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Bounds for the Order of Automorphism Groups of Cyclic Covering Fibrations of an Elliptic Surface

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Abstract. We study automorphism groups of fibered surfaces for finite cyclic covering fibrations of an elliptic surface. We estimate the order of a finite subgroup of automorphism groups in terms of the genus of the base curve, the covering degree and the square of the relative canonical divisor.

Introduction

We shall work over the complex number field $\mathbb{C}$. This is a continuation of our previous work [1] studying the order of automorphism groups of fibred algebraic surfaces.

The study of automorphism groups of projective varieties has a long history. Hurwitz's classical theorem states that if C is a smooth projective curve of genus $g(C) \geq 2$, then the order of the automorphism group is bounded by $84(g(C)-1)$. Xiao attempted to generalize this result to minimal surfaces of general type and gave the upper bound of the order of the automorphism groups ([10],[11]). While studies of the automorphism groups of surfaces exist in absolute settings as above, we can consider a similar problem in relative settings. For this purpose, we define basic notions of fibered surfaces and its automorphisms.

Let $f : S \to B$ be a surjective morphism from a complex smooth projective surface S to a smooth projective curve B with connected fibers. We call it a *fibration* or a *fibered surface* of genus g when a general fiber is a curve of genus g. A fibration is called *relatively minimal*, when any (-1)-curve is not contained in fibers. Here we call a smooth rational curve C with $C^2 = -n$ a $(-n)$-curve. A fibration is called *smooth* when all fibers are

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smooth, *isotrivial* when all of the smooth fibers are isomorphic, *locally trivial* when it is smooth and isotrivial. Assume that $f : S \to B$ is a relatively minimal fibration of genus g. We denote by $K_f = K_S - f^*K_B$ a relative canonical divisor.

An automorphism of the fibration $f : S \to B$ is a pair of automorphisms $(\kappa_S, \kappa_B) \in \mathrm{Aut}(S) \times \mathrm{Aut}(B)$ satisfying $f \circ \kappa_S = \kappa_B \circ f$, that is, the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \kappa_S\ } & S \\ {\scriptstyle f}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\ \kappa_B\ ]{} & B \end{array}$$

is commutative. We denote by $\mathrm{Aut}(f)$ the group of all automorphisms of f.

Our main objects are primitive cyclic covering fibrations:

Definition 0.1 ([5]). Let $f : S \to B$ be a relatively minimal fibration of genus $g \geq 2$. We call it a *primitive cyclic covering fibration* of type $(g, 1, n)$, when there are a (not necessarily relatively minimal) fibration $\tilde{\varphi} : \widetilde{W} \to B$ of genus 1 (*i.e.* elliptic surface) and a classical n-cyclic covering

$$\tilde{\theta} : \widetilde{S} = \mathrm{Spec}_{\widetilde{W}} \left(\bigoplus_{j=0}^{n-1} \mathcal{O}_{\widetilde{W}}(-j\widetilde{\mathfrak{d}}) \right) \to \widetilde{W}$$

branched over a smooth (not necessarily irreducible) curve $\widetilde{R} \in |n\widetilde{\mathfrak{d}}|$ and $\widetilde{\mathfrak{d}} \in \mathrm{Pic}(\widetilde{W})$ such that f is the relatively minimal model of $\tilde{f} = \tilde{\varphi} \circ \tilde{\theta}$.

Though it seems restrictive, it should be noticed that this class contains some classically important fibrations as subclasses. In fact, one sees immediately that any hyperelliptic (resp. bielliptic) fibrations of genus g are necessarily primitive cyclic covering fibrations of type $(g, 0, 2)$ (resp. of type $(g, 1, 2)$ when $g \geq 6$). Here, a bielliptic curve is a smooth projective curve admitting a double covering of an elliptic curve and a fibration is called *bielliptic* if a general fiber is a bielliptic curve. Bielliptic fibrations are an interesting class of fibrations. For instance, it seemed that the lower bound of slopes of fibrations increases monotonically with respect to its gonality. But it is not true. There exists a tetragonal fibration and its slope is less than the lower bound of the slope of trigonal fibrations. Such tetragonal fibrations are nothing more than bielliptic fibrations ([3],[8]).

Arakawa [2] and later Chen [4] studied the automorphism groups for hyperelliptic fibrations and gave the upper bound of the order of a finite subgroup of $\mathrm{Aut}(f)$ in terms of g, $g(B)$ (the genus of B) and K_f^2. In [1], we considered the case that $h = 0$, and gave an upper bound on the order of $\mathrm{Aut}(f)$, generalizing results due to Arakawa [2] and Chen [4] for hyperelliptic fibrations. We constructed an example that shows our bound is almost optimal in some cases.

In this paper, we study the case that $h = 1$, that is, cyclic covering fibrations of elliptic surfaces. In order to state our results, we need some further notation. Keeping the notation in Definition 0.1 with $h = 1$, let $\varphi : W \to B$ be the relatively minimal model of $\widetilde{\varphi} : \widetilde{W} \to B$, R the image of $\widetilde{R}$ by the natural birational map $\widetilde{W} \to W$, and Γ_p the fiber of φ over $p \in B$. If Γ_p is a smooth fiber, fixing a point $O_p \in \Gamma_p$, we can regard Γ_p as an abelian group with the unit element O_p. Put

$$\delta := \min\{\sharp \mathrm{Aut}(\Gamma_p, O_p) \mid p \in B \text{ and } \Gamma_p \text{ smooth } \},$$

where $\mathrm{Aut}(\Gamma_p, O_p) := \{\kappa \in \mathrm{Aut}(\Gamma_p) \mid \kappa(O_p) = O_p\}$.

Theorem 0.2 (Theorem 6.1). *Let $f : S \to B$ be a non-locally trivial primitive cyclic covering fibration of type $(g, 1, n)$ with $g \geq \max\{\frac{30n^2-47n+25}{n+1}, \frac{7}{2}n(n-1)+1\}$. Put*

$$\mu_n := \frac{12n^2\delta}{n^2-1}.$$

Assume furthermore that when $g(B) = 0$, f has at least 3 singular fibers. Let G be a finite subgroup of $\mathrm{Aut}(f)$. Then it holds

$$\sharp G \leq \begin{cases} 6(2g(B)-1)\mu_n K_f^2 & (g(B) \geq 1), \\ 5\mu_n K_f^2 & (g(B) = 0). \end{cases}$$

Theorem 0.3 (Theorem 6.2). *Let $f : S \to B$ be a non-locally trivial primitive cyclic covering fibration of type $(g, 1, n)$ with $g \geq \max\{\frac{30n^2-47n+25}{n+1}, \frac{7}{2}n(n-1)+1\}$. Put*

$$\mu'_n := \frac{6n^2\delta}{(n-1)(5n-4)}.$$

Assume furthermore that the branch locus R has singular points on at least three (resp. one) fibers when $g(B) = 0$ (resp. $g(B) \geq 1$). Let G be a finite subgroup of $\mathrm{Aut}(f)$. *Then it holds*

$$\sharp G \leq \begin{cases} 6(2g(B)-1)\mu'_n K_f^2 & (g(B) \geq 1), \\ 5\mu'_n K_f^2 & (g(B) = 0). \end{cases}$$

Theorem 0.4 (Corollary 6.3). *Let $f : S \to B$ be a non-locally trivial bielliptic fibration with $g \geq 17$. Assume furthermore that the branch locus R has singular points on at least three (resp. one) fibers when $g(B) = 0$ (resp. $g(B) \geq 1$). Let G be a finite subgroup of* $\mathrm{Aut}(f)$. *Then it holds*

$$\sharp G \leq \begin{cases} 24\delta(2g(B)-1)K_f^2 & (g(B) \geq 1), \\ 20\delta K_f^2 & (g(B) = 0). \end{cases}$$

Theorem 0.5 (Corollary 6.4). *Let $f : S \to B$ be a non-locally trivial primitive cyclic covering fibration of type $(g, 1, n)$ with $g \geq \max\{\frac{30n^2-47n+25}{n+1}, \frac{7}{2}n(n-1)+1\}$. Put $\mathrm{Aut}(S/B) := \{(\kappa_S, \mathrm{id}_B) \in \mathrm{Aut}(f)\}$. Assume that the branch locus R has a singular point. Then it holds*

$$\sharp\mathrm{Aut}(S/B) \leq \frac{6n^2\delta}{(n-1)(5n-4)}K_f^2.$$

We state the outline of the proof. Let G be a finite subgroup of $\mathrm{Aut}(f)$. Then G can be expressed as the extension of its horizontal part H by its vertical part K, that is, $1 \to K \to G \to H \to 1$ (exact). We can study H by using known results for the automorphism groups of curves. Hence it suffices to estimate K. Then K induces a subgroup $\widetilde{K}$ of the automorphism group of the elliptic surface $\varphi : W \to B$ preserving the branch locus R and our task is reduced to estimating the order of $\widetilde{K}$. For this purpose, we use the localization of the invariant K_f^2. It is known that K_f^2 can be localized to fibers of f ([6]). We refine the localized K_f^2 further by using the quantity defined at a point on a fiber, and obtain the new expression of it in Proposition 2.2. Then, we estimate the order of $\widetilde{K}$ from above with the localized K_f^2 multiplied by an explicit function in g and n (Propositions 5.3, 5.4 and 5.5).

In Section 1, we recall the basic properties of primitive cyclic covering fibrations mainly due to [5]. In Section 2, we recast Section 5 of [5] and consider the refined localization of K_f^2. In Section 3, we summarize the basic properties of elliptic surfaces as group manifolds. In Section 4, we descend the automorphism group to the elliptic surface W and analyze its action. In Section 5, we estimate the order of $\widetilde{K}$. In Section 6, we show our main result, Theorem 6.1, 6.2, Corollary 6.3 and 6.4. Section 7 is devoted to constructing an example of bielliptic fibrations with a large automorphism group.

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Contents

1. Primitive Cyclic Covering Fibrations

We recall the basic properties of primitive cyclic covering fibrations, most of which can be found in [5].

Definition 1.1. Let $f : S \to B$ be a relatively minimal fibration of genus $g \geq 2$. We call it a *primitive cyclic covering fibration* of type $(g, 1, n)$, when there are a (not necessarily relatively minimal) fibration $\tilde{\varphi}: \widetilde{W} \to B$ of genus 1 (*i.e.* elliptic surface) and a classical n-cyclic covering

$$\tilde{\theta}: \widetilde{S} = \mathrm{Spec}_{\widetilde{W}} \left(\bigoplus_{j=0}^{n-1} \mathcal{O}_{\widetilde{W}}(-j\tilde{\mathfrak{d}}) \right) \to \widetilde{W}$$

branched over a smooth (not necessarily irreducible) curve $\widetilde{R} \in |n\tilde{\mathfrak{d}}|$ and $\tilde{\mathfrak{d}} \in \mathrm{Pic}(\widetilde{W})$ such that f is the relatively minimal model of $\tilde{f} = \tilde{\varphi} \circ \tilde{\theta}$.

In addition, we employ the following notations. We denote by $\Sigma = \langle \tilde{\sigma} \rangle$ the covering transformation group of $\tilde{\theta}$ and by $\varphi: W \to B$ the relatively minimal model of $\tilde{\varphi}: \widetilde{W} \to B$ with the natural contraction map $\tilde{\psi}: \widetilde{W} \to W$. Furthermore, $\widetilde{F}$, F, $\widetilde{\Gamma}$ and Γ will denote general fibers of $\tilde{f}$, f, $\tilde{\varphi}$ and φ, respectively. We write fibers of $\tilde{f}$, f, $\tilde{\varphi}$ and φ over p as $\widetilde{F}_p$, F_p, $\widetilde{\Gamma}_p$ and Γ_p, respectively.

Remark 1.2. We note important properties of primitive cyclic covering fibrations in the following, which can be found in [5].

- There exists an automorphism $\sigma \in \mathrm{Aut}(S)$ which satisfies the following:
  - The natural morphism $\widetilde{S} \to S$ is a minimal succession of blowing-ups that resolves all isolated fixed points of σ.
  - The automorphism $\widetilde{\sigma}$ is induced by σ.
- Let $\widetilde{R}_v$ be the sum of $\widetilde{\varphi}$-vertical components of $\widetilde{R}$. The self-intersection number of each irreducible component of $\widetilde{R}_v$ is equals to $-an$ for some positive integer a.
- Any $\widetilde{\varphi}$-vertical (-1) curve intersects $\widetilde{R}$.

From now on, we let $f: S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$ and freely use the above notations and conventions.

Since the restriction map $\tilde{\theta}|_{\widetilde{F}}: \widetilde{F} \to \widetilde{\Gamma}$ is a classical n-cyclic covering branched over $\widetilde{R} \cap \widetilde{\Gamma}$, the Hurwitz formula for $\tilde{\theta}|_{\widetilde{F}}$ gives us

$$r := \widetilde{R}\widetilde{\Gamma} = \frac{2(g-1)}{n-1}. \tag{1.1}$$

From $\widetilde{R} \in |n\widetilde{\mathfrak{d}}|$, it follows that r is a multiple of n.

Since $\tilde{\psi} : \widetilde{W} \to W$ is a composite of blowing-ups, we can write $\tilde{\psi} = \psi_1 \circ \cdots \psi_N$, where $\psi_i : W_i \to W_{i-1}$ denotes the blowing-up at $z_i \in W_{i-1}$ ($i = 1, \cdots, N$), $W_0 = W$ and $W_N = \widetilde{W}$. We define reduced curves R_i inductively as $R_{i-1} = (\psi_i)_* R_i$ starting from $R_N = \widetilde{R}$ down to $R_0 = R$. We call R_i *the branch locus on* W_i. We also put $E_i = \psi_i^{-1}(z_i)$ and $m_i = \mathrm{mult}_{z_i} R_{i-1}$ ($i = 1, \cdots, N$).

Lemma 1.3 ([5], Lemma 1.5). *In the above situation, the following hold for any* $i = 1, \cdots, N$.

(1) *Either* $m_i \in n\mathbb{Z}_{\geq 1}$ *or* $n\mathbb{Z}_{\geq 1}+1$, *where* $\mathbb{Z}_{\geq 1}$ *is the set of positive integers. Furthermore,* $m_i \in n\mathbb{Z}_{\geq 1}$ *if and only if* E_i *is not contained in* R_i.

(2) $R_i = \psi_i^* R_{i-1} - n[\frac{m_i}{n}]E_i$, *where* $[t]$ *denotes the greatest integer not exceeding* t.

(3) *There exists a* $\mathfrak{d}_i \in \mathrm{Pic}(P_i)$ *such that* $\mathfrak{d}_i = \psi_i^* \mathfrak{d}_{i-1} - [\frac{m_i}{n}]E_i$ *and* $R_i \sim n\mathfrak{d}_i$, $\mathfrak{d}_N = \widetilde{\mathfrak{d}}$.

We say that a singular point of R_i is *of type* $n\mathbb{Z}$ (resp. $n\mathbb{Z}+1$) if its multiplicity is in $n\mathbb{Z}_{\geq 1}$ (resp. $n\mathbb{Z}_{\geq 1}+1$).

2. Localization of K_f^2

We recall the argument in Section 5 of [5] and give localizations of K_f^2. Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$.

First of all, let us recall the singularity indices introduced in [5]. For any fixed $p \in B$, we consider all singular points of the branch loci $R = R_0, \cdots, R_{N-1}$ over Γ_p. For any positive integer k, we let $\alpha_k(\Gamma_p)$ be the number of singular points of multiplicity either kn or $kn+1$ among them. We put $\alpha_k := \sum_{p \in B} \alpha_k(\Gamma_p)$ and call it the *k-th singularity index* of the fibration. For an effective vertical divisor T and $p \in B$, we denote by $T(p)$ the biggest subdivisor of T whose support is in the fiber over p. Then $T = \sum_{p \in B} T(p)$. We sometimes write $\sharp T$ as the number of irreducible components of T.

Let $j_{b,a}(\Gamma_p)$ be the number of irreducible curves with genus b and self-intersection number $-an$ contained in $\widetilde{R}_v(p)$.

We introduce the following indices:

$$j_{\bullet,a}(\Gamma_p) := \sum_{b\geq 0} j_{b,a}(\Gamma_p), \quad j_{b,\bullet}(\Gamma_p) := \sum_{a\geq 0} j_{b,a}(\Gamma_p), \quad j_{0,1} := \sum_{p\in B} j_{0,1}(\Gamma_p).$$

Let A be the sum of all $\widetilde{\varphi}$-vertical $(-n)$-curves contained in $\widetilde{R}$ and put $\widetilde{R}_0 := \widetilde{R} - A$. Then the 0-th singularity index is defined by $\alpha_0 := (K_{\tilde{\varphi}} + \widetilde{R}_0)\widetilde{R}_0$. Put

$$\alpha_0^+(\Gamma_p) := \widetilde{R}_h\widetilde{\Gamma}_p - \sharp(\mathrm{Supp}(\widetilde{R}_h) \cap \mathrm{Supp}(\widetilde{\Gamma}_p)),$$
$$\alpha_0(\Gamma_p) := \alpha_0^+(\Gamma_p) - 2\sum_{a\geq 2} j_{0,a}(\Gamma_p),$$

where $\widetilde{R}_h$ is the $\widetilde{\varphi}$-horizontal part of $\widetilde{R}$. We note that $\alpha_0 = \sum_{p\in B}\alpha_0(\Gamma_p)$. We introduce some indices depending only on Γ_p.

$\chi_\varphi(\Gamma_p) := e(\Gamma_p)/12$ where $e(\Gamma_p)$ is the topological Euler number of Γ_p

$\nu(\Gamma_p) := 1 - 1/l(\Gamma_p)$ where $l(\Gamma_p)$ is a multiplicity of Γ_p

$\nu := \sum_{p\in B}\nu(\Gamma_p)$

Then it holds that $e_\varphi = \sum_{p\in B} e(\Gamma_p)$ and $\chi_\varphi = \sum_{p\in B}\chi_\varphi(\Gamma_p)$.

Lemma 2.1 ([6], Lemma 2.1). *Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$. Then it holds*

$$K_f^2 = \sum_{k\geq 1}\left((n^2-1)k - n\right)\alpha_k + \frac{(n-1)^2}{n}(\alpha_0 - 2j_{0,1}) + \frac{n^2-1}{n}r(\chi_\varphi + \nu) + j_{0,1}.$$

Hence if we put

$$K_f^2(\Gamma_p) = \sum_{k\geq 1}\left((n^2-1)k - n\right)\alpha_k(\Gamma_p) + \frac{(n-1)^2}{n}(\alpha_0(\Gamma_p) - 2j_{0,1}(\Gamma_p)) + \frac{n^2-1}{n}r(\chi_\varphi(\Gamma_p) + \nu(\Gamma_p)) + j_{0,1}(\Gamma_p).$$

for $p \in B$, then we get $K_f^2 = \sum_{p\in B} K_f^2(\Gamma_p)$.

Let $\widehat{\varphi} : \widehat{W} \to B$ be any intermediate elliptic surface between $\widetilde{W}$ and W, and regard $\widetilde{\psi} : \widetilde{W} \to W$ as the composite of the natural birational morphisms $\widehat{\psi} : \widetilde{W} \to \widehat{W}$ and $\check{\psi} : \widehat{W} \to W$.

$$
\begin{array}{ccccc}
\widetilde{W} & \xrightarrow{\widehat{\psi}} & \widehat{W} & \xrightarrow{\check{\psi}} & W \\
& {\scriptstyle\widetilde{\varphi}}\searrow & \downarrow{\scriptstyle\widehat{\varphi}} & \swarrow{\scriptstyle\varphi} & \\
& & B & &
\end{array}
$$

We put $\widehat{R} = \widehat{\psi}_* \widetilde{R}$. The fiber of $\widehat{\varphi}$ over $p \in B$ will be denoted by $\widehat{\Gamma}_p$.

Let $\widehat{\alpha}_k(\Gamma_p)$ and $\check{\alpha}_k(\Gamma_p)$ be the number of the singular points contributing to $\alpha_k(\Gamma_p)$ appearing in $\widehat{\psi}$ and $\check{\psi}$, respectively. We note that the number of singular points of $\widehat{R}$ on $\widehat{\Gamma}_p$ is counted by $\widehat{\alpha}_k(\Gamma_p)$. Then we have $\alpha_k(\Gamma_p) = \widehat{\alpha}_k(\Gamma_p) + \check{\alpha}_k(\Gamma_p)$.

We note that an arbitrary irreducible component of $\widetilde{R}_v(p)$ is a proper transform of either an exceptional curve appearing from $\psi : \widetilde{W} \to W$ or an irreducible component of Γ_p. Let $\widehat{j}_{0,a}(\Gamma_p)$ (*resp.* $\check{j}_{0,a}(\Gamma_p)$) be the number of irreducible components of $\widetilde{R}_v(p)$ contributing to $j_{0,a}(\Gamma_p)$ which are proper transforms of exceptional curves appearing from $\widehat{\psi}$ (*resp.* $\check{\psi}$). Let $j'_{0,a}(\Gamma_p)$ be the number of irreducible components contributing to $j_{0,a}(\Gamma_p)$ which are proper transforms of irreducible components of Γ_p. Then we have $j_{0,a}(\Gamma_p) = \widehat{j}_{0,a}(\Gamma_p) + \check{j}_{0,a}(\Gamma_p) + j'_{0,a}(\Gamma_p)$. Put $\widehat{j}_{0,\bullet}(\Gamma_p) := \sum_{a\geq 1} \widehat{j}_{0,a}(\Gamma_p)$, $\check{j}_{0,\bullet}(\Gamma_p) := \sum_{a\geq 1} \check{j}_{0,a}(\Gamma_p)$ and $j'_{0,\bullet}(\Gamma_p) := \sum_{a\geq 1} j'_{0,a}(\Gamma_p)$.

Choose and fix $p \in B$ and $\widehat{z} \in \widehat{\Gamma}_p$. We consider the vertical part $\widetilde{R}_v = \sum_{p\in B} \widetilde{R}_v(p)$ of $\widetilde{R}$ with respect to $\widetilde{\varphi} : \widetilde{W} \to B$. We let $\widetilde{R}_v(p)_{\widehat{z}}$ be the biggest subdivisor of $\widetilde{R}_v(p)$ contracted to $\widehat{z}$ by $\widehat{\psi}$. Note that we have $\widetilde{R}_v(p)_{\widehat{z}} \neq 0$ only when there exists a singular point (of the branch loci) of $n\mathbb{Z}+1$ type and which is infinitely near to $\widehat{z}$ (including $\widehat{z}$ itself). Note also that $\widetilde{R}_v(p)_{\widehat{z}}$ is a disjoint union of non-singular rational curves each of which is a $(-an)$-curve for some positive integer a.

To decompose $\widetilde{R}_v(p)_{\widehat{z}}$, we define a family $\{L_i\}_i$ consisting of vertical irreducible curves in $\widetilde{R}_v(p)_{\widehat{z}}$ as follows.

(i) Choose and fix a (-1)-curve E_1 over $\widehat{z}$ on an elliptic surface between $\widetilde{W}$ and $\widehat{W}$, and let L_1 be the proper transform of E_1 on $\widetilde{W}$.

(ii) For $i \geq 2$, L_i is the proper transform of an exceptional (-1)-curve E_i that is contracted to a point x_i in E_k or its proper transform for some $k < i$.

Put $\mathfrak{L} := \{\{L_i\}_i \mid \{L_i\}_i$ satisfies (i) and (ii)$\}$. If $\{L_i\}_i$ and $\{L'_j\}_j$ ($\{L_i\}_i$,$\{L'_j\}_j \in \mathfrak{L}$) have a common curve, one can show the union of them $\{L_i, L'_j\}_{i,j}$ is in $\mathfrak{L}$. We define a partial order $\{L_i\}_i \leq \{L'_j\}_j$ if $\{L_i\}_i \subset \{L'_j\}_j$. Then $(\mathfrak{L}, \leq)$ is a partial order set. Applying Zorn's lemma, one should check that every chain admits the upper bound. Hence there exist maximal elements $\{L_{1,k}\}_{k=1}^{k_1}, \cdots, \{L_{\eta_z,k}\}_{k=1}^{k_{\eta_z}}$, where $\eta_{\widehat{z}}$ is the number of maximal elements. Any two of $\{L_{1,k}\}_{k=1}^{k_1}, \cdots, \{L_{\eta_z,k}\}_{k=1}^{k_{\eta_z}}$ have no common curves. We put $D_t := \sum_{k=1}^{k_t} L_{t,k}$ for $t = 1, \cdots, \eta_{\widehat{z}}$.

We can describe $\widetilde{R}_v(p)_{\widehat{z}}$ by using (the above) D_t's as follows. The divisor $\widetilde{R}_v(p)_{\widehat{z}}$ is decomposed into a disjoint sum consisting of such sums uniquely. We denote as

$$\widetilde{R}_v(p)_{\widehat{z}} = D_1 + \cdots + D_{\eta_z}, \quad D_t = \sum_{k=1}^{k_t} L_{t,k}. \tag{2.1}$$

Let $C_{t,k}$ be the exceptional (-1)-curve whose proper transform on $\widetilde{W}$ is $L_{t,k}$. We let $\iota^t(\Gamma_p)_{\widehat{z}}$ and $\kappa^t(\Gamma_p)_{\widehat{z}}$ denote the numbers of singular points of the branch loci $R = R_0, \cdots, R_{N-1}$ over $\widehat{z}$ of types $n\mathbb{Z}$ and $n\mathbb{Z}+1$, respectively, at which the proper transforms of two curves in $\{C_{t,k}\}_{k=1}^{k_t}$ meet, and put $\iota(\Gamma_p)_{\widehat{z}} = \sum_{t=1}^{\eta_z} \iota^t(\Gamma_p)_{\widehat{z}}$ and $\kappa(\Gamma_p)_{\widehat{z}} = \sum_{t=1}^{\eta_z} \kappa^t(\Gamma_p)_{\widehat{z}}$. By the definition of $\iota^t(\Gamma_p)_{\widehat{z}}$ and $\kappa^t(\Gamma_p)_{\widehat{z}}$, we note that $\iota(\Gamma_p)_{\widehat{z}} = \kappa(\Gamma_p)_{\widehat{z}} = 0$ if $\widehat{R}$ is smooth at $\widehat{z}$.

Furthermore, we localize some indices to points on $\widehat{\Gamma}_p$. Let $\widehat{\alpha}_k(\Gamma_p)_{\widehat{z}}$ be the number of singular points contributing to $\widehat{\alpha}_k(\Gamma_p)$ which is infinitely near to $\widehat{z}$. Let $\widehat{j}_{0,a}(\Gamma_p)_{\widehat{z}}$ (*resp.* $\widehat{j}^t_{0,a}(\Gamma_p)_{\widehat{z}}$ for $t = 1, \cdots \eta_{\widehat{z}}$) be the number of irreducible components of $\widetilde{R}_v(p)_{\widehat{z}}$ (*resp.* D_t) contributing to $\widehat{j}_{0,a}$. Put $\widehat{j}_{0,\bullet}(\Gamma_p)_{\widehat{z}} := \sum_{a \geq 1} \widehat{j}_{0,a}(\Gamma_p)_{\widehat{z}}$ and $\widehat{j}^t_{0,\bullet}(\Gamma_p)_{\widehat{z}} := \sum_{a \geq 1} \widehat{j}^t_{0,a}(\Gamma_p)_{\widehat{z}}$ for each $t = 1, \cdots \eta_{\widehat{z}}$. Note that $\widehat{j}_{0,\bullet}(\Gamma_p)_{\widehat{z}} = \sum_t \widehat{j}^t_{0,a}(\Gamma_p)_{\widehat{z}}$. We have $\widehat{\alpha}_k(\Gamma_p) =$

$\sum_{\widehat{z}\in\Gamma_p}\widehat{\alpha}_k(\Gamma_p)_{\widehat{z}}$ and $\widehat{j}_{0,a}(\Gamma_p)=\sum_{\widehat{z}\in\Gamma_p}\widehat{j}_{0,a}(\Gamma_p)_{\widehat{z}}$. We put

$$\tag{2.2}\begin{aligned}\alpha_0^{+}(\Gamma_p)_{\widehat{z}} &:= \begin{cases}(\widetilde{R}_h,\widetilde{\Gamma}_{p,\widehat{z}})-\sharp\left(\mathrm{Supp}(\widetilde{R}_h)\cap\mathrm{Supp}(\widetilde{\Gamma}_{p,\widehat{z}})\right) \\ \qquad\qquad (\text{if } \widehat{R} \text{ is singular at } \widehat{z}\ ), \\ \\ \left(\text{The ramification index of } \widehat{\varphi}|_{\widehat{R}_h}:\widehat{R}_h\to B \text{ at } \widehat{z}\right)-1 \\ \qquad\qquad (\text{if } \widehat{R} \text{ is smooth at } \widehat{z}\ ), \\ \\ 0 \qquad\qquad (\text{if } \widehat{z}\notin\widehat{R}),\end{cases}\\ \widehat{\alpha}_0(\Gamma_p)_{\widehat{z}} &:= \alpha_0^{+}(\Gamma_p)_{\widehat{z}}-2\sum_{a\geq 2}\widehat{j}_{0,a}(\Gamma_p)_{\widehat{z}},\end{aligned}$$

where $\widetilde{\Gamma}_{p,\widehat{z}}$ is the biggest subdivisor of $\widetilde{\Gamma}_p$ which is contracted to $\widehat{z}$ by $\widehat{\psi}$.

We put

$$\begin{aligned}K_f^2(\Gamma_p)_{\widehat{z}} := &\sum_{k\geq 1}\left((n^2-1)k-n\right)\widehat{\alpha}_k(\Gamma_p)_{\widehat{z}}\\ &+\frac{(n-1)^2}{n}(\widehat{\alpha}_0(\Gamma_p)_{\widehat{z}}-2\widehat{j}_{0,1}(\Gamma_p)_{\widehat{z}})+\widehat{j}_{0,1}(\Gamma_p)_{\widehat{z}}.\end{aligned}$$

Then we get the following:

Proposition 2.2. *Let $f:S\to B$ be a primitive cyclic covering fibration of type $(g,1,n)$. Let $\widetilde{\psi}=\check{\psi}\circ\widehat{\psi}$ be an arbitrary decomposition of $\widetilde{\psi}:\widetilde{W}\to W$ so that the commutative diagram*

$$\begin{array}{ccccc}\widetilde{W} & \xrightarrow{\hat{\psi}} & \widehat{W} & \xrightarrow{\check{\psi}} & W \\ & {\scriptstyle\tilde{\varphi}}\searrow & \downarrow{\scriptstyle\hat{\varphi}} & \swarrow{\scriptstyle\varphi} & \\ & & B. & & \end{array}$$

Denote a fiber of $\widehat{\varphi}$ over $p\in B$ by $\widehat{\Gamma}_p$. Then it holds that

$$K_f^2(\Gamma_p)=\sum_{\widehat{z}\in\widehat{\Gamma}_p}K_f^2(\Gamma_p)_{\widehat{z}}+\sum_{k\geq 1}\left((n^2-1)k-n\right)\check{\alpha}_k(\Gamma_p)$$

$$
\begin{aligned}
&- 2\frac{(n-1)^2}{n}\left(\check{j}_{0,\bullet}(\Gamma_p) + j'_{0,\bullet}(\Gamma_p)\right)\\
&+ \frac{n^2-1}{n} r(\chi_\varphi(\Gamma_p) + \nu(\Gamma_p)) + \check{j}_{0,1}(\Gamma_p) + j'_{0,1}(\Gamma_p).
\end{aligned}
$$

Proof. By definition, we have

$$
\begin{aligned}
\alpha_k(\Gamma_p) &= \sum_{\hat{z}\in\hat{\Gamma}_p} \widehat{\alpha}_k(\Gamma_p)_{\hat{z}} + \check{\alpha}_k(\Gamma_p),\\
j_{0,a}(\Gamma_p) &= \sum_{\hat{z}\in\hat{\Gamma}_p} \widehat{j}_{0,a}(\Gamma_p)_{\hat{z}} + \check{j}_{0,a}(\Gamma_p) + j'_{0,a}(\Gamma_p).
\end{aligned}
\tag{2.3}
$$

By (2.3), we have

$$
\begin{aligned}
K_f^2(\Gamma_p) =& \sum_{k\geq 1}\left((n^2-1)k - n\right)\alpha_k(\Gamma_p) + \frac{(n-1)^2}{n}(\alpha_0(\Gamma_p) - 2j_{0,1}(\Gamma_p))\\
&+ \frac{n^2-1}{n} r(\chi_\varphi(\Gamma_p) + \nu(\Gamma_p)) + j_{0,1}(\Gamma_p)\\
=& \sum_{k\geq 1}\left((n^2-1)k - n\right)\left(\sum_{\hat{z}\in\hat{\Gamma}_p} \widehat{\alpha}_k(\Gamma_p)_{\hat{z}} + \check{\alpha}_k(\Gamma_p)\right)\\
&+ \frac{(n-1)^2}{n}\Bigg(\sum_{\hat{z}\in\hat{\Gamma}_p} \widehat{\alpha}_0(\Gamma_p)_{\hat{z}} - 2\sum_{a\geq 2}\left(\check{j}_{0,a}(\Gamma_0) + j'_{0,a}(\Gamma_p)\right)\\
&-2\sum_{\hat{z}\in\hat{\Gamma}_p} \widehat{j}_{0,1}(\Gamma_p)_{\hat{z}} - 2\left(\check{j}_{0,1}(\Gamma_p) + j'_{0,1}(\Gamma_p)\right)\Bigg)\\
&+ \sum_{\hat{z}\in\hat{\Gamma}_p} \widehat{j}_{0,1}(\Gamma_p)_{\hat{z}} + \check{j}_{0,1}(\Gamma_p) + j'_{0,1}(\Gamma_p) + \frac{n^2-1}{n} r(\chi_\varphi(\Gamma_p) + \nu(\Gamma_p))\\
=& \sum_{\hat{z}\in\hat{\Gamma}_p} K_f^2(\Gamma_p)_{\hat{z}} + \sum_{k\geq 1}\left((n^2-1)k - n\right)\check{\alpha}_k(\Gamma_p)\\
&- 2\frac{(n-1)^2}{n}\left(\check{j}_{0,\bullet}(\Gamma_p) + j'_{0,\bullet}(\Gamma_p)\right)\\
&+ \frac{n^2-1}{n} r(\chi_\varphi(\Gamma_p) + \nu(\Gamma_p)) + \check{j}_{0,1}(\Gamma_p) + j'_{0,1}(\Gamma_p) \quad \square
\end{aligned}
$$

We consider and recast Lemma 5.2 in [5].

LEMMA 2.3. *The following hold:*
(1) $\iota(\Gamma_p)_{\mathfrak{z}} = \widehat{j}_{0,\bullet}(\Gamma_p)_{\mathfrak{z}} - \eta_{\mathfrak{z}}$.
(2) $\alpha_0^+(\Gamma_p)_{\mathfrak{z}} \geq (n-2)\left(\widehat{j}_{0,\bullet}(\Gamma_p)_{\mathfrak{z}} - \eta_{\mathfrak{z}} + 2\kappa(\Gamma_p)_{\mathfrak{z}}\right)$.
(3) $\sum_{k\geq 1}\widehat{\alpha}_k(\Gamma_p)_{\mathfrak{z}} \geq \sum_{a\geq 1}(an-2)\widehat{j}_{0,a}(\Gamma_p)_{\mathfrak{z}} + 2\eta_{\mathfrak{z}} - \kappa(\Gamma_p)_{\mathfrak{z}}$.

PROOF. For each t, we consider the graph $\mathbb{G}_t$ corresponding to D_t as follows. The vertex set $V(\mathbb{G}_t)$ and the edge set $E(\mathbb{G}_t)$ are respectively the sets of symbols $\{v_{t,k}\}_{k=1}^{\widehat{j}^t_{0,\bullet}(\Gamma_p)_{\mathfrak{z}}}$ and $\{e_x\}_x$, where x runs over all the singular points contributing to $\iota^t(\Gamma_p)_{\mathfrak{z}}$. If $C_{t,k}$ or a proper transform of it meets a proper transform of $C_{t,k'}$ at the singular point x of type $n\mathbb{Z}$, the edge e_x connects $v_{t,k}$ and $v_{t,k'}$. By the definition of D_t, we see that $\mathbb{G}_t$ is a connected tree. Counting the numbers of vertices and edges, we have $\iota^t(\Gamma_p)_{\mathfrak{z}} = \widehat{j}^t_{0,\bullet}(\Gamma_p)_{\mathfrak{z}} - 1$. Thus, we get (1) by summing it up for t.

When the component $L_{t,k}$ of D_t is a $(-an)$-curve, it is obtained by blowing $C_{t,k}$ up $an-1$ times. We recall that $\widehat{j}^t_{0,a}(\Gamma_p)_{\mathfrak{z}}$ is the number of irreducible components of D_t with self-intersection number $-an$. By $D_t := \sum_{k=1}^{k_t} L_{t,k}$, we need blowing-ups $\sum_{a\geq 1}(an-1)\widehat{j}^t_{0,a}(\Gamma_p)_{\mathfrak{z}}$ times to get D_t, disregarding overlaps. Taking into the account the duplication and the first blowing-up creating the (-1)-curve for $L_{t,1}$, we see that the number of blowing-ups to obtain D_t is not less than

$$\begin{aligned}&\sum_{a\geq 1}(an-1)\widehat{j}^t_{0,a}(\Gamma_p)_{\mathfrak{z}} - \iota^t(\Gamma_p)_{\mathfrak{z}} - \kappa^t(\Gamma_p)_{\mathfrak{z}} + 1\\ &= \sum_{a\geq 2}(an-2)\widehat{j}^t_{0,a}(\Gamma_p)_{\mathfrak{z}} + 2 - \kappa^t(\Gamma_p)_{\mathfrak{z}},\end{aligned}$$

since $\iota^t(\Gamma_p)_{\mathfrak{z}} = \widehat{j}^t_{0,\bullet}(\Gamma_p)_{\mathfrak{z}} - 1$. By $\widetilde{R}_v(p)_{\mathfrak{z}} = \sum_{t=1}^{\eta_{\mathfrak{z}}} D_t$, we need blowing-ups at least

$$\begin{aligned}&\sum_{t=1}^{\eta_{\mathfrak{z}}}\left(\sum_{a\geq 2}(an-2)\widehat{j}^t_{a}(\Gamma_p)_{\mathfrak{z}} + 2 - \kappa^t(\Gamma_p)_{\mathfrak{z}}\right)\\ &= \sum_{a\geq 2}(an-2)\widehat{j}_{0,a}(\Gamma_p)_{\mathfrak{z}} + 2\eta_{\mathfrak{z}} - \kappa(\Gamma_p)_{\mathfrak{z}}\end{aligned}$$

times to get $\widetilde{R}_v(p)_{\widehat{z}}$. This gives (3).

It remains to show (2). We may assume $\widehat{R}$ is singular at $\widehat{z}$. Let $\widetilde{\Gamma}_{p,\widehat{z}} = \sum m_i G_i$ be the irreducible decomposition. Then it follows from (2.2) that

$$\alpha_0^+(\Gamma_p)_{\widehat{z}} = \sum_i m_i \widetilde{R}_h G_i - \sharp(\mathrm{Supp}(\widetilde{R}_h) \cap \mathrm{Supp}(\cup_i G_i)) \geq \sum_i (m_i - 1)\widetilde{R}_h G_i.$$

We consider a directed graph $\mathbb{F}$ whose vertex set $V(\mathbb{F})$ is the set of symbols $\{v_x\}$, where x runs over all the singular points which contribute to either $\iota(\Gamma_p)_{\widehat{z}}$ or $\kappa(\Gamma_p)_{\widehat{z}}$. We define the directed edge from v_x to $v_{x'}$ if x' is a singular point infinitely near to x and any singular point between x and x' contributes to neither $\iota(\Gamma_p)_{\widehat{z}}$ nor $\kappa(\Gamma_p)_{\widehat{z}}$. Let $\mathbb{T}_1, \cdots, \mathbb{T}_s$ be connected components of the graph $\mathbb{F}$. We note that $\mathbb{T}_j$ $(j = 1, \cdots, s)$ is a directed and rooted tree graph. We denote the leaf set of $\mathbb{T}_j$ by $L(\mathbb{T}_j)$. By the definition of $\mathbb{F}$, any vertex in $L(\mathbb{T}_j)$ corresponds to a singular point contributing to $\iota(\Gamma_p)_{\widehat{z}}$. Let x_{last} be a singular point of the branch locus over x such that there exist no singular points over x_{last} and let $\widetilde{E}^x$ be the exceptional curve of the blow-up at x_{last}. Since $\widetilde{E}^x$ arises from a singular point of type $n\mathbb{Z}$, we see that $\widetilde{R}\widetilde{E}^x$ is a positive multiple of n, but it may be possible that $\widetilde{E}^x$ meets two vertical components of $\widetilde{R}$. Thus we have $\widetilde{R}_h\widetilde{E}^x \geq n-2$. Letting m^x be the multiplicity along $\widetilde{E}^x$ of $\widetilde{\Gamma}_p$, we have

$$\sum_i (m_i - 1)\widetilde{R}_h G_i \geq \sum_{j=1}^{s} \sum_{x \in L(\mathbb{T}_j)} (m^x - 1)\widetilde{R}_h\widetilde{E}^x.$$

We will show that

$$\sum_{x \in L(\mathbb{T}_j)} (m^x - 1) \geq \iota(\mathbb{T}_j) + 2\kappa(\mathbb{T}_j),$$

where $\iota(\mathbb{T}_j)$ (resp. $\kappa(\mathbb{T}_j)$) denotes the number of singular points of type $n\mathbb{Z}$ (resp. $n\mathbb{Z}+1$) in $\mathbb{T}_j$. Let $P^x(\mathbb{T}_j)$ be the set consisting of vertices appearing in the path connecting the root of $\mathbb{T}_j$ and $x \in L(\mathbb{T}_j)$. We denote the number of singular points of type $n\mathbb{Z}$ (resp. $n\mathbb{Z}+1$) in $P^x(\mathbb{T}_j)$ by $\iota(P^x(\mathbb{T}_j))$ (resp. $\kappa(P^x(\mathbb{T}_j))$). We claim that $m^x \geq 2\iota(P^x(\mathbb{T}_j)) + \kappa(P^x(\mathbb{T}_j))$. This can be seen as follows. Put $P^x(\mathbb{T}_j) = \{v_1, \cdots, v_l = v_x\}$ and let $m_1, \cdots, m_l$ be the multiplicities of the fiber $\widetilde{\Gamma}_p$ along the proper transforms of the exceptional curves arising from $v_1, \cdots, v_l$, respectively. If v_1 is of type $n\mathbb{Z}$, then we have

$m_2 \geq m_1+2$. If v_1 is of type $n\mathbb{Z}+1$, then we have $m_2 \geq m_1+1$. So we get $m^x = m_l \geq 2\iota(P^x(\mathbb{T}_j)) + \kappa(P^x(\mathbb{T}_j))$ inductively. Then,

$$\begin{aligned}\sum_{x\in L(\mathbb{T}_j)} (m^x-1) &\geq \sum_{x\in L(\mathbb{T}_j)} (2\iota(P^x(\mathbb{T}_j)) + \kappa(P^x(\mathbb{T}_j))) - \sharp L(\mathbb{T}_j)\\ &\geq 2\iota(\mathbb{T}_j) + 2\kappa(\mathbb{T}_j) - \sharp L(\mathbb{T}_j)\\ &\geq \iota(\mathbb{T}_j) + 2\kappa(\mathbb{T}_j)\end{aligned}$$

as wished.

Now, we get

$$\begin{aligned}\sum_{x\in L(\mathbb{T}_j)} (m^x-1)\widetilde{R}_h G_i &\geq \sum_{j=1}^{s}\sum_{x\in L(\mathbb{T}_j)} (m^x-1)\widetilde{R}_h\widetilde{E}^x\\ &\geq (n-2)\sum_{j=1}^{s}\sum_{x\in L(\mathbb{T}_j)}(m^x-1)\\ &\geq (n-2)(\iota(\Gamma_p)_{\hat{z}} + 2\kappa(\Gamma_p)_{\hat{z}}).\end{aligned}$$

Plugging the above inequalities to (2.2), we conclude that $\alpha_0^+(\Gamma_p)_{\hat{z}} \geq (n-2)(\iota(\Gamma_p)_{\hat{z}} + 2\kappa(\Gamma_p)_{\hat{z}})$. Then, since $\iota(\Gamma_p)_{\hat{z}} = j(\Gamma_p)_{\hat{z}} - \eta_{\hat{z}}$ by (1), we get (2). □

We can show the following inequality similarly to Lemma 4.7 of [6]

LEMMA 2.4. *If $n=2$, then it holds*

$$\kappa(\Gamma_p)_{\hat{z}} \leq \frac{2}{3}\sum_{a\geq 2}(a-1)\widehat{j}_{0,a}(\Gamma_p)_{\hat{z}}.$$

LEMMA 2.5. *Let the notation and the assumption as above. Then it holds that*

$$\widehat{\alpha}_0(\Gamma_p)_{\hat{z}} + B_n\sum_{k\geq 1}\widehat{\alpha}_k(\Gamma_p)_{\hat{z}} - \begin{cases} nB_n\widehat{j}_{0,1}(\Gamma_p)_{\hat{z}} & (n\geq 3)\\ \\ 0 & (n=2)\end{cases}$$

is non-negative, where $B_n = 2/n$ if $n\geq 4$, $B_3 = 1/2$ and $B_2 = 3/2$.

Proof. If $n \geq 3$, we can show the desired inequality similarly to Lemma 2.2 of [1].

For $n = 2$, we have

$$\begin{aligned}
&\widehat{\alpha}_0(\Gamma_p)_{\hat{z}} + B_2 \sum_{k\geq 1} \widehat{\alpha}_k(\Gamma_p)_{\hat{z}} \\
&= \alpha_0^+(\Gamma_p)_{\hat{z}} - 2\sum_{a\geq 2} \widehat{j}_{0,a}(\Gamma_p)_{\hat{z}} + B_2 \sum_{k\geq 1} \widehat{\alpha}_k(\Gamma_p)_{\hat{z}} \\
&\geq \alpha_0^+(\Gamma_p)_{\hat{z}} + \sum_{a\geq 2} \left(B_2(2a-2) - 2\right) \widehat{j}_{0,a}(\Gamma_p)_{\hat{z}} + 2B_2\eta_{\hat{z}} - B_2\kappa(\Gamma_p)_{\hat{z}} \\
&\geq \alpha_0^+(\Gamma_p)_{\hat{z}} + \sum_{a\geq 2} \left(\frac{4}{3}(a-1)B_2 - 2\right) \widehat{j}_{0,a}(\Gamma_p)_{\hat{z}},
\end{aligned}$$

where the first inequality is by (3) of Lemma 2.3, the second inequality is by Lemma 2.4. Since $B_2 = 3/2$, the coefficients of $\widehat{j}_{0,a}(\Gamma_p)_{\hat{z}}$ are non-negative for $a \geq 2$. □

Lemma 2.6. *The following holds for $p \in B$:*

$$K_f^2(\Gamma_p)_{\hat{z}} \geq \sum_{k\geq 1} \left((n^2-1)k - n - \frac{(n-1)^2}{n} B_n\right) \widehat{\alpha}_k(\Gamma_p)_{\hat{z}}.$$

In particular, $K_f^2(\Gamma_p)_{\hat{z}}$ is non-negative.

Proof. If $n \geq 4$, it is clear from Lemma 2.5.

If $n = 3$, we have

$$\widehat{\alpha}_0(\Gamma_p)_{\hat{z}} - 2\widehat{j}_{0,1}(\Gamma_p)_{\hat{z}} \geq -B_3 \sum_{k\geq 1} \widehat{\alpha}_k(\Gamma_p)_{\hat{z}} - \frac{1}{2}\widehat{j}_{0,1}(\Gamma_p)_{\hat{z}}$$

from Lemma 2.5. Hence we get

$$\begin{aligned}
K_f^2(\Gamma_p)_{\hat{z}} &\geq \sum_{k\geq 1} (8k-3)\, \widehat{\alpha}_k(\Gamma_p)_{\hat{z}} - \frac{4}{3}B_3 \sum_{k\geq 1} \widehat{\alpha}_k(\Gamma_p)_{\hat{z}} \\
&\quad - \frac{2}{3}\, \widehat{j}_{0,1}(\Gamma_p)_{\hat{z}} + \widehat{j}_{0,1}(\Gamma_p)_{\hat{z}} \\
&\geq \left(8k - 3 - \frac{4}{3}B_3\right) \widehat{\alpha}_k(\Gamma_p)_{\hat{z}}.
\end{aligned}$$

If $n=2$, we have

$$K_f^2(\Gamma_p)_{\hat{z}} = \sum_{k\geq 1} (3k-2)\,\widehat{\alpha}_k(\Gamma_p)_{\hat{z}} + \frac{1}{2}\widehat{\alpha}_0(\Gamma_p)_{\hat{z}}.$$

Hence we get

$$K_f^2(\Gamma_p)_{\hat{z}} \geq \sum_{k\geq 1}\left(3k-2-\frac{1}{2}B_2\right)\widehat{\alpha}_k(\Gamma_p)_{\hat{z}}$$

from Lemma 2.5. □

LEMMA 2.7. *The following holds for $p\in B$.*

$$\begin{aligned}
K_f^2(\Gamma_p) \geq & \sum_{\hat{z}\in\widehat{\Gamma}_p} \frac{(n-1)^2}{n}\alpha_0^+(\Gamma_p)_{\hat{z}} \\
&+ \sum_{\hat{z}\in\widehat{\Gamma}_p}\sum_{k\geq 1}\left((n^2-1)k-n-2\frac{(n-1)^2}{n}\right)\widehat{\alpha}_k(\Gamma_p)_{\hat{z}} \\
&+ \sum_{k\geq 1}\big((n^2-1)k-n\big)\,\check{\alpha}_k(\Gamma_p) - 2\frac{(n-1)^2}{n}\big(\check{j}_{0,\bullet}(\Gamma_p)+j'_{0,\bullet}(\Gamma_p)\big) \\
&+ \frac{n^2-1}{n}r(\chi_\varphi(\Gamma_p)+\nu(\Gamma_p)) + \check{j}_{0,1}(\Gamma_p) + j'_{0,1}(\Gamma_p).
\end{aligned}$$

PROOF. By the definition of $K_f^2(\Gamma_p)_{\hat{z}}$, we have

$$\begin{aligned}
K_f^2(\Gamma_p)_{\hat{z}} = & \sum_{k\geq 1}\big((n^2-1)k-n\big)\,\widehat{\alpha}_k(\Gamma_p)_{\hat{z}} \\
&+ \frac{(n-1)^2}{n}(\widehat{\alpha}_0(\Gamma_p)_{\hat{z}} - 2\widehat{j}_{0,1}(\Gamma_p)_{\hat{z}}) + \widehat{j}_{0,1}(\Gamma_p)_{\hat{z}}
\end{aligned}$$

for $\hat{z}\in\widehat{\Gamma}_p$. Since irreducible components of $\widehat{R}(p)_{\hat{z}}$ arise from singular points of type $n\mathbb{Z}+1$, we have $\widehat{j}_{0,\bullet}(\Gamma_p)_{\hat{z}} \leq \sum_{k\geq 1}\widehat{\alpha}_k(\Gamma_p)_{\hat{z}}$. Hence we get

$$K_f^2(\Gamma_p)_{\hat{z}} \geq \frac{(n-1)^2}{n}\alpha_0^+(\Gamma_p)_{\hat{z}} + \sum_{k\geq 1}\left((n^2-1)k-n-2\frac{(n-1)^2}{n}\right)\widehat{\alpha}_k(\Gamma_p)_{\hat{z}}.$$

Combining this with Proposition 2.2, we get the desired inequality. □

3. Automorphism Groups of Elliptic Surfaces

In this section, we summarize some properties of automorphism groups of elliptic surfaces. Let $\varphi : W \to B$ be a relatively minimal elliptic surface. We fix a point $p \in B$ and denote by $\Delta \subset B$ an analytic open neighborhood of p. We suppose that Γ_p is not multiple. Let $\varphi_\Delta : W_\Delta \to \Delta$ be a restriction of $\varphi : W \to B$ to Δ:

$$\begin{array}{ccc} W_\Delta & \hookrightarrow & W \\ {\scriptstyle \varphi_\Delta}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \Delta & \hookrightarrow & B. \end{array}$$

Replacing Δ by a smaller neighborhood of p if necessary, we may assume that φ_Δ has only one singular fiber Γ_p.

Let $W_{\Delta,\mathrm{sm}} := W_\Delta \setminus \mathrm{Sing}(\Gamma_p)$ and $\Gamma_{p,\mathrm{sm}} := \Gamma_p \setminus \mathrm{Sing}(\Gamma_p)$, where $\mathrm{Sing}(\Gamma_p)$ is the set of critical points of φ_Δ on Γ_p. Then there exists the natural map $\varphi_{\Delta,\mathrm{sm}} : W_{\Delta,\mathrm{sm}} \to \Delta$. The following theorem is due to Kodaira ([7]).

Theorem 3.1 (Kodaira). *There exist three holomorphic maps*

$$O_\Delta : \Delta \to W_{\Delta,\mathrm{sm}}, \quad \mu_\Delta : W_{\Delta,\mathrm{sm}} \times_\Delta W_{\Delta,\mathrm{sm}} \to W_{\Delta,\mathrm{sm}},$$
$$\iota_\Delta : W_{\Delta,\mathrm{sm}} \to W_{\Delta,\mathrm{sm}}$$

over Δ which satisfy the following conditions.

(1) *The fiber germ $\varphi_{\Delta,\mathrm{sm}} : W_{\Delta,\mathrm{sm}} \to \Delta$ is a group manifold over Δ.*

(2) *Let Γ_q $(q \in \Delta)$ be a fiber of φ_Δ and let $O_q := \Gamma_q \cap O_\Delta(\Delta)$. These three maps induce the commutative group law on $\Gamma_{q,\mathrm{sm}}$ whose unit element is O_q. The group $(\Gamma_{q,\mathrm{sm}}, O_q)$ is one of the followings.*

$$(\Gamma_{q,\mathrm{sm}}, O_q) \cong \begin{cases} \textit{The group law of elliptic curve} & (\mathrm{I}_0\textit{-type}) \\ \mathbb{G}_m \times G(\Gamma_q) & (\mathrm{I}_c\textit{-type}) \\ \mathbb{G}_a \times G(\Gamma_q) & (\textit{other}) \end{cases}$$

where

$$
\begin{aligned}
&G(\mathrm{I}_c) \cong \mathbb{Z}/c\mathbb{Z},\\
&G(\mathrm{I}^*_{2c}) \cong (\mathbb{Z}/2\mathbb{Z})^2,\\
&G(\mathrm{I}^*_{2c+1}) \cong \mathbb{Z}/4\mathbb{Z},\\
&G(\mathrm{II}) \cong G(\mathrm{II}^*) \cong \{0\},\\
&G(\mathrm{III}) \cong G(\mathrm{III}^*) \cong \mathbb{Z}/2\mathbb{Z},\\
&G(\mathrm{IV}) \cong G(\mathrm{IV}^*) \cong \mathbb{Z}/3\mathbb{Z}.
\end{aligned}
$$

Proof. Since Γ_p is not multiple, there exists a projective elliptic surface over a smooth projective curve $\varphi^\flat : W^\flat \to B^\flat$ which admits a global section and contains $\varphi_\Delta : W_\Delta \to \Delta$ as a fiber germ:

$$
\begin{array}{ccc}
W_\Delta & \hookrightarrow & W^\flat \\
{\scriptstyle \varphi_\Delta}\downarrow & & \downarrow{\scriptstyle \varphi^\flat} \\
\Delta & \hookrightarrow & B^\flat.
\end{array}
$$

We denote by $O_\flat : B^\flat \to W^\flat$ a global section of $\varphi^\flat$. Let $W^\flat_{\mathrm{sm}}$ be an open subset of $W^\flat$ consisting of all regular points of $\varphi^\flat$. From Theorem 9.1 of [7], there exist holomorphic maps $\mu_\flat : W^\flat_{\mathrm{sm}} \times_{B^\flat} W^\flat_{\mathrm{sm}} \to W^\flat_{\mathrm{sm}}$ and $\iota_\flat : W^\flat_{\mathrm{sm}} \to W^\flat_{\mathrm{sm}}$ over $B^\flat$ which satisfy conditions (1) and (2). These three maps $O_\flat$, $\mu_\flat$ and $\iota_\flat$ induce O_Δ, μ_Δ and ι_Δ by the universality of base change. $\square$

- We denote by $W_{\mathrm{sm}}(\Delta)$ the set of sections of $\varphi_{\Delta,\mathrm{sm}}$. Since $\varphi_{\Delta,\mathrm{sm}} : W_{\Delta,\mathrm{sm}} \to \Delta$ is a group manifold over Δ, $W_{\mathrm{sm}}(\Delta)$ is a commutative group with the unit element O_Δ. The symbol $\beta_1 \oplus \beta_2$ denotes the sum of β_1, $\beta_2 \in W_{\mathrm{sm}}(\Delta)$ and $\ominus\beta$ denotes the inverse element of $\beta \in W_{\mathrm{sm}}(\Delta)$.

- We have the following commutative diagram for a section $\beta \in W_{\mathrm{sm}}(\Delta)$:

$$
\begin{array}{ccccc}
W_{\Delta,\mathrm{sm}} & \xrightarrow[\beta\times_{W_{\Delta,\mathrm{sm}}}(\mathrm{pr}_1)]{} & W_{\Delta,\mathrm{sm}} \times_\Delta W_{\Delta,\mathrm{sm}} & \xrightarrow[\mathrm{pr}_2]{} & W_{\Delta,\mathrm{sm}} \\
{\scriptstyle \varphi_\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \varphi_\Delta} \\
\Delta & \xrightarrow[\beta]{} & W_{\Delta,\mathrm{sm}} & \xrightarrow[\varphi_\Delta]{} & \Delta
\end{array}
$$

Hence we have an automorphism $\tau_\beta := \mu_\Delta \circ \big(\beta \times_{W_{\Delta,\mathrm{sm}}} (\mathrm{pr}_1)\big) \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$ for each $\beta \in W_{\mathrm{sm}}(\Delta)$. Thus, there exists a natural injective homomorphism $W_{\mathrm{sm}}(\Delta) \to \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta); \beta \mapsto \tau_\beta$.

- We define the following groups.

$$\begin{aligned} &\mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta) := \{\kappa \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}) \mid \varphi_{\Delta,\mathrm{sm}} \circ \kappa = \varphi_{\Delta,\mathrm{sm}}\} \\ &\mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta) := \{\kappa \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta) \mid \kappa \circ O_\Delta = O_\Delta\} \\ &\mathrm{Aut}(\Gamma_{p,\mathrm{sm}}, O_p) := \{\kappa \in \mathrm{Aut}(\Gamma_{p,\mathrm{sm}}) \mid \kappa(O_p) = O_p\} \end{aligned} \tag{3.1}$$

Proposition 3.2. *There exists the exact sequence*

$$1 \to W_{\mathrm{sm}}(\Delta) \to \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta) \to \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta) \to 1.$$

In particular, it holds that

$$\mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta) \cong W_{\mathrm{sm}}(\Delta) \rtimes \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta).$$

Proof. It is sufficient to show the following two statements.

(1) There exist $\tau \in W_{\mathrm{sm}}(\Delta) \subset \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$ and $\varepsilon \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta)$ for an action $\kappa \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$ such that $\kappa = \tau \circ \varepsilon$.

(2) $W_{\mathrm{sm}}(\Delta) \lhd \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$.

(1) An action $\kappa \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$ determines a section $\kappa \circ O_\Delta \in W_{\mathrm{sm}}(\Delta)$. We consider the action $\tau_{\ominus(\kappa\circ O_\Delta)}$. Then we have $\varepsilon := \tau_{\ominus(\kappa\circ O_\Delta)} \circ \kappa \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta)$. Hence we get $\tau^{-1}_{\ominus(\kappa\circ O_\Delta)} \in W_{\mathrm{sm}}(\Delta)$ and $\varepsilon \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta)$ such that $\kappa = \tau^{-1}_{\ominus(\kappa\circ O_\Delta)} \circ \varepsilon$.

(2) We have to show $\kappa \circ \tau \circ \kappa^{-1} \in W_{\mathrm{sm}}(\Delta)$ for any $\kappa \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$ and $\tau \in W_{\mathrm{sm}}(\Delta)$. In order to show the above statement, it is sufficient to show $\varepsilon \circ \tau \circ \varepsilon^{-1} \in W_{\mathrm{sm}}(\Delta)$ for any $\varepsilon \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta)$ and $\tau \in W_{\mathrm{sm}}(\Delta)$. Suppose a section $\beta \in W_{\mathrm{sm}}(\Delta)$ corresponds to the action $\tau \in \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$ (namely $\tau = \tau_\beta$). Let Γ_q be an arbitrary smooth fiber of φ_Δ. Then it holds that $(\varepsilon \circ \tau_\beta \circ \varepsilon^{-1})|_{\Gamma_q} = (\tau_{\varepsilon\circ\beta})|_{\Gamma_q}$ since $\varepsilon|_{\Gamma_q}$ is compatible with the group law of Γ_q. Therefore we have $(\varepsilon\circ\tau_\beta\circ\varepsilon^{-1}) = \tau_{\varepsilon\circ\beta} \in W_{\mathrm{sm}}(\Delta)$. □

4. Automorphism of Fibered Surface

For a fibration $f : S \to B$, we define the automorphism group of f as

$$\mathrm{Aut}(f) := \{(\kappa_S, \kappa_B) \in \mathrm{Aut}(S) \times \mathrm{Aut}(B) \mid f \circ \kappa_S = \kappa_B \circ f\}.$$

Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$. Let Σ be the cyclic group of order n generated by σ, where σ is defined in Remark 1.2. Let G be an arbitrary finite subgroup of $\mathrm{Aut}(f)$. Since $\mathrm{Aut}(S/B) := \{(\kappa_S, \mathrm{id}) \in \mathrm{Aut}(f)\}$ is a finite group, we may assume $\sigma \in G$ to estimate the order of it. We have the exact sequence

$$1 \to K \to G \to H \to 1,$$

where $K := \{(\kappa_S, id) \in G\}$ and $H := \{\kappa_B \in \mathrm{Aut}(B) \mid (\kappa_S, \kappa_B) \in G, \exists \kappa_S \in \mathrm{Aut}(S)\}$.

Lemma 4.1. *Assume $r = \widetilde{R}\widetilde{\Gamma} \geq 4n$. Take a smooth fiber F of f and a point $z \in F$. Let κ_F be an automorphism of F. Then it holds that*

$$\kappa_F(\Sigma \cdot z) = \Sigma \cdot \kappa_F(z),$$

where $\Sigma \cdot z$ denotes the Σ-orbits of z.

Proof. The subgroup $\Sigma \subset \mathrm{Aut}(F)$ induces the quotient map $\theta : F \to \Gamma := F/\Sigma$ of degree n. From the assumption $r \geq 4n$, there exists an isomorphism $\kappa_\Gamma : \Gamma \to \Gamma$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\kappa_F} & F \\ \theta \downarrow & & \downarrow \theta \\ \Gamma & \xrightarrow[\kappa_\Gamma]{} & \Gamma \end{array}$$

commutes for any $\kappa_F \in \mathrm{Aut}(F)$ from Proposition 3.1 of [1]. Thus, we obtain $\kappa_F(\Sigma \cdot z) = \Sigma \cdot \kappa_F(z)$. $\square$

In what follows in section 4, we tacitly assume that $r \geq 4n$.

We can show the following by a similar argument as in Lemma 3.3 of [1].

Lemma 4.2. *Let F be a smooth fiber of $f : S \to B$. Regard Σ as a subgroup of* $\mathrm{Aut}(F)$*. Then Σ is a normal subgroup of* $\mathrm{Aut}(F)$*.*

Since we have $\Sigma \lhd K$, the action of $K \subset \mathrm{Aut}(S/B)$ on S can be lifted to the one on $\widetilde{S}$ and we can regard K as a subgroup of $\mathrm{Aut}(\widetilde{S}/B)$. If we put $\widetilde{K} = K/\Sigma$, then we have the exact sequence

$$1 \to \Sigma \to K \to \widetilde{K} \to 1.$$

Note that $\widetilde{K} \subset \mathrm{Aut}(\widetilde{W}/B)$ and $\widetilde{R}$ is $\widetilde{K}$-stable (namely $\widetilde{K}(\widetilde{R}) = \widetilde{R}$).

Lemma 4.3. *The action of $\widetilde{K}$ descends down faithfully on the relatively minimal model $\varphi : W \to B$. Hence we can regard $\widetilde{K}$ as a subgroup of* $\mathrm{Aut}(W/B)$*.*

Proof. Let $\widetilde{\Gamma}_p$ be an arbitrary singular fiber of $\widetilde{W} \to B$ and E any (-1)-curve contained in $\widetilde{\Gamma}_p$. Then its $\widetilde{K}$-orbits $\widetilde{K} \cdot E$ satisfies one of the following:

- $\widetilde{K} \cdot E$ consists of a disjoint union of (-1)-curves in $\widetilde{\Gamma}_p$.
- We can find two curves in $\widetilde{K} = \{E_1, \cdots, E_t\}$ meeting a point.

In the former case, contracting $\widetilde{K} \cdot E$ to points, the action of $\widetilde{K}$ descends down to the action on the new fiber obtained by the contraction.

We will show the latter case does not occur. Since the intersection form on $E_1 \cup E_2$ is negative semi-definite, $E_1^2 = E_2^2 = -1$ and $E_1E_2 > 0$, the only possibility is: $E_1E_2 = 1$ and $(E_1 + E_2)^2 = 0$. So we get

$$\mathrm{Supp}\, \Gamma_p = E_1 \cup E_2$$

by Zariski's lemma. This is impossible by the classification of singular fibers of elliptic surfaces. □

We fix a point $p \in B$ and denote by $\Delta \subset B$ an analytic open neighborhood of p. We suppose that Γ_p is not multiple. Let $\varphi_\Delta : W_\Delta \to \Delta$ be a restriction of $\varphi : W \to B$ to Δ. We note that there exists a natural inclusion $\widetilde{K} \subset \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta)$.

Definition 4.4. We call $\kappa \in \widetilde{K}$ a *translation* of $\widetilde{K}$ if κ satisfies either $\kappa(C_\Delta) \neq C_\Delta$ or $\kappa|_{C_\Delta} \neq \mathrm{id}_{C_\Delta}$ for any φ_Δ-horizontal local analytic branch C_Δ on $W_{\Delta,\mathrm{sm}}$. Let $T(\widetilde{K})_p$ be the set consisting of translations of $\widetilde{K}$ and $\mathrm{id}_{W_{\Delta,\mathrm{sm}}}$.

Proposition 4.5. *It holds that*

$$T(\widetilde{K})_p = \widetilde{K} \cap W_{\mathrm{sm}}(\Delta).$$

In particular, $T(\widetilde{K})_p$ is a subgroup of $\widetilde{K}$.

Proof. Since $W_{\mathrm{sm}}(\Delta)$ is a normal subgroup of $\mathrm{Aut}(W_{\mathrm{sm}}/\Delta)$, we deduce that $\widetilde{K} \cap W_{\Delta,\mathrm{sm}}(\Delta)$ is a normal subgroup of $\widetilde{K}$. We have an exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{K} \cap W_{\mathrm{sm}}(\Delta) & \longrightarrow & \widetilde{K} & \longrightarrow & \widetilde{K}/(\widetilde{K} \cap W_{\mathrm{sm}}(\Delta)) & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & W_{\mathrm{sm}}(\Delta) & \longrightarrow & \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta) & \longrightarrow & \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta) & \longrightarrow & 1.
\end{array}$$

We note that the injectivity of $\widetilde{K}/(\widetilde{K} \cap W_{\Delta,\mathrm{sm}}(\Delta)) \to \mathrm{Aut}(W_{\Delta,\mathrm{sm}}/\Delta, O_\Delta)$ is shown by a simple diagram chasing. Therefore we have the following exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{K} \cap W_{\mathrm{sm}}(\Delta) & \longrightarrow & \widetilde{K} & \longrightarrow & \widetilde{K}/(\widetilde{K} \cap W_{\mathrm{sm}}(\Delta)) & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & (\Gamma_{q,\mathrm{sm}}, O_q) & \longrightarrow & \mathrm{Aut}(\Gamma_{q,\mathrm{sm}}) & \longrightarrow & \mathrm{Aut}(\Gamma_{q,\mathrm{sm}}, O_q) & \longrightarrow & 1
\end{array}$$

for an arbitrary fiber Γ_q ($q \in \Delta$) of φ_Δ. Since $\widetilde{K}/(\widetilde{K} \cap W_{\Delta,\mathrm{sm}}(\Delta)) \to \mathrm{Aut}(\Gamma_{q,\mathrm{sm}}, O_q)$ is injective, we deduce that

$$\widetilde{K} \cap (\Gamma_q, O_q) \subset \widetilde{K} \cap W_{\mathrm{sm}}(\Delta)$$

for an arbitrary smooth fiber Γ_q.

Take a non-trivial arbitrary automorphism $\kappa \in T(\widetilde{K})_p$. Since κ is a translation of $\widetilde{K}$, $\kappa|_{\Gamma_q}$ has no fixed point for general $q \in \Delta$. It implies that $\kappa|_{\Gamma_q} \in \widetilde{K} \cap (\Gamma_q, O_q)$. By $\widetilde{K} \cap (\Gamma_q, O_q) \subset \widetilde{K} \cap W_{\mathrm{sm}}(\Delta)$, we have $\kappa \in$

$\widetilde{K} \cap W_{\mathrm{sm}}(\Delta)$. Therefore we obtain $T(\widetilde{K})_p \subset \widetilde{K} \cap W_{\mathrm{sm}}(\Delta)$. The converse $\widetilde{K} \cap W_{\mathrm{sm}}(\Delta) \subset T(\widetilde{K})_p$ is trivial. □

Corollary 4.6. *There exists an exact commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T(\widetilde{K})_p & \longrightarrow & \widetilde{K} & \longrightarrow & \widetilde{K}/T(\widetilde{K})_p & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & (\Gamma_{q,\mathrm{sm}}, O_q) & \longrightarrow & \mathrm{Aut}(\Gamma_{q,\mathrm{sm}}) & \longrightarrow & \mathrm{Aut}(\Gamma_{q,\mathrm{sm}}, O_q) & \longrightarrow & 1
\end{array}$$

for an arbitrary fiber Γ_q $(q \in \Delta)$ *of* φ_Δ.

The following two Corollaries are mentioned in [9].

Corollary 4.7. *If* $\varphi : W \to B$ *has a singular fiber* Γ_p $(p \in B)$ *except for type* $l\mathrm{I}_c$. *Then it holds that* $\sharp T(\widetilde{K})_p \leq 4$.

Proof. From Theorem 3.1 and Corollary 4.6, we have an injective homomorphism

$$T(\widetilde{K})_p \hookrightarrow \mathbb{G}_a \times G(\Gamma_{p,\mathrm{sm}}).$$

Since $\mathbb{G}_a$ is a torsion free group, we have $T(\widetilde{K})_p \hookrightarrow G(\Gamma_{p,\mathrm{sm}})$. By the assumption that Γ_p is not of type $l\mathrm{I}_c$, we have $\sharp G(\Gamma_{p,\mathrm{sm}}) \leq 4$. Thus, it holds that $\sharp T(\widetilde{K})_p \leq 4$. □

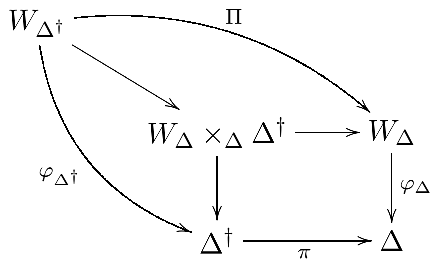
Corollary 4.8. *Let* Γ_p *be a singular fiber of type* I_c. *Then the only points of* Γ_p *with non-trivial stabilizer subgroup under the action of* $T(\widetilde{K})_p$ *are nodes of* Γ_p.

Corollary 4.9. *Assume* $\varphi : W \to B$ *is not isotrivial. Then it holds that* $\sharp(\widetilde{K}/T(\widetilde{K})_p) \leq 2$ *for any* $p \in B$. *In particular, it holds that* $\sharp\widetilde{K} \leq 2\sharp T(\widetilde{K})_p$.

Proof. Let $\varphi_\Delta : W_\Delta \to \Delta$ be a restriction of $\varphi : W \to B$ to $p \in \Delta$. Since $\varphi : W \to B$ is not isotrivial, there exists a smooth fiber Γ_q of $\varphi|_\Delta$ such that $\sharp(\mathrm{Aut}(\Gamma_q, O_q)) = 2$. From Corollary 4.6, we have the injective homomorphism $\widetilde{K}/T(\widetilde{K})_p \hookrightarrow \mathrm{Aut}(\Gamma_q, O_q)$. □

We suppose that Γ_p is the multiple fiber of type $l\mathrm{I}_c$. Let $\pi : \Delta^\dagger \to \Delta$ be a l-th root map branched at p. Then there exists a commutative diagram

(4.1)



where $W_{\Delta^\dagger}$ is the normalization of $W_\Delta \times_\Delta \Delta^\dagger$. We note that $W_{\Delta^\dagger}$ is non-singular. Since $\pi^{-1}(p)$ consists of only one point, denote this point by $p^\dagger$. Let $\Gamma_{p^\dagger}$ be a fiber of $\varphi_{\Delta^\dagger}$ over $p^\dagger$. We note that $\Gamma_{p^\dagger}$ is a singular fiber of type I_{lc}. Put $R_\dagger := \Pi^* R$. We note that $\Pi : W_{\Delta^\dagger} \to W_\Delta$ is an unramified covering of degree l.

Proposition 4.10. *There exists an injective homomorphism* $\mathrm{Aut}(W_\Delta/\Delta) \to \mathrm{Aut}(W_{\Delta^\dagger}/\Delta^\dagger)$.

Proof. An automorphism $\kappa : W_\Delta \to W_\Delta \in \mathrm{Aut}(W_\Delta/\Delta)$ induces the automorphism

$$\kappa \times_\Delta \Delta^\dagger : W_\Delta \times_\Delta \Delta^\dagger \to W_\Delta \times_\Delta \Delta^\dagger$$

by the universality of the fiber product. Hence there exists the injective homomorphism $\mathrm{Aut}(W_\Delta/\Delta) \to \mathrm{Aut}(W_\Delta \times_\Delta \Delta^\dagger/\Delta^\dagger)$. Furthermore, the universality of normalization induces the injective homomorphism

$$\mathrm{Aut}(W_\Delta \times_\Delta \Delta^\dagger/\Delta^\dagger) \hookrightarrow \mathrm{Aut}(W_{\Delta^\dagger}/\Delta^\dagger).$$

Thus, we get the desired homomorphism. □

Hence we can regard $\widetilde{K}$ as a subgroup of $\mathrm{Aut}(W_{\Delta^\dagger}/\Delta^\dagger)$. We denote by $\kappa^\dagger \in \mathrm{Aut}(W_{\Delta^\dagger}/\Delta^\dagger)$ the action which corresponds to $\kappa \in \mathrm{Aut}(W_\Delta/\Delta)$ and let $\widetilde{K}^\dagger := \{\kappa^\dagger \mid \kappa \in \widetilde{K}\}$.

Proposition 4.11. *Let $z^\dagger \in \Gamma_{p^\dagger}$ be a point and put $z = \Pi(z^\dagger) \in \Gamma_p$. Let $\kappa \in \mathrm{Stab}_{\widetilde{K}}(z)$. Then $\kappa^\dagger \in \mathrm{Stab}_{\widetilde{K}^\dagger}(z^\dagger)$. In particular, it holds $\sharp(\widetilde{K} \cdot z) = \sharp(\widetilde{K}^\dagger \cdot z^\dagger)$*

Proof. (1) The case that $\Gamma_{p^\dagger}$ is smooth at $z^\dagger$.

Then $\Gamma_{p,\mathrm{red}}$ is smooth at z. We take analytic local neighborhoods $(U;(x_1,x_2))$ of z on W_Δ, $(\Delta;y)$ of p and $(\Delta^\dagger;x_3)$ of $p^\dagger$ such that $\varphi^*y=x_1^l$ and $\pi^*y=x_3^l$. Then $U\times_\Delta\Delta^\dagger$ is defined by $x_1^l-x_3^l=0$ in $U\times\Delta^\dagger$. Put $H_i:=\{(x_1,x_2,\zeta_l^i x_1)\mid(x_1,x_2)\in U\}$ for $i=0,\cdots,l-1$, where ζ_l is a primitive l-th root of unity. Then we have

$$U\times_\Delta\Delta^\dagger=\bigcup_{i=0}^{l-1}H_i\subset U\times\Delta.$$

Thus, the natural projection $\sqcup_{i=0}^{l-1}H_i\to U\times_\Delta\Delta^\dagger$ is nothing more than the normalization of $U\times_\Delta\Delta^\dagger$. Let $\mathrm{id}\neq\kappa\in\mathrm{Stab}_{\widetilde{K}}(z)$. Replacing U by a smaller neighborhood of z if necessary, we may assume that $\kappa(U)=U$. We write $\kappa(x_1,x_2)=(\kappa_1(x_{1,x_2}),\kappa_2(x_1,x_2))$ on U. Then the induced automorphism $\kappa\times_\Delta\Delta^\dagger:U\times_\Delta\Delta^\dagger\to U\times_\Delta\Delta^\dagger$ by κ is written by $(\kappa\times_\Delta\Delta^\dagger)(x_1,x_2,x_3)=(\kappa_1(x_1,x_2),\kappa_2(x_1,x_2),x_3)$. It holds that $(\kappa\times_\Delta\Delta^\dagger)(H_i)=H_i$ for $i=0,\cdots,l-1$. Furthermore, we have a commutative diagram

$$\begin{array}{ccc} H_i & \xrightarrow{(\kappa\times_\Delta\Delta^\dagger)} & H_i \\ {\scriptstyle \Pi|_{H_i}}\downarrow & & \downarrow{\scriptstyle \Pi|_{H_i}} \\ U & \xrightarrow{\kappa} & U \end{array}$$

for $i=0,\cdots,l-1$. Thus, we have $\kappa^\dagger\in\mathrm{Stab}_{\widetilde{K}}(z^\dagger)$.

(2) The case that $\Gamma_{p^\dagger}$ has a node at $z^\dagger$.

Then $\Gamma_{p.\mathrm{red}}$ has a node at z. We take analytic local neighborhoods $(U;(x_1,x_2))$ of z on W_Δ, $(\Delta;y)$ of p and $(\Delta^\dagger;x_3)$ of $p^\dagger$ such that $\varphi^*y=(x_1x_2)^l$ and $\pi^*y=x_3^l$. Then $U\times_\Delta\Delta^\dagger$ is defined by $(x_1x_2)^l-x_3^l=0$ in $U\times\Delta^\dagger$. Put $H_i:=\{(x_1,x_2,\zeta_l^i x_1x_2)\mid(x_1,x_2)\in U\}$ for $i=0,\cdots,l-1$, where ζ_l is a primitive l-th root of unity. Then we have

$$U\times_\Delta\Delta^\dagger=\bigcup_{i=0}^{l-1}H_i\subset U\times\Delta.$$

Thus, the natural projection $\sqcup_{i=0}^{l-1}H_i\to U\times_\Delta\Delta^\dagger$ is the normalization of $U\times_\Delta\Delta^\dagger$. We can show $\kappa^\dagger\in\mathrm{Stab}_{\widetilde{K}}(z^\dagger)$ similarly to (1). $\square$

We can define $T(\widetilde{K}^{\dagger})_{p^{\dagger}}$ for $\widetilde{K}^{\dagger} \subset \mathrm{Aut}(W_{\Delta^{\dagger}}/\Delta^{\dagger})$ since $\Gamma_{p^{\dagger}}$ is not multiple. Hence we define

$$T(\widetilde{K})_p := \{\kappa \in \widetilde{K} \mid \kappa^{\dagger} \in T(\widetilde{K}^{\dagger})_{p^{\dagger}}\}$$

for a multiple fiber Γ_p. We note that $T(\widetilde{K})_p \cong T(\widetilde{K}^{\dagger})_{p^{\dagger}}$ and $\widetilde{K} \cong \widetilde{K}^{\dagger}$.

5. Estimation of the Order of $\sharp\widetilde{K}$

Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$ with $r = \frac{2(g-1)}{n-1} \geq 4n$.

Definition 5.1. Let $z \in R_h$ be a (smooth) ramification point of $\varphi|_{R_h}$. We call z a *good ramification point* when the ramification index of $\varphi|_{R_h}$ at z is greater than the multiplicity of the fiber of φ passing through z.

The goal in this section is to show Proposition 5.3, 5.4 and 5.5.

Definition 5.2. Put

$$\delta := \min\{\sharp\mathrm{Aut}(\Gamma_p, O_p) | \ \forall p \in \Delta \text{ with } \Gamma_p \text{ is smooth}\},$$

where $\mathrm{Aut}(\Gamma_p, O_p)$ is the same notation defined in (3.1). We note that $\delta = 2, 4$ or 6.

Proposition 5.3. *Let $f : S \to B$ be a primitive cyclic covering fibration of $(g, 1, n)$ with $r \geq 4n$. Let Γ_p be of type $l\mathrm{I}_c$. Assume R is smooth locally around Γ_p and has no good ramification points on Γ_p. Then it holds that*

$$K_f^2(\Gamma_p) \geq \frac{(n^2-1)}{12n\delta}\sharp\widetilde{K}.$$

Proof. Since R is smooth locally around Γ_p, we have $j'_{0,a}(\Gamma_p) = 0$ for any $a \geq 1$ if $n \geq 3$. Similarly, we have $j'_{0,a}(\Gamma_p) = 0$ for $a \geq 2$ if $n = 2$. Hence we have

$$K_f^2(\Gamma_p) = \frac{(n-1)^2}{n}\alpha_0^+(\Gamma_p) + \frac{n^2-1}{n}r\left(\chi_f(\Gamma_p) + 1 - \frac{1}{l}\right)$$

from Lemma 2.1. Since the ramification index of any ramification point of $\varphi|_{R_h}$ over p is l, we have

$$\alpha_0^+(\Gamma_p) = r(1 - \frac{1}{l}).$$

By Corollary 4.6, we have $\delta r \geq \sharp \widetilde{K}$. Thus, we have

$$\begin{aligned} K_f(\Gamma_p) &\geq \left(\frac{2(n-1)}{\delta} \left(1 - \frac{1}{l}\right) + \frac{n^2-1}{\delta n}\frac{c}{12} \right) \sharp \widetilde{K} \\ &\geq \frac{(n^2-1)}{12n\delta} \sharp \widetilde{K}. \ \square \end{aligned}$$

Proposition 5.4. *Let $f : S \to B$ be a primitive cyclic covering fibration of $(g, 1, n)$ with $r \geq \max\{60 + \frac{12}{n^2-1} - \frac{96}{n+1}, 7n\}$. Assume either Γ_p is not of type $l\mathrm{I}_c$ or R has a singular point on Γ_p. Then it holds that*

$$2n\delta K_f^2(\Gamma_p) \geq \frac{1}{3}(n-1)(5n-4)\sharp \widetilde{K}.$$

Proposition 5.5. *Let $f : S \to B$ be a primitive cyclic covering fibration of $(g, 1, n)$ with $r \geq 4n$. Let Γ_p is of type $l\mathrm{I}_c$. Assume R is smooth locally around Γ_p and has a good ramification point on Γ_p. Then it holds that*

$$2n\delta K_f^2(\Gamma_p) \geq (n-1)^2 \sharp \widetilde{K}.$$

5.1. The proof of Proposition 5.4

Lemma 5.6. *Let $f : S \to B$ be a primitive cyclic covering fibration of $(g, 1, n)$ with $r \geq 4n$. Assume Γ_p is a singular fiber except for type $l\mathrm{I}_c$. Then it holds that*

$$2n\delta K_f^2(\Gamma_p) \geq \frac{n^2-1}{12}\left(r - 12\frac{n-1}{n+1}\right)\sharp \widetilde{K}.$$

If $r \geq 7n$, it holds

$$2n\delta K_f^2(\Gamma_p) \geq \frac{1}{3}(n-1)(5n-4)\sharp \widetilde{K}.$$

PROOF. Let Γ_p be a singular fiber except for type $l\mathrm{I}_c$. We note that $e(\Gamma_p) \geq \sharp\Gamma_p + 1$ in this case. So we have $j'_{0,\bullet}(\Gamma_p) \leq \sharp\Gamma_p$ and $\chi_\varphi(\Gamma_p) \geq (\sharp\Gamma_p + 1)/12$. We have

$$K_f^2(\Gamma_p) \geq -2\frac{(n-1)^2}{n} j'_{0,\bullet}(\Gamma_p) + j'_{0,1}(\Gamma_p) + \frac{n^2-1}{n} r\chi_\varphi(\Gamma_p)$$

from Proposition 2.2 and Lemma 2.6. Hence we have

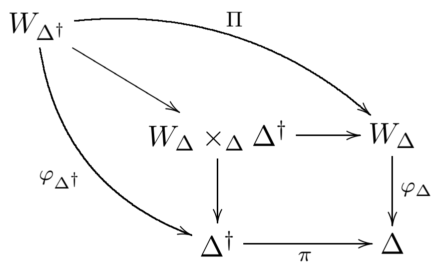
$$\begin{aligned} K_f^2(\Gamma_p) &\geq -2\frac{(n-1)^2}{n} j'_{0,\bullet}(\Gamma_p) + j'_{0,1}(\Gamma_p) + \frac{n^2-1}{12n} r(\sharp\Gamma_p + 1) \\ &= \left(\frac{n^2-1}{12n} r - \frac{2(n-1)^2}{n}\right) \sharp\Gamma_p + \frac{n^2-1}{12n} r \\ &\geq \frac{n^2-1}{6n} r - \frac{2(n-1)^2}{n} \text{ (by } \sharp\Gamma_p \geq 1) \\ &= \frac{n^2-1}{6n}\left(r - 12\frac{n-1}{n+1}\right). \end{aligned}$$

Thus, it is sufficient to show

$$2n\delta\frac{n^2-1}{6n}\left(r - 12\frac{n-1}{n+1}\right) - \frac{n^2-1}{12}\left(r - 12\frac{n-1}{n+1}\right)\sharp\widetilde{K} \geq 0.$$

Since Γ_p is not of type $l\mathrm{I}_c$, we have $\sharp\widetilde{K} \leq 4\delta$ from Corollary 4.7. Hence we get the desired inequality. □

We assume $\varphi : W \to B$ has at most singular fibers of type $l\mathrm{I}_c$ in what follows in this subsection. Let z be a singular point of R and $z \in \Gamma_p$ a fiber of type $l\mathrm{I}_c$ where l and c are non-negative integers. Recall the diagram (4.1).



We recall that $\pi^{-1}(p) = \{p^\dagger\}$ and $R_\dagger = \Pi^*(R|_{W_\Delta})$. Replacing Δ by a smaller neighborhood of p if necessary, we may assume that $\varphi_{\Delta^\dagger}$ has only one singular fiber $\Gamma_{p^\dagger}$ and $R_\dagger$ has a singular point only on $\Gamma_{p^\dagger}$.

Since $\Pi : W_{\Delta^\dagger} \to W_\Delta$ is an unramified covering, each singular point of $R_\dagger$ is analytically equivalent to the corresponding singular point of $R|_{W_\Delta}$. Hence we can consider a formal canonical resolution of $R_\dagger$ as follows.

Definition 5.7. Let $z_1^\dagger$ be a singular point of $R_\dagger$ on $\Gamma_{p^\dagger}$ and $(\psi_{\Delta^\dagger})_1 : W_{\Delta^\dagger,1} \to W_{\Delta^\dagger}$ the blowing-up at $z_1^\dagger$. We put

$$R_{\dagger,1} := (\psi_{\Delta^\dagger})^* R_\dagger - n \left[\frac{m_1}{n}\right] E_1,$$

where $E_1 := (\psi_{\Delta^\dagger})_1^{-1}(z_1^\dagger)$ and $m_1 := \mathrm{mult}_{z_1^\dagger}(R_\dagger)$. We define $(W_{\Delta^\dagger,i}, R_{\dagger,i})$ for $i = 2, \cdots, N^\dagger$ inductively as follows. Let $z_i^\dagger$ be a singular point of $R_{\dagger,i-1}$ and $(\psi_{\Delta^\dagger})_i : W_{\Delta^\dagger,i} \to W_{\Delta^\dagger,i-1}$ the blowing-up at $z_i^\dagger$. We put

$$R_{\dagger,i} := (\psi_{\Delta^\dagger})_i^* R_{\dagger,i-1} - n \left[\frac{m_i}{n}\right] E_i,$$

where $E_i := (\psi_{\Delta^\dagger})_i^{-1}(z_i^\dagger)$ and $m_i = \mathrm{mult}_{z_i^\dagger}(R_{\dagger,i-1})$. We continue this process until $R_{\dagger,i}$ is smooth. Since Π is an unramified covering, this process terminates. Let $(W_{\Delta^\dagger,N^\dagger}, R_{\dagger,N^\dagger})$ be the model which the above process terminates and put $\widetilde{\psi}_{\Delta^\dagger} := (\psi_{\Delta^\dagger})_1 \circ \cdots \circ (\psi_{\Delta^\dagger})_{N^\dagger}$. We call $R_{\dagger,i}$ $(i = 1, \cdots N^\dagger)$ the *branch locus* on $W_{\Delta^\dagger,i}$. We rewrite $(W_{\Delta^\dagger,N^\dagger}, R_{\dagger,N^\dagger})$ as $(\widetilde{W}_{\Delta^\dagger}, \widetilde{R}_\dagger)$.

Remark 5.8. The following hold.

- The multiplicity of an arbitrary singular point of $R_{\dagger,i}$ is in $n\mathbb{Z}$ or $n\mathbb{Z}+1$ for $i = 1, \cdots, N^\dagger$.
- The self-intersection number of any vertical component of $\widetilde{R}_\dagger$ which is contracted to a point by $\widetilde{\psi}_{\Delta^\dagger}$ is divisible by n.

Let $\widehat{\varphi}_{\Delta^\dagger} : \widehat{W}_{\Delta^\dagger} \to \Delta^\dagger$ be any intermediate elliptic fiber germ between $\widetilde{W}_{\Delta^\dagger}$ and $W_{\Delta^\dagger}$, and regard $\widetilde{\psi}_{\Delta^\dagger} : \widetilde{W}_{\Delta^\dagger} \to W_{\Delta^\dagger}$ as the composite of the natural birational morphisms $\widehat{\psi}_{\Delta^\dagger} : \widetilde{W}_{\Delta^\dagger} \to \widehat{W}_{\Delta^\dagger}$ and $\check{\psi} : \widehat{W}_{\Delta^\dagger} \to W_{\Delta^\dagger}$.

$$\begin{array}{ccccc} \widetilde{W}_{\Delta^\dagger} & \xrightarrow{\widehat{\psi}_{\Delta^\dagger}} & \widehat{W}_{\Delta^\dagger} & \xrightarrow{\check{\psi}_{\Delta^\dagger}} & W_{\Delta^\dagger} \\ & {\scriptstyle \widetilde{\varphi}_{\Delta^\dagger}} \searrow & \downarrow {\scriptstyle \widehat{\varphi}_{\Delta^\dagger}} & \swarrow {\scriptstyle \varphi_{\Delta^\dagger}} & \\ & & B & & \end{array}$$

We put $\widehat{R}_\dagger = (\widehat{\psi}_{\Delta^\dagger})_* \widetilde{R}_\dagger$. The fiber of $\widehat{\varphi}_{\Delta^\dagger}$ over $p \in \Delta^\dagger$ will be denoted by $\widehat{\Gamma}_{p^\dagger}$.

For any $\widehat{z}^\dagger \in \widehat{\Gamma}_{p^\dagger}$, we introduce indices as Section 2.

$$\alpha_0^+(\Gamma_{p^\dagger})_{\widehat{z}^\dagger} := \begin{cases} \left((\widetilde{R}_\dagger)_h, \widetilde{\Gamma}_{p^\dagger,\widehat{z}^\dagger}\right) - \sharp\left(\mathrm{Supp}(\widetilde{R}_\dagger)_h \cap \mathrm{Supp}(\widetilde{\Gamma}_{p^\dagger,\widehat{z}^\dagger})\right) \\ \qquad \text{(if } \widehat{R}_\dagger \text{ is singular at } \widehat{z}^\dagger \text{),} \\ \\ \left(\text{The ramification index of } (\widetilde{R}_\dagger)_h \to \Delta^\dagger \text{ at } \widehat{z}\right) - 1 \\ \qquad \text{(if } \widehat{R}_\dagger \text{ is smooth at } \widehat{z}^\dagger \text{),} \\ \\ 0 \qquad \text{(if } \widehat{z}^\dagger \notin \widehat{R}_\dagger\text{),} \end{cases}$$

where $\widetilde{\Gamma}_{p^\dagger,\widehat{z}^\dagger}$ is the biggest subdivisor of $\widetilde{\Gamma}_{p^\dagger}$ which is contracted to $\widehat{z}^\dagger$ by $\widehat{\psi}_{\Delta^\dagger}$.

- Let $\check{\alpha}_k(\Gamma_{p^\dagger})$ be the number of singular points of multiplicity either kn or $kn+1$ among all singular points of the branch loci appearing in $\check{\psi}_{\Delta^\dagger}$. If $\check{\psi}_{\Delta^\dagger} = \mathrm{id}_{W_{\Delta^\dagger}}$, we define $\check{\alpha}_k(\Gamma_{p^\dagger}) = 0$.
- Let $\widehat{\alpha}_k(\Gamma_{p^\dagger})_{\widehat{z}^\dagger}$ be the number of singular points of multiplicity either kn or $kn+1$ among all singular points of the branch loci which are infinitely near to $\widehat{z}^\dagger$. If $\check{\psi}_{\Delta^\dagger} = \mathrm{id}_{W_{\Delta^\dagger}}$, we rewrite $\widehat{\alpha}_k(\Gamma_{p^\dagger})_{\widehat{z}^\dagger}$ as $\alpha_k(\Gamma_{p^\dagger})_{\widehat{z}^\dagger}$.
- Put $j'_{0,a}(\Gamma_{p^\dagger}) := l j'_{0,a}(\Gamma_p)$ for $a \geq 1$. Put $j'_{0,\bullet}(\Gamma_{p^\dagger}) := \sum_{a\geq 1} j'_{0,a}(\Gamma_{p^\dagger})$
- Let $\check{j}_{0,a}(\Gamma_{p^\dagger})$ be the number of vertical irreducible components of $\widetilde{R}_\dagger$ that satisfy following two conditions.
  - The self-intersection number is $-an$.
  - It is a proper transform of an exceptional curve appearing from $\check{\psi}_{\Delta^\dagger}$.

  Put $\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) := \sum_{a\geq 1} \check{j}_{0,a}(\Gamma_{p^\dagger})$. If $\check{\psi}_{\Delta^\dagger} = \mathrm{id}_{W_{\Delta^\dagger}}$, we define $\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) = 0$.
- Let $\widehat{j}_{0,a}(\Gamma_{p^\dagger})_{\widehat{z}^\dagger}$ be the number of irreducible curves with self-intersection number $-an$ contained in $(\widetilde{R}_\dagger)_v(p^\dagger)$ and contracted to $\widehat{z}^\dagger$ by

$\widehat{\psi}_{\Delta^\dagger}$. Put $\widehat{j}_{0,\bullet}(\Gamma_{p^\dagger})_{\tilde{z}^\dagger} := \sum_{a\geq 1} \widehat{j}_{0,a}(\Gamma_{p^\dagger})_{\tilde{z}^\dagger}$. If $\check{\psi}_{\Delta^\dagger} = \mathrm{id}_{W_{\Delta^\dagger}}$, we rewrite $\widehat{j}_{0,a}(\Gamma_{p^\dagger})_{\tilde{z}^\dagger}$ as $j''_{0,a}(\Gamma_{p^\dagger})_{\tilde{z}^\dagger}$.

- Put $\widehat{\alpha}_0(\Gamma_{p^\dagger})_{\tilde{z}^\dagger} := \alpha_0^+(\Gamma_{p^\dagger})_{\tilde{z}^\dagger} - 2\sum_{a\geq 2} \widehat{j}_{0,a}(\Gamma_{p^\dagger})_{\tilde{z}^\dagger}$. If $\check{\psi}_{\Delta^\dagger} = \mathrm{id}_{W_{\Delta^\dagger}}$, we rewrite $\widehat{\alpha}_0(\Gamma_{p^\dagger})_{\tilde{z}^\dagger}$ as $\alpha_0(\Gamma_{p^\dagger})_{\tilde{z}^\dagger}$.
- $K_f^2(\Gamma_{p^\dagger})_{\tilde{z}^\dagger} := \sum_{k\geq 1} \big((n^2-1)k - n\big)\, \widehat{\alpha}_k(\Gamma_{p^\dagger})_{\tilde{z}^\dagger} + \frac{(n-1)^2}{n}(\widehat{\alpha}_0(\Gamma_{p^\dagger})_{\tilde{z}^\dagger} - 2\widehat{j}_{0,1}(\Gamma_p)_{\tilde{z}^\dagger}) + \widehat{j}_{0,1}(\Gamma_{p^\dagger})_{\tilde{z}^\dagger}$.
- $\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) := l\chi_{\varphi_\Delta}(\Gamma_p)$.

Using indices of the case $\check{\psi}_{\Delta^\dagger} = \mathrm{id}_{W_{\Delta^\dagger}}$, we put

$$K_f^2(\Gamma_{p^\dagger}) := \sum_{z^\dagger\in\Gamma_{p^\dagger}} K_f^2(\Gamma_{p^\dagger})_{z^\dagger} - 2\frac{(n-1)^2}{n} j'_{0,\bullet}(\Gamma_{p^\dagger}) + \frac{n^2-1}{n} r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) + j'_{0,1}(\Gamma_{p^\dagger}). \tag{5.1}$$

Proposition 5.9. *Let the notation and the assumption be as above. Then it holds that*

$$K_f^2(\Gamma_{p^\dagger}) = lK_f^2(\Gamma_p) - 2(n-1)r(l-1).$$

In particular, we have

$$lK_f^2(\Gamma_p) \geq K_f^2(\Gamma_{p^\dagger}).$$

Proof. By simple calculations, we have

$$\sum_{z^\dagger\in\Gamma_{p^\dagger}} K_f^2(\Gamma_{p^\dagger})_{z^\dagger} = l\sum_{z\in\Gamma_p} K_f^2(\Gamma_p)_z - \frac{(n-1)^2}{n}(l-1)r. \tag{5.2}$$

Applying Proposition 2.2 for $\check{\psi}_\Delta = \mathrm{id}_{W_\Delta}$, we have

$$l\sum_{z\in\Gamma_p} K_f^2(\Gamma_p)_z = lK_f^2(\Gamma_p) + 2\frac{(n-1)^2}{n} lj'_{0,\bullet}(\Gamma_p) - lj'_{0,1}(\Gamma_p) - \frac{n^2-1}{n} rl\chi_\varphi(\Gamma_p) - \frac{n^2-1}{n} r(l-1).$$

Substituting (5.2) to the above equation, we have

$$\sum_{z^\dagger\in\Gamma_{p^\dagger}} K_f^2(\Gamma_{p^\dagger}) = lK_f^2(\Gamma_p) + 2\frac{(n-1)^2}{n} lj'_{0,\bullet}(\Gamma_p) - lj'_{0,1}(\Gamma_p) - \frac{n^2-1}{n} rl\chi_\varphi(\Gamma_p) - 2r(l-1)(n-1) \tag{5.3}$$

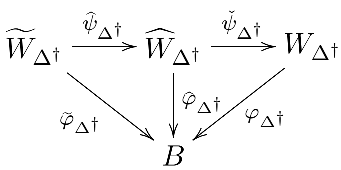
On the other hand, we have

$$K_f^2(\Gamma_{p^\dagger}) = \sum_{z^\dagger\in\Gamma_{p^\dagger}} K_f^2(\Gamma_{p^\dagger})_{z^\dagger} - \frac{2(n-1)^2}{n} j'_{0,\bullet}(\Gamma_{p^\dagger}) + \frac{n^2-1}{n} r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) + j'_{0,1}(\Gamma_{p^\dagger}).$$

Substituting (5.3) to the above equation, we get

$$K_f^2(\Gamma_{p^\dagger}) = lK_f^2(\Gamma_p) - 2(n-1)r(l-1). \ \square$$

PROPOSITION 5.10 (cf. Proposition 2.2). *Let $\widehat{\varphi}_{\Delta^\dagger} : \widehat{W}_{\Delta^\dagger} \to \Delta^\dagger$ be any intermediate elliptic fiber germ between $\widetilde{W}_{\Delta^\dagger}$ and $W_{\Delta^\dagger}$.*



Then it holds that

$$K_f^2(\Gamma_{p^\dagger}) = \sum_{\hat{z}^\dagger\in\widehat{\Gamma}_{p^\dagger}} K_f^2(\Gamma_{p^\dagger})_{\hat{z}^\dagger} + \sum_{k\geq 1}\left((n^2-1)k - n\right)\check{\alpha}_k(\Gamma_{p^\dagger}) - 2\frac{(n-1)^2}{n}\left(\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) + j'_{0,\bullet}(\Gamma_{p^\dagger})\right) + \frac{n^2-1}{n} r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) + \check{j}_{0,1}(\Gamma_{p^\dagger}) + j'_{0,1}(\Gamma_{p^\dagger}).$$

Proof. We can check it by simple calculations. □

We can show the following by a similar argument as in Lemma 2.6.

Lemma 5.11 (cf. Lemma 2.6). *It holds that*

$$K_f^2(\Gamma_{p^\dagger})_{\hat{z}^\dagger} \geq \sum_{k\geq 1}\left((n^2-1)k - n - \frac{(n-1)^2}{n}B_n\right)\widehat{\alpha}_k(\Gamma_{p^\dagger})_{\hat{z}^\dagger}.$$

In particular, $K_f^2(\Gamma_{p^\dagger})_{\hat{z}^\dagger}$ *is non-negative.*

Lemma 5.12 (cf. Lemma 2.7). *It holds that*

$$\begin{aligned} K_f^2(\Gamma_{p^\dagger}) \geq & \sum_{\hat{z}^\dagger \in \widehat{\Gamma}_{p^\dagger}} \frac{(n-1)^2}{n}\alpha_0^+(\Gamma_{p^\dagger})_{\hat{z}^\dagger} \\ & + \sum_{\hat{z}^\dagger \in \widehat{\Gamma}_{p^\dagger}} \sum_{k\geq 1}\left((n^2-1)k - n - 2\frac{(n-1)^2}{n}\right)\widehat{\alpha}_k(\Gamma_{p^\dagger})_{\hat{z}^\dagger} \\ & + \sum_{k\geq 1}\left((n^2-1)k-n\right)\check{\alpha}_k(\Gamma_{p^\dagger}) \\ & - 2\frac{(n-1)^2}{n}\left(\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) + j'_{0,\bullet}(\Gamma_{p^\dagger})\right) \\ & + \frac{n^2-1}{n} r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) + \check{j}_{0,1}(\Gamma_{p^\dagger}) + j'_{0,1}(\Gamma_{p^\dagger}). \end{aligned}$$

Proof. We can check it by simple calculations. □

Lemma 5.13. *Let* $z^\dagger \in R_\dagger$ *be a singular point of* $R_\dagger$ *and let* $\Pi(z^\dagger) = z$. *Then there exist* $l\cdot\sharp(T(\widetilde{K}^\dagger)_p\cdot z^\dagger)$ *singular points of* $R_\dagger$ *which are analytically equivalent to* $z\in R$.

Proof. We note that $\sharp\Pi^{-1}(T(\widetilde{K})_p\cdot z) = l\cdot\sharp(T(\widetilde{K})_p\cdot z)$ since $\Pi: W^\dagger \to W$ is an unramified covering of degree l. There exists an analytic neighborhood U of each $z^\dagger \in \Pi^{-1}(T(\widetilde{K})_p\cdot z)$ such that $\Pi|_U : U \to \Pi(U)$ is a biholomorphic map. Hence the singular point $z^\dagger \in R_\dagger$ is analytically equivalent to $z\in R$. Thus, there exist $l\cdot\sharp(T(\widetilde{K})_p\cdot z)$ singular points of $R_\dagger$

which are analytically equivalent to $z \in R$. On the other hand, we have $\sharp(T(\widetilde{K})_p \cdot z) = \sharp(T(\widetilde{K}^\dagger)_p \cdot z^\dagger)$ from Proposition 4.11. □

Let $(R_\dagger)_h$ be the local analytic curve consisting of $\varphi_{\Delta^\dagger}$-horizontal local analytic branches of $R_\dagger$. Take a singular point $z^\dagger \in R_\dagger$ on $\Gamma_{p^\dagger}$. We introduce the following notations for $z^\dagger$.

(1) We denote by $\mathrm{L}(R_\dagger, z^\dagger)$ and $\mathrm{L}((R_\dagger)_h, z^\dagger)$ the sets of local analytic branches of $R_\dagger$ and $(R_\dagger)_h$ at $z^\dagger$, respectively.

(2) Let $\mathrm{L}_{\mathrm{tr}}(R_\dagger, z^\dagger)$ and $\mathrm{L}_{\mathrm{tr}}((R_\dagger)_h, z^\dagger)$ be the subsets of $\mathrm{L}(R_\dagger, z^\dagger)$ and $\mathrm{L}((R_\dagger)_h, z^\dagger)$ which consist of local analytic branches that meet $\Gamma_{p^\dagger}$ transversally at $z^\dagger$, respectively.

(3) Let $\mathrm{L}_{\mathrm{ta}}(R_\dagger, z^\dagger)$ and $\mathrm{L}_{\mathrm{ta}}((R_\dagger)_h, z^\dagger)$ be the subsets of $\mathrm{L}(R_\dagger, z^\dagger)$ and $\mathrm{L}((R_\dagger)_h, z^\dagger)$ which consist of local analytic branches that are tangent to $\Gamma_{p^\dagger}$ at $z^\dagger$, respectively.

(4) For a local analytic branch Γ_i of $\Gamma_{p^\dagger}$ around $z^\dagger$, let $\mathrm{L}_{\mathrm{ta}}(R_\dagger, \Gamma_i, z^\dagger)$ and $\mathrm{L}_{\mathrm{ta}}((R_\dagger)_h, \Gamma_i, z^\dagger)$ be the subsets of $\mathrm{L}_{\mathrm{ta}}(R_\dagger, z^\dagger)$ and $\mathrm{L}_{\mathrm{ta}}((R_\dagger)_h, z^\dagger)$ which consist of local analytic branches that are tangent to Γ_i at $z^\dagger$, respectively.

If the local analytic branch Γ_i of $\Gamma_{p^\dagger}$ is contained in $R_\dagger$, we regard the local analytic branch Γ_i as belonging to $\mathrm{L}_{\mathrm{ta}}(R_\dagger, \Gamma_i, z^\dagger)$. Hence we have $\Gamma_i \in \mathrm{L}_{\mathrm{ta}}(R_\dagger, \Gamma_i, z^\dagger) \subset \mathrm{L}_{\mathrm{ta}}(R_\dagger, z^\dagger)$.

To show Proposition 5.4, we consider the following four conditions for $\Gamma_{p^\dagger}$ where $R_\dagger$ has a singular point.

(C1) There exists a singular point $z^\dagger \in R_\dagger$ on $\Gamma_{p^\dagger}$ such that $\mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(z^\dagger) = \{\mathrm{id}\}$.

(C2) The condition $(C1)$ does not hold and there exist a singular point $z^\dagger$ of $(R_\dagger)$ on $\Gamma_{p^\dagger}$ and a local analytic branch Γ_i of $\Gamma_{p^\dagger}$ around $z^\dagger$ such that $\mathrm{L}_{\mathrm{ta}}((R_\dagger)_h, \Gamma_i, z^\dagger) \neq \emptyset$.

(C3) Conditions $(C1)$ and $(C2)$ do not hold and there exists a singular point $z^\dagger \in R_\dagger$ on Γ_p such that $\in \mathrm{L}_{\mathrm{tr}}((R_\dagger)_h, z^\dagger) \neq \emptyset$.

(C4) Conditions $(C1)$, $(C2)$ and $(C3)$ do not hold.

Proposition 5.14. *Assume $r \geq 4n$ and $(C1)$ holds. Then it holds that*

$$2\delta n K_f^2(\Gamma_p) \geq 2n\left(n^2-1-n\right)\sharp\widetilde{K}.$$

Proof. Let $z^\dagger$ be a singular point of $R_\dagger$ on $\Gamma_{p^\dagger}$. By Lemma 5.13, there exist $\sharp T(\widetilde{K}^\dagger)_{p^\dagger} l$ singular points of the branch loci on $\Gamma_{p^\dagger}$ whose multiplicity is at least n.

If there exists a singular point of the branch loci over $z^\dagger \in R_\dagger$, there exist $2\sharp(T(\widetilde{K}^\dagger)_{p^\dagger})l$ singular points of $R_\dagger$ on Γ_p whose multiplicity is at least n. Applying Proposition 5.10 for $\check{\psi}_{\Delta^\dagger} = \mathrm{id}$ and Lemma 5.11, we have

$$\begin{aligned} K_f^2(\Gamma_{p^\dagger}) &\geq 2\left((n^2-1)-n-\frac{(n-1)^2}{n}B_n\right)\sharp T(\widetilde{K}^\dagger)_{p^\dagger} l \\ &\quad - 2\frac{(n-1)^2}{n} j'_{0,\bullet}(\Gamma_{p^\dagger}) + \frac{n^2-1}{n} r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) + j'_{0,1}(\Gamma_{p^\dagger}) \\ &\geq 2\left((n^2-1)-n-\frac{(n-1)^2}{n}B_n\right)\sharp T(\widetilde{K}^\dagger)_{p^\dagger} l. \end{aligned}$$

Hence it is sufficient to show

$$4\delta n\left((n^2-1)-n-\frac{(n-1)^2}{n}B_n\right)\sharp T(\widetilde{K}^\dagger)_{p^\dagger} - 2n\left(n^2-1-n\right)\sharp\widetilde{K} \geq 0.$$

From Corollary 4.6, we have $\sharp\widetilde{K} \leq \delta\sharp T(\widetilde{K}^\dagger)_{p^\dagger}$. Hence it is sufficient to show

$$2\left((n^2-1)-n-\frac{(n-1)^2}{n}B_n\right) - \left(n^2-1-n\right) \geq 0.$$

We can check it by a simple calculation.

Suppose that there exist no singular points over $z^\dagger \in R_\dagger$. We recall that there exist $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger})l$ singular points of $R_\dagger$ on Γ_p which is analytically equivalent to $z^\dagger \in R_\dagger$. Let $\check{\psi}_{\Delta^\dagger} : \widehat{W}_{\Delta^\dagger} \to W_{\Delta^\dagger}$ be the composite of blowing-ups at those $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger})l$ singular points. Since there exist no singular points over $z^\dagger \in R_\dagger$, we have $\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) = 0$. From Proposition 5.10 for $\check{\psi}_{\Delta^\dagger}$, we get

$$\begin{aligned} K_f^2(\Gamma_{p^\dagger}) = &\sum_{\hat{z}^\dagger \in \widehat{\Gamma}_{p^\dagger}} K_f^2(\Gamma_{p^\dagger})_{\hat{z}^\dagger} + \left(n^2-1-n\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger})l \\ &- 2\frac{(n-1)^2}{n} j'_{0,\bullet}(\Gamma_{p^\dagger}) + \frac{n^2-1}{n} r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) + j'_{0,1}(\Gamma_{p^\dagger}). \end{aligned}$$

Since we have $\sharp\Gamma_{p^\dagger} \geq j'_{0,1}(\Gamma_{p^\dagger})$ and $\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) \geq \sharp\Gamma_{p^\dagger}/12$, we have

$$\begin{aligned} K_f^2(\Gamma_{p^\dagger}) &\geq \left(n^2-1-n\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger})l + \frac{n^2-1}{n}\left(r - 24\frac{n-1}{n+1}\right)\sharp\Gamma_{p^\dagger} \\ &\geq \left(n^2-1-n\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger})l. \end{aligned}$$

Hence we have

$$K_f^2(\Gamma_p) \geq \left(n^2-1-n\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}).$$

Thus, it is sufficient to show

$$2\delta n\left(n^2-1-n\right)\sharp T(\widetilde{K}^\dagger)_{p^\dagger} - 2n\left(n^2-1-n\right)\sharp\widetilde{K} \geq 0.$$

From Corollary 4.6, we have $\sharp\widetilde{K} \leq \delta\sharp T(\widetilde{K}^\dagger)_{p^\dagger}$. Hence we get the desired inequality. □

REMARK 5.15. If (C1) does not hold, there exist no singular points of $R_\dagger$ on $\Gamma_{p,\mathrm{sm}}$ by Corollary 4.8.

PROPOSITION 5.16. *Assume* $r \geq \max\{60 + \frac{12}{n^2-1} - \frac{96}{n+1}, 4n\}$ *and* $(C2)$ *holds. Put*

$$C_{2,n} := \frac{1}{3}(n-1)(5n-4).$$

Then it holds that

$$2n\delta K_f^2(\Gamma_p) \geq C_{2,n}\sharp\widetilde{K}.$$

PROOF. Take $z^\dagger \in R_\dagger$ and $\Gamma_1 \subset \Gamma_{p^\dagger}$ such that $L_{\mathrm{ta}}((R_\dagger)_h, \Gamma_1, z^\dagger) \neq \emptyset$. For a local analytic branch $D \in L_{\mathrm{ta}}((R_\dagger)_h, \Gamma_1, z^\dagger)$, let $v(D)$ be the number of blowing-ups until the proper transform of D is not tangent to Γ_1. Since $L_{\mathrm{ta}}((R_\dagger)_h, \Gamma_1, z^\dagger) \neq \emptyset$ is a finite set, there exists $D_1 \in L_{\mathrm{ta}}((R_\dagger)_h, \Gamma_1, z^\dagger)$ such that $v(D_1)$ is a minimal value among $L_{\mathrm{ta}}((R_\dagger)_h, \Gamma_1, z^\dagger)$. Let $\mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(\Gamma_1, z^\dagger) := \{\kappa \in \mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(z^\dagger) \mid \kappa(\Gamma_1) = \Gamma_1\}$ and $\mathbb{D}_1$ be the $\mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(\Gamma_1, z^\dagger)$-orbit of D_1. We introduce some notations as follows.

- $m_{D_1} := \mathrm{mult}_{z^\dagger} D_1$
- $m_{\mathbb{D}_1} := \mathrm{mult}_{z^\dagger} \mathbb{D}_1$
- $m := \min\{m' \mid m' \geq m_{\mathbb{D}}, \quad m' \in n\mathbb{Z} \text{ or } n\mathbb{Z}+1\}$

Assume that we need blowing-ups v times until the proper transform of D_1 is not tangent to that of Γ_1. Since $\Gamma_{p^\dagger}$ has a node at $z^\dagger$, we have $(\mathbb{D}_1, \Gamma_{p^\dagger})_{z^\dagger} \leq (v+2)m$.

From Lemma 5.13, there exist $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$ local analytic curves of $R_\dagger$ which is analytically equivalent to $z^\dagger \in \mathbb{D}_1$. Let $z_j^\dagger \in \mathbb{D}_j$ be such local analytic curves for $j = 1, \cdots, (T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$, where we define $z_1^\dagger = z^\dagger$. Since $z_j^\dagger \in \mathbb{D}_j$ is analytically equivalent to $z_1^\dagger \in \mathbb{D}_1$, the proper transform of $\mathbb{D}_j$ is not tangent to the proper transform of $\Gamma_{p^\dagger}$ after v-times blowing-ups over $z_j^\dagger$ for each $j = 1, \cdots, (T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$.

We consider two cases separately.

(1) $n \geq 3$ or $n = 2$ and $\mathrm{mult}_{z^\dagger} R_\dagger \in 2\mathbb{Z}$.

Let $\check{\psi}_{\Delta^\dagger} : \widehat{W}_{\Delta^\dagger} \to W_{\Delta^\dagger}$ be the composite of the above v-times blowing-ups over $z_j^\dagger$ for each $j = 1, \cdots, (T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$. Let $\widehat{R}_\dagger$ be the branch locus on $\widehat{W}_{\Delta^\dagger}$ and $\widehat{\Gamma}_{p^\dagger}$ a fiber of $\widehat{\varphi} := \varphi_{\Delta^\dagger} \circ \check{\psi}_{\Delta^\dagger}$ over $p^\dagger$. We show

$$\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) \leq \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$$

if $n \geq 3$,

$$\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) \leq \check{j}_{0,1}(\Gamma_{p^\dagger}) + \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$$

if $n = 2$. We may assume $v \geq 2$. Let $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})_{z^\dagger}$ be the number of irreducible components contributing to $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})$ which is contracted to $z^\dagger$ by $\widetilde{\psi}_{\Delta^\dagger}$. If $n \geq 3$, it is sufficient to show $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})_{z^\dagger} \leq 2$. Let $(\check{\psi}_{\Delta^\dagger})_{z^\dagger} : (\widehat{W}_{\Delta^\dagger})_{z^\dagger} \to W_{\Delta^\dagger}$ be the composite of blowing-ups v-times over $z^\dagger$ and $\mathcal{E}_{z^\dagger}$ the set of exceptional curves of $(\check{\psi}_{\Delta^\dagger})_{z^\dagger}$. We note that $\sharp\mathcal{E}_{z^\dagger} = v$. For an exceptional curve $E \in \mathcal{E}_{z^\dagger}$, let $n(E)$ be the number such that E is the exceptional curve of the $n(E)$-th blowing-up in $(\check{\psi}_{\Delta^\dagger})_{z^\dagger}$. Let E_i be the exceptional curve in $\mathcal{E}_{z^\dagger}$ such that $n(E_i) = i$ for $i = 1, \cdots v$. We note that E_1 and E_v are exceptional curves of first and last blowing-ups in $(\check{\psi}_{\Delta^\dagger})_{z^\dagger}$, respectively. Then the proper transform of any local analytic branch in $\mathrm{L}((R_\dagger)_h, z^\dagger) \setminus \mathrm{L}_{\mathrm{ta}}(R_\dagger, \Gamma_i, z^\dagger)$ can be tangent to only E_1 among $\mathcal{E}_{z^\dagger}$. Furthermore, the proper transform of any local analytic branch in $\mathrm{L_{ta}}(R_\dagger, \Gamma_i, z^\dagger)$ can be tangent to only E_v from the choice of D_1. Thus, the self-intersection number of $\widehat{E}_i (i = 2, \cdots, v-1)$ is -2, where $\widehat{E}_i \subset (\widehat{W}_\Delta)_{z^\dagger}$ is the proper transform of E_i. On the other hand, the self-intersection number of each irreducible component of $\widetilde{R}_\dagger$ contributing $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})_{z^\dagger}$ is at most $-n$. Thus, if $n \geq 3$, a proper transform of $\mathcal{E}_{z^\dagger}$ which is contained in $\widetilde{R}_\dagger$ is either $\widehat{E}_1$ or $\widehat{E}_v$ at most. Hence we get $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})_{z^\dagger} \leq 2$ for $n \geq 3$. Furthermore, since the branch loci have no double points, we have $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})_{z^\dagger} \leq 1$ by chasing a resolution of $z^\dagger \in R_\dagger$. If $n = 2$, we can show $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})_{z^\dagger} \leq \check{j}_{0,1}(\Gamma_{p^\dagger})_{z^\dagger} + 1$ by chasing a resolution of $z^\dagger \in R_\dagger$. Thus, we can get the desired inequalities for $\check{j}_{0,\bullet}(\Gamma_{p^\dagger})$.

Since $\Gamma_{p^\dagger}$ has at least $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$ nodes, we have $\sharp\Gamma_{p^\dagger} \geq \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$. Hence we get

$$\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) \leq \sharp\Gamma_{p^\dagger}$$

if $n \geq 3$,

$$\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) \leq \sharp\Gamma_{p^\dagger} + \check{j}_{0,1}(\Gamma_{p^\dagger})$$

if $n = 2$. Put $\mathbb{D} := \mathrm{Stab}_{T(\widetilde{K}^\dagger)_{p^\dagger}}(z^\dagger) \cdot D_1$ and $\gamma := 1 - 1/(D_1, \Gamma_{p^\dagger})_{z^\dagger}$. Then we have

$$\sum_{\hat{z}^\dagger \in \widehat{\Gamma}_{p^\dagger}} \alpha_0^+(\Gamma_{p^\dagger})_{\hat{z}^\dagger} \geq \gamma(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l.$$

Since there exist $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)vl$-singular points of the branch loci on $\Gamma_{p^\dagger}$ which appear in $\check{\psi}_{\Delta^\dagger}$, we have

$$\sum_{k\geq 1} \left((n^2-1)k - n\right) \check{\alpha}_k(\Gamma_{p^\dagger}) \geq \left((n^2-1)\left[\frac{m}{n}\right] - n\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)vl.$$

Thus, we have

$$\begin{aligned} K_f^2(\Gamma_{p^\dagger}) \geq & \gamma\frac{(n-1)^2}{n}(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger}\sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l \\ & + \left((n^2-1)\left[\frac{m}{n}\right] - n\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)vl \\ & -2\frac{(n-1)^2}{n}\left(\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) + j'_{0,\bullet}(\Gamma_{p^\dagger})\right) \\ & + \frac{n^2-1}{n} r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) + \check{j}_{0,1}(\Gamma_{p^\dagger}) + j'_{0,1}(\Gamma_{p^\dagger}) \end{aligned}$$

by Lemma 5.12. By $\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) = e(\Gamma_{p^\dagger})/12 = \sharp\Gamma_{p^\dagger}/12 \geq \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l/12$, we have

$$\begin{aligned} K_f^2(\Gamma_{p^\dagger}) \geq & \gamma \frac{(n-1)^2}{n}(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l \\ & + \left((n^2-1)\left[\frac{m}{n}\right] - n\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)vl \\ & + \left(-4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l. \end{aligned}$$

By $K_f^2(\Gamma_p) \geq K_f^2(\Gamma_{p^\dagger})/l$, we get

$$\begin{aligned} K_f^2(\Gamma_p) \geq & \gamma \frac{(n-1)^2}{n}(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger) \\ & + \left((n^2-1)\left[\frac{m}{n}\right] - n\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)v \\ & + \left(-4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger). \end{aligned}$$

Hence it is sufficient to show

$$\begin{aligned} 2n\delta \Bigg\{ & \gamma \frac{(n-1)^2}{n}(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} + \left((n^2-1)\left[\frac{m}{n}\right] - n\right)v \\ & + \left(-4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right)\Bigg\} \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger) - C_{2,n}\sharp\widetilde{K} \geq 0. \end{aligned}$$

By $(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \geq \sharp\mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(z^\dagger)$, it is sufficient to show

$$\begin{aligned} 2n \Bigg\{ & \gamma \frac{(n-1)^2}{n}(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} + \left((n^2-1)\left[\frac{m}{n}\right] - n\right)v \\ & + \left(-4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right)\Bigg\} - C_{2,n}(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \geq 0. \end{aligned}$$

It is equivalent to

$$\begin{aligned} & 2n\left(\left((n^2-1)\left[\frac{m}{n}\right] - n\right)v - 4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right) \\ & \quad + \left(2\gamma(n-1)^2 - C_{2,n}\right)(\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \geq 0. \end{aligned}$$

By $2m(v+2) \geq 2(\mathbb{D}_1, \Gamma_{p^\dagger})_{z^\dagger} \geq (\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger}$, it suffices to show

$$2n\left(\left((n^2-1)\left[\frac{m}{n}\right]-n\right)v - 4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right) + 2m(v+2)\left(2\gamma(n-1)^2 - C_{2,n}\right) \geq 0. \tag{5.4}$$

Since it holds $(D_1, \Gamma_{p^\dagger})_{z^\dagger} \geq 3$, we have $\gamma = 1 - \frac{1}{(D_1, \Gamma_{p^\dagger})_{z^\dagger}} \geq 2/3$. Hence it suffices to show

$$2n\left(\left((n^2-1)\left[\frac{m}{n}\right]-n\right)v - 4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right) + m(v+2)\left(\frac{8}{3}(n-1)^2 - 2C_{2,n}\right) \geq 0. \tag{5.5}$$

The coefficient of v in (5.5)

$$2n\left((n^2-1)\left[\frac{m}{n}\right]-n\right) + \left(\frac{8}{3}(n-1)^2 - 2C_{2,n}\right)m$$

is non-negative. Hence we may assume $v=1$ to show (5.5). We will show

$$2n\left(\left((n^2-1)\left[\frac{m}{n}\right]-n\right) - 4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n}r\right) + m\left(8(n-1)^2 - 6C_{2,n}\right) \geq 0. \tag{5.6}$$

Since the coefficient of m is non-negative, we may assume $m = n+1$ to show (5.6). Hence it suffices to show

$$2n(n^2-1) + (n+1)\left(8(n-1)^2 - 6C_{2,n}\right) \geq 0,$$
$$\frac{n^2-1}{12n}r - 4\frac{(n-1)^2}{n} - n \geq 0.$$

We can check the former inequality by a simple calculation. The latter one clearly holds by the assumption $r \geq 60 + \frac{12}{n^2-1} - \frac{96}{n+1}$.

(2) $n=2$ and $\mathrm{mult}_{z^\dagger} R_\dagger \in 2\mathbb{Z}+1$.

Let $(\check{\psi}_{\Delta^\dagger})_0 : (\widehat{W}_{\Delta^\dagger})_0 \to W_{\Delta^\dagger}$ be the composite of the above blowing-ups v-times over $z_j^\dagger$ for each $j = 1, \cdots, (T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$. Let $(\widehat{R}_\dagger)_0$ be the branch locus on $(\widehat{W}_{\Delta^\dagger})_0$. Then all exceptional curves appearing from $(\check{\psi}_{\Delta^\dagger})_0$ are contained in $(\widehat{R}_\dagger)_0$ by the assumption (2). Thus, there exist $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot$

$z^\dagger)(v-1)l$ double points of $(\widehat{R}_\dagger)_0$. Let $\check{\psi}_{\Delta^\dagger}:\widehat{W}_{\Delta^\dagger}\to W_{\Delta^\dagger}$ be the composite of blowing-ups at those double points and $(\check{\psi}_{\Delta^\dagger})_0$. Let $\widehat{R}_\dagger$ be the branch locus on $\widehat{W}_{\Delta^\dagger}$ and $\widehat{\Gamma}_{p^\dagger}$ a fiber of $\widehat{\varphi}:=\varphi_{\Delta^\dagger}\circ\check{\psi}_{\Delta^\dagger}$ over $p^\dagger$. Since an arbitrary $\widehat{\varphi}_{\Delta^\dagger}$-vertical component of $\widehat{R}_\dagger$ appears from either exceptional curves in $(\check{\psi}_{\Delta^\dagger})_0$ or irreducible components of $\Gamma_{p^\dagger}$, we have

$$\check{j}_{0,\bullet}(\Gamma_{p^\dagger})\le \sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)vl.$$

Put $\mathbb{D}:=\mathrm{Stab}_{T(\widetilde{K}^\dagger)_{p^\dagger}}(z^\dagger)\cdot D_1$ and $\gamma:=1-1/(D_1,\Gamma_{p^\dagger})_{z^\dagger}$. Then we have

$$\sum_{\hat{z}\in\widehat{\Gamma}_p}\alpha_0^+(\Gamma_{p^\dagger})_{\hat{z}}\ge\gamma(\mathbb{D},\Gamma_{p^\dagger})_{z^\dagger}\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)l.$$

Since $R_\dagger$ has $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)vl$-singular points of multiplicity at least m and $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)(v-1)l$ double points which appear in $(\check{\psi}_{\Delta^\dagger})_0$, we have

$$\begin{aligned}\sum_{k\ge1}(3k-2)\,\check{\alpha}_k(\Gamma_{p^\dagger})\ge&\left(3\left[\frac{m}{2}\right]-2\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)vl\\&+\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)(v-1)l.\end{aligned}$$

Thus, we have

$$\begin{aligned}K_f^2(\Gamma_{p^\dagger})\ge&\frac{\gamma}{2}(\mathbb{D},\Gamma_{p^\dagger})_{z^\dagger}\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)l\\&+\left(3\left[\frac{m}{2}\right]-2\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)vl+\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)(v-1)l\\&-\left(\check{j}_{0,\bullet}(\Gamma_{p^\dagger})+j'_{0,\bullet}(\Gamma_{p^\dagger})\right)+\frac{3}{2}r\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger})+\check{j}_{0,1}(\Gamma_{p^\dagger})+j'_{0,1}(\Gamma_{p^\dagger})\end{aligned}$$

by Lemma 5.12. By $\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger})\ge\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)l/12$ and $2\sharp\Gamma_{p^\dagger}+(v-1)l\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)\ge\left(\check{j}_{0,\bullet}(\Gamma_{p^\dagger})+j'_{0,\bullet}(\Gamma_{p^\dagger})\right)$, we have

$$\begin{aligned}K_f^2(\Gamma_{p^\dagger})\ge&\frac{\gamma}{2}(\mathbb{D},\Gamma_{p^\dagger})_{z^\dagger}\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)l\\&+\left(3\left[\frac{m}{2}\right]-n\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)vl+\left(\frac{1}{8}r-2\right)\sharp(T(\widetilde{K}^\dagger)_{p^\dagger}\cdot z^\dagger)l.\end{aligned}$$

Hence it is sufficient to show

$$4\left(\left(3\left[\frac{m}{2}\right]-2\right)v+\frac{1}{8}r-2\right)+2m(v+2)\left(2\gamma-C_{2,2}\right)\ge0.$$

It clearly holds by (5.4). □

PROPOSITION 5.17. *Assume* $r \geq \max\{60 + \frac{12}{n^2-1} - \frac{96}{n+1}, 4n\}$ *and* $(C3)$ *holds. Put*

$$C_{3,n} := (n-1)(2n-1).$$

Then it holds that

$$2\delta n K_f^2(\Gamma_p) \geq C_{3,n} \sharp \widetilde{K}.$$

PROOF. Take $z^\dagger \in R_\dagger$ and $D \in L_{\mathrm{tr}}(R_\dagger, z^\dagger)$. Denote by $\mathbb{D}$ the $\mathrm{Stab}_{T(\widetilde{K}^\dagger)_{p^\dagger}}(z^\dagger)$-orbit of D. We introduce the following notations.

- $m_{\mathbb{D}} := \mathrm{mult}_{z^\dagger}\mathbb{D}$.
- $m := \min\{m' \mid m' \geq m_{\mathbb{D}}, m' \in n\mathbb{Z}_{>0} \text{ or } n\mathbb{Z}_{>0} + 1\}$.

From Lemma 5.13, there exist $\sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$ singular points of the branch loci on $\Gamma_{p^\dagger}$ which is analytically equivalent to $z^\dagger \in \mathbb{D}$. Let $\check{\psi}_{\Delta^\dagger} : \widehat{W}_{\Delta^\dagger} \to W_{\Delta^\dagger}$ be the composite of blowing-ups at $z_j^\dagger$ $(j = 1, \cdots, (T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l)$. Let $\widehat{R}_\dagger$ be the branch locus on $\widehat{W}_{\Delta^\dagger}$ and $\widehat{\Gamma}_{p^\dagger}$ a fiber of $\widehat{\varphi} := \varphi_{\Delta^\dagger} \circ \check{\psi}_{\Delta^\dagger}$ over $p^\dagger$. Put $\gamma := 1 - 1/(D_1, \Gamma_{p^\dagger})_{z^\dagger}$. Then we have

$$\begin{aligned} K_f^2(\Gamma_{p^\dagger}) \geq & \gamma \frac{(n-1)^2}{n} (\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l \\ & + \left((n^2-1)\left[\frac{m}{n}\right] - n\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l \\ & -2\frac{(n-1)^2}{n}\left(\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) + j'_{0,\bullet}(\Gamma_{p^\dagger})\right) + \frac{n^2-1}{n} r \chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}). \end{aligned}$$

By $\chi_{\varphi_{\Delta^\dagger}}(\Gamma_{p^\dagger}) \geq \sharp\Gamma_{p^\dagger}/12$, $\sharp\Gamma_{p^\dagger} \geq \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger)l$ and $2\sharp\Gamma_{p^\dagger} \geq \left(\check{j}_{0,\bullet}(\Gamma_{p^\dagger}) + j'_{0,\bullet}(\Gamma_{p^\dagger})\right)$, we have

$$\begin{aligned} K_f^2(\Gamma_p) \geq & \gamma \frac{(n-1)^2}{n} (\mathbb{D}, \Gamma_{p^\dagger})_{z^\dagger} \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger) \\ & + \left((n^2-1)\left[\frac{m}{n}\right] - n\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger) \\ & + \left(-4\frac{(n-1)^2}{n} + \frac{n^2-1}{12n} r\right) \sharp(T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger). \end{aligned}$$

Thus, it is sufficient to show

$$2n\left(\left((n^2-1)\left[\frac{m}{n}\right]-n\right)-4\frac{(n-1)^2}{n}+\frac{n^2-1}{12n}r\right) + \left(2\gamma(n-1)^2-C_{3,n}\right)(\mathbb{D},\Gamma_{p^\dagger})_{z^\dagger}\geq 0.$$

Since D is in $L_{\mathrm{tr}}((R_\dagger)_h, z^\dagger)$ and $\Gamma_{p^\dagger}$ has nodes at $z^\dagger$, we have $2m\geq 2m_{\mathbb{D}} = (\mathbb{D},\Gamma_{p^\dagger})_{z^\dagger}$ and $\gamma\geq 1/2$. Hence it suffices to show

$$2n\left(\left((n^2-1)\left[\frac{m}{n}\right]-n\right)-4\frac{(n-1)^2}{n}+\frac{n^2-1}{12n}r\right) + 2m\left((n-1)^2-C_{3,n}\right)\geq 0. \tag{5.7}$$

The coefficient of m in (5.7)

$$2(n^2-1)+2((n-1)^2-C_{3,n})$$

is non-negative. Hence we may assume $m=n+1$ to show (5.7). Thus, it suffices to show

$$2n(n^2-1)+2(n+1)\left((n-1)^2-C_{3,n}\right)\geq 0,$$
$$\frac{n^2-1}{12n}r-4\frac{(n-1)^2}{n}-n\geq 0.$$

We can check it by simple calculations. □

Proposition 5.18. *One of* (*C*1), (*C*2) *and* (*C*3) *holds.*

Proof. Supposing that (C1), (C2) and (C3) do not hold, we lead a contradiction. We note that $\Gamma_{p^\dagger}\not\subset R_\dagger$ in this assumption. Hence we have $\Gamma_p\not\subset R$. Take an arbitrary singular point $z\in R$. Let $z^\dagger\in R_\dagger$ be the singular point such that $\Pi(z^\dagger)=z$, where the morphism Π is in (4.1). Since (C1) does not hold, we have $\mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(z^\dagger)\neq\{\mathrm{id}\}$. By Corollary 4.8, the singular point $z^\dagger\in R_\dagger$ is on a node of $\Gamma_{p^\dagger}$. Hence the singular point $z\in R$ is on a node of Γ_p. Let Γ_1 and Γ_2 be local analytic branches of $(\Gamma)_{\mathrm{red}}$ at z. Since (C2) and (C3) do not hold, R consists of Γ_1 and Γ_2 locally around z. Hence it occurs only if $n=2$. If Γ_p is of type $l\mathrm{I}_1$, we have $(\Gamma_p)_{\mathrm{red}}\subset R$. But it is a contradiction. Hence we may assume that Γ_p is of type $l\mathrm{I}_c$ with

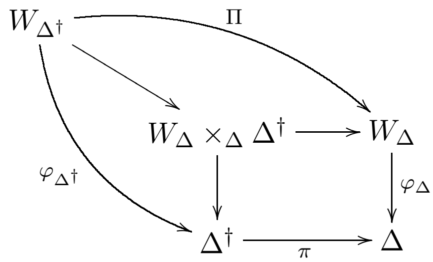
$c \geq 2$. Since R has the singular point z, we need to blow up at z to get $\widetilde{R}$. Let $\overline{\Gamma}_2$ be the irreducible component of $(\Gamma_p)_{\mathrm{red}}$ which contains Γ_2 and $\widetilde{\Gamma}_2$ the proper transform of Γ_2 in $\widetilde{R}$. By $n = 2$, it holds $\widetilde{\Gamma}_2^2 \leq -4$. Hence we need blowing-ups on $\overline{\Gamma}_2$ at least two times to get $\widetilde{R}$. Furthermore, since an arbitrary singular point of R is on a node of $(\Gamma_p)_{\mathrm{red}}$, there exists a singular point of R on the other node of $(\Gamma_p)_{\mathrm{red}}$ consisting of $\overline{\Gamma}_2$. Let $z_{23}^{\dagger}$ be the node and $\overline{\Gamma}_3$ another irreducible component of $(\Gamma_p)_{\mathrm{red}}$ passing through $z_{23}^{\dagger}$. We note that the node $z_{23}^{\dagger}$ of $(\Gamma_p)_{\mathrm{red}}$ consists of the intersection of $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$. Since (C2) and (C3) do not hold, R consists with $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ locally around z_{23}. Hence we have $\overline{\Gamma}_3 \subset R$. Inductively, we can show $(\Gamma_p)_{\mathrm{red}} \subset R$. But it is a contradiction. □

5.2. The proof of Proposition 5.5

Let Γ_p be of type $l\mathrm{I}_c$. Assume R is smooth locally around Γ_p and has a good ramification point on Γ_p. We will show

$$2n\delta K_f^2(\Gamma_p) \geq (n-1)^2 \sharp \widetilde{K}.$$

Let z be a good ramification point of $\varphi|_{R_h}$ and Γ_p the fiber through z of type $l\mathrm{I}_c$ where l and c are non-negative integers. Recall the diagram (4.1).



Since the ramification index of $\varphi|_{R_h} : R_h \to B$ at z is greater than l, the restriction map $\varphi_{\Delta^\dagger}|_{(R_\dagger)_h} : (R_\dagger)_h \to \Delta^\dagger$ is ramified at $z^\dagger \in \Pi^{-1}(z)$. We have

$$K_f^2(\Gamma_{p^\dagger}) \geq \frac{(n-1)^2}{n} \left(((R_\dagger)_h, \Gamma_{p^\dagger})_{z^\dagger} - 1 \right) \sharp (T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger) l.$$

Hence we have

$$K_f^2(\Gamma_p) \geq \frac{(n-1)^2}{n} \left(((R_\dagger)_h, \Gamma_{p^\dagger})_{z^\dagger} - 1 \right) \sharp (T(\widetilde{K}^\dagger)_{p^\dagger} \cdot z^\dagger).$$

So it is sufficient to show

$$2\delta(n-1)^2\left(((R_\dagger)_h,\Gamma_{p^\dagger})_{z^\dagger}-1\right)-\delta(n-1)^2\sharp(\mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(z^\dagger))\geq 0.$$

From $((R_\dagger)_h,\Gamma_{p^\dagger})_{z^\dagger}\geq\sharp(\mathrm{Stab}_{T(\tilde{K}^\dagger)_{p^\dagger}}(z^\dagger))$, it suffices to show

$$2\delta(n-1)^2\left(((R_\dagger)_h,\Gamma_{p^\dagger})_{z^\dagger}-1\right)-\delta(n-1)^2((R_\dagger)_h,\Gamma_{p^\dagger})_{z^\dagger}\geq 0.$$

By the assumption $((R_\dagger)_h,\Gamma_{p^\dagger})_{z^\dagger}\geq 2$, we have the desired inequality.

6. Upper Bound of the Order

Let $f:S\to B$ be a primitive cyclic covering fibration of type $(g,1,n)$ over an elliptic surface $\varphi:W\to B$. We consider the upper bound of the order of the automorphism groups of f. Let Γ_p be a fiber of φ such that $K_f^2(\Gamma_p)>0$. Put $r_p:=\sharp\mathrm{Stab}_H(p)$. Since $K_f^2(\Gamma)\geq 0$ for an arbitrary fiber Γ, we have

$$K_f^2\geq\sharp(H\cdot p)K_f^2(\Gamma_p).$$

By $\sharp G=\sharp K\sharp H=n\sharp\widetilde{K}\sharp H$, we have

$$\sharp G\leq\frac{n\sharp\widetilde{K}}{K_f^2(\Gamma_p)}r_pK_f^2.$$

Put

$$\mu_n:=\frac{12n^2\delta}{n^2-1},$$
$$\mu_n':=\frac{6n^2\delta}{(n-1)(5n-4)}.$$

If $K_f^2(\Gamma_p)>0$, we have

$$\frac{n\sharp\widetilde{K}}{K_f^2(\Gamma_p)}\leq\mu_n.$$

If R has a singular point on Γ_p, we have

$$\frac{n\sharp\widetilde{K}}{K_f^2(\Gamma_p)}\leq\mu_n'$$

from Proposition 5.3 and 5.4. We recall

$$\delta := \min\{\sharp \mathrm{Aut}(\Gamma_p, O_p) | \ \forall p \in \Delta \text{ with } \Gamma_p \text{ is smooth}\}.$$

We note that $\delta = 2, 4$ or 6.

THEOREM 6.1. *Let $f : S \to B$ be a non-locally trivial primitive cyclic covering fibration of type $(g, 1, n)$ with $g \geq \max\{\frac{30n^2-47n+25}{n+1}, \frac{7}{2}n(n-1)+1\}$. Put*

$$\mu_n := \frac{12n^2\delta}{n^2-1}.$$

Assume furthermore that when $g(B) = 0$, f has at least 3 singular fibers. Let G be a finite subgroup of $\mathrm{Aut}(f)$*. Then it holds*

$$\sharp G \leq \begin{cases} 6(2g(B)-1)\mu_n K_f^2 & (g(B) \geq 1), \\ 5\mu_n K_f^2 & (g(B) = 0). \end{cases}$$

PROOF. Put $r_p = \sharp \mathrm{Stab}_H(p)$ for $p \in B$. Let $\pi : B \to B/H$ be the quotient map of B by H. Note that r_p is the ramification index of π at p.

(i) The case of $g(B) \geq 2$.

We denote the genus of B/H by $g(B/H)$. From the Hurwitz formula, we get

$$2g(B) - 2 = \sharp H \left(2g(B/H) - 2 + \sum_{i=1}^{s} \frac{r_i - 1}{r_i} \right),$$

where s is the number of ramification points and r_i is the ramification index. Put

$$T := 2g(B/H) - 2 + \sum_{i=1}^{s} \frac{r_i - 1}{r_i},$$

which is positive. If $g(B/H) \geq 2$, then we get $\sharp H \leq g(B) - 1$ by $T \geq 2$, and it follows that $r_i \leq \sharp H \leq g(B) - 1$ for any $i = 1, \cdots s$.

Assume that $g(B/H) = 1$. Then we get $s > 0$ by $T > 0$. By $r_i \geq 2$ for any $i = 1, \cdots s$, we get $1 - 1/r_i \geq 1/2$. Therefore we obtain $r_i \leq \sharp H \leq 4(g(B) - 1)$ for any $i = 1, \cdots s$ by $T \geq 1/2$.

Assume that $g(B/H)=0$. When $s\geq 5$, we get $r_i\leq \sharp H\leq 4(g(B)-1)$ for any $i=1,\cdots s$ by $T\geq 1/2$. When $s=4$, one of r_i is not less than 3. So we get $r_i\leq \sharp H\leq 12(g(B)-1)$ for any $i=1,\cdots s$ by $T\geq 1/6$. When $s=3$, we may assume $r_1\geq r_2\geq r_3$. By the definition of T, we get

$$r_1\leq \sharp H=\frac{2g(B)-2}{1-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}},$$

so we obtain

$$r_1-1-\frac{r_1}{r_2}-\frac{r_1}{r_3}\leq 2g(B)-2.$$

If $r_3=2$, then $r_2\geq 3$, so we get

$$r_1-1-\frac{r_1}{3}-\frac{r_1}{2}\leq 2g(B)-2.$$

Hence we get $r_1\leq 6(2g(B)-1)$. If $r_2\geq r_3\geq 3$, we get $r_1\leq 3(2g(B)-1)$. Therefore we obtain $r_p\leq 6(2g(B)-1)$ for any $p\in B$.

(ii) The case of $g(B)=1$.

We do not have to consider a translation, since it has no fixed points. Then it is well known that the order of the automorphism groups of B which fixes a point is at most 6. So the order of the stabilizer of H is at most 6, and we get $r_p\leq 6$.

(iii) The case of $g(B)=0$

If H is neither a cyclic group nor a dihedral group, the order of the stabilizer of H is at most 5. So we may assume H is either a cyclic group or a dihedral group. It is well known that a rational pencil with at most two singular fibers is iso-trivial. If the number of singular fibers of f is at least three, then there exist a point $p\in B$ such that $\mu_n\geq n\sharp\widetilde{K}/K_f^2(\Gamma_p)$ and $r_p\leq 2$. Hence we may assume $r_p\leq 2$. □

Theorem 6.2. *Let $f:S\to B$ be a non-locally trivial primitive cyclic covering fibration of type $(g,1,n)$ with $g\geq \max\{\frac{30n^2-47n+25}{n+1},\frac{7}{2}n(n-1)+1\}$. Put*

$$\mu_n':=\frac{6n^2\delta}{(n-1)(5n-4)}.$$

Assume furthermore that the branch locus R has singular points on at least three (resp. one) fibers when $g(B) = 0$ (resp. $g(B) \geq 1$). Let G be a finite subgroup of $\mathrm{Aut}(f)$. *Then it holds*

$$\sharp G \leq \begin{cases} 6(2g(B)-1)\mu_n' K_f^2 & (g(B) \geq 1), \\ 5\mu_n' K_f^2 & (g(B) = 0). \end{cases}$$

Corollary 6.3. *Let $f : S \to B$ be a non-locally trivial bielliptic fibration with $g \geq 17$. Assume furthermore that the branch locus R has singular points on at least three (resp. one) fibers when $g(B) = 0$ (resp. $g(B) \geq 1$). Let G be a finite subgroup of* $\mathrm{Aut}(f)$. *Then it holds*

$$\sharp G \leq \begin{cases} 24\delta(2g(B)-1)K_f^2 & (g(B) \geq 1), \\ 20\delta K_f^2 & (g(B) = 0). \end{cases}$$

Corollary 6.4. *Let $f : S \to B$ be a non-locally trivial primitive cyclic covering fibration of type $(g, 1, n)$ with $g \geq \max\{\frac{30n^2-47n+25}{n+1}, \frac{7}{2}n(n-1)+1\}$. Put* $\mathrm{Aut}(S/B) := \{(\kappa_S, \mathrm{id}_B) \in \mathrm{Aut}(f)\}$. *Assume that the branch locus R has a singular point. Then it holds*

$$\sharp \mathrm{Aut}(S/B) \leq \frac{6n^2\delta}{(n-1)(5n-4)} K_f^2.$$

7. Example

We construct bielliptic fibrations $f : S \to B$ with a large automorphism group. Let (E, O) be an elliptic curve with the identity element $O \in E$. The symbol $P \oplus Q$ denotes the sum of P, $Q \in E$ as the group law. Furthermore, we define

$$[k]P := \underbrace{P \oplus \cdots \oplus P}_{k-summands}.$$

Let e be a positive even number and P a point of E of order e. We define the automorphisms of E as follows:

$$\tau_P : E \to E \; ; \; Q \mapsto P \oplus Q$$
$$\iota : E \to E \; ; \; Q \mapsto \ominus Q$$

where $\ominus Q$ denotes the inverse element of Q. Let $Q_1, Q_2 \in E$ be distinct points such that $[2]Q_1 = [2]Q_2 = P$. We note that the order of Q_1 and Q_2 is $2e$. We define divisors

$$\begin{aligned} \mathfrak{d}_i &:= Q_i + [3]Q_i + \cdots + [2e-1]Q_i \ (i = 1, 2), \\ R_E &:= \mathfrak{d}_1 + \mathfrak{d}_2. \end{aligned}$$

We recall that $P \oplus Q = R$ if and only if $P + Q \sim R + O$, where the symbol $\sim$ means the linearly equivalence. Since e is even, we have $\mathfrak{d}_1 \sim \mathfrak{d}_2 \sim eO$. Hence we consider a double covering

$$\pi : F := \mathrm{Spec}_E \left(\mathcal{O}_E \bigoplus \mathcal{O}_E(-\mathfrak{d}_1) \right) \to E$$

branched along $R_E \in |2\mathfrak{d}_1|$. Let K_E be the automorphism group on E generated by τ_P and ι, then K_E has order $2e$. Since $\mathfrak{d}_1$ and R_E are K_E-stable, there exists $\widetilde{\kappa} \in \mathrm{Aut}(F)$ for any $\kappa \in K_E$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\widetilde{\kappa}} & F \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\kappa]{} & E \end{array}$$

commutes.

Put $W := \mathbb{P}^1 \times E$ and let $\varphi : W \to \mathbb{P}^1$ be the projection. An action of $\kappa \in K_E$ induces the following action on W:

$$\kappa : W \to W \ : \ (x, y) \mapsto (x, \kappa(y))$$

Thus, we can regard K_E as a subgroup of $\mathrm{Aut}(W/\mathbb{P}^1)$. Let R_h and $\mathfrak{d}_h$ be the pullback of R_E and $\mathfrak{d}_1$ by the natural projection $W \to E$.

We recall that the icosahedral group I_{60} acts $\mathbb{P}^1$. We denote by p_1, $\cdots, p_{12}$ the I_{60}-orbit on $\mathbb{P}^1$ with length 12. Let Γ_i be the fiber of φ over p_i. We define the action on W for $h \in I_{60}$ as follows.

$$h : W \to W \ ; \ (x, y) \mapsto (h(x), y)$$

Hence we define an automorphism group

$$\widetilde{G} := I_{60} \times K_E \cong I_{60} K_E \subset \mathrm{Aut}(\varphi).$$

The order of $\widetilde{G}$ is $120e$. We define divisors

$$\mathfrak{d} := \mathfrak{d}_h + \sum_{i=1}^{6} \Gamma_i, \quad R := R_h + \sum_{i=1}^{12} \Gamma_i.$$

Since R_E and $2\mathfrak{d}_1$ are linearly equivalent, we have $R \in |2\mathfrak{d}|$. Hence we can take a double covering

$$\theta : S_0 := \mathrm{Spec}_W \left(\mathcal{O}_W \bigoplus \mathcal{O}_W(-\mathfrak{d}) \right) \to W$$

branched along $R \in |2\mathfrak{d}|$. Let S be the minimal resolution of S_0. Then the induced fibration $f : S \to \mathbb{P}^1$ is a bielliptic fibration with $g = l + 1$. By the assumption of $\widetilde{G}$, we have $\kappa^*\mathfrak{d} \sim \mathfrak{d}$ and $\kappa^* R = R$. Thus, any action κ induces an action $\widetilde{\kappa} \in \mathrm{Aut}(f)$. Hence we can construct a subgroup G of $\mathrm{Aut}(f)$ generated by $\widetilde{\kappa}$'s and the covering transformation. Then G is a finite subgroup with $\sharp G \geq 240e$.

On the other hand, we have

$$K_f^2(\Gamma_i) = 2e.$$

since R has e ordinary double points on each Γ_i. Thus, we have $K_f^2 = 24e$ and it holds that $\sharp G \geq 10K_f^2$.

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