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The Extension of Holomorphic Functions on a Non-Pluriharmonic Locus

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Abstract. Let $n \geq 4$ and let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let φ be a negative exhaustive smooth plurisubharmonic function on Ω. We show that any holomorphic function defined on a connected open neighborhood of the support of $(i\partial\overline{\partial}\varphi)^{n-3}$ can be extended to a holomorphic function on Ω.

1. Introduction

Hartogs's extension theorem is stated as follows:

Let Ω be an open subset in $\mathbb{C}^n$ ($n \geq 2$) and let $K \subset \Omega$ be a compact subset such that $\Omega \setminus K$ is connected. Then any holomorphic function on $\Omega \setminus K$ can be extended to a holomorphic function on Ω.

This is one of the major difference between the theory of one and several complex variables since any open subset is a domain of holomorphy in the case of one variable. In this paper, we give a new example of a subdomain such that any holomorphic function on the subdomain can be extended holomorphically to the entire domain.

Let T be a smooth form or a current in a domain in $\mathbb{C}^n$. We denote by $\operatorname{supp} T$ the support of T. Our main theorem is the following:

Theorem 1. *Let $n \geq 4$ and Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let φ be a negative smooth plurisubharmonic function on Ω such that $\varphi(z) \to 0$ when $z \to \partial\Omega$. Let $V \subset \Omega$ be a connected open neighborhood of $\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-3}$. Then any holomorphic function on V can be extended to a holomorphic function on Ω.*

If a holomorphic function is defined on a non-pluriharmonic locus, we can remove the assumption of the regularity of φ.

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THEOREM 2. *Let $n \geq 4$ and Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let φ be a negative continuous plurisubharmonic function on Ω such that $\varphi(z) \to 0$ when $z \to \partial\Omega$. Let $V \subset \Omega$ be a connected open neighborhood of* $\operatorname{supp} i\partial\overline{\partial}\varphi$. *Then any holomorphic function on V can be extended to a holomorphic function on Ω.*

We explain a motivation of Theorem 1. Let E be a compact subset of $\mathbb{C}^n$. We define the Shilov boundary of E by the smallest closed subset $\partial_S E$ of E such that, for each function f which is holomorphic on a neighborhood of E the equality $\max_E |f| = \max_{\partial_S E} |f|$ holds. Let $B_\varphi(r) = \{z \in \Omega \,|\, \varphi(z) < r\}$ and let $x \in B_\varphi(r)$. It is known that there exists a probability measure μ_x supported on $\partial_S \overline{B_\varphi(r)}$ such that $f(x) = \int f d\mu_x$ for any holomorphic functions on an open neighborhood of $\overline{B_\varphi(r)}$ (see [7]). This measure is called Jensen measure. Hence we may consider that Shilov boundaries of $\overline{B_\varphi(r)}$ $(r < 0)$ are important for the existence of holomorphic functions on Ω. On the other hand, [1] shows that there exists a complex foliation on $\Omega \setminus \operatorname{supp}(i\partial\overline{\partial}\varphi)^j$ $(1 \leq j \leq n)$ by complex submanifolds such that the restriction of φ on any leaf of the foliation is pluriharmonic. It follows that, for any $z \in \Omega \setminus \operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-1}$, there exists a complex curve through z contained in a level set of φ. Then z is not contained in the Shilov boundaries of level sets of φ. In this context, it might be interesting to ask whether one can extend holomorphic functions defined on $\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-1}$ to the holomorphic functions on Ω. In our theorem, we show that this question is true if $\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-1}$ is replaced by $\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-3}$.

The proof consists in solving $\overline{\partial}$ equation in the L^2-space defined by the degenerate Monge-Ampère measure. In Section 3, we prove Donnelly-Fefferman and Berndtsson type L^2-estimate ([6], [3]) . In Lemma 1 and Lemma 3, we use the argument in Theorem 2.3 of [4] to prove our L^2-estimate from (2) below. We solve $\overline{\partial}$ equations in the L^2-spaces defined by the complete Kähler metrics which converge to $-i\partial\overline{\partial}(\log(-\varphi))$, which is no longer a Kähler metric in general. To guarantee the weak convergence of solutions constructed in Section 3, we show an interior estimate of the solutions. Section 4 can be read independently of other sections.

After this work appeared in arXiv, Lee and Nagata [10] generalizes the main result of this paper. Their proof, which uses L^2 Serre duality, is simpler than ours. However, the estimates in Section 4 are not contained in [10] and we think it has some meaning to publish this paper.

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2. Preliminaries

First we introduce the set up and some notations. For details, we refer to [5]. Let Ω be a domain in $\mathbb{C}^n$ and let ω be a Kähler metric on Ω. Let ψ be a smooth function on Ω. By $L^2_{p,q}(\Omega, e^{\psi}, \omega)$ we denote the Hilbert space of (p,q)-forms α which satisfy

$$\|\alpha\|^2_{\psi,\omega} := \int_{\Omega} |\alpha|^2_{\omega} e^{\psi} dV_{\omega}.$$

Here $dV_{\omega} = (n!)^{-1}\omega^n$. For simplicity we put $L^2(\Omega, e^{\psi}, \omega) = L^2_{0,0}(\Omega, e^{\psi}, \omega)$. Let $A^2(\Omega, e^{\psi}, \omega)$ be the space of all holomorphic functions in $L^2(\Omega, e^{\psi}, \omega)$. Let $\overline{\partial}^*_{\psi}$ be the Hilbert space adjoint of linear, closed, densely defined operator

$$\overline{\partial} : L^2_{p,q}(\Omega, e^{\psi}, \omega) \to L^2_{p.q+1}(\Omega, e^{\psi}, \omega).$$

Let Λ_{ω} be the adjoint of multiplication by ω. If $q \geq 1$ and ω is a complete Kähler metric, the Bochner-Kodaira-Nakano inequality shows that

$$\|\overline{\partial}\alpha\|^2_{\psi,\omega} + \|\overline{\partial}^*_{\psi}\alpha\|^2_{\psi,\omega} \geq \int_{\Omega} \langle[-i\partial\overline{\partial}\psi, \Lambda_{\omega}]\alpha, \alpha\rangle_{\omega} e^{\psi} dV_{\omega}$$

for any $\alpha \in L^2_{p,q}(\Omega, e^{\psi}, \omega)$ which is contained in the both domains of $\overline{\partial}$ and $\overline{\partial}^*_{\psi}$. At each point $x \in \Omega$, we may choose an orthonormal basis $\sigma_1, \dots, \sigma_n$ for the holomorphic cotangent bundle with respect to ω such that $i\partial\overline{\partial}\psi = \lambda_1 i\sigma_1 \wedge \overline{\sigma_1} + \cdots + \lambda_n i\sigma_n \wedge \overline{\sigma_n}$. Let α be a $(0,q)$-form. We write $\alpha = \sum_{|J|=q} \alpha_J \overline{\sigma_J}$ where $J = (j_1, \dots, j_q)$ is a multi-index with $j_1 < \cdots < j_q$ and $\overline{\sigma_J} = \overline{\sigma_{j_1}} \wedge \cdots \wedge \overline{\sigma_{j_q}}$. Then

$$[-i\partial\overline{\partial}\psi, \Lambda_{\omega}]\alpha = \sum_{|J|=q} \left(\sum_{1\leq j\leq n, j\notin J} \lambda_j \right) \alpha_J \overline{\sigma_J}. \tag{1}$$

Assume that the operator $A_{\omega,\psi} = [-i\partial\overline{\partial}\psi, \Lambda_{\omega}]$ is positive definite on $L^2_{p,q}(\Omega, e^{\psi}, \omega)$, and that ω is a complete Kähler metric. Then, for any closed form $\alpha \in L^2_{p,q}(\Omega, e^{\psi}, \omega)$ which satisfies $\int_\Omega \langle A^{-1}_{\omega,\psi}\alpha, \alpha\rangle_\omega e^{\psi} dV_\omega < +\infty$, there exists $u \in L^2_{p,q-1}(\Omega, e^{\psi}, \omega)$ such that $\overline{\partial} u = \alpha$ and

$$\|u\|^2_{\psi,\omega} \leq \int_\Omega \langle A^{-1}_{\omega,\psi}\alpha, \alpha\rangle_\omega e^{\psi} dV_\omega. \tag{2}$$

3. Weighted L^2-Estimate

The purpose of this section is Proposition 1 below. Let $n \geq 4$. Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and let φ be a negative smooth plurisubharmonic function on Ω such that $\varphi(z) \to 0$ when $z \to \partial\Omega$. Let $\phi = -(\log(-\varphi))$. Then ϕ is an exhaustive smooth plurisubharmonic function such that $i\partial\overline{\partial}\phi \geq i\partial\phi \wedge \overline{\partial}\phi$. Let ψ be a smooth strongly plurisubharmonic function on a neighborhood of $\overline{\Omega}$. To prove our main theorem, we may assume that $\psi = |z|^2$. Let $\varepsilon > 0$ be a small positive number and let $\omega_\varepsilon = i\partial\overline{\partial}(\varepsilon\psi + \phi)$. Then ω_ε is a complete Kähler metric on Ω since ϕ is exhaustive and $|\partial\phi|_{\omega_\varepsilon} < 1$. Let $c > 0$ and let $A_{\varepsilon,c} = A_{\omega_\varepsilon, \psi + c\phi} = [-i\partial\overline{\partial}(\psi + c\phi), \Lambda_{\omega_\varepsilon}]$. We start by showing the following lemma:

Lemma 1. *Let $\delta, \delta' > 0$. Let α be a $\overline{\partial}$-closed $(0,1)$-form such that $\alpha \in L^2_{0,1}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$. Assume that $n > 1 + \delta$ and that $\varepsilon \leq \delta^{-1}$. Then there exists a function $u \in L^2(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$ such that $\overline{\partial} u = \alpha$ and*

$$\int_\Omega |u|^2 e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \leq C_{n,\delta} \int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}\alpha, \alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon}.$$

Here $C_{n,\delta}$ is a positive constant which depends only on n and δ.

Proof. Since $C\omega_\varepsilon \leq i\partial\overline{\partial}(\psi + \delta'\phi)$ for some $C > 0$, we have that $A^{-1}_{\varepsilon,\delta'} \leq C^{-1}$. By (2), there exists the solution $u \in L^2(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$ to $\overline{\partial} u = \alpha$ which is minimal in the $L^2(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$ norm. This means that $u \in A^2(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)^\perp$. Since $\phi(z) \to +\infty$ when $z \to \partial\Omega$, we have that $ue^{-\delta\phi} \in L^2(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon)$ and $A^2(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon) \subset A^2(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$. Hence $ue^{-\delta\phi} \in L^2(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon) \cap A^2(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon)^\perp$. It follows that $\overline{\partial}(ue^{-\delta\phi}) = (\alpha - \delta u\overline{\partial}\phi)e^{-\delta\phi} \in L^2_{0,1}(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon)$ since $|\overline{\partial}\phi|_{\omega_\varepsilon} < 1$ and $u \in L^2(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$. Then $ue^{-\delta\phi}$ is the minimal solution in the $L^2(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon)$ norm to $\overline{\partial}(ue^{-\delta\phi})$. We note that $A^{-1}_{\varepsilon,\delta+\delta'}$ is bounded from above in Ω. By (2), we have that

$$\begin{aligned}\int_\Omega |u|^2 e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} &= \int_\Omega |ue^{-\delta\phi}|^2 e^{\psi+(\delta+\delta')\phi} dV_{\omega_\varepsilon} \\ \leq \int_\Omega &\langle A^{-1}_{\varepsilon,\delta+\delta'}(\alpha-\delta u\overline{\partial}\phi), \alpha-\delta u\overline{\partial}\phi\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \\ \leq \left(1+\frac{1}{t}\right)&\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}\alpha,\alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \\ &+ (1+t)\delta^2 \int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}\overline{\partial}\phi, \overline{\partial}\phi\rangle_{\omega_\varepsilon} |u|^2 e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon}\end{aligned}$$

for every $t > 0$. Since $\varepsilon < \delta^{-1}$, we have that $\delta\omega_\varepsilon = \delta i\partial\overline{\partial}(\varepsilon\psi+\phi) \leq i\partial\overline{\partial}(\psi+(\delta+\delta')\phi)$. By (1), it follows that $\langle A_{\varepsilon,\delta+\delta'}\beta,\beta\rangle_{\omega_\varepsilon} \geq (n-1)\delta|\beta|^2_{\omega_\varepsilon}$ for any $(0,1)$-form β. Hence $\langle A^{-1}_{\varepsilon,\delta+\delta'}\overline{\partial}\phi,\overline{\partial}\phi\rangle_{\omega_\varepsilon} \leq \frac{1}{(n-1)\delta}|\overline{\partial}\phi|^2_{\omega_\epsilon} < \frac{1}{(n-1)\delta}$. By choosing t so small, there exists a constant C_1 which depends only on n and δ such that $(1+t)\frac{\delta}{n-1} < C_1 < 1$ since $n > 1+\delta$. Then we have

$$(1-C_1)\int_\Omega |u|^2 e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \leq C_2 \int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}\alpha,\alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon}.$$

Here $C_2 = \left(1+\frac{1}{t}\right)$ depends only on n and δ. This completes the proof. □

If there exists a sequence of $\overline{\partial}$-closed $(0,1)$-forms in $L^2_{0,1}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$ which approximates α, we can remove the assumption that $\alpha \in L^2_{0,1}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$ from Lemma 1.

Lemma 2. *Let $\delta, \delta' > 0$. and let α be a $\overline{\partial}$-closed $(0,1)$-form such that $\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}\alpha,\alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} < +\infty$. Assume that there exist $\overline{\partial}$-closed $(0,1)$-forms $\alpha_j \in L^2(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$ $(j = 1, 2, \dots)$ such that*

$$\lim_{j\to\infty}\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}(\alpha-\alpha_j), \alpha-\alpha_j\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} = 0.$$

Assume that $n > 1+\delta$ and that $\varepsilon \leq \delta^{-1}$. Then there exists $u \in L^2(\Omega, e^{\psi-(\delta-\delta')\phi}, \omega_\varepsilon)$ such that $\overline{\partial}u = \alpha$ and

$$\int_\Omega |u|^2 e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \leq C_{n,\delta}\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}\alpha,\alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon}.$$

Here $C_{n,\delta} > 0$ depends only on n and δ.

Proof. By Lemma 1, there exist u_j $(j = 1, 2, \dots)$ such that $\overline{\partial} u_j = \alpha_j$ and

$$\int_\Omega |u_j|^2 e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \le C_{n,\delta} \int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'} \alpha_j, \alpha_j\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon}$$

$$\le 2C_{n,\delta} \left(\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'} \alpha, \alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \right.$$
$$\left. + \int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'} (\alpha-\alpha_j), \alpha-\alpha_j\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \right).$$

Therefore we may choose a subsequence of $\{u_j\}_{j\in\mathbb{N}}$ converging weakly in $L^2(\Omega, e^{\psi-(\delta-\delta')\phi}, \omega_\varepsilon)$ to u. Since $\alpha_j \to \alpha$ $(j \to \infty)$ in the distribution sense, we have that $\overline{\partial} u = \alpha$ and

$$\int_\Omega |u|^2 e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \le 2C_{n,\delta} \int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'} \alpha, \alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon}. \square$$

Next, we construct a sequence which approximates α.

Lemma 3. *Let $\delta, \delta' > 0$. Let $\alpha \in L^2_{0,1}(\Omega, e^{\psi-(\delta-\delta')\phi}, \omega_\varepsilon)$ such that $\overline{\partial}\alpha = 0$. Assume that $n > 2 + \delta$ and that $\varepsilon \le \delta^{-1}$. Then there exist $\overline{\partial}$-closed $(0,1)$-forms $\alpha_j \in L^2_{0,1}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$ $(j = 1, 2, \dots)$ such that $\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}(\alpha - \alpha_j), \alpha-\alpha_j\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} \to 0$ when $j \to \infty$.*

Proof. First, note that $\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'} \alpha, \alpha\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} < +\infty$ since $\langle A^{-1}_{\varepsilon,\delta+\delta'} \alpha, \alpha\rangle_{\omega_\varepsilon} \le \frac{1}{(n-1)\delta} |\alpha|^2_{\omega_\varepsilon}$ by the proof of Lemma 1. Let $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 1$ for $t < 0$, $\chi(t) = 0$ for $t > 2$ and $|\chi'| \le 1$. Let $h_j = \chi(\phi - j) \in C^\infty_0(\Omega)$. Let N_1, resp. N_2, be the kernel space of linear, closed, densely defined operator $\overline{\partial} : L^2_{0,1}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon) \to L^2_{0,2}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$, resp. $\overline{\partial} : L^2_{0,1}(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon) \to L^2_{0,2}(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon)$. We have that $\overline{\partial}(h_j\alpha) \in L^2_{0,2}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon)$. By a reasoning analogous to that of the proof of Lemma 1, there exists $\beta_j \in L^2_{0,1}(\Omega, e^{\psi+\delta'\phi}, \omega_\varepsilon) \cap N_1^\perp$ such that $\overline{\partial}\beta_j = \overline{\partial}(h_j\alpha)$, $\beta_j e^{-\delta\phi} \in L^2_{0,1}(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon) \cap N_2^\perp$ and $\overline{\partial}(\beta_j e^{-\delta\phi}) \in L^2_{0,2}(\Omega, e^{\psi+(\delta+\delta')\phi}, \omega_\varepsilon)$. By (2), we have that

$$\int_\Omega |\beta_j|^2_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon} = \int_\Omega |\beta_j e^{-\delta\phi}|^2_{\omega_\varepsilon} e^{\psi+(\delta+\delta')\phi} dV_{\omega_\varepsilon}$$

$$\le \int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}(\overline{\partial}\beta_j - \delta\overline{\partial}\phi \wedge \beta_j), \overline{\partial}\beta_j - \delta\overline{\partial}\phi \wedge \beta_j\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi} dV_{\omega_\varepsilon}$$

$$
\begin{aligned}
&\le \left(1+\frac{1}{t}\right)\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}\overline{\partial}\beta_j,\overline{\partial}\beta_j\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}\\
&\quad + (1+t)\delta^2\int_\Omega \langle A^{-1}_{\varepsilon,\delta+\delta'}(\overline{\partial}\phi\wedge\beta_j),\overline{\partial}\phi\wedge\beta_j\rangle_{\omega_\varepsilon} e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}
\end{aligned}
$$

for every $t>0$. Here we regard $A_{\varepsilon,\delta+\delta'}$ as an endomorphism of the space of $(0,2)$-forms. Since $\varepsilon<\delta^{-1}$, we have that $\delta\omega_\varepsilon=\delta i\partial\overline{\partial}(\varepsilon\psi+\phi)\le i\partial\overline{\partial}(\psi+(\delta+\delta')\phi)$. By (1), it follows that $\langle A_{\varepsilon,\delta+\delta'}\gamma,\gamma\rangle_{\omega_\varepsilon}\ge(n-2)\delta|\gamma|^2_{\omega_\varepsilon}$ for any $(0,2)$-form γ. Hence $\langle A^{-1}_{\varepsilon,\delta+\delta'}(\overline{\partial}\phi\wedge\beta_j),\overline{\partial}\phi\wedge\beta_j\rangle_{\omega_\varepsilon}\le\frac{1}{(n-2)\delta}|\overline{\partial}\phi\wedge\beta_j|^2_{\omega_\epsilon}\le\frac{1}{(n-2)\delta}|\overline{\partial}\phi|^2_{\omega_\varepsilon}|\beta_j|^2_{\omega_\varepsilon}<\frac{1}{(n-2)\delta}|\beta_j|^2_{\omega_\varepsilon}$. By choosing t so small, there exists a constant C_1 which depends only on n and δ such that $(1+t)\frac{\delta}{n-2}<C_1<1$ since $n>2+\delta$. Then we have that

$$
\int_\Omega|\beta_j|^2_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}\le C_2\int_\Omega\langle A^{-1}_{\varepsilon,\delta+\delta'}\overline{\partial}\beta_j,\overline{\partial}\beta_j\rangle_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}
$$

where $C_2=(1-C_1)^{-1}\left(1+\frac{1}{t}\right)$ which depends only on n and δ. It follows that

$$
\begin{aligned}
&\int_\Omega\langle A^{-1}_{\varepsilon,\delta+\delta'}\overline{\partial}\beta_j,\overline{\partial}\beta_j\rangle_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}\le\frac{1}{(n-2)\delta}\int_\Omega|\overline{\partial}\beta_j|^2_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}\\
&=\frac{1}{(n-2)\delta}\int_\Omega|\overline{\partial}h_j\wedge\alpha|^2_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}\\
&\le\frac{1}{(n-2)\delta}\int_{\{j\le\phi\le j+2\}}|\alpha|^2_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}.
\end{aligned}
$$

Because $\alpha\in L^2_{0,1}(\Omega,e^{\psi-(\delta-\delta')\phi},\omega_\varepsilon)$, Lebesgue's dominated convergence theorem shows that the last term of the above inequality tends to 0 when j tends to $+\infty$. By the proof of Lemma 1, we have that $\langle A^{-1}_{\varepsilon,\delta+\delta'}\beta_j,\beta_j\rangle_{\omega_\varepsilon}\le\frac{1}{\delta(n-1)}|\beta_j|^2_{\omega_\varepsilon}$. Finally, we have that

$$
\begin{aligned}
&\lim_{j\to\infty}\int_\Omega\langle A^{-1}_{\varepsilon,\delta+\delta'}\beta_j,\beta_j\rangle_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}\\
&\le\lim_{j\to\infty}\frac{1}{\delta(n-1)}\int_\Omega|\beta_j|^2_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}=0.
\end{aligned}
$$

Let $\alpha_j=h_j\alpha-\beta_j$. Then $\alpha_j\in L^2_{0,1}(\Omega,e^{\psi+\delta'\phi},\omega_\varepsilon)$, $\overline{\partial}\alpha_j=0$, and

$$
\lim_{j\to\infty}\int_\Omega\langle A^{-1}_{\varepsilon,\delta+\delta'}(\alpha-\alpha_j),\alpha-\alpha_j\rangle_{\omega_\varepsilon}e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}
$$

$$\leq \lim_{j\to\infty} 2\int_\Omega \left((1-h_j)^2\langle A^{-1}_{\varepsilon,\delta+\delta'}\alpha,\alpha\rangle_{\omega_\varepsilon} + \langle A^{-1}_{\varepsilon,\delta+\delta'}\beta_j,\beta_j\rangle_{\omega_\varepsilon}\right) e^{\psi-(\delta-\delta')\phi}dV_{\omega_\varepsilon}$$
$$=0$$

by Lebesgue's dominated convergence theorem. □

LEMMA 4. *Let $\delta,\delta'>0$. Let $k<n-1$. Let α be a smooth $(0,1)$-form on Ω such that $\operatorname{supp}\alpha\subset\Omega\setminus\operatorname{supp}(i\partial\overline{\partial}\varphi)^k$. Then $\langle A^{-1}_{\varepsilon,\delta+\delta'}\alpha,\alpha\rangle_{\omega_\varepsilon}\leq (n-k-1)^{-1}|\alpha|^2_{i\partial\overline{\partial}\psi}$.*

PROOF. We have that $A^{-1}_{\varepsilon,\delta+\delta'}\leq A^{-1}_{\varepsilon,\delta}$. Hence it is enough to prove $\langle A^{-1}_{\varepsilon,\delta}\alpha,\alpha\rangle_{\omega_\varepsilon}\leq (n-k-1)^{-1}|\alpha|^2_{i\partial\overline{\partial}\psi}$. Because $i\partial\overline{\partial}\phi = i\frac{\partial\overline{\partial}\varphi}{-\varphi}+i\frac{\partial\varphi\wedge\overline{\partial}\varphi}{\varphi^2}$, we have that $\operatorname{supp}\alpha\subset\Omega\setminus\operatorname{supp}(i\partial\overline{\partial}\phi)^{k+1}$. Let $x\in\Omega\setminus\operatorname{supp}(i\partial\overline{\partial}\phi)^{k+1}$. At x, we choose an orthonormal basis $\theta_1,\dots,\theta_n$ for the holomorphic cotangent bundle with respect to $i\partial\overline{\partial}\psi$ such that $i\partial\overline{\partial}\phi = i\lambda_1\theta_1\wedge\overline{\theta_1}+\cdots+i\lambda_k\theta_k\wedge\overline{\theta_k}$ where $\lambda_j\geq 0$ for $1\leq j\leq k$. Then $\omega_\varepsilon = i\partial\overline{\partial}(\varepsilon\psi+\phi)=\sum_{j=1}^k i(\varepsilon+\lambda_j)\theta_j\wedge\overline{\theta_j}+\sum_{l=k+1}^n i\varepsilon\theta_l\wedge\overline{\theta_l}$ and $i\partial\overline{\partial}(\psi+\delta\phi)=\sum_{j=1}^k i(1+\delta\lambda_j)\theta_j\wedge\overline{\theta_j}+\sum_{l=k+1}^n i\theta_l\wedge\overline{\theta_l}$. Let $\sigma_j=\sqrt{\varepsilon+\lambda_j}\theta_j$ for $1\leq j\leq k$ and let $\sigma_l=\sqrt{\varepsilon}\theta_l$ for $k+1\leq l\leq n$. Then $\omega_\varepsilon=\sum_{j=1}^n i\sigma_j\wedge\overline{\sigma_j}$ and $i\partial\overline{\partial}(\psi+\delta\phi)=\sum_{j=1}^k i\frac{1+\delta\lambda_j}{\varepsilon+\lambda_j}\sigma_j\wedge\overline{\sigma_j}+\sum_{l=k+1}^n i\frac{1}{\varepsilon}\sigma_l\wedge\overline{\sigma_l}$. By (1), it follows that

$$\langle A^{-1}_{\varepsilon,\delta}\overline{\sigma_j},\overline{\sigma_s}\rangle_{\omega_\varepsilon} = \langle\Bigg(\sum_{\substack{1\leq m\leq k\\ m\neq j}}\frac{1+\delta\lambda_m}{\varepsilon+\lambda_m}+\frac{n-k}{\varepsilon}\Bigg)^{-1}\overline{\sigma_j},\overline{\sigma_s}\rangle_{\omega_\varepsilon}\leq\frac{\varepsilon}{n-k}\delta_{js}$$

for $j\leq k$, $1\leq s\leq n$, and

$$\langle A^{-1}_{\varepsilon,\delta}\overline{\sigma_l},\overline{\sigma_s}\rangle_{\omega_\varepsilon} = \langle\Bigg(\sum_{1\leq m\leq k}\frac{1+\delta\lambda_m}{\varepsilon+\lambda_m}+\frac{n-k-1}{\varepsilon}\Bigg)^{-1}\overline{\sigma_l},\overline{\sigma_s}\rangle_{\omega_\varepsilon}\leq\frac{\varepsilon}{n-k-1}\delta_{ls}$$

for $l\geq k+1$, $1\leq s\leq n$. Here δ_{js}, δ_{ls} are the Kronecker delta. Write $\alpha=\sum_{j=1}^n\alpha_j\overline{\theta_j}=\sum_{j=1}^k\frac{\alpha_j}{\sqrt{\varepsilon+\lambda_j}}\overline{\sigma_j}+\sum_{l=k+1}^n\frac{\alpha_l}{\sqrt{\varepsilon}}\overline{\sigma_l}$. We have that

$$\begin{aligned}\langle A^{-1}_{\varepsilon,\delta}\alpha,\alpha\rangle_{\omega_\varepsilon} &\leq \sum_{j=1}^k\frac{|\alpha_j|^2}{\varepsilon+\lambda_j}\frac{\varepsilon}{n-k}+\sum_{l=k+1}^n\frac{|\alpha_l|^2}{\varepsilon}\frac{\varepsilon}{n-k-1}\\ &\leq (n-k-1)^{-1}|\alpha|^2_{i\partial\overline{\partial}\psi}.\ \square\end{aligned}$$

LEMMA 5. *Assume that $\varphi \in C^\infty(\overline{\Omega})$ and that $d\varphi \neq 0$ on $\partial\Omega$. Let $p \in \partial\Omega$ and let $1 \leq k \leq n$ be an integer. Assume that $(i\partial\overline{\partial}\varphi)^k = 0$ in a neighborhood of p. If $\delta > k$, then $e^{\psi-\delta\phi}dV_{\omega_\varepsilon}$ is integrable around p in Ω.*

PROOF. Let $C_1, C_2, \ldots$ be sufficiently large positive constants. Since $\omega_\varepsilon = i\varepsilon\partial\overline{\partial}\psi + i\frac{\partial\overline{\partial}\varphi}{-\varphi} + i\frac{\partial\varphi\wedge\overline{\partial}\varphi}{\varphi^2}$, we have that $dV_{\omega_\varepsilon} = (n!)^{-1}\omega_\varepsilon^n \leq C_1(-\varphi)^{-k-1}(i\partial\overline{\partial}|z|^2)^n$ and $e^{\psi-\delta\phi}dV_{\omega_\varepsilon} \leq C_2(-\varphi)^{\delta-k-1}(i\partial\overline{\partial}|z|^2)^n$ around p. Let U be a small neighborhood of p. There exists a local coordinate system $(x_1, \ldots, x_{2n})$ on U such that $\varphi = x_1$. It follows that

$$\int_{\Omega\cap U} e^{\psi-\delta\phi}dV_{\omega_\varepsilon} \leq C_3\int_{\Omega\cap U} x_1^{\delta-k-1}dx_1\cdots dx_{2n} \leq C_4\int_0^1 x_1^{\delta-k-1}dx_1 < +\infty,$$

since $\delta > k$. □

PROPOSITION 1. *Let $\Omega \subset \mathbb{C}^n$ $(n \geq 4)$ be a bounded hyperconvex domain and let ψ be a smooth strongly plurisubharmonic function on a neighborhood of $\overline{\Omega}$. Let $\varphi \in C^\infty(\overline{\Omega})$ such that φ is negative plurisubharmonic on Ω, $\varphi(z) \to 0$ when $z \to \partial\Omega$, and $d\varphi \neq 0$ on $\partial\Omega$. Let α be a smooth $(0,1)$-form defined on an open neighborhood of $\overline{\Omega}$ such that $\overline{\partial}\alpha = 0$ in Ω and $\operatorname{supp}\alpha \subset \overline{\Omega} \setminus \operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-3}$ in $\overline{\Omega}$. Let δ be a positive constant such that $n-3 < \delta < n-2$. Then there exists $u \in C^\infty(\Omega)$ such that $\overline{\partial}u = \alpha$ and*

$$\int_\Omega |u|^2 e^{\psi-\delta\phi}dV_{\omega_\varepsilon} \leq C_{n,\delta}\int_\Omega |\alpha|^2_{i\partial\overline{\partial}\psi}e^{\psi-\delta\phi}dV_{\omega_\varepsilon} < +\infty$$

for sufficiently small $\varepsilon > 0$. Here $C_{n,\delta}$ is a positive constant which depends only on n and δ.

PROOF. Since $|\alpha|^2_{\omega_\varepsilon} \leq |\alpha|^2_{\varepsilon i\partial\overline{\partial}\psi}$, the norm $|\alpha|^2_{\omega_\varepsilon}$ is bounded from above in Ω. Then $|\alpha|^2_{\omega_\varepsilon}e^{\psi-\delta\phi}dV_{\omega_\varepsilon}$ is integrable by Lemma 5. Let $\delta' > 0$ be a sufficiently small positive number such that $\delta+\delta' < n-2$. We put $\delta'' = \delta+\delta'$. Then δ', δ'' depend only on n and δ. We have $\alpha \in L^2(\Omega, e^{\psi-(\delta''-\delta')\phi}, \omega_\varepsilon)$. By replacing δ with δ'' in Lemma 2, 3 and 4, it follows that there exists $u \in L^2(\Omega, e^{\psi-(\delta''-\delta')\phi}, \omega_\varepsilon)$ such that $\overline{\partial}u = \alpha$ and

$$\int_\Omega |u|^2 e^{\psi-(\delta''-\delta')\phi}dV_{\omega_\varepsilon} \leq C_{n,\delta}\int_\Omega |\alpha|^2_{i\partial\overline{\partial}\psi}e^{\psi-(\delta''-\delta')\phi}dV_{\omega_\varepsilon}.$$

Then we have the proposition since $\delta = \delta'' - \delta'$. The smoothness of u is known (see [5], [8]). □

4. Interior Estimate of Non-Negative Plurisubharmonic Functions

The purpose of this section is the following theorem:

THEOREM 3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and let φ be a negative continuous plurisubharmonic function on Ω such that $\varphi(z) \to 0$ when $z \to \partial\Omega$. Let $v \geq 0$ be a plurisubharmonic function on Ω. Then*

$$\int_{\{\varphi<r\}} i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}|z|^2)^{n-1} \leq C \int_\Omega vi\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1},$$

$$\int_{\{\varphi<r\}} v(i\partial\overline{\partial}|z|^2)^n \leq C \int_\Omega \big(v(i\partial\overline{\partial}\varphi)^n + vi\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1}\big)$$

for $r < 0$. Here $C = \big(1 + d(\Omega) + \sup|\varphi| + |r|^{-1}\big)^{C_n}$, $d(\Omega)$ is the diameter of Ω, and C_n is a positive constant which depends only on n.

In the above theorem, $i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}|z|^2)^{n-1}$, $i\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1}$, and $(i\partial\overline{\partial}\varphi)^n$ are defined in the sense of Bedford-Taylor (see [2], [9]).

LEMMA 6. *Let k be a non-negative integer. We assume the same hypothesis of Theorem 3, and we assume that $v, \varphi \in C^\infty(\overline{\Omega})$ and that $d\varphi \neq 0$ on $\partial\Omega$. Then*

$$\int_{\{\varphi<r\}} i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1}$$
$$\leq C_{n,k} \frac{(d(\Omega)^2 \sup|\varphi|)^{n-k-1}}{r^{2(n-k)}} \int_\Omega vi\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1}.$$

Here $C_{n,k}$ is a positive constant which depends only on n and k.

PROOF. By the Stokes theorem, we have that

$$\begin{aligned}
&\int_\Omega vi\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1} \\
&= -\int_\Omega \varphi i\partial v \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1} + \int_\Omega (-\varphi)v(i\partial\overline{\partial}\varphi)^n \\
&\geq -\frac{1}{2}\int_\Omega i\partial v \wedge \overline{\partial}\varphi^2 \wedge (i\partial\overline{\partial}\varphi)^{n-1} = \frac{1}{2}\int_\Omega \varphi^2 i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^{n-1},
\end{aligned}$$

and we have that

$$\int_\Omega \varphi^2 i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^{n-1} \leq 2\int_\Omega vi\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1}. \tag{3}$$

Without loss of generality, we may assume that $0 \in \partial\Omega$. Let $\eta = \frac{|r|}{2d(\Omega)^2}(|z|^2 - 2d(\Omega)^2)$. We have that η is smooth plurisubharmonic function such that $r < \eta < \frac{r}{2}$ in Ω. For sufficiently small $\epsilon > 0$, we put $\rho = \max_\epsilon\{\varphi, \eta\}$ where $\max_\epsilon$ is a regularized max function (see Chapter I, Section 5 of [5]). Then ρ is a smooth plurisubharmonic function on Ω such that $\rho = \varphi$ near $\{z \in \Omega \,|\, \varphi(z) = \frac{r}{3}\}$ and $\rho = \eta$ on $\{z \in \Omega \,|\, \varphi(z) < r\}$. After a slight perturbation of r, we may assume that $d\varphi \neq 0$ on $\{z \in \Omega \,|\, \varphi(z) = \frac{r}{3}\}$. By the Stokes theorem, we have that

$$\begin{aligned}
&\int_{\{\varphi<r\}} i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1} \\
&= \frac{2d(\Omega)^2}{|r|}\int_{\{\varphi<r\}} i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^k \wedge i\partial\overline{\partial}\rho \wedge (i\partial\overline{\partial}|z|^2)^{n-k-2} \\
&\leq \frac{2d(\Omega)^2}{|r|}\frac{3}{2|r|}\int_{\{\varphi<r\}} \left(\frac{r}{3} - \varphi\right) i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^k \wedge i\partial\overline{\partial}\rho \wedge (i\partial\overline{\partial}|z|^2)^{n-k-2} \\
&\leq \frac{3d(\Omega)^2}{|r|^2}\int_{\{\varphi<r/3\}} \left(\frac{r}{3} - \varphi\right) i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^k \wedge i\partial\overline{\partial}\left(\rho - \frac{r}{3}\right) \wedge (i\partial\overline{\partial}|z|^2)^{n-k-2} \\
&= \frac{3d(\Omega)^2}{|r|^2}\int_{\{\varphi<r/3\}} \left(\frac{r}{3} - \rho\right) i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^k \wedge i\partial\overline{\partial}\left(\varphi - \frac{r}{3}\right) \wedge (i\partial\overline{\partial}|z|^2)^{n-k-2} \\
&\leq \frac{3d(\Omega)^2 \sup|\varphi|}{|r|^2}\int_{\{\varphi<r/3\}} i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^{k+1} \wedge (i\partial\overline{\partial}|z|^2)^{n-k-2}.
\end{aligned}$$

By repeating the same process, we have that

$$\begin{aligned}
&\int_{\{\varphi<r\}} i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1} \\
&\leq 3^{(n-k-1)^2}\left(\frac{d(\Omega)^2 \sup|\varphi|}{|r|^2}\right)^{n-k-1}\int_{\{\varphi<r/3^{n-k-1}\}} i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^{n-1} \\
&\leq 3^{(n-k-1)^2}\left(\frac{d(\Omega)^2 \sup|\varphi|}{|r|^2}\right)^{n-k-1}\left(\frac{3^{n-k-1}}{|r|}\right)^2 \\
&\quad \times \int_{\{\varphi<r/3^{n-k-1}\}} \varphi^2 i\partial\overline{\partial}v \wedge (i\partial\overline{\partial}\varphi)^{n-1}
\end{aligned}$$

$$\leq 3^{(n-k-1)(n-k+1)} 2 \frac{(d(\Omega)^2 \sup|\varphi|)^{n-k-1}}{|r|^{2(n-k)}} \int_\Omega v i\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1}.$$

The last inequality follows from (3). This completes the proof. □

REMARK 1. To prove Theorem 1, the rest of this section is not necessary. Indeed, Lemma 6 shows that

$$\int_{\{\varphi<r\}} |\nabla F|^2 (i\partial\overline{\partial}|z|^2)^n \leq C \int_\Omega |F|^2 i\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1}$$

for holomorphic function F. Here C does not depend on F. This implies that the solutions constructed in Section 3 are bounded locally and we can prove Theorem 4 below.

LEMMA 7. *Let k be a non-negative integer. Under the same assumption of Lemma 6, we have that*

$$\begin{aligned} &\int_{\{\varphi<r\}} v(i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k} \\ &\leq C\left(\int_\Omega v(i\partial\overline{\partial}\varphi)^n + v i\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1}\right), \end{aligned}$$

where $C = \left(1 + d(\Omega) + \sup|\varphi| + |r|^{-1}\right)^{C_{n,k}}$, and $C_{n,k}$ is a positive constant which depends only on n and k.

PROOF. We prove the lemma by induction on $l = n-k$. It is clear for $l = 0$. Under the notation of the proof of Lemma 6, we have that

$$\begin{aligned} &\frac{|r|}{2d(\Omega)^2} \int_{\{\varphi<r\}} v(i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k} \\ &\leq \int_{\{\varphi<r/3\}} v(i\partial\overline{\partial}\varphi)^k \wedge i\partial\overline{\partial}\rho \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1} \\ &= \int_{\{\varphi=r/3\}} v i\overline{\partial}\rho \wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1} \\ &\quad - \int_{\{\varphi<r/3\}} i\partial v \wedge \overline{\partial}\rho \wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1} \\ &= \int_{\{\varphi=r/3\}} v i\overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1} \\ &\quad - \int_{\{\varphi<r/3\}} i\partial v \wedge \overline{\partial}\rho \wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1} \end{aligned}$$

$$= \int_{\{\varphi<r/3\}} i\partial v \wedge \overline{\partial}(\varphi-\rho)\wedge (i\partial\overline{\partial}\varphi)^k \wedge (i\partial\overline{\partial}|z|^2)^{n-k-1}$$
$$+ \int_{\{\varphi<r/3\}} v(i\partial\overline{\partial}\varphi)^{k+1}\wedge (i\partial\overline{\partial}|z|^2)^{n-k-1}.$$

The last term of the above inequality is bounded from above by the hypothesis of the induction. By Lemma 6, the second to last term of the above inequality is bounded from above by

$$C_{n,k}\frac{d(\Omega)^{2(n-k-1)}\sup|\varphi|^{n-k}}{r^{2(n-k)}}\int_{\Omega} vi\partial\varphi\wedge\overline{\partial}\varphi\wedge(i\partial\overline{\partial}\varphi)^{n-1},$$

since

$$\left|\int_{\{\varphi<r/3\}}(\rho-\varphi)i\partial\overline{\partial}v\wedge(i\partial\overline{\partial}\varphi)^k\wedge(i\partial\overline{\partial}|z|^2)^{n-k-1}\right|$$
$$\leq \sup|\varphi|\int_{\{\varphi<r/3\}} i\partial\overline{\partial}v\wedge(i\partial\overline{\partial}\varphi)^k\wedge(i\partial\overline{\partial}|z|^2)^{n-k-1}.$$

This completes the proof by the induction. □

PROOF OF THEOREM 3. We prove the first inequality. Let $\varepsilon>0$ be a small positive number. It is enough to prove the theorem with φ and Ω replaced by $\varphi+\varepsilon$ and $\{z\in\Omega\,|\,\varphi(z)+\varepsilon<0\}$. Hence we may assume that φ and v are plurisubharmonic functions defined on an open neighborhood of $\overline{\Omega}$. Let v_j be a decreasing sequence of smooth plurisubharmonic functions on an open neighborhood of $\overline{\Omega}$ which converge to v. Since $\int_{\{\varphi<r\}} i\partial\overline{\partial}v\wedge(i\partial\overline{\partial}|z|^2)^{n-1}\leq\liminf_{j\to\infty}\int_{\{\varphi<r\}} i\partial\overline{\partial}v_j\wedge(i\partial\overline{\partial}|z|^2)^{n-1}$, it is enough to prove the theorem for $v\in C^\infty(\overline{\Omega})$. Since φ is continuous, there exists a decreasing sequence φ_j of smooth plurisubharmonic functions on an open neighborhood of $\overline{\Omega}$ which converge to φ uniformly. Let $\Omega_j=\{z\in\Omega\,|\,\varphi_j(z)<0\}$. We may assume that $d\varphi_j\neq 0$ on $\partial\Omega_j$ by Sard's theorem. By Lemma 6, we have

$$\int_{\{\varphi_j<r\}} i\partial\overline{\partial}v\wedge(i\partial\overline{\partial}|z|^2)^{n-1}$$
$$\leq C_n\frac{(d(\Omega_j)^2\sup|\varphi_j|)^{n-1}}{r^{2n}}\int_{\Omega_j} vi\partial\varphi_j\wedge\overline{\partial}\varphi_j\wedge(i\partial\overline{\partial}\varphi_j)^{n-1}.$$

Since

$$\limsup_{j\to\infty}\int_{\overline{\Omega}} vi\partial\varphi_j\wedge\overline{\partial}\varphi_j\wedge(i\partial\overline{\partial}\varphi_j)^{n-1}\leq\int_{\overline{\Omega}} vi\partial\varphi\wedge\overline{\partial}\varphi\wedge(i\partial\overline{\partial}\varphi)^{n-1},$$

we have that

$$\begin{aligned}&\int_{\{\varphi<r\}} i\partial\overline{\partial}v\wedge(i\partial\overline{\partial}|z|^2)^{n-1}\\ &\leq C_n\frac{(d(\Omega)^2\sup|\varphi|)^{n-1}}{r^{2n}}\int_{\overline{\Omega}} vi\partial\varphi\wedge\overline{\partial}\varphi\wedge(i\partial\overline{\partial}\varphi)^{n-1}.\end{aligned}$$

Then the first inequality of Theorem 3 follows by the continuity of C in the theorem with respect to $d(\Omega), \sup|\varphi|$, and $|r|$. The second inequality can be proved by the same way. □

5. Proof of the Main Theorem

Theorem 4. *Let $\Omega\subset\mathbb{C}^n$ $(n\geq 4)$ be a bounded hyperconvex domain. Let $\varphi\in C^\infty(\overline{\Omega})$ such that φ is negative plurisubharmonic on Ω, $\varphi(z)\to 0$ when $z\to\partial\Omega$, and $d\varphi\neq 0$ on $\partial\Omega$. Let α be a smooth $(0,1)$-form defined on an open neighborhood of $\overline{\Omega}$ such that $\overline{\partial}\alpha=0$ in Ω and $\operatorname{supp}\alpha\subset\overline{\Omega}\setminus\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-3}$ in $\overline{\Omega}$. Then there exists a smooth function u on Ω such that $\overline{\partial}u=\alpha$ and $\int_\Omega|u|^2(i\partial\overline{\partial}\phi)^n=0$.*

Proof. We use the same notation as Proposition 1. Let ε_j be a decreasing sequence of positive numbers which converge to 0. We put $dV_j=(n!)^{-1}\omega_{\varepsilon_j}^n$. Then dV_j decreases to $dV_{i\partial\overline{\partial}\phi}=(n!)^{-1}(i\partial\overline{\partial}\phi)^n$. By Proposition 1, there exists a sequence u_j of smooth functions such that $\overline{\partial}u_j=\alpha$ and

$$\int_\Omega|u_j|^2e^{\psi-\delta\phi}dV_j\leq C_{n,\delta}\int_\Omega|\alpha|^2_{i\partial\overline{\partial}\psi}e^{\psi-\delta\phi}dV_j.$$

We have that $\operatorname{supp}\alpha\subset\Omega\setminus\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-3}\subset\Omega\setminus\operatorname{supp}(i\partial\overline{\partial}\phi)^n$. Hence the right hand side of the above inequality goes to 0 when $j\to\infty$ because of Lebesgue's dominated convergence theorem. Let $\Omega(r)=\{z\in\Omega\mid\varphi(z)<r\}$. Then $\int_{\Omega(r)}|u_j|^2dV_{i\partial\overline{\partial}\phi}$ goes to 0 when $j\to\infty$ for $r<0$. We take $h\in C^\infty(\Omega)$ such that $\overline{\partial}h=\alpha$ (see [5], [8]). Define $F_j=h-u_j$. Then F_j is a holomorphic function and $\int_{\Omega(r)} |F_j|^2 dV_{i\partial\overline{\partial}\phi}$ are bounded from above for all j. Since there exists a positive constant C such that $i\partial\varphi \wedge \overline{\partial}\varphi \wedge (i\partial\overline{\partial}\varphi)^{n-1} + (i\partial\overline{\partial}\varphi)^n \leq C(i\partial\overline{\partial}\phi)^n$ on $\Omega(r)$, Theorem 3 shows that $\int_{\Omega(r')} |F_j|^2 (i\partial\overline{\partial}|z|^2)^n$ and $\int_{\Omega(r')} |u_j|^2 (i\partial\overline{\partial}|z|^2)^n$ are bounded from above for all j when $r' < r$. We can thus find a weakly convergent subsequence u_{j_ν} in $L^2(\Omega(r'))$. Let u be the weak limit u. It follows that $\overline{\partial}u = \alpha$ on $\Omega(r')$ and $\int_{\Omega(r')} |u|^2 (i\partial\overline{\partial}\phi)^n = 0$. Then, by using a diagonal argument, we have the solution we are looking for. □

Proof of Theorem 1. Let $r < 0$ such that $|r|$ is sufficiently small and let $\Omega(r) = \{z \in \Omega \,|\, \varphi(z) < r\}$. We can choose r such that $d\varphi \neq 0$ on $\partial\Omega(r)$. Let $V(r) = \Omega(r) \cap V$. There exists $\delta > 0$ such that $d(\partial V(r) \setminus \partial\Omega(r), \mathrm{supp}\,(i\partial\overline{\partial}\varphi)^{n-3} \cap \Omega(r)) > 3\delta$. Here $d(A, B)$, $A, B \subset \mathbb{C}^n$ is the Euclidean distance between A and B. Let $U_j = \{z \in \Omega(r) \,|\, d(z, \mathrm{supp}\,(i\partial\overline{\partial}\varphi)^{n-3} \cap \Omega(r)) < j\delta\}$ $(j = 1, 2)$. We take a smooth function χ on $\Omega(r)$ such that $\chi = 1$ on U_1 and $\chi = 0$ on $\Omega(r) \setminus U_2$. Let f be a holomorphic function on V. Define $\alpha = \overline{\partial}(\chi f)$. We may assume that α is defined on an open neighborhood of $\overline{\Omega(r)}$ by a small perturbation of r. Then $\mathrm{supp}\,\alpha \subset \overline{\Omega(r)} \setminus \mathrm{supp}\,(i\partial\overline{\partial}\varphi)^{n-3}$ in $\overline{\Omega(r)}$. By Theorem 4, there exists $u \in C^\infty(\Omega(r))$ such that $\overline{\partial}u = \alpha$ and $\int_{\Omega(r)} |u|^2 (i\partial\overline{\partial}(-(\log(r-\varphi)))^n = 0$. (If $\Omega(r)$ is a disjoint union of bounded hyperconvex domain, we apply Theorem 4 to each component.) Then $u = 0$ on $\mathrm{supp}\,(i\partial\overline{\partial}\phi)^n \cap \Omega(r)$ since $\mathrm{supp}\,(i\partial\overline{\partial}\phi)^n = \mathrm{supp}\,(i\partial\overline{\partial}(-\log(r-\varphi)))^n$. Let $F_r = \chi f - u$. Then F_r is holomorphic on $\Omega(r)$ and $F_r = f$ on $\mathrm{supp}\,(i\partial\overline{\partial}\phi)^n \cap \Omega(r)$. We note that any component of $\Omega(r)$ intersects $\mathrm{supp}\,(i\partial\overline{\partial}\phi)^n$ by the comparison theorem (see [9]). By letting $r \to 0$, we obtain the holomorphic function F on Ω such that $F = f$ on $\mathrm{supp}\,(i\partial\overline{\partial}\phi)^n$ because of the identity theorem. Since $\mathrm{supp}\,(i\partial\overline{\partial}\phi)^n \subset V$ and V is connected, we have $F = f$ on V. □

Proof of Theorem 2. We use the same notation as the proof of Theorem 1. Let $p \in \mathrm{supp}\,(i\partial\overline{\partial}\phi)^n \subset V$. Let $h : \mathbb{C}^n \to \mathbb{R}^+$ be a smooth function of $|z|$ whose support is the unit ball and whose integral is equal to one. Define $h_\varepsilon = (1/\varepsilon^{2n})h(z/\varepsilon)$ for $\varepsilon > 0$. Let $\varphi_\varepsilon = \varphi * h_\varepsilon$ be a function on $\Omega(r)$ where $r < 0$ and $0 < \varepsilon << |r|$. Let $\phi_\varepsilon = -(\log(-\varphi_\varepsilon))$ and let W be a connected open neighborhood of p such that $W \subset V$. If ε is sufficiently small, then $\mathrm{supp}(i\partial\overline{\partial}\varphi_\varepsilon) \subset V$ in $\Omega(r)$ and $\mathrm{supp}\,(i\partial\overline{\partial}\phi_\varepsilon)^n \cap W \neq \emptyset$ by the continuity of the Monge-Ampère measure (see [2], [9]). Let $s < 0$ such that $W \subset \Omega(s)$. By taking $|r|$ and ε are sufficiently small, we may assume that there exists $t < 0$ such that $\Omega(s) \subset \Omega_\varepsilon(t) := \{z \in \Omega(r) \,|\, \varphi_\varepsilon(z) < t\}$ and $\overline{\Omega_\varepsilon(t)} \subset \Omega(r)$. Then there exists a holomorphic function F_t on $\Omega_\varepsilon(t)$ such that $F_t = f$ on $\operatorname{supp}(i\partial\overline{\partial}\phi_\varepsilon)^n$ by the same argument as the proof of Theorem 1. Then $F_t = f$ on W because $\operatorname{supp}(i\partial\overline{\partial}\phi_\varepsilon)^n \cap W \neq \emptyset$. Let $\Omega(s)_0$ be a component of $\Omega(s)$ which contains W. It follows that $F_t|_{\Omega(s)_0}$ does not depend on ε, r and t by the identity theorem. By letting $s \to 0$, there exists the holomoprhic function F on Ω such that $F = f$ on W. Since V is connected, we have $F = f$ on V. □

THEOREM 5. *Let $n \geq 4$ and Ω be a pseudoconvex domain in $\mathbb{C}^n$. Let φ be an exhaustive smooth plurisubharmonic function on Ω. Let $V \subset \Omega$ be a connected open neighborhood of* $\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-3}$. *Then any holomorphic function on V can be extended to a holomorphic function on Ω.*

THEOREM 6. *Let $n \geq 4$ and Ω be a pseudoconvex domain in $\mathbb{C}^n$. Let φ be an exhaustive continuous plurisubharmonic function on Ω. Let $V \subset \Omega$ be a connected open neighborhood of* $\operatorname{supp} i\partial\overline{\partial}\varphi$. *Then any holomorphic function on V can be extended to a holomorphic function on Ω.*

PROOF OF THEOREM 5 AND 6. Let $r \in \mathbb{R}$. Then $\Omega(r) = \{z \in \Omega \,|\, \varphi(z) < r\}$ is a bounded hyperconvex domain. It follows that $\operatorname{supp}(i\partial\overline{\partial}\varphi)^{n-3} \subset \operatorname{supp}(i\partial\overline{\partial}(-\log(r-\varphi)))^n$ and $\operatorname{supp}(i\partial\overline{\partial}(-\log(r'-\varphi)))^n \cap \Omega(r) = \operatorname{supp}(i\partial\overline{\partial}(-\log(r-\varphi)))^n$ for $r < r'$. Then the theorems follow from the same arguments as the proofs of Theorem 1 and 2. □

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 4)$ and let φ be an exhaustive continuous plurisubharmonic function on Ω. Let $\Omega(r) = \{z \in \Omega \,|\, \varphi(z) < r\}$. Then $\max\{\varphi, r\}$ is an exhaustive continuous plurisubharmonic function which is pluriharmonic on $\Omega(r)$. Hence any holomorphic function on a connected open neighborhood of $\Omega \setminus \Omega(r)$ can be extended to the holomorphic function on Ω. This is a special case of the Hartogs extension theorem. Finally we note that $\operatorname{supp}(i\partial\overline{\partial}\varphi)$ can be interpreted as an "ample divisor" on Ω and our results are associated with the Lefschetz hyperplane theorem (see [11]).

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