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# *A Remark on Quadratic Functionals of Brownian Motions*

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**Abstract.** It is a classical problem to give explicit formulae for characteristic functions of quadratic functionals of Brownian motions, and there are many works on this topic. Ikeda-Kusuoka-Manabe [4], [5] gave a new idea to solve this problem and showed some results from which almost all known results followed. In this paper the authors extend their results based on their idea.

## 1. Introduction and Result

Let $d \geqq 1$, $T > 0$, $W_0 = \{w \in C([0,T];\mathbf{R}^d);\ w(0) = 0\}$, and $\mathcal{B}(W_0^d)$ be the Borel algebra over $W_0^d$. Let $\mu$ be the Wiener measure on $(W_0^d, \mathcal{B}(W_0^d))$. Now let $a_i^k : [0,T] \to \mathbf{R}$, $b_i^k : [0,T] \to \mathbf{R}$, $i = 1, \dots, N$, $k = 1, \dots, d$, be continuous functions. Let $X : W_0^d \to \mathbf{R}$ be a random variable given by

$$X(w) = \sum_{i=1}^N \sum_{k,\ell=1}^d \int_0^T (\int_0^t b_i^\ell(s)dw^\ell(s))a_i^k(t)dw^k(t).$$

Here stochastic integrals are Ito integrals.

Let $\lambda \in \mathbf{C}$. Our concern is to compute the following.

$$E^\mu[\exp(\sum_{k=1}^d \int_0^T h^k(t)dw^k(t) + \lambda X)], \qquad h \in L^2([0,T];\mathbf{C}^d, dt).$$

Such a problem was first considered by Lévy [6] and then many people studied this problem and gave explicit formulae (e.g. [1], [3], [4], [5]). In the present paper, we consider general cases and show that we can reduce this problem to a problem of a linear ordinary differential equation by using ideas in [4] and [5], where symmetric linear operators are decomposed

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into summations of Volterra type operators and linear operators of finite dimensional range.

Let $\alpha^k : [0,T] \to \mathbf{R}^{2N}$, $k = 1, \dots, d$, be given by

$$\alpha_j^k(t) = \begin{cases} a_j^k(t), & j = 1, \dots, N, \\ b_{j-N}^k(t), & j = N+1, \dots, 2N. \end{cases}$$

Let $J : \mathbf{R}^{2N} \to \mathbf{R}^{2N}$ be a linear operator given by

$$J((z_i)_{i=1}^{2N})_j = \begin{cases} -z_{j+N}, & j = 1, \dots, N, \\ z_{j-N}, & j = N+1, \dots, 2N. \end{cases}$$

Let $\beta^k : [0,T] \to \mathbf{R}^{2N}$, $k = 1, \dots, d$, be given by $\beta^k(t) = J\alpha^k(t)$. Then we have

$$\beta_j^k(t) = \begin{cases} -b_j^k(t), & j = 1, \dots, N, \\ a_{j-N}^k(t), & j = N+1, \dots, 2N. \end{cases}$$

Let $c_{i,j} : [0,T] \to \mathbf{R}$, $i, j = 1, \dots, 2N$, be given by

$$c_{i,j}(t) = \sum_{k=1}^{d} \alpha_i^k(t)\beta_j^k(t), \quad t \in [0,T].$$

Also, let $e_{\lambda,i_1,i_2} : [0,T] \to \mathbf{C}$, $i_1, i_2 = 1, \dots, 2N$, $\lambda \in \mathbf{C}$, be the solution to the following ODE

$$(1) \qquad \frac{d}{dt} e_{\lambda,i_1,i_2}(t) = \lambda \sum_{j=1}^{2N} c_{i_1,j}(t) e_{\lambda,j,i_2}(t),$$
$$e_{\lambda,i_1,i_2}(0) = \delta_{i_1,i_2}, \qquad i_1, i_2 = 1, \dots, 2N.$$

Let $e_\lambda$ be a $2N \times 2N$-matrix valued function defined in $[0,T]$ given by $e_\lambda(t) = (e_{\lambda,i,j}(t))_{i,j=1,\dots,2N}$, and let $e_\lambda^0$ be an $N \times N$-matrix valued function defined in $[0,T]$ given by $e_\lambda^0(t) = (e_{\lambda,i,j}(t))_{i,j=1,\dots,N}$.

Let $\gamma_\lambda^k : [0,T] \to \mathbf{R}^{2N}$, $k = 1, \dots, d$, be continuous functions given by

$$\gamma_{\lambda,i}^k(t) = -\sum_{j=1}^{2N} e_{\lambda,j,i}(t)\beta_j^k(t), \qquad i = 1, \dots, 2N, \; t \in [0,T].$$

Now let $\Psi_\lambda : L^2([0,T];\mathbf{C}^d, dt) \to C([0,T];\mathbf{C}^{2N})$ be bounded linear operators given by

$$(\Psi_\lambda h)_i(t) = \sum_{k=1}^d \int_0^t \gamma_{\lambda,i}^k(s) h^k(s) ds, \qquad t \in [0,T],\ h \in L^2([0,T];\mathbf{C}^d, dt).$$

Also, let

$$\lambda_0 = \inf\{s \in \mathbf{R};\ E^\mu[\exp(sX)] < \infty\},$$

and

$$\lambda_1 = \sup\{s \in \mathbf{R};\ E^\mu[\exp(sX)] < \infty\}.$$

As is explained in Section 2, we see that $\lambda_0 < 0 < \lambda_1$.

Our main result is the following.

THEOREM 1. *Assume that $\lambda \in \mathbf{C}$ and $\lambda_0 < Re\ \lambda < \lambda_1$. Then we have the following.*
(1) *The $N \times N$-matrix $e_\lambda^0(T)$ is invertible.*
(2) *Let $e_\lambda^0(T)^{-1} = (e_\lambda^0(T)_{i,j}^{-1})_{i,j=1,\dots,N}$ be the inverse matrix of $e_\lambda^0(T)$. Let $d_{\lambda,i,j} \in \mathbf{C}$, $i,j = 1,\dots,N$, be given by*

$$d_{\lambda,i,j} = \sum_{r=1}^N e_\lambda^0(T)_{i,r}^{-1} e_{\lambda,r,N+j}(T).$$

*Then $d_{\lambda,i,j} = d_{\lambda,j,i}$, $i,j = 1,\dots,N$.*
(3) *For any $h \in L^2([0,T];\mathbf{C}^d, dt)$*

$$\begin{aligned} &E^\mu[\exp(\sum_{k=1}^d \int_0^T h^k(t) dw^k(t) + \lambda X)] \\ &= \det(e_\lambda^0(T))^{-1/2} \exp(-\frac{\lambda}{2} \sum_{i=1}^N \sum_{k=1}^d \int_0^T a_i^k(t) b_i^k(t) dt \\ &\qquad + \frac{1}{2} \int_0^T (\sum_{k=1}^d h_k(s)^2) ds + \frac{\lambda}{2} \mathcal{A}_\lambda(h,h)), \end{aligned}$$

*where*

$$\mathcal{A}_\lambda(h,h)$$
$$= -\int_0^T \sum_{i=1}^{2N} \frac{d}{dt}(\Psi_\lambda h)_i(t)(J(\Psi_\lambda h)(t)))_i dt + \sum_{i=1}^{N}(J(\Psi_\lambda h)(T))_i(\Psi_\lambda h)(T)_i$$
$$+ \sum_{i,j=1}^{N} d_{\lambda,ij}(\Psi_\lambda h)(T)_i(\Psi_\lambda h)(T)_j.$$

## 2. Preliminary Facts

In this section we assume that $E^\mu[\exp(X)] < \infty$. Let $\tilde{H}$ be the Cameron-Martin space of the Wiener space $(W_0, \mu)$, i.e.,

$$\tilde{H} = \{k \in W_0;\ k(t) \text{ is absolutely continuous in } t,\ \int_0^T |\frac{dk}{dt}(t)|^2 dt < \infty\},$$
$$(k_1, k_2)_{\tilde{H}} = \int_0^T \frac{dk_1}{dt}(t) \cdot \frac{dk_2}{dt}(t)dt.$$

Here $\cdot$ stands for the natural innner product in $\mathbf{R}^d$. Let $H = L^2([0,T]; \mathbf{R}^d, dt)$. Then the map $\Phi : \tilde{H} \to H$ corresponding $k$ to $\frac{dk}{dt}$ is an isomorphism.

Let $\mathcal{E} : H \times H \to \mathbf{R}$ be the symmetric bilinear form given by

$$\mathcal{E}(h_1, h_2)$$
$$= \sum_{i=1}^{N} \sum_{k,\ell=1}^{d} \int_0^T \int_0^T 1_{\{t>s\}} a_i^k(t) b_i^\ell(s) h_1^k(t) h_2^\ell(s) dtds$$
$$+ \sum_{i=1}^{N} \sum_{k,\ell=1}^{d} \int_0^T \int_0^T 1_{\{t>s\}} a_i^k(t) b_i^\ell(s) h_1^\ell(s) h_2^k(t) dtds$$

for $h_1, h_2 \in H$.

The associated symmetric bounded linear operator $E : H \to H$ to $\mathcal{E}$ is given by

$$(Eh)^k(t)$$
$$= \sum_{i=1}^{N} \sum_{\ell=1}^{d} \int_0^T 1_{\{t>s\}} a_i^k(t) b_i^\ell(s) h^\ell(s) ds + \sum_{i=1}^{N} \sum_{\ell=1}^{d} \int_0^T 1_{\{s>t\}} a_i^\ell(s) b_i^k(t) h^\ell(s) ds$$

$$= \sum_{i=1}^{N} \sum_{\ell=1}^{d} a_i^k(t) \int_0^t b_i^\ell(s) h^\ell(s) ds - \sum_{i=1}^{N} \sum_{\ell=1}^{d} b_i^k(t) \int_0^t a_i^\ell(s) h^\ell(s) ds$$
$$+ \sum_{i=1}^{N} \sum_{\ell=1}^{d} b_i^k(t) \int_0^T a_i^\ell(s) h^\ell(s) ds, \qquad t \in [0,T],\ k = 1, \dots, d,\ h \in H.$$

Then we see that $E$ is a Hilbert-Schmidt type symmetric linear operator in $H$. So there are a complete orthonormal basis $\{g_m\}_{m=1}^\infty$ in $H$ and a sequence $\{\eta_m\}_{m=1}^\infty$of real numbers such that

$$Eh = \sum_{m=1}^\infty \eta_m (h, g_m)_H g_m, \qquad h \in H,$$

and $\sum_{k=1}^\infty \eta_k^2 < \infty$. Moreover, we see that

$$X = \frac{1}{2} \sum_{m=1}^\infty \eta_m ((\sum_{k=1}^d \int_0^T g_m^k(t) dw^k(t))^2 - 1).$$

Note that $\sum_{k=1}^d \int_0^T g_m^k(t) dw^k(t)$, $m = 1, 2, \dots$ , is a sequence of independent normal distributed random variables.

Therefore for any $\lambda \in \mathbf{R}$ we see that $E^\nu[\exp(\lambda X)] < \infty$ if and only if $\sup_m \lambda \eta_m < 1$.

Let $\rho(E)$ be the set of eigenvalues of $E$. That is $\rho(E) = \{\eta_m;\ m = 1, 2, \dots\}$. Since we assume that $E^\mu[\exp(X)] < \infty$, we see that $\rho(E) \subset (-\infty, 1)$. Hence we see that $I_H - E : H \to H$ is invertible and positive-definite. Also, we have

$$E^\mu[\exp(\sum_{k=1}^d \int_0^T h^k(t) dw^k(t) + X)]$$
$$= \prod_{m=1}^\infty ((1 - \eta_m)^{-1/2} \exp(-\eta_m/2) \exp(\frac{1}{2}(1 - \eta_m)^{-1}(g_m, h)_H^2))$$
$$= \det_2(I_H - E)^{-1/2} \exp(\frac{1}{2}(h, (I_H - E)^{-1} h)_H), \qquad h \in H. \tag{2}$$

Here $\det_2$ is a regularized deteminant (see [2] for its definition and properties).

Let $V_{01} : H \to H$, and $A_r : H \to \mathbf{R}^N$, $r = 0, 1$, be given by

$$\begin{aligned}
&(V_{01}h)^k(t) \\
&= \sum_{i=1}^{N} \sum_{\ell=1}^{d} a_i^k(t) \int_0^t b_i^\ell(s) h^\ell(s) ds - \sum_{i=1}^{N} \sum_{\ell=1}^{d} b_i^k(t) \int_0^t a_i^\ell(s) h^\ell(s) ds, \\
&= \sum_{i=1}^{2N} \sum_{\ell=1}^{d} \beta_i^k(t) \int_0^t \alpha_i^\ell(s) h^\ell(s) ds, \qquad h \in H,\ k = 1, \dots, d, \\
&(A_0 h)_i = \sum_{\ell=1}^{d} \int_0^T a_i^\ell(s) h^\ell(s) ds, \qquad h \in H,\ i = 1, \dots, N,
\end{aligned}$$

and

$$(A_1 h)_i = \sum_{\ell=1}^{d} \int_0^T b_i^\ell(s) h^\ell(s) ds, \qquad h \in H,\ i = 1, \dots, N.$$

Note that $V_{01}$ is a Volterra type operator. So we see that a bounded linear map $I_H - V_{01} : H \to H$ is invertible. Also, we see that

$$E = V_{01} + A_1^* A_0.$$

PROPOSITION 2. *Assume that $E^\mu[\exp(X)] < \infty$. Then we have the following.*

(1) $\det_2(I_H - E)$

$$= \det(I_N - A_0(I_H - V_{01})^{-1} A_1^*) \exp(\mathrm{trace} A_0 A_1^*),$$

*where $I_N$ is the identity map in $\mathbf{R}^N$.*

*In particular, the $N \times N$-matrix $I_N - A_0(I_H - V_{01})^{-1} A_1^*$ is invertible .*

(2) $(I_H - E)^{-1}$

$$= (I_H - V_{01})^{-1} + (I_H - V_{01})^{-1} A_1^* (I_N - A_0 (I_H - V_{01})^{-1} A_1^*)^{-1} A_0 (I_H - V_{01})^{-1}.$$

PROOF. Let $z \in \mathbf{C}$ for which $|z|$ is sufficiently small. Then we have

$$\begin{aligned}
&\det_2(I_H - zE) \\
&= \det_2((I_H - zV_{01})(I_H - z(I_H - zV_{01})^{-1} A_1^* A_0)).
\end{aligned}$$

Since $V_{01}$ is a Volterra type operator, we have $\det_2(I_H - zV_{01}) = 1$. So we have

$$\begin{aligned}
&\det{}_2(I_H - zE)\\
&= \det{}_2(I_H - zV_{01})\det{}_2(I_H - z(I_H - zV_{01})^{-1}A_1^*A_0)\\
&\quad \times \exp(-\mathrm{trace}(z^2V_{01}(I_H - zV_{01})^{-1}A_1^*A_0))\\
&= \exp(-\sum_{k=2}^{\infty}\frac{1}{k}\mathrm{trace}((z(I_H - zV_{01})^{-1}A_1^*A_0)^k))\\
&\quad \times \exp(-\mathrm{trace}(zA_0((I_H - zV_{01})^{-1} - I_H)A_1^*))\\
&= \exp(-\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{trace}((zA_0(I_H - zV_{01})^{-1}A_1^*)^k))\exp(\mathrm{trace}(zA_0A_1^*))\\
&= \det(I_N - zA_0(I_H - zV_{01})^{-1}A_1^*)\exp(z\ \mathrm{trace}(A_0A_1^*)).
\end{aligned}$$

Here we use the fact that

$$\det{}_2((I_H+K_1)(I_H+K_2)) = \det{}_2(I_H+K_1)\det{}_2(I_H+K_2)\exp(-\mathrm{trace}(K_1K_2))$$

and

$$\mathrm{trace}(K_1K_2) = \mathrm{trace}(K_2K_1)$$

for any linear Hilbert-Schmidt type operators $K_i : H \to H$, $i = 1, 2$. We also use the fact that

$$\det{}_2(I_H - K) = \exp(-\sum_{k=2}^{\infty}\frac{1}{k}\mathrm{trace}(K^k))$$

for any linear Hilbert-Schmidt type operator $K : H \to H$ such that the Hilbert-Schmidt norm of $K$ is less than 1.

Also, we have

$$\begin{aligned}
(I_H - zE)^{-1} &= \{(I_H - zV_{01})(I_H - z(I_H - zV_{01})^{-1}A_1^*A_0)\}^{-1}\\
&= (\sum_{k=0}^{\infty}(z(I_H - zV_{01})^{-1}A_1^*A_0)^k)(I_H - zV_{01})^{-1}\\
&= \{I_H + \sum_{k=0}^{\infty} z(I_H - zV_{01})^{-1}A_1^*(zA_0(I_H - zV_{01})^{-1}A_1^*)^kA_0\}(I_H - zV_{01})^{-1}\\
&= (I_H - zV_{01})^{-1} + z(I_H - zV_{01})^{-1}A_1^*(I_N - zA_0(I_H - zV_{01})^{-1}A_1^*)^{-1}\\
&\quad \times A_0(I_H - zV_{01})^{-1}.
\end{aligned}$$

Note that $\det_2(I_H - zE)$ and $(I_H - zE)$ are holomorphic in $z$ over $\mathbf{C}$. So we see that $(I_H - zE)^{-1}$ is holomorphic in $z$ around $[0,1]$. So we have our assertion. □

## 3. Special Case

We also consider the case that $E^\mu[\exp(X)] < \infty$ in this section. Let $\tilde{e}_{i_1,i_2} : [0,T] \to \mathbf{R}$, $i_1, i_2 = 1, \dots, 2N$, be the solution to the following ODE

$$\frac{d}{dt}\tilde{e}_{i_1,i_2}(t) = \sum_{j=1}^{2N} c_{i_1,j}(t)\tilde{e}_{j,i_2}(t), \tag{3}$$

$$\tilde{e}_{i_1,i_2}(0) = \delta_{i_1,i_2}, \qquad i_1, i_2 = 1, \dots, 2N.$$

Let $\tilde{e}$ be a $2N \times 2N$-matrix valued function defined in $[0,T]$ given by $\tilde{e}(t) = (\tilde{e}_{i,j}(t))_{i,j=1,\dots,2N}$, and let $\tilde{e}^0$ be an $N \times N$-matrix valued function defined in $[0,T]$ given by $\tilde{e}^0(t) = (\tilde{e}_{i,j}(t))_{i,j=1,\dots,N}$.

Let $\tilde{\gamma}^k : [0,T] \to \mathbf{R}^{2N}$, $k = 1, \dots, d$, be continuous functions given by

$$\tilde{\gamma}_i^k(t) = -\sum_{j=1}^{2N} \tilde{e}_{j,i}(t)\beta_j^k(t), \qquad i = 1, \dots, 2N,\ t \in [0,T].$$

Now let $\tilde{\Psi} : H \to C([0,T];\mathbf{R}^{2N})$ be bounded linear operators given by

$$(\tilde{\Psi}h)_i(t) = \sum_{k=1}^{d} \int_0^t \tilde{\gamma}_i^k(s)h^k(s)ds, \qquad t \in [0,T],$$

for $h \in H$.

In this section we prove the following.

THEOREM 3. *Assume that $E^\mu[\exp(X)] < \infty$. Then we have the following.*
(1) *The $N \times N$-matrix $\tilde{e}^0(T)$ is invertible.*
(2) *Let $\tilde{e}^0(T)^{-1} = (e^0(T)^{-1}_{i,j})_{i,j=1,\dots,N}$ be the inverse matrix of $\tilde{e}^0(T)$. Let $\tilde{d}_{i,j} \in \mathbf{R}$, $i, j = 1, \dots, N$, be given by*

$$\tilde{d}_{i,j} = \sum_{r=1}^{N} \tilde{e}^0(T)^{-1}_{i,r}\tilde{e}_{r,N+j}(T).$$

*Then* $\tilde{d}_{i,j} = \tilde{d}_{j,i}$, $i,j = 1,\dots,N$.
(3) *For any* $h \in H$

$$E^{\mu}[\exp(\sum_{k=1}^{d}\int_0^T h^k(t)dw^k(t) + X)]$$
$$= \det(\tilde{e}^0(T))^{-1/2}\exp(-\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{d}\int_0^T a_i^k(t)b_i^k(t)dt + \frac{1}{2}||h||_H^2 + \frac{1}{2}\tilde{\mathcal{A}}(h,h)),$$

*where*

$$\tilde{\mathcal{A}}(h,h)$$
$$= -\int_0^T \sum_{i=1}^{2N}\frac{d}{dt}(\tilde{\Psi}h)_i(t)(J(\tilde{\Psi}h)(t)))_i dt + \sum_{i=1}^{N}(J(\tilde{\Psi}h)(T))_i(\tilde{\Psi}h)(T)_i$$
$$+ \sum_{i,j=1}^{N}\tilde{d}_{ij}(\tilde{\Psi}h)(T)_i(\tilde{\Psi}h)(T)_j.$$

We make some preparations to prove this theorem.

PROPOSITION 4. *Let* $\tilde{e}(t)^{-1}$ *be the inverse matrix of* $\tilde{e}(t)$, $t \in [0,T]$. *Then we see that*

$$\tilde{e}(t)^{-1} = -J\tilde{e}(t)^*J, \qquad t \in [0,T].$$

*Moreover,*

$$J(\tilde{\Psi}h)(t) = \sum_{k=1}^{d}\int_0^t \tilde{e}(s)^{-1}\alpha^k(s)h^k(s)ds, \qquad t \in [0,T].$$

PROOF. Remind that

$$\frac{d}{dt}\tilde{e}(t) = c(t)\tilde{e}(t), \qquad t \in [0,T].$$

Then we have

$$\frac{d}{dt}\tilde{e}(t)^{-1} = -\tilde{e}(t)^{-1}c(t), \qquad t \in [0,T].$$

Note that

$$(Jc(t))_{i,j} = \sum_{k=1}^{d} \beta_i^k(t)\beta_j^k(t) = (Jc(t))_{j,i} \qquad i,j = 1,\dots,N.$$

So we see that

$$c(t)^* = -(JJc(t))^* = Jc(t)J.$$

Therefore we have

$$\frac{d}{dt}(-J\tilde{e}(t)^*J) = -J(c(t)\tilde{e}(t))^*J = -(-J\tilde{e}(t)^*J)c(t).$$

Since $-J\tilde{e}(0)^*J = I_{2N}$, we have the first assertion by the uniqueness of a solution to ODE.

So we see that

$$J\tilde{\gamma}^k(t) = -J\tilde{e}(t)^*J\alpha^k(t) = \tilde{e}(t)^{-1}\alpha^k(t) \qquad k = 1,\dots,d.$$

This implies the second assertion. □

Then we have the following.

PROPOSITION 5. *(1)* $I_N - A_0(I_H - V_{01})^{-1}A_1^* = \tilde{e}^0(T)$, *and*

$$\det(I_N - A_0(I_H - V_{01})^{-1}A_1^*) = \det(\tilde{e}^0(T)).$$

*Moreover,* $\tilde{e}^0(T)$ *is invertible.*
(2) *For any* $h \in H$,

$$(A_0(I_H - V_{01})^{-1}h)_i = \sum_{j=1}^{2N} \tilde{e}_{i,j}(T)(J(\tilde{\Psi}h)(T))_j, \qquad i = 1,\dots,N.$$

(3) *For any* $h \in H$ *and* $v \in \mathbf{R}^N$,

$$(h, (I_H - V_{01})^{-1}A_1^*v)_H = \sum_{i=1}^{N} v_i(\tilde{\Psi}h)_i(T), \qquad k = 1,\dots,d,\ t \in [0,T].$$

(4) *For any* $h \in H$,

$$((I_H - V_{01})^{-1}h)^k(t) = h^k(t) - \sum_{j=1}^{2N} \tilde{\gamma}_j^k(t)(J(\tilde{\Psi}h)(t))_j,$$
$$k = 1, \ldots, d, \ t \in [0, T].$$

*In particular,*

$$(h_1, (I_H - V_{01})^{-1}h_2)_H = (h_1, h_2)_H - \int_0^T (\frac{d}{dt}(\tilde{\Psi}h_1)(t), J(\tilde{\Psi}h_2)(t))_{\mathbf{R}^{2N}} dt$$

*for* $h_1, h_2 \in H$.

Proof. Let $f \in C_0^\infty((0, T); \mathbf{R}^d) \subset H$ and let $\xi = (I_H - V_{01})^{-1}f$. Then we have

$$\xi = f + V_{01}\xi$$

Let

$$\eta_i(t) = \sum_{k=1}^{d} \int_0^t \alpha_i^k(s)\xi^k(s)ds, \quad i = 1, \ldots, 2N.$$

Then we have

$$(A_0(I_H - V_{01})^{-1}f)_i = \eta_i(T), \qquad i = 1, \ldots, N. \tag{4}$$

Also we have

$$\xi^k(t) = f^k(t) + \sum_{i=1}^{2N} \beta_i^k(t)\eta_i(t), \qquad k = 1, \ldots, d,$$

and so we have

$$\frac{d}{dt}\eta_i(t) = \sum_{k=1}^{d} \alpha_i^k(t)\xi^k(t)$$
$$= \sum_{k=1}^{d} \alpha_i^k(t)f^k(t) + \sum_{j=1}^{2N} c_{ij}(t)\eta_j(t), \qquad i = 1, \ldots, 2N.$$

Note that $\eta_i(0) = 0$, $i = 1, \dots, 2N$. So we see that

$$\eta_i(t) = \sum_{j_1,j_2=1}^{2N} \sum_{\ell=1}^{d} \tilde{e}_{i,j_1}(t) \int_0^t \tilde{e}(s)^{-1}_{j_1,j_2} \alpha^{\ell}_{j_2}(s) f^{\ell}(s) ds$$
$$= \sum_{j=1}^{2N} \tilde{e}_{i,j}(t) (J(\tilde{\Psi} f)(t))_j, \qquad i = 1, \dots, 2N,\ t \in [0,T].$$

This and Proposition 4 imply Assertion (2), since $C_0^{\infty}((0,T); \mathbf{R}^d)$ is dense in $H$.

Also, we see that

$$\xi^k(t) = f^k(t) + \sum_{j_1,j_2=1}^{2N} \beta^k_{j_1}(t) \tilde{e}_{j_1,j_2}(t) (J(\tilde{\Psi} f)(t))_{j_2},$$

for $k = 1, \dots, d$, $t \in [0,T]$. This implies the assertion (4).

Let $v \in \mathrm{R}^N$. Then we have by Proposition 4

$$(J(\tilde{\Psi} A_1^* v)(t))_i = \sum_{r=1}^{N} \sum_{j=1}^{2N} \sum_{\ell=1}^{d} v_r \int_0^t \tilde{e}(s)^{-1}_{i,j} \alpha^{\ell}_j(s) b^{\ell}_r(s) ds$$
$$= -\sum_{r=1}^{N} \sum_{j=1}^{2N} \sum_{\ell=1}^{d} v_r \int_0^t \tilde{e}(s)^{-1}_{i,j} c_{j,r}(s) ds$$
$$= \sum_{r=1}^{N} v_r \int_0^t \frac{d}{ds} \tilde{e}(s)^{-1}_{i,r} ds = \sum_{r=1}^{N} v_r \tilde{e}(t)^{-1}_{i,r} - v_i$$

Therefore by Assertion (4) we have

$$(h, (I_H - V_{01})^{-1} A_1^* v)$$
$$= (h, A_1^* v) + \sum_{j_1,j_2=1}^{2N} \sum_{i=1}^{N} v_i \int_0^T \beta^k_{j_1}(t) \tilde{e}(t)_{j_1,j_2} \tilde{e}(t)^{-1}_{j_2,i} h^k(t) dt$$
$$- \sum_{j=1}^{2N} \sum_{i=1}^{N} v_i \int_0^T \beta^k_j(t) \tilde{e}(t)_{j,i} h^k(t) dt$$
$$= \sum_{i=1}^{N} v_i \int_0^T \tilde{\gamma}^k_i(t) h^k(t) dt = \sum_{i=1}^{N} v_i (\tilde{\Psi} h)_i(T).$$

This implies the assertion (3).

So we have

$$
\begin{aligned}
&((I_N - A_0(I_H - V_{01})^{-1}A_1^*)v)_i \\
&= v_i + \sum_{k=1}^{d}\sum_{j_0=1}^{2N}\sum_{j_1=1}^{N} v_{j_1}\int_0^T a_i^k(t)\beta_{j_0}^k(t)\tilde{e}_{j_0,j_1}(t)dt \\
&= v_i + \sum_{j_0=1}^{2N}\sum_{j_1=1}^{N} v_{j_1}\int_0^T c(t)_{i,j_0}\tilde{e}(t)_{j_0,j_1}dt \\
&= v_i + \sum_{j=1}^{N} v_j(e(T)_{i,j} - \delta_{ij}) = \sum_{j=1}^{N}\tilde{e}(T)_{i,j}v_j.
\end{aligned}
$$

This and Proposition 2 (1) imply Assertion (1). □

PROPOSITION 6. (1) *$\tilde{d}_{i,j} = \tilde{d}_{j,i}$, for all $i, j = 1, \ldots N$.*
(2) *For any $h_1, h_2 \in H$,*

$$
\begin{aligned}
&(h_1, (I - E)^{-1}h_2)_H \\
&= (h_1, h_2)_H + \sum_{i=1}^{N} J(\tilde{\Psi}h_2)(T)_i(\tilde{\Psi}h_1)(T)_i \\
&\quad - \int_0^T (\frac{d}{dt}(\tilde{\Psi}h_1)(t), J(\tilde{\Psi}h_2)(t))_{\mathbf{R}^{2N}}dt \\
&\quad + \sum_{i,j=1}^{N}\tilde{d}_{ij}(\tilde{\Psi}h_2)(T))_i(\tilde{\Psi}h_1)(T)_j.
\end{aligned}
$$

PROOF. Note that $\tilde{e}(t)J\tilde{e}(t)^* = J$. This implies that for $i, j = 1, \ldots, N$,

$$
0 = (\tilde{e}(t)J\tilde{e}(t)^*)_{i,j} = -\sum_{r=1}^{N}\tilde{e}_{i,r}(t)\tilde{e}_{j,N+r}(t) + \sum_{r=1}^{N}\tilde{e}_{i,N+r}(t)\tilde{e}_{j,r}(t)
$$

Let $f_{i,j} : [0, T] \to \mathbf{R}$, $i, j = 1, \ldots, N$, be given by

$$
f_{i,j}(t) = \sum_{r=1}^{N}\tilde{e}_{i,r}(t)\tilde{e}_{j,N+r}(t),
$$

and let $F(t)$ be an $N \times N$-matrix given by $F(t) = (f_{i,j}(t))_{i,j=1,\dots,N}$. Then we have $F(t)^* = F(t)$. Since we have

$$(\tilde{d}_{i,j})_{i,j=1,\dots,N} = \tilde{e}^0(T)^{-1}(\tilde{e}^0(T)^{-1}F(T))^* = \tilde{e}^0(T)^{-1}F(T)(e^0(T)^{-1})^*,$$

we have Assertion (1).

By Propositions 2 and 5, we have for $h_1, h_2 \in H$,

$$\begin{aligned}
&(h_1, ((I_H - E)^{-1}h_2)_H \\
&= (h_1, (I_H - V_{01})^{-1}h_2)_H \\
&\quad + (h_1, (I_H - V_{01})^{-1}A_1^*(I_N - A_0(I_H - V_{01})^{-1}A_1^*)^{-1}A_0(I_H - V_{01})^{-1}h_2)_H \\
&= (h_1, h_2)_H - \int_0^T (\frac{d}{dt}(\tilde{\Psi}h_1)(t), J(\tilde{\Psi}h_2)(t))_{\mathbf{R}^{2N}}dt \\
&\quad + \sum_{i,j=1}^N \sum_{\ell=1}^{2N} \tilde{e}^0(T)^{-1}_{ij}\tilde{e}(T)_{j,\ell}(J(\tilde{\Psi}h_2)(T))_\ell(\tilde{\Psi}h_1)(T)_i \\
&= (h_1, h_2)_H - \int_0^T (\frac{d}{dt}(\tilde{\Psi}h_1)(t), J(\tilde{\Psi}h_2)(t))_{\mathbf{R}^{2N}}dt \\
&\quad + \sum_{i=1}^N (J(\tilde{\Psi}h_2)(T))_i(\tilde{\Psi}h_1)(T)_i \\
&\quad + \sum_{i,j=1}^N \tilde{d}_{ij}(\tilde{\Psi}h_2)(T)_i(\tilde{\Psi}h_1)(T)_j.
\end{aligned}$$

So we have Assertion (2). □

Now Theorem 3 is an easy consequence of Propositions 2, 5, 6 and Equation (2).

## 4. Proof of Theorem 1

Now let us prove Theorem 1.

First assume that $\lambda \in (0, \lambda_1)$. Let

$$\begin{aligned}
&\tilde{X}(w) = \lambda X(w) \\
&= \sum_{i=1}^N \sum_{k,\ell=1}^d \int_0^T (\int_0^t (\lambda^{1/2}b_i^\ell(s))dw^\ell(s))(\lambda^{1/2}a_i^k(t))dw^k(t).
\end{aligned}$$

Then we see that $E^{\mu}[\exp(\tilde{X})] < \infty$. So if we replace $a^k(t), b^k(t)$, $k = 1,\dots,N$, $t \in [0,T]$, by $\lambda^{1/2}a^k(t), \lambda^{1/2}b^k(t)$, $k = 1,\dots,N$, $t \in [0,T]$, and apply results in the previous section, we have the following.

$$\begin{aligned} \tilde{e}(t) = e_\lambda(t), \quad \tilde{e}^0(t) = e^0_\lambda(t), \qquad & t \in [0,T], \\ \tilde{\gamma}^k(t) = \lambda^{1/2}\gamma^k_\lambda(t), \qquad & t \in [0,T],\ k = 1,\dots,d, \end{aligned}$$

and

$$\tilde{\mathcal{A}}(h,h) = \lambda\mathcal{A}_\lambda(h,h), \qquad t \in [0,T],\ h \in H.$$

So we see by Theorem 3 that Assertions (1),(2) and (3) in Theorem 1 are valid for $\lambda \in (0,\lambda_1)$ and $h \in H$.

In particular, we see that

$$\begin{aligned} & \det(e^0_\lambda(T))E^{\mu}[\exp(\lambda X)]^2 \\ & = \exp(-\lambda\sum_{i=1}^{n}\sum_{k=1}^{d}\int_0^T a^k_i(t)b^k_i(t)dt) \end{aligned}$$

for any $\lambda \in (0,\lambda_0)$. Note that $e_{\lambda,i,j}(t)$, $i,j = 1,\dots,2N$, $t \in [0,T]$, is holomorphic in $\lambda$ over $\mathbf{C}$.

Let $D_0 = \{z \in \mathbf{C};\ \lambda_0 < Re\ z < \lambda_1\}$. Then $E^{\mu}[\exp(\lambda X)]$ is holomorphic in $\lambda$ over $D_0$. So we see that $\det e^0_\lambda(T) \neq 0$ for all $\lambda \in D_0$. This implies Assertion (1).

Then we see that $\det e^0_\lambda(T)^{-1}$ is holomorphic in $\lambda$ over $D_0$. So we see that $d_{\lambda,i,j}$, $i,j = 1,\dots,N$, is holomorphic in $\lambda$ over $D_0$. So we see that $d_{\lambda,i,j} = d_{\lambda,j,i}$, $i,j = 1,\dots,N$, for all $\lambda \in D_0$.

Note that $\gamma^k_\lambda(t)$, $k = 1,\dots,d$, $t \in [0,T]$, is holomorphic in $\lambda$ over $D_0$.

We already showed that for any $\lambda \in (0,\lambda_1)$, $h_0, h_1 \in H$, and $z \in \mathbf{R}$

$$\begin{aligned} & E^{\mu}[\exp(\sum_{k=1}^{d}\int_0^T (h^k_0(t) + zh^k_1(t))dw^k(t) + \lambda X)] \\ = & \det(e^0_\lambda(T))^{-1/2}\exp(-\frac{\lambda}{2}\sum_{i=1}^{N}\sum_{k=1}^{d}\int_0^T a^k_i(t)b^k_i(t)dt \\ & \qquad + \frac{1}{2}\int_0^T (\sum_{k=1}^{d}(h^k_0(s) + zh^k_1(s)^2 ds + \frac{\lambda}{2}\mathcal{A}_\lambda(h_0 + zh_1, h_0 + zh_1)). \end{aligned}$$

It is easy to see that

$$E^{\mu}[\exp(\sum_{k=1}^{d}\int_0^T (h_0^k(t)+zh_1^k(t))dw^k(t)+\lambda X)]$$

is holomorphic in $(\lambda, z)$ over $D_0\times\mathbf{C}$. Also, we see that $\mathcal{A}_\lambda(h_0+zh_1, h_0+zh_1)$ is holomorphic in $(\lambda, z)$ over $D_0\times\mathbf{C}$. So we obtain Assertion (3) for all $h=h_0+\sqrt{-1}h_1\in L^2((0,T);\mathbf{C}^d, dt)$ and $\lambda\in D_0$.

This completes the proof of Theorem 1.

## 5. Special Class

In this section we show that there is a special class where we can solve Equation (1) by using solutions of linear ODE's with constant coefficients even though $a_i^k(t), b_j^k(t)$, $i,j=1,\dots,d$, $k=1,\dots,N$, are not constants.

Let $K$ be a $2N\times 2N$ real matrix and $\tilde\alpha^k\in\mathbf{R}^{2N}$, $k=1,\dots,d$. We assume that the matrix $K$ satisfies

$$JKJ=K^*.$$

Note that the matrix $K$ satisfies this condition, if and only if

$$K_{22}=-K_{11}^*,\ K_{12}^*=K_{12}\text{ and }K_{21}^*=K_{21},$$

where $K_{ij}$, $i,j=1,2$ are $N\times N$ matrix such that

$$K=\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}.$$

Now let . $\alpha^k:[0,T]\to\mathbf{R}^{2N}$, $k=1,\dots,d$ be given by

$$\alpha^k(t)=\exp(tK)\tilde\alpha^k,\qquad t\in[0,T],\ k=1,\dots,d.$$

Let $a_i^k:[0,T]\to\mathbf{R}$ and $b_i^k:[0,T]\to\mathbf{R}$, $i=1,\dots,N$, $k=1,\dots,d$, be given by

$$a_i^k(t)=\alpha_i^k(t),\quad b_i^k(t)=\alpha_{N+i}^k(t),\qquad t\in[0,T].$$

Let $\beta^k(t)=J\alpha^k(t)$, $k=1,\dots,d$, $t\in[0,T]$. Also, let $c_{i,j}:[0,T]\to\mathbf{R}$, $i,j=1,\dots,2N$, be given by

$$c_{i,j}(t)=\sum_{k=1}^{d}\alpha_i^k(t)\beta_j^k(t),\quad t\in[0,T].$$

Then we have the following.

PROPOSITION 7. *Let $L$ be a $2N \times 2N$ real matrix given by*

$$L = \sum_{k=1}^{d} \alpha^k (J\alpha^k)^* = -\sum_{k=1}^{d} \alpha^k \alpha^{k*} J.$$

*Then $JLJ = L^*$.*

*Let $e_{\lambda,i_1,i_2} : [0,T] \to \mathbf{C}$, $i_1, i_2 = 1, \ldots, 2N$, $\lambda \in \mathbf{C}$, be the solution to the ODE* (1)*. Then we have*

$$e_\lambda(t) = \{e_{\lambda,i_1,i_2}(t)\}_{i_1,i_2=1,\ldots,2N} = \exp(tK)\exp(t(\lambda L - K)), \qquad t \in [0,T].$$

PROOF. The first assertion is obvious. Note that

$$J\exp(tK) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} J(KJ^2)^n = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (JKJ)^n J = \exp(-tK^*)J,$$

and so we have

$$\beta^k(t) = J\alpha^k(t) = J\exp(tK)\tilde{\alpha}^k = \exp(-tK^*)J\tilde{\alpha}^k.$$

Then we see that

$$c(t) = \sum_{k=1}^{d} \alpha^k(t)\beta^k(t)^* = \exp(tK)L\exp(-tK).$$

Since we have

$$\begin{aligned}\frac{d}{dt}(\exp(tK)\exp(t(\lambda L - K)) &= \lambda \exp(tK)L\exp(t(\lambda L - K)) \\ &= \lambda c(t)\exp(tK)\exp(t(\lambda L - K)),\end{aligned}$$

the uniqueness of the solution to the ordinary equation implies our assertion. □

*Example* 1. Let $d = 1$ and $N = 1$. Also, let

$$K = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \text{ and } \tilde{\alpha} = \begin{pmatrix} -2T \\ 1 \end{pmatrix}.$$

Then we see that $a_1(t) = 2(t - T)$ and $b_1(t) = 1$. So we have

$$X = \int_0^T (\int_0^t b_1(s)dw^1(s))a_1(t)dw^1(t) = -\int_0^T w^1(t)^2 dt + \frac{T^2}{2}.$$

In this case

$$L = \begin{pmatrix} 2T & -1 \\ 4T^2 & -2T \end{pmatrix}.$$

*Example* 2. Let $d = 2$ and $N = 2$. Also, let $K = 0$ as an $4 \times 4$ matrix,

$$\alpha^1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \ \alpha^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$a_1^1(t) = 0, \quad a_2^1(t) = -1, \quad a_1^2(t) = 1, \quad a_2^2(t) = 0$$

and

$$b_1^1(t) = 0, \quad b_2^1(t) = 1, \quad b_1^2(t) = 1, \quad b_2^2(t) = 0.$$

So

$$\begin{aligned} X &= \sum_{i=1}^2 \sum_{k,\ell=1}^2 \int_0^T (\int_0^t b_i^\ell(s)dw^\ell(s))a_i^k(t)dw^i(t) \\ &= \int_0^T w_2(t)dw_1(t) - \int_0^T w_1(t)dw_2(t). \end{aligned}$$

In this case

$$L = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

*Example* 3. Let $d = 2$ and $N \geqq 1$. Let $K_{ij}$, $i, j = 1, 2$ be $N \times N$-matrices given by

$$(K_{11})_{ij} = \begin{cases} -1, & \text{if } i = j + 1, \ j = 1, \ldots, N - 1, \\ 0, & \text{otherwise}, \end{cases}$$

$(K_{22})_{ij} = -(K_{11})_{ji}$, and $K_{12} = K_{21} = 0$. Let $K$ be $2N \times 2N$-matrices given by

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Also, let $\tilde{\alpha}^k \in \mathbf{R}^{2N}$, $k = 1, 2$, be given by

$$\tilde{\alpha}^1_i = \begin{cases} 1, & i = 1, \\ 0, & i = 2, \ldots, 2N, \end{cases}$$

and

$$\tilde{\alpha}^2_i = \begin{cases} 0, & i = 1, \ldots, 2N-1, \\ 1, & i = 2N. \end{cases}$$

Then we see that

$$a^1_i(t) = \frac{(-t)^{i-1}}{(i-1)!}, \qquad b^2_i(t) = \frac{t^{N-i}}{(N-i)!}, \qquad i = 1, \ldots, N,$$

and $a^2(t) = b^1(t) = 0$. So we have

$$\begin{aligned} X &= \sum_{i=1}^N \int_0^T (\int_0^t \frac{s^{N-i}}{(N-i)!} dw^2(s)) \frac{(-t)^{i-1}}{(i-1)!} dw^1(t) \\ &= \frac{(-1)^{N-1}}{(N-1)!} \int_0^T (\int_0^t (t-s)^{N-1} dw^2(s)) dw^1(t). \end{aligned}$$

In this case, $L = (L_{ij})_{i,j=1,\ldots,2N}$ is given by

$$L_{ij} = \begin{cases} 1, & \text{if } i = 1, j = N+1, \\ -1, & \text{if } i = 2N, j = N, \\ 0, & \text{otherwise.} \end{cases}$$

## References


[1] Aonghusa, P. M. and J. V. Pulé, An extension of Lévy's stochastic area formula, Stochastics Rep. **26** (1989), 247–255.

[2] Dunford, N. and J. T. Schwartz, *Linear Operators, Part II*, Interscience, New York, 1963.

[3] Helmes, K. and A. Schwane, Lévy's stochastic area formula in high dimensions, JJ. Funct. Anal. **54** (1983), 177–192.

[4] Ikeda, N., Kusuoka, S. and S. Manabe, Lévy's stochastic area formula for Gaussian processes, Comm. Pure Appl. Math. **47** (1994), 329–360.

[5] Ikeda, N., Kusuoka, S. and S. Manabe, Levy's stochastic area formula and related problems. in Stochastic analysis (Ithaca, NY, 1993), pp. 281–305, Proc. Sympos. Pure Math., **57**, Amer. Math. Soc., Providence, RI, 1995.

[6] Lévy, P., Wiener's random function and other Laplacian random functions, pp. 171–186, in Proceedings 2nd Berkley Symp. Math. Stat. Prob. Volume II, 1950.



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