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On the Stability of Pulled Back Parabolic Vector Bundles

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Abstract. Take an irreducible smooth projective curve X defined over an algebraically closed field of characteristic zero, and fix finitely many distinct point $D = \{x_1, \cdots, x_n\}$ of it; for each point $x \in D$ fix a positive integer N_x. Take a nonconstant map $f : Y \longrightarrow X$ from an irreducible smooth projective curve. We construct a natural subbundle $\mathcal{F} \subset f_*\mathcal{O}_Y$ using $(D, \{N_x\}_{x\in D})$. Let E_* be a stable parabolic vector bundle whose parabolic weights at each $x \in D$ are integral multiples of $\frac{1}{N_x}$. We prove that the pullback f^*E_* is also parabolic stable, if $\text{rank}(\mathcal{F}) = 1$.

1. Introduction

We begin by recalling the main result of [BP]. Let $f : Y \longrightarrow X$ be a surjective separable morphism between irreducible smooth projective curves defined over an algebraically closed field k. It is called genuinely ramified if the rank of the maximal semistable subbundle $F \subset f_*\mathcal{O}_Y$ is one. The main result of [BP] says that the pullback f^*E of every stable vector bundle E on X is also stable, provided f is genuinely ramified.

Our aim here is to prove an analogue of it for parabolic vector bundles, but under an extra assumption that the characteristic of the base field k is zero.

Let X be an irreducible smooth projective curve defined over an algebraically closed field k of characteristic zero. Fix finitely many points $D = \{x_1, \cdots, x_n\} \subset X$, and for each $x \in D$ fix a positive integer N_x. We consider the category of parabolic vector bundles E_* on X with parabolic divisor D such that all the parabolic weights of E_* at any $x \in D$ are integral multiples of $\frac{1}{N_x}$.

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Let $f : Y \longrightarrow X$ be a nonconstant morphism from an irreducible smooth projective curve Y. Using the above data $(D, \{N_x\}_{x\in D})$ we construct a natural subbundle

$$\mathcal{F} \subset f_*\mathcal{O}_Y . \tag{1.1}$$

This $\mathcal{F}$ is an analogue of the above mentioned maximal semistable subbundle $F \subset f_*\mathcal{O}_Y$ in the context of parabolic bundles. It should be clarified that $f_*\mathcal{O}_Y$ is parabolic polystable (see Proposition 2.1), so $\mathcal{F}$ is not related to the Harder–Narasimhan filtration or the socle filtration of $f_*\mathcal{O}_Y$.

We prove the following (see Theorem 5.1):

THEOREM 1.1. *Let*

$$f : Y \longrightarrow X$$

be a nonconstant map between irreducible smooth projective curves defined over an algebraically closed field k of characteristic zero. Take any stable parabolic vector bundle E_* on X with parabolic structure over D such that all the parabolic weights of E_* at each point $x \in D$ are integral multiples of $\frac{1}{N_x}$. If the rank of the vector bundle $\mathcal{F} \longrightarrow X$ in (1.1) *is one, then the parabolic vector bundle f^*E_* on Y is also stable.*

We also prove the following converse of Theorem 1.1 (see Lemma 4.3):

LEMMA 1.2. *Let*

$$f : Y \longrightarrow X$$

be a nonconstant map between irreducible smooth projective curves defined over an algebraically closed field k of characteristic zero. Assume that the rank of the holomorphic vector bundle $\mathcal{F} \longrightarrow X$ in (1.1) *is at least two. Then there is a stable parabolic vector bundle E_* on X with parabolic structure over D such that*

(1) *all the parabolic weights of E_* at each $x \in D$ are integral multiples of $\frac{1}{N_x}$, and*

(2) *the parabolic vector bundle f^*E_* on Y is not stable.*

Theorem 1.1 is first proved for $k = \mathbb{C}$; see Theorem 4.2. Then theorem 1.1 is deduced using Theorem 4.2.

2. Direct Image and Parabolic Structure

Let k be an algebraically closed field of characteristic zero.

Let X be an irreducible smooth projective curve defined over k. Fix a nonempty finite subset

$$D := \{x_1, \cdots, x_n\} \subset X$$

The reduced effective divisor $x_1 + \ldots + x_n$ on X will also be denoted by D. A quasiparabolic structure on a vector bundle E on X is a filtration of subspaces of the fiber E_{x_i} of E over x_i

$$E_{x_i} = E_{i,1} \supset E_{i,2} \supset \cdots \supset E_{i,l_i} \supset E_{i,l_i+1} = 0$$

for every $1 \leq i \leq n$. A parabolic structure on E is a quasiparabolic structure as above together with a string of rational numbers

$$0 \leq \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,l_i} < 1$$

for every $1 \leq i \leq n$. The above number $\alpha_{i,j}$ is called the parabolic weight of the subspace $E_{i,j}$. (See [MS], [MY], [Bh], [IIS], [In].)

A parabolic vector bundle is a vector bundle E equipped with a parabolic structure $(\{E_{i,j}\}, \{\alpha_{i,j}\})$. For notational convenience, $(E, (\{E_{i,j}\}, \{\alpha_{i,j}\}))$ will be denoted by E_*. The divisor D is known as the parabolic divisor of E_*.

The parabolic degree of E_* is defined to be

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{i=1}^{n} \sum_{j=1}^{l_i} \alpha_{i,j} \cdot \dim(E_{i,j}/E_{i,j+1}),$$

and $\frac{\text{par-deg}(E_*)}{\text{rank}(E)} \in \mathbb{Q}$ is denoted by par-$\mu(E_*)$.

Let $F \subset E$ be a subbundle. Then a parabolic structure on E produces a parabolic structure on F. The parabolic divisor for the induced parabolic structure on F is D itself. A subspace $0 \neq V \subset F_{x_i}$ appears in the quasiparabolic filtration of F_{x_i} if

$$F_{x_i} \bigcap E_{i,j+1} \neq V = F_{x_i} \bigcap E_{i,j}$$

for some $1 \leq j \leq l_i$. The parabolic weight of such a subspace V is $\alpha_{i,j}$. The vector bundle F with the induced parabolic structure will be denoted by F_*.

The parabolic bundle E_* is called *stable* (respectively, *semistable*) if

$$\text{par-}\mu(F_*) < \text{par-}\mu(E_*) \quad (\text{respectively, } \text{par-}\mu(F_*) \leq \text{par-}\mu(E_*))$$

for every subbundle $0 \neq F \subsetneq E$. The parabolic bundle E_* is called *polystable* if the following two conditions hold:

- E_* is parabolic semistable, and
- E_* is a direct sum of stable parabolic bundles.

Let Y be an irreducible smooth projective curve and

$$f : Y \longrightarrow X \tag{2.1}$$

a nonconstant morphism. Let

$$D_f := \{p_1, \cdots, p_m\} \subset X \tag{2.2}$$

be the subset over which the map f is ramified.

PROPOSITION 2.1. *The direct image $f_*\mathcal{O}_Y$ has a parabolic structure whose parabolic divisor is D_f defined in* (2.2). *This parabolic bundle given by $f_*\mathcal{O}_Y$ is parabolic polystable of parabolic degree zero.*

PROOF. We use a local model of the map f to describe the parabolic structure on $f_*\mathcal{O}_Y$. Take

- Y to be an open subset of $U \subset \mathbb{A}^1_k$ containing 0,
- X to be the image of U under the map $\mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ defined by $z \longmapsto z^d$, where d is a positive integer, and
- f to be the map $z \longmapsto z^d$.

Then the quasiparabolic filtration of the fiber $(f_*\mathcal{O}_U)_0 = (f_*\mathcal{O}_U)_{p_1}$ over $0 = p_1$ is given by the image of the fibers of the filtration of subsheaves

$$\begin{aligned} f_*\mathcal{O}_U &\supset f_*(\mathcal{O}_U(-p_1)) \supset f_*(\mathcal{O}_U(-2p_1)) \\ &\supset \cdots \supset f_*(\mathcal{O}_U(-(d-1)p_1)) \supset f_*(\mathcal{O}_U(-dp_1)), \end{aligned}$$

and the parabolic weight of the image of the fiber $(f_*(\mathcal{O}_U(-kp_1)))_{p_1}$ in $(f_*\mathcal{O}_U)_{p_1}$ is $\frac{k}{d}$. Note that the image of the fiber $(f_*(\mathcal{O}_U(-dp_1)))_{p_1}$ in $(f_*\mathcal{O}_U)_{p_1}$ is zero, because by the projection formula we have $f_*(\mathcal{O}_U(-dp_1)) = (f_*\mathcal{O}_U) \otimes \mathcal{O}_{f(U)}(-p_1)$.

Now consider the map f in (2.1). For each $z_i \in D_f$ (see (2.2)), let $\{y_i^1, \cdots, y_i^{b_i}\}$ be the reduced inverse image $f^{-1}(z_i)_{\text{red}}$. For $1 \leq j \leq b_i$, let U_i^j be the formal completion of y_i^j in Y. The restriction of f to U_i^j will be denoted by f_i^j. Now we have

$$(2.3) \qquad (f_*\mathcal{O}_Y)_{z_i} = \bigoplus_{j=1}^{b_i} \left((f_i^j)_*\mathcal{O}_{U_i^j}\right)_{z_i} .$$

Each direct summand $\left((f_i^j)_*\mathcal{O}_{U_i^j}\right)_{z_i}$ of the fiber $(f_*\mathcal{O}_Y)_{z_i}$ in (2.3) has a parabolic structure which is described above. The parabolic structure on $(f_*\mathcal{O}_X)_{z_i}$ is given by the direct sum of the parabolic structures on the direct summands in (2.3).

We will give an alternative description of the parabolic structure on $f_*\mathcal{O}_Y$. Let Z be an irreducible smooth projective curve and

$$(2.4) \qquad \phi : Z \longrightarrow Y$$

a nonconstant morphism, such that $f \circ \phi : Z \longrightarrow X$ is a (ramified) Galois covering. Let $\Gamma = \text{Gal}(f \circ \phi) \subset \text{Aut}(Z)$ be the Galois group.

Let $k[\Gamma]$ denote the algebra of functions on the finite group Γ. The left-translation action of Γ on itself produces an action of Γ on $k[\Gamma]$. On the other hand, the group $\Gamma \subset \text{Aut}(Z)$ has a tautological action on Z. Consider the diagonal action of Γ on $Z \times k[\Gamma]$. This action makes the trivial vector bundle

$$(2.5) \qquad Z \times k[\Gamma] \longrightarrow Z$$

a Γ–equivariant vector bundle on Z. Using the natural correspondence between equivariant bundles and parabolic bundles (see [Bis1], [Bo1], [Bo2]), this Γ–equivariant vector bundle $Z \times k[\Gamma]$ on Z produces a parabolic vector bundle on $Z/\Gamma = X$. This parabolic vector bundle on X will be denoted by W_*. The parabolic divisor par-div(W_*) for W_* is the subset of X over which the map $f \circ \phi$ is ramified, where ϕ is the map in (2.4). Note that $D_f \subset$ par-div(W_*), and par-div(W_*) may be larger than D_f.

The vector bundle underlying the parabolic vector bundle W_* is $(f \circ \phi)_*\mathcal{O}_Z$ [Bis2], [Par]. On the other hand, we have

$$f_*\mathcal{O}_Y \subset (f \circ \phi)_*\mathcal{O}_Z\,.$$

In fact, $f_*\mathcal{O}_Y$ is a subbundle of $(f \circ \phi)_*\mathcal{O}_Z$.

Hence the parabolic structure of W_* induces a parabolic structure on $f_*\mathcal{O}_Y$. Let

$$(f_*\mathcal{O}_Y)_* \longrightarrow X$$

denote the parabolic vector bundle with parabolic structure on $f_*\mathcal{O}_Y$ induced by W_*.

The parabolic structure on $(f_*\mathcal{O}_Y)_*$ over the complement par-div$(W_*) \setminus D_f$ is the trivial one, meaning $(f_*\mathcal{O}_Y)_*$ does not have any nonzero parabolic weight on the points of par-div$(W_*) \setminus D_f$.

It is straight-forward to check that $(f_*\mathcal{O}_Y)_*$ coincides with the parabolic bundle given by the parabolic structure on $f_*\mathcal{O}_Y$ constructed earlier. In particular, the above parabolic structure on $(f_*\mathcal{O}_Y)_*$ does not depend on the choice of the pair (Z, ϕ).

Since the vector bundle underlying the Γ–equivariant vector bundle in (2.5) is polystable (it is in fact trivial), the corresponding parabolic bundle $(f_*\mathcal{O}_Y)_*$ is polystable [BBN, p. 350–351, Theorem 4.3]. Since the degree of the Γ–equivariant vector bundle in (2.5) zero, it follows that the parabolic degree of $(f_*\mathcal{O}_Y)_*$ is zero [Bis1, p. 318, (3.12)]. □

We refer the reader to [Yo] for the definition of parabolic dual of parabolic vector bundles.

LEMMA 2.2. *The parabolic dual of the parabolic vector bundle $(f_*\mathcal{O}_Y)_*$, constructed in the proof of Proposition 2.1, is $(f_*\mathcal{O}_Y)_*$ itself.*

PROOF. If E_* is the parabolic vector bundle corresponding to an equivariant bundle V, then the parabolic vector bundle corresponding to the equivariant bundle V^* is the parabolic dual E_*^* of E_* [BBN].

Let

$$k[\Gamma] \otimes k[\Gamma] \longrightarrow k \tag{2.6}$$

be the pairing defined by

$$\left(\sum_{\gamma\in\Gamma} a_\gamma \gamma,\, \sum_{\gamma\in\Gamma} b_\gamma \gamma\right) \longmapsto \sum_{\gamma\in\Gamma} a_\gamma b_\gamma\,,$$

where a_γ, $b_\gamma \in k$. Consider the Γ–equivariant vector bundle $Z\times k[\Gamma] \longrightarrow Z$ in (2.5). The paring in (2.6) defines a homomorphism of coherent sheaves

$$(Z \times k[\Gamma]) \otimes (Z \times k[\Gamma]) \longrightarrow \mathcal{O}_Z$$

which is fiberwise nondegenerate. The resulting isomorphism of vector bundles

$$Z \times k[\Gamma] \xrightarrow{\sim} Z \times k[\Gamma]^* = (Z \times k[\Gamma])^*$$

is in fact Γ–equivariant. Therefore, we conclude that the parabolic dual of $(f_*\mathcal{O}_Y)_*$ is $(f_*\mathcal{O}_Y)_*$ itself. □

3. Construction of a Parabolic Subbundle

Let $V_* = (V, (\{V_{i,j}\}, \{\alpha_{i,j}\}))$ be a semistable parabolic bundle on X with parabolic divisor

$$\mathbb{D} := \{t_1, \cdots, t_r\} \subset X\,.$$

For any subbundle $F \subset V$, the parabolic vector bundle defined by F equipped with the parabolic structure induced by V_* will be denoted by F_*.

For each parabolic point $t \in \mathbb{D}$, we fix an integer $N_t \geq 1$. Assume that there is a subbundle

$$F \subset V$$

satisfying the following two conditions:

(1) All the parabolic weights of F_* at every $t \in \mathbb{D}$ are integral multiples of $\frac{1}{N_t}$. (If $N_t = 1$, then F_* does not have any nonzero parabolic weight at t.)

(2) par-$\mu(F_*) =$ par-$\mu(V_*)$.

Lemma 3.1. *There is a unique maximal subbundle*

$$\mathcal{F} \subset V$$

satisfying the following two conditions:

(1) *All the parabolic weights of $\mathcal{F}_*$ at every $t \in \mathbb{D}$ are integral multiples of $\frac{1}{N_t}$, and*

(2) $\text{par-}\mu(\mathcal{F}_*) = \text{par-}\mu(V_*)$.

Proof. Let F^1 and F^2 be two subbundles of V such that for $1 \leq j \leq 2$,

(1) all the parabolic weights of F^j_* at every $t \in \mathbb{D}$ are integral multiples of $\frac{1}{N_t}$, and

(2) $\text{par-}\mu(F^j_*) = \text{par-}\mu(V_*)$.

Since V_* is parabolic semistable, and $\text{par-}\mu(F^j_*) = \text{par-}\mu(V_*)$, it follows immediately that F^j_* is semistable for $j = 1, 2$. Consider the subsheaf $F^1 + F^2 \subset V$ equipped with the parabolic structure induced by the parabolic structure of V_*; the resulting parabolic bundle will be denoted by $(F^1+F^2)_*$. So

$$\text{par-}\mu((F^1 + F^2)_*) \leq \text{par-}\mu(V_*) \tag{3.1}$$

because V_* is parabolic semistable. On the other hand, $(F^1 + F^2)_*$ is a quotient of the direct sum $F^1_* \oplus F^2_*$, and $F^1_* \oplus F^2_*$ is parabolic semistable with

$$\text{par-}\mu((F^1 \oplus F^2)_*) = \text{par-}\mu(F^1_*) = \text{par-}\mu(F^2_*) = \text{par-}\mu(V_*)\,.$$

Hence we have

$$\text{par-}\mu((F^1 + F^2)_*) \geq \text{par-}\mu(V_*)\,.$$

Combining this with (3.1) we conclude that

$$\text{par-}\mu((F^1 + F^2)_*) = \text{par-}\mu(V_*). \tag{3.2}$$

We will show that $V/(F^1 + F^2)$ is torsionfree. To prove this, if T_0 is the torsion part of $V/(F^1 + F^2)$, consider $\mathbb{S} = q_0^{-1}(T_0)$, where

$q_0 := V \longrightarrow V/(F^1+F^2)$ is the quotient map. Let $\mathbb{S}_*$ denote the parabolic vector bundle given by $\mathbb{S}$ equipped with the parabolic structure induced by the parabolic structure of V_*. If $T_0 \neq 0$, then

$$\text{par-}\mu(\mathbb{S}_*) > \text{par-}\mu((F^1+F^2)_*) = \text{par-}\mu(V_*)$$

(see (3.2)). But this contradicts the given condition that V_* is parabolic semistable. Therefore, we conclude that $V/(F^1+F^2)$ is torsionfree. In other words,

$$F^3 := F^1 + F^2 \subset V$$

is a subbundle.

Consider the parabolic vector bundle F^3_* defined by F^3 equipped with the parabolic structure induced by the parabolic structure of V_*. Recall that for $1 \leq j \leq 2$, all the parabolic weights of F^j_* at every $t \in \mathbb{D}$ are integral multiples of $\frac{1}{N_t}$. This immediately implies that all the parabolic weights of F^3_* at each $t \in \mathbb{D}$ are also integral multiples of $\frac{1}{N_t}$.

In (3.2) we have seen that $\text{par-}\mu(F^3_*) = \text{par-}\mu(V_*)$.

Now take $\mathcal{F}$ to be the coherent subsheaf of V generated by all subbundles

$$F \subset V$$

such that

(1) all the parabolic weights of F_* at each $t \in \mathbb{D}$ are integral multiples of $\frac{1}{N_t}$, and

(2) $\text{par-}\mu(F_*) = \text{par-}\mu(V_*)$.

From the above observations on F^3_* it follows immediately that this coherent subsheaf $\mathcal{F} \subset V$ satisfies all the conditions in the statement of the lemma. □

As in (2.1), take any irreducible smooth projective curve Y together with a nonconstant morphism

$$f : Y \longrightarrow X .$$

As in (2.2), $D_f := \{p_1, \cdots, p_m\} \subset X$ denotes the subset over which f is ramified. Fix a divisor

$$D := \{x_1, \cdots, x_n\} \subset X . \tag{3.3}$$

Also, fix an integer

$$N_x \geq 1 \tag{3.4}$$

for each point $x \in D$.

PROPOSITION 3.2. *Let $(f_*\mathcal{O}_Y)_*$ be the parabolic bundle defined by $f_*\mathcal{O}_Y$ equipped with the natural parabolic structure (see Proposition 2.1). Then there is a unique maximal subbundle*

$$\mathcal{F} \subset f_*\mathcal{O}_Y$$

satisfying the following three conditions:

(1) *For any $x \in D \bigcap D_f$, all the parabolic weights of $\mathcal{F}_*$ at x are integral multiples of $\frac{1}{N_x}$ (see (3.4)),*

(2) *$\mathcal{F}_*$ does not have any nonzero parabolic weight over any point of $D_f \setminus (D \bigcap D_f)$, and*

(3) par-deg$(\mathcal{F}_*) = 0$.

PROOF. In Lemma 3.1, set

- $V_* = (f_*\mathcal{O}_Y)_*$, (so the parabolic divisor $\mathbb{D}$ in Lemma 3.1 is now D_f),
- $N_x = 1$ if $x \in D_f \setminus (D \bigcap D_f)$ (see (3.3) for D), and
- $N_x = N_x$ (see (3.4)) if $x \in D \bigcap D_f$.

Recall from Proposition 2.1 that par-deg$((f_*\mathcal{O}_Y)_*) = 0$. So we have

$$\text{par-}\mu((f_*\mathcal{O}_Y)_*) = 0.$$

Therefore, in view of Lemma 3.1 it suffices to show that there is a subbundle

$$F \subset f_*\mathcal{O}_Y$$

satisfying the following two conditions:

(1) All the parabolic weights of F_* at each $x \in D_f$ are integral multiples of $\frac{1}{N_x}$, and

(2) par-deg$(F_*) = 0$.

Since

$$\begin{aligned} H^0(Y, \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y)) &= H^0(Y, \mathrm{Hom}(f^*\mathcal{O}_X, \mathcal{O}_Y)) \\ &= H^0(X, \mathrm{Hom}(\mathcal{O}_X, f_*\mathcal{O}_Y)) \end{aligned}$$

(see [Ha, p. 110]), the identity map of $\mathcal{O}_Y$ produces a nonzero homomorphism

$$\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y .$$

This coherent subsheaf is actually a subbundle. Indeed, this follows immediately from the fact that for any $\psi \in H^0(U, \mathcal{O}_U)$, where $U \subset X$ is a Zariski open subset, the section $\psi \circ f \in H^0(f^{-1}(U), \mathcal{O}_{f^{-1}(U)})$ has the property that if $\psi(x) \neq 0$, then $\psi \circ f$ does not vanish on any point of $f^{-1}(x)$.

From the construction of the parabolic structure on $f_*\mathcal{O}_Y$ in Proposition 2.1 it follows immediately that the induced parabolic weight on $\mathcal{O}_X$ at any $x \in D_f$ is zero. Consequently, $F = \mathcal{O}_X$ satisfies the above two conditions. This proves the proposition. □

Equip the curve X with the following orbifold structure: For each point $x \in D$ the inertia group is $\mathbb{Z}/N_x\mathbb{Z}$, where N_x is the integer in (3.4). The curve X equipped with this orbifold structure will be denoted by $\mathcal{X}$. An *étale covering*

$$\varphi : Z \longrightarrow \mathcal{X} \tag{3.5}$$

is an irreducible smooth projective curve Z together with a nonconstant morphism

$$\varphi_0 : Z \longrightarrow X \tag{3.6}$$

such that the following conditions hold:

- the map φ_0 is unramified over $X \setminus D = X \setminus \{x_1, \cdots, x_n\}$, and
- for every $x \in D$, the order of ramification of φ_0 at each $z \in \varphi_0^{-1}(x)$ is a divisor of N_x.

An étale covering φ of $\mathcal{X}$ will be called *nontrivial* if degree$(\varphi_0) \geq 2$.

THEOREM 3.3. *Consider the map $f : Y \longrightarrow X$, and the corresponding subbundle*

$$\mathcal{F} \subset f_*\mathcal{O}_Y$$

in Proposition 3.2. Then the following two statements are equivalent:

(1) *There is a nontrivial étale covering*

$$\varphi : Z \longrightarrow \mathcal{X}$$

(see (3.5) *and* (3.6)*) and a morphism $\beta : Y \longrightarrow Z$, such that $\varphi_0 \circ \beta = f$.*

(2) *The rank of $\mathcal{F}$ is bigger than one.*

PROOF. First assume that there is a nontrivial étale covering

$$\varphi : Z \longrightarrow \mathcal{X},$$

and a morphism $\beta : Y \longrightarrow Z$, such that $\varphi_0 \circ \beta = f$. Consider the subbundle

$$(\varphi_0)_*\mathcal{O}_Z \subset f_*\mathcal{O}_Y . \tag{3.7}$$

The parabolic structure on $(\varphi_0)_*\mathcal{O}_Z$ constructed in Proposition 2.1 coincides with the one induced by the parabolic structure of $f_*\mathcal{O}_Y$ on the subbundle in (3.7). The parabolic bundle defined by this parabolic structure on $(\varphi_0)_*\mathcal{O}_Z$ will be denoted by $((\varphi_0)_*\mathcal{O}_Z)_*$.

Using the given condition that φ is an étale covering of $\mathcal{X}$ it is straightforward to verify that for every point $x \in D$, all the parabolic weights of $((\varphi_0)_*\mathcal{O}_Z)_*$ at x are integral multiples of $\frac{1}{N_x}$. Since φ_0 is unramified over the complement $X \setminus D$, the parabolic bundle $((\varphi_0)_*\mathcal{O}_Z)_*$ does not have any nonzero parabolic weights on $X \setminus D$. Also, from Proposition 2.1 we know that

$$\text{par-deg}(((\varphi_0)_*\mathcal{O}_Z)_*) = 0.$$

In view of these, from the uniqueness property of $\mathcal{F}$ in Proposition 3.2 we know that

$$(\varphi_0)_*\mathcal{O}_Z \subset \mathcal{F} . \tag{3.8}$$

Since degree$(\varphi_0) \geq 2$, from (3.8) we conclude that

$$\mathrm{rank}(\mathcal{F}) \geq 2\,.$$

To prove the converse, assume that

$$\mathrm{rank}(\mathcal{F}) \geq 2\,. \tag{3.9}$$

As before, $\mathcal{F}_*$ denotes the parabolic bundle defined by $\mathcal{F}$ equipped with the parabolic structure induced by the parabolic structure of $f_*\mathcal{O}_Y$.

The algebra structure of $\mathcal{O}_Y$ produces an algebra structure

$$\Phi \,:\, (f_*\mathcal{O}_Y)_* \otimes (f_*\mathcal{O}_Y)_* \longrightarrow (f_*\mathcal{O}_Y)_* \tag{3.10}$$

(see [Yo], [BBN] for the tensor product of parabolic vector bundles).

Since the parabolic weights of $\mathcal{F}_*$ at every $x \in D$ are integral multiples of $\frac{1}{N_x}$, we conclude that the parabolic weights of $\mathcal{F}_* \otimes \mathcal{F}_*$ at $x \in D$ are also integral multiples of $\frac{1}{N_x}$. From the given condition that par-deg$(\mathcal{F}_*) = 0$ it follows immediately that

$$\text{par-deg}(\mathcal{F}_* \otimes \mathcal{F}_*) = 0\,.$$

We note that $\mathcal{F}_* \otimes \mathcal{F}_*$ is parabolic semistable because $\mathcal{F}_*$ is so [BBN, p. 346, Proposition 3.2]. Let $\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_*$ be the parabolic vector bundle defined by $\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)$ equipped with the induced parabolic structure, where Φ is the homomorphism in (3.10). Since $\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_* \subset (f_*\mathcal{O}_Y)_*$ is a quotient parabolic bundle of the semistable parabolic bundle $\mathcal{F}_* \otimes \mathcal{F}_*$, we have

$$\text{par-}\mu(\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_*) \geq \text{par-}\mu(\mathcal{F}_* \otimes \mathcal{F}_*) = 0\,.$$

On the other hand,

$$\text{par-}\mu(\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_*) \leq \text{par-}\mu((f_*\mathcal{O}_Y)_*) = 0\,,$$

because $(f_*\mathcal{O}_Y)_*$ is polystable. Combining these, we have

$$\text{par-}\mu(\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_*) = 0.$$

Since $\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_*$ is a quotient of $\mathcal{F}_* \otimes \mathcal{F}_*$, and all the parabolic weights of $\mathcal{F}_* \otimes \mathcal{F}_*$ at every $x \in D$ are integral multiples of $\frac{1}{N_x}$, it follows that all the parabolic weights of $\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_*$ at every $x \in D$ are also integral multiples of $\frac{1}{N_x}$.

Therefore, from the uniqueness property of $\mathcal{F}$ we conclude that

$$\Phi(\mathcal{F}_* \otimes \mathcal{F}_*)_* \subset \mathcal{F}_* . \tag{3.11}$$

From (3.11) it follows that there is a unique étale covering

$$\varphi : Z \longrightarrow \mathcal{X} , \tag{3.12}$$

where $\mathcal{X}$ is the orbifold in (3.5), and a morphism $\beta : Y \longrightarrow Z$, such that following two hold:

(1) $\varphi_0 \circ \beta = f$, and

(2) the two subsheaves $(\varphi_0)_* \mathcal{O}_Z$ and $\mathcal{F}$ of $f_* \mathcal{O}_Y$ coincide.

From (3.9) and the above statement (2) we know that the étale covering φ in (3.12) is nontrivial. This completes the proof. □

Consider the set-up of Theorem 3.3. Let

$$Y' := Y \setminus f^{-1}(D) \subset Y$$

be the complement. Let

$$f' := f\big|_{Y'} : Y' \longrightarrow \mathcal{X} \tag{3.13}$$

be the restriction of f to Y'. The étale fundamental groups of Y' and $\mathcal{X}$ will be denoted by $\pi_1(Y')$ and $\pi_1(\mathcal{X})$ respectively. The following two statements are evidently equivalent:

(1) There is a nontrivial étale covering

$$\varphi : Z \longrightarrow \mathcal{X}$$

(see (3.5) and (3.6)) and a morphism $\beta : Y \longrightarrow Z$, such that $\varphi_0 \circ \beta = f$.

(2) The homomorphism of étale fundamental groups

$$(f'_{\text{et}})_* : \pi_1(Y') \longrightarrow \pi_1(\mathcal{X}) \tag{3.14}$$

induced by f' in (3.13) is not surjective.

Therefore, Theorem 3.3 gives the following:

COROLLARY 3.4. *Consider the map $f : Y \longrightarrow X$, and the corresponding subbundle*

$$\mathcal{F} \subset f_*\mathcal{O}_Y$$

in Proposition 3.2. Then the following two statements are equivalent:

(1) *The homomorphism of étale fundamental groups $(f'_{\text{et}})_*$ in* (3.14) *is not surjective.*

(2) *The rank of $\mathcal{F}$ is bigger than one.*

4. Complex Curves and Pullback of Stable Parabolic Bundles

Throughout this section we assume that $k = \mathbb{C}$. The topological fundamental group of any complex manifold or orbifold $\mathbf{N}$ will be denoted by $\pi_1^t(\mathbf{N})$; this is to distinguish it from the étale fundamental group of $\mathbf{N}$.

4.1. Homomorphism of topological fundamental groups

As before, $f : Y \longrightarrow X$ is a nonconstant holomorphic map between irreducible complex projective curves; the map f is ramified exactly over

$$D_f := \{p_1, \cdots, p_m\} \subset X .$$

Fix an integer $N_x \geq 1$ for each $x \in D$, and the resulting orbifold is denoted by $\mathcal{X}$. The curve Y' and the map f' are both as in (3.13).

PROPOSITION 4.1. *The following two statements are equivalent:*

(1) *the homomorphism of topological fundamental groups*

$$f'_* : \pi_1^t(Y') \longrightarrow \pi_1^t(\mathcal{X}) \tag{4.1}$$

induced by f' in (3.13) *is surjective.*

(2) *The rank of the holomorphic vector bundle $\mathcal{F} \longrightarrow X$ in Proposition 3.2 is one.*

Proof. First assume that the homomorphism

$$f'_* : \pi_1^t(Y') \longrightarrow \pi_1^t(\mathcal{X})$$

is surjective. The group $\pi_1(Y')$ (respectively, $\pi_1(\mathcal{X})$) is the profinite completion of $\pi_1^t(Y')$ (respectively, $\pi_1^t(\mathcal{X})$). Therefore, from the surjectivity of the above homomorphism f'_* it follows immediately that the homomorphism $(f'_{\mathrm{et}})_*$ in (3.14) is surjective. Now Corollary 3.4 says that

$$\mathrm{rank}(\mathcal{F}) = 1,$$

where $\mathcal{F} \longrightarrow X$ is the holomorphic vector bundle in Proposition 3.2.

To prove the converse, assume that

$$\mathrm{rank}(\mathcal{F}) = 1. \tag{4.2}$$

In view of (4.2), from Corollary 3.4 it follows that the homomorphism $(f'_{\mathrm{et}})_*$ in (3.14) is surjective. From the surjectivity of $(f'_{\mathrm{et}})_*$ it can be deduced that the homomorphism of topological fundamental groups

$$f'_* : \pi_1^t(Y') \longrightarrow \pi_1^t(\mathcal{X}) \tag{4.3}$$

induced by f' in (3.13) is surjective. To see this, first note that $\pi_1^t(\mathcal{X})$ is residually finite and $\pi_1^t(Y')$ is finitely generated as they are both surface groups. Now a result of Peter Scott, [Sc, p. 555, Theorem 3.3], says that for any finitely generated subgroup H of $\pi_1^t(\mathcal{X})$, and any $t \in \pi_1^t(\mathcal{X}) \setminus H$, there is a finite index subgroup

$$\widetilde{H} \subset \pi_1^t(\mathcal{X})$$

such that

$$t \notin \widetilde{H} \supset H$$

(see [Pat, p. 2892 Theorem 1.2] for an effective version of the theorem of Scott). Applying this to the image $f'_*(\pi_1^t(Y')) \subset \pi_1^t(\mathcal{X})$ we conclude that if f'_* is not surjective then the image $f'_*(\pi_1^t(Y')) \subset \pi_1^t(\mathcal{X})$ is contained in a proper subgroup

$$\Gamma \subsetneq \pi_1^t(\mathcal{X}) \tag{4.4}$$

of finite index.

Consider the finite étale covering

$$\varphi \,:\, Z \longrightarrow \mathcal{X}$$

given by the subgroup Γ in (4.4). Since $f'_*(\pi_1^t(Y')) \subset \Gamma$, and a morphism $\beta : Y \longrightarrow Z$, such that $\varphi_0 \circ \beta = f$. But this implies that

$$(f'_{\text{et}})_*(\pi_1(Y')) \subset \pi_1(Z) \subsetneq \pi_1(\mathcal{X}),$$

where $(f'_{\text{et}})_*$ is the homomorphism in (3.14). But this contradicts the fact that the homomorphism $(f'_{\text{et}})_*$ is surjective. Therefore, the homomorphism f'_* in (4.3) is surjective. □

4.2. Pullback of parabolic bundles

Let $E_* = (E, (\{E_{i,j}\}, \{\alpha_{i,j}\}))$ be a parabolic vector bundle with parabolic divisor $D = \{x_1, \cdots, x_n\}$. Take a nonconstant holomorphic map

$$f : Y \longrightarrow X$$

from an irreducible complex projective curve Y. Then, using f, the parabolic bundle E_* pulls back to a parabolic bundle f^*E_* on Y. We will briefly recall the construction of the parabolic bundle f^*E_*.

We first consider the case where $\text{rank}(E) = 1$. So for each $x_i \in D$ the parabolic weight of E_* is $\alpha_{i,1} = \alpha_i$. The parabolic divisor for f^*E_* is the reduced effective divisor $f^{-1}(D)_{\text{red}}$. For $1 \leq i \leq n$, let

$$f^{-1}(x_i) = \{y_{i,1}, \cdots, y_{i,b_i}\} \subset Y$$

be the inverse image, and let $m_{i,j}$ be the multiplicity of f at $y_{i,j}$ for every $1 \leq j \leq b_i$. For any $\lambda \in \mathbb{Q}$, let $\lfloor \lambda \rfloor$ be the integral part of λ, so $0 \leq \lambda - \lfloor \lambda \rfloor < 1$.

The holomorphic line bundle on Y underlying the parabolic line bundle f^*E_* is

$$F := (f^*E) \otimes \mathcal{O}_Y(\sum_{i=1}^{n} \sum_{j=1}^{b_i} \lfloor m_{i,j}\alpha_i \rfloor \cdot y_{i,j}),$$

and the parabolic weight of $F_{y_{i,j}}$ is $m_{i,j}\alpha_i - \lfloor m_{i,j}\alpha_i \rfloor$. Note that

$$\text{par-deg}(f^*E_*) = \text{degree}(f) \cdot \text{par-deg}(E_*)\,. \tag{4.5}$$

Any parabolic vector bundle E_* can locally be expressed as a direct sum of parabolic line bundles. In other words, X can be covered by Zariski open subsets $U_1, \cdots, U_m$ such that $E_*\big|_{U_j}$ is a direct sum of parabolic line bundles on U_j for all $1 \leq j \leq m$. We have described above the pullback of parabolic line bundles. The pullback of a direct sum of parabolic line bundles is the direct sum of the pulled back parabolic line bundles. Using the decomposition of $E_*\big|_{U_j}$ into a direct sum of parabolic line bundles we now have a description of the parabolic pullback f^*E_*. From (4.5) it follows that

$$\text{par-deg}(f^*E_*) \,=\, \text{degree}(f)\cdot\text{par-deg}(E_*) \tag{4.6}$$

for any parabolic vector bundle E_*.

For each point $x \in D$ fix an integer $N_x \geq 1$.

THEOREM 4.2. *Take any stable parabolic vector bundle E_* on X with parabolic structure over D such that all the parabolic weights of E_* at each $x \in D$ are integral multiples of $\frac{1}{N_x}$. If the rank of the holomorphic vector bundle $\mathcal{F} \longrightarrow X$ in Proposition 3.2 is one, then the parabolic vector bundle f^*E_* on Y is also stable.*

PROOF. Assume that

$$\text{rank}(\mathcal{F}) \,=\, 1\,. \tag{4.7}$$

Let E_* be a stable parabolic vector bundle of rank r on X with parabolic structure over D such that all the parabolic weights of E_* at each $x \in D$ are integral multiples of $\frac{1}{N_x}$. Consider the parabolic principal $\text{PGL}(r,\mathbb{C})$–bundle $\mathbb{P}(E_*)$ defined by E_*; see [BBN] for parabolic principal bundles. Since E_* is stable, we know that the parabolic principal $\text{PGL}(r,\mathbb{C})$–bundle $\mathbb{P}(E_*)$ is given by an irreducible homomorphism

$$\rho \,:\, \pi_1^t(\mathcal{X}) \,\longrightarrow\, \text{PU}(r) \tag{4.8}$$

[MS], [Biq]. Let $\mathbb{P}(f^*E_*)$ denote the parabolic principal $\text{PGL}(r,\mathbb{C})$–bundle on Y defined by the parabolic vector bundle f^*E_*. Since $\mathbb{P}(E_*)$ is given by the homomorphism ρ in (4.8), we conclude that the parabolic principal $\text{PGL}(r,\mathbb{C})$–bundle $\mathbb{P}(f^*E_*)$ is given by the homomorphism

$$\rho\circ f'_* \,:\, \pi_1^t(Y') \,\longrightarrow\, \text{PU}(r)\,, \tag{4.9}$$

where f'_* is the homomorphism in (4.1).

From (4.7) and Proposition 4.1 we know that the homomorphism f'_* in (4.9) is surjective. Therefore, from the property of the homomorphism ρ in (4.8) that it is irreducible we conclude that the homomorphism $\rho \circ f'_*$ in (4.9) is also irreducible. Since the parabolic principal $\mathrm{PGL}(r, \mathbb{C})$–bundle $\mathbb{P}(f^*E_*)$ is given by the irreducible projective unitary representation $\rho \circ f'_*$ in (4.9), we now conclude that the parabolic vector bundle f^*E_* is stable [MS], [Biq]. □

The following lemma is a converse of Theorem 4.2.

Lemma 4.3. *Assume that the rank of the holomorphic vector bundle $\mathcal{F} \longrightarrow X$ in Proposition 3.2 is at least two. Then there is a stable parabolic vector bundle E_* on X with parabolic structure over D such that*

(1) *all the parabolic weights of E_* at each $x \in D$ are integral multiples of $\frac{1}{N_x}$, and*

(2) *the parabolic vector bundle f^*E_* on Y is not stable.*

Proof. Since $\mathrm{rank}(\mathcal{F}) > 1$, from Proposition 4.1 we know that the homomorphism of topological fundamental groups

$$f'_* \,:\, \pi_1^t(Y') \longrightarrow \pi_1^t(\mathcal{X})$$

(see (4.1)) induced by f' in (3.13) is not surjective. Fix an irreducible representation

$$\rho \,:\, \pi_1^t(\mathcal{X}) \longrightarrow \mathrm{U}(r)\,,$$

for some $r \geq 2$, such that the composition of homomorphisms

$$\rho \circ f'_* \,:\, \pi_1^t(Y') \longrightarrow \mathrm{U}(r)$$

is not irreducible; such a ρ exists because f'_* is not surjective.

Let E_* be the parabolic vector bundle of rank r on X, with parabolic structure over D, given by ρ. We note that

(1) all the parabolic weights of E_* at each $x \in D$ are integral multiples of $\frac{1}{N_x}$, and

(2) the parabolic vector bundle E_* is stable, because ρ is irreducible [MS].

Since $\rho \circ f'_* \,:\, \pi_1^t(Y') \longrightarrow \mathrm{U}(r)$ is not irreducible, it follows that the parabolic vector bundle f^*E_* on Y is not stable. □

5. Algebraically Closed Fields of Characteristic Zero

Now let k be any algebraically closed fields of characteristic zero. As in Section 3, X is an irreducible smooth projective curve defined over k, and

$$D = \{x_1, \cdots, x_n\} \subset X$$

is a finite subset.

For each point $x \in D$ fix an integer $N_x \geq 1$. Consider the holomorphic vector bundle $\mathcal{F} \longrightarrow X$ in Proposition 3.2.

THEOREM 5.1. *Let*

$$f : Y \longrightarrow X$$

be a nonconstant map from an irreducible smooth projective curve Y. Take any stable parabolic vector bundle E_* on X with parabolic structure over D such that all the parabolic weights of E_* at each point $x \in D$ are integral multiples of $\frac{1}{N_x}$. If the rank of the vector bundle $\mathcal{F} \longrightarrow X$ in Proposition 3.2 is one, then the parabolic vector bundle f^*E_* on Y is also stable.

PROOF. Assume that

$$\operatorname{rank}(\mathcal{F}) = 1\,. \tag{5.1}$$

Take any stable parabolic vector bundle E_* on X with parabolic structure over D such that all the parabolic weights of E_* at each $x \in D$ are integral multiples of $\frac{1}{N_x}$. We need to show that the parabolic vector bundle f^*E_* on Y is stable.

Let $k_0 \subset k$ be an algebraically closed field of characteristic of finite transcendence degree over $\mathbb{Q}$ such that X, Y, D, f and E_* are defined over k_0. Fix an embedding of k_0 in $\mathbb{C}$. Let

$$X_{\mathbb{C}} := X \times_{k_0} \mathbb{C},\ Y_{\mathbb{C}} := Y \times_{k_0} \mathbb{C},\ f_{\mathbb{C}} := f \times_{k_0} \mathbb{C} \quad \text{and} \quad E^{\mathbb{C}}_* := E_* \otimes_{k_0} \mathbb{C} \tag{5.2}$$

be the base changes to $\mathbb{C}$ of X, Y, f and E_* respectively. Similarly, let

$$\mathcal{X}_{\mathbb{C}} := \mathcal{X} \times_{k_0} \mathbb{C},\ Y'_{\mathbb{C}} := Y' \times_{k_0} \mathbb{C} \quad \text{and} \quad f'_{\mathbb{C}} := f' \times_{k_0} \mathbb{C} \tag{5.3}$$

where Y' and f' are as in (3.13), be the base changes to $\mathbb{C}$ of $\mathcal{X}$, Y' and f' respectively.

We need the following lemma.

LEMMA 5.2. *The parabolic vector bundle* $E_*^{\mathbb{C}} = E_* \otimes_{k_0} \mathbb{C}$ *in* (5.2) *is stable.*

PROOF OF LEMMA 5.2. An equivariant vector bundle is equivariantly semistable if the underlying vector bundle is semistable, because the Harder–Narasimhan filtration of an equivariant bundle is preserved by the action of the group. It is known that the property of semistability of a vector bundle is preserved under field extensions (see [HL, p. 18, Corollary 1.3.8]). Now using the correspondence between the parabolic bundles and the equivariant bundles we conclude that for a semistable parabolic bundle V_* on X the parabolic bundle $V_* \times_{k_0} \mathbb{C}$ on $X_{\mathbb{C}}$ is also semistable. Therefore, the given condition that the parabolic bundle E_* is semistable implies that the parabolic bundle $E_*^{\mathbb{C}}$ is also semistable. Since the unique maximal polystable parabolic subbundle of the semistable parabolic bundle $E_*^{\mathbb{C}}$ (it is also known as the socle of $E_*^{\mathbb{C}}$ (see [HL, p. 23, Lemma 1.5.5]) is defined over k_0, and E_* is polystable, we conclude that the parabolic bundle $E_*^{\mathbb{C}}$ is polystable (see [HL, p. 24, Corollary 1.5.11].

For a parabolic vector bundle F_*, the sheaf of quasiparabolic structure preserving endomorphisms of the underlying vector bundle F will be denoted by $\mathrm{End}_P(F_*)$. A polystable parabolic vector bundle F_* is stable if and only if the space of global sections of $\mathrm{End}_P(F_*)$ is the base field. Since

$$\mathrm{End}_P(E_*^{\mathbb{C}}) = \mathrm{End}_P(E_*) \otimes_{k_0} \mathbb{C},$$

and E_* is stable, we have

$$H^0(X_{\mathbb{C}}, \mathrm{End}_P(E_*^{\mathbb{C}})) = H^0(X, \mathrm{End}_P(E_*)) \otimes_{k_0} \mathbb{C} = \mathbb{C}.$$

This implies that the polystable parabolic bundle $E_*^{\mathbb{C}}$ is stable. □

Continuing with the proof of Theorem 5.1, from Corollary 3.4 and (5.1) we know that the homomorphism of étale fundamental groups $(f'_{\mathrm{et}})_*$ in (3.14) is surjective. This implies that the homomorphism of étale fundamental groups

$$(f'_{\mathbb{C},\mathrm{et}})_* : \pi_1(Y'_{\mathbb{C}}) \longrightarrow \pi_1(\mathcal{X}_{\mathbb{C}})$$

induced by $f'_{\mathbb{C}}$ in (5.3) is surjective; both $Y'_{\mathbb{C}}$ and $\mathcal{X}$ are defined in (5.3). Hence from Theorem 4.2 and Lemma 5.2 we conclude that the parabolic vector bundle $f_{\mathbb{C}}^* E_*^{\mathbb{C}}$ on $Y_{\mathbb{C}}$ is stable, where $f_{\mathbb{C}}$ is the map in (5.2). This implies that the parabolic vector bundle f^*E is stable. □

Remark 5.3. We note that the main result of [BKP] implies that if the base field is algebraically closed of characteristic 0 then the conclusion to Theorem 5.1 holds under the strict condition that for every $x \in X$ the number N_x is coprime to the ramification indices of f at points above x. Theorem 5.1 is more general than this as illustrated by Example 5.4.

Example 5.4. Let $X = Y = \mathbb{P}^1_{\mathbb{C}}$ and $f : Y \longrightarrow X$ be the cyclic covering of degree 6 ramified at 0 and ∞. Let $N_0 = 2$ and $N_\infty = 3$; then the map $\pi_1(Y \setminus \{0, \infty\}) \longrightarrow \pi_1(\mathcal{X})$ is surjective, hence the rank of $\mathcal{F}$ is one so Theorem 5.1 applies. Though the hypothesis of [BKP, Theorem 5.1] does not hold in this example.

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