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An Extended KdV Hierarchy via an Energy Dependent Scattering

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Abstract. This paper formulates an extended KdV hierarchy involving a coupled KdV equation, the Boussinesq system as well as their higher order versions. Based upon an inverse scattering method on an energy dependent Schrödinger operator, N-soliton solutions in the extended hierarchy are constructed in a unified fashion. In even-order systems, each soliton is multi-peaked when a parameter exceeds the critical value. The classical solitons in the KdV hierarchy are embedded into those with the parameter being zero of even-order systems.

1. Introduction and Main Results

This paper gives a unified aspect of solitons to a class of soliton equations involving the KdV equations and the Boussinesq systems by carrying out rigorously a scheme proposed by Jaulent and Miodek [25]. In this section we only sketch the problem of interest here, main results and defer proofs of them, further details to succeeding sections.

Let L be an energy dependent Schrödinger operator

$$L = D^2 - (U + 2kQ), \tag{1.1}$$

where $D = \frac{d}{dx}$, k is a spectral parameter (wave number), U, Q are functions (potentials) in the Schwartz class $\mathcal{S}$ on $\boldsymbol{R}$, and let A_n, $n = 1, 2, \cdots$, be operators

$$A_n = \left(\sum_{j=0}^{n} p_j k^{n-j}\right) D - \frac{1}{2}\left(\sum_{j=1}^{n} p_j k^{n-j}\right)_x \quad \text{with} \quad p_0 = 1, \tag{1.2}$$

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with differential polynomials p_j in U, Q (namely, polynomials of U, Q and their x-derivatives) satisfying the Lax [35] commutator representation:

$$\frac{1}{ia_n}\frac{\partial}{\partial t}L = [A_n, L]. \tag{1.3}$$

Here a_n are nonzero constants concerned with scaling of time, on which an additional requirement will be imposed later. By a computation (see Section 2) with a replacement $D^2 = U + 2kQ - k^2$, we see that the polynomials p_j are determined as follows:

$$p_1 = Q, \quad p_2 = \frac{3}{2}Q^2 + \frac{1}{2}U, \tag{1.4}$$

$$2p'_{j+2} = 4Qp'_{j+1} + 2Q_x p_{j+1} + 2Up'_j + U_x p_j - \frac{1}{2}p'''_j, \quad j = 1, 2, \cdots. \tag{1.5}$$

Then equation (1.3) leads to

$$\text{QU}[n] \begin{cases} \frac{1}{ia_n}Q_t = p'_{n+1}, \\ \frac{1}{ia_n}U_t = 2Up'_n + U_x p_n - \frac{1}{2}p'''_n. \end{cases} \tag{1.6}$$

This is a couple of partial differential equations for each $n = 1, 2, \cdots$. We refer to (1.6) as a QU system and denote it by QU[n].

Example 1.1. The first four systems are written as:

$$\text{QU}[1] \begin{cases} \frac{1}{ia_1}Q_t = \left(\frac{3}{2}Q^2 + \frac{1}{2}U\right)_x, \\ \frac{1}{ia_1}U_t = 2Q_x U + QU_x - \frac{1}{2}Q_{xxx}. \end{cases}$$

$$\text{QU}[2] \begin{cases} \frac{1}{ia_2}Q_t = \left(\frac{5}{2}Q^3 + \frac{3}{2}QU - \frac{1}{4}Q_{xx}\right)_x, \\ \frac{1}{ia_2}U_t = -\frac{3}{2}QQ_{xxx} - \frac{9}{2}Q_x Q_{xx} + 6QQ_x U \\ \qquad\qquad + \frac{3}{2}Q^2 U_x + \frac{3}{2}UU_x - \frac{1}{4}U_{xxx}. \end{cases}$$

$$\text{QU}[3] \begin{cases} \frac{1}{ia_3}Q_t = \left(\frac{35}{8}Q^4 + \frac{15}{4}Q^2 U + \frac{3}{8}U^2 - \frac{5}{4}QQ_{xx} - \frac{5}{8}Q_x^2 - \frac{1}{8}U_{xx}\right)_x, \\ \frac{1}{ia_3}U_t = 2U\left(\frac{5}{2}Q^3 + \frac{3}{2}QU - \frac{1}{4}Q_{xx}\right)_x \\ \qquad\qquad + U_x\left(\frac{5}{2}Q^3 + \frac{3}{2}QU - \frac{1}{4}Q_{xx}\right) \\ \qquad\qquad - \frac{1}{2}\left(\frac{5}{2}Q^3 + \frac{3}{2}QU - \frac{1}{4}Q_{xx}\right)_{xxx}. \end{cases}$$

$$
\mathrm{QU}[4]\quad \begin{cases} \frac{1}{ia_4}Q_t = \big(\frac{63}{8}Q^5 + \frac{35}{4}Q^3U - \frac{5}{8}Q_xU_x - \frac{35}{8}Q^2Q_{xx} - \frac{5}{8}Q_{xx}U \\ \qquad\quad + \frac{15}{8}QU^2 - \frac{35}{8}QQ_x^2 - \frac{5}{8}QU_{xx} + \frac{1}{16}Q_{xxxx}\big)_x, \\ \frac{1}{ia_4}U_t = 2U\left(\frac{35}{8}Q^4 + \frac{15}{4}Q^2U + \frac{3}{8}U^2 - \frac{5}{4}QQ_{xx} - \frac{5}{8}Q_x^2 - \frac{1}{8}U_{xx}\right)_x \\ \qquad\quad + U_x\left(\frac{35}{8}Q^4 + \frac{15}{4}Q^2U + \frac{3}{8}U^2 - \frac{5}{4}QQ_{xx} - \frac{5}{8}Q_x^2 - \frac{1}{8}U_{xx}\right) \\ \qquad\quad - \frac{1}{2}\big(\frac{35}{8}Q^4 + \frac{15}{4}Q^2U + \frac{3}{8}U^2 \\ \qquad\qquad - \frac{5}{4}QQ_{xx} - \frac{5}{8}Q_x^2 - \frac{1}{8}U_{xx}\big)_{xxx}. \end{cases}
$$

Notice that QU[2] with $a_2 = -4i$ can be found in Jaulent and Miodek [25, equation (5.2)], where our Q becomes $\frac{1}{2}Q$. For a deduction of more general evolution systems, see Martínez Alonso [38].

If $Q \equiv 0$ then QU[n] with an even n gives the $\frac{n}{2}$-th order KdV equation, because, in the case $Q \equiv 0$, recursion relation (1.5) reads $p_{\mathrm{odd}} = 0$ as well as

$$2p'_{2(\nu+1)} = 2Up'_{2\nu} + U_x p_{2\nu} - \frac{1}{2}p'''_{2\nu},$$

which is no other than the recursion relation of differential polynomials (see, e.g., Marchenko [37, equation (4.1.9)]) for the KdV hierarchy. This implies that, for even n, if $Q \equiv 0$ then the first equation of (1.6) gives a trivial equation with both sides being 0 and the second equation of it becomes the $\frac{n}{2}$-th order KdV equation. Thus QU[even] can be viewed as a generalization with two unknowns of the KdV hierarchy. In particular QU[2] with $ia_2 = 4$ is a system involving the KdV equation.

A primary aim of this paper is to establish a method by which soliton solutions of QU[n] can be constructed. For the aim we employ an inverse scattering theory on the energy dependent Schrödinger operator (1.1). This subject, inverse scattering problem, is to reconstruct the potential (U, Q) in (1.1) from the scattering matrix

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \quad k \in \boldsymbol{R}, \tag{1.7}$$

composed of the transmission coefficient $s_{11}(k)$ $(= s_{22}(k))$ and the reflection coefficients $s_{12}(k)$, $s_{21}(k)$. The coefficient $s_{11}(k)$ can be analytically continued to a meromorphic function in the upper half plane $\boldsymbol{C}_+$ having at most finitely many (distinct) poles k_ℓ, $\ell = 1, \cdots, N$. We refer to the poles k_ℓ simply as bound states for L because k_ℓ^2 correspond to bound state energy levels. Moreover scattering is said to be reflectionless when

$$s_{21}(k) = s_{12}(k) = 0 \quad \text{for any } k \in \boldsymbol{R}. \tag{1.8}$$

In the standard case, namely, for Schrödinger operator $L_S = D^2 - U$, the inverse problem was solved in Deift and Trubowitz [9] completely, that is, by giving a necessary and sufficient condition for a matrix in the form (1.7) to be the scattering matrix for some potential U, based on reconstruction formulas derived by Kay and Moses [33], Marchenko [36], Faddeev [11]. The seminal work Gardner, Greene, Kruskal and Miura [13, 14] discovered that the KdV equation (QU[2] with $Q \equiv 0$ in our terminology) is an isospectral flow for L_S to solve the Cauchy problem for the equation with initial data in $\mathcal{S}$.

The inverse scattering problem on (1.1) was studied by Jaulent [21, 22], Jaulent and Jean [23, 24], Sattinger and Szmigielski [41], Aktosun, Klaus and van der Mee [1, 2], Kamimura [27, 28, 29, 30, 31]. Provided that U, Q are real and that there are no bound states ($N = 0$), a complete generalization of [9, Theorem 5.3] was obtained by [29]. Nevertheless, for potentials with bound states ($N > 0$), the generalization is still open. In the reflectionless case (1.8), the inverse scattering problem on (1.1) has been solved completely in [30] provided that U, Q are real, and in [31] provided that

$$U \text{ is real, } Q \text{ is purely imaginary.} \tag{1.9}$$

We employ the inverse scattering theory developed by the latter paper; so here and hereafter we assume (1.9) as well as $U, Q \in \mathcal{S}$.

Under assumption (1.9), the transmission coefficients $s_{11}^-(k)$ for potential $(U, -Q)$ and $s_{11}(k)$ for (U, Q) are identical (see [31, Proposition 2.3]) in the reflectionless scattering, and hence (U, Q) and $(U, -Q)$ admit identical poles (bound states) k_ℓ, $\ell = 1, \cdots, N$. In addition, every pole k_ℓ is simple (see [31, Corollary 3.7]). Thus, in the reflectionless scattering, the matrix $S(k)$ for $(U, \pm Q)$ with (1.9) is written as

$$S(k) = \begin{pmatrix} \prod_{\ell=1}^N \frac{k+k_\ell}{k-k_\ell} & 0 \\ 0 & \prod_{\ell=1}^N \frac{k+k_\ell}{k-k_\ell} \end{pmatrix}. \tag{1.10}$$

At each pole $k = k_\ell$, the Wronskian $W[f_+(x,k), f_-(x,k)]$ of Jost solutions $f_+(x,k)$, $f_-(x,k)$ of $(L + k^2)f = 0$ with asymptotic behaviors $f_\pm \sim e^{\pm ikx}$

as $x \to \pm\infty$ vanishes, and hence two solutions are connected by a nonzero constant d_ℓ^+ as $f_-(x,k_\ell) = d_\ell^+ f_+(x,k_\ell)$. The constants d_ℓ^+, $\ell = 1,\cdots,N$, are called the coupling constants for (U,Q). In a similar manner, coupling constants d_ℓ^-, $\ell = 1,\cdots,N$, for $(U,-Q)$ are defined.

The potential (U,Q) is not determined only by its bound states; we require a priori knowledge of certain additional information. The following nonzero constants are adopted in this paper:

$$c_\ell^\pm := -i\,\mathrm{Res}_{k=k_\ell} s_{11}(k) \times d_\ell^\pm, \quad \ell = 1,\cdots,N. \tag{1.11}$$

This is a generalized concept of the norming constants used in the scattering theory in the standard Schrödinger case (see, e.g., [9, p.158 and p.146]). Notice that, in the energy dependent case, the constants $c_\ell^\pm$ defined in (1.11) are complex numbers, in general.

The scattering transform (ST): $(U,Q) \mapsto \{0, k_\ell, c_\ell^\pm\}$ in which 0 indicates merely the reflectionless condition (1.8) and its inverse (IST) are completely characterized by functions $\varDelta^\pm$ defined by

$$\begin{aligned}\varDelta^\pm(x) := {}& \det(I - B^+B^-) \\ & + \begin{pmatrix} e^{ik_1x} & \cdots & e^{ik_Nx}\end{pmatrix}(I - B^\mp B^\pm)\tilde{}\,(B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp).\end{aligned} \tag{1.12}$$

Here $B^\pm$ are $N \times N$ matrices and $\boldsymbol{v}^\pm$ are N column vectors defined in terms of the scattering data $\{0, k_\ell, c_\ell^\pm\}$ by

$$B^\pm := \left(c_\ell^\pm \frac{e^{(ik_\ell + ik_j)x}}{ik_\ell + ik_j}\right), \quad \boldsymbol{v}^\pm := \left(c_\ell^\pm \frac{e^{ik_\ell x}}{ik_\ell}\right), \tag{1.13}$$

I is the $N \times N$ identity matrix, $\begin{pmatrix} e^{ik_1x} & \cdots & e^{ik_Nx}\end{pmatrix}$ is a $1 \times N$ matrix, and $(I - B^\mp B^\pm)\tilde{}$ denote the cofactor matrices of $I - B^\mp B^\pm$. In particular (IST): $\{0, k_\ell, c_\ell^\pm\} \mapsto (U,Q)$ is given by

$$\left\{\begin{aligned} & Q(x) = -\tfrac{1}{2i}\tfrac{d}{dx}\left(\log \varDelta^+(x) - \log \varDelta^-(x)\right), \\ & U(x) + Q(x)^2 = -\tfrac{1}{2}\tfrac{d^2}{dx^2}\left(\log \varDelta^+(x) + \log \varDelta^-(x)\right). \end{aligned}\right. \tag{1.14}$$

A precise formulation of the characterization is:

Proposition 1.2 ([31]). *A prescribed triplet $\{0, k_\ell, c_\ell^\pm\}$ is the scattering data for some $(U,Q) \in \mathcal{S} \times \mathcal{S}$ if and only if $\{0, k_\ell, c_\ell^\pm\}$ satisfies the following two conditions:*

(I) *there exists a permutation* $\sigma \in \mathfrak{S}_N$ *such that* $k_{\sigma(\ell)} = -\overline{k_\ell}$, $c^{\pm}_{\sigma(\ell)} = \overline{c^{\pm}_\ell}$;

(II) $\Delta^{\pm}(x) > 0$ *on* $\boldsymbol{R}$.

Under these conditions, (U, Q) *is uniquely determined by* (1.14).

This proposition gives a generalization of the reflectionless inverse scattering theory on the standard Schrödinger equation, because if $ik_\ell < 0$ (namely, σ is the identity permutation) and $c^+_\ell = c^-_\ell > 0$ for each ℓ then $\Delta^+(x) = \Delta^-(x) = (\det(I - B^+))^2 > 0$ (see [31, Corollary 3.8]) and hence, by (1.14),

$$Q(x) \equiv 0, \quad U(x) = -2\frac{d^2}{dx^2}\log\det(I - B^+). \tag{1.15}$$

This is an expression of reflectionless potentials of the standard Schrödinger equation (see [33], [14], Hirota [15], Wadati and Toda [47]).

Let us return to QU systems (1.6). In order to make (1.6) real systems, here and hereafter we impose the following condition on a_n:

$$\overline{a_n} = (-1)^{n-1}a_n, \quad n = 1, 2, \cdots, \tag{1.16}$$

that is, assume a_n are real for odd n and purely imaginary for even n. Then it turns out (see Proposition 2.4) that, for each $n = 1, 2, \cdots$, QU$[n]$ defined in (1.6) gives a real expression for physical variables $\frac{1}{i}Q$ and U. In addition (see Lemma 3.2), for each $n = 1, 2, \cdots$, QU$[n]$ gives an isospectral flow for L in (1.1), that is, if $(Q(x,t), U(x,t))$ with (1.9) satisfies QU$[n]$ with (1.16) then bound states for $L(t) = D^2 - (U(x,t) + 2kQ(x,t))$ are t-invariant. In this way we obtain an inverse scattering method in Figure 1.

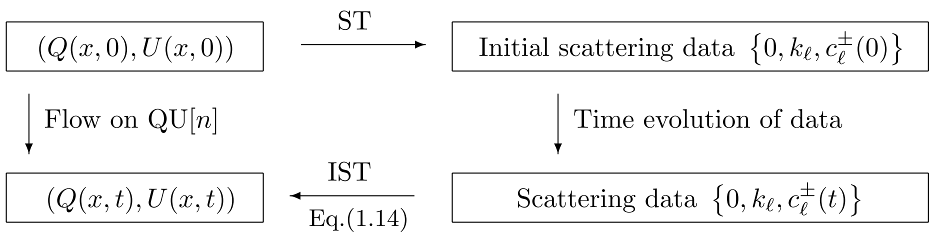

Fig. 1. Inverse scattering method.

The time evolution of the constants $c_\ell^\pm(t)$ is derived by a standard, asymptotic analysis (see Section 3) for the Jost solutions $f_\pm$ as

$$c_\ell^\pm(t) = c_\ell^\pm(0)e^{-2(\pm 1)^n a_n {k_\ell}^{n+1} t}, \tag{1.17}$$

under assumptions (1.9), (1.16). In particular if $ik_\ell < 0$ and a_n satisfies (1.16) then $-2(\pm 1)^n a_n {k_\ell}^{n+1}$ is real, and so that $c_\ell^\pm(0) > 0 \Rightarrow c_\ell^\pm(t) > 0$.

We are now in a position to state our main result.

THEOREM 1.3. *Assume that* $ik_\ell < 0$, $c_\ell^\pm(0) > 0$ *for* $\ell = 1, \cdots, N$ *and define* $c_\ell^\pm(t)$ *by* (1.17). *Then* $(Q(x,t), U(x,t))$ *defined by* (1.14) *is a solution on* $\boldsymbol{R}^2$ *of* QU[n] *with* (1.16).

This result guarantees that if $ik_\ell < 0$, $c_\ell^\pm(0) > 0$ then $(Q(x,t), U(x,t))$ obtained by the schema in Figure 1 give global solutions of QU[n] for $n = 1, 2, \cdots$. The proof is given in Section 6 after showing (see Corollary 6.2) that if $ik_\ell < 0$, $c_\ell^\pm > 0$ for $\ell = 1, \cdots, N$ then $\Delta^\pm$ satisfy condition (II) in Proposition 1.2.

Time evolution of $c_\ell^\pm(t)$ is quite different depending on whether n is odd or even:

$$n \text{ is odd} \Longrightarrow c_\ell^+(t)c_\ell^-(t) = c_\ell^+(0)c_\ell^-(0), \quad \ell = 1, \cdots, N, \tag{1.18}$$

$$n \text{ is even} \Longrightarrow \frac{c_\ell^+(t)}{c_\ell^-(t)} = \frac{c_\ell^+(0)}{c_\ell^-(0)}, \quad \ell = 1, \cdots, N. \tag{1.19}$$

By definition (1.11) and the first formula in (1.14) we see that $Q \equiv 0 \Leftrightarrow c_\ell^+ = c_\ell^-$. Accordingly we have

COROLLARY 1.4. *Let* $ik_\ell < 0$, $c_\ell^\pm(0) > 0$, *let* $c_\ell^\pm(t)$ *be functions in* (1.17), *and define* $(Q(x,t), U(x,t))$ *by* (1.14). *Then*

(1) *If* n *is odd then* $Q(x,t) \not\equiv 0$.

(2) *If* n *is even and* $c_\ell^+(0) = c_\ell^-(0)$ *for* $\ell = 1, \cdots, N$ *then* $(Q(x,t), U(x,t))$ *is a pair of* $Q \equiv 0$ *and an* N*-soliton solution* $U(x,t)$ *of the* $\frac{n}{2}$*-th order KdV equation.*

In light of (1.14), we rewrite (1.4), (1.5), (1.6) by a nonsingular transformation

(1.20)

$$iQ = -\frac{u}{4}, \quad U = -\frac{w}{4} + \frac{u^2}{16} \quad \left(\Longleftrightarrow u = -4iQ, \quad w = -4(U + Q^2)\right).$$

Then (1.4), (1.5) can be rewritten as

$$\begin{cases} p_1 = i\frac{u}{4}, \quad p_2 = -\frac{1}{16}u^2 - \frac{1}{8}w, \\ 2p'_{j+2} = iup'_{j+1} + \frac{1}{2}iu_x p_{j+1} - \frac{1}{2}\left(w - \frac{u^2}{4}\right)p'_j - \frac{1}{4}\left(w_x - \frac{1}{2}uu_x\right)p_j - \frac{1}{2}p'''_j. \end{cases}$$

This gives a recursive definition of differential polynomials p_j in terms of (u, w). By the polynomials, QU$[n]$ in (1.6) is transformed via (1.20) to

$$\text{(1.21)} \qquad \text{uw}[n] \quad \begin{cases} u_t - 4a_n\left(p_{n+1}\right)_x = 0, \\ w_t + a_n\left(8ip_{n+2} + 2up_{n+1}\right)_x = 0, \end{cases}$$

which gives an expression of QU$[n]$ in terms of (u, w).

We thus have the hierarchy (1.21) of coupled equations equivalent to QU$[n]$. We denote it by uw$[n]$. In the case $ik_\ell < 0$, in order for t-dependence of $c_\ell^\pm(t)$ to be simple, it is convenient to normalize a_n as

$$\text{(1.22)} \qquad a_n = (-2)^n i^{n+1}, \quad n = 1, 2, \cdots,$$

for instance $a_1 = 2$, $a_2 = -4i$ and so on. Then we get

$$\text{(1.23)} \qquad c_\ell^\pm(t) = c_\ell^\pm(0)e^{(\pm 1)^n(-2ik_\ell)^{n+1}t}, \quad \ell = 1, \cdots, N.$$

The first system of hierarchy (1.21) with $a_1 = 2$ is

$$\text{uw}[1] \quad \begin{cases} u_t + w_x + uu_x = 0, \\ w_t + u_{xxx} + (uw)_x = 0, \end{cases}$$

the second system with $a_2 = -4i$ is

$$\text{uw}[2] \quad \begin{cases} u_t + \left(\frac{3}{2}uw + u_{xx} + \frac{1}{4}u^3\right)_x = 0, \\ w_t + \left(\frac{3}{2}uu_{xx} + \frac{3}{4}{u_x}^2 + \frac{3}{4}u^2 w + \frac{3}{4}w^2 + w_{xx}\right)_x = 0, \end{cases}$$

the third system with $a_3 = -8$ is

$$\text{uw}[3] \quad \begin{cases} u_t + \left(\frac{3}{2}u^2 w + 2uu_{xx} + \frac{3}{4}w^2 + w_{xx} + \frac{3}{4}{u_x}^2 + \frac{1}{8}u^4\right)_x = 0, \\ w_t + \left(\frac{1}{2}u^3 w + \frac{3}{2}uw^2 + \frac{5}{2}u_x w_x + \frac{5}{2}u_{xx}w \right. \\ \qquad \left. + 2uw_{xx} + \frac{3}{2}u{u_x}^2 + \frac{3}{2}u^2 u_{xx} + u_{xxxx}\right)_x = 0, \end{cases}$$

the fourth system with $a_4 = 16i$ is

$$
\text{uw}[4] \quad \begin{cases} u_t + \frac{1}{16}\left(u^5 + 20u^3w + 30uw^2 + 30uu_x{}^2 + 40u_xw_x \right. \\ \qquad \left. +40u^2u_{xx} + 40u_{xx}w + 40uw_{xx} + 16u_{xxxx}\right)_x = 0, \\ w_t + \frac{1}{16}\left(5u^4w + 30u^2w^2 + 30u^2u_x{}^2 + 50u_x{}^2w + 100uu_xw_x \right. \\ \qquad +40u^2w_{xx} + 80u_xu_{xxx} + 40uu_{xxxx} + 20u^3u_{xx} \\ \qquad +100uu_{xx}w + 60u_{xx}{}^2 + 10w^3 + 40ww_{xx} \\ \qquad \left. +20w_x{}^2 + 16w_{xxxx}\right)_x = 0, \end{cases}
$$

and so on. It should be mentioned that each uw$[n]$ is a couple of conservation laws.

The following is a recast of Theorem 1.3 and Corollary 1.4 via (1.20):

Corollary 1.5. *Assume* $ik_\ell < 0$, $c_\ell^\pm(0) > 0$ *for* $\ell = 1, \cdots, N$ *and define* $c_\ell^\pm(t)$ *by* (1.23). *Then*

(1) *Each pair* (u, w) *defined by*

$$
(1.24) \qquad \begin{cases} u(x,t) = 2\frac{\partial}{\partial x}\left(\log \Delta^+(x,t) - \log \Delta^-(x,t)\right), \\ w(x,t) = 2\frac{\partial^2}{\partial x^2}\left(\log \Delta^+(x,t) + \log \Delta^-(x,t)\right), \end{cases}
$$

is a solution on $\boldsymbol{R}^2$ *of* uw$[n]$ *with* (1.22).

(2) *If* n *is odd then* $u(x,t) \not\equiv 0$.

(3) *If* n *is even and* $c_\ell^+(0) = c_\ell^-(0)$ *for* $\ell = 1, \cdots, N$ *then* $(u(x,t), w(x,t)) = (0, -4U)$, *where* U *is an* N*-soliton solution of the* $\frac{n}{2}$*-th order KdV equation in the form* (1.15) *with* (1.23).

The assertion (3) above implies that soliton solutions of the KdV hierarchy are embedded into the hierarchy uw[even] as waves $-\frac{1}{4}w$ with the no-motion field $u = 0$.

The system uw[1] is a dispersive, shallow water equation when the gravity force dominates over the capillary one (see Kamchatnov, Kraenkel and Umarov [26, page 356], see also [31]). In the equation, u denotes the horizontal velocity field and w denotes the height of the water surface above the horizontal bottom. Following Sachs [40] we use the symbols u, w and call uw[1] the Boussinesq system and, in addition, we view uw[odd] as higher order Boussinesq systems though any physical meaning can not be specified for $n = 3, 5, \cdots$. For other notation, terminology, and extensions of uw[1] ($\Leftrightarrow$ QU[1]), refer to Broer [7], El, Grimshaw and Kamchatnov [10], Kupershmidt [34], Antonowicz and Fordy [5, 6], Alber, Luther and Marsden [3], Alber, Luther and Miller [4].

Example 1.6 (uw[n], $N = 1$). We treat 1-soliton solutions of uw[n] with (1.22) under assumption $ik_1 < 0$. For $N = 1$, from definition (1.12) with cofactor matrix being the identity, we have

$$\Delta^{\pm}(x) = 1 - c_1^{\mp}\frac{e^{2ik_1x}}{ik_1} + c_1^{+}c_1^{-}\frac{e^{4ik_1x}}{(2ik_1)^2}.$$

Set $b = -2ik_1$ (> 0). Then formula (1.24) reads

$$u = 4\frac{e^{-bx}(c_1^{+} - c_1^{-})\left(1 - c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)}{\left(1 + \frac{2c_1^{-}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)\left(1 + \frac{2c_1^{+}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)},$$

$$w = 4be^{-bx}\frac{c_1^{+}\left(1 + \frac{2c_1^{-}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)^3 + c_1^{-}\left(1 + \frac{2c_1^{+}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)^3}{\left(1 + \frac{2c_1^{-}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)^2\left(1 + \frac{2c_1^{+}e^{-bx}}{b} + c_1^{+}c_1^{-}\frac{e^{-2bx}}{b^2}\right)^2}.$$

Here $c_1^{\pm} = c_1^{\pm}(0)e^{(\pm 1)^n b^{n+1}t}$ by (1.23). Let us introduce real parameters δ, ρ by relations

$$e^{b\delta} = \frac{b}{\sqrt{c_1^{+}(0)c_1^{-}(0)}}, \quad e^{b^{n+1}\rho} = \sqrt{\frac{c_1^{+}(0)}{c_1^{-}(0)}}. \tag{1.25}$$

Corresponding to (1.18), (1.19), we consider separately two cases.

Case 1: n =odd. In this case, 1-soliton solutions of uw[n] are expressed as follows:

$$u(x,t) = 2b\frac{\left(e^{b^{n+1}(t+\rho)} - e^{-b^{n+1}(t+\rho)}\right)\sinh b(x+\delta)}{\left(\cosh b(x+\delta) + e^{b^{n+1}(t+\rho)}\right)\left(\cosh b(x+\delta) + e^{-b^{n+1}(t+\rho)}\right)}, \tag{1.26}$$

$$w(x,t) = 2b^2\left(\frac{1 + e^{b^{n+1}(t+\rho)}\cosh b(x+\delta)}{\left(\cosh b(x+\delta) + e^{b^{n+1}(t+\rho)}\right)^2} + \frac{1 + e^{-b^{n+1}(t+\rho)}\cosh b(x+\delta)}{\left(\cosh b(x+\delta) + e^{-b^{n+1}(t+\rho)}\right)^2}\right). \tag{1.27}$$

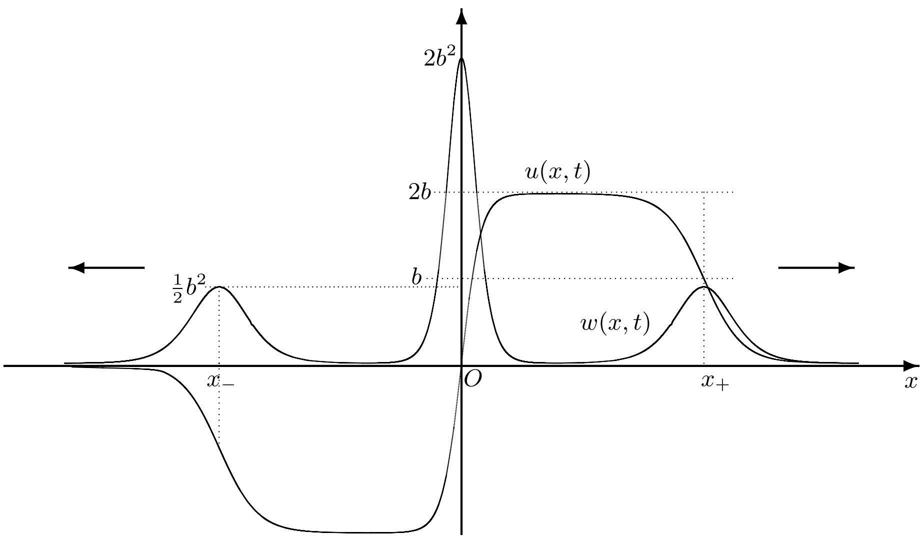

Fig. 2. Profile of 1-soliton solutions $(u(x,t),\ w(x,t))$ of uw[odd] for $\delta = \rho = 0$ at large t so that $t > \frac{1}{b^{n+1}} r_c$. The velocity $u(x,t)$ takes the value $\pm b$ approximately at the stationary points $x = x_\pm$ of the height $w(x,t)$, which behaves, near $x_\pm$, such as the solitary waves $\frac{1}{2}b^2 \mathrm{sech}^2 \frac{1}{2}(b(|x| - b^n t) - \log 2)$ with amplitude $\frac{1}{2}b^2$, speed b^n asymptotically as $t \to +\infty$.

By shifts of x, t we may assume $\delta = \rho = 0$. Then the followings hold (see Figure 2): (1) $0 < w(x,t) \leq 2b^2$. The maximum $2b^2$ is attained only at $x = 0$. (2) If $|t| \leq \frac{1}{b^{n+1}} r_c$ where r_c denotes a critical ratio

$$r_c := \log \frac{11 + 5\sqrt{5}}{2} \approx 2.406 \tag{1.28}$$

then $w(x,t)$ has only one maximum $2b^2$, while if $|t| > \frac{1}{b^{n+1}} r_c$ then $w(x,t)$ has a local maximum other than the maximum $2b^2$ at two points $x_\pm$, which have the asymptotic behaviors $b|x_\pm| = b^{n+1}|t| + \log 2 + o(1)$ as $|t| \to \infty$. The local maximum decreases monotonically from $\frac{6}{11}b^2$ to $\frac{1}{2}b^2$ as $|t|$ moves from $\frac{1}{b^{n+1}} r_c$ to ∞. (3) $|u(x,t)| < 2b$, and $|u(x_\pm, t)| \to b$ as $|t| \to \infty$.

Case 2: n =even. In this case, under assumption $ik_1 < 0$, 1-soliton solutions of uw[n] with (1.22) are expressed, in the (common) setting (1.25), as

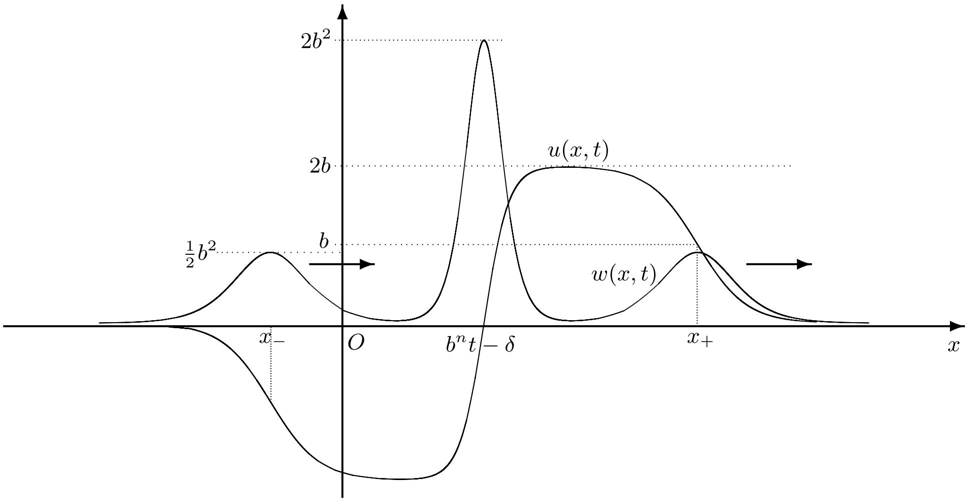

Fig. 3. Profile of 1-soliton solutions $(u(x,t), w(x,t))$ of uw[even] for large ρ so that $\rho > \frac{1}{b^{n+1}} r_c$. The function $u(x,t)$ takes the values $\pm b$ at the stationary points $x = x_\pm$ of $w(x,t)$ asymptotically as $\rho \to +\infty$. The stationary points $x_\pm$ have the asymptotic behaviors $|b\,(x_\pm - (b^n t - \delta))\,| = b^{n+1}\rho + \log 2 + o(1)$ as $\rho \to +\infty$..

follows:

$$
(1.29)\quad u(x,t) = 2b \frac{(e^{b^{n+1}\rho} - e^{-b^{n+1}\rho}) \sinh b(x - b^n t + \delta)}{\left(\cosh b(x - b^n t + \delta) + e^{b^{n+1}\rho}\right)\left(\cosh b(x - b^n t + \delta) + e^{-b^{n+1}\rho}\right)},
$$

$$
w(x,t) = 2b^2 \left(\frac{1 + e^{b^{n+1}\rho} \cosh b(x - b^n t + \delta)}{\left(\cosh b(x - b^n t + \delta) + e^{b^{n+1}\rho}\right)^2} + \frac{1 + e^{-b^{n+1}\rho} \cosh b(x - b^n t + \delta)}{\left(\cosh b(x - b^n t + \delta) + e^{-b^{n+1}\rho}\right)^2} \right).
$$

The solutions (u, w) are travelling waves with amplitudes $(2b, 2b^2)$ and the propagation velocity b^n.

For each $t \in \boldsymbol{R}$, the followings hold (see Figure 3): (1) $0 < w(x,t) \le 2b^2$. The maximum $2b^2$ is attained only at $x = b^n t - \delta$. (2) If $|\rho| \le \frac{1}{b^{n+1}} r_c$ then $w(x,t)$ has only one maximum $2b^2$, while if $|\rho| > \frac{1}{b^{n+1}} r_c$ then $w(x,t)$ has a local maximum other than $2b^2$ at two stationary points $x_\pm$. The local maximum decreases monotonically from $\frac{6}{11} b^2$ to $\frac{1}{2} b^2$ as ρ moves from

$\pm\frac{1}{b^{n+1}}r_c$ to $\pm\infty$. (3) $|u(x,t)| < 2b$, and $|u(x_\pm,t)| \to b$ as $|\rho| \to \infty$.

Notice that

$$u(x,t) \equiv 0 \Longleftrightarrow \rho = 0,$$

in which we have 1-soliton solutions $-\frac{1}{4}w(x,t) = -\frac{b^2}{2}\mathrm{sech}^2\frac{1}{2}b(x - b^n t + \delta)$ of the $\frac{n}{2}$-th order KdV equations.

Kaup [32] studied the inverse scattering problem on an energy dependent operator $L_m = D^2 + m^2 - (U + 2kQ)$ with $m \neq 0$ to construct soliton solutions of the Boussinesq system uw[1]. In connection with Kaup's solution, the problem on L_m has been studied by Tsutsumi [45], Sattinger and Szmigielski [42], van der Mee and Pivovarchik [46]. Kaup's solution was also derived in Hirota and Satsuma [19, 20] by the Hirota direct method (Hirota [16]). Apart from Kaup's solution, Hirota [17] constructed N-soliton solutions of uw[1] by using a reduction way from the first modified KP (Kadomtsev-Petviashvili) equation. For other solutions of uw[1], refer to Hirota [18], Sachs [40], Matveev and Yavor [39], Freeman, Gilson and Nimmo [12], Clarkson [8]. Unlike these solutions, the N-soliton solutions $(u(x,t), w(x,t))$ of uw[1] obtained as a consequence of Corollary 1.5(1), for instance 1-soliton (1.26), (1.27) with $n = 1$ that was found in [31], are solutions in $\mathcal{S} \times \mathcal{S}$ for each $t \in \boldsymbol{R}$.

The succeeding sections are organized as follows. In Section 2 we establish a recursion relation of differential polynomials p_n to provide QU systems QU$[n]$. In Section 3 we deduce the time evolutions of the constants $c_\ell^\pm$ to present an inverse scattering method for QU$[n]$. These two sections are mainly concerned with the forward scattering problem on energy dependent operator L. In Section 4 we find a characteristic of the inverse scattering method that if (Q,U) defined by its inversion formula satisfies the first equation of QU$[n]$ then the (Q,U) satisfies necessarily the second equation of it. For the proof a wave equation of a transformation kernel (the unknown of the Gelfand-Levitan-Marchenko equation) for L plays a fundamental role of connecting the polynomials p_n with the kernel. In Section 5 we show that (Q,U) obtained by the inverse scattering method satisfies the first equation of QU$[n]$ by using an algebraic computation with a linear equation associated with the scattering data. In Section 6 an expression of the functions $\Delta^\pm$ is obtained by means of the cofactor expansion. By using it, at the end of this section, we complete the proof of our main result. In

Section 7 we employ approaches developed for the KdV solitons together with the expression of $\Delta^{\pm}$ to investigate N-soliton solutions of uw$[n]$.

2. Differential Polynomials

This section is devoted to a deduction of QU system (1.6). Let $n = 1, 2, \cdots$, let $L = D^2 - (U + 2kQ)$, and define A_n by (1.2). We set

$$a = \sum_{j=0}^{n} p_j k^{n-j}, \quad b = -\frac{1}{2}\sum_{j=1}^{n} p'_j k^{n-j}. \tag{2.1}$$

Then $A_n = aD + b$ with a relation $a_x + 2b = 0$. The reason for the choice (1.2) of the operators A_n has to do with this relation. By $[aD, D^2] = [a, D^2]D$, $[a, D^2] = -2a_x D - a_{xx}$, $[aD, c] = ac_x$,

$$\begin{aligned}
&-[A_n, L] = -[aD + b,\ D^2 - (U + 2kQ)]\\
&= 2a_x D^2 + (a_x + 2b)_x D + b_{xx} + a(U_x + 2kQ_x)\\
&= 2\sum_{j=1}^{n} p'_j k^{n-j} D^2 - \frac{1}{2}\sum_{j=1}^{n} p'''_j k^{n-j} + \left(\sum_{j=0}^{n} p_j k^{n-j}\right)(U_x + 2kQ_x).
\end{aligned}$$

We now use the replacement $D^2 = U + 2kQ - k^2$. Then we have

$$\begin{aligned}
-[A_n, L] = {}& 2(Q_x - p'_1)k^{n+1} + (4Qp'_1 + 2Q_x p_1 + U_x - 2p'_2)k^n\\
&- 2\sum_{j=1}^{n-2} p'_{j+2} k^{n-j} + \sum_{j=1}^{n-1}(4Qp'_{j+1} + 2Q_x p_{j+1})k^{n-j}\\
&+ \sum_{j=1}^{n}\left(2Up'_j + U_x p_j - \frac{1}{2}p'''_j\right)k^{n-j}.
\end{aligned}$$

Since the left side in (1.3) is linear in k we take p_1, p_2 as (1.4). Then

$$\begin{aligned}
&\frac{1}{ia_n}(2kQ_t + U_t) = -[A_n, L]\\
&= \sum_{j=1}^{n-2}\left(4Qp'_{j+1} + 2Q_x p_{j+1} + 2Up'_j + U_x p_j - \frac{1}{2}p'''_j - 2p'_{j+2}\right)k^{n-j}
\end{aligned} \tag{2.2}$$

$$+\left(4Qp_n' + 2Q_x p_n + 2Up_{n-1}' + U_x p_{n-1} - \frac{1}{2}p_{n-1}'''\right)k$$
$$+\left(2Up_n' + U_x p_n - \frac{1}{2}p_n'''\right).$$

For the consistency of (2.2), recursion relation (1.5) is required.

It is not obvious that p_j determined by (1.5) become differential polynomials; we adopt, instead of an indefinite form (1.5), the following, definite definition:

$$p_n = Q\sum_{j=0}^{n-1} p_j\, p_{n-1-j} + \frac{1}{2}U\sum_{j=0}^{n-2} p_j\, p_{n-2-j}$$
$$-\begin{cases} \displaystyle\sum_{j=\frac{n-3}{2}}^{n-3} p_{j+2}\, p_{n-2-j} & \text{if } n \text{ is odd} \\ \displaystyle\sum_{j=\frac{n}{2}-1}^{n-3} p_{j+2}\, p_{n-2-j} + \frac{1}{2}(p_{\frac{n}{2}})^2 & \text{if } n \text{ is even} \end{cases}$$
$$+\frac{1}{4}\begin{cases} \displaystyle\sum_{j=\frac{n-1}{2}}^{n-3} p_j'\, p_{n-2-j}' & \text{if } n \text{ is odd} \\ \displaystyle\sum_{j=\frac{n}{2}}^{n-3} p_j' p_{n-2-j}' + \frac{1}{2}(p_{\frac{n}{2}-1}')^2 & \text{if } n \text{ is even} \end{cases}$$
$$-\frac{1}{4}\sum_{j=1}^{n-2} p_j''\, p_{n-2-j}, \qquad n = 3, 4, \cdots, \tag{2.3}$$

with the convention $p_{-1} = 0$, and the fourth term in the right side being 0 if $n = 3$, being $\frac{1}{4}\cdot\frac{1}{2}(p_1')^2$ if $n = 4$. Then we have the following:

Lemma 2.1. *A sequence* $\{p_n\}_{n=1}^{\infty}$ *of differential polynomials defined by* (1.4), (2.3) *satisfies* (1.5) *for* $j = -1, 0, 1, \cdots$.

Proof. Let n be odd. Then, by $p_0' = 0$,

$$\frac{1}{2}\left(\sum_{j=\frac{n-1}{2}}^{n-3} p_j' p_{n-2-j}' - \sum_{j=1}^{n-2} p_j''\, p_{n-2-j}\right)'$$

$$= \frac{1}{2}\left(\sum_{j=\frac{n-1}{2}}^{n-3} p_j'' p_{n-2-j}' + \sum_{j=\frac{n-1}{2}}^{n-3} p_j' p_{n-2-j}'' - \sum_{j=1}^{n-2} p_j''' \, p_{n-2-j} - \sum_{j=1}^{n-2} p_j'' \, p_{n-2-j}'\right)$$
$$= -\frac{1}{2}\sum_{j=1}^{n-2} p_j''' \, p_{n-2-j}.$$

Hence, by differentiating definition (2.3) multiplied by 2, we have, for odd n,

$$\begin{aligned} 2p_n' = {} & 4Q\sum_{j=0}^{n-1} p_j' \, p_{n-1-j} + 2Q_x\sum_{j=0}^{n-1} p_j \, p_{n-1-j} + 2U\sum_{j=0}^{n-2} p_j' \, p_{n-2-j} \\ & + U_x\sum_{j=0}^{n-2} p_j' \, p_{n-2-j} - 2\sum_{j=-1}^{n-3} p_{j+2} \, p_{n-2-j}' - \frac{1}{2}\sum_{j=1}^{n-2} p_j''' \, p_{n-2-j}. \end{aligned}$$

In analogy for odd n we verify that this equality holds also for even n. Thus we obtain, for any n,

$$\begin{aligned} \sum_{j=0}^{n-1} \Big(& 4Qp_{n-1-j}' + 2Q_x p_{n-1-j} + 2Up_{n-2-j}' \\ & + U_x p_{n-2-j} - \frac{1}{2}p_{n-2-j}''' - 2p_{n-j}' \Big) p_j = 0. \end{aligned}$$

Provided that (1.5) holds for $j = 0, 1, \cdots, n-3$, this shows that (1.5) holds for $j = n-2$, since $p_0 = 1$. Accordingly, by induction, we complete the proof. □

A few words are in order on definition (2.3). By induction we see that

(2.4) $$\{p_n\}_{n=1}^{\infty},\ \{q_n\}_{n=1}^{\infty} \text{ satisfy (1.4), (1.5),} \quad p_n,\, q_n \to 0,\ x \to \infty \Longrightarrow \{p_n\}_{n=1}^{\infty} = \{q_n\}_{n=1}^{\infty}.$$

It follows from this observation that definition (1.4), (2.3) and definition (1.4), (1.5) with $p_n \to 0$, $x \to \infty$ are equivalent.

By (1.5), the equality (2.2) is rewritten as

(2.5) $$\frac{1}{ia_n}(2kQ_t + U_t) = 2p_{n+1}'k + \left(2Up_n' + U_x p_n - \frac{1}{2}p_n'''\right),$$

which leads to (1.6). We now summarize the discussions above in:

PROPOSITION 2.2. *Let* $\{p_n\}_{n=1}^{\infty}$ *be the sequence of differential polynomials defined by* (1.4), (2.3), *and let* A_n *be operators defined by* (1.2). *Then the operator equation* (1.3) *for solutions* f *of* $(L+k^2)f=0$ *is equivalent to evolution system* (1.6).

We pick out some characters of $\{p_n\}_{n=1}^{\infty}$, which will be required later.

LEMMA 2.3. *Assume* (1.9) *and denote* p_n *for* $(U,-Q)$ *by* p_n^-. *Then*

(1) $\overline{p_n} = (-1)^n p_n$

(2) $p_n^- = (-1)^n p_n$

(3) $p_n^- = \overline{p_n}$

(4) *In the case* n *is odd, if* $Q=0$ *then* $p_n=0$.

PROOF. Assertions (1), (2) follow from (1.4), (1.5) by induction. Assertion (3) is immediate from (1), (2). Assertion (4) follows from (2), since if $Q=0$ then $p_n^- = p_n$. □

Under the assumption (1.9) we have

PROPOSITION 2.4. *Assume* (1.16). *Then, for each* $n=1,2,\cdots$, QU$[n]$ *is a couple of real equations.*

PROOF. From Lemma 2.3(1) and (1.16), we get $\overline{a_n p'_{n+1}} = a_n p'_{n+1}$. Hence $a_n p_{n+1}$ are real-valued functions for $n=1,2,\cdots$. This implies that the first equation $\frac{1}{i}Q_t = a_n p'_{n+1}$ in (1.6) is real for each n because Q is purely-imaginary. Similarly we can show that $R := ia_n(2Up'_n + U_x p_n - \frac{1}{2}p'''_n)$ are real-valued functions for $n=1,2,\cdots$. This implies that the second equation $U_t = R$ in (1.6) is real for each n. □

3. Time Evolution of Scattering Data

In this section we will show that systems QU[n] in (1.6) are isospectral flows for the operator L and derive the time evolutions of the constants $c_\ell^\pm$. Throughout this section we use the notation $\phi(x) \sim e(x)$ when $\phi'(x) = e'(x)[1 + o(1)]$ as well as $\phi(x) = e(x)[1 + o(1)]$.

Let M_n be operators defined by

$$M_n := \frac{1}{ia_n}\frac{\partial}{\partial t} - A_n, \quad n = 1, 2, \cdots, \tag{3.1}$$

namely,

$$M_n = \frac{1}{ia_n}\frac{\partial}{\partial t} - \left(\left(\sum_{j=0}^{n} p_j k^{n-j} \right) D - \frac{1}{2} \left(\sum_{j=1}^{n} p_j k^{n-j} \right)_x \right).$$

As for the standard case (see Tanaka [43], [37, Chapter 4.2]), we have:

LEMMA 3.1. *A pair* $(Q(x,t), U(x,t))$ *satisfies* QU[n] *if and only if the operator* M_n *transforms each solution* f *of* $(L + k^2)f = 0$ *to a solution of the same equation.*

PROOF. By Proposition 2.2, QU[n] is equivalent to (1.3), which is rewritten as the operator equation $[L, M_n] = 0$. Since, for each solution f of $(L + k^2)f = 0$, the equation $[L, M_n]f = 0$ is written as $(L + k^2)M_n f = 0$, we arrive at the conclusion in the lemma. □

On applying M_n to the Jost solutions $f_\pm(x, k, t)$ of $(L + k^2)f = 0$, whose asymptotic behaviors are $f_\pm \sim e^{\pm ikx}$ as $x \to \pm\infty$, we have solutions $M_n f_\pm$ of $(L + k^2)f = 0$. In view of definition (1.2) where $p_j \in \mathcal{S}$ for $j \geq 1$, these solution have the asymptotics

$$M_n f_\pm \sim \mp i k^{n+1} e^{\pm ikx}, \quad x \to \pm\infty.$$

Hence, by the uniqueness of the Jost solution, we find

$$M_n f_\pm = \mp i k^{n+1} f_\pm, \tag{3.2}$$

provided (Q, U) satisfies QU$[n]$. Conversely, if (3.2) holds then $(L+k^2)M_n f_\pm = 0$ and so, by means of $(L+k^2)\frac{\partial}{\partial t} f_\pm = -\left(\frac{\partial}{\partial t}L\right) f_\pm$, it follows that

$$\left(\frac{1}{ia_n}\frac{\partial}{\partial t}L - [A_n, L]\right) f_\pm = 0.$$

Since $f_\pm$ are nontrivial solutions, this implies that (2.5) holds. We thus see that

$$(Q(x,t), U(x,t)) \text{ satisfies QU}[n] \iff M_n f_\pm = \mp i k^{n+1} f_\pm. \tag{3.3}$$

Let $f_\pm^-$ be Jost solutions of $(L+k^2)f = 0$ with $-Q$ in place of Q. Under assumption (1.9), we have four solutions of $(L+k^2)f = 0$ for $k \in \boldsymbol{R}$:

$$f_+(x,k), \quad f_-(x,k), \quad \overline{f_+^-(x,k)}, \quad \overline{f_-^-(x,k)}.$$

By means of asymptotic behaviors, the Wronskians are computed as

$$W[f_\pm(x,k), \overline{f_\pm^-(x,k)}] = \mp 2ik. \tag{3.4}$$

The transmission coefficient $s_{11}(k)$ $(= s_{22}(k))$ and the reflection coefficients $s_{12}(k)$, $s_{21}(k)$ are defined as coefficients appearing in the linear combinations

$$\begin{aligned} s_{11}(k) f_+(x,k) &= \overline{f_-^-(x,k)} + s_{12}(k) f_-(x,k), \\ s_{22}(k) f_-(x,k) &= \overline{f_+^-(x,k)} + s_{21}(k) f_+(x,k). \end{aligned} \tag{3.5}$$

Use of (3.4) and (3.5) shows that

$$s_{11}(k) = -\frac{2ik}{W[f_+(x,k), f_-(x,k)]}. \tag{3.6}$$

Time evolutions of the scattering data of the family QU$[n]$ are derived as follows.

LEMMA 3.2. *Suppose that* $(Q, U) = (Q(x,t), U(x,t))$ *with* (1.9) *satisfies* QU$[n]$ *with* (1.16). *Then:*

(1) *Time evolutions of the scattering matrices*

$$S(k,t) = \begin{pmatrix} s_{11}(k,t) & s_{12}(k,t) \\ s_{21}(k,t) & s_{22}(k,t) \end{pmatrix},$$

$$S^-(k,t) = \begin{pmatrix} s_{11}^-(k,t) & s_{12}^-(k,t) \\ s_{21}^-(k,t) & s_{22}^-(k,t) \end{pmatrix},$$

of $(L+k^2)f=0$ *with* $(\pm Q, U)=(\pm Q(x,t), U(x,t))$ *are given by*

$$\begin{aligned} s_{11}(k,t) &= s_{11}(k,0), & s_{11}^-(k,t) &= s_{11}^-(k,0),\\ s_{12}(k,t) &= s_{12}(k,0)e^{2a_n k^{n+1}t}, & s_{12}^-(k,t) &= s_{12}^-(k,0)e^{2(-1)^n a_n k^{n+1}t},\\ s_{21}(k,t) &= s_{21}(k,0)e^{-2a_n k^{n+1}t}, & s_{21}^-(k,t) &= s_{21}^-(k,0)e^{-2(-1)^n a_n k^{n+1}t}. \end{aligned}$$

In particular reflectionless scattering is preserved in time evolution. Moreover poles k_ℓ, $\ell=1,\cdots,N$, *in* $\boldsymbol{C}_+$ *of* $s_{11}(k,t)$ *are* t*-invariant.*

(2) *In the reflectionless scattering, time evolutions of the constants* $c_\ell^\pm(t)$ *defined in* (1.1) *are given by*

$$c_\ell^+(t)=c_\ell^+(0)e^{-2a_n {k_\ell}^{n+1}t}, \quad c_\ell^-(t)=c_\ell^-(0)e^{-2(-1)^n a_n {k_\ell}^{n+1}t}. \tag{3.7}$$

Proof. (1) Set

$$a(k,t):=\frac{1}{s_{11}(k,t)}, \quad b_+(k,t):=\frac{s_{12}(k,t)}{s_{11}(k,t)}, \quad b_-(k,t):=\frac{s_{21}(k,t)}{s_{11}(k,t)}.$$

Then, from (3.5), we get

$$f_\pm(x,k,t)=a(k,t)\overline{f_\mp^-(x,k,t)}+b_\pm(k,t)f_\mp(x,k,t). \tag{3.8}$$

We apply the operator M_n in (3.1) to both sides. Then, by (3.2), we have

$$\mp ik^{n+1}f_\pm = M_n\left(a(k,t)\overline{f_\mp^-}+b_\pm(k,t)f_\mp\right).$$

In view of (3.2) and (1.2) with $p_j\in\mathcal{S}$ for $j\geq 1$,

$$\frac{1}{ia_n}\frac{\partial}{\partial t}f_\mp = A_n f_\mp \pm ik^{n+1}f_\mp \sim 0, \quad x\to\mp\infty,$$

for $k\in\boldsymbol{R}$. By replacement $Q\to -Q$, we have also $\frac{1}{ia_n}\frac{\partial}{\partial t}f_\mp^- \sim 0$, as $x\to\mp\infty$, and therefore

$$\frac{1}{ia_n}\frac{\partial}{\partial t}\overline{f_\mp^-}\sim 0, \quad x\to\mp\infty.$$

This leads to, for $k \in \boldsymbol{R}$,

$$\begin{aligned}
&\mp ik^{n+1} f_{\pm} = M_n \left(a(k,t)\overline{f_{\mp}^{-}} + b_{\pm}(k,t) f_{\mp} \right) \\
&\sim \frac{1}{ia_n}\dot{a}(k,t)e^{\pm ikx} + \frac{1}{ia_n}\dot{b}_{\pm}(k,t)e^{\mp ikx} \\
&\quad \mp ik^{n+1}a(k,t)e^{\pm ikx} \pm ik^{n+1}b_{\pm}(k,t)e^{\mp ikx} \\
&= \left(\frac{1}{ia_n}\dot{a}(k,t) \mp ik^{n+1}a(k,t) \right) e^{\pm ikx} + \left(\frac{1}{ia_n}\dot{b}_{\pm}(k,t) \pm ik^{n+1}b_{\pm}(k,t) \right) e^{\mp ikx} \\
&\sim \left(\frac{1}{ia_n}\dot{a}(k,t) \mp ik^{n+1}a(k,t) \right) \overline{f_{\mp}^{-}} + \left(\frac{1}{ia_n}\dot{b}_{\pm}(k,t) \pm ik^{n+1}b_{\pm}(k,t) \right) f_{\mp},
\end{aligned}$$

as $x \to \mp\infty$. Since, by (3.4), $\overline{f_{\mp}^{-}}$ and $f_{\mp}$ are linearly independent, this implies that

$$\begin{aligned}
\mp ik^{n+1} f_{\pm} = &\left(\frac{1}{ia_n}\dot{a}(k,t) \mp ik^{n+1}a(k,t) \right) \overline{f_{\mp}^{-}} \\
&+ \left(\frac{1}{ia_n}\dot{b}_{\pm}(k,t) \pm ik^{n+1}b_{\pm}(k,t) \right) f_{\mp}.
\end{aligned}$$

Comparing this with (3.8) we obtain

$$\dot{a}(k,t) = 0, \quad \dot{b}_{\pm}(k,t) = \pm 2a_n k^{n+1} b_{\pm}(k,t),$$

and so, we conclude that, for $k \in \boldsymbol{R}$,

$$\dot{s}_{11}(k,t) = 0, \quad \dot{s}_{12}(k,t) = 2a_n k^{n+1} s_{12}(k,t), \quad \dot{s}_{21}(k,t) = -2a_n k^{n+1} s_{21}(k,t).$$

Similarly, by applying the operator M_n in (3.1) to every term of

$$\overline{f_{\pm}^{-}(x,k,t)} = \overline{a^{-}(k,t)} f_{\mp}(x,k,t) + \overline{b_{\pm}^{-}(k,t)}\, \overline{f_{\mp}^{-}(x,k,t)},$$

where

$$a^{-}(k,t) = \frac{1}{s_{11}^{-}(k,t)}, \quad b_{+}^{-}(k,t) = \frac{s_{12}^{-}(k,t)}{s_{11}^{-}(k,t)}, \quad b_{-}^{-}(k,t) = \frac{s_{21}^{-}(k,t)}{s_{11}^{-}(k,t)},$$

we obtain

$$\overline{\dot{a}^{-}(k,t)} = 0, \quad \overline{\dot{b}_{\pm}^{-}(k,t)} = \mp 2a_n k^{n+1} \overline{b_{\pm}^{-}(k,t)}.$$

By taking the complex conjugate and using (1.16), we conclude that

$$\dot{a}^-(k,t) = 0, \quad \dot{b}^-_\pm(k,t) = \pm 2(-1)^n a_n k^{n+1} b^-_\pm(k,t).$$

This leads to

$$\begin{aligned}&\dot{s}^-_{11}(k,t) = 0, \quad \dot{s}^-_{12}(k,t) = 2(-1)^n a_n k^{n+1} s^-_{12}(k,t),\\ &\dot{s}^-_{21}(k,t) = -2(-1)^n a_n k^{n+1} s^-_{21}(k,t).\end{aligned}$$

We have thus proved assertion (1).

(2) We prove only the second formula in (3.7) because the first one can be derived similarly. The poles k_ℓ of $s_{11}(k,t) = s_{11}(k,0)$ are time-invariant. We apply M_n in (3.1) to both sides of

$$\overline{f^-_-(x,k_\ell,t)} = \overline{d^-_\ell(t)}\,\overline{f^-_+(x,k_\ell,t)}. \tag{3.9}$$

By Lemma 3.1, the function $M_n\overline{f^-_-(x,k_\ell,t)}$ is a solution of $(L+\overline{k_\ell}^2)f = 0$, whose asymptotics are

$$M_n\overline{f^-_-(x,k_\ell,t)} \sim -\overline{k_\ell}^n(\overline{f^-_-(x,k_\ell,t)})' \sim -\overline{k_\ell}^n i\overline{k_\ell}e^{i\overline{k_\ell}x} = -i\overline{k_\ell}^{n+1}e^{i\overline{k_\ell}x},$$

as $x \to -\infty$. Hence, by the uniqueness of the Jost solution and (3.9), we find that

$$M_n\overline{f^-_-(x,k_\ell,t)} = -i\overline{k_\ell}^{n+1}\overline{f^-_-(x,k_\ell,t)} = -i\overline{k_\ell}^{n+1}\overline{d^-_\ell(t)}\,\overline{f^-_+(x,k_\ell,t)}. \tag{3.10}$$

On the other hand, it follows that, as $x \to +\infty$,

$$\begin{aligned}&M_n\overline{d^-_\ell(t)}\,\overline{f^-_+(x,k_\ell,t)}\\ &\sim \frac{1}{ia_n}\overline{\dot{d}^-_\ell(t)}\overline{f^-_+(x,k_\ell,t)} - \overline{k_\ell}^n\overline{d^-_\ell(t)}(\overline{f^-_+(x,k_\ell,t)})'\\ &\sim \left(\frac{1}{ia_n}\overline{\dot{d}^-_\ell(t)} + i\overline{k_\ell}^{n+1}\overline{d^-_\ell(t)}\right)e^{-i\overline{k_\ell}x}.\end{aligned}$$

This implies that

$$\begin{aligned}M_n\overline{f^-_-(x,k_\ell,t)} &= M_n\overline{d^-_\ell(t)}\,\overline{f^-_+(x,k_\ell,t)}\\ &= \left(\frac{1}{ia_n}\overline{\dot{d}^-_\ell(t)} + i\overline{k_\ell}^{n+1}\overline{d^-_\ell(t)}\right)\overline{f^-_+(x,k_\ell,t)},\end{aligned}$$

and, combined with (3.10), yields

$$-i\overline{k_\ell}^{\,n+1}\overline{d_\ell^-(t)}\,\overline{f_+^-(x,k_\ell,t)} = \left(\frac{1}{ia_n}\overline{\dot{d}_\ell^-(t)} + i\overline{k_\ell}^{\,n+1}\overline{d_\ell^-(t)}\right)\overline{f_+^-(x,k_\ell,t)}.$$

Hence

$$\frac{1}{ia_n}\overline{\dot{d}_\ell^-(t)} = -2i\overline{k_\ell}^{\,n+1}\overline{d_\ell^-(t)}.$$

Taking the complex conjugate and using (1.16), we obtain

$$\dot{d}_\ell^-(t) = -2(-1)^n a_n {k_\ell}^{n+1} d_\ell^-(t).$$

This yields the second formula in (3.7). □

We thus have the inverse scattering method in Figure 1 in the following sense.

Proposition 3.3. *Let n be a natural number, let a_n be a nonzero constant satisfying* (1.16)*, and assume that $(Q(x,0),U(x,0)) \in \mathcal{S}\times\mathcal{S}$ with* (1.9) *is a reflectionless potential with N bound states k_ℓ. If the Cauchy problem* QU$[n]$ *with an initial value $(Q(x,0),U(x,0))$ admits a solution $(Q(x,t),U(x,t)) \in \mathcal{S}\times\mathcal{S}$ with* (1.9) *for each t, then the solution is obtained by the formula*

$$(3.11)\quad \begin{cases} Q(x,t) = -\frac{1}{2i}\frac{\partial}{\partial x}\left(\log \Delta^+(x,t) - \log\Delta^-(x,t)\right), \\ U(x,t) + Q(x,t)^2 = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\log\Delta^+(x,t) + \log\Delta^-(x,t)\right), \end{cases}$$

as long as $\Delta^\pm(x,t) > 0$. Here $\Delta^\pm(x,t)$ are functions defined by (1.12) *with* (1.17)*.*

Proof. By assumption, k_ℓ, $c_\ell^\pm(0)$ satisfies (I), (II) in Proposition 1.2. By Lemma 3.2, scattering transform of $(Q(x,t),U(x,t))$ is $\{0, k_\ell, c_\ell^\pm(t)\}$ with (1.17). It is easy to see that the constants $c_\ell^\pm(t)$ satisfy the condition in (I) of Proposition 1.2. This completes the proof. □

The formula (3.11) is based on Proposition 1.2, in which (1.14) has the following character.

Lemma 3.4. *The followings are equivalent.*

(i) $Q \equiv 0$. (ii) $c_\ell^+ = c_\ell^-$, $\ell = 1,\cdots N$. (iii) $\Delta^+ \equiv \Delta^-$.

PROOF. (i) $\Rightarrow$ (ii): If $Q(x) \equiv 0$ then $f_\pm^-(x,k) = f_\pm(x,k)$. Hence, by $f_-(x,k_\ell) = d_\ell^+ f_+(x,k_\ell)$, $f_-^-(x,k_\ell) = d_\ell^- f_+^-(x,k_\ell)$, we get $d_\ell^+ = d_\ell^-$ and so $c_\ell^+ = c_\ell^-$. (ii) $\Rightarrow$ (iii) is direct from definitions (1.12), (1.13). (iii) $\Rightarrow$ (i) follows from the first equality in (1.14). □

From the time evolutions in (3.7) and Lemma 3.4 we draw the following conclusion.

PROPOSITION 3.5. *Under the same assumptions as in Proposition* 3.3*, $Q(x,t) \equiv 0$ if and only if n is even and $c^+(0) = c^-(0)$.*

PROOF. If $Q(x,t) \equiv 0$ then, by Lemma 3.4, $c_\ell^+(t) \equiv c_\ell^-(t)$. But, in the case n is odd, $c_\ell^+(t) \equiv c_\ell^-(t)$ in (3.7) is impossible. Conversely if n is even and $c^+(0) = c^-(0)$ then, by (3.7), $c_\ell^+(t) \equiv c_\ell^-(t)$ and so, by Lemma 3.4, $Q(x,t) \equiv 0$. □

4. Reduction

Let a_n be nonzero constants satisfying (1.16). Following Figure 1 with (1.17), we take (Q,U) of L in (1.1) so that

$$(4.1) \qquad (Q(x,t),U(x,t)) \underset{\text{Eq.(3.11)}}{\longleftarrow} \left\{0, k_\ell, c_\ell^\pm(0)e^{-2(\pm 1)^n a_n k_\ell{}^{n+1} t}\right\}.$$

In this section we shall prove that if $(Q(x,t),U(x,t))$ satisfies the first equation of the evolution system QU$[n]$, namely $\frac{1}{ia_n}Q_t = p'_{n+1}$ of (1.6), then it satisfies the second equation of QU$[n]$ automatically, and hence, $(Q(x,t),U(x,t))$ becomes a solution of the system QU$[n]$.

For the proof we employ the transformation kernel representation

$$(4.2) \qquad f_\pm(x,k,t) = e^{\pm i\int_x^{\pm\infty} Q(r,t)dr} e^{\pm ikx} + \int_x^{\pm\infty} A_\pm(x,y,t)e^{\pm iky}dy, \quad k \in \overline{\boldsymbol{C}_+},$$

(see [24, page 110], where our k becomes $-k$) of the Jost solutions of $(L + k^2)f = 0$ in terms of functions $A_\pm(x,\cdot,t) \in L^1(x,\pm\infty) \cap L^\infty(x,\pm\infty)$.

We begin with an expression of the recursion formula (1.5) ($\Leftrightarrow$ (2.3)) in terms of the transformation kernel $A_+(x,y,t)$.

LEMMA 4.1. *The sequence* $\{p_n\}_{n=1}^{\infty}$ *defined by* (1.4) *and* (1.5) *satisfies, for* $n = 0, 1, \cdots,$

$$p_{n+2} = Qp_{n+1} - \frac{i}{2}p'_{n+1} - i\,p_{n+1}A_+(x,x,t)e^{-i\int_x^\infty Q dr} - \sum_{j=1}^{n} p_j\left(\left(i\frac{\partial}{\partial y}\right)^{n-j}\frac{\partial A_+}{\partial x}\right)(x,x,t)e^{-i\int_x^\infty Q dr} + \sum_{j=1}^{n}\frac{1}{2}p'_j\left(\left(i\frac{\partial}{\partial y}\right)^{n-j}A_+\right)(x,x,t)e^{-i\int_x^\infty Q dr} - \left(\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(i\frac{\partial}{\partial y}\right)^{n}A_+\right)(x,x,t)e^{-i\int_x^\infty Q dr}. \tag{4.3}$$

PROOF. The proof proceeds in three steps.

Step 1. We rewrite the fourth term of the right side in (4.3) by

$$\left(\left(i\frac{\partial}{\partial y}\right)^{n-j}\frac{\partial A_+}{\partial x}\right)(x,x,t) = \frac{\partial}{\partial x}\left(\left(\left(i\frac{\partial}{\partial y}\right)^{n-j}A_+\right)(x,x,t)\right) + i\left(\left(i\frac{\partial}{\partial y}\right)^{n-j+1}A_+\right)(x,x,t)$$

and use $p_0 = 1$ to have

$$p_{n+2} = Qp_{n+1} - \frac{i}{2}p'_{n+1} - \sum_{j=0}^{n} p_j\frac{\partial}{\partial x}\left(\left(\left(i\frac{\partial}{\partial y}\right)^{n-j}A_+\right)(x,x,t)\right)e^{-i\int_x^\infty Q dr} - \sum_{j=1}^{n+1} ip_j\left(\left(i\frac{\partial}{\partial y}\right)^{n-j+1}A_+\right)(x,x,t)e^{-i\int_x^\infty Q dr} + \sum_{j=1}^{n}\frac{1}{2}p'_j\left(\left(i\frac{\partial}{\partial y}\right)^{n-j}A_+\right)(x,x,t)e^{-i\int_x^\infty Q dr}.$$

By noting that $(\frac{\partial}{\partial x}\varphi)e^{-i\int_x^\infty Q dr} = (\frac{\partial}{\partial x} - iQ)(\varphi e^{-i\int_x^\infty Q dr})$ and setting

$$\phi_m := \left(\left(i\frac{\partial}{\partial y}\right)^{m}A_+\right)(x,x,t)e^{-i\int_x^\infty Q dr}, \quad m = 0, 1, 2, \cdots \tag{4.4}$$

recursion formula (4.3) is rewritten as

$$p_{n+2} = Qp_{n+1} - \frac{i}{2}p'_{n+1} - \sum_{j=0}^{n} p_j \phi'_{n-j} + \sum_{j=0}^{n} p_j iQ\phi_{n-j} - \sum_{j=0}^{n} ip_{j+1}\phi_{n-j} + \sum_{j=0}^{n} \frac{1}{2}p'_j \phi_{n-j}.$$

We use the convention $\phi_{-1} = -i$. Then, by noting $-2p_1 - ip'_0 + 2Qp_0 = 0$, recursion formula (4.3) can be written as

$$\sum_{j=1}^{n+1} \left(-2p_{j+1} - ip'_j + 2Qp_j\right) \phi_{n-j} + \sum_{j=0}^{n} 2ip_j \phi'_{n-j} = 0. \tag{4.5}$$

Hence our task becomes to show that $\{p_n\}_{n=1}^{\infty}$ satisfies (4.5).

Step 2. The transformation kernel $A_+ = A_+(x, y, t)$ satisfies a wave equation:

$$\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} + 2iQ(x)\frac{\partial}{\partial y}\right) A_+ = -U(x)A_+$$

(see [24, page 114]). Since

$$\begin{aligned}
&\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + iQ\right)\left(i\frac{\partial}{\partial y}\right)^m A_+ \\
&= \frac{i}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \left(i\frac{\partial}{\partial y}\right)^{m-1} A_+ \\
&+ \frac{i}{2}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} + 2iQ\frac{\partial}{\partial y}\right)\left(i\frac{\partial}{\partial y}\right)^{m-1} A_+,
\end{aligned}$$

we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + iQ\right)\left(i\frac{\partial}{\partial y}\right)^m A_+ \\
&= \frac{i}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \left(i\frac{\partial}{\partial y}\right)^{m-1} A_+ - \frac{i}{2}U(x)\left(i\frac{\partial}{\partial y}\right)^{m-1} A_+,
\end{aligned}$$

which leads to

$$\left(\frac{\partial}{\partial x} + iQ\right)\left(\left(\left(i\frac{\partial}{\partial y}\right)^m A_+\right)(x, x, t)\right)$$

$$= \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - U\right)\left(\left(\left(i\frac{\partial}{\partial y}\right)^{m-1} A_+\right)(x,x,t)\right).$$

This is written in terms of ϕ_m as

$$\phi_m' = \frac{i}{2}\left(\left(\frac{\partial}{\partial x} - iQ\right)^2 - U\right)\phi_{m-1}.$$

Accordingly ϕ_m admits a recursion formula

$$(4.6)\quad 2\phi_m' = i\left(\phi_{m-1}'' - 2iQ\phi_{m-1}' - (iQ_x + Q^2 + U)\phi_{m-1}\right), \quad m = 1, 2, \cdots .$$

By noting (see [29, equation (2.7)]) that

$$(4.7)\qquad \phi_0' = -\frac{1}{2}\left(iQ_x + Q^2 + U\right)$$

and remembering the convention $\phi_{-1} = -i$, it follows that (4.6) holds even for $m = 0$.

Step 3. In view of (2.4), to prove the lemma, it suffices, in place of to show $\{p_n\}_{n=1}^{\infty}$ determined by (1.4), (1.5) satisfies (4.5) directly, to prove that a sequence defined by (4.5) with the same functions for $n = 0, 1$ as p_0, p_1 satisfies (1.5). Hence we shall prove that a sequence $\{q_n\}_{n=0}^{\infty}$ of functions defined by $q_0 = 1$, $q_1 = Q$, and the recursion formula

$$(4.8)\quad \sum_{j=1}^{n+1}(-2q_{j+1} - iq_j' + 2Qq_j)\phi_{n-j} + \sum_{j=0}^{n} 2iq_j\phi_{n-j}' = 0, \quad n = 0, 1, \cdots ,$$

satisfies

$$(4.9)\quad 2q_{j+2}' = 4Qq_{j+1}' + 2Q_x q_{j+1} + 2Uq_j' + U_x q_j - \frac{1}{2}q_j''', \quad j = 1, 2, \cdots .$$

By $(4.8)_0$, (4.7), $q_2 = \frac{3}{2}Q^2 + \frac{1}{2}U$, and, by $(4.8)_1$, $(4.6)_1$, $q_3 = \frac{5}{2}Q^3 + \frac{3}{2}QU - \frac{1}{4}Q_{xx}$. It follows from these results that $(4.9)_1$ holds. We now assume that (4.9) holds for $j = 1, \cdots, n-1$. By setting

$$S_j := -2q_{j+1} - iq_j' + 2Qq_j, \quad T_j := 2iq_j,$$

$(4.8)_n$ is written as

$$\sum_{j=1}^{n+1} S_j \phi_{n-j} + \sum_{j=0}^{n} T_j \phi'_{n-j} = 0. \tag{4.10}$$

We compute $(4.8)'_n - \frac{i}{2}(4.8)''_{n-1} - Q(4.8)'_{n-1}$, which equals 0, as follows:

$$\begin{aligned}
&\sum_{j=1}^{n} \left(S'_{j+1} - \frac{i}{2} S''_j - Q S'_j \right) \phi_{n-1-j} + S'_1 \phi_{n-1} \\
&+ \sum_{j=0}^{n-1} \left(S_{j+1} - i S'_j - Q S_j \right) \phi'_{n-1-j} + \sum_{j=0}^{n-1} \left(T'_{j+1} - \frac{i}{2} T''_j - Q T'_j \right) \phi'_{n-1-j} \\
&- \frac{i}{2} \sum_{j=1}^{n} S_j \phi''_{n-1-j} + \sum_{j=-1}^{n-1} T_{j+1} \phi''_{n-1-j} - i \sum_{j=1}^{n-1} T'_j \phi''_{n-1-j} - Q \sum_{j=0}^{n-1} T_j \phi''_{n-1-j} \\
&- \frac{i}{2} \sum_{j=0}^{n-1} T_j \phi'''_{n-1-j} = 0.
\end{aligned}$$

From (4.6), (4.7) we get $\phi''_{n-1-j} = -2\phi'_0 \phi_{n-1-j} + 2iQ\phi'_{n-1-j} - 2i\phi'_{n-j}$. Hence this yields

$$\begin{aligned}
&\sum_{j=1}^{n} \left(S'_{j+1} - \frac{i}{2} S''_j - Q S'_j \right) \phi_{n-1-j} + S'_1 \phi_{n-1} \\
&+ \sum_{j=0}^{n-1} \left(-i S'_j \right) \phi'_{n-1-j} + \sum_{j=0}^{n-1} \left(T'_{j+1} - \frac{i}{2} T''_j - Q T'_j \right) \phi'_{n-1-j} \\
&+ \sum_{j=1}^{n} \left(i S_j - 2T_{j+1} + 2iT'_j + 2QT_j \right) \phi'_0 \phi_{n-1-j} \\
&+ \sum_{j=0}^{n-1} \left(4iQT_{j+1} + 2QT'_j - 2iQ^2 T_j - 2iT_{j+2} - 2T'_{j+1} \right) \phi'_{n-1-j} \\
&- \frac{i}{2} \sum_{j=0}^{n-1} T_j \phi'''_{n-1-j} - 4i\phi'_0 \phi_n + 4\phi'_{n+1} - 4Q\phi'_n = 0.
\end{aligned} \tag{4.11}$$

In a similar way, the last line can be computed as

$$
\begin{aligned}
&- \frac{i}{2}\sum_{j=0}^{n-1} T_j \phi'''_{n-1-j} - 4i\phi'_0\phi_n + 4\phi'_{n+1} - 4Q\phi'_n \\
&= \sum_{j=1}^{n} iT_j\phi''_0\phi_{n-1-j} + \sum_{j=1}^{n}(2T_{j+1} - 2QT_j)\phi'_0\phi_{n-1-j} \\
&+ \sum_{j=0}^{n-1}\left\{(Q_x + 2iQ^2)T_j - 4iQT_{j+1} + 2iT_{j+2}\right\}\phi'_{n-1-j} + \sum_{j=0}^{n-1}(iT_j)\phi'_0\phi'_{n-1-j} \\
&- 2\phi''_0\phi_{n-1}.
\end{aligned}
$$

Substituting this to (4.11) and noting that $S'_1 = 2\phi''_0$, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n}\left(S'_{j+1} - \frac{i}{2}S''_j - QS'_j + (iS_j + 2iT'_j)\phi'_0 + iT_j\phi''_0\right)\phi_{n-1-j} \\
&+ \sum_{j=0}^{n-1}\left(-iS'_j - T'_{j+1} - \frac{i}{2}T''_j + QT'_j + Q_xT_j + iT_j\phi'_0\right)\phi'_{n-1-j} = 0.
\end{aligned}
$$

It is easy to see from definition of S_j, T_j and (4.7) that

$$
\begin{aligned}
&S'_{j+1} - \frac{i}{2}S''_j - QS'_j + 2iT'_j\phi'_0 + iT_j\phi''_0 \\
&= -2q'_{j+2} + 4Qq'_{j+1} + 2Q_xq_{j+1} + 2Uq'_j + U_xq_j - \frac{1}{2}q'''_j
\end{aligned}
$$

and that

$$
-iS'_j - T'_{j+1} - \frac{i}{2}T''_j + QT'_j + Q_xT_j = 0.
$$

Consequently

$$
\begin{aligned}
&\sum_{j=1}^{n}\left(-2q'_{j+2} + 4Qq'_{j+1} + 2Q_xq_{j+1} + 2Uq'_j + U_xq_j - \frac{1}{2}q'''_j\right)\phi_{n-1-j} \\
&+ \left(\sum_{j=1}^{n} S_j\phi_{n-1-j} + \sum_{j=0}^{n-1} T_j\phi'_{n-1-j}\right) i\phi'_0 = 0.
\end{aligned}
$$

But originally, by (4.10), the second term vanishes. We thus find that

$$\sum_{j=1}^{n}\left(-2q'_{j+2}+4Qq'_{j+1}+2Q_xq_{j+1}+2Uq'_j+U_xq_j-\frac{1}{2}q'''_j\right)\phi_{n-1-j}=0.$$

In view of the induction assumption that (4.9) holds for $j=1,\cdots,n-1$ and $\phi_{-1}\neq 0$, this implies that (4.9) holds for $j=n$. The proof is complete. □

We now define $g_\pm(x,k,t)$ by $g_\pm = M_nf_\pm \pm ik^{n+1}f_\pm$, namely,

$$g_\pm(x,k,t):=\left(\frac{1}{ia_n}\frac{\partial}{\partial t}-A_n\right)f_\pm(x,k,t) \pm ik^{n+1}f_\pm(x,k,t),\quad k\in\overline{\boldsymbol{C}_+}. \tag{4.12}$$

Though these functions depend on n, we write them simply as $g_\pm$ with abbreviation of n because, in what follows, we consider n to be fixed.

Lemma 4.2. *The functions $g_\pm$ are expressed as, for $k\in\overline{\boldsymbol{C}_+}$,*

$$g_\pm(x,k,t)=\left(\pm i\int_x^{\pm\infty}\left(\frac{1}{ia_n}Q_t-p'_{n+1}\right)dy\,e^{\pm i\int_x^{\pm\infty}Q(r,t)dr}+\int_0^{\pm\infty}\Theta_\pm(x,x+z,t)e^{\pm ikz}dz\right)e^{\pm ikx},$$

with some bounded, integrable functions $\Theta_\pm(x,x+\cdot,t)\in L^1(0,\pm\infty)\cap L^\infty(0,\pm\infty)$.

Proof. The proof, which proceeds in two steps, is given for g_+.

Step 1. By induction on ν, we shall prove that, for $\nu=1,\cdots,n$,

$$\begin{aligned} g_+ = {} & \frac{1}{ia_n}i\int_x^\infty Q_t(y,t)dy\,e^{i\int_x^\infty Qdr}e^{ikx}+\frac{1}{ia_n}\int_x^\infty\frac{\partial A_+}{\partial t}e^{iky}dy \\ & - i\left(\sum_{j=\nu}^{n-1}p_{j+1}k^{n-j}\right)e^{i\int_x^\infty Qdr}e^{ikx} \\ & + \left(\sum_{j=\nu}^{n}p_jk^{n-j}\right)\left(ike^{i\int_x^\infty Qdr}e^{ikx}-f'_+\right)+\frac{1}{2}\left(\sum_{j=\nu}^{n}p'_jk^{n-j}\right)f_+ \end{aligned} \tag{4.13}$$

$$
\begin{aligned}
&- k^{n-\nu+1}\sum_{j=1}^{\nu-1} p_j \int_x^\infty \left(\left(i\frac{\partial}{\partial y}\right)^{\nu-1-j}\frac{\partial A_+}{\partial x}\right)e^{iky}dy\\
&+ k^{n-\nu+1}\sum_{j=1}^{\nu-1} \frac{1}{2}p_j' \int_x^\infty \left(\left(i\frac{\partial}{\partial y}\right)^{\nu-1-j} A_+\right)e^{iky}dy\\
&- k^{n-\nu+1}\int_x^\infty \left(\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(i\frac{\partial}{\partial y}\right)^{\nu-1} A_+\right)e^{iky}dy,
\end{aligned}
$$

where terms with $\sum_{j=\nu}^{n-1}$ when $n-1<\nu$ and those with $\sum_{j=1}^{\nu-1}$ when $\nu-1<j$ are supposed to be zero.

We differentiate representation (4.2) for f_+ in t and x to get

$$
\dot{f}_+ = i\int_x^\infty Q_t dr\, e^{i\int_x^\infty Q dr} e^{ikx} + \int_x^\infty \frac{\partial A_+}{\partial t} e^{iky}dy, \tag{4.14}
$$

$$
\begin{aligned}
f_+' &= ike^{i\int_x^\infty Q dr}e^{ikx} - iQe^{i\int_x^\infty Q dr}e^{ikx} - A_+(x,x,t)e^{ikx}\\
&\quad + \int_x^\infty \frac{\partial A_+}{\partial x}e^{iky}dy.
\end{aligned} \tag{4.15}
$$

By substituting these representations into (4.12), using the integration by parts

$$
ik^{n+1}\int_x^\infty A_+e^{iky}dy = -k^nA_+(x,x,t)e^{ikx} - k^n\int_x^\infty \frac{\partial A_+}{\partial y}e^{iky}dy,
$$

and taking $p_1 = Q$ into account, it follows that (4.13) holds for $\nu = 1$.

By setting

$$
\begin{aligned}
E_\nu &:= -i\left(\sum_{j=\nu}^{n-1} p_{j+1}k^{n-j}\right)e^{i\int_x^\infty Q dr}e^{ikx}\\
&\quad + \left(\sum_{j=\nu}^{n} p_jk^{n-j}\right)\left(ike^{i\int_x^\infty Q dr}e^{ikx} - f_+'\right) + \frac{1}{2}\left(\sum_{j=\nu}^{n} p_j'k^{n-j}\right)f_+,\\
I_\nu &:= -k^{n-\nu+1}\sum_{j=1}^{\nu-1} p_j\int_x^\infty \left(\left(i\frac{\partial}{\partial y}\right)^{\nu-1-j}\frac{\partial A_+}{\partial x}\right)e^{iky}dy
\end{aligned}
$$

$$+ k^{n-\nu+1}\sum_{j=1}^{\nu-1}\frac{1}{2}p_j'\int_x^\infty\left(\left(i\frac{\partial}{\partial y}\right)^{\nu-1-j}A_+\right)e^{iky}dy$$
$$- k^{n-\nu+1}\int_x^\infty\left(\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(i\frac{\partial}{\partial y}\right)^{\nu-1}A_+\right)e^{iky}dy,$$

formula $(4.13)_\nu$ is written as

$$g_+ = \frac{1}{ia_n}i\int_x^\infty Q_t(y,t)dy\, e^{i\int_x^\infty Qdr}e^{ikx} + \frac{1}{ia_n}\int_x^\infty \frac{\partial A_+}{\partial t}e^{iky}dy + E_\nu + I_\nu. \tag{4.16}$$

For $\nu = 1, \cdots, n-1$, this becomes

$$\begin{aligned} g_+ = {}& \frac{1}{ia_n}i\int_x^\infty Q_t(y,t)dy\, e^{i\int_x^\infty Qdr}e^{ikx} + \frac{1}{ia_n}\int_x^\infty \frac{\partial A_+}{\partial t}e^{iky}dy \\ & + E_{\nu+1} - ip_{\nu+1}k^{n-\nu}e^{i\int_x^\infty Qdr}e^{ikx} \\ & + p_\nu k^{n-\nu}\left(ike^{i\int_x^\infty Qdr}e^{ikx} - f_+'\right) + \frac{1}{2}p_\nu' k^{n-\nu}f_+ + I_\nu. \end{aligned} \tag{4.17}$$

By performing integrations by parts, I_ν is rewritten as

$$\begin{aligned} I_\nu = {}& -ik^{n-\nu}\sum_{j=1}^{\nu-1}p_j\left(\left(i\frac{\partial}{\partial y}\right)^{\nu-1-j}\frac{\partial A_+}{\partial x}\right)(x,x,t)\,e^{ikx} \\ & + ik^{n-\nu}\sum_{j=1}^{\nu-1}\frac{1}{2}p_j'\left(\left(i\frac{\partial}{\partial y}\right)^{\nu-1-j}A_+\right)(x,x,t)\,e^{ikx} \\ & - ik^{n-\nu}\left(\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(i\frac{\partial}{\partial y}\right)^{\nu-1}A_+\right)(x,x,t)\,e^{ikx} \\ & + I_{\nu+1} + p_\nu k^{n-\nu}\int_x^\infty\frac{\partial A_+}{\partial x}e^{iky}dy - \frac{1}{2}p_\nu' k^{n-\nu}\int_x^\infty A_+e^{iky}dy \end{aligned} \tag{4.18}$$

for $\nu = 1, \cdots, n$. This, together with (4.12), (4.14), (4.15), leads to

$$\begin{aligned} g_+ = {}& \frac{1}{ia_n}i\int_x^\infty Q_t(y,t)dy\, e^{i\int_x^\infty Qdr}e^{ikx} + \frac{1}{ia_n}\int_x^\infty \frac{\partial A_+}{\partial t}e^{iky}dy \\ & + ik^{n-\nu}\left\{Qp_\nu - \frac{i}{2}p_\nu' - ip_\nu A_+(x,x,t)\,e^{-i\int_x^\infty Qdr}\right. \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
&-\sum_{j=1}^{\nu-1} p_j \left(\left(i\frac{\partial}{\partial y} \right)^{\nu-1-j} \frac{\partial A_+}{\partial x} \right)(x,x,t)\, e^{-i\int_x^\infty Qdr} \\
&+\sum_{j=1}^{\nu-1} \frac{1}{2}p_j' \left(\left(i\frac{\partial}{\partial y} \right)^{\nu-1-j} A_+ \right)(x,x,t)\, e^{-i\int_x^\infty Qdr} \\
&-\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left(i\frac{\partial}{\partial y} \right)^{\nu-1} A_+ \right)(x,x,t)\, e^{-i\int_x^\infty Qdr} \\
&-p_{\nu+1} \Bigg\} e^{i\int_x^\infty Qdr} e^{ikx} + E_{\nu+1} + I_{\nu+1}.
\end{aligned}
$$

But, by virtue of Lemma 4.1, the term $\{\cdots\}$ in the above vanishes. Hence we arrive at

$$
g_+ = \frac{1}{ia_n} i \int_x^\infty Q_t(y,t)dy\, e^{i\int_x^\infty Qdr} e^{ikx} + \frac{1}{ia_n}\int_x^\infty \frac{\partial A_+}{\partial t} e^{iky} dy + E_{\nu+1} + I_{\nu+1}
$$

for $\nu = 1, \cdots, n-1$. In view of (4.16), this shows that $(4.13)_{\nu+1}$ holds, under the induction assumption $(4.13)_\nu$ holds. We have thus shown that (4.13) holds for $\nu = 1, \cdots, n$.

Step 2. We take $\nu = n$ in (4.13). Then

$$
\begin{aligned}
g_+ = &\frac{1}{ia_n} i \int_x^\infty Q_t(y,t)dy\, e^{i\int_x^\infty Qdr} e^{ikx} + \frac{1}{ia_n}\int_x^\infty \frac{\partial A_+}{\partial t} e^{iky} dy \\
&+ p_n \left(ike^{i\int_x^\infty Qdr} e^{ikx} - f_+' \right) + \frac{1}{2} p_n' f_+ + I_n.
\end{aligned}
$$

This equality is of the form (4.17) with changes $\nu \to n$, $E_{\nu+1} \to 0$, $p_{\nu+1} \to 0$. So, by using (4.18) with $\nu = n$, we follow the same steps as for (4.17) up to (4.19). Then, applying Lemma 4.1 to p_{n+1}, we obtain

$$
\begin{aligned}
g_+ &= i \left(\int_x^\infty \frac{1}{ia_n} Q_t(y,t)dy + p_{n+1} \right) e^{i\int_x^\infty Qdr} e^{ikx} \\
&+ \frac{1}{ia_n}\int_x^\infty \frac{\partial A_+}{\partial t} e^{iky} dy + I_{n+1} \\
&= i \int_x^\infty \left(\frac{1}{ia_n} Q_t(y,t) - p_{n+1}'(y,t) \right) dy\, e^{i\int_x^\infty Qdr} e^{ikx} \\
&+ \frac{1}{ia_n}\int_x^\infty \frac{\partial A_+}{\partial t} e^{iky} dy
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j=1}^{n} p_j \int_x^\infty \left(\left(i\frac{\partial}{\partial y}\right)^{n-j} \frac{\partial A_+}{\partial x}\right) e^{iky} dy \\
&+\sum_{j=1}^{n} \frac{1}{2} p_j' \int_x^\infty \left(\left(i\frac{\partial}{\partial y}\right)^{n-j} A_+\right) e^{iky} dy \\
&-\int_x^\infty \left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \left(i\frac{\partial}{\partial y}\right)^{n} A_+\right) e^{iky} dy,
\end{aligned}
$$

which can be represented in the form

$$
\begin{aligned}
g_+(x,y,t) = i \int_x^\infty \left(\frac{1}{ia_n} Q_t(y,t) - p_{n+1}'(y,t)\right) dy\, e^{i\int_x^\infty Q dr} e^{ikx} \\
+ \int_x^\infty \Theta_+(x,y,t) e^{iky} dy.
\end{aligned}
$$

This yields the expression for g_+ of the lemma. □

We are now in a position to prove the following

THEOREM 4.3. *Let n be any (fixed) natural number and let $a_n \neq 0$ satisfy* (1.16). *If a pair $(Q(x,t), U(x,t))$ obtained by the inverse scattering method* (4.1) *satisfies the first equation $\frac{1}{ia_n} Q_t = p_{n+1}'$ of the system* QU$[n]$ *then $(Q(x,t), U(x,t))$ is a solution of* QU$[n]$.

PROOF. Let $(Q(x,t), U(x,t))$ satisfy $\frac{1}{ia_n} Q_t = p_{n+1}'$. Then, by Lemma 4.2,

$$
g_\pm(x,k,t) = \int_0^{\pm\infty} \Theta_\pm(x, x+z, t) e^{\pm ikz} dz\, e^{\pm ikx}, \quad k \in \overline{\boldsymbol{C}_+}.
$$

This, combined with (4.2), i.e.,

$$
f_\pm(x,k,t) = \left(e^{\pm i \int_x^{\pm\infty} Q(r,t) dr} + \int_0^{\pm\infty} A_\pm(x, x+z, t) e^{\pm ikz} dz \right) e^{\pm ikx}
$$

where $A_\pm(x, x+\cdot, t) \in L^1(0,\pm\infty) \cap L^\infty(0,\pm\infty)$, gives a representation

$$
f_+(x,k,t) g_-(x,k,t) - f_-(x,k,t) g_+(x,k,t) = \int_0^\infty \Xi(x,z,t) e^{ikz} dz
$$

for $k \in \overline{\boldsymbol{C}_+}$, where $\Xi(x,\cdot,t) \in L^1(0,\infty) \cap L^\infty(0,\infty)$. This implies that the function in the left side is a bounded, analytic function on $\overline{\boldsymbol{C}_+}$ (in particular, a function in the Hardy space H^{2+}). By the Riemann-Lebesgue lemma it tends to zero as $|k| \to \infty$.

When k is a bound state k_ℓ, by $f_-(x,k_\ell,t) = d_\ell(t) f_+(x,k_\ell,t)$, we have

$$\dot{f}_-(x,k_\ell,t) = d_\ell(t)\dot{f}_+(x,k_\ell,t) + \dot{d}_\ell(t) f_+(x,k_\ell,t).$$

In our setting (4.1), the coupling constant $d_\ell(t)$ admits $\dot{d}_\ell(t) = -2a_n k_\ell^{n+1} d_\ell(t)$. This implies that

$$\left(\frac{1}{ia_n}\frac{\partial}{\partial t} - ik_\ell^{n+1}\right) f_-(x,k_\ell,t) = d_\ell(t)\left(\frac{1}{ia_n}\frac{\partial}{\partial t} + ik_\ell^{n+1}\right) f_+(x,k_\ell,t),$$

and so, by definition (4.12), that

$$g_-(x,k_\ell,t) = d_\ell(t) g_+(x,k_\ell,t).$$

Accordingly, for each $(x,t) \in \boldsymbol{R}^2$, a function

$$\varphi(k) := is_{11}(k)(f_+(x,k,t)g_-(x,k,t) - f_-(x,k,t)g_+(x,k,t)) \tag{4.20}$$

is also a bounded, analytic function on $\overline{\boldsymbol{C}_+}$, tending to zero as $|k| \to \infty$, because each bound state is a simple pole (see [31, Corollary 3.7]) of $s_{11}(k)$.

We next, by replacing Q by $-Q$, define

$$g_\pm^-(x,k,t) := \left(\frac{(-1)^n}{ia_n}\frac{\partial}{\partial t} - A_n^-\right) f_\pm^-(x,k,t) \pm ik^{n+1} f_\pm^-(x,k,t), \quad k \in \overline{\boldsymbol{C}_+},$$

where

$$A_n^- := \left(\sum_{j=0}^{n} p_j^- k^{n-j}\right) D - \frac{1}{2}\left(\sum_{j=1}^{n} p_j^- k^{n-j}\right)_x.$$

Then by the same discussion with $\dot{c}_\ell^-(t) = -2(-1)^n a_n k_\ell^{n+1} c_\ell(t)$ as in the above it follows that

$$\varphi^-(k) := is_{11}(k)(f_+^-(x,k,t)g_-^-(x,k,t) - f_-^-(x,k,t)g_+^-(x,k,t))$$

is a bounded, analytic function on $\overline{\boldsymbol{C}_+}$, tending to zero as $|k| \to \infty$. Notice (see [31, Proposition 2.3]) that $s_{11}(k)$ is invariant in the replacement: $Q \to -Q$.

We now consider $\varphi(k)$, $\varphi^-(k)$ for real k. By (3.5), in the reflection scattering, Jost solutions satisfy

$$\overline{s_{11}(k)f_\pm(x,k,t)} = f_\mp^-(x,k,t), \quad k \in \boldsymbol{R}.$$

In addition, by virtue of Lemma 2.3(3), $A_n^- = \overline{A_n}$, and moreover, by (1.16), $\overline{a_n} = (-1)^{n-1}a_n$. It follows from these properties that

$$\overline{s_{11}(k)g_\pm(x,k,t)} = g_\mp^-(x,k,t), \quad k \in \boldsymbol{R}.$$

As is readily seen by definition of φ, φ^- and condition $s_{11}(k)\overline{s_{11}(k)} = 1$ that follows from (3.5), this leads to the relation $\overline{\varphi(k)} = \varphi^-(k)$ on the real axis. This implies that the analytic function $\varphi(k)$ on the upper half plane $\overline{\boldsymbol{C}_+}$ can be analytically continued to the lower half plane $\overline{\boldsymbol{C}_-}$ as a function $\overline{\varphi^-(\overline{k})}$, because $\overline{\varphi^-(\overline{k})} = \varphi(k)$ for $k \in \boldsymbol{R}$. Accordingly $\varphi(k)$ is an entire function. But it is bounded and is going to 0 as $|k| \to \infty$, and hence, by Liouville's theorem, $\varphi(k) \equiv 0$. We have thus proved that

$$f_+(x,k,t)g_-(x,k,t) - f_-(x,k,t)g_+(x,k,t) = 0, \quad k \in \overline{\boldsymbol{C}_+}.$$

Differentiating this yields

$$\det\begin{pmatrix} f_+ & f_- \\ g'_+ & g'_- \end{pmatrix} = \det\begin{pmatrix} g_+ & g_- \\ f'_+ & f'_- \end{pmatrix}.$$

On the other hand, using the notation (2.1) with $a_x = -2b$, we find that

$$\begin{aligned}
&\det\begin{pmatrix} f_+ & f_- \\ g'_+ & g'_- \end{pmatrix} + \det\begin{pmatrix} g_+ & g_- \\ f'_+ & f'_- \end{pmatrix} \\
&= \det\begin{pmatrix} f_+ & f_- \\ \frac{1}{ia_n}\dot{f}'_+ + (b+ik^{n+1})f'_+ & \frac{1}{ia_n}\dot{f}'_- + (b-ik^{n+1})f'_- \end{pmatrix} \\
&\quad + \det\begin{pmatrix} \frac{1}{ia_n}\dot{f}_+ - (b-ik^{n+1})f_+ & \frac{1}{ia_n}\dot{f}_- - (b+ik^{n+1})f_- \\ f'_+ & f'_- \end{pmatrix} \\
&= \frac{1}{ia_n}\frac{\partial}{\partial t}W[f_+,f_-],
\end{aligned}$$

which vanishes because $s_{11}(k)$ is time-invariant (see (1.10)) and so, by (3.6), $\frac{\partial}{\partial t}W[f_+,f_-] = 0$. Consequently

$$\begin{pmatrix} f_+ & f_- \\ f'_+ & f'_- \end{pmatrix}\begin{pmatrix} g_- \\ -g_+ \end{pmatrix} = 0.$$

Since, in the reflectionless scattering, $f_+(x,k,t)$ and $f_-(x,k,t)$ are linearly independent for $k \in \boldsymbol{R} \setminus \{0\}$ (see [31, equation (2.7)]), this implies that $g_\pm(x,k,t) = 0$, namely, $M_n f_\pm = \mp i k^{n+1} f_\pm$. Therefore, by (3.3), $(Q(x,t), U(x,t))$ satisfies QU$[n]$. We complete the proof. □

5. The First Equation

In this section we shall prove that if $\Delta^\pm(x,t) > 0$ then $(Q(x,t), U(x,t))$ obtained by the inverse scattering method (4.1) satisfies the first equation of the system (1.6). For the proof we employ functions $J^\pm(x,y,t)$ defined by

$$(5.1) \qquad \begin{aligned} &J^\pm(x,y,t) \\ &:= (\, e^{ik_1 y} \ \cdots \ e^{ik_N y}\,)(I - B^\mp B^\pm)^{-1}(B^\mp \boldsymbol{v}^\pm - \boldsymbol{v}^\mp), \quad x \le y, \end{aligned}$$

where $B^\mp$, $\boldsymbol{v}^\mp$ are functions in (1.13) with (1.17):

$$c_\ell^\pm(t) = c_\ell^\pm(0) e^{-2(\pm 1)^n a_n {k_\ell}^{n+1} t}.$$

The functions $J^\pm(x,y,t)$ are defined for (x,t) such that

$$(5.2) \qquad D(x,t) := \det(I - B^\mp B^\pm) \ne 0.$$

Note that $\det(I - AB) = \det(I - BA)$ in general. One can show (see [31, Section 2]) that $1 + J^\pm(x,x,t) \ne 0$ and

$$(5.3) \qquad \Delta^\pm(x,t) = D(x,t)(1 + J^\pm(x,x,t))$$

for (x,t) such that (5.2). The determinant $D(x,t)$ is an analytic function and so, for each t, zeros x of $D(x,t)$ are discrete.

Let

$$(5.4) \qquad F^\pm(y,t) := -\sum_{\ell=1}^{N} c_\ell^\pm(t) e^{ik_\ell y}.$$

Then $J^\mp(x,y,t)$ satisfies a Gelfand-Levitan-Marchenko (GLM) equation:

$$(5.5) \qquad \begin{aligned} J^\mp(x,y,t) &+ \int_x^\infty J^\pm(x,r,t) F^\pm(r+y,t)\,dr \\ &+ \int_x^\infty F^\pm(r+y,t)\,dr = 0, \quad x \le y. \end{aligned}$$

In other words, $D(x,t)$ is the Fredholm determinant of integral equation (5.5), which is uniquely solved as (5.1) under the condition $D(x,t) \neq 0$.

Throughout this section we use the following notation:

$$
{}^t\boldsymbol{e} = \begin{pmatrix} e^{ik_1 x} & \cdots & e^{ik_N x} \end{pmatrix}, \quad \boldsymbol{b}^{\pm} := \begin{pmatrix} c_1^{\pm} e^{ik_1 x} \\ \vdots \\ c_N^{\pm} e^{ik_N x} \end{pmatrix},
$$

$$
K := \begin{pmatrix} ik_1 & & O \\ & \ddots & \\ O & & ik_N \end{pmatrix}.
$$

Moreover, for simplicity, we use the abbreviation $J^{\pm} := J^{\pm}(x,x,t)$.

The time evolution (1.17) of $c_\ell^{\pm}(t)$ is passed on to $F^{\pm}$ as a linear equation

$$
\left(\frac{1}{2a_{n-1}} \frac{\partial}{\partial t} - (\mp 1)^{n-1} i^n \frac{\partial^n}{\partial y^n} \right) F^{\pm} = 0, \quad n = 2, 3, \cdots.
$$

So, applying the differential operator above to (5.5) and using this linear equation, we obtain

$$
\begin{aligned}
\text{(5.6)} \qquad & \frac{1}{2a_{n-1}} J_t^{\mp}(x,y,t) + \int_x^{\infty} \frac{1}{2a_{n-1}} J_t^{\pm}(x,r,t) F^{\pm}(r+y,t)\,dr \\
& - (\mp 1)^{n-1} i^n \frac{\partial^n J^{\mp}}{\partial y^n}(x,y,t) = 0.
\end{aligned}
$$

Since

$$
\begin{aligned}
-(\mp 1)^{n-1} i^n \frac{\partial^n J^{\mp}}{\partial y^n}(x,y,t) &= \begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} \boldsymbol{w}^{\pm}, \\
\boldsymbol{w}^{\pm} &:= (\pm 1)^{n-1} (-iK)^n (I - B^{\pm} B^{\mp})^{-1} (B^{\pm} \boldsymbol{v}^{\mp} - \boldsymbol{v}^{\pm}),
\end{aligned}
$$

(5.6) is an equation with $\begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} \boldsymbol{w}^{\pm}$ in place of the last term

$$
\int_x^{\infty} F^{\pm}(r+y,t)\,dr = \begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} \boldsymbol{v}^{\pm}
$$

in (5.5). From this observation it follows that (5.6) is solved as, for $n = 2, 3, \cdots$,

$$
\frac{1}{2a_{n-1}} \frac{\partial}{\partial t} J^{\pm}(x,y,t) = \begin{pmatrix} e^{ik_1 y} & \cdots & e^{ik_N y} \end{pmatrix} (I - B^{\mp} B^{\pm})^{-1} (B^{\mp} \boldsymbol{w}^{\pm} - \boldsymbol{w}^{\mp}).
$$

Hence, by setting $y = x$, we have

$$\frac{1}{2a_{n-1}}\frac{\partial}{\partial t}J^{\pm} = (\pm 1)^{n-1}\,{}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1} \times \\ \left(B^{\mp}(-iK)^n(I - B^{\pm}B^{\mp})^{-1}\,(B^{\pm}\boldsymbol{v}^{\mp} - \boldsymbol{v}^{\pm}) \right.\\ \left. +(iK)^n(I - B^{\mp}B^{\pm})^{-1}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp})\right).$$

By using a relation

$$-K(I - B^{\mp}B^{\pm})^{-1}\,(B^{\mp}\boldsymbol{v}^{\pm} - \boldsymbol{v}^{\mp}) \\ = (I - B^{\mp}B^{\pm})^{-1}\left((1 + J^{\pm})B^{\mp}\boldsymbol{b}^{\pm} + (1 + J^{\mp})\boldsymbol{b}^{\mp}\right)$$

between $\boldsymbol{v}^{\pm}$ and $\boldsymbol{b}^{\pm}$ (see [31, equation (5.9)]) and the identity

$$B^{\pm}(I - B^{\mp}B^{\pm})^{-1} = (I - B^{\pm}B^{\mp})^{-1}B^{\pm}, \tag{5.7}$$

we arrive at

$$\frac{1}{2a_{n-1}}\frac{\partial}{\partial t}J^{\pm} = i^n(\pm 1)^{n-1}\,{}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1} \times \\ \{((-1)^{n-1}B^{\mp}K^{n-1} - K^{n-1}B^{\mp})(I - B^{\pm}B^{\mp})^{-1}\boldsymbol{b}^{\pm}(1 + J^{\pm}) \\ + ((-1)^{n-1}B^{\mp}K^{n-1}B^{\pm} - K^{n-1})(I - B^{\mp}B^{\pm})^{-1}\boldsymbol{b}^{\mp}(1 + J^{\mp})\}. \tag{5.8}$$

This gives an expression of $\frac{\partial}{\partial t}J^{\pm}$, which leads to

LEMMA 5.1. *We define, for $n = 1, 2, \cdots$,*

$$r_n := i^n\,{}^t\boldsymbol{e}\,(I - B^{-}B^{+})^{-1} \times \\ \left\{\left((-1)^{n-1}B^{-}K^{n-1} - K^{n-1}B^{-}\right)(I - B^{+}B^{-})^{-1}\boldsymbol{b}^{+} \right.\\ \left. + \left((-1)^{n-1}B^{-}K^{n-1}B^{+} - K^{n-1}\right)(I - B^{-}B^{+})^{-1}\boldsymbol{b}^{-}\frac{1 + J^{-}}{1 + J^{+}}\right\} \\ - i^n\,{}^t\boldsymbol{e}\,(I - B^{+}B^{-})^{-1} \times \\ \left\{\left(B^{+}K^{n-1} - (-1)^{n-1}K^{n-1}B^{+}\right)(I - B^{-}B^{+})^{-1}\boldsymbol{b}^{-} \right.\\ \left. + \left(B^{+}K^{n-1}B^{-} - (-1)^{n-1}K^{n-1}\right)(I - B^{+}B^{-})^{-1}\boldsymbol{b}^{+}\frac{1 + J^{+}}{1 + J^{-}}\right\}. \tag{5.9}$$

Then, for $n \geq 2$,

$$\frac{1}{2a_{n-1}}\frac{\partial}{\partial t}\left(\log \Delta^{+} - \log \Delta^{-}\right) = r_n. \tag{5.10}$$

In particular, if $\Delta^{\pm} > 0$ then the function $r_n(\cdot, t)$ is of the class $C^\infty(\boldsymbol{R})$ for $n \geq 2$.

Proof. By virtue of (5.3) we get, for (x,t) such that $D(x,t) \neq 0$,

$$\frac{1}{2a_{n-1}}\frac{\partial}{\partial t}\left(\log \Delta^+ - \log \Delta^-\right) = \frac{1}{2a_{n-1}}\frac{\partial}{\partial t}\left(\log(1+J^+) - \log(1+J^-)\right)$$
$$= \frac{1}{2a_{n-1}}\left(\frac{\frac{\partial}{\partial t}J^+}{1+J^+} - \frac{\frac{\partial}{\partial t}J^-}{1+J^-}\right).$$

Insertion of (5.8) now into the right side yields (5.10). Provided that $\Delta^{\pm} > 0$, the left side is analytic and zeros of $D(x,t)$ is discrete for each t. Hence this implies that r_n is continuously continued as a smooth function, namely, each point (x,t) such that (5.2) is an apparent singularity of r_n. The proof is complete. □

We wish to prove that $r_n = p_n$ for $n = 1, 2, \cdots$. By a linear algebraic computation we shall show that r_n satisfies the recursion relation in (1.5):

Lemma 5.2. *The sequence $\{r_n\}_{n=1}^\infty$ satisfies*

$$2r'_{n+2} = 4Qr'_{n+1} + 2Q_x r_{n+1} + 2Ur'_n + U_x r_n - \frac{1}{2}r'''_n, \quad n = 1, 2, \cdots. \tag{5.11}$$

Proof. We set

$$\rho_n = (-i)^n \frac{1}{2} r_n.$$

Then (5.11) is equivalent to the following recursion relation for the sequence $\{\rho_n\}_{n=1}^\infty$:

$$4\rho'_{n+2} = -8iQ\rho'_{n+1} - 4iQ_x\rho_{n+1} - 4U\rho'_n - 2U_x\rho_n + \rho'''_n, \qquad n = 1, 2, \cdots. \tag{5.12}$$

In what follows we give the proof of (5.12), which proceeds in four steps. In the proof we borrow

$$\frac{\frac{\partial}{\partial x}J^{\pm}}{1+J^{\mp}} = -2\,{}^t\boldsymbol{e}\,(I - B^{\mp}B^{\pm})^{-1}\boldsymbol{b}^{\mp}, \tag{5.13}$$

from [31, equation (5.1)].

Step 1. Let $\Phi_n^\pm$, $\Psi_n^\pm$ be $N \times N$ matrices defined by

$$\Phi_n^+ := (-1)^{n-1}K^{n-1}B^+ - B^+K^{n-1}, \quad \Phi_n^- := (-1)^{n-1}B^-K^{n-1} - K^{n-1}B^-,$$
$$\Psi_n^+ := (-1)^{n-1}K^{n-1} - B^+K^{n-1}B^-, \quad \Psi_n^- := (-1)^{n-1}B^-K^{n-1}B^+ - K^{n-1}.$$

In addition, we put

$$F_n^\pm := \frac{1}{2}\,{}^t\boldsymbol{e}\,(I - B^\pm B^\mp)^{-1}\Phi_n^\pm(I - B^\mp B^\pm)^{-1}\boldsymbol{b}^\mp,$$
$$G_n^\pm := {}^t\boldsymbol{e}(I - B^\pm B^\mp)^{-1}\Psi_n^\pm(I - B^\pm B^\mp)^{-1}\boldsymbol{b}^\pm.$$

Then definition (5.9) is rewritten in terms of ρ_n as

$$\rho_n = F_n^+ + F_n^- + \frac{1}{2}\,\frac{1+J^+}{1+J^-}G_n^+ + \frac{1}{2}\,\frac{1+J^-}{1+J^+}G_n^-. \tag{5.14}$$

A computation with

$$\begin{aligned} &({}^t\boldsymbol{e})' = {}^t\boldsymbol{e}K, \ \ (\boldsymbol{b}^\pm)' = K\boldsymbol{b}^\pm, \\ &((I - B^\mp B^\pm)^{-1})' = (I - B^\mp B^\pm)^{-1}(B^\mp B^\pm)'(I - B^\mp B^\pm)^{-1} \end{aligned} \tag{5.15}$$

shows that

$$\begin{aligned} 2\frac{\partial}{\partial x}F_n^- &= \frac{\partial}{\partial x}\big({}^t\boldsymbol{e}\,(I - B^-B^+)^{-1}\times \\ &\qquad \big((-1)^{n-1}B^-K^{n-1} - K^{n-1}B^-\big)\,(I - B^+B^-)^{-1}\boldsymbol{b}^+\big) \\ &= {}^t\boldsymbol{e}\,(I - B^-B^+)^{-1}\times \\ &\qquad \big[\,(I - B^-B^+)K(I - B^-B^+)^{-1}\times \\ &\qquad \big((-1)^{n-1}B^-K^{n-1} - K^{n-1}B^-\big) \\ &\qquad +((B^-)'B^+ + B^-(B^+)')(I - B^-B^+)^{-1}\times \\ &\qquad \big((-1)^{n-1}B^-K^{n-1} - K^{n-1}B^-\big) \\ &\qquad +(-1)^{n-1}(B^-)'K^{n-1} - K^{n-1}(B^-)' \\ &\qquad +\big((-1)^{n-1}B^-K^{n-1} - K^{n-1}B^-\big)\times \\ &\qquad (I - B^+B^-)^{-1}((B^+)'B^- + B^+(B^-)') \\ &\qquad +\big((-1)^{n-1}B^-K^{n-1} - K^{n-1}B^-\big)\times \\ &\qquad (I - B^+B^-)^{-1}K(I - B^+B^-)\,\big] \\ &\qquad \times(I - B^+B^-)^{-1}\boldsymbol{b}^+. \end{aligned} \tag{5.16}$$

The bracketed term $[\cdots\cdots]$ can be computed as

$$\begin{aligned}
[\cdots\cdots] &= (I-B^-B^+)K(I-B^-B^+)^{-1}\left((-1)^{n-1}B^-K^{n-1}-K^{n-1}B^-\right)\\
&+ B^-(B^+)'(I-B^-B^+)^{-1}\left((-1)^{n-1}B^-K^{n-1}-K^{n-1}B^-\right)\\
&- (B^-)'B^+(I-B^-B^+)^{-1}K^{n-1}B^-\\
&+ (-1)^{n-1}(B^-)'\left(B^+(I-B^-B^+)^{-1}B^-+I\right)K^{n-1}\\
&- K^{n-1}\left(I+B^-(I-B^+B^-)^{-1}B^+\right)(B^-)'\\
&+ (-1)^{n-1}B^-K^{n-1}(I-B^+B^-)^{-1}B^+(B^-)'\\
&+ \left((-1)^{n-1}B^-K^{n-1}-K^{n-1}B^-\right)(I-B^+B^-)^{-1}(B^+)'B^-\\
&+ \left((-1)^{n-1}B^-K^{n-1}-K^{n-1}B^-\right)(I-B^+B^-)^{-1}K(I-B^+B^-).
\end{aligned}$$

By using

$$B^{\pm}(I-B^{\mp}B^{\pm})^{-1}B^{\mp}+I=(I-B^{\pm}B^{\mp})^{-1}$$

and $(B^{\pm})'=B^{\pm}K+KB^{\pm}$, this can rewritten as

$$\begin{aligned}
[\cdots\cdots] &= (K+B^-KB^+)(I-B^-B^+)^{-1}\left((-1)^{n-1}B^-K^{n-1}-K^{n-1}B^-\right)\\
&\quad - (B^-)'B^+(I-B^-B^+)^{-1}K^{n-1}B^-\\
&\quad + (-1)^{n-1}(B^-)'(I-B^+B^-)^{-1}K^{n-1}\\
&\quad - K^{n-1}(I-B^-B^+)^{-1}(B^-)'\\
&\quad + (-1)^{n-1}B^-K^{n-1}(I-B^+B^-)^{-1}B^+(B^-)'\\
&\quad + \left((-1)^{n-1}B^-K^{n-1}-K^{n-1}B^-\right)\times\\
&\quad (I-B^+B^-)^{-1}(B^+KB^-+K).\\
&= 2(-1)^nB^-K^n+2K^nB^-\\
&\quad + 2(-1)^{n-1}(B^-)'(I-B^+B^-)^{-1}K^{n-1}\\
&\quad - 2K^{n-1}(I-B^-B^+)^{-1}(B^-)'\\
&\quad - 2(K+B^-KB^+)(I-B^-B^+)^{-1}K^{n-1}B^-\\
&\quad + 2(-1)^{n-1}B^-K^{n-1}(I-B^+B^-)^{-1}(K+B^+KB^-).
\end{aligned}$$

This, together with $(B^-)'=\boldsymbol{b}^{-}\,{}^t\boldsymbol{e}$, leads to

$$\begin{aligned}
\frac{\partial}{\partial x}F_n^- &= {}^t\boldsymbol{e}\,(I-B^-B^+)^{-1}\times\\
&\quad [\,(-1)^nB^-K^n+K^nB^-
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{n-1}\boldsymbol{b}^{-}\,{}^{t}\boldsymbol{e}(I-B^{+}B^{-})^{-1}K^{n-1}-K^{n-1}(I-B^{-}B^{+})^{-1}\boldsymbol{b}^{-}\,{}^{t}\boldsymbol{e}\\
&-(K+B^{-}KB^{+})(I-B^{-}B^{+})^{-1}K^{n-1}B^{-}\\
&+(-1)^{n-1}B^{-}K^{n-1}(I-B^{+}B^{-})^{-1}(K+B^{+}KB^{-})\,]\\
&\times(I-B^{+}B^{-})^{-1}\boldsymbol{b}^{+}.
\end{aligned}
$$

By using

$$
\begin{aligned}
K+B^{-}KB^{+}&=K(I-B^{-}B^{+})+(B^{-}K+KB^{-})B^{+},\\
K+B^{+}KB^{-}&=(I-B^{+}B^{-})K+B^{+}(B^{-}K+KB^{-})
\end{aligned}
$$

and (5.7), (5.13), this can be written as

$$
\begin{aligned}
\frac{\partial}{\partial x}F_n^{-} = {}&{}^{t}\boldsymbol{e}\,(I-B^{-}B^{+})^{-1}\times\\
&\big[\,(-1)^{n-1}\boldsymbol{b}^{-}\,{}^{t}\boldsymbol{e}(I-B^{+}B^{-})^{-1}K^{n-1}-K^{n-1}(I-B^{-}B^{+})^{-1}\boldsymbol{b}^{-}\,{}^{t}\boldsymbol{e}\\
&\ -\boldsymbol{b}^{-}\,{}^{t}\boldsymbol{e}B^{+}(I-B^{-}B^{+})^{-1}K^{n-1}B^{-}\\
&\ +(-1)^{n-1}B^{-}K^{n-1}(I-B^{+}B^{-})^{-1}B^{+}\boldsymbol{b}^{-}\,{}^{t}\boldsymbol{e}\,\big]\\
&\times(I-B^{+}B^{-})^{-1}\boldsymbol{b}^{+}\\
={}&-\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{-}}\,{}^{t}\boldsymbol{e}(I-B^{+}B^{-})^{-1}(-1)^{n-1}K^{n-1}(I-B^{+}B^{-})^{-1}\boldsymbol{b}^{+}\\
&+\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{+}}\,{}^{t}\boldsymbol{e}\,(I-B^{-}B^{+})^{-1}K^{n-1}(I-B^{-}B^{+})^{-1}\boldsymbol{b}^{-}\\
&+\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{-}}\,{}^{t}\boldsymbol{e}(I-B^{+}B^{-})^{-1}B^{+}K^{n-1}B^{-}(I-B^{+}B^{-})^{-1}\boldsymbol{b}^{+}\\
&-\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{+}}\,{}^{t}\boldsymbol{e}\,(I-B^{-}B^{+})^{-1}(-1)^{n-1}B^{-}K^{n-1}B^{+}(I-B^{-}B^{+})^{-1}\boldsymbol{b}^{-}.
\end{aligned}
$$

Accordingly

$$
\frac{\partial}{\partial x}F_n^{-}=-\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{-}}G_n^{+}-\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{+}}G_n^{-}. \tag{5.17}
$$

By the change $Q\to -Q$, we also have

$$
\frac{\partial}{\partial x}F_n^{+}=-\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{-}}G_n^{+}-\frac{1}{2}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{+}}G_n^{-},
$$

and so

$$\frac{\partial}{\partial x}\frac{F_n^+ + F_n^-}{2} = -\frac{1}{2}\frac{\frac{\partial}{\partial x}J^+}{1+J^-}G_n^+ - \frac{1}{2}\frac{\frac{\partial}{\partial x}J^-}{1+J^+}G_n^-. \tag{5.18}$$

In a similar way to for the deduction of (5.17) we obtain

$$\frac{1}{2}\frac{\partial}{\partial x}G_n^+ = -G_{n+1}^+ - \frac{\frac{\partial}{\partial x}J^-}{1+J^+}F_n^- - \frac{\frac{\partial}{\partial x}J^-}{1+J^+}F_n^+, \tag{5.19}$$

and also, by the change $Q \to -Q$, we have

$$\frac{1}{2}\frac{\partial}{\partial x}G_n^- = G_{n+1}^- - \frac{\frac{\partial}{\partial x}J^+}{1+J^-}F_n^+ - \frac{\frac{\partial}{\partial x}J^+}{1+J^-}F_n^-, \tag{5.20}$$

We now set

$$F_n = \frac{F_n^+ + F_n^-}{2}.$$

Then (5.14), (5.18), (5.19), (5.20) are simply expressed as

$$\rho_n = 2F_n + \frac{1}{2}\frac{1+J^+}{1+J^-}G_n^+ + \frac{1}{2}\frac{1+J^-}{1+J^+}G_n^-, \tag{5.21}$$

$$\begin{cases} F_n{}' = -\dfrac{1}{2}\dfrac{\frac{\partial}{\partial x}J^+}{1+J^-}G_n^+ - \dfrac{1}{2}\dfrac{\frac{\partial}{\partial x}J^-}{1+J^+}G_n^-, \\ \dfrac{1}{2}(G_n^+)' = -2\dfrac{\frac{\partial}{\partial x}J^-}{1+J^+}F_n - G_{n+1}^+, \\ \dfrac{1}{2}(G_n^-)' = -2\dfrac{\frac{\partial}{\partial x}J^+}{1+J^-}F_n + G_{n+1}^-. \end{cases} \tag{5.22}$$

Step 2. In order to compute derivatives of ρ_n, we employ the formulas:

$$-2iQ = \frac{\frac{\partial}{\partial x}J^+}{1+J^+} - \frac{\frac{\partial}{\partial x}J^-}{1+J^-}, \tag{5.23}$$

$$-2U = \left(\frac{\frac{\partial}{\partial x}J^+}{1+J^+} + \frac{\frac{\partial}{\partial x}J^-}{1+J^-}\right)' - \frac{1}{2}\left(\frac{\frac{\partial}{\partial x}J^+}{1+J^+} + \frac{\frac{\partial}{\partial x}J^-}{1+J^-}\right)^2, \tag{5.24}$$

$$\frac{1}{2}\left(\frac{1+J^\pm}{1+J^\mp}\right)' = \mp iQ\,\frac{1+J^\pm}{1+J^\mp}. \tag{5.25}$$

Formula (5.23) is immediate from (3.11) and (5.3) as

$$-2iQ = \frac{\partial}{\partial x}\left(\log \Delta^+ - \log \Delta^-\right) = \frac{\partial}{\partial x}\left(\log(1+J^+) - \log(1+J^-)\right),$$

formula (5.24) is borrowed from [31, equation (3.9)], and formula (5.25) follows from (5.23) because

$$\left(\frac{1+J^\pm}{1+J^\mp}\right)' = \pm\frac{1+J^\pm}{1+J^\mp}\left(\frac{\frac{\partial}{\partial x}J^+}{1+J^+} - \frac{\frac{\partial}{\partial x}J^-}{1+J^-}\right).$$

By using (5.21), (5.22), (5.25), (5.23), the derivative of ρ_n is computed as

$$\begin{aligned}\rho_n' = {} & -2\Omega F_n - \frac{1}{2}\Omega\frac{1+J^+}{1+J^-}G_n^+ - \frac{1}{2}\Omega\frac{1+J^-}{1+J^+}G_n^- \\ & - \frac{1+J^+}{1+J^-}G_{n+1}^+ + \frac{1+J^-}{1+J^+}G_{n+1}^-,\end{aligned} \tag{5.26}$$

where we put

$$\Omega := \frac{\frac{\partial}{\partial x}J^+}{1+J^+} + \frac{\frac{\partial}{\partial x}J^-}{1+J^-}.$$

This coupled with (5.21) yields

$$\begin{aligned} & -4\rho_{n+2}' - 8iQ\rho_{n+1}' - 4iQ_x\rho_{n+1} - 2U\rho_n' \\ & = 4\Omega UF_n + \Omega\frac{1+J^+}{1+J^-}UG_n^+ + \Omega\frac{1+J^-}{1+J^+}UG_n^- \\ & \quad + 8\left(\Omega\, 2iQ - iQ_x\right)F_{n+1} + 2\left(\Omega\, 2iQ - iQ_x + U\right)\frac{1+J^+}{1+J^-}G_{n+1}^+ \\ & \quad + 2\left(\Omega\, 2iQ - iQ_x - U\right)\frac{1+J^-}{1+J^+}G_{n+1}^- \\ & \quad + 8\Omega F_{n+2} + 2\left(\Omega + 4iQ\right)\frac{1+J^+}{1+J^-}G_{n+2}^+ \\ & \quad + 2\left(\Omega - 4iQ\right)\frac{1+J^-}{1+J^+}G_{n+2}^- \\ & \quad + 4\frac{1+J^+}{1+J^-}G_{n+3}^+ - 4\frac{1+J^-}{1+J^+}G_{n+3}^-.\end{aligned} \tag{5.27}$$

Step 3. Differentiating (5.26), using (5.22), (5.25), (5.24), (5.23), (5.21), and noting (5.24) $\Leftrightarrow \Omega' = -2U + \frac{1}{2}\Omega^2$ show that

$$
\begin{aligned}
&\rho_n'' - 2U\rho_n \\
&= \Omega^2 F_n + \frac{1}{4}\Omega^2 \frac{1+J^+}{1+J^-} G_n^+ + \frac{1}{4}\Omega^2 \frac{1+J^-}{1+J^+} G_n^- + 8iQF_{n+1} \\
&\quad + 2\frac{\frac{\partial}{\partial x}J^-}{1+J^-}\frac{1+J^+}{1+J^-}G_{n+1}^+ - 2\frac{\frac{\partial}{\partial x}J^+}{1+J^+}\frac{1+J^-}{1+J^+}G_{n+1}^- \\
&\quad + 2\frac{1+J^+}{1+J^-}G_{n+2}^+ + 2\frac{1+J^-}{1+J^+}G_{n+2}^-.
\end{aligned}
$$

We differentiate this and use (5.22). Then, with the aid of (5.24), (5.25), (5.23), the coefficients of F_n, $G_n^\pm$, $G_{n+1}^\pm$ can be computed respectively as

$$
\begin{aligned}
&2\Omega\,\Omega' - \Omega^3 = -4\Omega U, \\
&-\frac{1}{2}\Omega^2 \frac{\frac{\partial}{\partial x}J^\pm}{1+J^\mp} + \frac{1}{4}\left(\Omega^2 \frac{1+J^\pm}{1+J^\mp}\right)' = -\Omega\frac{1+J^\pm}{1+J^\mp}U, \\
&2\left(\frac{\frac{\partial}{\partial x}J^\mp}{1+J^\mp}\frac{1+J^\pm}{1+J^\mp}\right)' - 4iQ\frac{\frac{\partial}{\partial x}J^\pm}{1+J^\mp} \mp \frac{1}{2}\Omega^2\frac{1+J^\pm}{1+J^\mp} \\
&\qquad = 2\left(-\Omega\,2iQ + iQ_x \mp U\right)\frac{1+J^\pm}{1+J^\mp}.
\end{aligned}
$$

Consequently we find

$$
\begin{aligned}
&\left(\rho_n'' - 2U\rho_n\right)' \\
&= -4\Omega U F_n - \Omega\frac{1+J^+}{1+J^-}UG_n^+ - \Omega\frac{1+J^-}{1+J^+}UG_n^- + 8\left(iQ_x - \Omega\,2iQ\right)F_{n+1} \\
&\quad + 2\left(-\Omega\,2iQ + iQ_x - U\right)\frac{1+J^+}{1+J^-}G_{n+1}^+ \\
&\quad + 2\left(-\Omega\,2iQ + iQ_x + U\right)\frac{1+J^-}{1+J^+}G_{n+1}^- \\
&\quad - 8\Omega F_{n+2} + 2\left(\left(\frac{1+J^+}{1+J^-}\right)' - 2\frac{\frac{\partial}{\partial x}J^-}{1+J^-}\frac{1+J^+}{1+J^-}\right)G_{n+2}^+ \\
&\quad + 2\left(\left(\frac{1+J^-}{1+J^+}\right)' - 2\frac{\frac{\partial}{\partial x}J^+}{1+J^+}\frac{1+J^-}{1+J^+}\right)G_{n+2}^- \\
&\quad - 4\frac{1+J^+}{1+J^-}G_{n+3}^+ + 4\frac{1+J^-}{1+J^+}G_{n+3}^-.
\end{aligned}
\tag{5.28}
$$

Step 4. Adding (5.28) to (5.27) and observing

$$(\Omega \pm 4iQ)\frac{1+J^{\pm}}{1+J^{\mp}} + \left(\frac{1+J^{\pm}}{1+J^{\mp}}\right)' - 2\frac{\frac{\partial}{\partial x}J^{\mp}}{1+J^{\mp}}\frac{1+J^{\pm}}{1+J^{\mp}} = 0,$$

we conclude that

$$-4\rho'_{n+2} - 8iQ\rho'_{n+1} - 4iQ_x\rho_{n+1} - 2U\rho'_n + \left(\rho''_n - 2U\rho_n\right)' = 0.$$

This is nothing but (5.12). We have thus proved (5.11). □

We are now in a position to prove

THEOREM 5.3. *Let $\Delta^{\pm}$ be functions in* (1.12) *with* (1.17) *and assume that $\Delta^{\pm}(x,t) > 0$. Then (Q,U) defined by* (3.11) *satisfies the first equation*

$$\frac{1}{ia_n}Q_t = p'_{n+1} \tag{5.29}$$

of system QU$[n]$ *for each $n = 1, 2, \cdots$.*

PROOF. In the case $n = 1$, (5.9) becomes

$$r_1 = -i\left({}^t\boldsymbol{e}\,(I - B^-B^+)^{-1}\boldsymbol{b}^-\frac{1+J^-}{1+J^+} - {}^t\boldsymbol{e}\,(I - B^+B^-)^{-1}\boldsymbol{b}^+\frac{1+J^+}{1+J^-}\right).$$

Use of (5.13), (5.23) now shows that

$$r_1 = -\frac{1}{2i}\left(\frac{\frac{\partial}{\partial x}J^+}{1+J^+} - \frac{\frac{\partial}{\partial x}J^-}{1+J^-}\right) = Q = p_1.$$

As was shown in Lemma 5.1, if $\Delta^{\pm} > 0$ then the function $r_n(\cdot,t)$ is of the class $C^{\infty}(\boldsymbol{R})$ for $n \geq 2$. The equality above tells us that $r_1(\cdot,t)$ is also of the class. Therefore the recursion relation (5.11) holds on $\boldsymbol{R}$ for each t.

In the case $n = 2$, since $B^{\pm}K + KB^{\pm} = \boldsymbol{b}^{\pm}\,{}^t\boldsymbol{e}$, expression (5.9) gives

$$\begin{aligned} r_2 = {} & 2\,{}^t\boldsymbol{e}\,(I - B^-B^+)^{-1}\boldsymbol{b}^-\,{}^t\boldsymbol{e}(I - B^+B^-)^{-1}\boldsymbol{b}^+ \\ & + {}^t\boldsymbol{e}\,(I - B^-B^+)^{-1}\left(K + B^-KB^+\right)(I - B^-B^+)^{-1}\boldsymbol{b}^-\frac{1+J^-}{1+J^+} \\ & + {}^t\boldsymbol{e}\,(I - B^+B^-)^{-1}\left(K + B^+KB^-\right)(I - B^+B^-)^{-1}\boldsymbol{b}^+\frac{1+J^+}{1+J^-}. \end{aligned}$$

The first term in the right is rewritten via (5.13). Also, by differentiating (5.13) and using (5.15) as in (5.16), we have

$$-\frac{1}{4}\frac{\partial}{\partial x}\frac{\frac{\partial}{\partial x}J^{\pm}}{1+J^{\mp}} = {}^{t}\boldsymbol{e}\,(I-B^{\mp}B^{\pm})^{-1}(K+B^{\mp}K\,B^{\pm})(I-B^{\mp}B^{\pm})^{-1}\,\boldsymbol{b}^{\mp}.$$

Hence r_2 is expressed as

$$\begin{aligned} r_2 &= \frac{1}{2}\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{-}}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{+}} - \frac{1}{4}\left(\frac{\partial}{\partial x}\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{-}}\right)\frac{1+J^{-}}{1+J^{+}} \\ &\quad - \frac{1}{4}\left(\frac{\partial}{\partial x}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{+}}\right)\frac{1+J^{+}}{1+J^{-}} \\ &= \frac{\frac{\partial}{\partial x}J^{+}}{1+J^{-}}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{+}} - \frac{1}{4}\left(\frac{\frac{\partial^2}{\partial x^2}J^{+}}{1+J^{+}} + \frac{\frac{\partial^2}{\partial x^2}J^{-}}{1+J^{-}}\right). \end{aligned}$$

On the other hand, a direct computation shows (see [31, page 704]) that

$$U+Q^2 = -\frac{1}{2}\left(\frac{\frac{\partial^2}{\partial x^2}J^{+}}{1+J^{+}} + \frac{\frac{\partial^2}{\partial x^2}J^{-}}{1+J^{-}}\right) + \frac{1}{2}\left(\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{+}} + \frac{\frac{\partial}{\partial x}J^{-}}{1+J^{-}}\right)^2,$$

which together with (5.23) yields

$$U+Q^2+2Q^2 = -\frac{1}{2}\left(\frac{\frac{\partial^2}{\partial x^2}J^{+}}{1+J^{+}} + \frac{\frac{\partial^2}{\partial x^2}J^{-}}{1+J^{-}}\right) + 2\frac{\frac{\partial}{\partial x}J^{+}}{1+J^{+}}\frac{\frac{\partial}{\partial x}J^{-}}{1+J^{-}}.$$

Hence $p_2 = \frac{1}{2}(U+3Q^2) = r_2$.

This, combined with Lemma 5.2, shows that $r_n = p_n$ for $n \geq 1$, and so, by (5.10), that

$$\frac{1}{2a_n}\frac{\partial}{\partial t}\left(\log \Delta^{+} - \log \Delta^{-}\right) = p_{n+1}$$

for $n \geq 1$. Differentiating this in x and taking (3.11) into consideration we get the desired result (5.29). □

By combining Theorem 5.3 with Theorem 4.3 we draw the following conclusion.

Theorem 5.4. *Let* a_n *satisfy* (1.16). *Assume that* k_ℓ, $c_\ell^{\pm}(0)$ *satisfy conditions* (I), (II) *in Proposition* 1.2 *and define* $c_\ell^{\pm}(t)$ *by* (1.17). *Then*

(Q, U) *defined by* (3.11) *satisfies the system* QU$[n]$ *for each natural number* n *as long as* $\Delta^{\pm}(x,t) > 0$ *on* $\boldsymbol{R}$.

The schema of inverse scattering method in Figure 1 based upon Proposition 1.2 has been thus carried out as long as $\Delta^{\pm} > 0$.

6. Expansion Expression of $\Delta^{\pm}$

For further studies on soliton solutions of QU$[n]$, in particular, to consider under what conditions the situation $\Delta^{\pm} > 0$ can be guaranteed, we require a more direct expression of $\Delta^{\pm}$.

Proposition 6.1. *Assume* $\operatorname{Im} k_\ell > 0$, $\ell = 1, \cdots, N$, *set* $\alpha_\ell := ik_\ell$, $X_\ell := e^{2ik_\ell x} = e^{2\alpha_\ell x}$, *and let* V *denote the difference product*

$$V(z_1, \cdots, z_M) = \prod_{m<n} (z_m - z_n)$$

with the convention $V(z_1) = V(\phi) = 1$. *Then the functions* $\Delta^{\pm}(x)$ *are expressed as polynomials of* $X_1, \cdots, X_N$ *of the order* $2N$ *in the following expansion forms:*

$$\begin{aligned}
\Delta^{\pm} = 1 &+ \sum_{p=1}^{N} \left(-\frac{1}{\alpha_p}\right) c_p^{\mp} X_p \\
&+ \sum_{r=2}^{N} \sum_{p_1<\cdots<p_r} \sum_{q_1<\cdots<q_{r-1}} \left(-\frac{\alpha_{q_1}\cdots\alpha_{q_{r-1}}}{\alpha_{p_1}\cdots\alpha_{p_r}}\right) \times \\
&\left(\frac{V(\alpha_{p_1}, \cdots, \alpha_{p_r}) V(\alpha_{q_1}, \cdots, \alpha_{q_{r-1}})}{\prod_{m=p_1,\cdots,p_r} \prod_{n=q_1,\cdots,q_{r-1}} (\alpha_m + \alpha_n)}\right)^2 \times \\
&\left(\prod_{m=p_1,\cdots,p_r} c_m^{\mp} X_m\right) \left(\prod_{m=q_1,\cdots,q_{r-1}} c_m^{\pm} X_m\right) \\
&+ \sum_{r=1}^{N} \sum_{p_1<\cdots<p_r} \sum_{q_1<\cdots<q_r} \left(\frac{\alpha_{p_1}\cdots\alpha_{p_r}}{\alpha_{q_1}\cdots\alpha_{q_r}}\right) \times
\end{aligned} \tag{6.1}$$

$$\left(\frac{V(\alpha_{p_1},\cdots,\alpha_{p_r})V(\alpha_{q_1},\cdots,\alpha_{q_r})}{\prod_{m=p_1,\cdots,p_r}\prod_{n=q_1,\cdots,q_r}(\alpha_m+\alpha_n)}\right)^2\times$$
$$\left(\prod_{m=p_1,\cdots,p_r} c_m^{\mp}X_m\right)\left(\prod_{m=q_1,\cdots,q_r} c_m^{\pm}X_m\right).$$

Here the highest order term of $\Delta^{\pm}$ is given by $r=N$ in the last term as

$$\left(\frac{V(\alpha_1,\cdots,\alpha_N)^2}{\prod_{m=1}^N\prod_{n=1}^N(\alpha_m+\alpha_n)}\right)^2\left(\prod_{m=1}^N c_m^+c_m^-X_m^2\right).$$

PROOF. We shall prove the expression of Δ^+. Set

$$A=(a_{\ell j})=\left(\frac{c_\ell^-}{\alpha_\ell+\alpha_j}\right),\quad B=(b_{\ell j})=\left(\frac{c_\ell^+}{\alpha_\ell+\alpha_j}\right),\quad \boldsymbol{v}_0^{\pm}=\left(\frac{c_\ell^{\pm}}{\alpha_\ell}\right),$$
$$\boldsymbol{a}_j=\begin{pmatrix}a_{1j}\\ \vdots\\ a_{Nj}\end{pmatrix},\quad \boldsymbol{b}_j=\begin{pmatrix}b_{1j}\\ \vdots\\ b_{Nj}\end{pmatrix},$$
$$G=\begin{pmatrix}\frac{1}{\alpha_1+\alpha_1}&\cdots&\frac{1}{\alpha_1+\alpha_N}\\ \vdots&\cdots&\vdots\\ \frac{1}{\alpha_N+\alpha_1}&\cdots&\frac{1}{\alpha_N+\alpha_N}\end{pmatrix},\quad \boldsymbol{e}_j=\begin{pmatrix}0\\ \cdot\\ 1\\ \cdot\\ 0\end{pmatrix},$$

and use the notation

$${}^t\!\begin{pmatrix}\boldsymbol{a}_1X_1\\ \vdots\\ \boldsymbol{a}_NX_N\end{pmatrix}\boldsymbol{b}_j:=\sum_{\ell=1}^N\boldsymbol{a}_\ell X_\ell\, b_{\ell j}.$$

Then $\det(I-B^+B^-)$ is written as

$$\det(I-B^+B^-)=(-1)^N\left(\prod_{m=1}^N X_m\right)\times$$
$$\left|-X_1^{-1}\boldsymbol{e}_1+{}^t\!\begin{pmatrix}\boldsymbol{a}_1X_1\\ \vdots\\ \boldsymbol{a}_NX_N\end{pmatrix}\boldsymbol{b}_1\cdots\cdots-X_N^{-1}\boldsymbol{e}_N+{}^t\!\begin{pmatrix}\boldsymbol{a}_1X_1\\ \vdots\\ \boldsymbol{a}_NX_N\end{pmatrix}\boldsymbol{b}_N\right|. \tag{6.2}$$

Also, $-(\, e^{ik_1x} \;\cdots\; e^{ik_Nx}\,)(I - B^- B^+)\tilde{}\,(B^- \boldsymbol{v}^+ - \boldsymbol{v}^-)$ is expressed as

$$\Gamma := (-1)^N \prod_{m=1}^{N} X_m \times$$
$$\sum_{j=1}^{N} \left| -X_1^{-1}\boldsymbol{e}_1 + {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots - \boldsymbol{v}_0^- + {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{v}_0^+ \cdots \right.$$
$$\left. -X_N^{-1}\boldsymbol{e}_N + {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|.$$

Accordingly

$$\Delta^+ = \det(I - B^+ B^-) - \Gamma. \tag{6.3}$$

By introducing Γ_1, Γ_2 by

$$\Gamma_1 := (-1)^N \prod_{m=1}^{N} X_m \times \tag{6.4}$$
$$\sum_{j=1}^{N} \left| -X_1^{-1}\boldsymbol{e}_1 + {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots - \boldsymbol{v}_0^- \cdots \right.$$
$$\left. -X_N^{-1}\boldsymbol{e}_N + {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|,$$

$$\Gamma_2 := (-1)^N \prod_{m=1}^{N} X_m \times \tag{6.5}$$
$$\sum_{j=1}^{N} \left| -X_1^{-1}\boldsymbol{e}_1 + {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{v}_0^+ \cdots \right.$$
$$\left. -X_N^{-1}\boldsymbol{e}_N + {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|,$$

where $\boldsymbol{v}_0^{\mp}$ is in the j-th column, Γ is decomposed as $\Gamma = \Gamma_1 + \Gamma_2$. Note that $\boldsymbol{a}_\ell$, $\boldsymbol{b}_\ell$, $\boldsymbol{v}_0^{\pm}$ are independent of X_ℓ and hence Γ_1, Γ_2 are polynomials in $X_1, \cdots X_N$ of the forms

$$\Gamma_1 = \sum_{0 \le \sigma_1, \cdots, \sigma_N \le 2} C_{\sigma_1, \cdots, \sigma_N} X_1^{\sigma_1} \cdots X_N^{\sigma_N}, \quad \sum_{\ell=1}^{N} \sigma_\ell = \text{odd},$$

$$\Gamma_2 = \sum_{0 \le \sigma_1, \cdots, \sigma_N \le 2} C_{\sigma_1, \cdots, \sigma_N} X_1^{\sigma_1} \cdots X_N^{\sigma_N}, \quad \sum_{\ell=1}^{N} \sigma_\ell = \text{even}.$$

In view of (6.2), $\det(I - B^+B^-)$ consists of terms of even orders, namely, of $X_1^{\sigma_1} \cdots X_N^{\sigma_N}$ with $\sum \sigma_\ell$ being even. Hence the sum of odd order terms in Δ^+ which we denote by Δ^+_{odd} is given by $-\Gamma_1$: $\Delta^+_{\text{odd}} = -\Gamma_1$

With the aid of (6.3), (6.4), for $s = 1, \cdots, N-1$, the sum of terms of order $2N - (2s+1)$ in Δ^+ is computed as

$$\begin{aligned}
\Delta^+_{2N-(2s+1)} &= (-1)^{N-1} \left(\prod_{m=1}^{N} X_m \right) \times \\
&\sum_{j=1}^{N} \sum_{k_1 < \cdots < k_s; k_s \neq j} \left| {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots - X_{k_1}^{-1} \boldsymbol{e}_{k_1} \cdots - \boldsymbol{v}_0^- \cdots \right. \\
&\qquad \left. - X_{k_s}^{-1} \boldsymbol{e}_{k_s} \cdots {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_N \right| \\
&= (-1)^{N-1} (-1)^{s+1} \left(\prod_{m=1}^{N} X_m \right) \times \\
&\sum_{k_1 < \cdots < k_s} X_{k_1}^{-1} \cdots X_{k_s}^{-1} \sum_{j \neq k_1, \cdots, k_s} \times \\
&\left| {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots \boldsymbol{e}_{k_1} \cdots \boldsymbol{v}_0^- \cdots \boldsymbol{e}_{k_s} \cdots {}^t\!\begin{pmatrix} \boldsymbol{a}_1 X_1 \\ \vdots \\ \boldsymbol{a}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|
\end{aligned}$$

$$= (-1)^{N+s}\left(\prod_{m=1}^{N} X_m\right) \times$$

$$\sum_{k_1<\cdots<k_s} X_{k_1}^{-1}\cdots X_{k_s}^{-1} \sum_{j\neq k_1,\cdots,k_s} \times$$

$$\left| {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots \overset{k_1}{\vee} \cdots \check{\boldsymbol{v}}_0^- \cdots \overset{k_s}{\vee} \cdots {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|,$$

where $\check{\boldsymbol{a}}$ denotes a vector obtained by deleting the $k_1, \cdots, k_s$-th rows of $\boldsymbol{a}$, and $\overset{k}{\vee}$ is the symbol for the delete of the k-th element. Taking the order into account, we have

$$\sum_{j\neq k_1,\cdots,k_s} \left| {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots \overset{k_1}{\vee} \cdots \check{\boldsymbol{v}}_0^- \cdots \overset{k_s}{\vee} \cdots {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|$$

$$= \sum_{\lambda_1<\cdots<\lambda_{s+1}} C_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_{s+1}} \left(\prod_{m=1}^{N} X_m\right) X_{\lambda_1}^{-1}\cdots X_{\lambda_{s+1}}^{-1},$$

with constants $C_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_{s+1}}$. Because these constants can be obtained by putting

$$X_{\lambda_1} = \cdots = X_{\lambda_{s+1}} = 0, \quad X_m = 1 \quad \text{for} \quad m \neq \lambda_1, \cdots, \lambda_{s+1}$$

these coefficients are expressed as

$$C_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_{s+1}}$$

$$= \sum_{j\neq k_1,\cdots,k_s} \left| \sum_{m\neq\lambda_1,\cdots,\lambda_{s+1}} \check{\boldsymbol{a}}_m b_{m1} \cdots \overset{k_1}{\vee} \cdots \boldsymbol{v}_0^- \cdots \overset{k_s}{\vee} \cdots \sum_{m\neq\lambda_1,\cdots,\lambda s+1}^{N} \check{\boldsymbol{a}}_m b_{mN} \right|.$$

Thus

$$\Delta_{2N-(2s+1)}^{+} = (-1)^{N+s}\left(\prod_{m=1}^{N} X_m^2\right) \sum_{k_1<\cdots<k_s} X_{k_1}^{-1}\cdots X_{k_s}^{-1} \times \sum_{\lambda_1<\cdots<\lambda_{s+1}} C_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_{s+1}} X_{\lambda_1}^{-1}\cdots X_{\lambda_{s+1}}^{-1}. \tag{6.6}$$

By expanding $C^{\lambda_1\cdots\lambda_{s+1}}_{k_1\cdots k_s}$ with respect to the j-th column, we get

$$C^{\lambda_1\cdots\lambda_{s+1}}_{k_1\cdots k_s} = \sum_{j\neq k_1,\cdots,k_s}\sum_{\ell\neq k_1,\cdots,k_s}(-1)^{\ell+j}\frac{c^-_\ell}{\alpha_\ell}\times \left|\sum_{m\neq\lambda_1,\cdots,\lambda_{s+1}}\check{\check{\boldsymbol{a}}}_m b_{m1}\cdots \overset{k_1}{\vee}\cdots\overset{j}{\vee}\cdots\overset{k_s}{\vee}\cdots\sum_{m\neq\lambda_1,\cdots,\lambda_{s+1}}\check{\check{\boldsymbol{a}}}_m b_{mN}\right|,$$

where $\check{\check{\boldsymbol{a}}}$ denotes a vector obtained by deleting the $\ell, k_1,\cdots,k_s$-th rows of $\boldsymbol{a}$. This becomes

$$C^{\lambda_1\cdots\lambda_{s+1}}_{k_1\cdots k_s} = \sum_{j\neq k_1,\cdots,k_s}\sum_{\ell\neq k_1,\cdots,k_s}(-1)^{\ell+j}\frac{c^-_\ell}{\alpha_\ell}\times \left|(\check{\check{\boldsymbol{a}}}_1\cdots\overset{\lambda_1}{\vee}\cdots\overset{\lambda_{s+1}}{\vee}\cdots\check{\check{\boldsymbol{a}}}_N)(\check{\check{\boldsymbol{b}}}_1\cdots\overset{k_1}{\vee}\cdot\overset{j}{\vee}\cdot\overset{k_s}{\vee}\cdots\check{\check{\boldsymbol{b}}}_N)\right|,$$

where $\check{\check{\boldsymbol{b}}}$ denotes the vector obtained by deleting the $\lambda_1,\cdots,\lambda_{s+1}$-th rows from $\boldsymbol{b}$. Let $A^{\lambda_1\cdots\lambda_{s+1}}_{\ell k_1\cdots k_s}$ denotes the $(N-(s+1))\times(N-(s+1))$-matrix obtained by deleting the $\ell, k_1,\cdots,k_s$-th rows and the $\lambda_1,\cdots,\lambda_{s+1}$-th columns from an $N\times N$-matrix A. Then we have

$$\begin{aligned}
C^{\lambda_1\cdots\lambda_{s+1}}_{k_1\cdots k_s} &= \sum_{j\neq k_1,\cdots,k_s}\sum_{\ell\neq k_1,\cdots,k_s}(-1)^{\ell+j}\frac{c^-_\ell}{\alpha_\ell}\left|A^{\lambda_1\cdots\lambda_{s+1}}_{\ell k_1\cdots k_s}B^{k_1\cdots k_s j}_{\lambda_1\cdots\lambda_{s+1}}\right| \\
&= \sum_{j\neq k_1,\cdots,k_s}\sum_{\ell\neq k_1,\cdots,k_s}(-1)^{\ell+j}\frac{c^-_\ell}{\alpha_\ell}\left|A^{\lambda_1\cdots\lambda_{s+1}}_{\ell k_1\cdots k_s}\right|\left|B^{k_1\cdots k_s j}_{\lambda_1\cdots\lambda_{s+1}}\right| \\
&= \left(\prod_{m\neq k_1,\cdots,k_s}c^-_m\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_{s+1}}c^+_m\right) \\
&\quad\times\sum_{j\neq k_1,\cdots,k_s}\sum_{\ell\neq k_1,\cdots,k_s}\frac{(-1)^{\ell+\lambda_{s+1}}}{\alpha_\ell}\left|G^{\lambda_1\cdots\lambda_{s+1}}_{\ell k_1\cdots k_s}\right|(-1)^{\lambda_{s+1}+j}\left|G^{k_1\cdots k_s j}_{\lambda_1\cdots\lambda_{s+1}}\right| \\
&= \left(\prod_{m\neq k_1,\cdots,k_s}c^-_m\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_{s+1}}c^+_m\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \begin{array}{r|ccccc|}
 & \frac{1}{\alpha_1+\alpha_1} & \cdot \overset{\lambda_1}{\vee} \cdot\cdot \overset{\lambda_s}{\vee} \cdot\cdot & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\alpha_N} \\
k_1> & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots \\
k_s> & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \frac{1}{\alpha_N+\alpha_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\alpha_N}
\end{array} \\
&\times \sum_{j\neq k_1,\cdots,k_s} (-1)^{\lambda_{s+1}+j} \left| G^{k_1\cdots k_s j}_{\lambda_1\cdots\lambda_{s+1}} \right| \\
= &\left(\prod_{m\neq k_1,\cdots,k_s} c^-_m \right) \left(\prod_{m\neq \lambda_1,\cdots,\lambda_{s+1}} c^+_m \right) \\
&\times \begin{array}{r|ccccc|}
 & \frac{1}{\alpha_1+\alpha_1} & \cdot \overset{\lambda_1}{\vee} \cdot\cdot \overset{\lambda_s}{\vee} \cdot\cdot & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\alpha_N} \\
k_1> & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots \\
k_s> & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \frac{1}{\alpha_N+\alpha_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\alpha_N}
\end{array} \\
&\times \begin{array}{|cccc|l}
\frac{1}{\alpha_1+\alpha_1} & \cdot \overset{k_1}{\vee} \cdot\cdot \overset{k_s}{\vee} & \cdots & \frac{1}{\alpha_1+\alpha_N} & \\
\vdots & \vdots & \vdots & \vdots & <\lambda_1 \\
\vdots & \vdots & \vdots & \vdots & <\lambda_s \\
1 & 1 & \cdots & 1 & \leftarrow\lambda_{s+1} \\
\vdots & \vdots & \vdots & \vdots & \\
\frac{1}{\alpha_N+\alpha_1} & \cdots & \cdots & \frac{1}{\alpha_N+\alpha_N} &
\end{array} .
\end{aligned}
$$

But, by [31, Lemma 3.1],

$$
\begin{array}{r|ccccc|}
 & \frac{1}{\alpha_1+\alpha_1} & \cdot \overset{\lambda_1}{\vee} \cdot\cdot \overset{\lambda_s}{\vee} \cdot\cdot & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_1+\alpha_N} \\
k_1> & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots \\
k_s> & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \frac{1}{\alpha_N+\alpha_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\alpha_N}
\end{array}
$$

$$= (-1)^{N-s-\lambda_{s+1}} \left(\frac{\alpha_{k_1} \cdots \alpha_{k_s}}{\alpha_{\lambda_1} \cdots \alpha_{\lambda_{s+1}}} \right)$$
$$\times \frac{V(\alpha_1, \cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots, \alpha_N) V(\alpha_1, \cdots \overset{\lambda_1}{\vee} \cdots \overset{\lambda_{s+1}}{\vee} \cdots, \alpha_N)}{\displaystyle\prod_{m \neq k_1, \cdots, k_s} \prod_{n \neq \lambda_1, \cdots, \lambda_{s+1}} (\alpha_m + \alpha_n)},$$

and

$$\begin{vmatrix} \frac{1}{\alpha_1+\alpha_1} & \cdot \overset{k_1}{\vee} \cdot\cdot \overset{k_s}{\vee} & \cdots & \frac{1}{\alpha_1+\alpha_N} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha_N+\alpha_1} & \cdots & \cdots & \frac{1}{\alpha_N+\alpha_N} \end{vmatrix} \begin{matrix} <\lambda_1 \\ \\ <\lambda_s \\ \leftarrow \lambda_{s+1} \\ \\ \\ \end{matrix}$$
$$= (-1)^{\lambda_{s+1}+1} \frac{V(\alpha_1, \cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots, \alpha_N) V(\alpha_1, \cdots \overset{\lambda_1}{\vee} \cdots \overset{\lambda_{s+1}}{\vee} \cdots, \alpha_N)}{\displaystyle\prod_{m \neq k_1, \cdots, k_s} \prod_{n \neq \lambda_1, \cdots, \lambda_{s+1}} (\alpha_m + \alpha_n)}.$$

Therefore

$$C_{k_1 \cdots k_s}^{\lambda_1 \cdots \lambda_{s+1}} = (-1)^{N-s+1} \left(\prod_{m \neq k_1, \cdots, k_s} c_m^- \right) \left(\prod_{m \neq \lambda_1, \cdots, \lambda_{s+1}} c_m^+ \right)$$
$$\times \left(\frac{\alpha_{k_1} \cdots \alpha_{k_s}}{\alpha_{\lambda_1} \cdots \alpha_{\lambda_{s+1}}} \right)$$
$$\times \left(\frac{V(\alpha_1, \cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots, \alpha_N) V(\alpha_1, \cdots \overset{\lambda_1}{\vee} \cdots \overset{\lambda_{s+1}}{\vee} \cdots, \alpha_N)}{\displaystyle\prod_{m \neq k_1, \cdots, k_s} \prod_{n \neq \lambda_1, \cdots, \lambda_{s+1}} (\alpha_m + \alpha_n)} \right)^2.$$

This, together with (6.6), shows that, for $s = 1, \cdots, N-1$,

$$(6.7)\quad \Delta^+_{2N-(2s+1)} = \sum_{k_1<\cdots<k_s} \sum_{\lambda_1<\cdots<\lambda_{s+1}} \left(\prod_{m\neq k_1,\cdots,k_s} c^-_m \right) \left(\prod_{m\neq \lambda_1,\cdots,\lambda_{s+1}} c^+_m \right)$$
$$\times \left(-\frac{\alpha_{k_1}\cdots\alpha_{k_s}}{\alpha_{\lambda_1}\cdots\alpha_{\lambda_{s+1}}} \right)$$
$$\times \left(\frac{V(\alpha_1, \cdot\cdot \overset{k_1}{\vee} \cdot\cdot \overset{k_s}{\vee} \cdot\cdot, \alpha_N) V(\alpha_1, \cdot\cdot \overset{\lambda_1}{\vee} \cdot\cdot \overset{\lambda_{s+1}}{\vee} \cdot\cdot, \alpha_N)}{\prod_{m\neq k_1,\cdots,k_s} \prod_{n\neq \lambda_1,\cdots,\lambda_{s+1}} (\alpha_m + \alpha_n)} \right)^2$$
$$\times \left(\prod_{m=1}^N X_m^2 \right) X_{k_1}^{-1} \cdots X_{k_s}^{-1} X_{\lambda_1}^{-1} \cdots X_{\lambda_{s+1}}^{-1}.$$

In particular, in the case $s = N-1$, we get

$$\Delta^+_1 = \sum_{p=1}^N c^-_p \left(-\frac{1}{\alpha_p} \right) X_p$$

by setting $p := \{k_1, \cdots, k_s\}^c$. In addition, for $s = 0$, formula (6.7) reads

$$\Delta^+_{2N-1} = \sum_{\lambda=1}^N \left(\prod_{m=1}^N c^-_m \right) \left(\prod_{m\neq\lambda}^N c^+_m \right) \left(-\frac{1}{\alpha_\lambda} \right)$$
$$\times \left(\frac{V(\alpha_1, \cdots, \alpha_N) V(\alpha_1, \cdots \overset{\lambda}{\vee} \cdots, \alpha_N)}{\prod_{m=1}^N \prod_{n\neq\lambda}^N (\alpha_m + \alpha_n)} \right)^2 \left(\prod_{m=1}^N X_m^2 \right) X_\lambda^{-1},$$

which is verified by an easier computation than that for $s \geq 1$. We have thus proved that

$$(6.8)\quad \Delta^+_{\mathrm{odd}} = \sum_{s=0}^{N-1} \sum_{k_1<\cdots<k_s} \sum_{\lambda_1<\cdots<\lambda_{s+1}} \left(\prod_{m\neq k_1,\cdots,k_s} c^-_m \right) \left(\prod_{m\neq \lambda_1,\cdots,\lambda_{s+1}} c^+_m \right)$$
$$\times \left(-\frac{\alpha_{k_1}\cdots\alpha_{k_s}}{\alpha_{\lambda_1}\cdots\alpha_{\lambda_{s+1}}} \right)$$

$$\times \left(\frac{V(\alpha_1, \cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots, \alpha_N) V(\alpha_1, \cdots \overset{\lambda_1}{\vee} \cdots \overset{\lambda_{s+1}}{\vee} \cdots, \alpha_N)}{\prod_{m \neq k_1, \cdots, k_s} \prod_{n \neq \lambda_1, \cdots, \lambda_{s+1}} (\alpha_m + \alpha_n)} \right)^2$$

$$\times \left(\prod_{m=1}^{N} X_m^2 \right) X_{k_1}^{-1} \cdots X_{k_s}^{-1} X_{\lambda_1}^{-1} \cdots X_{\lambda_{s+1}}^{-1}.$$

We next seek an expression of Δ_{even}^{+}. It follows from (6.2) that, for $s = 1, \cdots, N-1$, the sum of terms of order $2N - 2s$ in $\det(I - B^+B^-)$ is written as

$$\det(I - B^+B^-)_{2N-2s} = (-1)^{N+s} \left(\prod_{m=1}^{N} X_m \right) \times$$

$$\sum_{k_1 < \cdots < k_s} X_{k_1}^{-1} \cdots X_{k_s}^{-1} \left| {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|,$$

where $\check{\boldsymbol{a}}$ is the same symbol as before. Because

$$\left| {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_1 \cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots {}^t\!\begin{pmatrix} \check{\boldsymbol{a}}_1 X_1 \\ \vdots \\ \check{\boldsymbol{a}}_N X_N \end{pmatrix} \boldsymbol{b}_N \right|$$
$$= \sum_{\lambda_1 < \cdots < \lambda_s} C_{k_1 \cdots k_s}^{\lambda_1 \cdots \lambda_s} X_1 \cdots \overset{\lambda_1}{\vee} \cdots \overset{\lambda_s}{\vee} \cdots X_N$$

with

$$C_{k_1 \cdots k_s}^{\lambda_1 \cdots \lambda_s} = \left| \sum_{m \neq \lambda_1, \cdots, \lambda_s} \check{\boldsymbol{a}}_m b_{m1} \cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots \sum_{m \neq \lambda_1, \cdots, \lambda_s} \check{\boldsymbol{a}}_m b_{mN} \right|$$
$$= |A_{k_1 \cdots k_s}^{\lambda_1 \cdots \lambda_s} B_{\lambda_1 \cdots \lambda_s}^{k_1 \cdots k_s}|$$
$$= \left(\prod_{m \neq k_1, \cdots, k_s} c_m^- \right) \left(\prod_{m \neq \lambda_1, \cdots, \lambda_s} c_m^+ \right) \left| G_{k_1 \cdots k_s}^{\lambda_1 \cdots \lambda_s} \right| \left| G_{\lambda_1 \cdots \lambda_s}^{k_1 \cdots k_s} \right|,$$

it follows from the symmetry of the matrix G that

$$\det(I - B^+B^-)_{2N-2s} = (-1)^{N+s}\left(\prod_{m=1}^{N} X_m^2\right) \times \sum_{k_1<\cdots<k_s}\sum_{\lambda_1<\cdots<\lambda_s}\left(\prod_{m\neq k_1,\cdots,k_s} c_m^-\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_s} c_m^+\right) \times \left|G_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_s}\right|^2 X_{k_1}^{-1}\cdots X_{k_s}^{-1}X_{\lambda_1}^{-1}\cdots X_{\lambda_s}^{-1}. \tag{6.9}$$

On the other hand, by (6.5), the sum of terms of the same order in Γ_2 is computed as

$$\begin{aligned}(\mathit{\Gamma}_2)_{2N-2s} &= (-1)^{N+s}\left(\prod_{m=1}^{N} X_m\right)\sum_{j=1}^{N}\sum_{k_1,\cdots,k_s\neq j} \times \\ &\left| {}^t\!\begin{pmatrix} \boldsymbol{a}_1X_1 \\ \vdots \\ \boldsymbol{a}_NX_N \end{pmatrix}\boldsymbol{b}_1 \cdots X_{k_1}^{-1}\boldsymbol{e}_{k_1}\cdots \right. \\ &\left. {}^t\!\begin{pmatrix} \boldsymbol{a}_1X_1 \\ \vdots \\ \boldsymbol{a}_NX_N \end{pmatrix}\boldsymbol{v}_0^+ \cdots X_{k_s}^{-1}\boldsymbol{e}_{k_s}\cdots {}^t\!\begin{pmatrix} \boldsymbol{a}_1X_1 \\ \vdots \\ \boldsymbol{a}_NX_N \end{pmatrix}\boldsymbol{b}_N \right| \\ &= (-1)^{N+s}\left(\prod_{m=1}^{N} X_m\right)\sum_{k_1<\cdots<k_s} X_{k_1}^{-1}\cdots X_{k_s}^{-1} \times \\ &\sum_{j\neq k_1,\cdots,k_s}\left| {}^t\!\begin{pmatrix} \boldsymbol{a}_1X_1 \\ \vdots \\ \boldsymbol{a}_NX_N \end{pmatrix}\boldsymbol{b}_1 \cdots \overset{k_1}{\vee} \cdots \right. \\ &\left. {}^t\!\begin{pmatrix} \boldsymbol{a}_1X_1 \\ \vdots \\ \boldsymbol{a}_NX_N \end{pmatrix}\boldsymbol{v}_0^+ \cdots \overset{k_s}{\vee} \cdots {}^t\!\begin{pmatrix} \boldsymbol{a}_1X_1 \\ \vdots \\ \boldsymbol{a}_NX_N \end{pmatrix}\boldsymbol{b}_N \right| \\ &= (-1)^{N+s}\left(\prod_{m=1}^{N} X_m^2\right) \times\end{aligned}$$

$$\sum_{k_1<\cdots<k_s}\sum_{\lambda_1<\cdots<\lambda_s}\tilde{C}_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_s}X_{k_1}^{-1}\cdots X_{k_s}^{-1}X_{\lambda_1}^{-1}\cdots X_{\lambda_s}^{-1},$$

with

$$\begin{aligned}
&\tilde{C}_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_s}\\
&=\sum_{j\neq k_1,\cdots,k_s}\Bigg|\sum_{m\neq\lambda_1,\cdots,\lambda_s}\check{\boldsymbol{a}}_m b_{m1}\cdots\overset{k_1}{\vee}\cdots\\
&\qquad\sum_{m\neq\lambda_1,\cdots,\lambda_s}\check{\boldsymbol{a}}_m\frac{c_m^+}{\alpha_m}\cdots\overset{k_s}{\vee}\cdots\sum_{m\neq\lambda_1,\cdots,\lambda_s}\check{\boldsymbol{a}}_m b_{mN}\Bigg|\\
&=\sum_{j\neq k_1,\cdots,k_s}\left|A_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_s}\begin{pmatrix}\frac{c_1^+}{\alpha_1+\alpha_1} & \cdot\overset{k_1}{\vee}\cdot & \frac{c_1^+}{\alpha_1} & \cdot\overset{k_s}{\vee}\cdot & \frac{c_1^+}{\alpha_1+\alpha_N}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{c_N^+}{\alpha_N+\alpha_1} & \cdots & \frac{c_N^+}{\alpha_N} & \cdots & \frac{c_N^+}{\alpha_N+\alpha_N}\end{pmatrix}\begin{matrix}\\ <\lambda_1\\ \\ <\lambda_s\\ \\ \end{matrix}\right|\\
&=\left(\prod_{m\neq k_1,\cdots,k_s}c_m^-\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_s}c_m^+\right)\left|G_{k_1\cdots k_s}^{\lambda_1\cdots\lambda_s}\right|\times\\
&\qquad\sum_{j\neq k_1,\cdots,k_s}\begin{vmatrix}\frac{1}{\alpha_1+\alpha_1} & \cdot\overset{k_1}{\vee}\cdot & \frac{1}{\alpha_1} & \cdot\overset{k_s}{\vee}\cdot & \frac{c_1^+}{\alpha_1+\alpha_N}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{1}{\alpha_N+\alpha_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\alpha_N}\end{vmatrix}\begin{matrix}\\ <\lambda_1\\ \\ <\lambda_s\\ \\ \end{matrix}.
\end{aligned}$$

But, by [31, Lemma 3.1(3)],

$$\sum_{j\neq k_1,\cdots,k_s}\begin{vmatrix}\frac{1}{\alpha_1+\alpha_1} & \cdot\overset{k_1}{\vee}\cdot & \frac{1}{\alpha_1} & \cdot\overset{k_s}{\vee}\cdot & \frac{c_1^+}{\alpha_1+\alpha_N}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{1}{\alpha_N+\alpha_1} & \cdots & \frac{1}{\alpha_N} & \cdots & \frac{1}{\alpha_N+\alpha_N}\end{vmatrix}\begin{matrix}\\ <\lambda_1\\ \\ <\lambda_s\\ \\ \end{matrix}$$

$$= \left|G^{\lambda_1\cdots\lambda_s}_{k_1\cdots k_s}\right| \left(1 + (-1)^{N-(s+1)} \frac{\alpha_{\lambda_1}\cdots\alpha_{\lambda_s}}{\alpha_{k_1}\cdots\alpha_{k_s}}\right).$$

Hence

$$(\Gamma_2)_{2N-2s} = \left(\prod_{m=1}^{N} X_m^2\right) \sum_{k_1<\cdots<k_s} \sum_{\lambda_1<\cdots<\lambda_s} \times$$
$$\left(\prod_{m\neq k_1,\cdots,k_s} c_m^-\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_s} c_m^+\right) \times$$
$$\left|G^{\lambda_1\cdots\lambda_s}_{k_1\cdots k_s}\right|^2 \left((-1)^{N+s} - \frac{\alpha_{\lambda_1}\cdots\alpha_{\lambda_s}}{\alpha_{k_1}\cdots\alpha_{k_s}}\right) X_{k_1}^{-1}\cdots X_{k_s}^{-1} X_{\lambda_1}^{-1}\cdots X_{\lambda_s}^{-1}.$$

This, together with (6.3), (6.9), leads to

$$\Delta^+_{2N-2s} = \left(\prod_{m=1}^{N} X_m^2\right) \sum_{k_1<\cdots<k_s} \sum_{\lambda_1<\cdots<\lambda_s} \times$$
$$\left(\prod_{m\neq k_1,\cdots,k_s} c_m^-\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_s} c_m^+\right) \times$$
$$\left|G^{\lambda_1\cdots\lambda_s}_{k_1\cdots k_s}\right|^2 \left(\frac{\alpha_{\lambda_1}\cdots\alpha_{\lambda_s}}{\alpha_{k_1}\cdots\alpha_{k_s}}\right) X_{k_1}^{-1}\cdots X_{k_s}^{-1} X_{\lambda_1}^{-1}\cdots X_{\lambda_s}^{-1}.$$

Since, by [31, Lemma 3.1(1)],

$$\left|G^{\lambda_1\cdots\lambda_s}_{k_1\cdots k_s}\right| = \frac{V(\alpha_1,\cdots \overset{k_1}{\vee} \cdots \overset{k_s}{\vee} \cdots \alpha_N) V(\alpha_1,\cdots \overset{\lambda_1}{\vee} \cdots \overset{\lambda_s}{\vee} \cdots,\alpha_N)}{\prod\limits_{m\neq k_1,\cdots,k_s} \prod\limits_{n\neq\lambda_1,\cdots.\lambda_s} (\alpha_m+\alpha_n)},$$

we obtain, for $s = 0,\cdots,N-1$,

$$\Delta^+_{2N-2s} = \sum_{k_1<\cdots<k_s} \sum_{\lambda_1<\cdots<\lambda_s} \left(\prod_{m\neq k_1,\cdots,k_s} c_m^-\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_s} c_m^+\right)$$
$$\times \left(\frac{\alpha_{\lambda_1}\cdots\alpha_{\lambda_s}}{\alpha_{k_1}\cdots\alpha_{k_s}}\right) \left(\frac{V(\alpha_1,\cdot\cdot \overset{k_1}{\vee} \cdot\cdot \overset{k_s}{\vee} \cdot\cdot,\alpha_N) V(\alpha_1,\cdot\cdot \overset{\lambda_1}{\vee} \cdot\cdot \overset{\lambda_s}{\vee} \cdot\cdot,\alpha_N)}{\prod\limits_{m\neq k_1,\cdots,k_s} \prod\limits_{n\neq\lambda_1,\cdots,\lambda_s} (\alpha_m+\alpha_n)}\right)^2$$

$$\times \left(\prod_{m=1}^{N} X_m^2\right) X_{k_1}^{-1} \cdots X_{k_s}^{-1} X_{\lambda_1}^{-1} \cdots X_{\lambda_s}^{-1},$$

which is, in the case $s = 0$,

$$\Delta_{2N}^{+} = \left(\prod_{m=1}^{N} c_m^{+} c_m^{-}\right) |G|^2 \left(\prod_{m=1}^{N} X_m^2\right).$$

Since the constant term is 1 we get

$$\Delta_{\text{even}}^{+} = 1 + \sum_{s=0}^{N-1} \sum_{k_1<\cdots<k_s} \sum_{\lambda_1<\cdots<\lambda_s} \left(\prod_{m\neq k_1,\cdots,k_s} c_m^{-}\right) \left(\prod_{m\neq \lambda_1,\cdots,\lambda_s} c_m^{+}\right)$$
$$\times \left(\frac{\alpha_{\lambda_1}\cdots\alpha_{\lambda_s}}{\alpha_{k_1}\cdots\alpha_{k_s}}\right) \left(\frac{V(\alpha_1,\cdot\cdot \overset{k_1}{\vee} \cdot\cdot \overset{k_s}{\vee} \cdot\cdot,\alpha_N) V(\alpha_1,\cdot\cdot \overset{\lambda_1}{\vee} \cdot\cdot \overset{\lambda_s}{\vee} \cdot\cdot,\alpha_N)}{\prod_{m\neq k_1,\cdots,k_s} \prod_{n\neq \lambda_1,\cdots,\lambda_s} (\alpha_m+\alpha_n)}\right)^2$$
$$\times \left(\prod_{m=1}^{N} X_m^2\right) X_{k_1}^{-1} \cdots X_{k_s}^{-1} X_{\lambda_1}^{-1} \cdots X_{\lambda_s}^{-1}.$$

This, combined with (6.8), shows that

$$\Delta^{+} = 1$$
$$+ \sum_{s=0}^{N-1} \sum_{k_1<\cdots<k_s} \sum_{\lambda_1<\cdots<\lambda_{s+1}} \left(\prod_{m\neq k_1,\cdots,k_s} c_m^{-}\right) \left(\prod_{m\neq \lambda_1,\cdots,\lambda_{s+1}} c_m^{+}\right) \times$$
$$\left(-\frac{\alpha_{k_1}\cdots\alpha_{k_s}}{\alpha_{\lambda_1}\cdots\alpha_{\lambda_{s+1}}}\right) \times$$
$$\left(\frac{V(\alpha_1,\cdot\cdot \overset{k_1}{\vee} \cdot\cdot \overset{k_s}{\vee} \cdot\cdot,\alpha_N) V(\alpha_1,\cdot\cdot \overset{\lambda_1}{\vee} \cdot\cdot \overset{\lambda_{s+1}}{\vee} \cdot\cdot,\alpha_N)}{\prod_{m\neq k_1,\cdots,k_s} \prod_{n\neq \lambda_1,\cdots,\lambda_{s+1}} (\alpha_m+\alpha_n)}\right)^2 \times$$
$$\left(\prod_{m=1}^{N} X_m^2\right) X_{k_1}^{-1} \cdots X_{k_s}^{-1} X_{\lambda_1}^{-1} \cdots X_{\lambda_{s+1}}^{-1}$$

$$
+\sum_{s=0}^{N-1}\sum_{k_1<\cdots<k_s}\sum_{\lambda_1<\cdots<\lambda_s}\left(\prod_{m\neq k_1,\cdots,k_s} c_m^-\right)\left(\prod_{m\neq\lambda_1,\cdots,\lambda_s} c_m^+\right)\times
$$
$$
\left(\frac{\alpha_{\lambda_1}\cdots\alpha_{\lambda_s}}{\alpha_{k_1}\cdots\alpha_{k_s}}\right)\times
$$
$$
\left(\frac{V(\alpha_1,\cdot\cdot\overset{k_1}{\vee}\cdot\cdot\overset{k_s}{\vee}\cdot\cdot,\alpha_N)V(\alpha_1,\cdot\cdot\overset{\lambda_1}{\vee}\cdot\cdot\overset{\lambda_s}{\vee}\cdot\cdot,\alpha_N)}{\displaystyle\prod_{m\neq k_1,\cdots,k_s}\prod_{n\neq\lambda_1,\cdots,\lambda_s}(\alpha_m+\alpha_n)}\right)^2\times
$$
$$
\left(\prod_{m=1}^N X_m^2\right)X_{k_1}^{-1}\cdots X_{k_s}^{-1}X_{\lambda_1}^{-1}\cdots X_{\lambda_s}^{-1}.
$$

Rewriting this by

$$
r=N-s,\quad \{k_1,\cdots,k_s\}^c=\{p_1,\cdots,p_r\},
$$
$$
\{\lambda_1,\cdots,\lambda_{s+1}\}^c=\{q_1,\cdots,q_{r-1}\}\quad\text{or}\quad\{\lambda_1,\cdots,\lambda_s\}^c=\{q_1,\cdots,q_r\},
$$

we arrive at expression (6.1) for Δ^+. □

As a direct consequence of Proposition 6.1, we have:

COROLLARY 6.2. *If* $ik_\ell<0$, $c_\ell^\pm>0$ *for* $\ell=1,\cdots,N$ *then* $\Delta^\pm>0$.

We can now establish our main result.

PROOF OF THEOREM 1.3. Immediate from Theorem 5.4 and Corollary 6.2 because if $ik_\ell<0$, $c_\ell^\pm(0)>0$ then $c_\ell^\pm(t)$ defined in (1.17) remain positive. □

7. N-Soliton Solutions

We conclude this paper with three propositions on N-soliton solutions of uw$[n]$.

PROPOSITION 7.1. *Let* n *be a (fixed) natural number, assume* $ik_\ell<0$, $c_\ell^\pm(0)>0$ *for* $\ell=1,\cdots,N$, *and define* $c_\ell^\pm(t)$ *by* (1.23). *Then the function* $w(x,t)$ *in each solution* (u,w) *of* uw$[n]$ *with* (1.22) *is positive on* $\boldsymbol{R}^2$.

PROOF (Cf: [44, Section 9]). By means of Proposition 6.1 the functions $\Delta^{\pm}(x,t)$ with $c_\ell^{\pm}(t)$ are expressed in the forms

$$\Delta^{\pm}(x,t) = 1 + \sum_{\sigma_1,\cdots,\sigma_N} C^{\pm}_{\sigma_1,\cdots,\sigma_N} X_1^{\sigma_1}\cdots X_N^{\sigma_N},$$

where $C^{\pm}_{\sigma_1,\cdots,\sigma_N}$ are positive functions of t with $0 \le \sigma_\ell \le 2$, $(\sigma_1,\cdots,\sigma_N) \neq (0,\cdots,0)$. From this forms we have

$$\begin{aligned}
&\frac{1}{4}\left(\Delta^{\pm}\Delta^{\pm}_{xx} - (\Delta^{\pm}_x)^2\right)\\
&= \left(1 + \sum_{\sigma_1,\cdots,\sigma_N} C^{\pm}_{\sigma_1,\cdots,\sigma_N} X_1^{\sigma_1}\cdots X_N^{\sigma_N}\right)\\
&\quad \times \left(\sum_{\sigma_1,\cdots,\sigma_N} C^{\pm}_{\sigma_1,\cdots,\sigma_N}\left(\sum_{\ell=1}^N \alpha_\ell\sigma_\ell\right)^2 X_1^{\sigma_1}\cdots X_N^{\sigma_N}\right)\\
&\quad - \left(\sum_{\sigma_1,\cdots,\sigma_N} C^{\pm}_{\sigma_1,\cdots,\sigma_N}\left(\sum_{\ell=1}^N \alpha_\ell\sigma_\ell\right) X_1^{\sigma_1}\cdots X_N^{\sigma_N}\right)^2\\
&= \left(\sum_{\sigma_1,\cdots,\sigma_N} C^{\pm}_{\sigma_1,\cdots,\sigma_N}\left(\sum_{\ell=1}^N \alpha_\ell\sigma_\ell\right)^2 X_1^{\sigma_1}\cdots X_N^{\sigma_N}\right)\\
&\quad + \frac{1}{2}\sum_{\sigma_1,\cdots,\sigma_N;\tau_1,\cdots,\tau_N} C^{\pm}_{\sigma_1,\cdots,\sigma_N} C^{\pm}_{\tau_1,\cdots,\tau_N}\\
&\quad \times \left(\sum_{\ell=1}^N \alpha_\ell\sigma_\ell - \sum_{\ell=1}^N \alpha_\ell\tau_\ell\right)^2 X_1^{\sigma_1+\tau_1}\cdots X_N^{\sigma_N+\tau_N}.
\end{aligned}$$

This shows that $\Delta^{\pm}\Delta^{\pm}_{xx} - (\Delta^{\pm}_x)^2$ are positive functions on $\boldsymbol{R}^2$. Hence, by (1.24),

$$\begin{aligned}
w(x,t) &= 2\frac{\partial^2}{\partial x^2}\left(\log\Delta^+ + \log\Delta^-\right)\\
&= 2\left(\frac{\Delta^+\Delta^+_{xx} - (\Delta^+_x)^2}{(\Delta^+)^2} + \frac{\Delta^-\Delta^-_{xx} - (\Delta^-_x)^2}{(\Delta^-)^2}\right) > 0
\end{aligned}$$

on $\boldsymbol{R}^2$. The proof is complete. □

Proposition 7.2. *Let n be a (fixed) even natural number, set*

$$w^{\rm s}(x,t,b,\delta,\rho) := 2b^2\left(\frac{1+e^{b^{n+1}\rho}\cosh b(x-b^nt+\delta)}{\left(\cosh b(x-b^nt+\delta)+e^{b^{n+1}\rho}\right)^2}\right.$$
$$\left.+\frac{1+e^{-b^{n+1}\rho}\cosh b(x-b^nt+\delta)}{\left(\cosh b(x-b^nt+\delta)+e^{-b^{n+1}\rho}\right)^2}\right)$$

(see (1.29)*), and assume that $b_\ell := -2ik_\ell > 0$ which are ordered as $b_1 > \cdots > b_N$. Then N-soliton solutions $w(x,t)$ of* uw$[n]$ *with* (1.22) *obtained by Corollary* 1.5 *admit the asymptotic behaviors*

$$\lim_{t\to\pm\infty} w(x,t) = \begin{cases} \displaystyle\sum_{\ell=1}^{N} w^{\rm S}\left(x,t,b_\ell,\delta_\ell-\frac{1}{b_\ell}\sum_{j=1}^{\ell-1}A_{\ell j},\,\rho_\ell\right), & t\to+\infty, \\ \displaystyle\sum_{\ell=1}^{N} w^{\rm S}\left(x,t,b_\ell,\delta_\ell-\frac{1}{b_\ell}\sum_{j=\ell+1}^{N}A_{\ell j},\,\rho_\ell\right), & t\to-\infty, \end{cases} \tag{7.1}$$

where δ_ℓ, ρ_ℓ, $A_{\ell j}$ are defined by

$$\begin{aligned} e^{b_\ell\delta_\ell} &= \frac{b_\ell}{\sqrt{c_\ell^+(0)c_\ell^-(0)}}, \\ e^{b_\ell^{n+1}\rho_\ell} &= \sqrt{\frac{c_\ell^+(0)}{c_\ell^-(0)}} \quad \left(\Longleftrightarrow c_\ell^\pm(0) = b_\ell e^{\pm b_\ell^{n+1}\rho_\ell}e^{-b_\ell\delta_\ell}\right), \end{aligned} \tag{7.2}$$

$$e^{\frac{1}{2}A_{\ell j}} = \left|\frac{b_\ell-b_j}{b_\ell+b_j}\right| \quad \left(\Longleftrightarrow A_{\ell j} = 2\log\left|\frac{b_\ell-b_j}{b_\ell+b_j}\right|\right). \tag{7.3}$$

Proof (Cf: Wadati and Toda [47]). By (1.23) we take

$$c_\ell^\pm(t) = c_\ell^\pm(0)e^{b_\ell^{n+1}t}. \tag{7.4}$$

Also we use the notation

$$y_\ell := x - {b_\ell}^n t, \quad \ell = 1,\cdots,N. \tag{7.5}$$

In a coordinate y_1 we let $t \to +\infty$. Then, by

$$y_p = y_1 + (b_1^n - b_p^n)t, \quad p = 2, \cdots, N,$$

we see that $e^{-b_p y_p} \to 0$ as $t \to +\infty$ for $p \geq 2$. Hence, by virtue of Proposition 6.1,

$$\begin{aligned}
\Delta^{\pm}(x,t) &\to 1 + \frac{2c_1^{\mp}(0)}{b_1} e^{-b_1 y_1} + \frac{c_1^{+}(0)c_1^{-}(0)}{{b_1}^2} e^{-2b_1 y_1} \\
&= 1 + 2e^{\mp b_1^{n+1}\rho_1} e^{-b_1(y_1+\delta_1)} + e^{-2b_1(y_1+\delta_1)}
\end{aligned} \tag{7.6}$$

as $t \to +\infty$. This, together with the same computation as one for (1.29), shows that

$$\begin{aligned}
\lim_{t\to+\infty} w(x,t) &= 2 \lim_{t\to+\infty} \frac{\partial^2}{\partial x^2} \big(\log \Delta^{+}(x,t) + \log \Delta^{-}(x,t)\big) \\
&= w^{\mathrm{S}}(x,t,b_1,\delta_1,\rho_1)
\end{aligned}$$

in the coordinate y_1.

We next let $\ell \geq 2$ be fixed and consider the behavior as $t \to +\infty$ in the coordinate y_ℓ. By

$$y_p = y_\ell + (b_\ell^n - b_p^n)t, \quad p = 1, \cdots, N,$$

we see that $e^{-b_1 y_1}, \cdots, e^{-b_{\ell-1} y_{\ell-1}} \to \infty$, $e^{-b_{\ell+1} y_{\ell+1}}, \cdots, e^{-b_N y_N} \to 0$ as $t \to +\infty$. It follows that the primary terms of $\Delta^{\pm}(x,t)$ emerge in the three cases in (6.1):

$$\begin{aligned}
&r = \ell - 1 \quad \text{and} \quad (p_1, \cdots, p_{\ell-1}) = (1, \cdots, \ell-1), \\
&\qquad\qquad\qquad\quad\ (q_1, \cdots, q_{\ell-1}) = (1, \cdots, \ell-1); \\
&r = \ell \quad \text{and} \quad (p_1, \cdots, p_\ell) = (1, \cdots, \ell), \ \ (q_1, \cdots, q_{\ell-1}) = (1, \cdots, \ell-1); \\
&r = \ell \quad \text{and} \quad (p_1, \cdots, p_\ell) = (1, \cdots, \ell), \ \ (q_1, \cdots, q_\ell) = (1, \cdots, \ell).
\end{aligned}$$

Since $c_m^{\pm} X_m = c_m^{\pm}(0) e^{-b_m y_m}$ by (7.4), (7.5), these terms are respectively given by

$$\left(\frac{V(\alpha_1,\cdots,\alpha_{\ell-1})^2}{\prod_{m=1}^{\ell-1}\prod_{n=1}^{\ell-1}(\alpha_m+\alpha_n)}\right)^2\left(\prod_{m=1}^{\ell-1}c_m^{\mp}(0)e^{-b_m y_m}\right)\left(\prod_{m=1}^{\ell-1}c_m^{\pm}(0)e^{-b_m y_m}\right),$$

$$\frac{2}{b_\ell}\left(\frac{V(\alpha_1,\cdots,\alpha_\ell)V(\alpha_1,\cdots,\alpha_{\ell-1})}{\prod_{m=1}^{\ell}\prod_{n=1}^{\ell-1}(\alpha_m+\alpha_n)}\right)^2$$
$$\times\left(\prod_{m=1}^{\ell}c_m^{\mp}(0)e^{-b_m y_m}\right)\left(\prod_{m=1}^{\ell-1}c_m^{\pm}(0)e^{-b_m y_m}\right),$$

$$\left(\frac{V(\alpha_1,\cdots,\alpha_\ell)^2}{\prod_{m=1}^{\ell}\prod_{n=1}^{\ell}(\alpha_m+\alpha_n)}\right)^2\left(\prod_{m=1}^{\ell}c_m^{\mp}(0)e^{-b_m y_m}\right)\left(\prod_{m=1}^{\ell}c_m^{\pm}(0)e^{-b_m y_m}\right).$$

We denote the first term by $e^{\lambda(x,t)}$, where $\lambda(x,t)$ is linear in x because of (7.5). Then, by

$$V(\alpha_1,\cdots,\alpha_\ell)=(\alpha_1-\alpha_\ell)\cdots(\alpha_{\ell-1}-\alpha_\ell)V(\alpha_1,\cdots,\alpha_{\ell-1}),$$
$$\prod_{m=1}^{\ell}\prod_{n=1}^{\ell-1}(\alpha_m+\alpha_n)=(\alpha_1+\alpha_\ell)\cdots(\alpha_{\ell-1}+\alpha_\ell)\prod_{m=1}^{\ell-1}\prod_{n=1}^{\ell-1}(\alpha_m+\alpha_n),$$
$$\prod_{m=1}^{\ell}\prod_{n=1}^{\ell}(\alpha_m+\alpha_n)=2\alpha_\ell(\alpha_1+\alpha_\ell)^2\cdots(\alpha_{\ell-1}+\alpha_\ell)^2\prod_{m=1}^{\ell-1}\prod_{n=1}^{\ell-1}(\alpha_m+\alpha_n),$$

we find that

$$\Delta^{\pm}(x,t)=e^{\lambda(x,t)}\left(1+\frac{2}{b_\ell}\prod_{j=1}^{\ell-1}\left(\frac{b_\ell-b_j}{b_\ell+b_j}\right)^2c_\ell^{\mp}(0)e^{-b_\ell y_\ell}\right.$$

$$+\frac{1}{b_\ell^2}\prod_{j=1}^{\ell-1}\left(\frac{b_\ell-b_j}{b_\ell+b_j}\right)^4 c_\ell^+(0)c_\ell^-(0)e^{-2b_\ell y_\ell}\Bigg)[1+o(1)]$$

and so, by (7.2), (7.3), that

$$\Delta^\pm(x,t)=e^{\lambda(x,t)}\left(1+2e^{\mp b_\ell^{n+1}\rho_\ell}\,e^{-b_\ell\left(y_\ell+\delta_\ell-\frac{1}{b_\ell}\sum_{j=1}^{\ell-1}A_{\ell j}\right)}\right.$$
$$\left.+\,e^{-2b_\ell\left(y_\ell+\delta_\ell-\frac{1}{b_\ell}\sum_{j=1}^{\ell-1}A_{\ell j}\right)}\right)[1+o(1)]$$

as $t\to+\infty$. By observing that $\frac{\partial^2}{\partial x^2}\log e^{\lambda(x,t)}=0$ and comparing the above with (7.6), it follows that

$$\lim_{t\to+\infty}w(x,t)=w^{\mathrm{S}}\left(x,t,b_\ell,\,\delta_\ell-\frac{1}{b_\ell}\sum_{j=1}^{\ell-1}A_{\ell j},\,\rho_\ell\right)$$

in the coordinate y_ℓ with $\ell\ge 2$.

Similarly we have, in a coordinate y_ℓ,

$$\lim_{t\to-\infty}w(x,t)=w^{\mathrm{S}}\left(x,t,b_\ell,\,\delta_\ell-\frac{1}{b_\ell}\sum_{j=\ell+1}^{N}A_{\ell j},\,\rho_\ell\right)$$

for $\ell\le N-1$ and $=w^{\mathrm{S}}(x,t,b_N,\delta_N,\rho_N)$ for $\ell=N$. The proof is complete. □

Example 7.3. Consider the system uw[2] with $a_2=-4i$ under the initial data

$$\left(b_1,b_2,c_1^\pm(0),c_2^\pm(0)\right)=\left(3,2,9e^{\pm 27},6e^{\pm 8}\right),$$

namely

$$(\delta_1,\delta_2,\rho_1,\rho_2)=\left(-\frac{1}{3}\log 3,-\frac{1}{2}\log 3,1,1\right).$$

This Cauchy problem can be solved by computing $\Delta^\pm(x,t)$ for $c_1^\pm=9e^{\pm 27}e^{27t}$, $c_2^\pm=6e^{\pm 8}e^{8t}$ with the aid of Proposition 6.1 and then applying (3.11) to the resulting functions. The profile of $w(x,4)$ at $t=4$ of the solution $w(x,t)$ is given in Figure 4. Though the profile of $w(x,t)$ during an interaction region $0\le t\le 3$ is rather complicated, time $t=4$ is

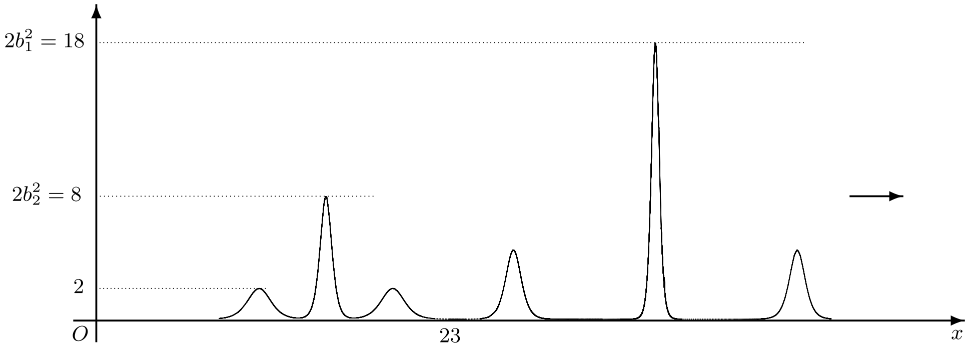

Fig. 4. A two-soliton of uw[2].

sufficiently large in this case so that $w(x,4)$ is well-splitted into two pure solitons w^{S} for $\ell = 1,2$ in the formula (7.1). Since $\rho_\ell > \frac{1}{b_\ell^3} r_c$, $\ell = 1,2$, both w^{S} are multi-peaked. Here r_c is the critical ratio defined in (1.28).

PROPOSITION 7.4. *On solutions (u,w) obtained in Corollary 1.5, there exist infinitely many conserved densities. The first four conserved quantities are written as:*

$$\int_{-\infty}^{\infty} u(x,t)dx = 0, \tag{7.7}$$

$$\int_{-\infty}^{\infty} w(x,t)dx = 8\sum_{\ell=1}^{N}(-2ik_\ell), \tag{7.8}$$

$$\int_{-\infty}^{\infty} u(x,t)w(x,t)dx = 0, \tag{7.9}$$

$$\int_{-\infty}^{\infty} \left(w(x,t)^2 + u(x,t)^2 w(x,t) - u'(x,t)^2\right) dx = \frac{32}{3}\sum_{\ell=1}^{N}(-2ik_\ell)^3. \tag{7.10}$$

PROOF (Cf. Zakharov and Faddeev [48]). Let $|k|$ be sufficiently large and write the Jost solution $f_+(x,k)$ of $(L+k^2)f = 0$ in the form

$$f_+(x,k) = e^{ikx - \int_x^\infty \sigma(x,k)dx} \tag{7.11}$$

(see [37, page 330]). Insertion of this form into $(L+k^2)f=0$ yields a Riccati equation

$$\sigma' + \sigma^2 + 2ik\sigma - (U+2kQ) = 0. \tag{7.12}$$

On the other hand, by virtue of (3.5) with $s_{12} \equiv 0$,

$$f_+(x,k)e^{-ikx} = s_{11}(k)^{-1}\overline{f_-^-(x,k)e^{ikx}} \to s_{11}(k)^{-1}$$

as $x \to -\infty$. Hence, by (7.11), we have $\int_{-\infty}^{\infty} \sigma(x,k)dx = \log s_{11}(k)$. This implies that $\sigma(x.k)$ is a conserved density. In view of (1.10), $\log s_{11}(k)$ is expanded for large $|k|$ as

$$\log s_{11}(k) = \sum_{m=0}^{\infty} \frac{C_m}{(2ik)^m},$$

where

$$\begin{cases} C_{2j} = 0, \quad j = 0, 1, \cdots, \\ C_{2j+1} = \frac{2}{2j+1}\sum_{\ell=1}^{N}(2ik_\ell)^{2j+1}, \quad j = 0, 1, \cdots. \end{cases} \tag{7.13}$$

Therefore

$$\int_{-\infty}^{\infty} \sigma(x,k)dx = \sum_{m=0}^{\infty} \frac{C_m}{(2ik)^m} \tag{7.14}$$

for large $|k|$.

We expand $\sigma(x,k)$ asymptotically as

$$\sigma(x,k) = \sum_{m=0}^{\infty} \frac{\sigma_m(x)}{(2ik)^m}.$$

Then, by (7.12), we obtain

$$\sum_{m=-1}^{\infty} \frac{\sigma_{m+1}}{(2ik)^m} + \sum_{m=0}^{\infty} \frac{\sigma_m'}{(2ik)^m} + \sum_{m=0}^{\infty} \frac{\sum_{\nu=0}^{m} \sigma_\nu \sigma_{m-\nu}}{(2ik)^m} - (U+2kQ) = 0, \tag{7.15}$$

and also, by (7.14), we get

$$\int_{-\infty}^{\infty} \sigma_m(x) = C_m, \quad m = 0, 1, 2, \cdots, \tag{7.16}$$

where C_m are given by (7.13).

In the case $m = -1$, (7.15) yields $\sigma_0 = -iQ = \frac{u}{4}$. This, together with (7.16), leads to (7.7). In the case $m = 0$, (7.15) and (1.20) yield $\sigma_1 = -\frac{1}{4}(u' + w)$, and so, by (7.16) with $m = 1$, lead to (7.8). For $m = 1, 2, \cdots$, (7.15) is written as

$$\sigma_{m+1} = -\sigma'_m - \sum_{\nu=0}^{m} \sigma_\nu \sigma_{m-\nu}.$$

This proves the existence of infinitely many conserved densities

$$\sigma_2 = \frac{1}{4}\left(u' + w + \frac{1}{4}u^2\right)' + \frac{1}{8}uw,$$
$$\sigma_3 = -\left(\sigma_2 + \frac{1}{48}u^3 - \frac{1}{8}uu' - \frac{1}{8}uw\right)' - \frac{1}{16}\left(w^2 + u^2 w - (u')^2\right),$$

and so on. The trace formulas (7.9), (7.10) are deduced from (7.16) with $m = 2, 3$. □

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