+1} be a general Kummer surface. Choosing a finite morphism $g : Y_{r+1} \to \mathbb{P}^2$, we have the diagram

$$\begin{array}{ccc}
\bigotimes_{i=1}^{r} \mathrm{CH}^{\bullet}_{\mathbb{Q}}(Y_i) \otimes \mathrm{CH}^{\bullet}_{\mathbb{Q}}(\prod_{i=r+1}^{m} Y_i) & \twoheadrightarrow & \mathrm{CH}^{k-1}_{\mathbb{Q}}(\prod_{i=1}^{m} Y_i) \\
\Big\downarrow g_* & & \Big\downarrow g_* \\
\bigotimes_{i=1}^{r} \mathrm{CH}^{\bullet}_{\mathbb{Q}}(Y_i) \otimes \mathrm{CH}^{\bullet}_{\mathbb{Q}}(\mathbb{P}^2 \times \prod_{i=r+2}^{m} Y_i) & \longrightarrow & \mathrm{CH}^{k-1}_{\mathbb{Q}}(\prod_{i=1}^{r} Y_i \times \mathbb{P}^2 \times \prod_{i=r+2}^{m} Y_i)
\end{array}$$

Clearly, we see from the above diagram that its bottom row is also surjective.

Combining this with the Künneth decomposition (3.11), we have surjections

$$\bigotimes_{i=1}^{r} \mathrm{CH}^{\bullet}_{\mathbb{Q}}(Y_i) \otimes \mathrm{CH}^{\bullet}_{\mathbb{Q}}(\prod_{i=r+2}^{m} Y_i) \twoheadrightarrow \mathrm{CH}^{d}_{\mathbb{Q}}(\prod_{i=1}^{r} Y_i \times \prod_{i=r+2}^{m} Y_i) \tag{3.12}$$

for $d = k-1, k-2, k-3$. Then we obtain the surjection

(3.13)
$$\bigotimes_{i=1}^{r} \mathrm{CH}^{\bullet}_{\mathbb{Q}}(Y_i) \otimes \mathrm{CH}^{\bullet}_{\mathbb{Q}}(P \times \prod_{i=r+2}^{m} Y_i) \twoheadrightarrow \mathrm{CH}^{k-1}_{\mathbb{Q}}(\prod_{i=1}^{r} Y_i \times P \times \prod_{i=r+2}^{m} Y_i)$$

from (3.11) and (3.12) for every irreducible component $P \subset Z_t$ and all $t \in \phi^{-1}(\Sigma)$.

Combining (3.10) and (3.13), we see that the map (2.1) is surjective in Theorem 2.1 for every irreducible component V of W_t and all $t \in \Gamma$ with

$$W = Z \times \prod_{i=r+2}^{m} Y_i.$$

So we can apply the theorem and obtain a higher Chow class

$$\eta \in \mathrm{CH}^{k}_{\mathbb{Q}}(\prod_{i=1}^{r} Y_i \times Z \times \prod_{i=r+2}^{m} Y_i, 1)$$

and pre-higher Chow cycles $\alpha_0, \alpha_1, ..., \alpha_r$ as in the theorem such that

$$\eta - \sum_{i=0}^{r} \alpha_i = \xi$$

on $Y_1 \times ... \times Y_r \times Z_U \times Y_{r+2} \times ... \times Y_m$. For $\alpha_0, \alpha_1, ..., \alpha_r$ given in Theorem 2.1, it follows from the explicit regulator formula applied to the precycles that

$$r_{k,1}(\eta_t) - r_{k,1}(\xi_t) \in \Big(\bigotimes_{i=1}^{r} H^2_{\mathrm{tr}}(Y_i) \otimes H^2_{\mathrm{tr}}(Z_t) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^{m} Y_i) \Big)^{\perp}$$

for all $t \in U$. Then a standard monodromy argument shows that $r_{k,1}(\xi_t)$ is trivial for $t \in U$ general (see, for example, [CL06]). We will sketch this argument at the end of this section.

It remains to prove Proposition 3.2. This is achieved by finding an explicit resolution of the singularities of $R \times_B \Gamma$.

Proof of Proposition 3.2. The problem is local at every point $\phi^{-1}(\Sigma)$. Let us replace Γ by a disk centered at a point $0 \in \phi^{-1}(\Sigma)$. So $S \times_B \Gamma$ and $T \times_B \Gamma$ have singularities of type $xy = t^{2m}$ at the nodes of $S_{\phi(0)}$ and $T_{\phi(0)}$, respectively, where $2m$ is the ramification index of ϕ at 0. Let $\widehat{S}$ and $\widehat{T}$ be the minimal resolution of $S \times_B \Gamma$ and $T \times_B \Gamma$, respectively.

The central fiber

$$\widehat{S}_0 = C_0 \cup C_1 \cup C_2 \cup ... \cup C_{2m-1}$$

of $\widehat{S}/\Gamma$ is a union of $2m$ smooth rational curves of simple normal crossings whose dual graph is a circle, where C_0 is the proper transform of $S_{\phi(0)}$ and $C_i \cap C_{i+1} \neq \emptyset$ for $i = 0, 1, ..., 2m-1$ with $C_{2m} = C_0$.

It is easy to see that the involution $\sigma_S : S \to S$ lifts to an involution $\widehat{\sigma}_S : \widehat{S} \to \widehat{S}$ whose action on $\widehat{S}_0$ is given by

$$\begin{aligned}&\widehat{\sigma}_S(C_0 \cap C_1) = C_{2m-1} \cap C_0,\ \widehat{\sigma}_S(C_1 \cap C_2) = C_{2m-2} \cap C_{2m-1},\\ &\widehat{\sigma}_S(C_2 \cap C_3) = C_{2m-3} \cap C_{2m-2},\ ..., \widehat{\sigma}_S(C_{m-1} \cap C_m) = C_m \cap C_{m+1}\end{aligned}$$

In the case of $m = 1$, $\widehat{\sigma}_S$ switches the two intersections of C_0 and C_1. The fixed locus $\widehat{\sigma}_S$ consists of four disjoint sections P_1, P_2, P_3, P_4 of $\widehat{S}/\Gamma$ with P_1 and P_2 meeting C_0 and P_3 and P_4 meeting C_m.

The exact same holds for $\widehat{T}$:

$$\widehat{T}_0 = D_0 \cup D_1 \cup D_2 \cup ... \cup D_{2m-1}$$

is a union of $2m$ smooth rational curves of simple normal crossings whose dual graph is a circle and the involution $\sigma_T : T \to T$ lifts to an involution $\widehat{\sigma}_T : \widehat{T} \to \widehat{T}$ whose fixed locus consists of four disjoint sections Q_1, Q_2, Q_3, Q_4 of $\widehat{T}/\Gamma$.

Let $\widehat{\sigma} = \widehat{\sigma}_S \times \widehat{\sigma}_T$ be the involution on $\widehat{S} \times_\Gamma \widehat{T}$. Then the singular locus of $\widehat{S} \times_\Gamma \widehat{T}/\widehat{\sigma}$ consists of the images of the 16 sections $P_i \times_\Gamma Q_j$ and $4m^2$ isolated points $(C_a \cap C_{a+1}) \times (D_b \cap D_{b+1})$. At each point among

$$(C_a \cap C_{a+1}) \times (D_b \cap D_{b+1}),$$

$\widehat{\sigma}_S \times \widehat{\sigma}_T$ has a 3-fold rational double point $xy = zw = t$; the same is true for $\widehat{S} \times_\Gamma \widehat{T}/\widehat{\sigma}$ at the images of $(C_a \cap C_{a+1}) \times (D_b \cap D_{b+1})$. So we can easily resolve the singularities of $\widehat{S} \times_\Gamma \widehat{T}/\widehat{\sigma}$ by blowing it up along its singular locus. Let Z be the resulting blowup. Clearly, all components of Z_0 are smooth rational surfaces. So we have obtained a resolution of $R \times_B \Gamma$ with the required property via the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & \widehat{S} \times_\Gamma \widehat{T}/\widehat{\sigma} \arrow[d] \\ & (S \times_B T/\sigma) \times_B \Gamma = R \times_B \Gamma \end{tikzcd} \qquad \square$$

We will outline the monodromy argument. To set this up, suppose that we have a smooth projective family Z/U of Kummer surfaces of maximal moduli over a smooth quasi-projective surface U and a higher Chow class

$$\xi \in \mathrm{CH}^k_{\mathbb{Q}}(\prod_{i=1}^r Y_i \times Z \times \prod_{i=r+2}^m Y_i, 1).$$

We want to show that $r_{k,1}(\xi_b)$ is trivial for $b \in U$ general.

Given our construction of the one parameter family $S \times_B T$, after a base change, we can find a morphism $C \to U$ from a smooth quasi-projective curve C to U whose image passing through a general point of U with the following property. The one-parameter family $Z_C = Z \times_U C$ over C can be extended to a family $Z_{\overline{C}}$ of Kummer surfaces over the completion $\overline{C}$ of C such that $Z_{\overline{C}}$ is smooth and the pullback ξ_C of ξ to Z_C can be extended to a higher Chow class $\eta_{\overline{C}} \in \mathrm{CH}^k_{\mathbb{Q}}(Z_{\overline{C}}, 1)$ satisfying

(3.14)
$$r_{k,1}(\eta_t) - r_{k,1}(\xi_t) \in \Big(\bigotimes_{i=1}^r H^2_{\mathrm{tr}}(Y_i) \otimes H^2_{\mathrm{tr}}(Z_t) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i)\Big)^\perp$$

for all $t \in C$. Actually, (3.14) holds for the full regulator $\mathrm{cl}_{k,1}$. That is,

(3.15)
$$\widetilde{\mathrm{cl}}_{k,1}(\eta_t) - \widetilde{\mathrm{cl}}_{k,1}(\xi_t) \in \Big(\bigotimes_{i=1}^r H^2_{\mathrm{tr}}(Y_i) \otimes H^2_{\mathrm{tr}}(Z_t) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i)\Big)^\perp$$

where $\widetilde{\mathrm{cl}}_{k,1}(\eta_t)$ and $\widetilde{\mathrm{cl}}_{k,1}(\xi_t)$ are local lifts of $\mathrm{cl}_{k,1}(\eta_t)$ and $\mathrm{cl}_{k,1}(\xi_t)$, respectively. The Gauss-Manin connection ∇ on $Y_1 \times ... \times Y_r \times Z_C \times Y_{r+2} \times ... \times Y_m/C$ acts on $\widetilde{\mathrm{cl}}_{k,1}(\eta_t)$ and $\widetilde{\mathrm{cl}}_{k,1}(\xi_t)$ (see, for example, [CDKL16]).

Let us fix a class

$$\omega \in \bigotimes_{i=1}^{r} H^2_{\mathrm{tr}}(Y_i) \otimes H^{4m-2k-2r}\Big(\prod_{i=r+2}^{m} Y_i\Big). \tag{3.16}$$

Then by (3.15),

$$\pi_*(\widetilde{\mathrm{cl}}_{k,1}(\eta_t) \wedge \omega - \widetilde{\mathrm{cl}}_{k,1}(\xi_t) \wedge \omega) \in H^2_{\mathrm{tr}}(Z_t)^{\perp} \tag{3.17}$$

for all $t \in C$, where π is the projection $Y_1 \times ... \times Y_r \times Z \times Y_{r+2} \times ... \times Y_m \to Z$.

It follows from (3.17) that

$$\nabla\big(\pi_*(\widetilde{\mathrm{cl}}_{k,1}(\eta_t) \wedge \omega - \widetilde{\mathrm{cl}}_{k,1}(\xi_t) \wedge \omega)\big) = 0 \tag{3.18}$$

for the Gauss-Manin connection ∇ on Z_C/C. Since $\widetilde{\mathrm{cl}}_{k,1}(\eta_t)$ is the restriction of $\widetilde{\mathrm{cl}}_{k,1}(\eta)$ defined on the smooth projective variety $Z_{\overline{C}}$, we have

$$\nabla\big(\pi_*(\widetilde{\mathrm{cl}}_{k,1}(\eta_t) \wedge \omega)\big) = 0. \tag{3.19}$$

Combining (3.18) and (3.19), we obtain

$$\nabla\big(\pi_*(\widetilde{\mathrm{cl}}_{k,1}(\xi_t) \wedge \omega)\big) = 0 \tag{3.20}$$

on Z_C/C.

By our construction of $S \times_B T$, we can choose two such curves C_i with two points $p_i \in C_i$ and maps $f_i : C_i \to U$ for $i = 1, 2$ satisfying that $f_1(p_1) = f_2(p_2) = b$ and the differential maps df_i of f_i on the tangent spaces of C_i at p_i satisfy that

$$T_{C_1,p_1} \oplus T_{C_2,p_2} \xrightarrow{df_1 \oplus df_2} T_{U,b} \tag{3.21}$$

is surjective. By shrinking U, let us assume that (3.21) holds for every $b \in U$.

Then by (3.21), we see that (3.20) actually holds on Z/U. Namely,

$$\nabla\big(\pi_*(\widetilde{\mathrm{cl}}_{k,1}(\xi_b) \wedge \omega)\big) = 0 \tag{3.22}$$

on Z/U for the Gauss-Manin connection ∇ on Z/U. And since Z/U is a complete family of Kummer surfaces, (3.22) implies that

$$\pi_*(\widetilde{\mathrm{cl}}_{k,1}(\xi_b)\wedge\omega)\in H^2_{\mathrm{tr}}(Z_b)^{\perp} \tag{3.23}$$

for all $b\in U$. And since (3.23) holds for all ω in the space (3.16), we conclude that

$$r_{k,1}(\xi_b)\in\Big(\bigotimes_{i=1}^{r}H^2_{\mathrm{tr}}(Y_i)\otimes H^2_{\mathrm{tr}}(Z_b)\otimes H^{4m-2k-2r}(\prod_{i=r+2}^{m}Y_i)\Big)^{\perp}$$

for all $b\in U$.

4. Bloch-Beilinson Conjecture on Abel-Jacobi Maps

The following conjecture stated in [Lew01], can be thought of as a variant of the Bloch-Beilinson conjecture:

CONJECTURE 4.1. *Let $V/\overline{\mathbb{Q}}$ be a smooth quasiprojective variety. Then the Abel-Jacobi map $\Phi_{k,\mathbb{Q}}:\mathrm{CH}^k_{\mathrm{hom}}(V/\overline{\mathbb{Q}};\mathbb{Q})\to J^k(V(\mathbb{C}))\otimes\mathbb{Q}$ is injective.*

Here the definition of the Abel-Jacobi map for smooth quasiprojective varieties, which is an extension of Griffiths' prescription, involves Carlson's extension class interpretation of intermediate jacobians ([Car80]). A detailed description of this map for example can be found in [Jan90, §9]. We now make use of the following result:

THEOREM 4.2 ([Lew01]). *Assume given a smooth projective variety $X/\mathbb{C}$. Then for all k, there is a filtration*

$$\begin{aligned}\mathrm{CH}^k(X;\mathbb{Q})=F^0\supset F^1\supset\cdots\supset F^{\ell}\supset F^{\ell+1}&\\ \supset\cdots\supset F^k\supset F^{k+1}=F^{k+2}=\cdots,&\end{aligned}$$

which satisfies the following

(i) $F^1=\mathrm{CH}^k_{\mathrm{hom}}(X;\mathbb{Q})$

(ii) $F^2\subset\ker\Phi_{k,\mathbb{Q}}:\mathrm{CH}^k_{\mathrm{hom}}(X;\mathbb{Q})\to J^k(X)\otimes\mathbb{Q}$.

(iii) $F^{\ell}\bullet F^r\subset F^{\ell+r}$, *where $\bullet$ is the intersection product.*

(iv) *F^ℓ is preserved under push-forwards f_* and pull-backs f^*, where $f : X \to Y$ is a morphism of smooth projective varieties. [In short, F^ℓ is preserved under the action of correspondences between smooth projective varieties.]*

(v) $\mathrm{Gr}_F^\ell := F^\ell/F^{\ell+1}$ *factors through the Grothendieck motive. More specifically, let us assume that the Künneth components of the diagonal class* $[\Delta] = \bigoplus_{p+q=2n}[\Delta(p,q)] \in H^{2n}(X \times X, \mathbb{Q})$ *are algebraic. Then*

$$\Delta(2n-2k+r, 2k-r)_*\Big|_{\mathrm{Gr}_F^\ell \mathrm{CH}^k(X;\mathbb{Q})} = \begin{cases} \text{Identity} & \text{if } r = \ell \\ 0 & \text{otherwise} \end{cases}$$

(vi) *Let $D^k(X) := \bigcap_\ell F^\ell$. If Conjecture 4.1 above holds, then $D^k(X) = 0$.*

Using Theorem 4.2, it was proved in [CL06, Lemma 3.2] that if Conjecture 4.1 holds, $\mathrm{CH}^2_{\mathbb{Q}}(X \times Y)$ has Künneth decomposition for a product $X \times Y$ of two smooth projective surfaces satisfying $H^1(X) = H^1(Y) = 0$ and

$$(4.1)\quad (H^2(X,\mathbb{Q}) \otimes H^2(Y,\mathbb{Q})) \cap H^{2,2}(X \times Y) = H^{1,1}(X,\mathbb{Q}) \otimes H^{1,1}(Y,\mathbb{Q}).$$

Let us verify (4.1) for a very general Kummer surface X and a Kummer surface Y with $\mathrm{rank}_{\mathbb{Z}} \mathrm{Pic}(Y) \le 19$. Actually, we have

Proposition 4.3. *Let $\pi : X \to B$ be a non-isotrivial smooth family of K3 surfaces over a smooth variety B and let Y be a smooth K3 surface. Then*

$$(H^2(X_b,\mathbb{Q}) \otimes H^2(Y,\mathbb{Q})) \cap H^{2,2}(X_b \times Y) = H^{1,1}(X_b,\mathbb{Q}) \otimes H^{1,1}(Y,\mathbb{Q})$$

for $b \in B$ very general. In particular, the identity (4.1) *holds for the product of a very general Kummer surface and an arbitrary smooth K3 surface.*

Proof. It suffices to prove

$$(4.2)\qquad (H^{1,1}(X_b,\mathbb{Q})^\perp \otimes H^{1,1}(Y,\mathbb{Q})^\perp) \cap H^{2,2}(X_b \times Y) = 0$$

for $b \in B$ very general, where $H^{1,1}(X_b, \mathbb{Q})^\perp$ and $H^{1,1}(Y, \mathbb{Q})^\perp$ are the orthogonal complements of $H^{1,1}(X_b, \mathbb{Q})$ and $H^{1,1}(Y, \mathbb{Q})$ in $H^2(X_b, \mathbb{Q})$ and $H^2(Y, \mathbb{Q})$, respectively.

We may take B to be a polydisk and assume that the Kodaira-Spencer map

$$T_{B,b} \longrightarrow H^1(T_{X_b})$$

is nonzero at all $b \in B$.

If (4.2) fails, after shrinking B, there exists

$$\eta \in H^0(B, (R^2\pi_*\mathbb{Q})_{\mathrm{tr}}) \otimes H^{1,1}(Y, \mathbb{Q})^\perp$$

such that

$$\eta_b \neq 0 \in H^{2,2}(X_b \times Y)$$

for all $b \in B$, where $(R^2\pi_*\mathbb{Q})_{\mathrm{tr}}$ is the subsheaf of $R^2\pi_*\mathbb{Q}$ orthogonal to the relative algebraic cycles of X/B.

Since η_b is orthogonal to

$$F^1H^2(X_b) \otimes H^{2,0}(Y) = (H^{1,1}(X_b) \oplus H^{2,0}(X_b)) \otimes H^{2,0}(Y),$$

we have

$$\langle \eta, \gamma \otimes \omega_Y \rangle = 0$$

for all $\gamma \in H^0(B, F^1R^2\pi_*\mathbb{C})$, where $\omega_Y \in H^{2,0}(Y)$ is a nonvanishing holomorphic 2-form on Y. Applying the Gauss-Manin connection, we obtain

$$\langle \eta, \nabla\gamma \otimes \omega_Y \rangle = 0$$

where we observe that $\nabla\eta = 0$. Since the Kodaira-Spencer map of π is nonzero, we have

$$\nabla(F^1R^2\pi_*\mathbb{C}) \not\subset F^1R^2\pi_*\mathbb{C} \otimes \Omega_B$$

due to the fact that the pairing $H^{1,1}(X_b) \otimes H^1(T_{X_b}) \to H^{0,2}(X_b)$ is nondegenerate. Thus, we conclude

$$\langle \eta_b, \xi_b \otimes \omega_Y \rangle = 0$$

for all $\xi_b \in H^2(X_b)$ and $b \in B$. That is,

$$\eta_b \in (H^2(X_b) \otimes H^{2,0}(Y))^\perp.$$

But we know that

$$\begin{aligned}
&(H^{1,1}(X_b,\mathbb{Q})^\perp \otimes H^{1,1}(Y,\mathbb{Q})^\perp) \cap (H^2(X_b)\otimes H^{2,0}(Y))^\perp \\
&= H^{1,1}(X_b,\mathbb{Q})^\perp \otimes (H^{1,1}(Y,\mathbb{Q})^\perp \cap H^{1,1}(Y)) \\
&= H^{1,1}(X_b,\mathbb{Q})^\perp \otimes (H^{1,1}(Y,\mathbb{Q})^\perp \cap H^{1,1}(Y,\mathbb{Q})) = 0.
\end{aligned}$$

This leads to $\eta_b = 0$, which is a contradiction. □

Combining the above proposition and [CL06, Lemma 3.2], we are able to apply Theorem 1.2 to the case $(k,r,m,n) = (3,1,2,4)$ and conclude that the Hodge-$\mathscr{D}$-conjecture fails for the real regulator $r_{3,1}$ on a product of four very general elliptic curves, if the Bloch-Beilinson Conjecture 4.1 holds.

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