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Navier–Stokes Equations in a Curved Thin Domain, Part I: Uniform Estimates for the Stokes Operator

By Tatsu-Hiko Miura

Abstract. In the series of this paper and the forthcoming papers [47, 48] we study the Navier–Stokes equations in a three-dimensional curved thin domain around a given closed surface under Navier's slip boundary conditions. We focus on the study of the Stokes operator for the curved thin domain in this paper. The uniform norm equivalence for the Stokes operator and a uniform difference estimate for the Stokes and Laplace operators are established in which constants are independent of the thickness of the curved thin domain. To prove these results we show a uniform Korn inequality and a uniform a priori estimate for the vector Laplace operator on the curved thin domain based on a careful analysis of vector fields and surface quantities on the boundary. We also present examples of curved thin domains and vector fields for which the uniform Korn inequality is not valid but a standard Korn inequality holds with a constant that blows up as the thickness of a thin domain tends to zero.

Contents

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1. Introduction

1.1. Problem and main results

Let Γ be a closed surface in $\mathbb{R}^3$ with unit outward normal vector field n. Also, let g_0 and g_1 be functions on Γ satisfying $g := g_1 - g_0 \geq c$ on Γ with some constant $c > 0$. For a sufficiently small $\varepsilon > 0$ we define a curved thin domain Ω_ε in $\mathbb{R}^3$ with small thickness of order ε by

$$\Omega_\varepsilon := \{y + rn(y) \mid y \in \Gamma,\, \varepsilon g_0(y) < r < \varepsilon g_1(y)\} \tag{1.1}$$

and write $\Gamma_\varepsilon := \Gamma_\varepsilon^0 \cup \Gamma_\varepsilon^1$ and n_ε for the boundary of Ω_ε and its unit outward normal vector field, where Γ_ε^0 and Γ_ε^1 are the inner and outer boundaries given by $\Gamma_\varepsilon^i := \{y + \varepsilon g_i(y)n(y) \mid y \in \Gamma\}$ for $i = 0, 1$. In the series of this paper and the forthcoming papers [47, 48] we consider the Navier–Stokes equations with Navier's slip boundary conditions

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon - \nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ \operatorname{div} u^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ u^\varepsilon \cdot n_\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ [\sigma(u^\varepsilon, p^\varepsilon)n_\varepsilon]_{\mathrm{tan}} + \gamma_\varepsilon u^\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ u^\varepsilon|_{t=0} = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon. \end{cases} \tag{1.2}$$

Here $\nu > 0$ is the viscosity coefficient independent of ε and $\gamma_\varepsilon \geq 0$ is the friction coefficient on Γ_ε given by

$$\gamma_\varepsilon := \gamma_\varepsilon^i \quad \text{on} \quad \Gamma_\varepsilon^i,\ i = 0, 1, \tag{1.3}$$

where γ_ε^0 and γ_ε^1 are nonnegative constants depending on ε. Also,

$$\sigma(u^\varepsilon,p^\varepsilon) := 2\nu D(u^\varepsilon) - p^\varepsilon I_3, \quad [\sigma(u^\varepsilon,p^\varepsilon)n_\varepsilon]_{\tan} := P_\varepsilon[\sigma(u^\varepsilon,p^\varepsilon)n_\varepsilon]$$

are the stress tensor and the tangential component of the stress vector on Γ_ε, where $D(u^\varepsilon) := \{\nabla u^\varepsilon + (\nabla u^\varepsilon)^T\}/2$ is the strain rate tensor, I_3 is the 3×3 identity matrix, $n_\varepsilon \otimes n_\varepsilon$ is the tensor product of n_ε with itself, and $P_\varepsilon := I_3 - n_\varepsilon \otimes n_\varepsilon$ is the orthogonal projection onto the tangent plane of Γ_ε. Note that $[\sigma(u^\varepsilon,p^\varepsilon)n_\varepsilon]_{\tan} = 2\nu P_\varepsilon D(u^\varepsilon)n_\varepsilon$ is independent of p^ε and the slip boundary conditions can be expressed as

$$u^\varepsilon \cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u^\varepsilon)n_\varepsilon + \gamma_\varepsilon u^\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon. \tag{1.4}$$

Hereafter we mainly refer to (1.4) as the slip boundary conditions.

The aims of our study are to establish the global-in-time existence of a strong solution to (1.2) for large data and to study the behavior of the strong solution as $\varepsilon \to 0$. In this paper, however, we focus on the study of the Stokes operator A_ε associated with the Stokes problem in Ω_ε under the slip boundary conditions

$$\begin{cases} -\nu\Delta u + \nabla p = f, \quad \operatorname{div} u = 0 & \text{in} \quad \Omega_\varepsilon, \\ u\cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u)n_\varepsilon + \gamma_\varepsilon u = 0 & \text{on} \quad \Gamma_\varepsilon \end{cases} \tag{1.5}$$

and provide fundamental results on A_ε for the aims of our study. The goal of this paper is to show the uniform norm equivalence for A_ε and its square root of the form

$$\begin{gathered} c^{-1}\|u\|_{H^k(\Omega_\varepsilon)} \leq \|A_\varepsilon^{k/2}u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^k(\Omega_\varepsilon)}, \\ u \in D(A_\varepsilon^{k/2}),\ k = 1,2 \end{gathered} \tag{1.6}$$

and the uniform difference estimate for A_ε and $-\nu\Delta$ of the form

$$\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)}, \quad u \in D(A_\varepsilon) \tag{1.7}$$

with a constant $c > 0$ independent of ε (see Section 2 for the precise statements).

The estimates (1.6) and (1.7) play a fundamental role in the second part [47] of our study. In [47] we prove the global existence of a strong solution u^ε to (1.2) for large data u_0^ε and f^ε such that

$$\|u_0^\varepsilon\|_{H^1(\Omega_\varepsilon)},\ \|f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))} = O(\varepsilon^{-1/2})$$

when ε is sufficiently small. We also derive estimates for u^ε with constants explicitly depending on ε which are essential for the last paper [48]. To get the global existence we show that the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon^{1/2}u^\varepsilon$ is bounded uniformly in time by a standard energy method. A key tool for the proof is a good estimate for the trilinear term $((u^\varepsilon\cdot\nabla)u^\varepsilon, A_\varepsilon u^\varepsilon)_{L^2(\Omega_\varepsilon)}$ which implies a differential inequality in time for the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon^{1/2}u^\varepsilon$ similar to the one for the two-dimensional Navier–Stokes equations. To derive that estimate we require (1.6) and (1.7). Note that we have the $H^1(\Omega_\varepsilon)$-norm of u, not its $H^2(\Omega_\varepsilon)$-norm, in the right-hand side of (1.7), which is important in order to get a good estimate for the trilinear term.

Let us also mention the last part [48] of our study. We consider the thin-film limit for (1.2) and study the behavior of the strong solution u^ε as $\varepsilon\to 0$ in [48]. Using the results of this paper and [47] we show that the average in the thin direction of u^ε converges on Γ as $\varepsilon\to 0$. Moreover, we derive limit equations on Γ for (1.2) by characterizing the limit of the average of u^ε as a solution to the limit equations. When the thickness of Ω_ε is ε (i.e. $g\equiv 1$) and there is no friction between the fluid and the boundary Γ_ε (i.e. $\gamma_\varepsilon = 0$), the limit equations derived in [48] agree with the Navier–Stokes equations on a Riemannian manifold

$$
\begin{cases} \partial_t v + \overline{\nabla}_v v - \nu\{\Delta_B v + \mathrm{Ric}(v)\} + \nabla_\Gamma q = f & \text{on} \quad \Gamma\times(0,\infty), \\ \mathrm{div}_\Gamma v = 0 & \text{on} \quad \Gamma\times(0,\infty) \end{cases} \tag{1.8}
$$

introduced in [12, 43, 72] and studied in many works (see e.g. [6, 9, 30, 32, 42, 51, 56, 57, 60, 66]). Here $\overline{\nabla}_v v$ is the covariant derivative of v along itself, Ric is the Ricci curvature of Γ, and Δ_B, ∇_Γ, and div_Γ are the Bochner Laplacian, the tangential gradient, and the surface divergence on Γ (see [48] for details). We emphasize that the last paper [48] provides the first result on a rigorous derivation of the surface Navier–Stokes equations on a general closed surface in $\mathbb{R}^3$ by the thin-film limit and that for [48] the results of this paper and [47] are essential.

1.2. Ideas of the proofs

Let us explain the ideas of the proofs of (1.6) and (1.7) (see Section 7 for details). Since the bilinear form for (1.5) is of the form

$$
a_\varepsilon(u_1,u_2) = 2\nu\bigl(D(u_1),D(u_2)\bigr)_{L^2(\Omega_\varepsilon)} + \sum_{i=0,1}\gamma_\varepsilon^i(u_1,u_2)_{L^2(\Gamma_\varepsilon^i)}
$$

due to the slip boundary conditions (see Lemma 7.1), we show that a_ε is bounded and coercive uniformly in ε on an appropriate function space on Ω_ε in order to get (1.6) with $k=1$ (see Theorem 2.4). To this end, we use the trace inequality

$$\|\varphi\|_{L^2(\Gamma_\varepsilon^i)} \leq c\varepsilon^{-1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}, \quad \varphi \in H^1(\Omega_\varepsilon),\ i=0,1$$

with a constant $c>0$ independent of ε, which follows from a more precise inequality given in Lemma 4.1, and the uniform Korn inequality

$$\|u\|_{H^1(\Omega_\varepsilon)} \leq c\|D(u)\|_{L^2(\Omega_\varepsilon)} \tag{1.9}$$

for $u \in H^1(\Omega_\varepsilon)^3$ which satisfies the impermeable boundary condition

$$u \cdot n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon \tag{1.10}$$

and the condition that there exists a constant $\beta \in [0,1)$ independent of ε such that

$$|(u,\bar{v})_{L^2(\Omega_\varepsilon)}| \leq \beta\|u\|_{L^2(\Omega_\varepsilon)}\|\bar{v}\|_{L^2(\Omega_\varepsilon)} \tag{1.11}$$

for every Killing vector field v on Γ (see Section 2) satisfying

$$v \cdot \nabla_\Gamma g = 0 \quad \text{on} \quad \Gamma, \tag{1.12}$$

where $\bar{v}$ is the constant extension of v in the normal direction of Γ and ∇_Γ is the tangential gradient on Γ (see Section 3.1). We prove (1.9) under the conditions (1.10) and (1.11) in Theorem 5.6. Moreover, we observe in Theorem 5.7 that, if every Killing vector field on Γ satisfying (1.12) is the restriction on Γ of an infinitesimal rigid displacement of $\mathbb{R}^3$, i.e. a vector filed on $\mathbb{R}^3$ of the form

$$w(x) = a \times x + b, \quad x \in \mathbb{R}^3 \tag{1.13}$$

with $a, b \in \mathbb{R}^3$, then (1.9) holds under the conditions (1.10) and, instead of (1.11),

$$|(u,w)_{L^2(\Omega_\varepsilon)}| \leq \beta\|u\|_{L^2(\Omega_\varepsilon)}\|w\|_{L^2(\Omega_\varepsilon)} \tag{1.14}$$

for every vector field w of the form (1.13) satisfying

$$w|_\Gamma \cdot n = w|_\Gamma \cdot \nabla_\Gamma g = 0 \quad \text{on} \quad \Gamma, \tag{1.15}$$

where $\beta \in [0,1)$ is again a constant independent of ε. The proof of (1.9) consists of two steps. First we estimate ∇u to derive

$$\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq 4\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 + c\|u\|_{L^2(\Omega_\varepsilon)}^2 \tag{1.16}$$

in Lemma 5.1. To this end, we apply integration by parts twice to get

$$\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq 2\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 - \int_{\Gamma_\varepsilon} (u\cdot\nabla)u\cdot n_\varepsilon \, d\mathcal{H}^2$$

and estimate the last term by reducing the order of the derivatives of u on Γ_ε with the aid of (1.10) and interpolating integrals over the inner and outer boundaries Γ_ε^0 and Γ_ε^1. Next we prove the uniform estimate

$$\|u\|_{L^2(\Omega_\varepsilon)}^2 \leq \alpha\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 + c\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 \tag{1.17}$$

for a given $\alpha > 0$ in Lemma 5.3 by contradiction as in the case of a flat thin domain studied in [19]. We dilate Ω_ε to a domain with fixed thickness and show that a sequence of vector fields failing to satisfy (1.17) converges to the constant extension of a Killing vector field v on Γ satisfying (1.12) as $\varepsilon \to 0$. Then we take v in (1.11) or (1.14), send $\varepsilon \to 0$, and use $\beta < 1$ to get a contradiction. Note that both steps are based on a careful analysis of surface quantities of Γ_ε.

To establish (1.7) we follow the idea of the works [17, 18] on a flat thin domain. Using (1.4) we derive the integration by parts formula

$$\int_{\Omega_\varepsilon} \operatorname{curl}\operatorname{curl} u\cdot\Phi\,dx = -\int_{\Omega_\varepsilon} \operatorname{curl} G(u)\cdot\Phi\,dx + \int_{\Omega_\varepsilon} \{\operatorname{curl} u + G(u)\}\cdot\operatorname{curl}\Phi\,dx$$

for $\Phi \in L^2(\Omega_\varepsilon)^3$ with $\operatorname{curl}\Phi \in L^2(\Omega_\varepsilon)^3$, where $G(u)$ is a vector field on Ω_ε whose $H^1(\Omega_\varepsilon)$-norm is uniformly bounded by that of u (see Lemmas 7.2 and 7.3). Then we combine this formula and the Helmholtz–Leray decomposition for $-\nu\Delta u$ on Ω_ε to get (1.7). Here the uniform estimate for $G(u)$ plays an important role, but its proof involves a complicated calculations of surface quantities of Γ_ε^0 and Γ_ε^1 since we construct $G(u)$ by interpolating surface quantities of Γ_ε^0 and those of Γ_ε^1.

To prove (1.6) with $k = 2$ we employ (1.7) and the uniform a priori estimate for the vector Laplace operator

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq c\left(\|\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \tag{1.18}$$

for $u \in H^2(\Omega_\varepsilon)^3$ satisfying (1.4) (see Theorem 6.1). The proof of (1.18) proceeds as in that of (1.16), but calculations are more involved. We first show that the above u is approximated by H^3 vector fields on Ω_ε satisfying (1.4) to assume $u \in H^3(\Omega_\varepsilon)^3$ (see Lemma 6.3). Then we use integration by parts twice to get (see Appendix A for notations)

$$\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}^2 = \|\Delta u\|_{L^2(\Omega_\varepsilon)}^2 + \int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2.$$

Thus we intend to show the uniform estimate for the last term

$$\begin{aligned} &\left|\int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2\right| \\ &\qquad \leq c\left(\|u\|_{H^1(\Omega_\varepsilon)}^2 + \|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}\right). \end{aligned} \tag{1.19}$$

To this end, we first reduce the second order derivatives of u on Γ_ε to the first order ones by using (1.4). In this step we employ formulas for the covariant derivatives of tangential vector fields on Γ_ε given in Appendix D to carry out calculations on Γ_ε without a change of variables. Then we interpolate integrals of u and its first order derivatives over Γ_ε^0 and Γ_ε^1 to get (1.19). For this purpose, we apply estimates for the sum and difference of surface quantities of Γ_ε^0 and those of Γ_ε^1 given in Section 3.2. However, the proofs of those estimates involve complicated calculations of differential geometry of surfaces (see Appendix C).

1.3. Literature overview

The study of the Navier–Stokes equations in thin domains has a long history. A main subject is to prove the global existence of a strong solution for large data depending on the smallness of the thickness of a thin domain, since a thin domain in $\mathbb{R}^3$ with very small thickness is almost two-dimensional. It is also important to study the behavior of a solution as the thickness of a thin domain tends to zero in order to understand the dependence of a solution on the thin and other directions. Raugel and

Sell [61, 62, 63] first studied the Navier–Stokes equations in a thin product domain $Q \times (0, \varepsilon)$ in $\mathbb{R}^3$ with a rectangle Q and a sufficiently small $\varepsilon > 0$ under the purely periodic or mixed Dirichlet-periodic boundary conditions and obtained the global existence of a strong solution. Temam and Ziane [74] generalized the results of [61, 62, 63] to a thin product domain $\omega \times (0, \varepsilon)$ in $\mathbb{R}^3$ around a bounded domain ω in $\mathbb{R}^2$ under combinations of the Dirichlet, periodic, and Hodge boundary conditions. They also proved that the average in the thin direction of a solution to the original equations under suitable boundary conditions converges towards a solution to the two-dimensional Navier–Stokes equations in ω as $\varepsilon \to 0$. For further results on the Navier–Stokes equations in three-dimensional thin product domains we refer to [22, 23, 24, 34, 35, 49, 50] and the references cited therein.

Thin product domains appearing in the above cited papers are flat in the sense that they shrink to domains in $\mathbb{R}^2$ as $\varepsilon \to 0$ and their top and bottom boundaries are flat, but in physical problems we frequently encounter nonflat thin domains (see [64] for examples of them). Temam and Ziane [75] first dealt with a nonflat thin domain in the study of the Navier–Stokes equations. Under the Hodge boundary conditions they proved the global existence of a strong solution to the Navier–Stokes equations in a thin spherical shell $\{x \in \mathbb{R}^3 \mid a < |x| < a + \varepsilon a\}$, $a > 0$ and the convergence of its average towards a solution of limit equations on a sphere as $\varepsilon \to 0$. Iftimie, Raugel, and Sell [25] considered a flat thin domain with a nonflat top boundary

$$\{(x', x_3) \in \mathbb{R}^3 \mid x' \in (0,1)^2,\, 0 < x_3 < \varepsilon g(x')\}, \quad g\colon (0,1)^2 \to \mathbb{R}$$

under the horizontally periodic and vertically slip boundary conditions and obtained the global existence of a strong solution. They also compared the strong solution with a solution to limit equations in $(0,1)^2$. Hoang [18, 20] and Hoang and Sell [19] generalized the existence result of [25] to a flat thin domain with nonflat top and bottom boundaries (in [20] two-phase flows were studied).

Let us also mention the slip boundary conditions (1.4) and the Stokes problem (1.5). The slip boundary conditions introduced by Navier [52] state that the fluid slips on the boundary with velocity proportional to the tangential component of the stress vector. These conditions are considered as an appropriate model for flows with free boundaries and for flows past chemically reacting walls in which the usual no-slip boundary condition is not valid (see [76]). They also arise in the study of the atmosphere and ocean dynamics [38, 39, 40] and in the homogenization of the no-slip boundary condition on a rough boundary [16, 26]. The Stokes problem (1.5) under the slip boundary conditions for a general bounded domain in $\mathbb{R}^3$ was first studied by Solonnikov and Ščadilov [69] in the L^2-setting. Beirão da Veiga [4] considered the generalized system for (1.5) and proved the H^2-regularity estimate for a solution. The L^p-theory for (1.5) in a bounded domain in $\mathbb{R}^3$ were established by Amrouche and Rejaiba [2]. Note that the main results (1.6) and (1.7) of this paper are not covered by [2, 4, 69] since we show that the constant c in these estimates does not depend on the thickness of the curved thin domain.

In this paper and the forthcoming papers [47, 48] we deal with the curved thin domain Ω_ε of the form (1.1) which degenerates into the closed surface Γ as $\varepsilon \to 0$. Curved thin domains around hypersurfaces and lower dimensional manifolds were considered in the study of eigenvalues of the Laplace operator [28, 33, 67, 77] and of reaction-diffusion equations [58, 59, 78]. The series of our works gives the first study of the Navier–Stokes equations in a curved thin domain in $\mathbb{R}^3$ whose limit set is a general closed surface. Our aim is not just to generalize the shape of a thin domain, but to provide the first result on a rigorous derivation of the surface Navier–Stokes equations (1.8) by the thin-film limit.

Although the main purpose of this paper is to present preliminary results for the study of (1.2), we show new results on the uniform Korn inequality (1.9). Korn's inequality is a basic tool in the theory of linear elasticity and fluid mechanics and has been studied in various contexts (see [21] and the references cited therein). The uniform Korn inequality (1.9) in a curved thin domain in $\mathbb{R}^k$ with $k \geq 2$ around a closed hypersurface was first given by Lewicka and Müller [37]. In [37, Theorem 2.2] they proved (1.9) under the conditions (1.10) and (1.11) (see also [37, Theorem 2.1] for other conditions). Their proof was based on a uniform Korn inequality in a thin cylinder and Korn's inequality on a hypersurface for which Killing vector fields on the hypersurface play a fundamental role. In this paper we present another proof of (1.9) under the same conditions by following the idea of the work [19] on a flat thin domain. Moreover, we prove (1.9) by imposing (1.10) and the new condition (1.14) under the assumption that every Killing vector field on Γ satisfying (1.12) is the restriction on Γ of an infinitesimal rigid displacement of $\mathbb{R}^3$. This assumption is valid for many kinds of closed surfaces in $\mathbb{R}^3$ (see Remark 2.1). In particular, we can use (1.14) instead of (1.11) for curved thin domains around the unit sphere in $\mathbb{R}^3$. We also note that we take a vector field w defined on $\mathbb{R}^3$ itself in (1.14), not the constant extension of a vector field on Γ as in (1.11). This fact is crucial in order to relate the Stokes operator A_ε properly to the Stokes problem (1.5) (see Remark 2.10). In Section 5.2 we further show that the conditions (1.11) and (1.14) are more strict than the condition for a standard Korn inequality related to the axial symmetry of a domain by giving examples of both axially symmetric and not axially symmetric curved thin domains.

We also mention that we use some techniques to avoid the analysis of vector fields on the boundary Γ_ε under local coordinate systems. In the proof of (1.19) we need to compute the second order derivatives of $u \in H^3(\Omega_\varepsilon)^3$ on Γ_ε to reduce the order of the derivatives. To carry out such calculations we usually take a local coordinate system of Γ_ε or transform a part of Γ_ε into the boundary of a half-space, but here these choices will result in too complicated calculations which we can hardly complete. Instead we use a local orthonormal frame for the tangent bundle of Γ_ε and formulas for the covariant derivatives of tangential vector fields on Γ_ε given in Appendix D to work without a change of variables. The most important tool is the Gauss formula $(X \cdot \nabla)Y = \overline{\nabla}^\varepsilon_X Y + (W_\varepsilon X \cdot Y)n_\varepsilon$ for tangential vector fields X and Y on Γ_ε, which expresses the directional derivative $(X\cdot\nabla)Y$ in $\mathbb{R}^3$ in terms of the covariant derivative $\overline{\nabla}^\varepsilon_X Y$ on Γ_ε and the second fundamental form $(W_\varepsilon X\cdot Y)n_\varepsilon$ of Γ_ε (see Lemma D.1). It enables us to apply formulas of differential geometry to quantities on Γ_ε expressed in a fixed coordinate system of $\mathbb{R}^3$ and to write resulting expressions in the same coordinate system. Such an idea was also used in [10] to express intrinsically defined differential operators on a hypersurface such as the Lamè operator in terms of the global coordinate system of the ambient Euclidean space. This method is useful to deduce properties of functions on a domain from their behavior on the boundary since it avoids a change of variables. It also provides an easy and understandable way to compute vector fields on surfaces without introducing local coordinate systems and differential forms. We expect that the methods used here and in [10] will be applicable to other problems involving complicated calculations of vector fields on surfaces, especially to partial differential equations for vector fields on stationary or moving surfaces such as the surface Navier–Stokes and Stokes equations (see e.g. [27, 31, 53, 65]).

1.4. Organization of this paper

The rest of this paper is organized as follows. In Section 2 we provide the main results of this paper. Notations and basic results on a closed surface and a curved thin domain are presented in Section 3. Section 4 gives fundamental inequalities and formulas for functions on the curved thin domain and its boundary. In Section 5 we establish the uniform Korn inequality (1.9) and compare it with a standard Korn inequality. We also derive the uniform a priori estimate for the vector Laplace operator (1.18) in Section 6. Using the results of Sections 4–6 we prove our main results in Section 7. Appendix A fixes notations on vectors and matrices. Some auxiliary results related to the closed surface are shown in Appendix B. In Appendix C we provide the proofs of lemmas in Section 3 and Lemmas 5.4, 5.5, and 7.2 involving elementary but long calculations of differential geometry of surfaces. Appendix D presents formulas for the covariant derivatives of tangential vector fields on the closed surface used in Section 6. In Appendix E we show some properties of infinitesimal rigid displacements of $\mathbb{R}^3$ related to the axial symmetry of the closed surface and the curved thin domain.

Most results of this paper were obtained in the doctoral thesis of the author [45]. In this paper, however, we newly prove the uniform Korn inequality (1.9) under the condition (1.14) and give Appendix E to study properties of infinitesimal rigid displacements of $\mathbb{R}^3$ related to the axial symmetry of a closed surface and a curved thin domain. By these new results we can add the condition (A3) in Assumption 2.3 to consider some curved thin domains excluded in [45]. The most important example of a curved thin domain newly included in this paper is the thin spherical shell $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 1 < |x| < 1 + \varepsilon\}$ under the perfect slip boundary conditions (1.4) with $\gamma_\varepsilon = 0$. This kind of curved thin domain was studied by Temam and Ziane [75] under different boundary conditions (see Remark 2.9). We also add Section 5.2 in which we discuss the difference between the uniform Korn inequality and a standard Korn inequality.

2. Main Results

In this section we present the main results of this paper. The proofs of theorems in this section will be given in Section 7.

To state the main results we first fix some notations (see also Section 3). Let Γ be a two-dimensional closed (i.e. compact and without boundary), connected, and oriented surface in $\mathbb{R}^3$ with unit outward normal vector field n and $g_0, g_1 \in C^4(\Gamma)$. We assume that Γ is of class C^5 and there exists a constant $c > 0$ such that

$$g := g_1 - g_0 \geq c \quad \text{on} \quad \Gamma. \tag{2.1}$$

Note that we do not assume $g_0 \leq 0$ or $g_1 \geq 0$ on Γ. For a sufficiently small $\varepsilon \in (0,1]$ let Ω_ε be the curved thin domain in $\mathbb{R}^3$ of the form (1.1) and

$$L^2_\sigma(\Omega_\varepsilon) := \{u \in L^2(\Omega_\varepsilon)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega_\varepsilon,\ u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}$$

the standard L^2-solenoidal space on Ω_ε. By integration by parts we observe that the bilinear form for the Stokes probelm (1.5) is given by

$$a_\varepsilon(u_1, u_2) := 2\nu\big(D(u_1), D(u_2)\big)_{L^2(\Omega_\varepsilon)} + \sum_{i=0,1} \gamma^i_\varepsilon (u_1, u_2)_{L^2(\Gamma^i_\varepsilon)} \tag{2.2}$$

for $u_1, u_2 \in H^1(\Omega_\varepsilon)^3$ (see Lemma 7.1). Here $D(u) := \{\nabla u + (\nabla u)^T\}/2$ is the strain rate tensor for a vector field u on Ω_ε and γ^0_ε and γ^1_ε are the friction coefficients appearing in (1.3). Clearly, a_ε is symmetric.

To make a_ε uniformly in ε bounded and coercive on an appropriate function space, we define function spaces and impose assumptions on γ^0_ε, γ^1_ε, and Γ. Let

$$\mathcal{R} := \{w(x) = a \times x + b,\ x \in \mathbb{R}^3 \mid a, b \in \mathbb{R}^3,\ w|_\Gamma \cdot n = 0 \text{ on } \Gamma\} \tag{2.3}$$

be the space of all infinitesimal rigid displacements of $\mathbb{R}^3$ whose restrictions on Γ are tangential. Note that $\mathcal{R}$ is of finite dimension and that $\mathcal{R} \neq \{0\}$ if and only if Γ is axially symmetric, i.e. invariant under a rotation by any angle around some line (see Lemma E.1). Let ∇_Γ the tangential gradient operator on Γ (see Section 3.1 for its definition). We define subspaces of $\mathcal{R}$ by

$$\begin{aligned} &\mathcal{R}_i := \{w \in \mathcal{R} \mid w|_\Gamma \cdot \nabla_\Gamma g_i = 0 \text{ on } \Gamma\}, \quad i = 0,1, \\ &\mathcal{R}_g := \{w \in \mathcal{R} \mid w|_\Gamma \cdot \nabla_\Gamma g = 0 \text{ on } \Gamma\} \quad (g = g_1 - g_0). \end{aligned} \tag{2.4}$$

Note that $\mathcal{R}_0 \cap \mathcal{R}_1 \subset \mathcal{R}_g$. It turns out (see Lemmas E.6 and E.7) that Ω_ε is axially symmetric around the same line for all $\varepsilon \in (0,1]$ if $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \{0\}$, while Ω_ε is not axially symmetric around any line for all $\varepsilon > 0$ sufficiently small if $\mathcal{R}_g = \{0\}$.

Next we define the surface strain rate tensor by $D_\Gamma(v) := P(\nabla_\Gamma v)_S P$ on Γ for a (not necessarily tangential) vector field v on Γ, where $P := I_3 - n \otimes n$ is the orthogonal projection onto the tangent plane of Γ and $(\nabla_\Gamma v)_S := \{\nabla_\Gamma v + (\nabla_\Gamma v)^T\}/2$ is the symmetric part of the tangential gradient matrix of v (see Section 3.1 for details). Then we set

$$
\begin{aligned}
\mathcal{K}(\Gamma) &:= \{v \in H^1(\Gamma)^3 \mid v \cdot n = 0,\ D_\Gamma(v) = 0 \text{ on } \Gamma\}, \\
\mathcal{K}_g(\Gamma) &:= \{v \in \mathcal{K}(\Gamma) \mid v \cdot \nabla_\Gamma g = 0 \text{ on } \Gamma\}.
\end{aligned} \tag{2.5}
$$

If Γ is of class C^4, then $v \in \mathcal{K}(\Gamma)$ is in fact of class C^1 (see Lemma B.8) and $\overline{\nabla}_X v \cdot Y + X \cdot \overline{\nabla}_Y v = 0$ on Γ for all tangential vector fields X and Y on Γ, where $\overline{\nabla}_X v := P(X \cdot \nabla_\Gamma)v$ denotes the covariant derivative of v along X. Such a vector field generates a one-parameter group of isometries of Γ and is called a Killing vector field on Γ. It is known that $\mathcal{K}(\Gamma)$ is a Lie algebra of dimension at most three. For details of Killing vector fields we refer to [29, 55].

Remark 2.1. For $w(x) = a \times x + b$, $x \in \mathbb{R}^3$ with $a, b \in \mathbb{R}^3$, direct calculations show that $D_\Gamma(w) = 0$ on Γ. Hence w is Killing on Γ if it tangential on Γ, i.e. $\mathcal{R}|_\Gamma := \{w|_\Gamma \mid w \in \mathcal{R}\} \subset \mathcal{K}(\Gamma)$. The set $\mathcal{R}|_\Gamma$ represents the extrinsic infinitesimal symmetry of the embedded surface Γ, while $\mathcal{K}(\Gamma)$ describes the intrinsic one of the abstract Riemannian manifold Γ. It is known that $\mathcal{R}|_\Gamma = \mathcal{K}(\Gamma)$ if Γ is a surface of revolution (see also Lemma E.3). The same relation holds if Γ is closed and convex since any isometry between two closed and convex surfaces in $\mathbb{R}^3$ is a motion in $\mathbb{R}^3$ (a rotation and a translation) or a motion and a reflection by the Cohn-Vossen theorem (see [71]). However, it is not known whether $\mathcal{R}|_\Gamma$ agrees with $\mathcal{K}(\Gamma)$ for a general (nonconvex and not axially symmetric) closed surface. In particular, the existence of a closed surface in $\mathbb{R}^3$ that is not axially symmetric but admits a nontrivial Killing vector field, i.e. $\mathcal{R} = \{0\}$ but $\mathcal{K}(\Gamma) \neq \{0\}$, is an open problem (see [37, Remark 3.2]).

We make the following assumptions on the friction coefficients γ_ε^0 and

γ_ε^1, the closed surface Γ, and the functions g_0 and g_1 (see also Remarks 2.9 and 2.10).

Assumption 2.2. *There exists a constant $c > 0$ such that*

$$\gamma_\varepsilon^0 \le c\varepsilon, \quad \gamma_\varepsilon^1 \le c\varepsilon \tag{2.6}$$

for all $\varepsilon \in (0,1]$.

Assumption 2.3. *Either of the following conditions is satisfied:*

(A1) *There exists a constant $c > 0$ such that*

$$\gamma_\varepsilon^0 \ge c\varepsilon \quad \text{for all} \quad \varepsilon \in (0,1] \quad \text{or} \quad \gamma_\varepsilon^1 \ge c\varepsilon \quad \text{for all} \quad \varepsilon \in (0,1].$$

(A2) *The relation $\mathcal{K}_g(\Gamma) = \{0\}$ holds.*

(A3) *The relations $\mathcal{R}_g = \mathcal{R}_0 \cap \mathcal{R}_1$ and $\mathcal{R}_g|_\Gamma = \mathcal{K}_g(\Gamma)$ hold, where $\mathcal{R}_g|_\Gamma := \{w|_\Gamma \mid w \in \mathcal{R}_g\}$, and $\gamma_\varepsilon^0 = \gamma_\varepsilon^1 = 0$ for all $\varepsilon \in (0,1]$.*

These assumptions are imposed in this section and Section 7. We also impose Assumption 2.2 in Section 6. Under Assumptions 2.2 and 2.3 we define subspaces of $L^2(\Omega_\varepsilon)^3$ and $H^1(\Omega_\varepsilon)^3$ by

$$\begin{aligned} \mathcal{H}_\varepsilon &:= \begin{cases} L^2_\sigma(\Omega_\varepsilon) & \text{if (A1) or (A2) is satisfied,} \\ L^2_\sigma(\Omega_\varepsilon) \cap \mathcal{R}_g^\perp & \text{if (A3) is satisfied,} \end{cases} \\ \mathcal{V}_\varepsilon &:= \mathcal{H}_\varepsilon \cap H^1(\Omega_\varepsilon)^3, \end{aligned} \tag{2.7}$$

where $\mathcal{R}_g^\perp$ is the orthogonal complement of $\mathcal{R}_g$ in $L^2(\Omega_\varepsilon)^3$. Here we consider vector fields in $\mathcal{R}_g$ defined on the whole space $\mathbb{R}^3$ as elements of $L^2(\Omega_\varepsilon)^3$ just by restricting them on $\overline{\Omega}_\varepsilon$. Note that $\mathcal{R}_0 \cap \mathcal{R}_1 \subset L^2_\sigma(\Omega_\varepsilon)$ by Lemma E.8 and thus $\mathcal{R}_g \subset L^2_\sigma(\Omega_\varepsilon)$ under the condition (A3). Also, $\mathcal{H}_\varepsilon$ and $\mathcal{V}_\varepsilon$ are closed in $L^2(\Omega_\varepsilon)^3$ and $H^1(\Omega_\varepsilon)^3$. By $\mathbb{P}_\varepsilon$ we denote the orthogonal projection from $L^2(\Omega_\varepsilon)^3$ onto $\mathcal{H}_\varepsilon$. Note that $\mathbb{P}_\varepsilon$ may be slightly different from the standard Helmholtz–Leray projection from $L^2(\Omega_\varepsilon)^3$ onto $L^2_\sigma(\Omega_\varepsilon)$ under the condition (A3).

Now let us present the main results of this paper. The first result is the uniform boundedness and coerciveness of the bilinear form a_ε given by (2.2) on $\mathcal{V}_\varepsilon$.

THEOREM 2.4. *Under Assumptions 2.2 and 2.3, there exist constants* $\varepsilon_0 \in (0,1]$ *and* $c > 0$ *such that*

$$c^{-1}\|u\|^2_{H^1(\Omega_\varepsilon)} \leq a_\varepsilon(u,u) \leq c\|u\|^2_{H^1(\Omega_\varepsilon)} \tag{2.8}$$

for all $\varepsilon \in (0,\varepsilon_0]$ *and* $u \in \mathcal{V}_\varepsilon$.

Throughout this section we fix the constant ε_0 given in Theorem 2.4 and take $\varepsilon \in (0,\varepsilon_0]$. By Theorem 2.4 the bilinear form a_ε is bounded, coercive, and symmetric on the Hilbert space $\mathcal{V}_\varepsilon$. Hence by the Lax–Milgram theorem there exists a bounded linear operator A_ε from $\mathcal{V}_\varepsilon$ into its dual space $\mathcal{V}_\varepsilon'$ such that

$$_{\mathcal{V}_\varepsilon'}\langle A_\varepsilon u_1, u_2\rangle_{\mathcal{V}_\varepsilon} = a_\varepsilon(u_1,u_2), \quad u_1,u_2 \in \mathcal{V}_\varepsilon,$$

where $_{\mathcal{V}_\varepsilon'}\langle\cdot,\cdot\rangle_{\mathcal{V}_\varepsilon}$ is the duality product between $\mathcal{V}_\varepsilon'$ and $\mathcal{V}_\varepsilon$. We consider A_ε as an unbounded operator on $\mathcal{H}_\varepsilon$ with domain $D(A_\varepsilon) = \{u \in \mathcal{V}_\varepsilon \mid A_\varepsilon u \in \mathcal{H}_\varepsilon\}$. Then the Lax–Milgram theory shows that A_ε is a positive self-adjoint operator on $\mathcal{H}_\varepsilon$ and its square root $A_\varepsilon^{1/2}$ is well-defined on $D(A_\varepsilon^{1/2}) = \mathcal{V}_\varepsilon$. Moreover,

$$(A_\varepsilon u_1, u_2)_{L^2(\Omega_\varepsilon)} = (A_\varepsilon^{1/2}u_1, A_\varepsilon^{1/2}u_2)_{L^2(\Omega_\varepsilon)} \tag{2.9}$$

for all $u_1 \in D(A_\varepsilon)$ and $u_2 \in \mathcal{V}_\varepsilon$, and

$$\begin{aligned} \|A_\varepsilon^{1/2}u\|^2_{L^2(\Omega_\varepsilon)} &= a_\varepsilon(u,u) \\ &= 2\nu\|D(u)\|^2_{L^2(\Omega_\varepsilon)} + \gamma_\varepsilon^0\|u\|^2_{L^2(\Gamma_\varepsilon^0)} + \gamma_\varepsilon^1\|u\|^2_{L^2(\Gamma_\varepsilon^1)} \end{aligned} \tag{2.10}$$

for all $u \in \mathcal{V}_\varepsilon$ (see e.g. [5, 70] for details). From a regularity result for a solution to the Stokes problem (1.5) (see [2, 4, 69]) it also follows that

$$D(A_\varepsilon) = \{u \in \mathcal{V}_\varepsilon \cap H^2(\Omega_\varepsilon)^3 \mid 2\nu P_\varepsilon D(u)n_\varepsilon + \gamma_\varepsilon u = 0 \text{ on } \Gamma_\varepsilon\} \tag{2.11}$$

and $A_\varepsilon u = -\nu\mathbb{P}_\varepsilon\Delta u$ for $u \in D(A_\varepsilon)$. We call A_ε the Stokes operator associated with (1.5) or the Stokes operator for Ω_ε under the slip boundary conditions.

Let us give basic estimates for $A_\varepsilon^{1/2}$ with constants independent of ε.

LEMMA 2.5. *Under Assumptions 2.2 and 2.3, let* ε_0 *be the constant given in Theorem 2.4. There exists a constant* $c > 0$ *such that*

$$c^{-1}\|u\|_{H^1(\Omega_\varepsilon)} \leq \|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)} \tag{2.12}$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and* $u \in \mathcal{V}_\varepsilon$. *Moreover, if* $u \in D(A_\varepsilon)$, *then we have*

$$\|A_\varepsilon^{1/2} u\|_{L^2(\Omega_\varepsilon)} \leq c\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}. \tag{2.13}$$

PROOF. The inequality (2.12) is an immediate consequence of (2.8) and (2.10). Also, by (2.9) and Hölder's inequality,

$$\|A_\varepsilon^{1/2} u\|_{L^2(\Omega_\varepsilon)}^2 = (u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)} \leq \|u\|_{L^2(\Omega_\varepsilon)} \|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}$$

for $u \in D(A_\varepsilon)$. By this inequality and (2.12) we get (2.13). □

Since $A_\varepsilon = -\nu \mathbb{P}_\varepsilon \Delta$ on $\mathcal{H}_\varepsilon$ and $\mathbb{P}_\varepsilon$ is the orthogonal projection from $L^2(\Omega_\varepsilon)^3$ onto $\mathcal{H}_\varepsilon$, we easily observe that

$$\begin{aligned} \|A_\varepsilon u + \nu \Delta u\|_{L^2(\Omega_\varepsilon)} &= \nu \|\Delta u - \mathbb{P}_\varepsilon \Delta u\|_{L^2(\Omega_\varepsilon)} \\ &\leq \nu \|\Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^2(\Omega_\varepsilon)} \end{aligned}$$

for all $u \in D(A_\varepsilon)$ with a constant $c > 0$ independent of ε. The next theorem shows that the right-hand side can be replaced by the $H^1(\Omega_\varepsilon)$-norm of u under the slip boundary conditions (1.4).

THEOREM 2.6. *Under Assumptions 2.2 and 2.3, let* ε_0 *be the constant given in Theorem 2.4. There exists a constant* $c > 0$ *such that*

$$\|A_\varepsilon u + \nu \Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)} \tag{2.14}$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and* $u \in D(A_\varepsilon)$.

The inequality (2.14) is useful to derive a good estimate for the trilinear term $((u \cdot \nabla)u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)}$, $u \in D(A_\varepsilon)$, which is essential for the proof of the global existence of a strong solution to the Navier–Stokes equations (1.2). For details, we refer to [47].

Finally, we present the uniform norm equivalence for A_ε.

THEOREM 2.7. *Under Assumptions 2.2 and 2.3, let* ε_0 *be the constant given in Theorem 2.4. There exists a constant* $c > 0$ *such that*

$$c^{-1}\|u\|_{H^2(\Omega_\varepsilon)} \leq \|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^2(\Omega_\varepsilon)} \tag{2.15}$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and* $u \in D(A_\varepsilon)$.

As a consequence of Lemma 2.5 and Theorem 2.7 we obtain an interpolation inequality for a vector field in $D(A_\varepsilon)$.

COROLLARY 2.8. *Under Assumptions 2.2 and 2.3, let* ε_0 *be the constant given in Theorem 2.4. Then there exists a constant* $c > 0$ *such that*

$$\|u\|_{H^1(\Omega_\varepsilon)} \leq c\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2} \tag{2.16}$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and* $u \in D(A_\varepsilon)$.

PROOF. Let $u \in D(A_\varepsilon)$. From (2.9) and (2.12) it follows that

$$\|u\|_{H^1(\Omega_\varepsilon)}^2 \leq c\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}^2 = c(A_\varepsilon u, u)_{L^2(\Omega_\varepsilon)} \leq c\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}\|u\|_{L^2(\Omega_\varepsilon)}.$$

By this inequality and (2.15) we obtain (2.16). □

We conclude this section with two remarks on Assumption 2.3.

REMARK 2.9. The conditions of Assumption 2.3 are valid in the following cases:

(A1) When at least one of γ_ε^0 and γ_ε^1 is bounded from below by ε, we may consider any closed surface Γ. In this case, however, the perfect slip (i.e. $\gamma_\varepsilon = 0$) of the fluid on Γ_ε is not allowed.

(A2) It is known (see e.g. [68, Proposition 2.2]) that there exists no nontrivial Killing vector field on Γ (i.e. $\mathcal{K}(\Gamma) = \{0\}$) if the genus of Γ is greater than one. In this case $\mathcal{K}_g(\Gamma) = \{0\}$ for any $g = g_1 - g_0$ and we may take any nonnegative γ_ε^0 and γ_ε^1 (bounded above by ε). Note that, if $\mathcal{K}_g(\Gamma) = \{0\}$, then $\mathcal{R}_g = \{0\}$ and the curved thin domain Ω_ε is not axially symmetric around any line for all $\varepsilon > 0$ sufficiently small (see Lemma E.7).

(A3) As mentioned in Remark 2.1, if Γ is a surface of revolution or it is closed and convex then $\mathcal{R}|_\Gamma = \mathcal{K}(\Gamma)$ and thus $\mathcal{R}_g|_\Gamma = \mathcal{K}_g(\Gamma)$ for any $g = g_1 - g_0$. Also, the relation $\mathcal{R}_0 \cap \mathcal{R}_1 = \mathcal{R}_g$ holds if, for example, g_0 or g_1 is constant. In this case we only consider the perfect slip boundary conditions

$$u \cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u) n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon. \tag{2.17}$$

A typical but important example of this case is the thin spherical shell $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 1 < |x| < 1+\varepsilon\}$ ($\Gamma = S^2$, $g_0 \equiv 0$, $g_1 \equiv 1$) around the unit sphere S^2 in $\mathbb{R}^3$ considered by Temam and Ziane [75] under the Hodge boundary conditions

$$u \cdot n_\varepsilon = 0, \quad \operatorname{curl} u \times n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon. \tag{2.18}$$

Note that, if $u \cdot n_\varepsilon = 0$ on Γ_ε, then we have

$$2P_\varepsilon D(u) n_\varepsilon - \operatorname{curl} u \times n_\varepsilon = 2W_\varepsilon u \quad \text{on} \quad \Gamma_\varepsilon,$$

see [41, Section 2] and Lemma B.10. Here W_ε is the Weingarten map (or the shape operator) of Γ_ε representing the curvatures of Γ_ε (see Section 3.2). Hence the perfect slip boundary conditions (2.17) are different from the Hodge boundary conditions (2.18) by the curvatures of the boundary.

We also note that, if $\Gamma = \mathbb{T}^2$ is the flat torus, then

$$\begin{aligned} \mathcal{R}_i &= \{(a_1, a_2, 0)^T \in \mathbb{R}^2 \times \{0\} \mid a_1 \partial_1 g_i + a_2 \partial_2 g_i = 0 \text{ on } \mathbb{T}^2\}, \quad i = 0, 1, \\ \mathcal{R}_g &= \mathcal{K}_g(\Gamma) = \{(a_1, a_2, 0)^T \in \mathbb{R}^2 \times \{0\} \mid a_1 \partial_1 g + a_2 \partial_2 g = 0 \text{ on } \mathbb{T}^2\} \end{aligned}$$

and the conditions (A2) and (A3) were imposed in [18] and [19, 25], respectively, which studied the Naiver–Stokes equations in a flat thin domain around $\Gamma = \mathbb{T}^2$.

Remark 2.10. For a function η on Γ let $\bar{\eta}$ be its constant extension in the normal direction of Γ, $\overline{\mathcal{K}}_g(\Gamma) := \{\bar{v} \mid v \in \mathcal{K}_g(\Gamma)\}$, and

$$\mathbb{H}_\varepsilon := L^2_\sigma(\Omega_\varepsilon) \cap \overline{\mathcal{K}}_g(\Gamma)^\perp, \quad \mathbb{V}_\varepsilon := \mathbb{H}_\varepsilon \cap H^1(\Omega_\varepsilon)^3.$$

Then we see by the uniform Korn inequality given in Theorem 5.6 that the bilinear form a_ε is uniformly coercive on $\mathbb{V}_\varepsilon$ even if Assumption 2.3 is not imposed. Since we can also show that a_ε is uniformly bounded on $\mathbb{V}_\varepsilon$ under Assumption 2.2 as in Theorem 2.4, we obtain a bounded linear operator $\mathbb{A}_\varepsilon$ from $\mathbb{V}_\varepsilon$ into its dual space induced by a_ε. This $\mathbb{A}_\varepsilon$, however, is not properly related to the Stokes problem (1.5). To see this, let $u \in \mathbb{V}_\varepsilon$ such that $f := \mathbb{A}_\varepsilon u \in \mathbb{H}_\varepsilon$. Then

$$a_\varepsilon(u,\varphi) = (f,\varphi)_{L^2(\Omega_\varepsilon)} \quad \text{for all} \quad \varphi \in \mathbb{V}_\varepsilon. \tag{2.19}$$

If (2.19) was valid for all $\varphi \in L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ then we could recover the Stokes problem (1.5) from (2.19) by a standard argument (see [8, 5, 70, 73]), but we cannot verify it because of the condition $\varphi \in \overline{\mathcal{K}}_g(\Gamma)^\perp$ for the test function φ. Indeed, let $\varphi \in L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ and assume that it can be decomposed into $\varphi = \Phi + \bar{v}$ with some $\Phi \in \mathbb{V}_\varepsilon$ and $\bar{v} \in \overline{\mathcal{K}}_g(\Gamma)$ (this is possible if $\overline{\mathcal{K}}_g(\Gamma) \subset L^2_\sigma(\Omega_\varepsilon)$, but such a relation is not always valid since $\bar{v} \in \overline{\mathcal{K}}_g(\Gamma)$ does not satisfy $\bar{v} \cdot n_\varepsilon = 0$ on Γ_ε in general). Then since (2.19) is valid for $\Phi \in \mathbb{V}_\varepsilon$ and $(f,\bar{v})_{L^2(\Omega_\varepsilon)} = 0$ by $f \in \mathbb{H}_\varepsilon$, to verify (2.19) for $\varphi = \Phi + \bar{v}$ we need to show that

$$a_\varepsilon(u,\bar{v}) = 2\nu\big(D(u),D(\bar{v})\big)_{L^2(\Omega_\varepsilon)} + \gamma^0_\varepsilon(u,\bar{v})_{L^2(\Gamma^0_\varepsilon)} + \gamma^1_\varepsilon(u,\bar{v})_{L^2(\Gamma^1_\varepsilon)}$$

vanishes. However, the second and third terms on the right-hand side do not vanish unless $\gamma^0_\varepsilon = \gamma^1_\varepsilon = 0$. The first term also does not vanish in general, since for the constant extension $\bar{v}$ of a vector field v on Γ we observe by (3.16) that

$$D(\bar{v})(x) = \frac{1}{2}\Big[\{I_3 - rW(y)\}^{-1}\nabla_\Gamma v(y) + \{\nabla_\Gamma v(y)\}^T\{I_3 - rW(y)\}^{-1}\Big]$$

for $x = y + rn(y) \in \Omega_\varepsilon$ with $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$, where W is the Weingarten map of Γ (see Section 3.1), and $D(\bar{v})$ does not vanish on Ω_ε just by $D_\Gamma(v) = 0$ on Γ (even if $\Gamma = S^2$ and $\bar{v}(x) = e_3 \times (x/|x|)$ is the constant extension of $v(y) = e_3 \times y \in \mathcal{K}(S^2)$ with $e_3 = (0,0,1)^T$). Thus we fail to show (2.19) for $\varphi \in L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ and it is not clear whether u is a solution to the Stokes problem (1.5) with $f = \mathbb{A}_\varepsilon u$. This observation implies that the operator $\mathbb{A}_\varepsilon$ is not appropriate for the study of the Navier–Stokes equations (1.2).

The above problem does not occur if we impose Assumption 2.3 and consider the bilinear form a_ε on the function space $\mathcal{V}_\varepsilon$ given by (2.7). In this case, for $u \in D(A_\varepsilon)$ and $f := A_\varepsilon u \in \mathcal{H}_\varepsilon$ we a priori have

$$a_\varepsilon(u,\varphi) = (f,\varphi)_{L^2(\Omega_\varepsilon)} \quad \text{for all} \quad \varphi \in \mathcal{V}_\varepsilon. \tag{2.20}$$

Under the condition (A1) or (A2) we have $\mathcal{V}_\varepsilon = L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ and thus (2.20) implies that u is indeed a solution to the Stokes problem (1.5) with $f = A_\varepsilon u$. If we impose the condition (A3), then $\mathcal{V}_\varepsilon$ may be smaller than $L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$. In this case, however, since $\mathcal{R}_g = \mathcal{R}_0 \cap \mathcal{R}_1$ is of finite dimension and contained in $L^2_\sigma(\Omega_\varepsilon)$ by Lemma E.8, each $\varphi \in L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ can be decomposed into $\varphi = \Phi + w$ with some $\Phi \in \mathcal{V}_\varepsilon$ and $w \in \mathcal{R}_g$. Then (2.20) holds for $\Phi \in \mathcal{V}_\varepsilon$ and, since $w \in \mathcal{R}_g$ is of the form $w(x) = a \times x + b$, $x \in \mathbb{R}^3$ with $a, b \in \mathbb{R}^3$, we easily get $D(w) = 0$ on $\mathbb{R}^3$. From this fact and $\gamma^0_\varepsilon = \gamma^1_\varepsilon = 0$ by the condition (A3) it follows that $a_\varepsilon(u, w) = 0$. Thus (2.20) is also valid for all $\varphi \in L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ under the condition (A3).

3. Preliminaries

We fix notations on a closed surface and a curved thin domain and give their basic properties. Notations on vectors and matrices are given in Appendix A.

Some lemmas in this section are proved just by calculations involving differential geometry. We give the proofs of them in Appendix C to avoid making this section too long. Also, some results in this section are not used in the following sections but essential for the second and third parts [47, 48] of our study. We include them here since they easily follow from other results used in this paper or we can prove them just by a few discussions along with the proofs of the other results.

Throughout this paper we denote by c a general positive constant independent of the parameter ε. Also, we fix a coordinate system of $\mathbb{R}^3$ and write x_i, $i = 1, 2, 3$ for the i-th component of a point $x \in \mathbb{R}^3$ under this coordinate system.

3.1. Closed surface

Let Γ be a two-dimensional closed, connected, and oriented surface in $\mathbb{R}^3$. We assume that Γ is of class C^ℓ with $\ell \geq 2$. By n and d we denote the unit outward normal vector field of Γ and the signed distance function from Γ increasing in the direction of n. Also, let κ_1 and κ_2 be the principal curvatures of Γ. From the C^ℓ-regularity of Γ it follows that $n \in C^{\ell-1}(\Gamma)^3$ and $\kappa_1, \kappa_2 \in C^{\ell-2}(\Gamma)$. In particular, κ_1 and κ_2 are bounded on the compact set Γ. Hence we can take a tubular neighborhood

$N := \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x, \Gamma) < \delta\}$, $\delta > 0$ of Γ such that for each $x \in N$ there exists a unique point $\pi(x) \in \Gamma$ satisfying

$$x = \pi(x) + d(x)n(\pi(x)), \quad \nabla d(x) = n(\pi(x)). \tag{3.1}$$

Moreover, d and π are of class C^ℓ and $C^{\ell-1}$ on $\overline{N}$ (see [15, Section 14.6] for details). By the boundedness of κ_1 and κ_2 we also have

$$c^{-1} \leq 1 - r\kappa_i(y) \leq c \quad \text{for all} \quad y \in \Gamma,\ r \in (-\delta, \delta),\ i = 1, 2 \tag{3.2}$$

if we take $\delta > 0$ sufficiently small.

Let us define differential operators on Γ. We set $P := I_3 - n \otimes n$ and $Q := n \otimes n$ on Γ. Then P and Q are the orthogonal projections onto the tangent plane and the normal direction of Γ and satisfy $|P| = 2$, $|Q| = 1$ (here we use the Frobenius norm for matrices as indicated in Appendix A), and

$$\begin{gathered} I_3 = P + Q, \quad PQ = QP = 0, \quad P^T = P^2 = P, \quad Q^T = Q^2 = Q, \\ |a|^2 = |Pa|^2 + |Qa|^2, \quad |Pa| \leq |a|, \quad Pa \cdot n = 0, \quad a \in \mathbb{R}^3 \end{gathered}$$

on Γ. Also, $P, Q \in C^{\ell-1}(\Gamma)^{3\times 3}$ by the C^ℓ-regularity of Γ. For $\eta \in C^1(\Gamma)$ we define the tangential gradient and the tangential derivatives of η as

$$\nabla_\Gamma \eta(y) := P(y)\nabla\tilde{\eta}(y), \quad \underline{D}_i\eta(y) := \sum_{j=1}^3 P_{ij}(y)\partial_j\tilde{\eta}(y) \tag{3.3}$$

for $y \in \Gamma$ and $i = 1, 2, 3$ so that $\nabla_\Gamma \eta = (\underline{D}_1\eta, \underline{D}_2\eta, \underline{D}_3\eta)^T$. Here $\tilde{\eta}$ is a C^1-extension of η to N with $\tilde{\eta}|_\Gamma = \eta$. Note that

$$P\nabla_\Gamma \eta = \nabla_\Gamma \eta, \quad n \cdot \nabla_\Gamma \eta = 0 \quad \text{on} \quad \Gamma. \tag{3.4}$$

Also, $\nabla_\Gamma \eta$ agrees with the gradient on a Riemannian manifold expressed under a local coordinate system (see Lemma B.2). Hence the values of $\nabla_\Gamma \eta$ and $\underline{D}_i\eta$ are independent of the choice of an extension $\tilde{\eta}$. In particular, for the constant extension $\bar{\eta} := \eta \circ \pi$ of η in the normal direction of Γ, we have

$$\nabla\bar{\eta}(y) = \nabla_\Gamma \eta(y), \quad \partial_i\bar{\eta}(y) = \underline{D}_i\eta(y), \quad y \in \Gamma,\ i = 1, 2, 3 \tag{3.5}$$

since $\nabla\pi(y) = P(y)$ for $y \in \Gamma$ by (3.1) and $d(y) = 0$. In what follows, the notation $\bar{\eta}$ with an overline always stands for the constant extension of a function η on Γ in the normal direction of Γ. The tangential Hessian matrix of $\eta \in C^2(\Gamma)$ and the Laplace–Beltrami operator are given by $\nabla_\Gamma^2\eta := (\underline{D}_i\underline{D}_j\eta)_{i,j}$ and $\Delta_\Gamma\eta := \mathrm{tr}[\nabla_\Gamma^2\eta] = \sum_{i=1}^3 \underline{D}_i^2\eta$ on Γ. Note that $\nabla_\Gamma^2\eta$ is not symmetric in general (see Lemma 3.2).

For a (not necessarily tangential) vector field $v \in C^1(\Gamma)^3$ we define the tangential gradient matrix and the surface divergence of v by

$$\nabla_\Gamma v := \begin{pmatrix} \underline{D}_1 v_1 & \underline{D}_1 v_2 & \underline{D}_1 v_3 \\ \underline{D}_2 v_1 & \underline{D}_2 v_2 & \underline{D}_2 v_3 \\ \underline{D}_3 v_1 & \underline{D}_3 v_2 & \underline{D}_3 v_3 \end{pmatrix}, \quad \mathrm{div}_\Gamma v := \mathrm{tr}[\nabla_\Gamma v] = \sum_{i=1}^3 \underline{D}_i v_i \tag{3.6}$$

on Γ with $v = (v_1, v_2, v_3)^T$ (note that in several papers the indices i, j of $\underline{D}_i v_j$ in $\nabla_\Gamma v$ are reversed, see also Appendix A for our notation of ∇u for $u\colon \mathbb{R}^3 \to \mathbb{R}^3$) and the surface strain rate tensor for v by

$$D_\Gamma(v) := P(\nabla_\Gamma v)_S P \quad \text{on} \quad \Gamma, \quad (\nabla_\Gamma v)_S = \frac{\nabla_\Gamma v + (\nabla_\Gamma v)^T}{2}. \tag{3.7}$$

Also, for $v \in C^1(\Gamma)^3$ and $\eta \in C(\Gamma)^3$ we set

$$(\eta \cdot \nabla_\Gamma)v := \begin{pmatrix} \eta \cdot \nabla_\Gamma v_1 \\ \eta \cdot \nabla_\Gamma v_2 \\ \eta \cdot \nabla_\Gamma v_3 \end{pmatrix} = (\nabla_\Gamma v)^T \eta \quad \text{on} \quad \Gamma.$$

Note that for any C^1-extension $\tilde{v}$ of v to N with $\tilde{v}|_\Gamma = v$ we have

$$\nabla_\Gamma v = P\nabla\tilde{v}, \quad (\eta \cdot \nabla_\Gamma)v = [(P\eta) \cdot \nabla]\tilde{v} \quad \text{on} \quad \Gamma. \tag{3.8}$$

Next we define the Weingarten map W and (twice) the mean curvature H of Γ by $W := -\nabla_\Gamma n$ and $H := \mathrm{tr}[W] = -\mathrm{div}_\Gamma n$ on Γ. Note that W and H are of class $C^{\ell-2}$ and thus bounded on Γ.

LEMMA 3.1. *The Weingarten map W is symmetric and*

$$Wn = 0, \quad PW = WP = W \quad \textit{on} \quad \Gamma. \tag{3.9}$$

Also, if $v \in C^1(\Gamma)^3$ is tangential, i.e. $v \cdot n = 0$ on Γ, then

$$(\nabla_\Gamma v)n = Wv, \quad \nabla_\Gamma v = P(\nabla_\Gamma v)P + (Wv) \otimes n \quad \textit{on} \quad \Gamma. \tag{3.10}$$

PROOF. We see by (3.1) and (3.5) that $W = -\nabla^2 d$ is symmetric. Also, we have (3.9) and (3.10) by applying ∇_Γ to $|n|^2 = 1$ and $v \cdot n = 0$ on Γ and using (3.4) and $I_3 = P + Q$. $\Box$

By (3.9) we see that W has the eigenvalue zero associated with the eigenvector n. Its other eigenvalues are the principal curvatures κ_1 and κ_2 and thus $H = \kappa_1 + \kappa_2$ on Γ (see e.g. [15, 36]).

The Weingarten map W appears when we exchange the tangential derivatives and compute the gradient of the constant extension of a function on Γ.

LEMMA 3.2. *For $\eta \in C^2(\Gamma)$ and $i, j = 1, 2, 3$ we have*

$$\underline{D}_i\underline{D}_j\eta - \underline{D}_j\underline{D}_i\eta = [W\nabla_\Gamma\eta]_i n_j - [W\nabla_\Gamma\eta]_j n_i \quad \text{on} \quad \Gamma. \tag{3.11}$$

Here $[W\nabla_\Gamma\eta]_i$ is the i-th component of the vector field $W\nabla_\Gamma\eta$.

For the proof of Lemma 3.2, see e.g. [44, Lemma 2.2].

LEMMA 3.3. *The matrix*

$$I_3 - d(x)\overline{W}(x) = I_3 - rW(y)$$

is invertible for all $x = y + rn(y) \in N$ with $y \in \Gamma$ and $r \in (-\delta, \delta)$, and

$$\{I_3 - rW(y)\}^{-1}P(y) = P(y)\{I_3 - rW(y)\}^{-1}. \tag{3.12}$$

Moreover, there exists a constant $c > 0$ such that

$$c^{-1}|a| \leq \left|\{I_3 - rW(y)\}^k a\right| \leq c|a|, \quad k = \pm 1, \tag{3.13}$$

$$\left|I_3 - \{I_3 - rW(y)\}^{-1}\right| \leq c|r| \tag{3.14}$$

for all $y \in \Gamma$, $r \in (-\delta, \delta)$, and $a \in \mathbb{R}^3$.

LEMMA 3.4. *For all $x \in N$ we have*

$$\nabla\pi(x) = \left\{I_3 - d(x)\overline{W}(x)\right\}^{-1}\overline{P}(x). \tag{3.15}$$

Let $\eta \in C^1(\Gamma)$. Then its constant extension $\bar\eta = \eta \circ \pi$ satisfies

$$\nabla\bar\eta(x) = \left\{I_3 - d(x)\overline{W}(x)\right\}^{-1}\overline{\nabla_\Gamma\eta}(x), \quad x \in N \tag{3.16}$$

and there exists a constant $c>0$ independent of η such that

$$c^{-1}\left|\overline{\nabla_\Gamma\eta}(x)\right| \leq |\nabla\bar{\eta}(x)| \leq c\left|\overline{\nabla_\Gamma\eta}(x)\right|, \tag{3.17}$$

$$\left|\nabla\bar{\eta}(x)-\overline{\nabla_\Gamma\eta}(x)\right| \leq c\left|d(x)\overline{\nabla_\Gamma\eta}(x)\right| \tag{3.18}$$

for all $x\in N$. If Γ is of class C^3 and $\eta\in C^2(\Gamma)$, then we have

$$|\nabla^2\bar{\eta}(x)| \leq c\left(\left|\overline{\nabla_\Gamma\eta}(x)\right|+\left|\overline{\nabla_\Gamma^2\eta}(x)\right|\right), \quad x\in N. \tag{3.19}$$

Lemmas 3.3 and 3.4 are proved in Appendix C. Note that

$$\nabla\bar{n}(x)=-\left\{I_3-d(x)\overline{W}(x)\right\}^{-1}\overline{W}(x), \quad x\in N \tag{3.20}$$

by (3.16) and $W=-\nabla_\Gamma n$ on Γ.

Let us give integration by parts formulas on Γ (see also [11, Theorem 2.10] and [15, Lemma 16.1] for different proofs).

Lemma 3.5. *For $v\in C^1(\Gamma)^3$ we have*

$$\int_\Gamma \mathrm{div}_\Gamma v\,d\mathcal{H}^2=-\int_\Gamma (v\cdot n)H\,d\mathcal{H}^2, \tag{3.21}$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. Moreover,

$$\int_\Gamma (\eta\underline{D}_i\xi+\xi\underline{D}_i\eta)\,d\mathcal{H}^2=-\int_\Gamma \eta\xi Hn_i\,d\mathcal{H}^2 \tag{3.22}$$

for $\eta,\xi\in C^1(\Gamma)$ and $i=1,2,3$.

Proof. If $X\in C^1(\Gamma)^3$ is tangential, then we can show $\int_\Gamma \mathrm{div}_\Gamma X\,d\mathcal{H}^2=0$ by a standard localization argument and an expression of $\mathrm{div}_\Gamma X$ under local coordinates (see Lemma B.3). By this equality and $\mathrm{div}_\Gamma[(v\cdot n)n]=-(v\cdot n)H$ on Γ we get (3.21). Also, we have (3.22) by setting $v=\eta\xi e_i$ in (3.21), where $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbb{R}^3$. $\Box$

Based on (3.22), we introduce the Sobolev spaces on Γ as in [11]. For $p\in[1,\infty]$ and $i=1,2,3$ we say that $\eta\in L^p(\Gamma)$ has the i-th weak tangential derivative if there exists $\eta_i\in L^p(\Gamma)$ such that

$$\int_\Gamma \eta_i\xi\,d\mathcal{H}^2=-\int_\Gamma \eta(\underline{D}_i\xi+\xi Hn_i)\,d\mathcal{H}^2 \tag{3.23}$$

for all $\xi \in C^1(\Gamma)$. In this case we write $\underline{D}_i\eta = \eta_i$ and define

$$W^{1,p}(\Gamma) := \{\eta \in L^p(\Gamma) \mid \underline{D}_i\eta \in L^p(\Gamma) \text{ for all } i = 1,2,3\},$$
$$\|\eta\|_{W^{1,p}(\Gamma)} := \begin{cases} \left(\|\eta\|^p_{L^p(\Gamma)} + \|\nabla_\Gamma\eta\|^p_{L^p(\Gamma)}\right)^{1/p} & \text{if} \quad p \in [1,\infty), \\ \|\eta\|_{L^\infty(\Gamma)} + \|\nabla_\Gamma\eta\|_{L^\infty(\Gamma)} & \text{if} \quad p = \infty. \end{cases}$$

Here $\nabla_\Gamma\eta := (\underline{D}_1\eta, \underline{D}_2\eta, \underline{D}_3\eta)^T$ is the weak tangential gradient of $\eta \in W^{1,p}(\Gamma)$. This notation is consistent with (3.3) for a C^1 function on Γ. Also, for $\eta \in W^{1,p}(\Gamma)$ and $v \in C^1(\Gamma)^3$, we have

$$\int_\Gamma \nabla_\Gamma\eta \cdot v\, d\mathcal{H}^2 = -\int_\Gamma \eta\{\mathrm{div}_\Gamma v + (v\cdot n)H\}\, d\mathcal{H}^2 \tag{3.24}$$

by (3.23). We also define the second order Sobolev space

$$W^{2,p}(\Gamma) := \{\eta \in W^{1,p}(\Gamma) \mid \underline{D}_i\underline{D}_j\eta \in L^p(\Gamma) \text{ for all } i,j = 1,2,3\},$$
$$\|\eta\|_{W^{2,p}(\Gamma)} := \begin{cases} \left(\|\eta\|^p_{W^{1,p}(\Gamma)} + \|\nabla^2_\Gamma\eta\|^p_{L^p(\Gamma)}\right)^{1/p} & \text{if} \quad p \in [1,\infty), \\ \|\eta\|_{W^{1,\infty}(\Gamma)} + \|\nabla^2_\Gamma\eta\|_{L^\infty(\Gamma)} & \text{if} \quad p = \infty \end{cases}$$

and $W^{m,p}(\Gamma)$ with $m \geq 2$ similarly, and write $W^{0,p}(\Gamma) := L^p(\Gamma)$ and $H^m(\Gamma) := W^{m,2}(\Gamma)$ for $p \in [1,\infty]$ and $m \geq 0$. Here $\nabla^2_\Gamma\eta := (\underline{D}_i\underline{D}_j\eta)_{i,j}$ for $\eta \in W^{2,p}(\Gamma)$. Note that $W^{m,p}(\Gamma)$ is a Banach space. In Lemma B.4 we see that the Sobolev spaces introduced here are equivalent to the standard ones used in the literature of differential geometry, e.g. [3, 29]. Thus we have the following density result by standard localization and mollification arguments (note that we can consider C^m functions on Γ only for $m \leq \ell$ since Γ is of class C^ℓ).

Lemma 3.6. *Let $m = 0,1,\dots,\ell$ and $p \in [1,\infty)$. Then $C^\ell(\Gamma)$ is dense in $W^{m,p}(\Gamma)$.*

By Lemma 3.6 we can apply the results for C^m functions on Γ given in this subsection to $W^{m,p}$ functions with $m = 1,2$ and $p \in [1,\infty)$.

Let $\mathcal{X}(\Gamma)$ be a function space on Γ such as $C^m(\Gamma)$ and $W^{m,p}(\Gamma)$. We define the space of all tangential vector fields on Γ whose components belong to $\mathcal{X}(\Gamma)$ by $\mathcal{X}(\Gamma, T\Gamma) := \{v \in \mathcal{X}(\Gamma)^3 \mid v \cdot n = 0 \text{ on } \Gamma\}$. Then $W^{m,p}(\Gamma, T\Gamma)$

is a closed subspace of $W^{m,p}(\Gamma)^3$ for $m \geq 0$ and $p \in [1,\infty]$. Also, for $v \in W^{1,p}(\Gamma, T\Gamma)$ with $p \in [1,\infty]$ we have

$$\int_\Gamma \mathrm{div}_\Gamma v \, d\mathcal{H}^2 = -\int_\Gamma (v \cdot n) H \, d\mathcal{H}^2 = 0 \tag{3.25}$$

by (3.23) with $\xi \equiv 1$ (note that $\nabla_\Gamma \xi = 0$ on Γ if ξ is constant). When $m \leq \ell - 1$ and $p \neq \infty$, an element of $W^{m,p}(\Gamma, T\Gamma)$ is approximated by $C^{\ell-1}$ tangential vector fields on Γ.

LEMMA 3.7. *Let $m = 0, 1, \dots, \ell - 1$ and $p \in [1,\infty)$. Then $C^{\ell-1}(\Gamma, T\Gamma)$ is dense in $W^{m,p}(\Gamma, T\Gamma)$.*

PROOF. Let $v \in W^{m,p}(\Gamma, T\Gamma) \subset W^{m,p}(\Gamma)^3$. By Lemma 3.6 we can take a sequence $\{\tilde{v}_k\}_{k=1}^\infty$ in $C^\ell(\Gamma)^3$ that converges to v strongly in $W^{m,p}(\Gamma)^3$. For each $k \in \mathbb{N}$ let $v_k := P\tilde{v}_k$ on Γ. Then $v_k \in C^{\ell-1}(\Gamma, T\Gamma)$ since P is of class $C^{\ell-1}$ on Γ. Moreover, since v is tangential on Γ, we have $v - v_k = P(v - \tilde{v}_k)$ on Γ and thus

$$\|v - v_k\|_{W^{m,p}(\Gamma)} \leq c\|v - \tilde{v}_k\|_{W^{m,p}(\Gamma)} \to 0 \quad \text{as} \quad k \to \infty$$

by the $C^{\ell-1}$-regularity of P on Γ and the strong convergence of $\{\tilde{v}_k\}_{k=1}^\infty$ to v in $W^{m,p}(\Gamma)^3$. Hence the claim is valid. $\Box$

3.2. Curved thin domain

From now on, we assume that the closed surface Γ is of class C^5. Let $g_0, g_1 \in C^4(\Gamma)$ such that $g := g_1 - g_0$ satisfies (2.1). For $\varepsilon \in (0,1]$ we define a curved thin domain Ω_ε in $\mathbb{R}^3$ by (1.1), i.e. $\Omega_\varepsilon = \{y + rn(y) \mid y \in \Gamma,\ \varepsilon g_0(y) < r < \varepsilon g_1(y)\}$. Since g_0 and g_1 are bounded on Γ, there exists $\tilde{\varepsilon} \in (0,1]$ such that $\tilde{\varepsilon}|g_i| < \delta$ on Γ for $i = 0, 1$, where $\delta > 0$ is the radius of the tubular neighborhood N of Γ given in Section 3.1. Hence $\overline{\Omega}_\varepsilon \subset N$ and the lemmas in Section 3.1 are applicable in $\overline{\Omega}_\varepsilon$ for all $\varepsilon \in (0, \tilde{\varepsilon}]$. In what follows, we assume $\tilde{\varepsilon} = 1$ by replacing g_i with $\tilde{\varepsilon} g_i$ for $i = 0, 1$.

Let Γ_ε^0 and Γ_ε^1 be the inner and outer boundaries of Ω_ε defined as $\Gamma_\varepsilon^i = \{y + \varepsilon g_i(y) n(y) \mid y \in \Gamma\}$, $i = 0, 1$. Then the whole boundary of Ω_ε is $\Gamma_\varepsilon := \Gamma_\varepsilon^0 \cup \Gamma_\varepsilon^1$. Note that Γ_ε is of class C^4 by the C^5-regularity of Γ and $g_0, g_1 \in C^4(\Gamma)$. We use this fact in the proof of a uniform a priori estimate for the vector Laplace operator (see Section 6).

Let us give surface quantities on Γ_ε. We define vector fields on Γ by

$$\tau_\varepsilon^i(y) := \{I_3 - \varepsilon g_i(y)W(y)\}^{-1}\nabla_\Gamma g_i(y), \tag{3.26}$$

$$n_\varepsilon^i(y) := (-1)^{i+1}\frac{n(y) - \varepsilon\tau_\varepsilon^i(y)}{\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}} \tag{3.27}$$

for $y \in \Gamma$ and $i = 0, 1$. Then τ_ε^i is tangential on Γ by (3.4), (3.12), and $Pa \cdot n = 0$ on Γ for $a \in \mathbb{R}^3$. Also, τ_ε^i and n_ε^i are bounded on Γ uniformly in ε along with their first and second order tangential derivatives.

LEMMA 3.8. *There exists a constant $c > 0$ independent of ε such that*

$$|\tau_\varepsilon^i(y)| \leq c, \quad |\underline{D}_k\tau_\varepsilon^i(y)| \leq c, \quad |\underline{D}_l\underline{D}_k\tau_\varepsilon^i(y)| \leq c, \tag{3.28}$$

$$|\tau_\varepsilon^i(y) - \nabla_\Gamma g_i(y)| \leq c\varepsilon, \quad |\nabla_\Gamma\tau_\varepsilon^i(y) - \nabla_\Gamma^2 g_i(y)| \leq c\varepsilon \tag{3.29}$$

for all $y \in \Gamma$, $i = 0, 1$, and $k, l = 1, 2, 3$. We also have

$$|n_\varepsilon^i(y)| = 1, \quad |\underline{D}_k n_\varepsilon^i(y)| \leq c, \quad |\underline{D}_l\underline{D}_k n_\varepsilon^i(y)| \leq c, \tag{3.30}$$

$$|n_\varepsilon^0(y) + n_\varepsilon^1(y)| \leq c\varepsilon, \quad |\nabla_\Gamma n_\varepsilon^0(y) + \nabla_\Gamma n_\varepsilon^1(y)| \leq c\varepsilon \tag{3.31}$$

for all $y \in \Gamma$, $i = 0, 1$, and $k, l = 1, 2, 3$.

We present the proof of Lemmas 3.8 in Appendix C.

Let n_ε be the unit outward normal vector field of Γ_ε. For $i = 0, 1$ the direction of n_ε on Γ_ε^i is the same as that of $(-1)^{i+1}\bar{n}$ since the signed distance function d from Γ increases in the direction of n.

LEMMA 3.9. *Let $\bar{n}_\varepsilon^i = n_\varepsilon^i \circ \pi$ be the constant extension of n_ε^i. Then*

$$n_\varepsilon(x) = \bar{n}_\varepsilon^i(x), \quad x \in \Gamma_\varepsilon^i,\ i = 0, 1. \tag{3.32}$$

PROOF. We observe in Lemma B.5 that, if we define

$$\tau_h := (I_3 - hW)^{-1}\nabla_\Gamma h, \quad n_h := \frac{n - \tau_h}{\sqrt{1+|\tau_h|^2}} \quad \text{on} \quad \Gamma \tag{3.33}$$

for $h \in C^1(\Gamma)$ satisfying $|h| < \delta$ on Γ, then the constant extension of n_h is a unit normal vector field of the parametrized surface

$$\Gamma_h := \{y + h(y)n(y) \mid y \in \Gamma\}. \tag{3.34}$$

Setting $h = \varepsilon g_i$ in Lemma B.5 and noting that the direction of n_ε on Γ_ε^i is the same as that of $(-1)^{i+1}\bar{n}$ for $i = 0, 1$ we obtain (3.32). □

As in Section 3.1, we set $P_\varepsilon := I_3 - n_\varepsilon \otimes n_\varepsilon$ and $Q_\varepsilon := n_\varepsilon \otimes n_\varepsilon$ on Γ_ε and define the tangential gradient and the tangential derivatives of $\varphi \in C^1(\Gamma_\varepsilon)$ by $\nabla_{\Gamma_\varepsilon}\varphi := P_\varepsilon \nabla\tilde{\varphi}$ and $\underline{D}_i^\varepsilon \varphi := \sum_{j=1}^3 [P_\varepsilon]_{ij}\partial_j\tilde{\varphi}$ on Γ_ε, $i = 1, 2, 3$, where $\tilde{\varphi}$ is any C^1-extension of φ to an open neighborhood of Γ_ε with $\tilde{\varphi}|_{\Gamma_\varepsilon} = \varphi$. For $u = (u_1, u_2, u_3)^T \in C^1(\Gamma_\varepsilon)^3$ we define the tangential gradient matrix and the surface divergence of u by

$$\nabla_{\Gamma_\varepsilon} u := \begin{pmatrix} \underline{D}_1^\varepsilon u_1 & \underline{D}_1^\varepsilon u_2 & \underline{D}_1^\varepsilon u_3 \\ \underline{D}_2^\varepsilon u_1 & \underline{D}_2^\varepsilon u_2 & \underline{D}_2^\varepsilon u_3 \\ \underline{D}_3^\varepsilon u_1 & \underline{D}_3^\varepsilon u_2 & \underline{D}_3^\varepsilon u_3 \end{pmatrix}, \quad \mathrm{div}_{\Gamma_\varepsilon} u := \mathrm{tr}[\nabla_{\Gamma_\varepsilon} u] = \sum_{i=1}^3 \underline{D}_i^\varepsilon u_i$$

on Γ_ε. Also, for $u \in C^1(\Gamma_\varepsilon)^3$ and $\varphi \in C(\Gamma_\varepsilon)^3$ we write

$$(\varphi \cdot \nabla_{\Gamma_\varepsilon})u := \begin{pmatrix} \varphi \cdot \nabla_{\Gamma_\varepsilon} u_1 \\ \varphi \cdot \nabla_{\Gamma_\varepsilon} u_2 \\ \varphi \cdot \nabla_{\Gamma_\varepsilon} u_3 \end{pmatrix} = (\nabla_{\Gamma_\varepsilon} u)^T \varphi \quad \text{on} \quad \Gamma_\varepsilon.$$

Note that, as in the case of Γ, we have

$$\nabla_{\Gamma_\varepsilon} u = P_\varepsilon \nabla\tilde{u}, \quad (\varphi \cdot \nabla_{\Gamma_\varepsilon})u = [(P_\varepsilon\varphi) \cdot \nabla]\tilde{u} \quad \text{on} \quad \Gamma_\varepsilon \tag{3.35}$$

for any C^1-extension $\tilde{u}$ of u to an open neighborhood of Γ_ε with $\tilde{u}|_{\Gamma_\varepsilon} = u$. We also define the Weingarten map W_ε and (twice) the mean curvature H_ε of Γ_ε as $W_\varepsilon := -\nabla_{\Gamma_\varepsilon} n_\varepsilon$ and $H_\varepsilon := \mathrm{tr}[W_\varepsilon] = -\mathrm{div}_{\Gamma_\varepsilon} n_\varepsilon$ on Γ_ε. Then by Lemma 3.1 we have

$$W_\varepsilon^T = P_\varepsilon W_\varepsilon = W_\varepsilon P_\varepsilon = W_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon. \tag{3.36}$$

The weak tangential derivatives of functions on Γ_ε and the Sobolev spaces $W^{m,p}(\Gamma_\varepsilon)$ are also defined as in Section 3.1.

By the expression (3.27) of the unit outward normal n_ε to Γ_ε, we can compare the surface quantities of Γ_ε with those of Γ.

Lemma 3.10. *There exists a constant $c > 0$ independent of ε such that*

$$\left| n_\varepsilon(x) - (-1)^{i+1}\left\{\bar{n}(x) - \varepsilon\overline{\nabla_\Gamma g_i}(x)\right\}\right| \leq c\varepsilon^2, \tag{3.37}$$

$$\left|P_\varepsilon(x) - \overline{P}(x)\right| \leq c\varepsilon, \quad \left|Q_\varepsilon(x) - \overline{Q}(x)\right| \leq c\varepsilon, \tag{3.38}$$

$$\left|W_\varepsilon(x) - (-1)^{i+1}\overline{W}(x)\right| \leq c\varepsilon, \quad \left|H_\varepsilon(x) - (-1)^{i+1}\overline{H}(x)\right| \leq c\varepsilon, \tag{3.39}$$

$$\left|\underline{D}_j^\varepsilon W_\varepsilon(x) - (-1)^{i+1}\overline{\underline{D}_j W}(x)\right| \leq c\varepsilon \tag{3.40}$$

for all $x \in \Gamma_\varepsilon^i$, $i = 0, 1$, *and* $j = 1, 2, 3$. *In particular,* W_ε, H_ε, *and* $\underline{D}_j^\varepsilon W_\varepsilon$ *with* $j = 1, 2, 3$ *are uniformly bounded in* ε *on* Γ_ε *(note that* $|P_\varepsilon| = 2$ *and* $|Q_\varepsilon| = 1$ *on* Γ_ε*).*

We can also compare the surface quantities of Γ_ε^0 and Γ_ε^1.

LEMMA 3.11. *There exists a constant* $c > 0$ *independent of* ε *such that*

$$|F_\varepsilon(y + \varepsilon g_1(y)n(y)) - F_\varepsilon(y + \varepsilon g_0(y)n(y))| \leq c\varepsilon, \tag{3.41}$$

$$|G_\varepsilon(y + \varepsilon g_1(y)n(y)) + G_\varepsilon(y + \varepsilon g_0(y)n(y))| \leq c\varepsilon \tag{3.42}$$

for all $y \in \Gamma$, $F_\varepsilon = P_\varepsilon, Q_\varepsilon$, *and* $G_\varepsilon = W_\varepsilon, H_\varepsilon, \underline{D}_j^\varepsilon W_\varepsilon$ *with* $j = 1, 2, 3$.

The proofs of Lemmas 3.10 and 3.11 are given in Appendix C. Note that, in (3.42), the sum of the surface quantities on Γ_ε^0 and Γ_ε^1 related to n_ε, not the difference of them, is of order ε. This is because the direction of n_ε on Γ_ε^0 is the opposite to that of n_ε on Γ_ε^1.

Next we give a change of variables formula for an integral over Ω_ε. For functions φ on Ω_ε and η on Γ_ε^i, $i = 0, 1$ we use the notations

$$\varphi^\sharp(y, r) := \varphi(y + rn(y)), \qquad y \in \Gamma,\ r \in (\varepsilon g_0(y), \varepsilon g_1(y)), \tag{3.43}$$

$$\eta_i^\sharp(y) := \eta(y + \varepsilon g_i(y)n(y)), \quad y \in \Gamma. \tag{3.44}$$

Let $J = J(y, r)$ be a function given by

$$J(y, r) := \det[I_3 - rW(y)] = \{1 - r\kappa_1(y)\}\{1 - r\kappa_2(y)\} \tag{3.45}$$

for $y \in \Gamma$ and $r \in (-\delta, \delta)$. By (3.2) and $\kappa_1, \kappa_2 \in C^3(\Gamma)$ we have

$$c^{-1} \leq J(y, r) \leq c, \quad \left|\frac{\partial J}{\partial r}(y, r)\right| \leq c, \tag{3.46}$$

$$|J(y, r) - 1| \leq c|r| \tag{3.47}$$

for all $y \in \Gamma$ and $r \in (-\delta, \delta)$. Note that J is the Jacobian appearing in the change of variables formula

$$\int_{\Omega_\varepsilon} \varphi(x)\, dx = \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y + rn(y))J(y, r)\, dr\, d\mathcal{H}^2(y) \tag{3.48}$$

for a function φ on Ω_ε (see e.g. [15, Section 14.6]). The formula (3.48) can be seen as a co-area formula. From (3.46) and (3.48) it immediately follows that

$$c^{-1}\|\varphi\|^p_{L^p(\Omega_\varepsilon)} \le \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r)|^p \, dr \, d\mathcal{H}^2(y) \le c\|\varphi\|^p_{L^p(\Omega_\varepsilon)} \tag{3.49}$$

for $\varphi \in L^p(\Omega_\varepsilon)$, $p \in [1,\infty)$. In the sequel we frequently use this inequality and the following estimates for the constant extension $\bar\eta = \eta\circ\pi$ of a function η on Γ.

LEMMA 3.12. *For $p \in [1,\infty)$ we have $\eta \in L^p(\Gamma)$ if and only if $\bar\eta \in L^p(\Omega_\varepsilon)$, and there exists a constant $c > 0$ independent of ε and η such that*

$$c^{-1}\varepsilon^{1/p}\|\eta\|_{L^p(\Gamma)} \le \|\bar\eta\|_{L^p(\Omega_\varepsilon)} \le c\varepsilon^{1/p}\|\eta\|_{L^p(\Gamma)}. \tag{3.50}$$

Moreover, $\eta \in W^{1,p}(\Gamma)$ if and only if $\bar\eta \in W^{1,p}(\Omega_\varepsilon)$ and we have

$$c^{-1}\varepsilon^{1/p}\|\eta\|_{W^{1,p}(\Gamma)} \le \|\bar\eta\|_{W^{1,p}(\Omega_\varepsilon)} \le c\varepsilon^{1/p}\|\eta\|_{W^{1,p}(\Gamma)}. \tag{3.51}$$

PROOF. Since $\bar\eta^\sharp(y,r) = \eta(y)$ for $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$,

$$\int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\bar\eta^\sharp(y,r)|^p \, dr \, d\mathcal{H}^2(y) = \varepsilon \int_\Gamma g(y)|\eta(y)|^p \, d\mathcal{H}^2(y).$$

By this equality, (2.1), and (3.49), we get (3.50). Similarly,

$$c^{-1}\varepsilon^{1/p}\|\nabla_\Gamma\eta\|_{L^p(\Gamma)} \le \|\nabla\bar\eta\|_{L^p(\Omega_\varepsilon)} \le c\varepsilon^{1/p}\|\nabla_\Gamma\eta\|_{L^p(\Gamma)}$$

by (2.1), (3.17), and (3.49). This inequality and (3.50) yield (3.51). □

We also give a change of variables formula for an integral over Γ^i_ε.

LEMMA 3.13. *For $\varphi \in L^1(\Gamma^i_\varepsilon)$, $i = 0,1$ let $\varphi^\sharp_i$ be given by (3.44). Then*

$$\int_{\Gamma^i_\varepsilon} \varphi(x) \, d\mathcal{H}^2(x) = \int_\Gamma \varphi^\sharp_i(y) J(y, \varepsilon g_i(y)) \sqrt{1+\varepsilon^2|\tau^i_\varepsilon(y)|^2} \, d\mathcal{H}^2(y) \tag{3.52}$$

with τ_ε^i *given by* (3.26). *Moreover, if* $\varphi \in L^p(\Gamma_\varepsilon^i)$, $p \in [1,\infty)$, *then* $\varphi_i^\sharp \in L^p(\Gamma)$ *and*

$$c^{-1}\|\varphi\|_{L^p(\Gamma_\varepsilon^i)} \leq \|\varphi_i^\sharp\|_{L^p(\Gamma)} \leq c\|\varphi\|_{L^p(\Gamma_\varepsilon^i)}, \tag{3.53}$$

where $c > 0$ *is a constant independent of* ε *and* φ.

Proof. In Lemma B.6 we show the change of variables formula

$$\int_{\Gamma_h} \varphi(x)\,d\mathcal{H}^2(x) = \int_\Gamma \varphi_h^\sharp(y) J(y,h(y))\sqrt{1+|\tau_h(y)|^2}\,d\mathcal{H}^2(y)$$

for $\varphi \in L^1(\Gamma_h)$, where τ_h and Γ_h are given by (3.33) and (3.34) with $h \in C^1(\Gamma)$ satisfying $|h| < \delta$ on Γ and $\varphi_h^\sharp(y) := \varphi(y+h(y)n(y))$ for $y \in \Gamma$. Setting $h = \varepsilon g_i$, $i = 0,1$ in the above formula we obtain (3.52). Also, (3.53) follows from (3.28), (3.46), and (3.52). □

4. Fundamental Inequalities and Formulas

Let us give fundamental inequalities and formulas for functions on Ω_ε and Γ_ε. For a function φ on Ω_ε and $x \in \Omega_\varepsilon$ with $y = \pi(x) \in \Gamma$ let

$$\partial_n\varphi(x) := (\bar{n}(x)\cdot\nabla)\varphi(x) = \frac{d}{dr}\bigl(\varphi(y+rn(y))\bigr)\Big|_{r=d(x)} \tag{4.1}$$

be the derivative of φ in the normal direction of Γ. Note that $\partial_n\bar{\eta} = (\bar{n}\cdot\nabla)\bar{\eta} = 0$ in Ω_ε for the constant extension $\bar{\eta} = \eta\circ\pi$ of $\eta \in C^1(\Gamma)$.

First we show Poincaré and trace type inequalities on Ω_ε.

Lemma 4.1. *There exists a constant* $c > 0$ *such that*

$$\|\varphi\|_{L^p(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/p}\|\varphi\|_{L^p(\Gamma_\varepsilon^i)} + \varepsilon\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}\right), \tag{4.2}$$

$$\|\varphi\|_{L^p(\Gamma_\varepsilon^i)} \leq c\left(\varepsilon^{-1/p}\|\varphi\|_{L^p(\Omega_\varepsilon)} + \varepsilon^{1-1/p}\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}\right) \tag{4.3}$$

for all $\varepsilon \in (0,1]$, $\varphi \in W^{1,p}(\Omega_\varepsilon)$ *with* $p \in [1,\infty)$, *and* $i = 0,1$.

Proof. We prove (4.2) and (4.3) for $i = 0$. The proofs for $i = 1$ are the same. We use the notations (3.43) and (3.44). Since

$$\varphi^\sharp(y,\varepsilon g_0(y)) = \varphi_0^\sharp(y), \quad \frac{\partial\varphi^\sharp}{\partial r}(y,r) = (\partial_n\varphi)^\sharp(y,r)$$

for $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$ by (4.1), we have

$$\varphi^\sharp(y,r) = \varphi_0^\sharp(y) + \int_{\varepsilon g_0(y)}^r (\partial_n \varphi)^\sharp(y,\tilde{r})\, d\tilde{r}. \tag{4.4}$$

From (4.4) and Hölder's inequality it follows that

$$|\varphi^\sharp(y,r)| \leq |\varphi_0^\sharp(y)| + c\varepsilon^{1-1/p} \left(\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n \varphi)^\sharp(y,\tilde{r})|^p \, d\tilde{r} \right)^{1/p}.$$

Noting that the right-hand side is independent of r, we integrate the p-th power of both sides of this inequality with respect to r to get

$$\begin{aligned} &\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r)|^p \, dr \\ &\qquad \leq c \left(\varepsilon |\varphi_0^\sharp(y)|^p + \varepsilon^p \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n \varphi)^\sharp(y,\tilde{r})|^p \, d\tilde{r} \right), \quad y \in \Gamma. \end{aligned} \tag{4.5}$$

We integrate both sides with respect to y and use (3.49) to have

$$\|\varphi\|_{L^p(\Omega_\varepsilon)}^p \leq c \left(\varepsilon \|\varphi_0^\sharp\|_{L^p(\Gamma)}^p + \varepsilon^p \|\partial_n \varphi\|_{L^p(\Omega_\varepsilon)}^p \right).$$

Applying (3.53) to the first term on the right-hand side we obtain (4.2).

Next let us prove (4.3). As in the proof of (4.5), we use (4.4) to get

$$\begin{aligned} &g(y)|\varphi_0^\sharp(y)|^p \\ &\qquad \leq c \left(\varepsilon^{-1} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r)|^p \, dr + \varepsilon^{p-1} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n \varphi)^\sharp(y,\tilde{r})|^p \, d\tilde{r} \right) \end{aligned}$$

for all $y \in \Gamma$. Here the function $g(y)$ on the left-hand side comes from the integration with respect to r. Integrating both sides of the above inequality with respect to y and using (2.1) and (3.49) we obtain

$$\|\varphi_0^\sharp\|_{L^p(\Gamma)}^p \leq c \left(\varepsilon^{-1} \|\varphi\|_{L^p(\Omega_\varepsilon)}^p + \varepsilon^{p-1} \|\partial_n \varphi\|_{L^p(\Omega_\varepsilon)}^p \right).$$

We apply (3.53) to the left-hand side of this inequality to get (4.3). $\square$

Next we present two results for a vector field $u\colon \Omega_\varepsilon \to \mathbb{R}^3$ satisfying the impermeable boundary condition

$$u \cdot n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon. \tag{4.6}$$

LEMMA 4.2. *For $i = 0, 1$ let $u \in L^2(\Gamma_\varepsilon^i)^3$ satisfy (4.6) on Γ_ε^i. Then*

$$u \cdot \bar{n} = \varepsilon u \cdot \bar{\tau}_\varepsilon^i, \quad |u \cdot \bar{n}| \leq c\varepsilon |u| \quad \text{on} \quad \Gamma_\varepsilon^i, \tag{4.7}$$

where τ_ε^i is given by (3.26) and $c > 0$ is a constant independent of ε and u.

PROOF. The first equality of (4.7) follows from (3.27), (3.32), and (4.6) on Γ_ε^i. Also, we get the second inequality of (4.7) by the first one and (3.28). □

LEMMA 4.3. *There exists a constant $c > 0$ such that*

$$\left| \int_{\Gamma_\varepsilon} (u \cdot \nabla) u \cdot n_\varepsilon \, d\mathcal{H}^2 \right| \leq c \left(\|u\|_{L^2(\Omega_\varepsilon)}^2 + \|u\|_{L^2(\Omega_\varepsilon)} \|\nabla u\|_{L^2(\Omega_\varepsilon)} \right) \tag{4.8}$$

for all $\varepsilon \in (0, 1]$ and $u \in H^2(\Omega_\varepsilon)^3$ satisfying (4.6).

Note that the second order derivatives of u do not appear in (4.8), although we need $u \in H^2(\Omega_\varepsilon)^3$ to confirm $\nabla u|_{\Gamma_\varepsilon} \in L^2(\Omega_\varepsilon)^{3\times 3}$. We use (4.8) in the proof of Lemma 5.1 below, where in fact $u \in C^2(\overline{\Omega}_\varepsilon)^3$, and also in the last part of our study [48, Lemma 4.14], where we only have $u \in H^2(\Omega_\varepsilon)^3$.

PROOF. Noting that u is tangential on Γ_ε by (4.6), we use (3.35) to get

$$(u \cdot \nabla) u \cdot n_\varepsilon = (u \cdot \nabla_{\Gamma_\varepsilon}) u \cdot n_\varepsilon = u \cdot \nabla_{\Gamma_\varepsilon}(u \cdot n_\varepsilon) - u \cdot (u \cdot \nabla_{\Gamma_\varepsilon}) n_\varepsilon$$

on Γ_ε. The first term on the right-hand side vanishes by (4.6) (note that the tangential gradient on Γ_ε depends only on the values of a function on Γ_ε). Also, $(u \cdot \nabla_{\Gamma_\varepsilon}) n_\varepsilon = -W_\varepsilon u$ by $-\nabla_{\Gamma_\varepsilon} n_\varepsilon = W_\varepsilon = W_\varepsilon^T$. Hence $(u \cdot \nabla) u \cdot n_\varepsilon = u \cdot W_\varepsilon u$ on Γ_ε and

$$\int_{\Gamma_\varepsilon} (u \cdot \nabla) u \cdot n_\varepsilon \, d\mathcal{H}^2 = \int_{\Gamma_\varepsilon} u \cdot W_\varepsilon u \, d\mathcal{H}^2 = \sum_{i=0,1} \int_{\Gamma_\varepsilon^i} u \cdot W_\varepsilon u \, d\mathcal{H}^2. \tag{4.9}$$

To estimate the right-hand side we interpolate the integrals over Γ_ε^0 and Γ_ε^1 to produce an integral over Ω_ε. Let

$$
\begin{aligned}
&F_i(y) := \sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}\, W_{\varepsilon,i}^\sharp(y), \quad i=0,1,\\
(4.10)\qquad &F(y,r) := \frac{1}{\varepsilon g(y)}\big\{\big(r-\varepsilon g_0(y)\big)F_1(y) - \big(\varepsilon g_1(y)-r\big)F_0(y)\big\},\\
&\varphi(y,r) := u^\sharp(y,r)\cdot F(y,r)u^\sharp(y,r)J(y,r)
\end{aligned}
$$

for $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$ with τ_ε^i, $i=0,1$ and J given by (3.26) and (3.45). Here and hereafter we use the notations (3.43) and (3.44) and sometimes suppress the arguments y and r. By (4.10) we have

$$
[u\cdot W_\varepsilon u]_i^\sharp(y)\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}J(y,\varepsilon g_i(y)) = (-1)^{i+1}\varphi(y,\varepsilon g_i(y))
$$

for $y\in\Gamma$ and $i=0,1$. From this relation and (3.52) we deduce that

$$
\begin{aligned}
&\sum_{i=0,1}\int_{\Gamma_\varepsilon^i}[u\cdot W_\varepsilon u](x)\,d\mathcal{H}^2(x)\\
(4.11)\qquad &= \int_\Gamma\{\varphi(y,\varepsilon g_1(y))-\varphi(y,\varepsilon g_0(y))\}\,d\mathcal{H}^2(y)\\
&= \int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\frac{\partial\varphi}{\partial r}(y,r)\,dr\,d\mathcal{H}^2(y).
\end{aligned}
$$

To estimate the integrand on the last line we use (3.46) to get

$$
(4.12)\qquad \left|\frac{\partial\varphi}{\partial r}\right| \le c\left\{\left(|F|+\left|\frac{\partial F}{\partial r}\right|\right)|u^\sharp|^2+|F||u^\sharp||(\nabla u)^\sharp|\right\}.
$$

By (3.28) and the uniform boundedness in ε of W_ε on Γ_ε (see Lemma 3.10), we see that F_0 and F_1 are bounded on Γ uniformly in ε. Thus

$$
(4.13)\qquad |F(y,r)| \le \frac{c}{\varepsilon g(y)}\big\{\big(r-\varepsilon g_0(y)\big)+\big(\varepsilon g_1(y)-r\big)\big\} = c
$$

for $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$. Also,

$$
(4.14)\qquad \left|\frac{\partial F}{\partial r}\right| \le c\varepsilon^{-1}\left(|W_{\varepsilon,1}^\sharp+W_{\varepsilon,0}^\sharp|+\sum_{i=0,1}\left(\sqrt{1+\varepsilon^2|\tau_\varepsilon^i|^2}-1\right)|W_{\varepsilon,i}^\sharp|\right)
$$

by $\partial F/\partial r = (\varepsilon g)^{-1}(F_1 + F_0)$ and (4.10). We observe by the mean value theorem for the function $\sqrt{1+s}$, $s \geq 0$ and (3.28) that

$$0 \leq \sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2} - 1 \leq \frac{\varepsilon^2}{2}|\tau_\varepsilon^i(y)|^2 \leq c\varepsilon^2, \quad y \in \Gamma. \tag{4.15}$$

We apply this inequality, (3.42) with $G_\varepsilon = W_\varepsilon$, and the uniform boundedness in ε of W_ε on Γ_ε to the right-hand side of (4.14) to obtain

$$\left|\frac{\partial F}{\partial r}(y,r)\right| \leq c \quad \text{for all} \quad y \in \Gamma,\, r \in [\varepsilon g_0(y), \varepsilon g_1(y)]. \tag{4.16}$$

From (4.12), (4.13), and (4.16), we deduce that

$$\left|\frac{\partial \varphi}{\partial r}(y,r)\right| \leq c\Big(|u^\sharp(y,r)|^2 + \Big[|u^\sharp||(\nabla u)^\sharp|\Big](y,r)\Big) \tag{4.17}$$

for $y \in \Gamma$ and $r \in [\varepsilon g_0(y), \varepsilon g_1(y)]$. Thus, by (4.9), (4.11), and (4.17),

$$\left|\int_{\Gamma_\varepsilon} (u\cdot\nabla)u \cdot n_\varepsilon \, d\mathcal{H}^2\right| \leq c\int_\Gamma \int_{\varepsilon g_0}^{\varepsilon g_1} \Big(|u^\sharp|^2 + |u^\sharp||(\nabla u)^\sharp|\Big)\, dr\, d\mathcal{H}^2$$

and we apply (3.49) and Hölder's inequality to the right-hand side to obtain (4.8). $\square$

Let us give two formulas on Γ_ε from the slip boundary conditions

$$u \cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u) n_\varepsilon + \gamma_\varepsilon u = 0 \quad \text{on} \quad \Gamma_\varepsilon \tag{4.18}$$

which are crucial for the proofs of a uniform a priori estimate for the vector Laplace operator on Ω_ε (see Theorem 6.1) and the uniform difference estimate for the Stokes and Laplace operators (2.14).

LEMMA 4.4. *For $i = 0, 1$ let $u \in H^2(\Omega_\varepsilon)^3$ satisfy (4.18) on Γ_ε^i. Then*

$$P_\varepsilon(n_\varepsilon \cdot \nabla)u = -W_\varepsilon u - \frac{\gamma_\varepsilon}{\nu}u \quad \textit{on} \quad \Gamma_\varepsilon^i, \tag{4.19}$$

$$n_\varepsilon \times \operatorname{curl} u = -n_\varepsilon \times \left\{n_\varepsilon \times \left(2W_\varepsilon u + \frac{\gamma_\varepsilon}{\nu}u\right)\right\} \quad \textit{on} \quad \Gamma_\varepsilon^i. \tag{4.20}$$

Proof. Taking the tangential gradient of $u \cdot n_\varepsilon = 0$ on Γ_ε^i we have

$$(\nabla_{\Gamma_\varepsilon} u) n_\varepsilon = -(\nabla_{\Gamma_\varepsilon} n_\varepsilon) u = W_\varepsilon u \quad \text{on} \quad \Gamma_\varepsilon^i. \tag{4.21}$$

By $2D(u) = \nabla u + (\nabla u)^T$, (3.35), and (4.21), we have $2P_\varepsilon D(u) n_\varepsilon = W_\varepsilon u + P_\varepsilon (n_\varepsilon \cdot \nabla) u$ on Γ_ε^i. This equality and (4.18) give (4.19).

To prove (4.20) we see that the vector field $n_\varepsilon \times \operatorname{curl} u$ is tangential on Γ_ε^i. By this fact, (3.35), (4.19), and (4.21),

$$\begin{aligned} n_\varepsilon \times \operatorname{curl} u &= P_\varepsilon (n_\varepsilon \times \operatorname{curl} u) = P_\varepsilon \{ (\nabla u) n_\varepsilon - (\nabla u)^T n_\varepsilon \} \\ &= (\nabla_{\Gamma_\varepsilon} u) n_\varepsilon - P_\varepsilon (n_\varepsilon \cdot \nabla) u = 2 W_\varepsilon u + \frac{\gamma_\varepsilon}{\nu} u \end{aligned}$$

on Γ_ε^i. The equality (4.20) follows from the the above equality and the identity $a \times (a \times b) = (a \cdot b) a - |a|^2 b$ with $a = n_\varepsilon$ and $b = 2 W_\varepsilon u + \nu^{-1} \gamma_\varepsilon u$, since $n_\varepsilon \cdot u = n_\varepsilon \cdot W_\varepsilon u = 0$ and $|n_\varepsilon|^2 = 1$ on Γ_ε^i. □

5. Korn's Inequality on a Curved Thin Domain

In this section we establish the uniform Korn inequality (1.9) on Ω_ε that is essential for the uniform coerciveness of the bilinear form a_ε given by (2.2). We also compare it with a standard Korn inequality for simple examples of Ω_ε.

5.1. Uniform Korn inequality on a curved thin domain

The goal of this subsection is to show the uniform Korn inequality under suitable assumptions for a vector field u on Ω_ε. First we give a uniform L^2-estimate for ∇u on Ω_ε.

Lemma 5.1. *There exists a constant $c_{K,1} > 0$ independent of ε such that*

$$\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq 4 \|D(u)\|_{L^2(\Omega_\varepsilon)}^2 + c_{K,1} \|u\|_{L^2(\Omega_\varepsilon)}^2 \tag{5.1}$$

for all $\varepsilon \in (0,1]$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (4.6).

Let us prove an auxiliary density result.

Lemma 5.2. *Let $u \in H^1(\Omega_\varepsilon)^3$ satisfy* (4.6). *Then there exists a sequence $\{u_k\}_{k=1}^\infty$ in $C^2(\overline{\Omega}_\varepsilon)^3$ such that u_k satisfies* (4.6) *for each $k \in \mathbb{N}$ and $\lim_{k\to\infty} \|u - u_k\|_{H^1(\Omega_\varepsilon)} = 0$.*

PROOF. We follow the idea of the proof of [5, Theorem IV.4.7], but here it is not necessary to localize a vector field on Ω_ε. For $x \in N$ we define

$$\tilde{n}(x) := \frac{1}{\varepsilon\bar{g}(x)}\bigl\{\bigl(d(x) - \varepsilon\bar{g}_0(x)\bigr)\bar{n}_\varepsilon^1(x) + \bigl(\varepsilon\bar{g}_1(x) - d(x)\bigr)\bar{n}_\varepsilon^0(x)\bigr\},$$

where n_ε^0 and n_ε^1 are given by (3.27) and $\bar{\eta} = \eta \circ \pi$ denotes the constant extension of a function η on Γ. Then $\tilde{n} \in C^2(N)^3$ by the regularity of Γ, g_0, and g_1. Moreover, $\tilde{n} = n_\varepsilon$ on Γ_ε by Lemma 3.9. Hence if $u \in H^1(\Omega_\varepsilon)^3$ satisfies (4.6), then we have $u \cdot \tilde{n} \in H_0^1(\Omega_\varepsilon)$ and $w := u - (u \cdot \tilde{n})\tilde{n} \in H^1(\Omega_\varepsilon)^3$. Since Γ_ε is of class C^4, there exist sequences $\{\varphi_k\}_{k=1}^\infty$ in $C_c^\infty(\Omega_\varepsilon)$ and $\{w_k\}_{k=1}^\infty$ in $C^\infty(\overline{\Omega}_\varepsilon)^3$ such that

$$\lim_{k\to\infty} \|u \cdot \tilde{n} - \varphi_k\|_{H^1(\Omega_\varepsilon)} = \lim_{k\to\infty} \|w - w_k\|_{H^1(\Omega_\varepsilon)} = 0.$$

Here $C_c^\infty(\Omega_\varepsilon)$ is the space of all smooth and compactly supported functions on Ω_ε. Therefore, setting $u_k := \varphi_k\tilde{n} + w_k - (w_k \cdot \tilde{n})\tilde{n} \in C^2(\overline{\Omega}_\varepsilon)^3$ we see that $u_k \cdot n_\varepsilon = u_k \cdot \tilde{n} = \varphi_k = 0$ on Γ_ε for each $k \in \mathbb{N}$ and

$$\begin{aligned}\|u - u_k\|_{H^1(\Omega_\varepsilon)} &= \|(u \cdot \tilde{n} - \varphi_k)\tilde{n} + (w - w_k) - \{(w - w_k) \cdot \tilde{n}\}\tilde{n}\|_{H^1(\Omega_\varepsilon)} \\ &\le c_\varepsilon \left(\|u \cdot \tilde{n} - \varphi_k\|_{H^1(\Omega_\varepsilon)} + \|w - w_k\|_{H^1(\Omega_\varepsilon)}\right) \to 0\end{aligned}$$

as $k \to \infty$ (note that $u = (u \cdot \tilde{n})\tilde{n} + w$ and $w \cdot \tilde{n} = 0$ in Ω_ε and c_ε is independent of k). $\square$

PROOF OF LEMMA 5.1. It is sufficient to show (5.1) for all $u \in C^2(\overline{\Omega}_\varepsilon)^3$ satisfying (4.6) by Lemma 5.2 and a density argument. Then we can carry out integration by parts twice to get

$$\begin{aligned}&\int_{\Omega_\varepsilon} \nabla u : (\nabla u)^T\,dx \\ &\qquad = \int_{\Omega_\varepsilon} (\operatorname{div} u)^2\,dx + \int_{\Gamma_\varepsilon} \{(u \cdot \nabla)u \cdot n_\varepsilon - (u \cdot n_\varepsilon)\operatorname{div} u\}\,d\mathcal{H}^2.\end{aligned}$$

Since $(\operatorname{div} u)^2 \ge 0$ in Ω_ε and $u \cdot n_\varepsilon = 0$ on Γ_ε, the above equality implies

$$\int_{\Omega_\varepsilon} \nabla u : (\nabla u)^T\,dx \ge \int_{\Gamma_\varepsilon} (u \cdot \nabla)u \cdot n_\varepsilon\,d\mathcal{H}^2.$$

By this inequality and $|\nabla u|^2 = 2|D(u)|^2 - \nabla u : (\nabla u)^T$ in Ω_ε, we have

$$\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \leq 2\|D(u)\|^2_{L^2(\Omega_\varepsilon)} - \int_{\Gamma_\varepsilon} (u \cdot \nabla)u \cdot n_\varepsilon \, d\mathcal{H}^2.$$

Noting that $u \in C^2(\overline{\Omega}_\varepsilon)^3$ satisfies (4.6), we apply (4.8) to the last term and use $ab \leq (a^2+b^2)/2$ for $a, b \geq 0$ to obtain

$$\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \leq 2\|D(u)\|^2_{L^2(\Omega_\varepsilon)} + c\|u\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega_\varepsilon)}.$$

Hence (5.1) follows. □

Next we show a uniform L^2-estimate for u by the L^2-norms of ∇u and $D(u)$ on Ω_ε. Recall that for a function η on Γ we denote by $\bar{\eta} = \eta \circ \pi$ its constant extension in the normal direction of Γ.

LEMMA 5.3. *For given $\alpha > 0$ and $\beta \in [0,1)$ there exist constants $\varepsilon_K = \varepsilon_K(\alpha,\beta) \in (0,1]$ and $c_{K,2} = c_{K,2}(\alpha,\beta) > 0$ such that*

$$\|u\|^2_{L^2(\Omega_\varepsilon)} \leq \alpha\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} + c_{K,2}\|D(u)\|^2_{L^2(\Omega_\varepsilon)} \tag{5.2}$$

for all $\varepsilon \in (0, \varepsilon_K]$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (4.6) and

$$\left|(u, \bar{v})_{L^2(\Omega_\varepsilon)}\right| \leq \beta\|u\|_{L^2(\Omega_\varepsilon)}\|\bar{v}\|_{L^2(\Omega_\varepsilon)} \quad \text{for all} \quad v \in \mathcal{K}_g(\Gamma). \tag{5.3}$$

Here $\mathcal{K}_g(\Gamma)$ is the function space on Γ given by (2.5).

A geometric interpretation of the condition (5.3) for the uniform Korn inequality is given in [37, Remark 3.1]. Also, we observe in Section 5.2 that (5.3) is indeed necessary for the uniform Korn inequality.

To prove Lemma 5.3 we transform integrals over Ω_ε into those over the domain Ω_1 with fixed thickness by using the following lemmas (note that we assume $\overline{\Omega}_1 \subset N$ by scaling g_0 and g_1).

LEMMA 5.4. *For $\varepsilon \in (0,1]$ let*

$$\Phi_\varepsilon(X) := \pi(X) + \varepsilon d(X)\bar{n}(X), \quad X \in \Omega_1. \tag{5.4}$$

Then Φ_ε is a bijection from Ω_1 onto Ω_ε and

$$\int_{\Omega_\varepsilon} \varphi(x)\, dx = \varepsilon \int_{\Omega_1} \xi(X) J(\pi(X), d(X))^{-1} J(\pi(X), \varepsilon d(X))\, dX \tag{5.5}$$

for a function φ *on* Ω_ε, *where* $\xi := \varphi \circ \Phi_\varepsilon$ *on* Ω_1 *and* J *is given by* (3.45). *Moreover, if* $\varphi \in L^2(\Omega_\varepsilon)$, *then* $\xi \in L^2(\Omega_1)$ *and there exist constants* $c_1, c_2 > 0$ *independent of* ε *and* φ *such that*

$$c_1\varepsilon^{-1}\|\varphi\|^2_{L^2(\Omega_\varepsilon)} \leq \|\xi\|^2_{L^2(\Omega_1)} \leq c_2\varepsilon^{-1}\|\varphi\|^2_{L^2(\Omega_\varepsilon)}. \tag{5.6}$$

If in addition $\varphi \in H^1(\Omega_\varepsilon)$, *then* $\xi \in H^1(\Omega_1)$ *and*

$$\varepsilon^{-1}\|\nabla\varphi\|^2_{L^2(\Omega_\varepsilon)} \geq c\left(\left\|\overline{P}\nabla\xi\right\|^2_{L^2(\Omega_1)} + \varepsilon^{-2}\|\partial_n\xi\|^2_{L^2(\Omega_1)}\right), \tag{5.7}$$

where $\partial_n\xi = (\bar{n}\cdot\nabla)\xi$ *on* Ω_1 *and* $c > 0$ *is a constant independent of* ε *and* φ.

Lemma 5.5. *For* $\varepsilon \in (0,1]$ *let* $\Phi_\varepsilon\colon \Omega_1 \to \Omega_\varepsilon$ *be the bijection given by* (5.4). *Also, let* $u \in H^1(\Omega_\varepsilon)^3$. *Then* $U := u \circ \Phi_\varepsilon \in H^1(\Omega_1)^3$ *and* (5.7) *holds with* φ *and* ξ *replaced by* u *and* U, *respectively. Moreover,*

$$\begin{aligned}\varepsilon^{-1}\|D(u)\|^2_{L^2(\Omega_\varepsilon)} & \\ &\geq c\left(\left\|\overline{P}F_\varepsilon(U)_S\overline{P}\right\|^2_{L^2(\Omega_1)} + \varepsilon^{-2}\|\partial_n(U\cdot\bar{n})\|^2_{L^2(\Omega_1)}\right),\end{aligned} \tag{5.8}$$

where $F_\varepsilon(U)_S = \{F_\varepsilon(U) + F_\varepsilon(U)^T\}/2$ *is the symmetric part of*

$$F_\varepsilon(U) := \left(I_3 - \varepsilon d\overline{W}\right)^{-1}\left(I_3 - d\overline{W}\right)\nabla U \quad \text{on} \quad \Omega_1 \tag{5.9}$$

and $c > 0$ *is a constant independent of* ε *and* u.

We give the proofs of Lemmas 5.4 and 5.5 in Appendix C.

Proof of Lemma 5.3. Following the idea in the case of a flat thin domain [19, Lemma 4.14], we prove (5.2) by contradiction.

Assume to the contrary that there exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ of positive numbers with $\lim_{k\to\infty}\varepsilon_k = 0$ and vector fields $u_k \in H^1(\Omega_{\varepsilon_k})^3$ satisfying (4.6) on Γ_{ε_k}, (5.3), and

$$\|u_k\|^2_{L^2(\Omega_{\varepsilon_k})} > \alpha\|\nabla u_k\|^2_{L^2(\Omega_{\varepsilon_k})} + k\|D(u_k)\|^2_{L^2(\Omega_{\varepsilon_k})}, \quad k \in \mathbb{N}. \tag{5.10}$$

For each $k \in \mathbb{N}$ let Φ_{ε_k} be the bijection from Ω_1 onto Ω_{ε_k} given by (5.4) and $U_k := u_k \circ \Phi_{\varepsilon_k} \in H^1(\Omega_1)^3$. Also, let $F_{\varepsilon_k}(U_k)$ be given by (5.9). We divide both sides of (5.10) by ε_k and use (5.6)–(5.8) to get

$$\|U_k\|_{L^2(\Omega_1)}^2 > c\alpha \left(\left\|\overline{P}\nabla U_k\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n U_k\|_{L^2(\Omega_1)}^2 \right) + ck \left(\left\|\overline{P}F_{\varepsilon_k}(U_k)_S\overline{P}\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n(U_k \cdot \bar{n})\|_{L^2(\Omega_1)}^2 \right).$$

Since $\|U_k\|_{L^2(\Omega_1)} > 0$, we may assume

$$\|U_k\|_{L^2(\Omega_1)} = 1, \quad k \in \mathbb{N} \tag{5.11}$$

by replacing U_k with $U_k/\|U_k\|_{L^2(\Omega_1)}$. Then

$$\left\|\overline{P}\nabla U_k\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n U_k\|_{L^2(\Omega_1)}^2 < c\alpha^{-1}, \tag{5.12}$$

$$\left\|\overline{P}F_{\varepsilon_k}(U_k)_S\overline{P}\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n(U_k \cdot \bar{n})\|_{L^2(\Omega_1)}^2 < ck^{-1} \tag{5.13}$$

and $\{U_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega_1)^3$ by (5.11), (5.12), and

$$|\nabla U_k|^2 = \left|\overline{P}\nabla U_k\right|^2 + \left|\overline{Q}\nabla U_k\right|^2, \quad \left|\overline{Q}\nabla U_k\right| = |\bar{n} \otimes \partial_n U_k| = |\partial_n U_k|$$

in Ω_1. By these facts and the compact embedding $H^1(\Omega_1) \hookrightarrow L^2(\Omega_1)$, we see that $\{U_k\}_{k=1}^\infty$ converges (up to a subsequence) to some U strongly in $L^2(\Omega_\varepsilon)^3$ and weakly in $H^1(\Omega_\varepsilon)^3$. Thus, by (5.11),

$$\|U\|_{L^2(\Omega_1)} = \lim_{k\to\infty} \|U_k\|_{L^2(\Omega_1)} = 1. \tag{5.14}$$

Moreover, since $\{U_k\}_{k=1}^\infty$ converges to U weakly in $H^1(\Omega_1)^3$ and

$$\lim_{k\to\infty} \|\partial_n U_k\|_{L^2(\Omega_1)} = 0 \tag{5.15}$$

by (5.12), we have $\partial_n U = 0$ in Ω_1, i.e. U is independent of the normal direction of Γ. Hence, setting $v(y) := U(y + g_0(y)n(y))$ for $y \in \Gamma$, we can consider U as the constant extension of v, i.e. $U = \bar{v}$ in Ω_1.

Now we claim that $v \in \mathcal{K}_g(\Gamma)$. If this claim is valid, then

$$|(u_k, \bar{v})_{L^2(\Omega_{\varepsilon_k})}| \le \beta\|u_k\|_{L^2(\Omega_{\varepsilon_k})}\|\bar{v}\|_{L^2(\Omega_{\varepsilon_k})}, \quad k \in \mathbb{N} \tag{5.16}$$

with $\beta \in [0,1)$ since we assume that u_k satisfies (5.3). We express this inequality in terms of U_k and send $k \to \infty$. Let

$$\varphi_k(X) := J(\pi(X), d(X))^{-1} J(\pi(X), \varepsilon_k d(X)), \quad X \in \Omega_1. \tag{5.17}$$

Then by (5.5) and $U_k = u_k \circ \Phi_{\varepsilon_k}$ on Ω_1 we have

$$(u_k, \bar{v})_{L^2(\Omega_{\varepsilon_k})} = \varepsilon_k \int_{\Omega_1} U_k \cdot (\bar{v} \circ \Phi_{\varepsilon_k}) \varphi_k \, dX.$$

Here $\bar{v} \circ \Phi_{\varepsilon_k} = \bar{v}$ in Ω_1 since $\pi \circ \Phi_{\varepsilon_k} = \pi$ in Ω_1 by (5.4). Moreover, since

$$\varphi_k(X) - J(\pi(X), d(X))^{-1} = J(\pi(X), d(X))^{-1} \{J(\pi(X), \varepsilon_k d(X)) - 1\}$$

for $X \in \Omega_1$, we observe by (3.46), (3.47), and $|d| \leq c$ in Ω_1 that

$$|\varphi_k(X) - J(\pi(X), d(X))^{-1}| \leq c\varepsilon_k |d(X)| \leq c\varepsilon_k \to 0 \tag{5.18}$$

as $k \to \infty$ uniformly in $X \in \Omega_1$. By these facts and the strong convergence of $\{U_k\}_{k=1}^\infty$ to $U = \bar{v}$ in $L^2(\Omega_1)^3$, we have

$$\begin{aligned} \lim_{k\to\infty} \varepsilon_k^{-1} (u_k, \bar{v})_{L^2(\Omega_{\varepsilon_k})} &= \lim_{k\to\infty} \int_{\Omega_1} (U_k \cdot \bar{v}) \varphi_k \, dX \\ &= \int_{\Omega_1} |\bar{v}|^2 J(\pi(\cdot), d(\cdot))^{-1} \, dX. \end{aligned}$$

Moreover, the last term is of the form

$$\int_\Gamma \int_{g_0(y)}^{g_1(y)} |v(y)|^2 J(y,r)^{-1} J(y,r) \, dr \, d\mathcal{H}^2(y) = \int_\Gamma g(y) |v(y)|^2 \, d\mathcal{H}^2(y)$$

by (3.48) with $\varepsilon = 1$. Therefore,

$$\lim_{k\to\infty} \varepsilon_k^{-1} (u_k, \bar{v})_{L^2(\Omega_{\varepsilon_k})} = \|g^{1/2} v\|_{L^2(\Gamma)}^2. \tag{5.19}$$

By the same arguments we have

$$\lim_{k\to\infty} \varepsilon_k^{-1} \|u_k\|_{L^2(\Omega_{\varepsilon_k})}^2 = \lim_{k\to\infty} \varepsilon_k^{-1} \|\bar{v}\|_{L^2(\Omega_{\varepsilon_k})}^2 = \|g^{1/2} v\|_{L^2(\Gamma)}^2. \tag{5.20}$$

We divide both sides of (5.16) by ε_k, send $k \to \infty$, and use (5.19) and (5.20) to get $\|g^{1/2} v\|_{L^2(\Gamma)}^2 \leq \beta \|g^{1/2} v\|_{L^2(\Gamma)}^2$. By this inequality, $\beta < 1$, and (2.1), we find that $v = 0$ on Γ and thus $U = \bar{v} = 0$ in Ω_1, which contradicts (5.14). Hence (5.2) is valid.

It remains to show $v \in \mathcal{K}_g(\Gamma)$. First we note that $v \in H^1(\Gamma)^3$ by $\bar{v} = U \in H^1(\Omega_1)^3$ and Lemma 3.12. To verify the other conditions for $v \in \mathcal{K}_g(\Gamma)$, we use the impermeable condition (4.6) on Γ_{ε_k} for u_k and the relations (5.13) and (5.15) for U_k.

Let us prove $v \cdot n = 0$ on Γ. We have

$$\|U_k - \bar{v}\|_{L^2(\Gamma_1)} \leq c\left(\|U_k - \bar{v}\|_{L^2(\Omega_1)} + \|\partial_n U_k\|_{L^2(\Omega_1)}\right), \quad k \in \mathbb{N}$$

by (4.3) with $\varepsilon = 1$ and $\partial_n \bar{v} = 0$ in Ω_1. Thus

$$\lim_{k\to\infty} \|U_k - \bar{v}\|_{L^2(\Gamma_1)} = 0 \tag{5.21}$$

by the strong convergence of $\{U_k\}_{k=1}^\infty$ to $\bar{v} = U$ in $L^2(\Omega_1)^3$ and (5.15). On the other hand, since u_k satisfies (4.6) on Γ_{ε_k}, we can use (4.7) to get $|u_k \cdot \bar{n}| \leq c\varepsilon_k |u_k|$ on Γ_{ε_k}, or equivalently, $|U_k \cdot \bar{n}| \leq c\varepsilon_k |U_k|$ on Γ_1 for each $k \in \mathbb{N}$. By this inequality, (4.3) with $\varepsilon = 1$, and the boundedness of $\{U_k\}_{k=1}^\infty$ in $H^1(\Omega_1)^3$, we find that

$$\|U_k \cdot \bar{n}\|_{L^2(\Gamma_1)} \leq c\varepsilon_k \|U_k\|_{L^2(\Gamma_1)} \leq c\varepsilon_k \|U_k\|_{H^1(\Omega_1)} \to 0$$

as $k \to \infty$. Combining this with (5.21) we obtain $\bar{v} \cdot \bar{n} = 0$ on Γ_1, which yields $v \cdot n = 0$ on Γ since $\bar{v}$ and $\bar{n}$ are the constant extensions of v and n in the normal direction of Γ.

Next we verify $D_\Gamma(v) = 0$ on Γ. Since $F_{\varepsilon_k}(U_k)$ is given by (5.9), $\{U_k\}_{k=1}^\infty$ converges to $U = \bar{v}$ weakly in $H^1(\Omega_1)^3$, and

$$\left|\left\{I_3 - \varepsilon_k d(X)\overline{W}(X)\right\}^{-1} - I_3\right| \leq c\varepsilon_k |d(X)| \leq c\varepsilon_k \to 0$$

as $k \to \infty$ uniformly in $X \in \Omega_1$ by (3.14), we have

$$\lim_{k\to\infty} F_{\varepsilon_k}(U_k) = \left(I_3 - d\overline{W}\right)\nabla\bar{v} = \overline{\nabla_\Gamma v} \quad \text{weakly in} \quad L^2(\Omega_1)^{3\times 3}.$$

Here the last equality follows from (3.16). Thus we get

$$\lim_{k\to\infty} \overline{P} F_{\varepsilon_k}(U_k)_S \overline{P} = \overline{P}\left(\overline{\nabla_\Gamma v}\right)_S \overline{P} = \overline{D_\Gamma(v)} \quad \text{weakly in} \quad L^2(\Omega_1)^{3\times 3}.$$

Moreover, $\lim_{k\to\infty} \left\|\overline{P}F_{\varepsilon_k}(U_k)_S\overline{P}\right\|_{L^2(\Omega_1)} = 0$ by (5.13). Hence $\overline{D_\Gamma(v)} = 0$ in Ω_1 and we obtain $D_\Gamma(v) = 0$ on Γ.

Lastly, let us prove $v \cdot \nabla_\Gamma g = 0$ on Γ. Hereafter we use the notations (3.43) and (3.44) (with $\varepsilon = 1$). For each $k \in \mathbb{N}$, since u_k satisfies (4.6) on Γ_{ε_k}, we have $u_k \cdot \bar{\tau}^i_{\varepsilon_k} = \varepsilon_k^{-1} u_k \cdot \bar{n}$ on $\Gamma^i_{\varepsilon_k}$, $i = 0, 1$ by (4.7). This equality yields $U_k \cdot \bar{\tau}^i_{\varepsilon_k} = \varepsilon_k^{-1} U_k \cdot \bar{n}$ on Γ^i_1, or equivalently, $U^\sharp_{k,i} \cdot \tau^i_{\varepsilon_k} = \varepsilon_k^{-1} U^\sharp_{k,i} \cdot n$ on Γ for $i = 0, 1$. Hence

$$\|U^\sharp_{k,1} \cdot \tau^1_{\varepsilon_k} - U^\sharp_{k,0} \cdot \tau^0_{\varepsilon_k}\|_{L^2(\Gamma)} = \varepsilon_k^{-1} \|U^\sharp_{k,1} \cdot n - U^\sharp_{k,0} \cdot n\|_{L^2(\Gamma)}. \tag{5.22}$$

Moreover, since $\bar{n}^\sharp(y, r) = n(y)$ for $y \in \Gamma$ and $r \in (g_0(y), g_1(y))$,

$$\begin{aligned} (U^\sharp_{k,1} \cdot n)(y) - (U^\sharp_{k,0} \cdot n)(y) &= \int_{g_0(y)}^{g_1(y)} \frac{\partial}{\partial r}\Big((U_k \cdot \bar{n})^\sharp(y, r)\Big)\, dr \\ &= \int_{g_0(y)}^{g_1(y)} [\partial_n(U_k \cdot \bar{n})]^\sharp(y, r)\, dr. \end{aligned}$$

By this equality, Hölder's inequality, (3.49), and (5.13), we have

$$\begin{aligned} \|U^\sharp_{k,1} \cdot n - U^\sharp_{k,0} \cdot n\|^2_{L^2(\Gamma)} &= \int_\Gamma \left(\int_{g_0(y)}^{g_1(y)} [\partial_n(U_k \cdot \bar{n})]^\sharp(y, r)\, dr\right)^2 d\mathcal{H}^2(y) \\ &\leq c\|\partial_n(U_k \cdot \bar{n})\|^2_{L^2(\Omega_1)} \leq c\varepsilon_k^2 k^{-1}. \end{aligned}$$

Applying this inequality to the right-hand side of (5.22) we get

$$\|U^\sharp_{k,1} \cdot \tau^1_{\varepsilon_k} - U^\sharp_{k,0} \cdot \tau^0_{\varepsilon_k}\|_{L^2(\Gamma)} \leq ck^{-1/2} \to 0 \quad \text{as} \quad k \to \infty. \tag{5.23}$$

Also, by (3.28), (3.29), and (3.53), we see that

$$\begin{aligned} &\|U^\sharp_{k,i} \cdot \tau^i_{\varepsilon_k} - v \cdot \nabla_\Gamma g_i\|_{L^2(\Gamma)} \\ &\qquad \leq \|(U^\sharp_{k,i} - v) \cdot \tau^i_{\varepsilon_k}\|_{L^2(\Gamma)} + \|v \cdot (\tau^i_{\varepsilon_k} - \nabla_\Gamma g_i)\|_{L^2(\Gamma)} \\ &\qquad \leq c\left(\|U^\sharp_{k,i} - v\|_{L^2(\Gamma)} + \varepsilon_k\|v\|_{L^2(\Gamma)}\right) \\ &\qquad \leq c\left(\|U_k - \bar{v}\|_{L^2(\Gamma^i_1)} + \varepsilon_k\|v\|_{L^2(\Gamma)}\right). \end{aligned}$$

Since the right-hand side tends to zero as $k \to \infty$ by (5.21),

$$\lim_{k\to\infty} \|U^\sharp_{k,i} \cdot \tau^i_{\varepsilon_k} - v \cdot \nabla_\Gamma g_i\|_{L^2(\Gamma)} = 0, \quad i = 0, 1.$$

By this equality, (5.23), and $g = g_1 - g_0$ on Γ, we obtain $\|v \cdot \nabla_\Gamma g\|_{L^2(\Gamma)} = 0$. Hence $v \cdot \nabla_\Gamma g = 0$ on Γ and we conclude that $v \in \mathcal{K}_g(\Gamma)$. $\Box$

By Lemmas 5.1 and 5.3 we get the uniform Korn inequality on Ω_ε.

Theorem 5.6. *For $\beta \in [0,1)$ there exist $\varepsilon_{K,\beta} \in (0,1]$ and $c_{K,\beta} > 0$ such that*

$$\|u\|^2_{H^1(\Omega_\varepsilon)} \leq c_{K,\beta}\|D(u)\|^2_{L^2(\Omega_\varepsilon)} \tag{5.24}$$

for all $\varepsilon \in (0, \varepsilon_{K,\beta}]$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (4.6) *and* (5.3).

Proof. Let $c_{K,1} > 0$ be the constant given in Lemma 5.1. Also, let $\varepsilon_K \in (0,1]$ and $c_{K,2} > 0$ be the constants given in Lemma 5.3 with $\alpha := 1/2c_{K,1}$. For $\varepsilon \in (0, \varepsilon_K]$ let $u \in H^1(\Omega_\varepsilon)^3$ satisfy (4.6) and (5.3). By (5.1) and (5.2) we have

$$\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \leq (4 + c_{K,1}c_{K,2})\|D(u)\|^2_{L^2(\Omega_\varepsilon)} + c_{K,1}\alpha\|\nabla u\|^2_{L^2(\Omega_\varepsilon)}.$$

Since $\alpha = 1/2c_{K,1}$, the above inequality implies that

$$\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \leq c_{\beta,1}\|D(u)\|^2_{L^2(\Omega_\varepsilon)}, \quad c_{\beta,1} := 2(4 + c_{K,1}c_{K,2}). \tag{5.25}$$

From this inequality and (5.2), we further deduce that

$$\|u\|^2_{L^2(\Omega_\varepsilon)} \leq c_{\beta,2}\|D(u)\|^2_{L^2(\Omega_\varepsilon)}, \quad c_{\beta,2} := 2(2c_{K,1}^{-1} + c_{K,2}). \tag{5.26}$$

By (5.25) and (5.26) we get (5.24) with $\varepsilon_{K,\beta} := \varepsilon_K$ and $c_{K,\beta} := c_{\beta,1} + c_{\beta,2}$. $\Box$

Let $\mathcal{R}_g$ be the space of infinitesimal rigid displacements of $\mathbb{R}^3$ given by (2.4). We show that (5.24) holds under another condition if $\mathcal{K}_g(\Gamma)$ agrees with $\mathcal{R}_g|_\Gamma := \{w|_\Gamma \mid w \in \mathcal{R}_g\}$ (see also Remark 2.1).

Theorem 5.7. *Suppose that $\mathcal{R}_g|_\Gamma = \mathcal{K}_g(\Gamma)$. Then for $\beta \in [0,1)$ there exist constants $\varepsilon_{K,\beta} \in (0,1]$ and $c_{K,\beta} > 0$ such that the inequality* (5.24) *holds for all $\varepsilon \in (0, \varepsilon_{K,\beta}]$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying* (4.6) *and*

$$\left|(u,w)_{L^2(\Omega_\varepsilon)}\right| \leq \beta\|u\|_{L^2(\Omega_\varepsilon)}\|w\|_{L^2(\Omega_\varepsilon)} \quad \text{for all} \quad w \in \mathcal{R}_g. \tag{5.27}$$

Note that $w \in \mathcal{R}_g$ in (5.27) has an explicit form $w(x) = a \times x + b$ for $x \in \mathbb{R}^3$, which is essential for the proof of Theorem 5.7.

PROOF. The proof is the same as that of Theorem 5.6 if we show that the statement of Lemma 5.3 is still valid under the condition (5.27) instead of (5.3). Assume to the contrary that there exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ of positive numbers convergent to zero and vector fields $u_k \in H^1(\Omega_{\varepsilon_k})^3$, $k \in \mathbb{N}$ satisfying (4.6) on Γ_{ε_k}, (5.10), and (5.27). Let Φ_{ε_k} be the bijection from Ω_1 onto Ω_{ε_k} given by (5.4) and $U_k := u_k \circ \Phi_{\varepsilon_k} \in H^1(\Omega_1)^3$. Then, after replacing U_k with $U_k/\|U_k\|_{L^2(\Omega_1)}$, we can show as in the proof of Lemma 5.3 that $\{U_k\}_{k=1}^\infty$ converges (up to a subsequence) strongly in $L^2(\Omega_1)^3$ to the constant extension $\bar{v}$ of some $v \in \mathcal{K}_g(\Gamma)$ and

$$\|\bar{v}\|_{L^2(\Omega_1)} = \lim_{k\to\infty} \|U_k\|_{L^2(\Omega_1)} = 1. \tag{5.28}$$

Now we can take $w \in \mathcal{R}_g$ such that $w|_\Gamma = v$ on Γ by the assumption $\mathcal{R}_g|_\Gamma = \mathcal{K}_g(\Gamma)$. Then since u_k satisfies (5.27) and $w \in \mathcal{R}_g$,

$$\left|(u_k, w)_{L^2(\Omega_{\varepsilon_k})}\right| \leq \beta \|u_k\|_{L^2(\Omega_{\varepsilon_k})} \|w\|_{L^2(\Omega_{\varepsilon_k})}, \quad k \in \mathbb{N}. \tag{5.29}$$

Let φ_k be the function on Ω_1 given by (5.17). Then by (5.5) we get

$$(u_k, w)_{L^2(\Omega_{\varepsilon_k})} = \varepsilon_k \int_{\Omega_1} U_k \cdot (w \circ \Phi_{\varepsilon_k})\varphi_k \, dX.$$

Since $w \in \mathcal{R}_g$ is of the form $w(x) = a \times x + b$ with $a, b \in \mathbb{R}^3$,

$$\begin{aligned} w(\Phi_{\varepsilon_k}(X)) &= a \times \{\pi(X) + \varepsilon_k d(X)\bar{n}(X)\} + b \\ &= w(\pi(X)) + \varepsilon_k d(X)\{a \times \bar{n}(X)\} \end{aligned}$$

for $X \in \Omega_1$. Hence by $|d| \leq c$ and $|\bar{n}| = 1$ in Ω_1 we have

$$|w \circ \Phi_{\varepsilon_k} - w \circ \pi| = \varepsilon_k |d(a \times \bar{n})| \leq c\varepsilon_k \to 0$$

as $k \to \infty$ uniformly on Ω_1. By this fact, (5.18), and the strong convergence of $\{U_k\}_{k=1}^\infty$ to $\bar{v}$ in $L^2(\Omega_1)^3$, we get

$$\lim_{k\to\infty} \varepsilon_k^{-1}(u_k, w)_{L^2(\Omega_{\varepsilon_k})} = \int_{\Omega_1} \bar{v} \cdot (w \circ \pi) J(\pi(\cdot), d(\cdot))^{-1} \, dX.$$

To the right-hand side we further apply (3.48) with $\varepsilon = 1$. Then since $\pi(y + rn(y)) = y$ for $y \in \Gamma$ and $r \in (g_0(y), g_1(y))$ and since $w|_\Gamma = v$ on Γ, we obtain

$$\lim_{k\to\infty} \varepsilon_k^{-1}(u_k, w)_{L^2(\Omega_{\varepsilon_k})} = \int_\Gamma g(y)v(y) \cdot w(y)\, d\mathcal{H}^2(y) = \|g^{1/2}v\|^2_{L^2(\Gamma)}.$$

In the same way we can show that

$$\lim_{k\to\infty} \varepsilon_k^{-1}\|u_k\|^2_{L^2(\Omega_{\varepsilon_k})} = \lim_{k\to\infty} \varepsilon_k^{-1}\|w\|^2_{L^2(\Omega_{\varepsilon_k})} = \|g^{1/2}v\|^2_{L^2(\Gamma)}.$$

Thus, as in the proof of Lemma 5.3, we can derive $v = 0$ on Γ by dividing both sides of (5.29) by ε_k, sending $k \to \infty$, and using the above equalities, $\beta < 1$, and (2.1). This implies $\bar{v} = 0$ on Ω_1, which contradicts (5.28). Hence the statement of Lemma 5.3 holds under the condition (5.27) instead of (5.3). □

REMARK 5.8. The uniform Korn inequality (5.24) was first established by Lewicka and Müller [37, Theorem 2.2] under the condition (5.3). They combined a uniform Korn inequality on a thin cylinder and Korn's inequality on a surface to prove (5.24). In Theorem 5.6 we gave a more direct proof of (5.24) under the same condition.

The condition (5.27) under the assumption $\mathcal{R}_g|_\Gamma = \mathcal{K}_g(\Gamma)$ is a new condition for the uniform Korn inequality (5.24). Note that we take a vector field $w \in \mathcal{R}_g$ defined on $\mathbb{R}^3$ itself in (5.27), not its restriction on Γ as in [37]. Due to this fact, Theorem 5.7 under the assumption $\mathcal{R}_g = \mathcal{K}_g(\Gamma)$ gives an improvement of [37, Theorem 2.3] which shows Korn's inequality with a constant of order ε^{-1}.

As we mentioned in Remark 2.1, we have $\mathcal{R}|_\Gamma = \mathcal{K}(\Gamma)$ and thus $\mathcal{R}_g|_\Gamma = \mathcal{K}_g(\Gamma)$ for any g if Γ is axially symmetric or it is closed and convex. In particular, Theorem 5.7 is applicable for curved thin domains around the unit sphere S^2 in $\mathbb{R}^3$.

5.2. Difference between the uniform and standard Korn inequalities

In this subsection we discuss the difference between the uniform Korn inequality (5.24) and a standard Korn inequality related to the axial symmetry of a domain.

For a fixed $\varepsilon \in (0,1]$ let

$$\mathcal{R}_\varepsilon := \{w(x) = a \times x + b,\ x \in \mathbb{R}^3 \mid a, b \in \mathbb{R}^3,\ w|_{\Gamma_\varepsilon} \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}.$$

The set $\mathcal{R}_\varepsilon$ stands for the axial symmetry of Ω_ε, i.e. $\mathcal{R}_\varepsilon \neq \{0\}$ if and only if Ω_ε is axially symmetric around some line (see Lemma E.1). It appears in the following standard Korn inequality with a constant depending on a domain (see also [2, 4, 69]).

Lemma 5.9. *For fixed $\varepsilon \in (0,1]$ and $\beta \in [0,1)$ there exists a constant $c_\varepsilon > 0$ depending on ε (and β) such that*

$$\|u\|_{H^1(\Omega_\varepsilon)} \leq c_\varepsilon \|D(u)\|_{L^2(\Omega_\varepsilon)} \tag{5.30}$$

for all $u \in H^1(\Omega_\varepsilon)^3$ satisfying (4.6) *and*

$$\left|(u, w)_{L^2(\Omega_\varepsilon)}\right| \leq \beta \|u\|_{L^2(\Omega_\varepsilon)} \|w\|_{L^2(\Omega_\varepsilon)} \quad \text{for all} \quad w \in \mathcal{R}_\varepsilon. \tag{5.31}$$

Proof. The proof is much easier than those of Theorems 5.6 and 5.7 since we fix ε and do not use a change of variables. By (5.1) it suffices to show $\|u\|_{L^2(\Omega_\varepsilon)} \leq c_\varepsilon \|D(u)\|_{L^2(\Omega_\varepsilon)}$ for all $u \in H^1(\Omega_\varepsilon)^3$ satisfying (4.6) and (5.31). Assume to the contrary that there exists $u_k \in H^1(\Omega_\varepsilon)^3$ that satisfies (4.6), (5.31), $\|u_k\|_{L^2(\Omega_\varepsilon)} = 1$, and $\|D(u_k)\|_{L^2(\Omega_\varepsilon)} < k^{-1}$ for $k \in \mathbb{N}$. Then $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega_\varepsilon)$ by (5.1) and thus converges (up to a subsequence) to some u strongly in $L^2(\Omega_\varepsilon)^3$ and weakly in $H^1(\Omega_\varepsilon)^3$ by the compact embedding $H^1(\Omega_\varepsilon) \hookrightarrow L^2(\Omega_\varepsilon)$. Moreover, $\{u_k|_{\Gamma_\varepsilon}\}_{k=1}^\infty$ converges to $u|_{\Gamma_\varepsilon}$ strongly in $L^2(\Gamma_\varepsilon)^3$ by the trace inequality (5.32) given below (note that we fix ε). By these facts, we get $D(u) = 0$ in Ω_ε and $u \cdot n_\varepsilon = 0$ on Γ_ε, which shows $u \in \mathcal{R}_\varepsilon$, and also $\|u\|_{L^2(\Omega_\varepsilon)} = 1$. Now we have (5.31) with u and w replaced by u_k and u by the assumption on u_k and $u \in \mathcal{R}_\varepsilon$. Then we send $k \to \infty$ to get $1 \leq \beta$, which contradicts $\beta \in [0,1)$. Hence the claim is valid. $\Box$

Lemma 5.10. *There exists a constant $c > 0$ such that*

$$\|\varphi\|_{L^2(\Gamma_\varepsilon^i)} \leq c\left(\varepsilon^{-1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)} + \|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\partial_n\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2}\right) \tag{5.32}$$

for all $\varepsilon \in (0,1]$ and $\varphi \in H^1(\Omega_\varepsilon)$, where $\partial_n\varphi = (\bar{n} \cdot \nabla)\varphi$.

Proof. The proof is the same as that of (4.3), and here we use

$$|\varphi_i^\sharp(y)|^2 = |\varphi^\sharp(y,r)|^2 + \int_r^{\varepsilon g_i(y)} \frac{\partial}{\partial \tilde{r}}\Big(|\varphi^\sharp(y,\tilde{r})|^2\Big) d\tilde{r}$$

and $\partial|\varphi^\sharp|^2/\partial r = 2\varphi^\sharp(\partial_n\varphi)^\sharp$ for $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$ instead of (4.4), where we used the notations (3.43)–(3.44). $\Box$

The constant c_ε in (5.30) may blow up as $\varepsilon \to 0$ (see [19, Corollary 4.11] for the case of a flat thin domain). To see this, we use a vector field introduced in [37, Section 4] as a counterexample to (5.24).

Lemma 5.11. *Suppose that $\mathcal{K}_g(\Gamma) \neq \{0\}$. Let $v \in \mathcal{K}_g(\Gamma)$, $v \not\equiv 0$ and*

$$v^\varepsilon(x) := \left\{I_3 - d(x)\overline{W}(x)\right\}\bar{v}(x) + \varepsilon\left\{\bar{v}(x)\cdot\overline{\nabla_\Gamma g_0}(x)\right\}\bar{n}(x) \tag{5.33}$$

for $x \in N$. Then v^ε satisfies (4.6) *for all $\varepsilon \in (0,1]$. Moreover, there exist constants $c_v^1, c_v^2 > 0$ and $\varepsilon_v \in (0,1]$ depending on v such that*

$$\|v^\varepsilon\|_{H^1(\Omega_\varepsilon)} \geq c_v^1\varepsilon^{1/2}, \quad \|D(v^\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq c_v^2\varepsilon^{3/2} \tag{5.34}$$

for all $\varepsilon \in (0,\varepsilon_v]$.

This result was shown in [37, Section 4]. An alternative proof in our notations can be found in the arXiv version of this paper [46].

From Lemma 5.11 it immediately follows that c_ε blows up if $\mathcal{K}_g(\Gamma) \neq \{0\}$ and Ω_ε is not axially symmetric for all $\varepsilon \in (0,1]$ sufficiently small. Let us give an example.

Lemma 5.12. *Let $\Gamma = S^2$ be the unit sphere in $\mathbb{R}^3$ and*

$$g_0(y) = y_3, \quad g_1(y) = y_2 + 2, \quad y = (y_1,y_2,y_3) \in S^2.$$

Then for all $\varepsilon \in (0,1]$ the curved thin domain

$$\Omega_\varepsilon = \{ry \mid y \in S^2,\, 1+\varepsilon y_3 < r < 1+\varepsilon(y_2+2)\}$$

is not axially symmetric around any line. Also, there exist constants $c_b > 0$ and $\varepsilon_b \in (0,1]$ such that the constant c_ε given in Lemma 5.9 with any $\beta \in [0,1)$ satisfies

$$c_\varepsilon \geq c_b\varepsilon^{-1} \tag{5.35}$$

for all $\varepsilon \in (0, \varepsilon_b]$ *and thus* $c_\varepsilon \to \infty$ *as* $\varepsilon \to 0$.

PROOF. First note that $g(y) = y_2 - y_3 + 2 \geq 2 - \sqrt{2}$ for all $y \in S^2$. Using the spherical coordinate system

$$S^2 = \{(\sin\vartheta_1 \cos\vartheta_2, \sin\vartheta_1 \sin\vartheta_2, \cos\vartheta_1) \mid \vartheta_1 \in [0,\pi],\, \vartheta_2 \in [0, 2\pi]\}$$

we can express the inner boundary of Ω_ε as

$$\begin{gathered}\Gamma_\varepsilon^0 = \{(\varphi(\vartheta_1)\cos\vartheta_2, \varphi(\vartheta_1)\sin\vartheta_2, \psi(\vartheta_1)) \mid \vartheta_1 \in [0,\pi],\, \vartheta_2 \in [0,2\pi]\}, \\ \varphi(\vartheta_1) := (1 + \varepsilon\cos\vartheta_1)\sin\vartheta_1, \quad \psi(\vartheta_1) := (1+\varepsilon\cos\vartheta_1)\cos\vartheta_1.\end{gathered}$$

Thus Γ_ε^0 is axially symmetric around the x_3-axis. Since Γ_ε^0 is not a sphere, it is not axially symmetric around other lines (see Remark E.4). Similarly, we see that the outer boundary Γ_ε^1 is axially symmetric only around the x_2-axis. Hence Ω_ε is not axially symmetric around any line, i.e. $\mathcal{R}_\varepsilon = \{0\}$ for all $\varepsilon \in (0, 1]$ (see Lemma E.1).

Next let us prove (5.35). Since $\Gamma = S^2$, we have

$$\mathcal{R} = \{w(x) = a \times x,\, x \in \mathbb{R}^3 \mid a \in \mathbb{R}^3\}, \quad \mathcal{K}(S^2) = \mathcal{R}|_{S^2}. \tag{5.36}$$

Let $v \in \mathcal{K}(S^2)$ be of the form $v(y) = a \times y$, $y \in S^2$ with $a \in \mathbb{R}^3$. Then since $\nabla_\Gamma g(y) = P(y)(e_2 - e_3)$ for $y \in S^2$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$, and v is tangential on S^2,

$$v(y) \cdot \nabla_\Gamma g(y) = v(y) \cdot (e_2 - e_3) = \{(e_2 - e_3) \times a\} \cdot y, \quad y \in S^2.$$

Hence $v \cdot \nabla_\Gamma g = 0$ on S^2 if and only if $a = \alpha(e_2 - e_3)$ with $\alpha \in \mathbb{R}$, i.e.

$$\mathcal{K}_g(S^2) = \{\alpha v_0 \mid \alpha \in \mathbb{R}\} \neq \{0\}, \quad v_0(y) := (e_2 - e_3) \times y, \quad y \in S^2.$$

Let v^ε be the vector field given by (5.33) with $v = v_0$. Also, let c_v^1, c_v^2, and ε_v be the constants given in Lemma 5.11 and $\varepsilon_b := \varepsilon_v$. Then v^ε satisfies (4.6) and (5.34) for all $\varepsilon \in (0, \varepsilon_b]$ by Lemma 5.11. Moreover, since $\mathcal{R}_\varepsilon = \{0\}$, the condition (5.31) with any $\beta \in [0, 1)$ is automatically satisfied. Hence we can apply (5.30) and (5.34) to v^ε to get

$$c_v^1 \varepsilon^{1/2} \leq \|v^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq c_\varepsilon \|D(v^\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq c_\varepsilon c_v^2 \varepsilon^{3/2}$$

for all $\varepsilon \in (0, \varepsilon_b]$. This inequality yields (5.35) with $c_b := c_v^1 / c_v^2$. □

By Lemmas 5.11 and 5.12, we find that the axial asymmetry of a curved thin domain is not sufficient for the uniform Korn inequality (5.24). Next we give an example of an axially symmetric curved thin domain for which c_ε blows up.

Lemma 5.13. *Let $\Gamma = S^2$ be the unit sphere in $\mathbb{R}^3$ and*

$$g_0(y) = y_3^2, \quad g_1(y) = y_3^2 + 1, \quad y = (y_1, y_2, y_3) \in S^2.$$

Then for all $\varepsilon \in (0,1]$ the curved thin domain

$$\Omega_\varepsilon = \{ry \mid y \in S^2,\ 1 + \varepsilon y_3^2 < r < 1 + \varepsilon(y_3^2 + 1)\} \tag{5.37}$$

is axially symmetric only around the x_3-axis. Moreover, there exist constants $c_b > 0$ and $\varepsilon_b \in (0,1]$ such that the constant c_ε given in Lemma 5.9 with any $\beta \in [0,1)$ satisfies (5.35) for all $\varepsilon \in (0,\varepsilon_b]$ and thus $c_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Proof. Since $g = g_1 - g_0 = 1$ on S^2, we have $\nabla_\Gamma g = 0$ on S^2 and

$$\begin{gathered} \mathcal{R}_g = \mathcal{R} = \{w(x) = a \times x,\ x \in \mathbb{R}^3 \mid a \in \mathbb{R}^3\}, \\ \mathcal{K}_g(S^2) = \mathcal{K}(S^2) = \mathcal{R}|_{S^2}. \end{gathered} \tag{5.38}$$

Also, as in the proof of Lemma 5.12, we see that the inner and outer boundaries of Ω_ε are axially symmetric only around the x_3-axis. Hence Ω_ε is axially symmetric only around the x_3-axis and

$$\mathcal{R}_\varepsilon = \{\alpha w_3 \mid \alpha \in \mathbb{R}\}, \quad w_3(x) = e_3 \times x, \quad x \in \mathbb{R}^3 \tag{5.39}$$

for all $\varepsilon \in (0,1]$ by Lemma E.1, where $e_3 = (0,0,1)^T$.

Next we prove (5.35). Let $e_1 = (1,0,0)^T$ and

$$w_1(x) := e_1 \times x, \quad x \in \mathbb{R}^3, \quad v_1 := w_1|_{S^2} \in \mathcal{K}_g(S^2) \setminus \{0\},$$

and v^ε be the vector field given by (5.33) with $v = v_1$. Then v^ε satisfies (4.6) and (5.34) by Lemma 5.11. Let us show that v^ε satisfies the condition (5.31) for (5.30). For $y \in S^2$ we have

$$\begin{gathered} n(y) = y, \quad W(y) = -\nabla_\Gamma n(y) = -P(y), \quad W(y)v_1(y) = -v_1(y), \\ \nabla_\Gamma g_0(y) = 2y_3 P(y) e_3, \quad v_1(y) \cdot \nabla_\Gamma g_0(y) = 2y_3 v_1(y) \cdot e_3 = 2y_2 y_3 \end{gathered}$$

since v_1 is tangential on S^2 and $g_0(y) = y_3^2$. Also, $d(x) = |x| - 1$ and $\bar{\eta}(x) = \eta(x/|x|)$ for $x \in N$. By these formulas, v^ε is of the form

$$v^\varepsilon(x) = |x| \left(e_1 \times \frac{x}{|x|} \right) + 2\varepsilon \frac{x_2 x_3}{|x|^2} \frac{x}{|x|} = w_1(x) + 2\varepsilon \frac{x_2 x_3}{|x|^2} \frac{x}{|x|} \tag{5.40}$$

for $x \in N$ and thus $v^\varepsilon(x) \cdot w_3(x) = -x_1 x_3$ and $(v^\varepsilon, w_3)_{L^2(\Omega_\varepsilon)} = 0$. Hence v^ε satisfies (5.31) with any $\beta \in [0,1)$ by (5.39). Since v^ε also satisfies (4.6), we can apply (5.30) to v^ε. Thus we get (5.35) by (5.30) and (5.34) as in the proof of Lemma 5.12. □

As we observed in the above proof, the vector field v^ε of the form (5.40) satisfies the condition (5.31) for the standard Korn inequality (5.30), but the uniform Korn inequality (5.24) is not valid for v^ε. Let us directly show that v^ε does not satisfy the condition (5.3) or (5.27) with any $\beta \in [0,1)$ for (5.24).

Lemma 5.14. *Let Ω_ε and v^ε be the curved thin domain and the vector field of the form* (5.37) *and* (5.40), *respectively. Then for each $\beta \in [0,1)$ there exists a constant $\varepsilon_\beta \in (0,1]$ such that v^ε does not satisfy* (5.3) *or* (5.27) *for all $\varepsilon \in (0, \varepsilon_\beta]$.*

Proof. Let v^ε be the vector field of the form (5.40), i.e.

$$v^\varepsilon(x) = w_1(x) + \varepsilon u(x), \quad w_1(x) = e_1 \times x, \quad u(x) = \frac{2x_2 x_3}{|x|^2} \frac{x}{|x|}$$

for $x \in N$. Then $w_1 \in \mathcal{R}_g$ and $v_1 := w_1|_{S^2} \in \mathcal{K}_g(S^2)$ by (5.38). Since $w_1 \cdot u = 0$ in N, we have

$$\begin{aligned} (v^\varepsilon, w_1)_{L^2(\Omega_\varepsilon)} &= \|w_1\|^2_{L^2(\Omega_\varepsilon)}, \\ \|v^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &= \|w_1\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^2 \|u\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{5.41}$$

Also, by $\bar{v}_1(x) = v_1(x/|x|) = |x|^{-1} w_1(x)$ for $x \in N$,

$$(v^\varepsilon, \bar{v}_1)_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \frac{|w_1(x)|^2}{|x|}\, dx, \quad \|\bar{v}_1\|^2_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \frac{|w_1(x)|^2}{|x|^2}\, dx. \tag{5.42}$$

Since $w_1 \not\equiv 0$, the first equalities of (5.41) and (5.42) show that v^ε does not satisfy (5.3) or (5.27) with $\beta = 0$ for all $\varepsilon \in (0,1]$. Now let $\beta \in (0,1)$. For $x = ry \in \Omega_\varepsilon$ with $y \in S^2$ we have

$$(5.43) \qquad 1 \le |x| = r \le 1 + 2\varepsilon, \quad r \in \bigl(1 + \varepsilon y_3^2, 1 + \varepsilon(y_3^2 + 1)\bigr)$$

and we deduce from (5.42) and (5.43) that

$$(5.44) \qquad (v^\varepsilon, \bar{v}_1)_{L^2(\Omega_\varepsilon)} \ge \frac{1}{1+2\varepsilon} \|w_1\|_{L^2(\Omega_\varepsilon)}^2, \quad \|\bar{v}_1\|_{L^2(\Omega_\varepsilon)}^2 \le \|w_1\|_{L^2(\Omega_\varepsilon)}^2.$$

Also, we easily get $\|w_1\|_{L^2(\Omega_\varepsilon)}^2 \ge c_1\varepsilon$ and $\|u\|_{L^2(\Omega_\varepsilon)}^2 \le c_2\varepsilon$ with constants $c_1, c_2 > 0$ independent of ε by the change of variables and (5.43). Hence $\|u\|_{L^2(\Omega_\varepsilon)}^2 \le c_3 \|w_1\|_{L^2(\Omega_\varepsilon)}^2$ with $c_3 := c_2/c_1$, and we see by this inequality and the second equality of (5.41) that

$$(5.45) \qquad \|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \le (1 + c_3\varepsilon^2)^{1/2} \|w_1\|_{L^2(\Omega_\varepsilon)}.$$

Now since $\beta \in (0,1)$ there exists a constant $\varepsilon_\beta \in (0,1]$ such that

$$(5.46) \qquad \beta(1 + c_3\varepsilon^2)^{1/2}(1 + 2\varepsilon) < 1, \quad \beta(1 + c_3\varepsilon^2)^{1/2} < \frac{1}{1+2\varepsilon} < 1$$

for all $\varepsilon \in (0, \varepsilon_\beta]$. Then by (5.41), (5.45), and (5.46) we get

$$\begin{aligned} \beta \|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \|w_1\|_{L^2(\Omega_\varepsilon)} &\le \beta(1 + c_3\varepsilon^2)^{1/2} \|w_1\|_{L^2(\Omega_\varepsilon)}^2 \\ &< \|w_1\|_{L^2(\Omega_\varepsilon)}^2 = (v^\varepsilon, w_1)_{L^2(\Omega_\varepsilon)} \end{aligned}$$

and, by (5.44), (5.45), and (5.46),

$$\begin{aligned} \beta \|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\bar{v}_1\|_{L^2(\Omega_\varepsilon)} &\le \beta(1 + c_3\varepsilon^2)^{1/2} \|w_1\|_{L^2(\Omega_\varepsilon)}^2 \\ &< \frac{1}{1+2\varepsilon} \|w_1\|_{L^2(\Omega_\varepsilon)}^2 \le (v^\varepsilon, \bar{v}_1)_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

Hence v^ε fails to satisfy (5.3) with $v = v_1 \in \mathcal{K}_g(S^2)$ and (5.27) with $w = w_1 \in \mathcal{R}_g$ for all $\varepsilon \in (0, \varepsilon_\beta]$. $\square$

Lemmas 5.13 and 5.14 show that the conditions (5.3) and (5.27) for the uniform Korn inequality (5.24) can be more strict than the condition (5.31) for the standard Korn inequality (5.30) even if a curved thin domain is axially symmetric. Note that it may also happen that the condition (5.27) is the same as (5.31). For example, if $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 1 < |x| < 1+\varepsilon\}$ is a thin spherical shell, then

$$\mathcal{R}_g = \mathcal{R}_\varepsilon = \mathcal{R} = \{w(x) = a \times x,\, x \in \mathbb{R}^3 \mid a \in \mathbb{R}^3\}$$

for all $\varepsilon \in (0,1]$ and thus the condition (5.27) is the same as (5.31).

Remark 5.15. The authors of [37] constructed the vector field of the form (5.33) and proved (5.34) under the assumption $\mathcal{K}_g(\Gamma) \neq \{0\}$ to show that the uniform Korn inequality (5.24) fails to hold without the condition (5.3). Based on that result they mentioned at the end of [37, Section 4] that the constant c_ε in (5.30) blows up as $\varepsilon \to 0$ even if the limit surface Γ is not axially symmetric. Indeed, if Γ is not axially symmetric, then Ω_ε is also not axially symmetric for all $\varepsilon \in (0,1]$ sufficiently small (see Lemma E.7) and we get (5.35) as in the proof of Lemma 5.12. However, as we mentioned in Remark 2.1, it is not known whether there exists a closed surface in $\mathbb{R}^3$ that is not axially symmetric but admits a nontrivial Killing vector field, i.e. $\mathcal{R} = \{0\}$ but $\mathcal{K}(\Gamma) \neq \{0\}$. To avoid this problem, we presented the concrete examples of curved thin domains around the unit sphere in $\mathbb{R}^3$ for which the relations (5.36) hold and c_ε blows up as $\varepsilon \to 0$ in Lemmas 5.12 and 5.13. Note that the comment at the end of [37, Section 4] is valid for curved thin domains in $\mathbb{R}^2$, since every smooth closed curve in $\mathbb{R}^2$ has a nontrivial tangential vector field of constant length as a nontrivial Killing vector field.

6. Uniform a Priori Estimate for the Vector Laplace Operator

The purpose of this section is to prove the following uniform a priori estimate for the vector Laplace operator on Ω_ε under the slip boundary conditions (4.18).

Theorem 6.1. *Under Assumption 2.2, there exists a constant $c > 0$ independent of ε such that*

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq c\left(\|\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \tag{6.1}$$

for all $\varepsilon \in (0,1]$ and $u \in H^2(\Omega_\varepsilon)^3$ satisfying (4.18).

First we give an approximation result for a vector field in $H^2(\Omega_\varepsilon)^3$ satisfying (4.18). To this end, we consider the problem

$$(6.2)\qquad \begin{cases} -\nu\{\Delta u + \nabla(\operatorname{div} u)\} + u = f & \text{in} \quad \Omega_\varepsilon, \\ u \cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u) n_\varepsilon + \gamma_\varepsilon u = 0 & \text{on} \quad \Gamma_\varepsilon \end{cases}$$

for a given data $f\colon \Omega_\varepsilon \to \mathbb{R}^3$. The bilinear form for (6.2) is given by

$$\tilde{a}_\varepsilon(u_1, u_2) := 2\nu\big(D(u_1), D(u_2)\big)_{L^2(\Omega_\varepsilon)} + (u_1, u_2)_{L^2(\Omega_\varepsilon)} + \sum_{i=0,1} \gamma_\varepsilon^i (u_1, u_2)_{L^2(\Gamma_\varepsilon^i)}$$

for $u_1, u_2 \in H^1(\Omega_\varepsilon)^3$ (see Lemma 7.1).

LEMMA 6.2. *For $\varepsilon \in (0,1]$ let $f \in L^2(\Omega_\varepsilon)^3$. Suppose that the inequalities (2.6) are valid. Then there exist a unique solution $u \in H^2(\Omega_\varepsilon)^3$ to (6.2) and a constant c_ε depending on ε such that*

$$(6.3)\qquad \|u\|_{H^2(\Omega_\varepsilon)} \leq c_\varepsilon \|f\|_{L^2(\Omega_\varepsilon)}.$$

If in addition $f \in H^1(\Omega_\varepsilon)^3$ then $u \in H^3(\Omega_\varepsilon)^3$.

Note that it does not matter how the constant c_ε in (6.3) depends on ε since we apply Lemma 6.2 just for approximation of a vector field on Ω_ε (see Lemma 6.3).

PROOF. Let $H^1_{n,0}(\Omega_\varepsilon) := \{u \in H^1(\Omega_\varepsilon)^3 \mid u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}$, which is a closed subspace of $H^1(\Omega_\varepsilon)^3$. By (5.1), we easily find that

$$(6.4)\qquad \|u\|^2_{H^1(\Omega_\varepsilon)} \leq c\tilde{a}_\varepsilon(u,u), \quad u \in H^1_{n,0}(\Omega_\varepsilon)$$

with a constant $c > 0$ independent of ε. On the other hand, for $i = 0, 1$ we have $\gamma_\varepsilon^i \|u\|^2_{L^2(\Gamma_\varepsilon^i)} \leq c\|u\|^2_{H^1(\Omega_\varepsilon)}$ by (2.6) and (4.3). From this inequality and $|D(u)| \leq |\nabla u|$ in Ω_ε it follows that

$$(6.5)\qquad \tilde{a}_\varepsilon(u,u) \leq c\|u\|^2_{H^1(\Omega_\varepsilon)}, \quad u \in H^1_{n,0}(\Omega_\varepsilon).$$

Since $\tilde{a}_\varepsilon$ is bounded and coercive on $H^1_{n,0}(\Omega_\varepsilon)$ by (6.4) and (6.5), the Lax–Milgram theorem shows that there exists a unique weak solution $u \in$

$H^1_{n,0}(\Omega_\varepsilon)$ to (6.2) in the sense that $\tilde{a}_\varepsilon(u,\Phi) = (f,\Phi)_{L^2(S^2)}$ for all $\Phi \in H^1_{n,0}(\Omega_\varepsilon)$. Moreover, since $f \in L^2(\Omega_\varepsilon)^3$, we can get $u \in H^2(\Omega_\varepsilon)^3$ and (6.3) by a standard localization argument and a method of the difference quotient. Here we omit the proof since it is the same as the proofs of [4, Theorem 1.2] and [69, Theorem 2] which established the H^2-regularity of a weak solution to the Stokes problem in a general bounded domain under the slip boundary conditions.

The H^3-regularity of u for $f \in H^1(\Omega_\varepsilon)^3$ is proved by induction and a localization argument as in the case of a general second order elliptic equation shown in [14, Section 6.3, Theorem 5]. Note that here we require the C^4-regularity of Γ_ε, see the arguments in [4, 69]. $\Box$

Using Lemma 6.2, we show that a vector field in $H^2(\Omega_\varepsilon)^3$ is approximated by those in $H^3(\Omega_\varepsilon)^3$ under the slip boundary conditions (4.18).

Lemma 6.3. *For $\varepsilon \in (0,1]$ let $u \in H^2(\Omega_\varepsilon)^3$ satisfy* (4.18) *and suppose that the inequalities* (2.6) *are valid. Then there exists a sequence $\{u_k\}_{k=1}^\infty$ in $H^3(\Omega_\varepsilon)^3$ such that u_k satisfies* (4.18) *for each $k \in \mathbb{N}$ and $\lim_{k\to\infty}\|u - u_k\|_{H^2(\Omega_\varepsilon)} = 0$.*

Proof. Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy (4.18) and $f := -\nu\{\Delta u + \nabla(\operatorname{div} u)\} + u \in L^2(\Omega_\varepsilon)^3$. Then we can take a sequence $\{f_k\}_{k=1}^\infty$ in $C_c^\infty(\Omega_\varepsilon)^3$ that converges to f strongly in $L^2(\Omega_\varepsilon)^3$. For each $k \in \mathbb{N}$ let $u_k \in H^3(\Omega_\varepsilon)^3$ be a unique solution to (6.2) with data $f_k \in C_c^\infty(\Omega_\varepsilon)^3$ given by Lemma 6.2. Then u_k satisfies (4.18). Moreover, since $u - u_k$ is a unique weak solution to (6.2) with data $f - f_k$,

$$\|u - u_k\|_{H^2(\Omega_\varepsilon)} \leq c_\varepsilon \|f - f_k\|_{L^2(\Omega_\varepsilon)} \to 0 \quad \text{as} \quad k \to \infty$$

by (6.3) and the strong convergence of $\{f_k\}_{k=1}^\infty$ to f in $L^2(\Omega_\varepsilon)^3$ (note that c_ε does not depend on k). $\Box$

Now let us prove Theorem 6.1. As in Section 3.1, for a function space $\mathcal{X}(\Gamma_\varepsilon)$ on Γ_ε we denote the space of all tangential vector fields on Γ_ε of class $\mathcal{X}$ by $\mathcal{X}(\Gamma_\varepsilon, T\Gamma_\varepsilon) := \{u \in \mathcal{X}(\Gamma_\varepsilon)^3 \mid u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}$. We define the covariant derivative $\overline{\nabla}^\varepsilon_v u := P_\varepsilon(v \cdot \nabla)\tilde{u} = P_\varepsilon(v \cdot \nabla_{\Gamma_\varepsilon})u$ on Γ_ε for $u \in H^1(\Gamma_\varepsilon, T\Gamma_\varepsilon)$ and $v \in L^2(\Gamma_\varepsilon, T\Gamma_\varepsilon)$, where $\tilde{u}$ is any H^1-extension of u to an open neighborhood of Γ_ε with $\tilde{u}|_{\Gamma_\varepsilon} = u$. We use the formulas for the covariant derivatives given in Appendix D.

PROOF OF THEOREM 6.1. Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy (4.18). Since $\|u\|^2_{H^2(\Omega_\varepsilon)} = \|u\|^2_{H^1(\Omega_\varepsilon)} + \|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)}$, it is sufficient for (6.1) to show that

$$\|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)} \le c\left(\|\Delta u\|^2_{L^2(\Omega_\varepsilon)} + \|u\|^2_{H^1(\Omega_\varepsilon)}\right). \tag{6.6}$$

Moreover, by Lemma 6.3 and a density argument, we may assume that u belongs to $H^3(\Omega_\varepsilon)^3$ and satisfies (4.18), and thus $u|_{\Gamma_\varepsilon} \in H^2(\Gamma_\varepsilon, T\Gamma_\varepsilon)$. By $u \in H^3(\Omega_\varepsilon)^3$ we can carry out integration by parts twice to get

$$\begin{aligned} \|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)} &= \|\Delta u\|^2_{L^2(\Omega_\varepsilon)} \\ &\quad + \int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2. \end{aligned} \tag{6.7}$$

Here $(n_\varepsilon \cdot \nabla)\nabla u$ denotes a 3×3 matrix whose (i,j)-entry is given by

$$[(n_\varepsilon \cdot \nabla)\nabla u]_{ij} := n_\varepsilon \cdot \nabla(\partial_i u_j), \quad i,j = 1,2,3.$$

Let us estimate the boundary integral in (6.7). Our goal is to show

$$\begin{aligned} &\left|\int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2\right| \\ &\qquad \le c\left(\|u\|^2_{H^1(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}\right). \end{aligned} \tag{6.8}$$

Since u satisfies (4.18) and u and $W_\varepsilon u$ are tangential on Γ_ε, we have

$$(\nabla u)^T n_\varepsilon = (n_\varepsilon \cdot \nabla)u = -W_\varepsilon u - \tilde{\gamma}_\varepsilon u + \xi_\varepsilon n_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon \tag{6.9}$$

by (4.19), where $\tilde{\gamma}_\varepsilon := \gamma_\varepsilon/\nu$ and $\xi_\varepsilon := (n_\varepsilon \cdot \nabla)u \cdot n_\varepsilon = \nabla u : Q_\varepsilon$.

The first step for (6.8) is to reduce the second order derivatives of u on Γ_ε coming from $(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u$ to the zeroth and first order ones by using (6.9) and the formulas for the covariant derivatives on Γ_ε given in Appendix D. More precisely, we show that

$$\int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2 = \sum_{k=1}^{4} \int_{\Gamma_\varepsilon} \varphi_k\, d\mathcal{H}^2, \tag{6.10}$$

where

$$
\begin{aligned}
&\varphi_1 := -2\{\nabla_{\Gamma_\varepsilon} W_\varepsilon \cdot u + (\nabla u)W_\varepsilon + \tilde{\gamma}_\varepsilon \nabla u\} : P_\varepsilon(\nabla u)P_\varepsilon, \\
&\varphi_2 := W_\varepsilon \nabla u : (\nabla u)P_\varepsilon + H_\varepsilon(\nabla u : Q_\varepsilon)^2 \\
&\qquad\qquad - 2(u \cdot \mathrm{div}_{\Gamma_\varepsilon} W_\varepsilon + 2\nabla u : W_\varepsilon)(\nabla u : Q_\varepsilon), \\
&\varphi_3 := -(W_\varepsilon^3 u - H_\varepsilon W_\varepsilon^2 u) \cdot u, \\
&\varphi_4 := -\tilde{\gamma}_\varepsilon(2W_\varepsilon^2 u - 2H_\varepsilon W_\varepsilon u - \tilde{\gamma}_\varepsilon H_\varepsilon u) \cdot u.
\end{aligned} \tag{6.11}
$$

In (6.11), $\nabla_{\Gamma_\varepsilon} W_\varepsilon \cdot u$ is a 3×3 matrix with (i,j)-entry

$$
[\nabla_{\Gamma_\varepsilon} W_\varepsilon \cdot u]_{ij} := \sum_{k=1}^{3} (\underline{D}_i^\varepsilon [W_\varepsilon]_{jk}) u_k, \quad i,j = 1,2,3 \tag{6.12}
$$

and $\mathrm{div}_{\Gamma_\varepsilon} W_\varepsilon$ is a vector field with j-th component

$$
[\mathrm{div}_{\Gamma_\varepsilon} W_\varepsilon]_j := \sum_{i=1}^{3} \underline{D}_i^\varepsilon [W_\varepsilon]_{ij}, \quad j = 1,2,3. \tag{6.13}
$$

Using a partition of unity on Γ_ε we may assume that $u|_{\Gamma_\varepsilon}$ is compactly supported in a relatively open subset O of Γ_ε on which we can take a local orthonormal frame $\{\tau_1, \tau_2\}$ (see Appendix D). Then

$$
\begin{aligned}
&\nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\} \\
&\qquad = (\nabla u)^T : [\{(n_\varepsilon \cdot \nabla)\nabla u\}^T - \Delta u \otimes n_\varepsilon] = \eta_1 + \eta_2 + \eta_3
\end{aligned} \tag{6.14}
$$

on O since $\{\tau_1, \tau_2, n_\varepsilon\}$ is an orthonormal basis of $\mathbb{R}^3$, where

$$
\eta_i := (\nabla u)^T \tau_i \cdot [\{(n_\varepsilon \cdot \nabla)\nabla u\}^T \tau_i - (\Delta u \otimes n_\varepsilon)\tau_i], \quad i = 1,2, \tag{6.15}
$$

$$
\eta_3 := (\nabla u)^T n_\varepsilon \cdot [\{(n_\varepsilon \cdot \nabla)\nabla u\}^T n_\varepsilon - (\Delta u \otimes n_\varepsilon) n_\varepsilon]. \tag{6.16}
$$

In what follows, we carry out calculations on O. By (D.2) and $\tau_i \cdot n_\varepsilon = 0$,

$$
\begin{aligned}
(\nabla u)^T \tau_i &= (\tau_i \cdot \nabla)u = \overline{\nabla}_i^\varepsilon u + (W_\varepsilon u \cdot \tau_i) n_\varepsilon, \\
(\Delta u \otimes n_\varepsilon)\tau_i &= (\tau_i \cdot n_\varepsilon)\Delta u = 0,
\end{aligned} \tag{6.17}
$$

where $\overline{\nabla}_i^\varepsilon := \overline{\nabla}_{\tau_i}^\varepsilon$, $i = 1,2$. For $j = 1,2,3$ let τ_i^j and n_ε^j be the j-th components of τ_i and n_ε. Then the j-th component of $\{(n_\varepsilon \cdot \nabla)\nabla u\}^T \tau_i$ is of the form

$$\begin{aligned}\sum_{k,l=1}^{3} n_\varepsilon^k(\partial_k\partial_l u_j)\tau_i^l &= \sum_{k=1}^{3} n_\varepsilon^k(\tau_i\cdot\nabla)(\partial_k u_j) = \sum_{k=1}^{3} n_\varepsilon^k(\tau_i\cdot\nabla_{\Gamma_\varepsilon})(\partial_k u_j) \\ &= \sum_{k=1}^{3}\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})(n_\varepsilon^k\partial_k u_j) - (\tau_i\cdot\nabla_{\Gamma_\varepsilon} n_\varepsilon^k)\partial_k u_j\} \\ &= (\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(n_\varepsilon\cdot\nabla)u_j\} - \{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})n_\varepsilon\cdot\nabla\}u_j\end{aligned}$$

by (3.35) and $P_\varepsilon\tau_i = \tau_i$ (also note that the tangential derivatives depend only on the values of functions on Γ_ε). Hence

$$\{(n_\varepsilon\cdot\nabla)\nabla u\}^T\tau_i = (\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(n_\varepsilon\cdot\nabla)u\} - \{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})n_\varepsilon\cdot\nabla\}u.$$

By (6.9), (D.2), $-\nabla_{\Gamma_\varepsilon}n_\varepsilon = W_\varepsilon = W_\varepsilon^T$, and

$$(\tau_i\cdot\nabla_{\Gamma_\varepsilon})n_\varepsilon = (\nabla_{\Gamma_\varepsilon}n_\varepsilon)^T\tau_i = -W_\varepsilon\tau_i, \tag{6.18}$$

we further observe that

$$\begin{aligned}\{(n_\varepsilon\cdot\nabla)\nabla u\}^T\tau_i = -\overline{\nabla}_i^\varepsilon(W_\varepsilon u) - \tilde{\gamma}_\varepsilon\overline{\nabla}_i^\varepsilon u + \overline{\nabla}_{W_\varepsilon\tau_i}^\varepsilon u - \xi_\varepsilon W_\varepsilon\tau_i \\ + \{(-\tilde{\gamma}_\varepsilon W_\varepsilon u + \nabla_{\Gamma_\varepsilon}\xi_\varepsilon)\cdot\tau_i\}n_\varepsilon.\end{aligned} \tag{6.19}$$

Note that the first four terms on the right-hand side of (6.19) are tangential on Γ_ε. From (6.15), (6.17), and (6.19) we deduce that

$$\begin{aligned}\eta_i = -\left\{\overline{\nabla}_i^\varepsilon(W_\varepsilon u) + \tilde{\gamma}_\varepsilon\overline{\nabla}_i^\varepsilon u - \overline{\nabla}_{W_\varepsilon\tau_i}^\varepsilon u + \xi_\varepsilon W_\varepsilon\tau_i\right\}\cdot\overline{\nabla}_i^\varepsilon u \\ + (W_\varepsilon u\cdot\tau_i)\{(-\tilde{\gamma}_\varepsilon W_\varepsilon u + \nabla_{\Gamma_\varepsilon}\xi_\varepsilon)\cdot\tau_i\}\end{aligned}$$

for $i = 1,2$. Since $W_\varepsilon u$ and $\nabla_{\Gamma_\varepsilon}\xi_\varepsilon$ are tangential on Γ_ε and $\{\tau_1,\tau_2\}$ is an orthonormal basis of the tangent plane of Γ_ε, by the above equality and (D.9)–(D.11) we obtain

$$\begin{aligned}\eta_1+\eta_2 = -\{\nabla_{\Gamma_\varepsilon}(W_\varepsilon u) + \tilde{\gamma}_\varepsilon\nabla_{\Gamma_\varepsilon}u - W_\varepsilon\nabla_{\Gamma_\varepsilon}u\} : (\nabla_{\Gamma_\varepsilon}u)P_\varepsilon \\ - \xi_\varepsilon(\nabla_{\Gamma_\varepsilon}u : W_\varepsilon) + W_\varepsilon u\cdot(-\tilde{\gamma}_\varepsilon W_\varepsilon u + \nabla_{\Gamma_\varepsilon}\xi_\varepsilon).\end{aligned} \tag{6.20}$$

To calculate η_3 we see that the j-th component of $\{(n_\varepsilon\cdot\nabla)\nabla u\}^T n_\varepsilon$ for $j=1,2,3$ is of the form

$$\begin{aligned}\sum_{k,l=1}^3 n_\varepsilon^k(\partial_k\partial_l u_j)n_\varepsilon^l &= \mathrm{tr}[Q_\varepsilon\nabla^2 u_j] = \mathrm{tr}[\nabla^2 u_j]-\mathrm{tr}[P_\varepsilon\nabla^2 u_j]\\ &= \Delta u_j-\sum_{i=1,2}P_\varepsilon(\nabla^2 u_j)\tau_i\cdot\tau_i - P_\varepsilon(\nabla^2 u)n_\varepsilon\cdot n_\varepsilon\\ &= \Delta u_j-\sum_{i=1,2}\{(\tau_i\cdot\nabla)\nabla u_j\}\cdot\tau_i\end{aligned}$$

by $P_\varepsilon^T=P_\varepsilon$, $P_\varepsilon\tau_i=\tau_i$, and $P_\varepsilon n_\varepsilon=0$. From this equality and

$$\begin{aligned}\{(\tau_i\cdot\nabla)\nabla u_j\}\cdot\tau_i &= \{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\nabla u_j\}\cdot\tau_i\\ &= (\tau_i\cdot\nabla_{\Gamma_\varepsilon})(\nabla u_j\cdot\tau_i)-\nabla u_j\cdot(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\\ &= (\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(\tau_i\cdot\nabla)u_j\}-\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\cdot\nabla\}u_j\end{aligned}$$

by (3.35) and $P_\varepsilon\tau_i=\tau_i$, we deduce that

$$\begin{aligned}&\{(n_\varepsilon\cdot\nabla)\nabla u\}^T n_\varepsilon\\ &\qquad= \Delta u-\sum_{i=1,2}[(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(\tau_i\cdot\nabla)u\}-\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\cdot\nabla\}u].\end{aligned}$$

Moreover, since $(\Delta u\otimes n_\varepsilon)n_\varepsilon=(n_\varepsilon\cdot n_\varepsilon)\Delta u=\Delta u$, it follows that

$$\begin{aligned}&\{(n_\varepsilon\cdot\nabla)\nabla u\}^T n_\varepsilon-(\Delta u\otimes n_\varepsilon)n_\varepsilon\\ &\qquad= -\sum_{i=1,2}[(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(\tau_i\cdot\nabla)u\}-\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\cdot\nabla\}u].\end{aligned}\tag{6.21}$$

By (6.9), (6.18), and (D.2) we also observe that

$$\begin{aligned}&(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(\tau_i\cdot\nabla)u\}=(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\left\{\overline{\nabla}_i^\varepsilon u+(W_\varepsilon u\cdot\tau_i)n_\varepsilon\right\}\\ &\quad= \overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-(W_\varepsilon u\cdot\tau_i)W_\varepsilon\tau_i+\left\{W_\varepsilon\overline{\nabla}_i^\varepsilon u\cdot\tau_i+\tau_i\cdot\nabla_{\Gamma_\varepsilon}(W_\varepsilon u\cdot\tau_i)\right\}n_\varepsilon\end{aligned}$$

and

$$\begin{aligned}&\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\cdot\nabla\}u=\left[\left\{\overline{\nabla}_i^\varepsilon\tau_i+(W_\varepsilon\tau_i\cdot\tau_i)n_\varepsilon\right\}\cdot\nabla\right]u\\ &\quad= \overline{\nabla}_{\overline{\nabla}_i^\varepsilon\tau_i}^\varepsilon u-(W_\varepsilon\tau_i\cdot\tau_i)(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)+\left(W_\varepsilon u\cdot\overline{\nabla}_i^\varepsilon\tau_i+\xi_\varepsilon W_\varepsilon\tau_i\cdot\tau_i\right)n_\varepsilon.\end{aligned}$$

We substitute these expressions for (6.21) and use (D.7), (D.9), and

$$\begin{aligned}\sum_{i=1,2}\left\{\tau_i\cdot\nabla_{\Gamma_\varepsilon}(W_\varepsilon u\cdot\tau_i)-W_\varepsilon u\cdot\overline{\nabla}_i^\varepsilon\tau_i\right\}&=\sum_{i=1,2}\overline{\nabla}_i^\varepsilon(W_\varepsilon u)\cdot\tau_i\\&=\mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u),\end{aligned}$$

which follows from (D.5) and (D.8), and

$$\sum_{i=1,2}(W_\varepsilon u\cdot\tau_i)W_\varepsilon\tau_i=W_\varepsilon\sum_{i=1,2}(W_\varepsilon u\cdot\tau_i)\tau_i=W_\varepsilon^2u$$

due to the facts that $W_\varepsilon u$ is tangential on Γ_ε and that $\{\tau_1,\tau_2\}$ is an orthonormal basis of the tangent plane of Γ_ε. Then we have

$$\begin{aligned}&\{(n_\varepsilon\cdot\nabla)\nabla u\}^Tn_\varepsilon-(\Delta u\otimes n_\varepsilon)n_\varepsilon\\&\quad=-\sum_{i=1,2}\left(\overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-\overline{\nabla}^\varepsilon_{\overline{\nabla}_i^\varepsilon\tau_i}u\right)+W_\varepsilon^2u-H_\varepsilon W_\varepsilon u-\tilde{\gamma}_\varepsilon H_\varepsilon u\\&\qquad\qquad-\{\nabla_{\Gamma_\varepsilon}u:W_\varepsilon+\mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u)-\xi_\varepsilon H_\varepsilon\}n_\varepsilon.\end{aligned}\tag{6.22}$$

Hence by (6.9), (6.16), and (6.22) we get

$$\begin{aligned}\eta_3=&\sum_{i=1,2}\left(\overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-\overline{\nabla}^\varepsilon_{\overline{\nabla}_i^\varepsilon\tau_i}u\right)\cdot(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)\\&-(W_\varepsilon^2u-H_\varepsilon W_\varepsilon u-\tilde{\gamma}_\varepsilon H_\varepsilon u)\cdot(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)\\&-\xi_\varepsilon\{\nabla_{\Gamma_\varepsilon}u:W_\varepsilon+\mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u)-\xi_\varepsilon H_\varepsilon\}.\end{aligned}\tag{6.23}$$

Now we observe by (3.35) and direct calculations that

$$\nabla_{\Gamma_\varepsilon}u:(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon=P_\varepsilon(\nabla u):P_\varepsilon(\nabla u)P_\varepsilon=\nabla u:P_\varepsilon^TP_\varepsilon(\nabla u)P_\varepsilon.$$

Since $P_\varepsilon^T=P_\varepsilon^2=P_\varepsilon$, the above equality implies that

$$\nabla_{\Gamma_\varepsilon}u:(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon=\nabla u:P_\varepsilon(\nabla u)P_\varepsilon.\tag{6.24}$$

By the same calculations with (3.35) and (3.36) we have

$$\begin{aligned}\nabla_{\Gamma_\varepsilon}(W_\varepsilon u):(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon&=\{\nabla_{\Gamma_\varepsilon}W_\varepsilon\cdot u+(\nabla_{\Gamma_\varepsilon}u)W_\varepsilon\}:(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon\\&=\{\nabla_{\Gamma_\varepsilon}W_\varepsilon\cdot u+(\nabla u)W_\varepsilon\}:P_\varepsilon(\nabla u)P_\varepsilon,\end{aligned}\tag{6.25}$$

where the matrix $\nabla_{\Gamma_\varepsilon} W_\varepsilon \cdot u$ is given by (6.12), and

$$W_\varepsilon(\nabla_{\Gamma_\varepsilon} u) : (\nabla_{\Gamma_\varepsilon} u)P_\varepsilon = W_\varepsilon(\nabla u) : (\nabla u)P_\varepsilon, \\ \nabla_{\Gamma_\varepsilon} u : W_\varepsilon = \nabla u : W_\varepsilon. \tag{6.26}$$

Also, it is easy to get (recall that $\mathrm{div}_{\Gamma_\varepsilon} W_\varepsilon$ is given by (6.13))

$$\begin{aligned} W_\varepsilon u \cdot \nabla_{\Gamma_\varepsilon} \xi_\varepsilon &= \mathrm{div}_{\Gamma_\varepsilon}(\xi_\varepsilon W_\varepsilon u) - \xi_\varepsilon \mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u), \\ \mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u) &= u \cdot \mathrm{div}_{\Gamma_\varepsilon} W_\varepsilon + \nabla_{\Gamma_\varepsilon} u : W_\varepsilon \\ &= u \cdot \mathrm{div}_{\Gamma_\varepsilon} W_\varepsilon + \nabla u : W_\varepsilon. \end{aligned} \tag{6.27}$$

By (6.14), (6.20), (6.23)–(6.27), $W_\varepsilon^T = W_\varepsilon$, and $\xi_\varepsilon = \nabla u : Q_\varepsilon$, we get

$$\begin{aligned} &\int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2 \\ &\quad = \sum_{i=1,2} \int_{\Gamma_\varepsilon} \left(\overline{\nabla}_i^\varepsilon \overline{\nabla}_i^\varepsilon u - \overline{\nabla}^\varepsilon_{\overline{\nabla}_i^\varepsilon \tau_i} u\right) \cdot (W_\varepsilon u + \tilde{\gamma}_\varepsilon u)\, d\mathcal{H}^2 \\ &\qquad + \int_{\Gamma_\varepsilon} \left(\frac{1}{2}\varphi_1 + \sum_{k=2}^4 \varphi_k\right) d\mathcal{H}^2 + \int_{\Gamma_\varepsilon} \mathrm{div}_{\Gamma_\varepsilon}(\xi_\varepsilon W_\varepsilon u)\, d\mathcal{H}^2, \end{aligned} \tag{6.28}$$

with $\varphi_1, \dots, \varphi_4$ given by (6.11). Moreover, we apply (D.13) to the first term on the right-hand side and use (D.10), (6.24), and (6.25) to have

$$\sum_{i=1,2} \int_{\Gamma_\varepsilon} \left(\overline{\nabla}_i^\varepsilon \overline{\nabla}_i^\varepsilon u - \overline{\nabla}^\varepsilon_{\overline{\nabla}_i^\varepsilon \tau_i} u\right) \cdot (W_\varepsilon u + \tilde{\gamma}_\varepsilon u)\, d\mathcal{H}^2 = \frac{1}{2}\int_{\Gamma_\varepsilon} \varphi_1\, d\mathcal{H}^2.$$

Also, since $\xi_\varepsilon W_\varepsilon u = (\nabla u : Q_\varepsilon) W_\varepsilon u \in W^{1,1}(\Gamma_\varepsilon, T\Gamma_\varepsilon)$ by $u \in H^3(\Omega_\varepsilon)^3$ and $Q_\varepsilon, W_\varepsilon \in C^2(\Gamma_\varepsilon)^{3\times 3}$, we can apply (3.25) to $\xi_\varepsilon W_\varepsilon u$ to find that the last term of (6.28) vanishes. Hence we obtain (6.10).

The second step for (6.8) is to show that

$$\left|\int_{\Gamma_\varepsilon} \varphi_k\, d\mathcal{H}^2\right| \leq c\left(\|u\|^2_{H^1(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}\right) \tag{6.29}$$

for $k = 1, 2$ and

$$\left|\int_{\Gamma_\varepsilon} \varphi_k\, d\mathcal{H}^2\right| \leq c\|u\|^2_{H^1(\Omega_\varepsilon)}, \quad k = 3, 4 \tag{6.30}$$

with a constant $c>0$ independent of ε. The estimate (6.30) for $k=4$ is an easy consequence of (2.6), (4.3), and the uniform boundedness of W_ε and H_ε on Γ_ε (see Lemma 3.10):

$$\left|\int_{\Gamma_\varepsilon}\varphi_4\,d\mathcal{H}^2\right|\leq c\varepsilon\|u\|_{L^2(\Gamma_\varepsilon)}^2\leq c\|u\|_{H^1(\Omega_\varepsilon)}^2.$$

Let us prove (6.29) for $k=1$. Here the idea is the same as that of the proof of Lemma 4.3: using (3.52), we rewrite the boundary integral

$$\begin{aligned}\int_{\Gamma_\varepsilon}\varphi_1(x)\,d\mathcal{H}^2(x)&=\sum_{i=0,1}\int_{\Gamma_\varepsilon^i}\varphi_1(x)\,d\mathcal{H}^2(x)\\&=\sum_{i=0,1}\int_\Gamma\varphi_1(y+\varepsilon g_i(y)n(y))J(y,\varepsilon g_i(y))\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}\,d\mathcal{H}^2(y)\end{aligned}$$

as the integral over Γ of the form

$$\begin{aligned}\int_\Gamma\{\Phi_1(y+\varepsilon g_1(y)n(y))-\Phi_1(y+\varepsilon g_0(y)n(y))\}\,d\mathcal{H}^2(y)&\\=\int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\frac{d}{dr}\Big(\Phi_1(y+rn(y))\Big)\,dr\,d\mathcal{H}^2(y),&\end{aligned}$$

where Φ_1 is a function on Ω_ε such that

$$\begin{aligned}&\Phi_1(y+\varepsilon g_i(y)n(y))\\&\qquad=(-1)^{i+1}\varphi_1(y+\varepsilon g_i(y)n(y))J(y,\varepsilon g_i(y))\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}\end{aligned}$$

for $y\in\Gamma$ and $i=0,1$. Then we estimate the last integral by showing an appropriate estimate for Φ_1 on Ω_ε. The point is that the sign of Φ_1 on Γ_ε^0 is opposite to that of Φ_1 on Γ_ε^1, which enables us to write the sum of the integrals of φ_1 over Γ_ε^0 and Γ_ε^1 as the integral over Γ of the difference of the values of Φ_1 on Γ_ε^0 and Γ_ε^1. Also, to get an appropriate estimate for Φ_1, we use the comparison results for the surface quantities of Γ_ε^0 and Γ_ε^1 given in Lemma 3.11.

In what follows, we use the notations (3.43) and (3.44). Also, we always take the arguments $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$ and sometimes suppress them. We define

$$F(y,r) := \frac{1}{\varepsilon g(y)}\bigl\{\bigl(r-\varepsilon g_0(y)\bigr)W^\sharp_{\varepsilon,1}(y) - \bigl(\varepsilon g_1(y)-r\bigr)W^\sharp_{\varepsilon,0}(y)\bigr\},$$

$$\tilde\gamma(y,r) := \frac{1}{\varepsilon g(y)}\bigl\{\bigl(r-\varepsilon g_0(y)\bigr)\tilde\gamma^1_\varepsilon - \bigl(\varepsilon g_1(y)-r\bigr)\tilde\gamma^0_\varepsilon\bigr\},$$

where $\tilde\gamma^i_\varepsilon := \gamma^i_\varepsilon/\nu$, $i=0,1$, and

$$G^l_{jk}(y,r) := \frac{1}{\varepsilon g(y)}\bigl\{\bigl(r-\varepsilon g_0(y)\bigr)\bigl(\underline{D}^\varepsilon_j[W_\varepsilon]_{kl}\bigr)^\sharp_1(y) - \bigl(\varepsilon g_1(y)-r\bigr)\bigl(\underline{D}^\varepsilon_j[W_\varepsilon]_{kl}\bigr)^\sharp_0(y)\bigr\}$$

for $j,k,l=1,2,3$. Then

$$\begin{aligned}(6.31)\quad &[\nabla_{\Gamma_\varepsilon}W_\varepsilon\cdot u + (\nabla u)W_\varepsilon + \tilde\gamma_\varepsilon\nabla u]^\sharp_i(y)\\ &\qquad = (-1)^{i+1}[G\cdot u^\sharp + (\nabla u)^\sharp F + \tilde\gamma(\nabla u)^\sharp](y,\varepsilon g_i(y))\end{aligned}$$

for $i=0,1$, where $G\cdot u^\sharp$ is a 3×3 matrix with (j,k)-entry

$$[G\cdot u^\sharp]_{jk} := \sum_{l=1}^3 G^l_{jk}u^\sharp_l,\quad j,k=1,2,3.$$

Moreover, by (2.1), (2.6), (3.42) for W_ε and $\underline{D}^\varepsilon_j W_\varepsilon$ with $j=1,2,3$,

$$(6.32)\qquad |r-\varepsilon g_i(y)| \le \varepsilon g(y) \le c\varepsilon,\quad i=0,1$$

and the uniform boundedness in ε of W_ε and $\underline{D}^\varepsilon_j W_\varepsilon$ on Γ_ε (see Lemma 3.10), we have

$$(6.33)\qquad |\eta(y,r)| + \left|\frac{\partial\eta}{\partial r}(y,r)\right| \le c,\quad \eta = F, G^l_{jk}, \tilde\gamma$$

with a constant $c>0$ independent of ε. We also define

$$R(y,r) := \frac{1}{\varepsilon g(y)}\bigl\{\bigl(r-\varepsilon g_0(y)\bigr)P^\sharp_{\varepsilon,1}(y) + \bigl(\varepsilon g_1(y)-r\bigr)P^\sharp_{\varepsilon,0}(y)\bigr\},$$

$$S_i(y) := \sqrt{1+\varepsilon^2|\tau^i_\varepsilon(y)|^2}\,P^\sharp_{\varepsilon,i}(y),\quad i=0,1,$$

$$S(y,r) := \frac{1}{\varepsilon g(y)}\bigl\{\bigl(r-\varepsilon g_0(y)\bigr)S_1(y) + \bigl(\varepsilon g_1(y)-r\bigr)S_0(y)\bigr\},$$

where τ_ε^0 and τ_ε^1 are given by (3.26). Then, for $i=0,1$,

$$\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}\,[P_\varepsilon(\nabla u)P_\varepsilon]_i^\sharp(y)=[R(\nabla u)^\sharp S](y,\varepsilon g_i(y)). \tag{6.34}$$

Moreover, by (3.41) for P_ε, (4.15), (6.32), and $|P_\varepsilon|=2$ on Γ_ε, we have

$$|\eta(y,r)|+\left|\frac{\partial\eta}{\partial r}(y,r)\right|\le c,\quad \eta=R,S. \tag{6.35}$$

Now let J be given by (3.45) and

$$\Phi_1(y,r):=-2[\{G\cdot u^\sharp+(\nabla u)^\sharp F+\tilde\gamma(\nabla u)^\sharp\}:R(\nabla u)^\sharp S](y,r)J(y,r).$$

Then we see by (6.11), (6.31), and (6.34) that

$$\Phi_1(y,\varepsilon g_i(y))=(-1)^{i+1}\varphi_{1,i}^\sharp(y)J(y,\varepsilon g_i(y))\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}$$

for $y\in\Gamma$ and $i=0,1$ (note that here we use (3.44) for φ_1). Hence

$$\begin{aligned}\int_{\Gamma_\varepsilon}\varphi_1(x)\,d\mathcal{H}^2(x)&=\sum_{i=0,1}\int_{\Gamma_\varepsilon^i}\varphi_1(x)\,d\mathcal{H}^2(x)\\&=\int_\Gamma\{\Phi_1(y,\varepsilon g_1(y))-\Phi_1(y,\varepsilon g_0(y))\}\,d\mathcal{H}^2(y)\\&=\int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\frac{\partial\Phi_1}{\partial r}(y,r)\,dr\,d\mathcal{H}^2(y)\end{aligned}$$

by (3.52). Moreover, it follows from (3.46), (6.33), and (6.35) that

$$\left|\frac{\partial\Phi_1}{\partial r}\right|\le c\{|u^\sharp|^2+|(\nabla u)^\sharp|^2+(|u^\sharp|+|(\nabla u)^\sharp|)|(\nabla^2u)^\sharp|\}$$

with some constant $c>0$ independent of ε (here we also used Young's inequality). By the above relations, (3.49), and Hölder's inequality,

$$\begin{aligned}&\left|\int_{\Gamma_\varepsilon}\varphi_1\,d\mathcal{H}^2\right|\\&\quad\le c\int_\Gamma\int_{\varepsilon g_0}^{\varepsilon g_1}\{|u^\sharp|^2+|(\nabla u)^\sharp|^2+(|u^\sharp|+|(\nabla u)^\sharp|)|(\nabla^2u)^\sharp|\}\,dr\,d\mathcal{H}^2\\&\quad\le c\left(\|u\|_{H^1(\Omega_\varepsilon)}^2+\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2u\|_{L^2(\Omega_\varepsilon)}\right).\end{aligned}$$

Thus the inequality (6.29) for $k = 1$ is valid. By the same arguments we can prove (6.29) for $k = 2$ and (6.30) for $k = 3$.

Finally, we obtain (6.8) by (6.10), (6.29), and (6.30), and we apply (6.8) to (6.7) and then use $ab \leq (a^2 + b^2)/2$ for $a, b \geq 0$ to get

$$\|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)} \leq \|\Delta u\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla^2 u\|^2_{L^2(\Omega_\varepsilon)} + c\|u\|^2_{H^1(\Omega_\varepsilon)},$$

which yields (6.6). Hence the inequality (6.1) is valid. □

7. Proofs of the Main Results

In this section we establish Theorems 2.4, 2.6, and 2.7. First we give an integration by parts formula related to the slip boundary conditions (4.18).

Lemma 7.1. *For $u_1 \in H^2(\Omega_\varepsilon)^3$ and $u_2 \in H^1(\Omega_\varepsilon)^3$ we have*

$$\begin{aligned} \int_{\Omega_\varepsilon} \{\Delta u_1 + \nabla(\operatorname{div} u_1)\} \cdot u_2\, dx \\ = -2\int_{\Omega_\varepsilon} D(u_1) : D(u_2)\, dx + 2\int_{\Gamma_\varepsilon} [D(u_1)n_\varepsilon] \cdot u_2\, d\mathcal{H}^2. \end{aligned} \tag{7.1}$$

In particular,

$$\begin{aligned} \nu \int_{\Omega_\varepsilon} \Delta u_1 \cdot u_2\, dx = -2\nu \int_{\Omega_\varepsilon} D(u_1) : D(u_2)\, dx \\ - \sum_{i=0,1} \gamma_\varepsilon^i \int_{\Gamma_\varepsilon^i} u_1 \cdot u_2\, d\mathcal{H}^2 \end{aligned}$$

if u_1 satisfies $\operatorname{div} u_1 = 0$ in Ω_ε and (4.18), and if u_2 satisfies (4.6).

Proof. Since $\Delta u_1 + \nabla(\operatorname{div} u_1) = 2\operatorname{div}[D(u_1)]$ in Ω_ε,

$$\int_{\Omega_\varepsilon} \{\Delta u_1 + \nabla(\operatorname{div} u_1)\} \cdot u_2\, dx = 2\int_{\Omega_\varepsilon} \operatorname{div}[D(u_1)] \cdot u_2\, dx. \tag{7.2}$$

Moreover, for $A \in H^1(\Omega_\varepsilon)^{3\times 3}$ and $u \in H^1(\Omega_\varepsilon)^3$ we have

$$\int_{\Omega_\varepsilon} \operatorname{div} A \cdot u\, dx = \int_{\Gamma_\varepsilon} (A^T n_\varepsilon) \cdot u\, d\mathcal{H}^2 - \int_{\Omega_\varepsilon} A : \nabla u\, dx$$

by integration by parts. Applying this with $A = D(u_1)$ and $u = u_2$ to (7.2) and using $D(u_1)^T = D(u_1)$ and $D(u_1) : \nabla u_2 = D(u_1) : D(u_2)$, we obtain (7.1). $\square$

By Lemma 7.1, we see that the bilinear form for the Stokes problem (1.5) is given by (2.2), i.e. $a_\varepsilon(u_1, u_2) = 2\nu\big(D(u_1), D(u_2)\big)_{L^2(\Omega_\varepsilon)} + \sum_{i=0,1} \gamma_\varepsilon^i (u_1, u_2)_{L^2(\Gamma_\varepsilon^i)}$ for $u_1, u_2 \in H^1(\Omega_\varepsilon)^3$. Now we impose Assumptions 2.2 and 2.3 and define the function spaces $\mathcal{H}_\varepsilon$ and $\mathcal{V}_\varepsilon$ by (2.7). Let us show that a_ε is uniformly bounded and coercive on $\mathcal{V}_\varepsilon$.

PROOF OF THEOREM 2.4. Let $u \in \mathcal{V}_\varepsilon$. Then $\gamma_\varepsilon^i \|u\|_{L^2(\Gamma_\varepsilon^i)}^2 \leq c\|u\|_{H^1(\Omega_\varepsilon)}^2$ for $i = 0, 1$ by (2.6) in Assumption 2.2 and (4.3). Combining this with $|D(u)| \leq |\nabla u|$ in Ω_ε, we get the right-hand inequality of (2.8).

Let us prove the left-hand inequality of (2.8). First we suppose that the condition (A1) of Assumption 2.3 is satisfied. Without loss of generality, we may assume $\gamma_\varepsilon^0 \geq c\varepsilon$ for all $\varepsilon \in (0, 1]$. For $u \in \mathcal{V}_\varepsilon$ we use (4.2) with $i = 0$ and then apply $\gamma_\varepsilon^0 \geq c\varepsilon$ and (5.1) (note that $u \in \mathcal{V}_\varepsilon$ satisfies (4.6)) to get

$$\begin{aligned}
\|u\|_{L^2(\Omega_\varepsilon)}^2 &\leq c\left(\gamma_\varepsilon^0 \|u\|_{L^2(\Gamma_\varepsilon^0)}^2 + \varepsilon^2 \|D(u)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^2 \|u\|_{L^2(\Omega_\varepsilon)}^2\right) \\
&\leq c_1 a_\varepsilon(u, u) + c_2 \varepsilon^2 \|u\|_{L^2(\Omega_\varepsilon)}^2
\end{aligned}$$

with constants $c_1, c_2 > 0$ independent of ε. We set $\varepsilon_1 := 1/\sqrt{2c_2}$ and take $\varepsilon \in (0, \varepsilon_1]$ in the above inequality to get $\|u\|_{L^2(\Omega_\varepsilon)}^2 \leq 2c_1 a_\varepsilon(u, u)$. By this inequality and (5.1) we also have $\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq c a_\varepsilon(u, u)$. These two inequalities imply the left-hand inequality of (2.8).

Next we suppose that the condition (A2) or (A3) of Assumption 2.3 is satisfied. Then $u \in \mathcal{V}_\varepsilon$ satisfies (4.6) and (5.3) (resp. (5.27)) with $\beta = 0$ under the condition (A2) (resp. (A3)). Hence Theorems 5.6 and 5.7 imply that there exist $\varepsilon_{K,0} \in (0, 1]$ and $c_{K,0} > 0$ such that

$$\|u\|_{H^1(\Omega_\varepsilon)}^2 \leq c_{K,0} \|D(u)\|_{L^2(\Omega_\varepsilon)}^2 \leq c_{K,0} a_\varepsilon(u, u)$$

for all $\varepsilon \in (0, \varepsilon_{K,0}]$, i.e. the left-hand inequality of (2.8) holds.

Therefore, the theorem is valid with $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_{K,0}\}$. $\square$

As in Section 2, we fix the constant ε_0 given in Theorem 2.4 and denote by A_ε the Stokes operator for Ω_ε under the slip boundary conditions for $\varepsilon \in (0, \varepsilon_0]$.

Next we derive the uniform difference estimate (2.14) for A_ε and $-\nu\Delta$. For this purpose, we give an integration by parts formula for the curl of a vector field on Ω_ε. Let n_ε^0 and n_ε^1 be the vector fields on Γ given by (3.27) and

$$W_\varepsilon^i(x) := -\{I_3 - \bar{n}_\varepsilon^i(x) \otimes \bar{n}_\varepsilon^i(x)\}\nabla\bar{n}_\varepsilon^i(x), \quad x \in N,\ i = 0,1. \tag{7.3}$$

Here $\bar{n}_\varepsilon^i = n_\varepsilon^i \circ \pi$, $i = 0,1$ is the constant extension of n_ε^i. Let

(7.4)

$$\begin{aligned}
\tilde{n}_1(x) &:= \frac{1}{\varepsilon\bar{g}(x)}\bigl\{\bigl(d(x) - \varepsilon\bar{g}_0(x)\bigr)\bar{n}_\varepsilon^1(x) - \bigl(\varepsilon\bar{g}_1(x) - d(x)\bigr)\bar{n}_\varepsilon^0(x)\bigr\}, \\
\tilde{n}_2(x) &:= \frac{1}{\varepsilon\bar{g}(x)}\left\{\bigl(d(x) - \varepsilon\bar{g}_0(x)\bigr)\frac{\gamma_\varepsilon^1}{\nu}\bar{n}_\varepsilon^1(x) + \bigl(\varepsilon\bar{g}_1(x) - d(x)\bigr)\frac{\gamma_\varepsilon^0}{\nu}\bar{n}_\varepsilon^0(x)\right\}, \\
\widetilde{W}(x) &:= \frac{1}{\varepsilon\bar{g}(x)}\bigl\{\bigl(d(x) - \varepsilon\bar{g}_0(x)\bigr)W_\varepsilon^1(x) - \bigl(\varepsilon\bar{g}_1(x) - d(x)\bigr)W_\varepsilon^0(x)\bigr\}
\end{aligned}$$

for $x \in N$. From these definitions and Lemma 3.9 it follows that

$$\tilde{n}_1 = (-1)^{i+1}n_\varepsilon, \quad \tilde{n}_2 = \frac{\gamma_\varepsilon}{\nu}n_\varepsilon, \quad \widetilde{W} = (-1)^{i+1}W_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon^i \tag{7.5}$$

for $i = 0,1$. For $u\colon \Omega_\varepsilon \to \mathbb{R}^3$ we define $G(u)\colon \Omega_\varepsilon \to \mathbb{R}^3$ by

$$\begin{gathered}
G(u) := G_1(u) + G_2(u), \\
G_1(u) := 2\tilde{n}_1 \times \widetilde{W}u, \quad G_2(u) := \tilde{n}_2 \times u.
\end{gathered} \tag{7.6}$$

Lemma 7.2. *Suppose that the inequalities* (2.6) *are valid. Then*

$$|G(u)| \leq c|u|, \quad |\nabla G(u)| \leq c(|u| + |\nabla u|) \quad \text{in} \quad \Omega_\varepsilon \tag{7.7}$$

for all $u \in C^1(\Omega_\varepsilon)^3$, *where* $c > 0$ *is a constant independent of* ε *and* u.

Lemma 7.2 is proved just by direct calculations and the application of the results given in Section 3. We give its proof in Appendix C.

Lemma 7.3. *The integration by parts formula*

$$\int_{\Omega_\varepsilon} \operatorname{curl}\operatorname{curl} u \cdot \Phi\,dx = -\int_{\Omega_\varepsilon} \operatorname{curl} G(u) \cdot \Phi\,dx + \int_{\Omega_\varepsilon} \{\operatorname{curl} u + G(u)\} \cdot \operatorname{curl}\Phi\,dx \tag{7.8}$$

holds for all $u \in H^2(\Omega_\varepsilon)^3$ *satisfying* (4.18) *and* $\Phi \in L^2(\Omega_\varepsilon)^3$ *with* $\operatorname{curl}\Phi \in L^2(\Omega_\varepsilon)^3$.

The proof of (7.8) is the same as in the case of a flat thin domain (see the proofs of [17, Lemma 2.3] and [18, Lemma 5.2]). Here we give it for the completeness.

Proof. By standard cut-off, dilatation, and mollification arguments, we can show as in the proof of [73, Chapter 1, Theorem 1.1] that for $\Phi \in L^2(\Omega_\varepsilon)^3$ with $\operatorname{curl}\Phi \in L^2(\Omega_\varepsilon)^3$ there exists a sequence $\{\Phi_k\}_{k=1}^\infty$ in $C^\infty(\overline{\Omega}_\varepsilon)^3$ such that

$$\lim_{k\to\infty} \|\Phi - \Phi_k\|_{L^2(\Omega_\varepsilon)} = \lim_{k\to\infty} \|\operatorname{curl}\Phi - \operatorname{curl}\Phi_k\|_{L^2(\Omega_\varepsilon)} = 0.$$

Thus, by a density argument, it suffices to show (7.8) for $\Phi \in C^\infty(\overline{\Omega}_\varepsilon)^3$.

Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy (4.18) and $\Phi \in C^\infty(\overline{\Omega}_\varepsilon)^3$. Then

$$\begin{aligned}(7.9)\quad &\int_{\Omega_\varepsilon} \operatorname{curl}\operatorname{curl} u \cdot \Phi\, dx \\ &\qquad = \int_{\Gamma_\varepsilon} (n_\varepsilon \times \operatorname{curl} u)\cdot \Phi\, d\mathcal{H}^2 + \int_{\Omega_\varepsilon} \operatorname{curl} u \cdot \operatorname{curl}\Phi\, dx\end{aligned}$$

by integration by parts. Since u satisfies (4.18),

$$\begin{aligned} n_\varepsilon \times \operatorname{curl} u &= -n_\varepsilon \times \left\{ n_\varepsilon \times \left(2W_\varepsilon u + \frac{\gamma_\varepsilon}{\nu} u\right)\right\} \\ &= -n_\varepsilon \times \left(2\tilde{n}_1 \times \widetilde{W} u + \tilde{n}_2 \times u\right) = -n_\varepsilon \times G(u)\end{aligned}$$

on Γ_ε by (4.20), (7.5), and (7.6). Hence integration by parts yields

$$\begin{aligned}\int_{\Gamma_\varepsilon} (n_\varepsilon \times \operatorname{curl} u)\cdot\Phi\, d\mathcal{H}^2 &= -\int_{\Gamma_\varepsilon} \{n_\varepsilon \times G(u)\}\cdot\Phi\, d\mathcal{H}^2 \\ &= \int_{\Omega_\varepsilon} \{G(u)\cdot \operatorname{curl}\Phi - \operatorname{curl} G(u)\cdot\Phi\}\, dx.\end{aligned}$$

Substituting this for (7.9) we obtain (7.8). $\square$

Now let us prove (2.14). We follow the idea of the proof of a similar estimate for a flat thin domain given in [17, Theorem 2.1] and [18, Corollary

5.3]. Main tools are the integration by parts formula (7.8) and the standard Helmholtz–Leray projection from $L^2(\Omega_\varepsilon)^3$ onto $L^2_\sigma(\Omega_\varepsilon)$ which we denote by $\mathbb{L}_\varepsilon$. It is well known (see [5, 8, 70, 73]) that for each $u \in L^2(\Omega_\varepsilon)^3$ the Helmholtz–Leray decomposition

$$u = \mathbb{L}_\varepsilon u + \nabla q \quad \text{in} \quad L^2(\Omega_\varepsilon)^3, \quad \mathbb{L}_\varepsilon u \in L^2_\sigma(\Omega_\varepsilon), \quad \nabla q \in L^2_\sigma(\Omega_\varepsilon)^\perp$$

holds, where $q \in H^1(\Omega_\varepsilon)$ is a weak solution to the Neumann problem of Poisson's equation

$$\Delta q = \operatorname{div} u \quad \text{in} \quad \Omega_\varepsilon, \quad \frac{\partial q}{\partial n_\varepsilon} = u \cdot n_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon.$$

Note that $\mathbb{L}_\varepsilon$ may differ from the orthogonal projection $\mathbb{P}_\varepsilon$ from $L^2(\Omega_\varepsilon)^3$ onto the closed subspace $\mathcal{H}_\varepsilon$ given by (2.7) under the condition (A3) of Assumption 2.3. In this case we require a little more discussions to establish (2.14).

Proof of Theorem 2.6. We first show that

$$\|\nu\Delta u - \nu\mathbb{L}_\varepsilon\Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)} \tag{7.10}$$

for all $\varepsilon \in (0, \varepsilon_0]$ and $u \in D(A_\varepsilon)$ with a constant $c > 0$ independent of ε and u. It follows from the Helmholtz–Leray decomposition

$$\nu\Delta u = \nu\mathbb{L}_\varepsilon\Delta u + \nabla q \quad \text{in} \quad L^2(\Omega_\varepsilon)^3, \quad (\nu\mathbb{L}_\varepsilon\Delta u, \nabla q)_{L^2(\Omega_\varepsilon)} = 0$$

with $q \in H^1(\Omega_\varepsilon)$ and $\Delta u = -\operatorname{curl}\operatorname{curl} u$ in Ω_ε by $\operatorname{div} u = 0$ that

$$\begin{aligned}\|\nu\Delta u - \nu\mathbb{L}_\varepsilon\Delta u\|^2_{L^2(\Omega_\varepsilon)} &= (\nu\Delta u - \nu\mathbb{L}_\varepsilon\Delta u, \nabla q)_{L^2(\Omega_\varepsilon)} \\ &= -\nu(\operatorname{curl}\operatorname{curl} u, \nabla q)_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Noting that $\operatorname{curl}\nabla q = 0$ in Ω_ε, we apply (7.8) with $\Phi = \nabla q$ to the last term to get

$$\begin{aligned}-\nu(\operatorname{curl}\operatorname{curl} u, \nabla q)_{L^2(\Omega_\varepsilon)} &= \nu(\operatorname{curl} G(u), \nabla q)_{L^2(\Omega_\varepsilon)} \\ &\leq c\|\nabla G(u)\|_{L^2(\Omega_\varepsilon)}\|\nabla q\|_{L^2(\Omega_\varepsilon)}\end{aligned}$$

with $G(u)$ given by (7.6). Since the inequalities (2.6) hold by Assumption 2.2, we can apply (7.7) to the right-hand side of this inequality. Hence (note that $\nabla q = \nu\Delta u - \nu\mathbb{L}_\varepsilon\Delta u$)

$$\begin{aligned}\|\nu\Delta u - \nu\mathbb{L}_\varepsilon\Delta u\|_{L^2(\Omega_\varepsilon)}^2 &\le c\|\nabla G(u)\|_{L^2(\Omega_\varepsilon)}\|\nabla q\|_{L^2(\Omega_\varepsilon)} \\ &\le c\|u\|_{H^1(\Omega_\varepsilon)}\|\nu\Delta u - \nu\mathbb{L}_\varepsilon\Delta u\|_{L^2(\Omega_\varepsilon)}\end{aligned}$$

and (7.10) is valid. If the condition (A1) or (A2) of Assumption 2.3 is satisfied, then $\mathbb{L}_\varepsilon$ agrees with the orthogonal projection $\mathbb{P}_\varepsilon$ onto $\mathcal{H}_\varepsilon = L^2_\sigma(\Omega_\varepsilon)$. Hence

$$A_\varepsilon u = -\nu\mathbb{P}_\varepsilon\Delta u = -\nu\mathbb{L}_\varepsilon\Delta u \quad \text{in} \quad L^2(\Omega_\varepsilon)^3, \quad u \in D(A_\varepsilon) \tag{7.11}$$

and (2.14) follows from (7.10).

Next we suppose that the condition (A3) of Assumption 2.3 is satisfied. Then $A_\varepsilon u = -\nu\mathbb{P}_\varepsilon\Delta u \in \mathcal{H}_\varepsilon = L^2_\sigma(\Omega_\varepsilon) \cap \mathcal{R}_g^\perp$ for $u \in D(A_\varepsilon)$, where $\mathcal{R}_g$ is the space of infinitesimal rigid displacements of $\mathbb{R}^3$ given by (2.4). In this case, however, we still have (7.11). To see this, let $w \in \mathcal{R}_g$. Then w belongs to $L^2_\sigma(\Omega_\varepsilon)$ by the assumption $\mathcal{R}_g = \mathcal{R}_0 \cap \mathcal{R}_1$ and Lemma E.8 and thus $(\mathbb{L}_\varepsilon\Delta u, w)_{L^2(\Omega_\varepsilon)} = (\Delta u, w)_{L^2(\Omega_\varepsilon)}$ since $\mathbb{L}_\varepsilon$ is the orthogonal projection from $L^2(\Omega_\varepsilon)^3$ onto $L^2_\sigma(\Omega_\varepsilon)$. Moreover, under the assumptions $\mathcal{R}_g = \mathcal{R}_0 \cap \mathcal{R}_1$ and $\gamma_\varepsilon^0 = \gamma_\varepsilon^1 = 0$, the vector fields $u \in D(A_\varepsilon)$ and $w \in \mathcal{R}_g$ satisfy (note that w is of the form $w(x) = a \times x + b$)

$$\begin{gathered}\operatorname{div} u = 0, \quad D(w) = 0 \quad \text{in} \quad \Omega_\varepsilon, \\ u \cdot n_\varepsilon = 0, \quad P_\varepsilon D(u)n_\varepsilon = 0, \quad w \cdot n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon\end{gathered}$$

by (2.11) and Lemma E.5. These equalities and (7.1) yield

$$(\Delta u, w)_{L^2(\Omega_\varepsilon)} = -2\big(D(u), D(w)\big)_{L^2(\Omega_\varepsilon)} + 2(D(u)n_\varepsilon, w)_{L^2(\Gamma_\varepsilon)} = 0.$$

Hence $(\mathbb{L}_\varepsilon\Delta u, w)_{L^2(\Omega_\varepsilon)} = 0$ for all $w \in \mathcal{R}_g$, i.e. $\mathbb{L}_\varepsilon\Delta u \in L^2_\sigma(\Omega_\varepsilon) \cap \mathcal{R}_g^\perp = \mathcal{H}_\varepsilon$. Now since the Helmholtz–Leray decomposition

$$\begin{gathered}\Delta u = \mathbb{L}_\varepsilon\Delta u + \nabla\tilde{q} \quad \text{in} \quad L^2(\Omega_\varepsilon)^3, \\ \mathbb{L}_\varepsilon\Delta u \in \mathcal{H}_\varepsilon, \quad \nabla\tilde{q} \in L^2_\sigma(\Omega_\varepsilon)^\perp \subset \mathcal{H}_\varepsilon^\perp\end{gathered}$$

holds and $\mathbb{P}_\varepsilon$ is the orthogonal projection from $L^2(\Omega_\varepsilon)^3$ onto $\mathcal{H}_\varepsilon$, it follows that $\mathbb{P}_\varepsilon\Delta u = \mathbb{P}_\varepsilon\mathbb{L}_\varepsilon\Delta u = \mathbb{L}_\varepsilon\Delta u$ in $L^2(\Omega_\varepsilon)^3$ for $u \in D(A_\varepsilon)$. Thus the relation (7.11) holds and (2.14) again follows from (7.10). □

Finally, we prove the uniform norm equivalence (2.15) for A_ε.

Proof of Theorem 2.7. Let $u \in D(A_\varepsilon)$. Since u satisfies the slip boundary conditions (4.18) by (2.11), we can use (2.14) and (6.1) to get

$$\begin{aligned}\|u\|_{H^2(\Omega_\varepsilon)} &\leq c\left(\|\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \\ &\leq c\left(\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \\ &\leq c\left(\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right).\end{aligned}$$

Applying (2.12) and (2.13) to the second term on the last line we obtain the left-hand inequality of (2.15). Also, by (2.14),

$$\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \leq \|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} + \|\nu\Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^2(\Omega_\varepsilon)}.$$

Hence the right-hand inequality of (2.15) holds. □

Appendix A. Notations on Vectors and Matrices

In this appendix we fix notations on vectors and matrices. For $l, m \in \mathbb{N}$, we consider a vector $a \in \mathbb{R}^m$ as a column vector and express $a \in \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{l\times m}$ as

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = (a_1, \cdots, a_m)^T, \quad A = (A_{ij})_{i,j} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{l1} & \cdots & A_{lm} \end{pmatrix}.$$

We denote the i-th component of a by a_i or sometimes a^i or $[a]_i$, and write A_{ij} or $[A]_{ij}$ for the (i,j)-entry of A. Also, we denote the transpose of A by A^T and, when $l = m$, the symmetric part of A by $A_S := (A + A^T)/2$ and the $m \times m$ identity matrix by I_m. The tensor product of vectors $a \in \mathbb{R}^l$ and $b \in \mathbb{R}^m$ is given by

$$a \otimes b := \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_m \\ \vdots & & \vdots \\ a_l b_1 & \cdots & a_l b_m \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_l \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

For three-dimensional vector fields $u = (u_1, u_2, u_3)^T$ and φ on an open set in $\mathbb{R}^3$ let

$$\nabla u := \begin{pmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 \\ \partial_3 u_1 & \partial_3 u_2 & \partial_3 u_3 \end{pmatrix}, \quad (\varphi \cdot \nabla) u := \begin{pmatrix} \varphi \cdot \nabla u_1 \\ \varphi \cdot \nabla u_2 \\ \varphi \cdot \nabla u_3 \end{pmatrix} = (\nabla u)^T \varphi,$$

and $|\nabla^2 u|^2 := \sum_{i,j,k=1}^3 |\partial_i \partial_j u_k|^2$, where $\partial_i = \partial/\partial x_i$. Also, for a 3×3 matrix-valued function $A = (A_{ij})_{i,j}$ on an open set in $\mathbb{R}^3$ we set

$$\operatorname{div} A := \begin{pmatrix} [\operatorname{div} A]_1 \\ [\operatorname{div} A]_2 \\ [\operatorname{div} A]_3 \end{pmatrix}, \quad [\operatorname{div} A]_j := \sum_{i=1}^3 \partial_i A_{ij}, \quad j = 1, 2, 3.$$

We define the inner product and the norm of matrices $A, B \in \mathbb{R}^{3\times 3}$ by $A : B := \operatorname{tr}[A^T B] = \sum_{i=1}^3 AE_i \cdot BE_i$ and $|A| := \sqrt{A : A}$, where $\{E_1, E_2, E_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Note that $A : B$ does not depend on a choice of $\{E_1, E_2, E_3\}$. In particular, taking the standard basis of $\mathbb{R}^3$ we get $A : B = \sum_{i,j=1}^3 A_{ij} B_{ij}$ and thus $A : B = B : A = A^T : B^T$ and $AB : C = A : CB^T = B : A^T C$ for $A, B, C \in \mathbb{R}^{3\times 3}$. Also, for $a, b \in \mathbb{R}^3$ we have $|a \otimes b| = |a||b|$.

Appendix B. Auxiliary Results Related to a Closed Surface

This appendix gives some auxiliary results related to a closed surface. The results of this appendix are well known or easily proved by direct calculations, so we briefly explain the proofs and omit details here. For detailed calculations, see the arXiv version of this paper [46].

Let Γ be a closed, connected, and oriented surface in $\mathbb{R}^3$ of class C^ℓ with $\ell \geq 2$. We use the notations given in Section 3.1. First we give some properties of the Riemannian metric of Γ.

Lemma B.1. *Let U be an open set in $\mathbb{R}^2$, $\mu\colon U \to \Gamma$ a C^ℓ local parametrization of Γ, and $\mathcal{K}$ a compact subset of U. Then there exists a constant $c > 0$ such that*

$$|\partial_{s_i} \mu(s)| \leq c, \quad |\partial_{s_i} \partial_{s_j} \mu(s)| \leq c \quad \text{for all} \quad s \in \mathcal{K},\ i, j = 1, 2. \tag{B.1}$$

We define the Riemannian metric $\theta = (\theta_{ij})_{i,j}$ *of* Γ *by*

$$\theta(s) := \nabla_s\mu(s)\{\nabla_s\mu(s)\}^T, \quad \nabla_s\mu := \begin{pmatrix} \partial_{s_1}\mu_1 & \partial_{s_1}\mu_2 & \partial_{s_1}\mu_3 \\ \partial_{s_2}\mu_1 & \partial_{s_2}\mu_2 & \partial_{s_2}\mu_3 \end{pmatrix} \tag{B.2}$$

for $s \in U$ *and denote by* $\theta^{-1} = (\theta^{ij})_{i,j}$ *the inverse matrix of* θ *(note that* θ *and* θ^{-1} *are symmetric). Then*

$$|\theta^k(s)| \leq c, \quad |\partial_{s_i}\theta^k(s)| \leq c, \quad c^{-1} \leq \det\theta(s) \leq c \tag{B.3}$$

for all $s \in \mathcal{K}$*,* $i = 1, 2$*, and* $k = \pm 1$*. Moreover,*

$$c^{-1}|a|^2 \leq \theta^{-1}(s)a \cdot a \leq c|a|^2 \tag{B.4}$$

for all $s \in \mathcal{K}$ *and* $a \in \mathbb{R}^2$*.*

Proof. The inequalities (B.1) and (B.3) follow from the C^ℓ-regularity of μ on U, $\partial_{s_i}\theta^{-1} = -\theta^{-1}(\partial_{s_i}\theta)\theta^{-1}$ and $\det\theta > 0$ in U, and the compactness of $\mathcal{K}$ in U. Also, for $s \in U$ and $a = (a_1, a_2)^T \in \mathbb{R}^2$, we see that $X(s,a) := \sum_{i,j=1}^2 \theta^{ij}(s)a_i\partial_{s_j}\mu(s)$ vanishes if and only if $a = 0$. Hence $|X(s,a)|^2 = \theta^{-1}(s)a \cdot a$ is strictly positive for $a \neq 0$ and it is bounded from above and below by positive constants on the compact set $\mathcal{K} \times S^1$, where S^1 is the unit circle in $\mathbb{R}^2$. Hence (B.4) follows. □

Next we see that the differential operators on Γ given in Section 3.1 agree with those defined in differential geometry.

Lemma B.2. *Let* U *be an open set in* $\mathbb{R}^2$ *and* $\mu\colon U \to \Gamma$ *a* C^ℓ *local parametrization of* Γ*. For* $\eta \in C^1(\Gamma)$ *let* $\eta^\flat := \eta \circ \mu$ *on* U*. Then the tangential gradient of* η *defined by* (3.3) *is expressed as*

$$\nabla_\Gamma\eta(\mu(s)) = \sum_{i,j=1}^2 \theta^{ij}(s)\partial_{s_i}\eta^\flat(s)\partial_{s_j}\mu(s), \quad s \in U. \tag{B.5}$$

Proof. Noting that $\{\partial_{s_1}\mu(s), \partial_{s_2}\mu(s), n(\mu(s))\}$ is a basis of $\mathbb{R}^3$ for $s \in U$, we can easily get (B.5) by $\partial_{s_i}\eta^\flat(s) = \partial_{s_i}\mu(s) \cdot \nabla_\Gamma\eta(\mu(s))$ for $i = 1, 2$ and $n \cdot \nabla_\Gamma\eta = 0$ on Γ. □

Lemma B.3. *Let U be an open set in $\mathbb{R}^2$ and $\mu\colon U \to \Gamma$ a C^ℓ local parametrization of Γ. The surface divergence of $X \in C^1(\Gamma, T\Gamma)$ defined by (3.6) is locally of the form*

$$\operatorname{div}_\Gamma X(\mu(s)) = \frac{1}{\sqrt{\det\theta(s)}}\sum_{i=1}^{2}\partial_{s_i}\Bigl(X^i(s)\sqrt{\det\theta(s)}\Bigr), \quad s \in U, \tag{B.6}$$

where $X^i(s) := \sum_{j=1}^2 \theta^{ij}(s)\partial_{s_j}\mu(s)\cdot X(\mu(s))$ for $i = 1, 2$.

Proof. We write $\eta^\flat(s) := \eta(\mu(s))$, $s \in U$ for a function η on Γ and suppress the argument $s \in U$. Let $X = (X_1, X_2, X_3)^T$ so that $\operatorname{div}_\Gamma X = \sum_{k=1}^3 \underline{D}_k X_k$ on Γ. Then it follows from (B.5) for $\eta = X_k$ that

$$(\operatorname{div}_\Gamma X)^\flat = \sum_{k=1}^{3}\sum_{i,j=1}^{2}\theta^{ij}(\partial_{s_i}X_k^\flat)\partial_{s_j}\mu_k = \sum_{i,j=1}^{2}\theta^{ij}\partial_{s_i}X^\flat\cdot\partial_{s_j}\mu.$$

Since we can show $X^\flat = \sum_{i=1}^2 X^i\partial_{s_i}\mu$ as in the proof of Lemma B.2, we substitute this for the above equality. Also, we compute the right-hand side of (B.6) by using Jacobi's formula $\partial_{s_i}(\det\theta) = \operatorname{tr}(\theta^{-1}\partial_{s_i}\theta)\det\theta$ and $\operatorname{tr}(\theta^{-1}\partial_{s_i}\theta) = 2\sum_{j,k=1}^2 \theta^{jk}\partial_{s_i}\partial_{s_k}\mu\cdot\partial_{s_j}\mu$. Then, by comparing the resulting expressions, we find that (B.6) is valid. □

Let us consider the equivalence of the Sobolev spaces on Γ.

Lemma B.4. *Let $p \in [1, \infty]$ and $m = 0, 1, \dots, \ell$.*

(a) *Let $\mu\colon U \to \Gamma$ be a C^ℓ local parametrization of Γ with an open subset U of $\mathbb{R}^2$. Also, let η be a function on Γ compactly supported in $\mu(U)$ and $\eta^\flat := \eta\circ\mu$ on U. Then $\eta \in W^{m,p}(\Gamma)$ if and only if $\eta^\flat \in W^{m,p}(U)$, and*

$$c^{-1}\|\eta^\flat\|_{W^{m,p}(U)} \le \|\eta\|_{W^{m,p}(\Gamma)} \le c\|\eta^\flat\|_{W^{m,p}(U)}. \tag{B.7}$$

In particular, if $\eta \in W^{1,p}(\Gamma)$, then (B.5) holds on U and

$$c^{-1}\|\nabla_s\eta^\flat\|_{L^p(U)} \le \|\nabla_\Gamma\eta\|_{L^p(\Gamma)} \le c\|\nabla_s\eta^\flat\|_{L^p(U)}, \tag{B.8}$$

where $\nabla_s \eta^\flat = (\partial_{s_1}\eta^\flat, \partial_{s_2}\eta^\flat)^T$ *is the gradient of* $\eta^\flat$ *in* $s \in \mathbb{R}^2$*.*

(b) *Let* η *be a function on* Γ*. Then* $\eta \in W^{m,p}(\Gamma)$ *if and only if* $(\xi\eta) \circ \mu \in W^{m,p}(U)$ *for any* C^ℓ *local parametrization* $\mu\colon U \to \Gamma$ *with an open subset* U *of* $\mathbb{R}^2$ *and any* $\xi \in C^\ell(\Gamma)$ *compactly supported in* $\mu(U)$*.*

This result seems to be well known, but we cannot find the literature which gives an explicit proof based on the definition (3.23) of the weak tangential derivative. Here we give the outline of the proof when $m = 0, 1, 2$ for the readers' convenience.

Proof. The statement (b) follows from (a) and a localization argument with a partition of unity on Γ, which consists of C^ℓ functions on Γ since Γ is of class C^ℓ. Let us show (a) when $m = 0, 1, 2$ (the higher order case can be shown similarly). In what follows, we write $\xi^\flat := \xi \circ \mu$ on U for a function ξ on Γ. Also, let $\mathcal{K}$ be a compact subset of U such that η is supported in $\mu(\mathcal{K})$.

When $m = 0$, we have (B.7) for $p \neq \infty$ by the definition of the surface integral and (B.3). Also, $\|\eta\|_{L^\infty(\Gamma)} = \|\eta^\flat\|_{L^\infty(U)}$ if $p = \infty$.

Let $m = 1$ and $\eta \in W^{1,p}(\Gamma)$. For $i = 1, 2$ and $\varphi \in C_c^1(U)$ we set

$$X(\mu(s)) := \frac{\varphi(s)}{\sqrt{\det \theta(s)}} \partial_{s_i}\mu(s), \quad s \in U$$

and extend X to Γ by zero outside $\mu(U)$. Then $X \in C^1(\Gamma, T\Gamma)$ and

$$\begin{aligned} -\int_U \eta^\flat \partial_{s_i}\varphi \, ds &= -\int_\Gamma \eta \operatorname{div}_\Gamma X \, d\mathcal{H}^2 = \int_\Gamma \nabla_\Gamma \eta \cdot X \, d\mathcal{H}^2 \\ &= \int_U \{\partial_{s_i}\mu \cdot (\nabla_\Gamma \eta)^\flat\} \varphi \, ds \end{aligned}$$

by (3.24), (B.6), and $X \cdot n = 0$ on Γ. Hence

$$\partial_{s_i}\eta^\flat = \partial_{s_i}\mu \cdot (\nabla_\Gamma \eta)^\flat \quad \text{on} \quad U,\ i = 1, 2 \tag{B.9}$$

and we get $\eta^\flat \in W^{1,p}(U)$ and the left-hand inequality of (B.8) by (B.1) and (B.3), since $\eta \in W^{1,p}(\Gamma)$ is supported in $\mu(\mathcal{K})$.

Conversely, let $\eta^\flat \in W^{1,p}(U)$. We define $v(\mu(s))$ by the right-hand side of (B.5) for $s \in U$ and extend v to Γ by zero outside of $\mu(U)$. Then

$v \in L^p(\Gamma)^3$ by (B.1) and (B.3), since $\eta^\flat$ is supported in $\mathcal{K}$. For $k=1,2,3$ let v_k be the k-th component of v. We prove $\underline{D}_k\eta = v_k$ on Γ by verifying (3.23). Let $\xi \in C^1(\Gamma)$ be a test function in (3.23). Since η is supported in $\mu(\mathcal{K})$, we may assume that ξ is so. We set $Y := \xi Pe_k$ on Γ, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Then Y belongs to $C^1(\Gamma, T\Gamma)$ and is supported in $\mu(\mathcal{K})$, and $\mathrm{div}_\Gamma Y = \underline{D}_k\xi + \xi H n_k$ on Γ. Hence, by (B.6) for Y and integration by parts with respect to s_i,

$$-\int_\Gamma \eta(\underline{D}_k\xi + \xi H n_k)\, d\mathcal{H}^2 = \sum_{i=1}^2 \int_U (\partial_{s_i}\eta^\flat) Y^i \sqrt{\det\theta}\, ds.$$

Moreover, $Y^i = \sum_{j=1}^2 \theta^{ij}\partial_{s_j}\mu \cdot Y^\flat = \xi^\flat \sum_{j=1}^2 \theta^{ij}\partial_{s_j}\mu_k$ on U, where μ_k is the k-th component of μ, since $Y = \xi Pe_i$ on Γ and $\partial_{s_j}\mu$ is tangent to Γ. Hence

$$-\int_\Gamma \eta(\underline{D}_k\xi + \xi H n_k)\, d\mathcal{H}^2 = \int_U \left(\sum_{i,j=1}^2 \theta^{ij}(\partial_{s_i}\eta^\flat)\partial_{s_j}\mu_k\right) \xi^\flat \sqrt{\det\theta}\, ds$$

and the right-hand side is equal to $\int_\Gamma v_k\xi\, d\mathcal{H}^2$ since v is given by the right-hand side of (B.5). Therefore, $\underline{D}_k\eta = v_k$ on Γ for $k=1,2,3$ and we obtain $\eta \in W^{1,p}(\Gamma)$ and the right-hand inequality of (B.8) by (B.1) and (B.3), since $\eta^\flat \in W^{1,p}(U)$ is supported in $\mathcal{K}$.

Now let $m=2$ and $\eta \in W^{2,p}(\Gamma)$. To prove $\eta^\flat \in W^{2,p}(U)$, it suffices to consider $\partial_{s_i}\partial_{s_j}\eta^\flat$ for $i,j=1,2$. For $\varphi \in C_c^1(U)$ we set

$$Z_k(\mu(s)) := \frac{\varphi(s)\partial_{s_j}\mu_k(s)}{\sqrt{\det\theta(s)}}\partial_{s_i}\mu(s), \quad s \in U,\ k=1,2,3$$

and extend Z_k to Γ by zero outside $\mu(U)$ to get $Z_k \in C^1(\Gamma, T\Gamma)$. Note that here Z_k does not mean the k-th component of a vector field Z. Then, by (B.6) for Z_k and (B.9), we can show that

$$\begin{aligned} -\int_U (\partial_{s_j}\eta^\flat)\partial_{s_i}\varphi\, ds &= -\sum_{k=1}^3 \int_U (\underline{D}_k\eta)^\flat(\partial_{s_j}\mu_k)\partial_{s_i}\varphi\, ds \\ &= J + \int_U \{\partial_{s_i}\partial_{s_j}\mu \cdot (\nabla_\Gamma\eta)^\flat\}\varphi\, ds, \end{aligned}$$

where $J = -\sum_{k=1}^3 \int_\Gamma (\underline{D}_k\eta)\,\mathrm{div}_\Gamma Z_k\, d\mathcal{H}^2$. Moreover, we have

$$J = \sum_{k=1}^3 \int_\Gamma \nabla_\Gamma(\underline{D}_k\eta)\cdot Z_k\, d\mathcal{H}^2 = \int_U [\partial_{s_i}\mu\cdot\{(\nabla_\Gamma^2\eta)^\flat\partial_{s_j}\mu\}]\varphi\, ds$$

by (3.24) with $v = Z_k$ and $Z_k\cdot n = 0$ on Γ. Hence

$$-\int_U (\partial_{s_j}\eta^\flat)\partial_{s_i}\varphi\, ds = \int_U [\partial_{s_i}\mu\cdot\{(\nabla_\Gamma^2\eta)^\flat\partial_{s_j}\mu\} + \partial_{s_i}\partial_{s_j}\mu\cdot(\nabla_\Gamma\eta)^\flat]\varphi\, ds$$

for all $\varphi\in C_c^1(U)$, which shows that

$$\partial_{s_i}\partial_{s_j}\eta^\flat = \partial_{s_i}\mu\cdot\{(\nabla_\Gamma^2\eta)^\flat\partial_{s_j}\mu\} + \partial_{s_i}\partial_{s_j}\mu\cdot(\nabla_\Gamma\eta)^\flat \quad \text{on} \quad U.$$

Therefore, $\eta^\flat\in W^{2,p}(U)$ and the left-hand inequality of (B.7) holds by (B.1) and (B.3), since $\eta\in W^{2,p}(\Gamma)$ is supported in $\mu(\mathcal{K})$.

Conversely, suppose that $\eta^\flat\in W^{2,p}(U)$. Let us show $\eta\in W^{2,p}(\Gamma)$. It is sufficient to deal with $\underline{D}_k\underline{D}_l\eta$ for $k,l = 1,2,3$. By the proof in the case $m=1$, we observe that $\underline{D}_l\eta$ is supported in $\mu(U)$ and $(\underline{D}_l\eta)^\flat = \sum_{i,j=1}^2 \theta^{ij}(\partial_{s_i}\eta^\flat)\partial_{s_j}\mu_l$ on U. Thus, using again (B.6) for $Y = \xi Pe_k$, we can get $-\int_\Gamma \underline{D}_l\eta(\underline{D}_k\xi + \xi Hn_k)\, d\mathcal{H}^2 = \int_\Gamma A_{kl}\xi\, d\mathcal{H}^2$ for all $\xi\in C^1(\Gamma)$ as in the case $m=1$, where we set

$$A_{kl}^\flat := \sum_{i,j,i',j'=1}^2 \theta^{i'j'}[\partial_{s_{i'}}\{\theta^{ij}(\partial_{s_i}\eta^\flat)\partial_{s_j}\mu_l\}]\partial_{s_{j'}}\mu_k \quad \text{on} \quad U$$

and extend A_{kl} to Γ by zero outside $\mu(U)$. Hence $\underline{D}_k\underline{D}_l\eta = A_{kl}$ on Γ by (3.23), and we get $\eta\in W^{2,p}(\Gamma)$ and the right-hand inequality of (B.7) by (B.1) and (B.3), since $\eta^\flat\in W^{2,p}(U)$ is supported in $\mathcal{K}$. $\Box$

We give two lemmas related to a parametrized surface used in the proofs of Lemmas 3.9 and 3.13. For $h\in C^1(\Gamma)$ with $|h|<\delta$ on Γ let

$$\Gamma_h := \{y + h(y)n(y) \mid y\in\Gamma\}\subset\mathbb{R}^3. \tag{B.10}$$

Note that $\Gamma_h\subset N$ by $|h|<\delta$ on Γ (see Section 3.1). We also define

$$\tau_h := (I_3 - hW)^{-1}\nabla_\Gamma h, \quad n_h := \frac{n-\tau_h}{\sqrt{1+|\tau_h|^2}} \quad \text{on} \quad \Gamma. \tag{B.11}$$

Note that $\tau_h \cdot n = 0$ on Γ. We assume that the orientation of Γ_h is the same as that of Γ.

LEMMA B.5. *The constant extension of n_h in the normal direction of Γ gives the unit outward normal vector field of Γ_h.*

PROOF. Let $\bar{n}_h = n_h \circ \pi$ be the constant extension of n_h. Since $|n_h| = 1$ on Γ and the direction of n_h is the same as that of n, it is sufficient to show that $\bar{n}_h$ is perpendicular to the tangent plane of Γ_h.

Let $\mu \colon U \to \Gamma$ be a local parametrization of Γ with an open set U of $\mathbb{R}^2$ and $\mu_h(s) := \mu(s) + h(\mu(s))n(\mu(s))$ for $s \in U$. Then μ_h is a local parametrization of Γ_h and $\{\partial_{s_1}\mu_h(s), \partial_{s_2}\mu_h(s)\}$ is a basis of the tangent plane of Γ_h at $\mu_h(s)$. Hence to show that $\bar{n}_h$ is perpendicular to the tangent plane of Γ_h it suffices to prove

$$\bar{n}_h(\mu_h(s)) \cdot \partial_{s_k}\mu_h(s) = 0, \quad s \in U,\ k = 1, 2. \tag{B.12}$$

Moreover, since $\bar{n}_h(\mu_h(s)) = n_h(\mu(s))$ for $s \in U$ by $\pi(\mu_h(s)) = \mu(s) \in \Gamma$, the equality (B.12) is equivalent to $n^\flat \cdot \partial_{s_k}\mu_h = \tau_h^\flat \cdot \partial_{s_k}\mu_h$ on U for $k = 1, 2$, where $\eta^\flat := \eta \circ \mu$ for a function η on Γ, and this equality holds since both sides are equal to $\partial_{s_k}\mu \cdot (\nabla_\Gamma h)^\flat$ by

$$\partial_{s_k}\mu_h = (I_3 - h^\flat W^\flat)\partial_{s_k}\mu + \{\partial_{s_k}\mu \cdot (\nabla_\Gamma h)^\flat\}n^\flat \quad \text{on} \quad U$$

and by (B.11), $\tau_h^\flat \cdot n^\flat = 0$, and $(W^T)^\flat = W^\flat$. $\square$

LEMMA B.6. *For $\varphi \in L^1(\Gamma_h)$ we have the change of variables formula*

$$\int_{\Gamma_h} \varphi(x)\, d\mathcal{H}^2(x) = \int_\Gamma \varphi_h^\sharp(y) J(y, h(y)) \sqrt{1 + |\tau_h(y)|^2}\, d\mathcal{H}^2(y), \tag{B.13}$$

where $\varphi_h^\sharp(y) := \varphi(y + h(y)n(y))$ for $y \in \Gamma$. Also, J and τ_h are given by (3.45) *and* (B.11).

To prove Lemma B.6 we use the tangential gradient of $\varphi \in C^1(\Gamma_h)$ given by

$$\nabla_{\Gamma_h}\varphi(x) := \{I_3 - \bar{n}_h(x) \otimes \bar{n}_h(x)\}\nabla\tilde{\varphi}(x), \quad x \in \Gamma_h, \tag{B.14}$$

where $\tilde{\varphi}$ is an arbitrary extension of φ to N satisfying $\tilde{\varphi}|_{\Gamma_h} = \varphi$.

PROOF. By a localization argument, it is sufficient to show

$$\sqrt{\det\theta_h} = J(\mu, h\circ\mu)\sqrt{(1+|\tau_h\circ\mu|^2)\det\theta} \quad \text{on} \quad U, \tag{B.15}$$

where $\mu\colon U \to \Gamma$ is a local parametrization of Γ with an open subset U of $\mathbb{R}^2$, $\mu_h := \mu + (hn)\circ\mu$ is a local parametrization of Γ_h on U, and θ and θ_h are the Riemannian metrics of Γ and Γ_h given by (B.2).

In what follows, we write $\eta^\flat(s) := \eta(\mu(s))$, $s \in U$ for a function η on Γ and suppress the argument $s \in U$. Let $\bar{h} = h\circ\pi$ be the constant extension of h. First we prove

$$(1-|(\nabla_{\Gamma_h}\bar{h})\circ\mu_h|^2)\det\theta_h = J(\mu, h^\flat)^2\det\theta. \tag{B.16}$$

Since $\nabla_s\mu_h = \nabla_s\mu(I_3 - h^\flat W^\flat) + \nabla_s h^\flat \otimes n^\flat$ by $\mu_h = \mu + h^\flat n^\flat$,

$$\theta_h = \nabla_s\mu_h(\nabla_s\mu_h)^T = \nabla_s\mu(I_3 - h^\flat W^\flat)^2(\nabla_s\mu)^T + \nabla_s h^\flat \otimes \nabla_s h^\flat$$

by $(\nabla_s\mu)n^\flat = 0$, $W^\flat n^\flat = 0$, and $|n^\flat|^2 = 1$. Hence

$$\det(\theta_h - \nabla_s h^\flat \otimes \nabla_s h^\flat) = \det[\nabla_s\mu(I_3 - h^\flat W^\flat)^2(\nabla_s\mu)^T].$$

Let θ_h^{-1} be the inverse matrix of θ_h. Then

$$\det(\theta_h - \nabla_s h^\flat \otimes \nabla_s h^\flat) = \{1 - (\theta_h^{-1}\nabla_s h^\flat)\cdot\nabla_s h^\flat\}\det\theta_h$$

by $\det(I_2 + a\otimes b) = 1 + a\cdot b$ for $a, b \in \mathbb{R}^2$. Moreover, since $h^\flat = h\circ\mu = \bar{h}\circ\mu_h$, we have $(\theta_h^{-1}\nabla_s h^\flat)\cdot\nabla_s h^\flat = |(\nabla_{\Gamma_h}\bar{h})\circ\mu_h|^2$ by the local expression (B.5) of $\nabla_{\Gamma_h}\bar{h}$ on Γ_h. Also, setting

$$A := \begin{pmatrix} \nabla_s\mu \\ (n^\flat)^T \end{pmatrix}, \quad A_h := \begin{pmatrix} \nabla_s\mu(I_3 - h^\flat W^\flat) \\ (n^\flat)^T \end{pmatrix},$$

we have $A_h = A(I_3 - h^\flat W^\flat)$ and thus, by calculating $\det[A_h A_h^T]$,

$$\det[\nabla_s\mu(I_3 - h^\flat W^\flat)^2(\nabla_s\mu)^T] = J(\mu, h^\flat)^2\det\theta.$$

Hence (B.16) follows from the above relations.

Let τ_h and n_h be given by (B.11). Next we show

$$1 - \left|\nabla_{\Gamma_h}\bar{h}(y + h(y)n(y))\right|^2 = \frac{1}{1+|\tau_h(y)|^2}, \quad y \in \Gamma. \tag{B.17}$$

Hereafter we suppress the argument y of functions on Γ. We see by (3.16), $d(y+hn)=h$, and $\pi(y+hn)=y$ that $\nabla\bar{h}(y+hn)=\tau_h$. By this equality, (B.14), $\bar{n}_h(y+hn)=n_h$, and $\tau_h\cdot n=0$, we can get (B.17). Hence (B.15) holds by (B.16) and (B.17). $\square$

Remark B.7. To show Lemma B.6 we used (3.16) which will be proved in Appendix C. Note that we do not apply Lemma B.6 to show (3.16).

Let us give a regularity result for a Killing vector field on Γ.

Lemma B.8. *If Γ is of class C^ℓ with $\ell\geq 3$ and $v\in H^1(\Gamma,T\Gamma)$ satisfies $D_\Gamma(v)=0$ on Γ, where $D_\Gamma(v)$ is given by (3.7), then v is of class $C^{\ell-3}$ on Γ. In particular, v is smooth if Γ is smooth.*

Proof. Let $v\in H^1(\Gamma,T\Gamma)$ satisfy $D_\Gamma(v)=0$ on Γ. Then

$$\nabla_\Gamma v+(\nabla_\Gamma v)^T=(Wv)\otimes n+n\otimes(Wv) \tag{B.18}$$

on Γ by (3.10), $P^T=P$, and $D_\Gamma(v)=0$, and we get $\operatorname{div} v=0$ on Γ by taking the trace of (B.18). Moreover, by (B.18) we have

$$\nabla_\Gamma v_i=-\underline{D}_i v+n_iWv+[Wv]_in \quad \text{on} \quad \Gamma,\ i=1,2,3,$$

where $\underline{D}_iv=(\underline{D}_iv_1,\underline{D}_iv_2,\underline{D}_iv_3)^T$. Using this relation and (3.23), and applying (3.11), $\operatorname{div}_\Gamma v=0$, $v\cdot n=0$, and $W^T=W$ on Γ, we can get

$$(\nabla_\Gamma v_i,\nabla_\Gamma\xi)_{L^2(\Gamma)}=-(\nabla_\Gamma(n_iH)\cdot v,\xi)_{L^2(\Gamma)} \quad \text{for all} \quad \xi\in C^2(\Gamma).$$

Since $C^2(\Gamma)$ is dense in $H^1(\Gamma)$ by Lemma 3.6, this equality shows that v_i is a weak solution to $\Delta_\Gamma v_i=\nabla_\Gamma(n_iH)\cdot v$ on Γ, which is expressed under a local parametrization $\mu\colon U\to\Gamma$ by

$$\frac{1}{\sqrt{\det\theta}}\sum_{k,l=1}^{2}\partial_{s_k}\Big(\theta^{kl}(\partial_{s_l}v_i^\flat)\sqrt{\det\theta}\Big)=\sum_{j=1}^{3}b_i^jv_j^\flat \quad \text{on} \quad U,$$

where U is an open set in $\mathbb{R}^2$, θ and $\theta^{-1}=(\theta^{kl})_{k,l}$ are the Riemannian metric of Γ and its inverse, and $v_i^\flat:=v_i\circ\mu$ and $b_i^j:=[\underline{D}_j(n_iH)]\circ\mu$ on U. Then since θ and θ^{-1} are of class $C^{\ell-1}$ and b_i^j is of class $C^{\ell-3}$ on U by the C^ℓ-regularity of Γ (see Section 3.1), we have $v_i^\flat\in H^{\ell-1}(U')$ with any bounded open subset U' of U with $\overline{U'} \subset U$ by the elliptic regularity theorem (see [14, 15]) and a bootstrap argument. Hence $v_i^\flat \in C^{\ell-3}(U)$ by the Sobolev embedding theorem (see [1]), which shows that $v = (v_1, v_2, v_3)^T$ is of class $C^{\ell-3}$ on Γ. $\square$

REMARK B.9. To prove Lemma B.8 we used Lemma 3.6, which is shown by Lemma B.4 and localization and mollification arguments. Note that of course we do not use Lemma B.8 to get Lemma 3.6.

Finally, we show that the perfect slip boundary conditions

$$u \cdot n = 0, \quad 2PD(u)n = 0 \quad \text{on} \quad \Gamma$$

are different by the curvatures of Γ from the Hodge boundary conditions

$$u \cdot n = 0, \quad \operatorname{curl} u \times n = 0 \quad \text{on} \quad \Gamma$$

for $u \colon \Omega \to \mathbb{R}^3$, where Ω is a bounded domain in $\mathbb{R}^3$ with $\partial\Omega = \Gamma$.

LEMMA B.10. *Suppose that $u \in C^1(\overline{\Omega})^3$ satisfies $u \cdot n = 0$ on Γ. Then*

$$2PD(u)n - \operatorname{curl} u \times n = 2Wu \quad \text{on} \quad \Gamma. \tag{B.19}$$

PROOF. Since $2D(u) = \nabla u + (\nabla u)^T$ and $\operatorname{curl} u \times n = (\nabla u)^T n - (\nabla u)n$ under our notation for ∇u (see Appendix A), and since $\operatorname{curl} u \times n$ is tangential on Γ, i.e. $\operatorname{curl} u \times n = P(\operatorname{curl} u \times n)$ on Γ, we have

$$2PD(u)n - \operatorname{curl} u \times n = 2P(\nabla u)n = 2(\nabla_\Gamma u)n = 2Wu$$

on Γ by (3.8) and (3.10), i.e. (B.19) is valid. $\square$

Appendix C. Proofs of Auxiliary Lemmas

The purpose of this appendix is to give the proofs of the lemmas in Section 3 and Lemmas 5.4, 5.5, and 7.2 involving elementary but long calculations of differential geometry of the surfaces Γ, Γ_ε^0, and Γ_ε^1.

As in Appendix B, let Γ be a closed, connected, and oriented surface in $\mathbb{R}^3$ of class C^ℓ with $\ell \geq 2$. First we prove the lemmas in Section 3.1.

Proof of Lemma 3.3. Since W has the eigenvalues zero, κ_1, and κ_2,

$$\det[I_3 - rW(y)] = \{1 - r\kappa_1(y)\}\{1 - r\kappa_2(y)\} > 0$$

for $y \in \Gamma$ and $r \in (-\delta, \delta)$ by (3.2). Hence $I_3 - rW(y)$ is invertible. Also, (3.12) follows from (3.9).

Let us prove (3.13) and (3.14). We fix and suppress $y \in \Gamma$. Since W is real and symmetric by Lemma 3.1 and has the eigenvalues κ_1, κ_2, and zero with $Wn = 0$, we can take an orthonormal basis $\{\tau_1, \tau_2, n\}$ of $\mathbb{R}^3$ such that $W\tau_i = \kappa_i \tau_i$, $i = 1, 2$. Then

$$(I_3 - rW)^k \tau_i = (1 - r\kappa_i)^k \tau_i, \quad (I_3 - rW)^k n = n$$

for $r \in (-\delta, \delta)$, $i = 1, 2$, and $k = \pm 1$. By these relations and (3.2), we can easily get (3.13) and (3.14). □

Proof of Lemma 3.4. For $x \in N$ we have $\pi(x) = x - d(x)\bar{n}(\pi(x))$ and $-\nabla \bar{n}(\pi(x)) = \overline{W}(x)$ by (3.1), $n \circ \pi = \bar{n} \circ \pi$ in N, (3.5) with $\pi(x) \in \Gamma$, and $-\nabla_\Gamma n = W$ on Γ. There relations and $\nabla d = \bar{n} \circ \pi$ in N imply

$$\nabla \pi(x) = \overline{P}(x) + d(x)\nabla\pi(x)\overline{W}(x), \quad \text{i.e.} \quad \nabla\pi(x)A(x) = \overline{P}(x),$$

where $A := I_3 - d\overline{W}$ is invertible in N by Lemma 3.3. Hence we have (3.15) by the above equality and (3.12).

Let $\eta \in C^1(\Gamma)$ and $\bar{\eta} = \eta \circ \pi$ be its constant extension. Then

$$\nabla \bar{\eta}(x) = \nabla\pi(x)\nabla\bar{\eta}(\pi(x)) = A(x)^{-1}\overline{P}(x)\overline{\nabla_\Gamma \eta}(\pi(x))$$

by $\bar{\eta}(x) = \bar{\eta}(\pi(x))$ and $\pi(x) \in \Gamma$ for $x \in N$, (3.5), and (3.15). Hence (3.16) follows from the above equality, (3.4), and $\overline{\nabla_\Gamma \eta}(\pi(x)) = \overline{\nabla_\Gamma \eta}(x)$. We also have (3.17) and (3.18) by (3.13), (3.14), and (3.16).

Now let Γ be of class C^3 and $\eta \in C^2(\Gamma)$. For $i = 1, 2, 3$ we differentiate both sides of (3.16) with respect to x_i to get

$$\partial_i \nabla \bar{\eta} = \left(\partial_i A^{-1}\right) \overline{\nabla_\Gamma \eta} + A^{-1}\partial_i\left(\overline{\nabla_\Gamma \eta}\right) \quad \text{in} \quad N. \tag{C.1}$$

To estimate the right-hand side we differentiate $A^{-1}A = I_3$ with respect to x_i and use $A = I_3 + d\overline{W}$ and $\nabla d = \bar{n}$ to get

$$\partial_i A^{-1} = A^{-1}\left(\bar{n}_i \overline{W} + d\partial_i \overline{W}\right)A^{-1} \quad \text{in} \quad N. \tag{C.2}$$

The right-hand side of (C.2) is bounded on N by (3.13), (3.17), and the C^1-regularity of W since Γ is of class C^3. By this fact, (3.13), (3.17), and (C.1), we obtain (3.19). □

Next we assume that Γ is of class C^5 and prove the formulas and inequalities in Section 3.2 for the surface quantities of Γ_ε^0 and Γ_ε^1.

Proof of Lemma 3.8. First note that, since $W \in C^3(\Gamma)^{3\times 3}$ by the C^5-regularity of Γ and $g_0, g_1 \in C^4(\Gamma)$, they are bounded on Γ along with their first and second order tangential derivatives.

Let τ_ε^i and n_ε^i, $i = 0, 1$ be the vector fields on Γ given by (3.26) and (3.27). The first inequalities of (3.28) and (3.29) immediately follow from (3.13) and (3.14). To show the second inequalities, we set

$$R_\varepsilon^i(y) := \{I_3 - \varepsilon g_i(y) W(y)\}^{-1}, \quad y \in \Gamma \tag{C.3}$$

and apply $\underline{D}_k$, $k = 1, 2, 3$ to $R_\varepsilon^i(I_3 - \varepsilon g_i W) = I_3$ on Γ to get

$$\underline{D}_k R_\varepsilon^i = \varepsilon R_\varepsilon^i \{(\underline{D}_k g_i) W + g_i \underline{D}_k W\} R_\varepsilon^i \quad \text{on} \quad \Gamma. \tag{C.4}$$

Hence it follows from (3.13) that

$$|\underline{D}_k R_\varepsilon^i| \le c\varepsilon \quad \text{on} \quad \Gamma \tag{C.5}$$

with a constant $c > 0$ independent of ε. We apply (3.13), (3.14), and (C.5) to $\underline{D}_k \tau_\varepsilon^i = (\underline{D}_k R_\varepsilon^i)\nabla_\Gamma g_i + R_\varepsilon^i(\underline{D}_k \nabla_\Gamma g)$ to get $|\underline{D}_k \tau_\varepsilon^i| \le c$ and

$$|\underline{D}_k \tau_\varepsilon^i - \underline{D}_k \nabla_\Gamma g| \le |(\underline{D}_k R_\varepsilon^i)\nabla_\Gamma g_i| + |(R_\varepsilon^i - I_3)(\underline{D}_k \nabla_\Gamma g)| \le c\varepsilon$$

on Γ for $k = 1, 2, 3$. Hence the second inequalities of (3.28) and (3.29) are valid. We further apply $\underline{D}_l$, $l = 1, 2, 3$ to (C.4) and use (3.13) and (C.5) to get $|\underline{D}_l \underline{D}_k R_\varepsilon^i| \le c\varepsilon$ on Γ. Using this, (3.13), and (C.5) to

$$\begin{aligned}\underline{D}_l \underline{D}_k \tau_\varepsilon^i = (\underline{D}_l \underline{D}_k R_\varepsilon^i)\nabla_\Gamma g_i &+ (\underline{D}_k R_\varepsilon^i)(\underline{D}_l \nabla_\Gamma g_i) \\ &+ (\underline{D}_l R_\varepsilon^i)(\underline{D}_k \nabla_\Gamma g_i) + R_\varepsilon^i(\underline{D}_l \underline{D}_k \nabla_\Gamma g_i),\end{aligned}$$

we obtain the third inequality of (3.28).

Next we show (3.30) and (3.31). We have the first equality of (3.30) by (3.27) and the other inequalities by (3.28). To prove (3.31) let

$$\eta_\varepsilon^i := \frac{1}{\sqrt{1 + \varepsilon^2 |\tau_\varepsilon^i|^2}}, \quad i = 0, 1, \quad \varphi_\varepsilon := \eta_\varepsilon^1 - \eta_\varepsilon^0, \quad \tau_\varepsilon := -\eta_\varepsilon^1 \tau_\varepsilon^1 + \eta_\varepsilon^0 \tau_\varepsilon^0$$

so that $n_\varepsilon^0+n_\varepsilon^1=\varphi_\varepsilon n+\varepsilon\tau_\varepsilon$ on Γ. We deduce from (3.28) that $|\tau_\varepsilon|\leq c$ and $|\nabla_\Gamma\tau_\varepsilon|\leq c$ on Γ with a constant $c>0$ independent of ε. Also,

$$|\varphi_\varepsilon|\leq\frac{\varepsilon^2}{2}\big||\tau_\varepsilon^1|^2-|\tau_\varepsilon^0|^2\big|\leq c\varepsilon^2\quad\text{on}\quad\Gamma$$

by the mean value theorem for $(1+s)^{-1/2}$, $s\geq 0$ and (3.28). Since

$$\nabla_\Gamma\eta_\varepsilon^i=-\frac{\varepsilon^2(\nabla_\Gamma\tau_\varepsilon^i)\tau_\varepsilon^i}{(1+\varepsilon^2|\tau_\varepsilon^i|^2)^{3/2}}\quad\text{on}\quad\Gamma,\ i=0,1,$$

we have $|\nabla_\Gamma\varphi_\varepsilon|\leq c\varepsilon^2$ on Γ by (3.28). Applying the above inequalities to $n_\varepsilon^0+n_\varepsilon^1=\varphi_\varepsilon n+\varepsilon\tau_\varepsilon$ we obtain (3.31). $\square$

Proof of Lemma 3.10. Throughout the proof we write c for a general positive constant independent of ε and denote by $\bar\eta=\eta\circ\pi$ the constant extension of a function η on Γ. First note that $\bar n$, $\overline{P}$, and $\overline{W}$ are bounded on N independently of ε along their first and second order derivatives by (3.17), (3.19), and the C^5-regularity of Γ. In the sequel we use this fact without mention.

For $i=0,1$ let τ_ε^i and n_ε^i be given by (3.26) and (3.27), and

$$\varphi_\varepsilon^i(x):=\frac{1}{\sqrt{1+\varepsilon^2|\bar\tau_\varepsilon^i(x)|^2}}-1,\quad x\in N.$$

By the mean value theorem for $(1+s)^{-1/2}$, $s\geq 0$ and (3.28) we have

$$|\varphi_\varepsilon^i(x)|\leq\frac{\varepsilon^2}{2}|\bar\tau_\varepsilon^i(x)|^2\leq c\varepsilon^2,\quad x\in N. \tag{C.6}$$

We also differentiate φ_ε^i and use (3.17), (3.19), and (3.28) to get

$$|\partial_x^\alpha\varphi_\varepsilon^i(x)|\leq c\varepsilon^2,\quad x\in N,\ |\alpha|=1,2, \tag{C.7}$$

where $\partial_x^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{Z}^3$ with $\alpha_j\geq 0$, $j=1,2,3$. The inequality (3.37) follows from (3.28), (3.29), and (C.6), since

$$(-1)^{i+1}n_\varepsilon-\Big(\bar n-\varepsilon\overline{\nabla_\Gamma g_i}\Big)=\varphi_\varepsilon^i(\bar n-\varepsilon\bar\tau_\varepsilon^i)-\varepsilon\Big(\bar\tau_\varepsilon^i-\overline{\nabla_\Gamma g_i}\Big)$$

on Γ_ε by (3.27) and (3.32). We also have (3.38) by (3.37) and the definitions of P, Q, P_ε, and Q_ε.

Next we prove (3.39). For $x \in N$ we set

$$\Phi_\varepsilon^i(x) := (-1)^{i+1}\left\{\varphi_\varepsilon^i(x)\bar{n}(x) - \frac{\varepsilon\bar{\tau}_\varepsilon^i(x)}{\sqrt{1+\varepsilon^2|\bar{\tau}_\varepsilon^i(x)|^2}}\right\}.$$

Then it follows from (3.17), (3.19), (3.28), (C.6), and (C.7) that

$$|\partial_x^\alpha \Phi_\varepsilon^i(x)| \leq c\varepsilon, \quad x \in N,\ |\alpha| = 0,1,2. \tag{C.8}$$

Since $\bar{n}_\varepsilon^i = (-1)^{i+1}\bar{n} + \Phi_\varepsilon^i$ in N, we see by (3.20) that

$$\nabla\bar{n}_\varepsilon^i = (-1)^i\left(I_3 - d\overline{W}\right)^{-1}\overline{W} + \nabla\Phi_\varepsilon^i \quad \text{in} \quad N.$$

Moreover, since $\bar{n}_\varepsilon^i$ is an extension of $n_\varepsilon|_{\Gamma_\varepsilon^i}$ to N, the Weingarten map of Γ_ε^i is given by $W_\varepsilon = -P_\varepsilon\nabla\bar{n}_\varepsilon^i$ on Γ_ε^i. Thus the above equality yields

$$W_\varepsilon = P_\varepsilon\left\{(-1)^{i+1}\overline{R_\varepsilon^i}\,\overline{W} - \nabla\Phi_\varepsilon^i\right\} \quad \text{on} \quad \Gamma_\varepsilon^i, \tag{C.9}$$

where R_ε^i is given by (C.3), and it follows from (3.9) that

$$\left|W_\varepsilon - (-1)^{i+1}\overline{W}\right| \leq \left|\left(P_\varepsilon - \overline{P}\right)\overline{R_\varepsilon^i}\,\overline{W}\right| + \left|\overline{P}\left(\overline{R_\varepsilon^i} - I_3\right)\overline{W}\right| + |P_\varepsilon\nabla\Phi_\varepsilon^i|$$

on Γ_ε^i. By this inequality, (3.13), (3.14), (3.38), and (C.8), we get the first inequality of (3.39). Also, the second inequality of (3.39) follows from the first one since $H = \mathrm{tr}[W]$ and $H_\varepsilon = \mathrm{tr}[W_\varepsilon]$.

Let us show (3.40). Based on (C.9) we define an extension of $W_\varepsilon|_{\Gamma_\varepsilon^i}$ to N by $\widetilde{W_\varepsilon^i} := \overline{P_\varepsilon^i}\left\{(-1)^{i+1}\overline{R_\varepsilon^i}\,\overline{W} - \nabla\Phi_\varepsilon^i\right\}$ in N, where $P_\varepsilon^i := I_3 - n_\varepsilon^i \otimes n_\varepsilon^i$ on Γ. Let

$$F_{\varepsilon,1}^i := (-1)^{i+1}\left(\overline{P_\varepsilon^i} - \overline{P}\right)\overline{R_\varepsilon^i}\,\overline{W},$$
$$F_{\varepsilon,2}^i := (-1)^{i+1}\overline{P}\left(\overline{R_\varepsilon^i} - I_3\right)\overline{W}, \quad F_{\varepsilon,3}^i := -\overline{P_\varepsilon^i}\nabla\Phi_\varepsilon^i$$

in N so that $\widetilde{W_\varepsilon^i} = (-1)^{i+1}\overline{W} + \sum_{k=1}^3 F_{\varepsilon,k}^i$ in N by (3.9). Then

$$\partial_j\widetilde{W_\varepsilon^i} = \sum_{k=1}^3(-1)^{i+1}\left[\left(I_3 - d\overline{W}\right)^{-1}\right]_{jk}\overline{\underline{D}_kW} + \sum_{k=1}^3\partial_jF_{\varepsilon,k}^i \tag{C.10}$$

in N for $j=1,2,3$ by (3.16). To estimate the last term, we see that

$$\overline{P}^i_\varepsilon - \overline{P} = (-1)^i(\bar{n}\otimes\Phi^i_\varepsilon + \Phi^i_\varepsilon\otimes\bar{n}) - \Phi^i_\varepsilon\otimes\Phi^i_\varepsilon \quad \text{in} \quad N$$

by $\bar{n}^i_\varepsilon = (-1)^{i+1}\bar{n}+\Phi^i_\varepsilon$ in N and the definitions of P and P^i_ε. Hence

$$\left|\overline{P}^i_\varepsilon - \overline{P}\right| \le c\varepsilon, \quad \left|\partial_j\overline{P}^i_\varepsilon - \partial_j\overline{P}\right| \le c\varepsilon \quad \text{in} \quad N$$

by (C.8), and we see by these inequalities, (3.13), (3.14), (3.17), (C.5), and (C.8) that $|\partial_j F^i_{\varepsilon,k}| \le c\varepsilon$ in N. Applying this inequality and (3.14) to (C.10), we get

$$\left|\partial_j\widetilde{W}^i_\varepsilon(x) - (-1)^{i+1}\overline{\underline{D}_jW}(x)\right| \le c(|d(x)|+\varepsilon), \quad x\in N. \tag{C.11}$$

Now we observe that $\underline{D}^\varepsilon_j W_\varepsilon = \sum_{k=1}^3 [P_\varepsilon]_{jk}\partial_k\widetilde{W}^i_\varepsilon$ on Γ^i_ε since $\widetilde{W}^i_\varepsilon$ is an extension of $W_\varepsilon|_{\Gamma^i_\varepsilon}$ to N. From this fact and $\underline{D}_jW=\sum_{k=1}^3 P_{jk}\underline{D}_kW$ on Γ by (3.4), we deduce that

$$\begin{aligned}&\left|\underline{D}^\varepsilon_j W_\varepsilon - (-1)^{i+1}\overline{\underline{D}_jW}\right| \\ &\qquad \le \sum_{k=1}^3\left(\left|\left[P_\varepsilon-\overline{P}\right]_{jk}\partial_k\widetilde{W}^i_\varepsilon\right| + \left|\overline{P}_{jk}\left\{\partial_k\widetilde{W}^i_\varepsilon - (-1)^{i+1}\overline{\underline{D}_kW}\right\}\right|\right)\end{aligned}$$

on Γ^i_ε for $j=1,2,3$. Applying (3.38) and (C.11) with $|d|=\varepsilon|\bar{g}_i|\le c\varepsilon$ on Γ^i_ε to the right-hand side, we obtain (3.40). □

Proof of Lemma 3.11. We fix and suppress the argument y of functions on Γ. Since $\overline{P}(y+\varepsilon g_0 n)=\overline{P}(y+\varepsilon g_1 n)=P$ for the constant extension $\overline{P}=P\circ\pi$ of P, we have

$$|P_\varepsilon(y+\varepsilon g_1 n) - P_\varepsilon(y+\varepsilon g_0 n)| \le \sum_{i=0,1}\left|\left[P_\varepsilon-\overline{P}\right](y+\varepsilon g_i n)\right|.$$

To the right-hand side we apply (3.38) to get (3.41) for $F_\varepsilon = P_\varepsilon$. Using (3.38)–(3.40) we can show the other inequalities in the same way. □

Now let us give the proofs of Lemmas 5.4 and 5.5.

Proof of Lemma 5.4. Let Φ_ε be the mapping given by (5.4), i.e.

$$\Phi_\varepsilon(X) := \pi(X)+\varepsilon d(X)\bar{n}(X), \quad X\in\Omega_1.$$

We see by (3.1) that Φ_ε is a bijection from Ω_1 onto Ω_ε with inverse

$$\Phi_\varepsilon^{-1}(x) := \pi(x) + \varepsilon^{-1}d(x)\bar{n}(x), \quad x \in \Omega_\varepsilon.$$

Moreover, by $\nabla d = \bar{n}$ in N, (3.9), (3.15), and (3.20), we have

$$\nabla\Phi_\varepsilon(X) = \left\{I_3 - d(X)\overline{W}(X)\right\}^{-1}\left\{I_3 - \varepsilon d(X)\overline{W}(X)\right\}\overline{P}(X) + \varepsilon\overline{Q}(X)$$

for $X \in \Omega_1$. Since W has the eigenvalues zero, κ_1, and κ_2 with $Wn = 0$ on Γ, we can take an orthonormal basis $\{\tau_1, \tau_2, \bar{n}(X)\}$ such that

$$\overline{W}(X)\tau_i = \bar{\kappa}_i(X)\tau_i = \kappa_i(\pi(X))\tau_i, \quad \overline{P}(X)\tau_i = \tau_i, \quad \overline{Q}(X)\tau_i = 0$$

for each $X \in \Omega_1$ and $i = 1, 2$. Then for $i = 1, 2$ we have

$$[\nabla\Phi_\varepsilon(X)]\tau_i = \{1 - d(X)\kappa_i(\pi(X))\}^{-1}\{1 - \varepsilon d(X)\kappa_i(\pi(X))\}\tau_i.$$

Also, $[\nabla\Phi_\varepsilon(X)]\bar{n}(X) = \varepsilon\bar{n}(X)$ by $Pn = 0$ and $Qn = n$ on Γ. Thus

$$\det\nabla\Phi_\varepsilon(X) = \varepsilon J(\pi(X), d(X))^{-1}J(\pi(X), \varepsilon d(X)), \quad X \in \Omega_1$$

and we obtain (5.5). Moreover, when $\varphi \in L^2(\Omega_\varepsilon)$,

$$\|\varphi\|_{L^2(\Omega_\varepsilon)}^2 = \varepsilon\int_{\Omega_1}|\xi(X)|^2 J(\pi(X), d(X))^{-1}J(\pi(X), \varepsilon d(X))\,dX$$

with $\xi := \varphi \circ \Phi_\varepsilon$ on Ω_1 by (5.5) and thus (5.6) follows from (3.46).

Let $\varphi \in H^1(\Omega_\varepsilon)$. Then the right-hand inequality of (5.6) yields

$$\varepsilon^{-1}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}^2 \geq c\|(\nabla\varphi)\circ\Phi_\varepsilon\|_{L^2(\Omega_1)}^2. \tag{C.12}$$

To estimate the right-hand side we observe that

$$\nabla\Phi_\varepsilon^{-1}(x) = \left\{I_3 - d(x)\overline{W}(x)\right\}^{-1}\left\{I_3 - \varepsilon^{-1}d(x)\overline{W}(x)\right\}\overline{P}(x) + \varepsilon^{-1}\overline{Q}(x)$$

for $x \in \Omega_\varepsilon$ by $\nabla d = \bar{n}$ in N, (3.9), (3.15), and (3.20). In this equality, we set $x = \Phi_\varepsilon(X)$ with $X \in \Omega_1$. Then it follows from $d(\Phi_\varepsilon(X)) = \varepsilon d(X)$ and $\pi(\Phi_\varepsilon(X)) = \pi(X)$ that

$$\begin{aligned} &\nabla\Phi_\varepsilon^{-1}(\Phi_\varepsilon(X)) = \Lambda_\varepsilon(X)\overline{P}(X) + \varepsilon^{-1}\overline{Q}(X), \quad X \in \Omega_1, \\ &\Lambda_\varepsilon(X) := \left\{I_3 - \varepsilon d(X)\overline{W}(X)\right\}^{-1}\left\{I_3 - d(X)\overline{W}(X)\right\}. \end{aligned} \tag{C.13}$$

Let $\xi = \varphi \circ \Phi_\varepsilon$ on Ω_1. In what follows, we carry out calculations in Ω_1 and omit the argument $X \in \Omega_1$. We have

$$(\nabla\varphi) \circ \Phi_\varepsilon = [(\nabla\Phi_\varepsilon^{-1}) \circ \Phi_\varepsilon]\nabla\xi = \overline{P}\Lambda_\varepsilon\nabla\xi + \varepsilon^{-1}(\partial_n\xi)\bar{n}$$

by (C.13), $\overline{Q}\nabla\xi = (\partial_n\xi)\bar{n}$, and $\Lambda_\varepsilon\overline{P} = \overline{P}\Lambda_\varepsilon$, which follows from (3.9) and (3.12). Then we see by $\overline{P}\Lambda_\varepsilon = \Lambda_\varepsilon\overline{P}$ and (3.13) that

$$|(\nabla\varphi) \circ \Phi_\varepsilon|^2 = \left|\overline{P}\Lambda_\varepsilon\nabla\xi\right|^2 + \varepsilon^{-2}|(\partial_n\xi)\bar{n}|^2 \geq c\left|\overline{P}\nabla\xi\right|^2 + \varepsilon^{-2}|\partial_n\xi|^2.$$

By this inequality and (C.12) we get (5.7). Also, $\xi \in H^1(\Omega_1)$ by (5.6) and (5.7) since $|\nabla\xi|^2 = \left|\overline{P}\nabla\xi\right|^2 + \left|\overline{Q}\nabla\xi\right|^2$ and $\left|\overline{Q}\nabla\xi\right| = |\partial_n\xi|$. □

To prove Lemma 5.5 we present an auxiliary result.

LEMMA C.1. *Let $\tau_1, \tau_2, n_0 \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3\times3}$ satisfy*

$$|n_0| = 1, \quad n_0 \cdot \tau_1 = n_0 \cdot \tau_2 = 0, \quad An_0 = A^T n_0 = 0. \tag{C.14}$$

Then for $B := A + \tau_1 \otimes n_0 + n_0 \otimes \tau_2 + cn_0 \otimes n_0$ with $c \in \mathbb{R}$ we have

$$|B|^2 = |A|^2 + |\tau_1|^2 + |\tau_2|^2 + |c|^2. \tag{C.15}$$

PROOF. By direct calculations with $|n_0| = 1$, $n_0 \cdot \tau_1 = 0$, and $A^T n_0 = 0$,

$$\begin{aligned} B^T B = A^T A + \tau_2 \otimes \tau_2 + (|\tau_1|^2 + |c|^2) n_0 \otimes n_0 \\ + (A^T\tau_1) \otimes n_0 + n_0 \otimes (A^T\tau_1) + c(\tau_2 \otimes n_0 + n_0 \otimes \tau_2). \end{aligned}$$

Hence $|B|^2 = \mathrm{tr}[B^T B]$ is of the form

$$|B|^2 = |A|^2 + |\tau_2|^2 + (|\tau_1|^2 + |c|^2)|n_0|^2 + 2(A^T\tau_1) \cdot n_0 + 2c(\tau_2 \cdot n_0).$$

Since $|n_0| = 1$, $\tau_2 \cdot n_0 = 0$, and $(A^T\tau_1) \cdot n_0 = \tau_1 \cdot (An_0) = 0$ by $An_0 = 0$, we conclude by the above equality that (C.15) is valid. □

PROOF OF LEMMA 5.5. Let Φ_ε be the bijection from Ω_1 onto Ω_ε given by (5.4). For $u \in H^1(\Omega_\varepsilon)^3$ we see by Lemma 5.4 that $U := u \circ \Phi_\varepsilon \in H^1(\Omega_1)^3$ and (5.7) holds with φ and ξ replaced by u and U.

Let us show (5.8). We carry out calculations in Ω_1 and suppress the argument $X \in \Omega_1$ unless otherwise stated. Noting that

$$\overline{Q}\nabla U = \bar{n} \otimes [(\nabla U)^T \bar{n}], \quad (\nabla U)^T \bar{n} = (\bar{n} \cdot \nabla) U = \partial_n U,$$

we observe by (C.13) and $\Lambda_\varepsilon \overline{P} = \overline{P} \Lambda_\varepsilon$ that

$$(\nabla u) \circ \Phi_\varepsilon = \overline{P} F_\varepsilon(U) + \varepsilon^{-1} \bar{n} \otimes \partial_n U, \quad F_\varepsilon(U) := \Lambda_\varepsilon \nabla U.$$

Moreover, by $I_3 = P + Q$ on Γ and $\partial_n \bar{\eta} = 0$ in Ω_1 with $\eta = n, P$,

$$\begin{aligned} \overline{P} F_\varepsilon(U) &= \overline{P} F_\varepsilon(U) \overline{P} + \overline{P} F_\varepsilon(U) \overline{Q} = \overline{P} F_\varepsilon(U) \overline{P} + \left[\overline{P} F_\varepsilon(U) \bar{n}\right] \otimes \bar{n}, \\ \partial_n U &= \partial_n \left[\overline{P} U + (U \cdot \bar{n}) \bar{n}\right] = \overline{P} \partial_n U + \{\partial_n (U \cdot \bar{n})\} \bar{n}. \end{aligned}$$

By these relations and $D(u) = (\nabla u)_S = \{\nabla u + (\nabla u)^T\}/2$, we get

$$\begin{gathered} D(u) \circ \Phi_\varepsilon = A + \tau_1 \otimes n_0 + n_0 \otimes \tau_2 + \varepsilon^{-1} \{\partial_n (U \cdot \bar{n})\} n_0 \otimes n_0, \\ A := \overline{P} F_\varepsilon(U)_S \overline{P}, \quad \tau_1 = \tau_2 := \frac{1}{2} \overline{P} \{F_\varepsilon(U) \bar{n} + \varepsilon^{-1} \partial_n U\}, \quad n_0 := \bar{n}. \end{gathered}$$

Moreover, since $Pn = P^T n = 0$ on Γ, we see that A, τ_1, τ_2, and n_0 satisfy (C.14). Hence we can apply (C.15) to $B = D(u) \circ \Phi_\varepsilon$ to get

$$|D(u) \circ \Phi_\varepsilon|^2 \geq |A|^2 + \varepsilon^{-2} |\partial_n (U \cdot \bar{n})|^2$$

in Ω_1. We deduce from this inequality and (5.6) that

$$\varepsilon^{-1} \|D(u)\|_{L^2(\Omega_\varepsilon)}^2 \geq c \left(\|A\|_{L^2(\Omega_1)}^2 + \varepsilon^{-2} \|\partial_n (U \cdot \bar{n})\|_{L^2(\Omega_1)}^2 \right).$$

Since $A = \overline{P} F_\varepsilon(U)_S \overline{P}$ and $F_\varepsilon(U) = \Lambda_\varepsilon \nabla U$ is of the form (5.9), we have (5.8) by the above inequality. □

Finally, we prove Lemma 7.2 for the vector field $G(u)$ given by (7.6).

Proof of Lemma 7.2. For a function η on Γ we denote by $\bar{\eta} = \eta \circ \pi$ its constant extension in the normal direction of Γ. Let n_ε^0 and n_ε^1 be the vector fields on Γ given by (3.27) and W_ε^0, W_ε^1, $\tilde{n}_1$, $\tilde{n}_2$, and $\widetilde{W}$ the functions on N given by (7.3) and (7.4). We observe by (3.17), (3.19), (3.30), and $g_0, g_1 \in C^4(\Gamma)$ that

$$|\partial_x^\alpha \bar{n}_\varepsilon^i(x)| \leq c, \quad |\partial_x^\alpha \bar{g}_i(x)| \leq c, \quad x \in N,\, i = 0,1,\, |\alpha| = 0,1,2, \tag{C.16}$$

where $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T \in \mathbb{Z}^3$ with $\alpha_j \geq 0$, $j = 1,2,3$ and $c > 0$ is a constant independent of ε. Also,

$$|W_\varepsilon^i(x)| \leq c, \quad |\partial_k W_\varepsilon^i(x)| \leq c, \quad x \in N,\, i = 0,1,\, k = 1,2,3 \tag{C.17}$$

by (C.16). It follows from (2.6), (C.16), (C.17) and

$$0 \leq d(x) - \varepsilon \bar{g}_0(x) \leq \varepsilon \bar{g}(x), \quad 0 \leq \varepsilon \bar{g}_1(x) - d(x) \leq \varepsilon \bar{g}(x) \tag{C.18}$$

for $x \in \Omega_\varepsilon$ that

$$|\tilde{n}_1| \leq c, \quad |\tilde{n}_2| \leq c\varepsilon, \quad \left|\widetilde{W}\right| \leq c \quad \text{in} \quad \Omega_\varepsilon. \tag{C.19}$$

Applying (C.19) to (7.6) we obtain the first inequality of (7.7).

To prove the second inequality of (7.7), we estimate the first order derivatives of $\tilde{n}_1$, $\tilde{n}_2$, and $\widetilde{W}$. We differentiate $\tilde{n}_1$ to get

$$\nabla \tilde{n}_1 = \frac{1}{\varepsilon \bar{g}} \bar{n} \otimes (\bar{n}_\varepsilon^0 + \bar{n}_\varepsilon^1) + A_1 \quad \text{in} \quad N$$

by $\nabla d = \bar{n}$, where A_1 is a 3×3 matrix-valued function given by

$$\begin{aligned} A_1 := -\frac{\nabla \bar{g}}{\bar{g}} \otimes \tilde{n}_1 - \frac{1}{\bar{g}}(\nabla \bar{g}_0 \otimes \bar{n}_\varepsilon^1 + \nabla \bar{g}_1 \otimes \bar{n}_\varepsilon^0) \\ + \frac{1}{\varepsilon \bar{g}}\{(d - \varepsilon \bar{g}_0)\nabla \bar{n}_\varepsilon^1 - (\varepsilon \bar{g}_1 - d)\nabla \bar{n}_\varepsilon^0\}. \end{aligned}$$

We observe by (2.1), (C.16), (C.18), and (C.19) that A_1 is bounded on Ω_ε uniformly in ε. By this fact, (2.1), and (3.31), we have

$$|\nabla \tilde{n}_1| \leq \frac{1}{\varepsilon \bar{g}} |\bar{n}_\varepsilon^0 + \bar{n}_\varepsilon^1| + |A_1| \leq c \quad \text{in} \quad \Omega_\varepsilon. \tag{C.20}$$

Similarly, by $\nabla d = \bar{n}$ in N, (2.1), and (C.16)–(C.19),

$$\begin{aligned} \nabla \tilde{n}_2 &= \frac{1}{\varepsilon \bar{g}} \bar{n} \otimes \left(\frac{\gamma_\varepsilon^1}{\nu} \bar{n}_\varepsilon^1 - \frac{\gamma_\varepsilon^0}{\nu} \bar{n}_\varepsilon^0\right) + A_2, \\ \partial_k \widetilde{W} &= \frac{1}{\varepsilon \bar{g}} \bar{n}_k (W_\varepsilon^0 + W_\varepsilon^1) + B_k, \quad k = 1,2,3 \end{aligned}$$

in N, where A_2 and B_k are the matrix-valued functions bounded on Ω_ε uniformly in ε. Thus we use (2.1), (2.6), and (3.30) to $\nabla\tilde{n}_2$ to get

$$|\nabla\tilde{n}_2| \leq \frac{c(\gamma_\varepsilon^0+\gamma_\varepsilon^1)}{\varepsilon\bar{g}} + |A_2| \leq c \quad \text{in} \quad \Omega_\varepsilon. \tag{C.21}$$

Also, we see by (3.16) that

$$\begin{aligned} W_\varepsilon^0 + W_\varepsilon^1 = \{(\bar{n}_\varepsilon^0+\bar{n}_\varepsilon^1)\otimes\bar{n}_\varepsilon^0 - \bar{n}_\varepsilon^1\otimes(\bar{n}_\varepsilon^0+\bar{n}_\varepsilon^1)\}\Bigl(I_3 - d\overline{W}\Bigr)^{-1}\overline{\nabla_\Gamma n_\varepsilon^0} \\ - (I_3 - \bar{n}_\varepsilon^1\otimes\bar{n}_\varepsilon^1)\Bigl(I_3 - d\overline{W}\Bigr)^{-1}\Bigl(\overline{\nabla_\Gamma n_\varepsilon^0} + \overline{\nabla_\Gamma n_\varepsilon^1}\Bigr) \end{aligned}$$

and thus $|W_\varepsilon^0 + W_\varepsilon^1| \leq c\varepsilon$ in N by (3.13), (3.30), and (3.31). Hence

$$\Bigl|\partial_k\widetilde{W}\Bigr| \leq \frac{1}{\varepsilon\bar{g}}|W_\varepsilon^0 + W_\varepsilon^1| + |B_k| \leq c \quad \text{in} \quad \Omega_\varepsilon,\ k=1,2,3. \tag{C.22}$$

Here we also used (2.1) and the uniform in ε boundedness of B_k on Ω_ε in the second inequality. Noting that $G(u)$ is given by (7.6), we apply (C.19)–(C.22) to $\nabla G(u)$ to obtain the second inequality of (7.7). $\square$

Appendix D. Formulas for the Covariant Derivatives

In this appendix we present formulas for the covariant derivatives of tangential vector fields on an embedded surface in $\mathbb{R}^3$ used in the proof of Theorem 6.1.

Let Γ be a closed, connected, and oriented surface in $\mathbb{R}^3$ of class C^3. We use the notations given in Section 3.1. For $X \in C^1(\Gamma, T\Gamma)$ and $Y \in C(\Gamma, T\Gamma)$ we define the covariant derivative of X along Y by

$$\overline{\nabla}_Y X := P(Y\cdot\nabla)\widetilde{X} \quad \text{on} \quad \Gamma, \tag{D.1}$$

where $\widetilde{X}$ is a C^1-extension of X to an open neighborhood of Γ with $\widetilde{X}|_\Gamma = X$. Since Y is tangential on Γ, we have $(Y\cdot\nabla)\widetilde{X} = (Y\cdot\nabla_\Gamma)X$ on Γ by (3.8). Thus the value of $\overline{\nabla}_Y X$ does not depend on the choice of an extension of X.

Lemma D.1. *For $X \in C^1(\Gamma, T\Gamma)$ and $Y \in C(\Gamma, T\Gamma)$ we have*

$$(Y\cdot\nabla)\widetilde{X} = (Y\cdot\nabla_\Gamma)X = \overline{\nabla}_Y X + (WX\cdot Y)n \quad \textit{on} \quad \Gamma, \tag{D.2}$$

where $\widetilde{X}$ *is any* C^1*-extension of* X *to an open neighborhood of* Γ*.*

PROOF. We have $(Y\cdot\nabla_\Gamma)X\cdot n = WX\cdot Y$ on Γ by applying $Y\cdot\nabla_\Gamma$ to $X\cdot n=0$. By this relation and (D.1) we get (D.2). □

The formula (D.2) is called the Gauss formula (see e.g. [7, 36]). Let us give fundamental relations of the covariant derivative.

LEMMA D.2. *The following equalities hold on* Γ*:*

- *For* $X\in C^1(\Gamma,T\Gamma)$*,* $Y,Z\in C(\Gamma,T\Gamma)$*, and* $\eta,\xi\in C(\Gamma)$*,*

$$\overline{\nabla}_{\eta Y+\xi Z}X = \eta\overline{\nabla}_Y X + \xi\overline{\nabla}_Z X. \tag{D.3}$$

- *For* $X\in C^1(\Gamma,T\Gamma)$*,* $Y\in C(\Gamma,T\Gamma)$*, and* $\eta\in C^1(\Gamma)$*,*

$$\overline{\nabla}_Y(\eta X) = (Y\cdot\nabla_\Gamma\eta)X + \eta\overline{\nabla}_Y X. \tag{D.4}$$

- *For* $X,Y\in C^1(\Gamma,T\Gamma)$ *and* $Z\in C(\Gamma,T\Gamma)$*,*

$$Z\cdot\nabla_\Gamma(X\cdot Y) = \overline{\nabla}_Z X\cdot Y + X\cdot\overline{\nabla}_Z Y. \tag{D.5}$$

- *For* $X,Y\in C^1(\Gamma,T\Gamma)$ *and* $\eta\in C^2(\Gamma)$*,*

$$X\cdot\nabla_\Gamma(Y\cdot\nabla_\Gamma\eta) - Y\cdot\nabla_\Gamma(X\cdot\nabla_\Gamma\eta) = \left(\overline{\nabla}_X Y - \overline{\nabla}_Y X\right)\cdot\nabla_\Gamma\eta. \tag{D.6}$$

PROOF. We easily get (D.3)–(D.5) by (D.1) and (D.2). Also, writing the left-hand side of (D.6) as the sum of $\{(X\cdot\nabla_\Gamma)Y-(Y\cdot\nabla_\Gamma)X\}\cdot\nabla_\Gamma\eta$ and $\sum_{i,j=1}^3 X_iY_j(\underline{D}_i\underline{D}_j\eta-\underline{D}_j\underline{D}_i\eta)$, and using (3.11), (D.2), and $Z\cdot n=0$ on Γ for $Z=\nabla_\Gamma\eta, X, Y$, we can show (D.6). □

Lemma D.1 shows that $\overline{\nabla}$ is the Riemannian (or Levi-Civita) connection on Γ (see e.g. [7, 36]). Note that (D.6) stands for the torsion-free condition $[X,Y]=\overline{\nabla}_X Y-\overline{\nabla}_Y X$, where $[X,Y]:=XY-YX$ is the Lie bracket of X and Y.

Let O be a relatively open subset of Γ. If O is sufficiently small, then by the C^3-regularity of Γ we can take C^2 vector fields τ_1 and τ_2 on O such that $\{\tau_1(y), \tau_2(y)\}$ is an orthonormal basis of the tangent plane of Γ at each $y \in \Gamma$. We call the pair $\{\tau_1, \tau_2\}$ of such vector fields a local orthonormal frame for the tangent bundle of Γ on O, or simply a local orthonormal frame on O. Note that

$$H = \mathrm{tr}[W] = W\tau_1 \cdot \tau_1 + W\tau_2 \cdot \tau_2 \quad \text{on} \quad O \tag{D.7}$$

since $\{\tau_1, \tau_2, n\}$ is an orthonormal basis of $\mathbb{R}^3$ and $Wn = 0$ on Γ. We express several quantities related to the tangential gradient matrix of tangential vector fields on Γ in terms of the covariant derivatives and the local orthonormal frame.

LEMMA D.3. *Let $\{\tau_1, \tau_2\}$ be a local orthonormal frame for the tangent bundle of Γ on a relatively open subset O of Γ. For $X, Y \in C^1(\Gamma, T\Gamma)$ we have*

$$\mathrm{div}_\Gamma X = \sum_{i=1,2} \overline{\nabla}_i X \cdot \tau_i, \tag{D.8}$$

$$\nabla_\Gamma X : W = \sum_{i=1,2} \overline{\nabla}_i X \cdot W\tau_i = \sum_{i=1,2} W\overline{\nabla}_i X \cdot \tau_i, \tag{D.9}$$

$$\nabla_\Gamma X : (\nabla_\Gamma Y)P = \sum_{i=1,2} \overline{\nabla}_i X \cdot \overline{\nabla}_i Y, \tag{D.10}$$

$$W\nabla_\Gamma X : (\nabla_\Gamma Y)P = \sum_{i=1,2} \overline{\nabla}_{W\tau_i} X \cdot \overline{\nabla}_i Y \tag{D.11}$$

on O, where $\overline{\nabla}_i := \overline{\nabla}_{\tau_i}$ for $i = 1, 2$.

PROOF. We carry out calculations on O. By (3.4) and (D.2) we have

$$\begin{aligned} &(\nabla_\Gamma X)^T \tau_i = (\tau_i \cdot \nabla_\Gamma)X = \overline{\nabla}_i X + (WX \cdot \tau_i)n, \quad i = 1, 2, \\ &(\nabla_\Gamma X)^T n = (n \cdot \nabla_\Gamma)X = 0. \end{aligned} \tag{D.12}$$

Since $\{\tau_1, \tau_2, n\}$ forms an orthonormal basis of $\mathbb{R}^3$,

$$\mathrm{div}_\Gamma X = \mathrm{tr}[\nabla_\Gamma X] = \sum_{i=1,2} (\nabla_\Gamma X)^T \tau_i \cdot \tau_i + (\nabla_\Gamma X)^T n \cdot n.$$

The equality (D.8) follows from this equality and (D.12). We also have (D.9) by applying (D.12), $W^T = W$, and $Wn = 0$ to

$$\nabla_\Gamma X : W = (\nabla_\Gamma X)^T : W^T = \sum_{i=1,2} (\nabla_\Gamma X)^T \tau_i \cdot W^T \tau_i + (\nabla_\Gamma X)^T n \cdot W^T n.$$

Next we observe by (D.12), $P^T = P$, $Pn = 0$, and $P\overline{\nabla}_i Y = \overline{\nabla}_i Y$ that

$$[(\nabla_\Gamma Y)P]^T \tau_i = P[(\nabla_\Gamma Y)^T \tau_i] = P\Big\{\overline{\nabla}_i Y + (WY \cdot \tau_i)n\Big\} = \overline{\nabla}_i Y$$

for $i = 1, 2$. We apply this equality, (D.12), and $\overline{\nabla}_i Y \cdot n = 0$ to

$$\begin{aligned} \nabla_\Gamma X : (\nabla_\Gamma Y)P &= (\nabla_\Gamma X)^T : [(\nabla_\Gamma Y)P]^T \\ &= \sum_{i=1,2} (\nabla_\Gamma X)^T \tau_i \cdot [(\nabla_\Gamma Y)P]^T \tau_i + (\nabla_\Gamma X)^T n \cdot [(\nabla_\Gamma Y)P]^T n \end{aligned}$$

to find that (D.10) holds. Similarly, we can prove (D.11) if we use

$$[W(\nabla_\Gamma X)]^T \tau_i = (W\tau_i \cdot \nabla_\Gamma)X = \overline{\nabla}_{W\tau_i} X + (WX \cdot W\tau_i)n$$

and $[W(\nabla_\Gamma X)]^T n = 0$ by $W^T = W$, $Wn = 0$, and (D.2). □

We also give an integration by parts formula for the covariant derivatives along vector fields of a local orthonormal frame.

Lemma D.4. *Let $\{\tau_1, \tau_2\}$ be a local orthonormal frame for the tangent bundle of Γ on a relatively open subset O of Γ and $\overline{\nabla}_i := \overline{\nabla}_{\tau_i}$ for $i = 1, 2$. Suppose that $X \in C^2(\Gamma, T\Gamma)$ and $Y \in C^1(\Gamma, T\Gamma)$ are compactly supported in O. Then we have*

$$\text{(D.13)} \quad \sum_{i=1,2} \int_\Gamma \Big(\overline{\nabla}_i \overline{\nabla}_i X - \overline{\nabla}_{\overline{\nabla}_i \tau_i} X\Big) \cdot Y \, d\mathcal{H}^2 = -\sum_{i=1,2} \int_\Gamma \overline{\nabla}_i X \cdot \overline{\nabla}_i Y \, d\mathcal{H}^2.$$

Proof. The proof is the same as that of [55, Proposition 34], so we omit it (see also the arXiv version of this paper [46] for details). □

Remark D.5. Since $C^2(\Gamma, T\Gamma)$ is dense in $H^m(\Gamma, T\Gamma)$ for $m = 0, 1, 2$ by Lemma 3.7 and the C^3-regularity of Γ, the formulas given in this appendix are also valid if we replace $C^m(\Gamma, T\Gamma)$ with $H^m(\Gamma, T\Gamma)$.

Appendix E. Infinitesimal Rigid Displacements on a Closed Surface

In this appendix we show several results on infinitesimal rigid displacements of $\mathbb{R}^3$ related to the axial symmetry of a closed surface and a curved thin domain.

Let Γ be a C^2 closed, connected, and oriented surface in $\mathbb{R}^3$ and $\mathcal{R}$ the set of the form (2.3) which consists of infinitesimal rigid displacements of $\mathbb{R}^3$ with tangential restrictions on Γ.

Lemma E.1. *Let $w(x) = a \times x + b \in \mathcal{R}$. If $w \not\equiv 0$, then $a \neq 0$, $a \cdot b = 0$, and Γ is axially symmetric around the line parallel to the vector a and passing through the point $b_a := |a|^{-2}(a \times b)$. Conversely, if Γ is axially symmetric around the line parallel to $a \neq 0$ and passing through $\tilde{b} \in \mathbb{R}^3$, then $\tilde{w}(x) = a \times (x - \tilde{b}) \in \mathcal{R} \setminus \{0\}$.*

This result is well known, so here we give the outline of the proof. See the arXiv version of this paper [46] for details.

Proof. Suppose first that $w(x) = a \times x + b \in \mathcal{R}$ and $w \not\equiv 0$. If $a = 0$, then $b \cdot n = w \cdot n = 0$ on Γ and thus $b = 0$ since Γ is closed. Hence $a \neq 0$ if $w \not\equiv 0$. Also, the flow map $x(\cdot, t)\colon \mathbb{R}^3 \to \mathbb{R}^3$ generated by $w \not\equiv 0$, i.e. a solution to $(\partial x/\partial t)(X, t) = w(x(X, t))$ with $x(X, 0) = X \in \mathbb{R}^3$ is given by the combination of a translation by $t(b \cdot E_3)E_3$ with $E_3 := a/|a|$ and a rotation through the angle $|a|t$ around the axis parallel to E_3 and passing through b_a. Thus the relation $a \cdot b = 0$ and the axial symmetry of Γ follow from the compactness of Γ and the fact that Γ is preserved under the action of $x(\cdot, t)$ if $w \in \mathcal{R}$. Similarly, we can show the converse statement for $\tilde{w}(x) = a \times (x - \tilde{b})$. □

Lemma E.2. *Let $w(x) = a \times x + b \in \mathcal{R}$ satisfy $w \not\equiv 0$. Then*

$$W(y)w(y) = \lambda(y)w(y), \quad a \times n(y) = -\lambda(y)w(y), \quad y \in \Gamma \tag{E.1}$$

with some $\lambda(y) \in \mathbb{R}$. Here $W = -\nabla_\Gamma n$ is the Weingarten map of Γ.

Proof. Since $w(x) = a \times x + b \in \mathcal{R}$ and $w \not\equiv 0$, Lemma E.1 implies that $a \neq 0$, $a \cdot b = 0$, and Γ is axially symmetric around the line parallel to a

and passing through $b_a = |a|^{-2}(a \times b)$. Also, since $a \times b_a = -b$ by $a \cdot b = 0$, we have $w(x) = a \times (x - b_a)$. Hence, by a translation along b_a and a rotation of coordinates, we may assume that Γ is axially symmetric around the x_3-axis and $w(x) = \alpha(e_3 \times x)$, where $\alpha = |a| > 0$ and $e_3 = (0,0,1)^T$. We may further assume $\alpha = 1$, i.e. $a = e_3$ and $w(x) = e_3 \times x$ by replacing w with $\alpha^{-1}w$. Under these assumptions, Γ is represented as a surface of revolution

$$\begin{aligned} \Gamma &= \{\mu(s,\vartheta) \mid s \in [0,L],\ \vartheta \in [0,2\pi]\}, \\ \mu(s,\vartheta) &= (\varphi(s)\cos\vartheta, \varphi(s)\sin\vartheta, \psi(s)). \end{aligned} \tag{E.2}$$

Here $\gamma(s) = (\varphi(s), 0, \psi(s))$ is a C^2 curve parametrized by the arc length $s \in [0,L]$, $L > 0$ such that $\varphi(s) > 0$ for $s \neq 0, L$. We may further assume that for $s = 0, L$ if $\varphi(s) = 0$ then $\psi'(s) = 0$, otherwise Γ is not of class C^2 at the point

$$\mu(s,\vartheta) = (\varphi(s)\cos\vartheta, \varphi(s)\sin\vartheta, \psi(s)) = (0,0,\psi(s)), \quad \vartheta \in [0,2\pi].$$

By the arc length parametrization of γ we have

$$\{\varphi'(s)\}^2 + \{\psi'(s)\}^2 = 1, \quad s \in [0,L]. \tag{E.3}$$

Let $y = \mu(s,\vartheta) \in \Gamma$. Hereafter we suppress the arguments of μ and its derivatives. We deduce from (E.3) and

$$\partial_s\mu = \begin{pmatrix} \varphi'(s)\cos\vartheta \\ \varphi'(s)\sin\vartheta \\ \psi'(s) \end{pmatrix}, \quad \partial_\vartheta\mu = \begin{pmatrix} -\varphi(s)\sin\vartheta \\ \varphi(s)\cos\vartheta \\ 0 \end{pmatrix} = w(y), \tag{E.4}$$

where the last equality is due to $w(y) = e_3 \times y$, that

$$\partial_s\mu \times \partial_\vartheta\mu = \varphi(s) \begin{pmatrix} -\psi'(s)\cos\vartheta \\ -\psi'(s)\sin\vartheta \\ \varphi'(s) \end{pmatrix}, \quad |\partial_s\mu \times \partial_\vartheta\mu| = \varphi(s).$$

Suppose that $\varphi(s) > 0$. Without loss of generality, we may assume that the direction of $\partial_s\mu \times \partial_\vartheta\mu$ is the same as that of $n(y)$. Then

$$n(y) = n(\mu(s,\vartheta)) = \frac{\partial_s\mu \times \partial_\vartheta\mu}{|\partial_s\mu \times \partial_\vartheta\mu|} = \begin{pmatrix} -\psi'(s)\cos\vartheta \\ -\psi'(s)\sin\vartheta \\ \varphi'(s) \end{pmatrix} \tag{E.5}$$

and we differentiate both sides with respect to ϑ and use $-\nabla_\Gamma n = W = W^T$ and (E.4) to get $W(y)w(y) = \lambda(y)w(y)$ with $\lambda(y) = \psi'(s)/\varphi(s)$. We also have $e_3 \times n(y) = -\lambda(y)w(y)$ by (E.4) and (E.5). Hence (E.1) is valid when $\varphi(s) > 0$ (note that we assume $a = e_3$).

Now for $s = 0, L$ suppose that $\varphi(s) = 0$. Then $\psi'(s) = 0$ by our assumption and thus the tangent plane of Γ at the point

$$y = \mu(s,\vartheta) = (\varphi(s)\cos\vartheta, \varphi(s)\sin\vartheta, \psi(s)) = (0, 0, \psi(s)), \quad \vartheta \in [0, 2\pi]$$

is orthogonal to the x_3-axis. Hence $n(y) = \pm e_3$ and $a \times n(y) = 0$ by the assumption $a = e_3$. Moreover, $w(y) = 0$ by (E.4) and $\varphi(s) = 0$. By these facts we conclude that (E.1) holds for any $\lambda(y) \in \mathbb{R}$. $\Box$

LEMMA E.3. *If Γ is of class C^5 and $\mathcal{R} \neq \{0\}$, then $\mathcal{K}(\Gamma) = \mathcal{R}|_\Gamma$. Here $\mathcal{K}(\Gamma)$ is the space of Killing vector fields on Γ given by* (2.5).

PROOF. For $w(x) = a \times x + b$ we have $\nabla w + (\nabla w)^T = 0$ in $\mathbb{R}^3$ and thus $D_\Gamma(w) = 0$ on Γ by $\nabla_\Gamma w = P\nabla w$ and $P^2 = P$. Thus, if w is tangential on Γ, then $w \in \mathcal{K}(\Gamma)$, i.e. $\mathcal{R}|_\Gamma \subset \mathcal{K}(\Gamma)$.

Suppose that Γ is a sphere in $\mathbb{R}^3$. By a translation we may assume that Γ is centered at the origin. Then $\mathcal{R}|_\Gamma = \{w(y) = a \times y,\, y \in \Gamma \mid a \in \mathbb{R}^3\}$ is a three-dimensional subspace of $\mathcal{K}(\Gamma)$, while the dimension of $\mathcal{K}(\Gamma)$ is at most three (see [55, Theorem 35]). Thus $\mathcal{K}(\Gamma) = \mathcal{R}|_\Gamma$.

Next suppose that Γ is not a sphere. Since Γ is axially symmetric by $\mathcal{R} \neq \{0\}$ and Lemma E.1, as in the proof of Lemma E.2 we may assume that Γ is axially symmetric around the x_3-axis, i.e.

$$\{w(y) = c(e_3 \times y),\, y \in \Gamma \mid c \in \mathbb{R}\} \subset \mathcal{R}|_\Gamma, \quad e_3 = (0,0,1)^T. \tag{E.6}$$

We may further assume that Γ is a surface of revolution of the form (E.2) with C^5 functions φ and ψ satisfying (E.3) and $\varphi(s) > 0$ for $s \neq 0, L$. Then the Gaussian curvature of Γ is given by

$$K(\mu(s,\vartheta)) = -\frac{\varphi''(s)}{\varphi(s)}, \quad s \in (0, L),\, \vartheta \in [0, 2\pi], \tag{E.7}$$

see e.g. [54, Section 5.7]. We use this formula later. Also,

$$\varphi'(s)\varphi''(s) + \psi'(s)\psi''(s) = 0, \quad s \in (0, L) \tag{E.8}$$

by (E.3). Let $X \in \mathcal{K}(\Gamma)$ be of the form

$$X(\mu(s,\vartheta)) = X^s(s,\vartheta)\partial_s\mu(s,\vartheta) + X^\vartheta(s,\vartheta)\partial_\vartheta\mu(s,\vartheta)$$

for $s \in [0,L]$ and $\vartheta \in [0,2\pi]$. Note that $X \in C^2(\Gamma, T\Gamma)$ by Lemma B.8, since Γ is of class C^5. Also, for all $Y, Z \in C(\Gamma, T\Gamma)$ we have

$$(Y\cdot\nabla_\Gamma)X\cdot Z + Y\cdot(Z\cdot\nabla_\Gamma)X = 2D_\Gamma(X)Y\cdot Z = 0 \quad \text{on} \quad \Gamma$$

by $PY = Y$, $PZ = Z$, and $D_\Gamma(X) = 0$ on Γ. Noting that

$$(\partial_s\mu\cdot\nabla_\Gamma)X = (\partial_s X^s)\partial_s\mu + X^s\partial_s^2\mu + (\partial_s X^\vartheta)\partial_\vartheta\mu + X^\vartheta\partial_s\partial_\vartheta\mu$$

by $(\partial_s\mu\cdot\nabla_\Gamma)X = \partial_s(X\circ\mu)$ and a similar relation holds for $(\partial_\vartheta\mu\cdot\nabla_\Gamma)X$, we substitute $\partial_s\mu$ and $\partial_\vartheta\mu$ for Y and Z in the above equality and then use (E.3), (E.4), and (E.8) to find that

$$\partial_s X^s = 0, \quad \partial_\vartheta X^s + \varphi^2\partial_s X^\vartheta = 0, \quad \varphi^2\partial_\vartheta X^\vartheta + \varphi\varphi' X^s = 0. \tag{E.9}$$

If $X^s \equiv 0$ then $X^\vartheta \equiv c$ is constant by the second and third equations of (E.9) (note that $\varphi > 0$ on $(0,L)$ and X is of class C^2). In this case,

$$X(y) = c\partial_\vartheta\mu(s,\vartheta) = c(e_3\times y) \in \mathcal{R}|_\Gamma, \quad y = \mu(s,\vartheta) \in \Gamma$$

by (E.4) and (E.6). Let us show that each $X \in \mathcal{K}(\Gamma)$ is of this form (here the arguments are essentially the same as in [13, Section 74]). Assume to the contrary that $X^s \not\equiv 0$. By the first equation of (E.9), $X^s = X^s(\vartheta)$ is independent of s. Since X^s continuous and $X^s \not\equiv 0$, it does not vanish on some open interval $I \subset [0,2\pi]$. Also,

$$\partial_s X^\vartheta(s,\vartheta) = -\frac{\partial_\vartheta X^s(\vartheta)}{\{\varphi(s)\}^2}, \quad \partial_\vartheta X^\vartheta(s,\vartheta) = -\frac{\varphi'(s)X^s(\vartheta)}{\varphi(s)}$$

for $s \in (0,L)$ and $\vartheta \in [0,2\pi]$ by the second and third equations of (E.9) and $\varphi(s) > 0$ for $s \neq 0, L$. Since X is of class C^2, we have $\partial_\vartheta\partial_s X^\vartheta = \partial_s\partial_\vartheta X^\vartheta$. Thus the above equations imply that

$$\frac{\partial_\vartheta^2 X^s(\vartheta)}{X^s(\vartheta)} = \varphi(s)\varphi''(s) - \{\varphi'(s)\}^2, \quad s \in (0,L),\ \vartheta \in I.$$

Noting that the left-hand side is independent of s and φ is of class C^5, we differentiate both sides of this equality with respect to s to get

$$\varphi(s)\varphi'''(s) - \varphi'(s)\varphi''(s) = 0, \quad s \in (0, L).$$

Now we observe by this equality and (E.7) that

$$\frac{\partial}{\partial s}\big(K(\mu(s,\vartheta))\big) = -\frac{\varphi(s)\varphi'''(s) - \varphi'(s)\varphi''(s)}{\{\varphi(s)\}^2} = 0$$

for $s \in (0, L)$ and $\vartheta \in [0, 2\pi]$, which shows that K is constant on the whole surface Γ since K and μ are continuous on Γ and $[0, L] \times [0, 2\pi]$. Hence Γ is a sphere by Liebmann's theorem (see e.g. [54, Section 6.3, Theorem 3.7]), which contradicts our assumption that Γ is not a sphere. Thus $\mathcal{K}(\Gamma)$ contains only vector fields of the form $w(y) = c(e_3 \times y)$, $y \in \Gamma$ with $c \in \mathbb{R}$, which means that $\mathcal{K}(\Gamma) \subset \mathcal{R}|_\Gamma$ by (E.6). Since $\mathcal{R}|_\Gamma$ is a subspace of $\mathcal{K}(\Gamma)$, we conclude that $\mathcal{K}(\Gamma) = \mathcal{R}|_\Gamma$. $\Box$

Remark E.4. By the proof of Lemma E.3 we see that

- $\mathcal{R} = \{0\}$ if Γ is not axially symmetric,
- the dimension of $\mathcal{R}$ is one if Γ is axially symmetric but not a sphere, and
- the dimension of $\mathcal{R}$ is three if Γ is a sphere.

In particular, if Γ is axially symmetric around some line and it is not a sphere, then it is not axially symmetric around other lines.

Now we assume again that Γ is of class C^2 and take $g_0, g_1 \in C^1(\Gamma)$ satisfying (2.1). Let $\mathcal{R}_0$, $\mathcal{R}_1$, and $\mathcal{R}_g$ be the subspaces of $\mathcal{R}$ given by (2.4) and Ω_ε the curved thin domain of the form (1.1) with boundary Γ_ε. As in Section 3.2, we scale g_i to assume $|g_i| < \delta$ on Γ for $i = 0, 1$, where δ is the radius of the tubular neighborhood N of Γ given in Section 3.1, and thus $\overline{\Omega_\varepsilon} \subset N$ for all $\varepsilon \in (0, 1]$.

Lemma E.5. *For an infinitesimal rigid displacement $w(x) = a \times x + b$ of $\mathbb{R}^3$ with $a, b \in \mathbb{R}^3$ the following conditions are equivalent:*

(a) *For all* $\varepsilon \in (0,1]$ *the restriction of* w *on* Γ_ε *satisfies* $w|_{\Gamma_\varepsilon} \cdot n_\varepsilon = 0$ *on* Γ_ε.

(b) *There exists a sequence* $\{\varepsilon_k\}_{k=1}^\infty$ *of positive numbers such that*

$$\lim_{k\to\infty} \varepsilon_k = 0, \quad w|_{\Gamma_{\varepsilon_k}} \cdot n_{\varepsilon_k} = 0 \quad \textit{on} \quad \Gamma_{\varepsilon_k} \quad \textit{for all} \quad k \in \mathbb{N}.$$

(c) *The vector field* w *belongs to* $\mathcal{R}_0 \cap \mathcal{R}_1$.

PROOF. For $\varepsilon \in (0,1]$ and $i = 0,1$ let τ_ε^i be given by (3.26). Then

$$n_\varepsilon(y + \varepsilon g_i(y)n(y)) = (-1)^{i+1} \frac{n(y) - \varepsilon\tau_\varepsilon^i(y)}{\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}}, \quad y + \varepsilon g_i(y)n(y) \in \Gamma_\varepsilon^i$$

with $y \in \Gamma$ by Lemma 3.9. Moreover, for $w(x) = a \times x + b$ we have

$$\begin{aligned} &w(y + \varepsilon g_i(y)n(y)) = w(y) + \varepsilon g_i(y)\{a \times n(y)\}, \\ &\{a \times n(y)\} \cdot n(y) = 0, \quad y \in \Gamma. \end{aligned}$$

Hence the condition $w|_{\Gamma_\varepsilon^i} \cdot n_\varepsilon = 0$ on Γ_ε^i is equivalent to

$$w|_\Gamma \cdot n - \varepsilon w|_\Gamma \cdot \tau_\varepsilon^i - \varepsilon^2 g_i(a \times n) \cdot \tau_\varepsilon^i = 0 \quad \text{on} \quad \Gamma. \tag{E.10}$$

Let us prove the lemma. The condition (a) clearly implies (b). We show that (b) yields (c). Suppose that (b) is satisfied. Then, by (E.10),

$$w|_\Gamma \cdot n - \varepsilon_k w|_\Gamma \cdot \tau_{\varepsilon_k}^i - \varepsilon_k^2 g_i(a \times n) \cdot \tau_{\varepsilon_k}^i = 0 \quad \text{on} \quad \Gamma$$

for $k \in \mathbb{N}$ and $i = 0,1$. Letting $k \to \infty$ in this equality we get $w|_\Gamma \cdot n = 0$ on Γ by (3.28). Hence $w \in \mathcal{R}$ and $w|_\Gamma \cdot \tau_{\varepsilon_k}^i + \varepsilon_k g_i(a \times n) \cdot \tau_{\varepsilon_k}^i = 0$ on Γ. Since $\{\tau_{\varepsilon_k}^i\}_{k=1}^\infty$ converges to $\nabla_\Gamma g_i$ uniformly on Γ by (3.29), we send $k \to \infty$ in this equality to get $w|_\Gamma \cdot \nabla_\Gamma g_i = 0$ on Γ for $i = 0,1$. Thus $w \in \mathcal{R}_0 \cap \mathcal{R}_1$, i.e. (c) is valid.

Let us show that (c) implies (a). If $w \equiv 0$ then (a) is trivial. Suppose that $w \not\equiv 0$ belongs to $\mathcal{R}_0 \cap \mathcal{R}_1$. Let $\varepsilon \in (0,1]$ and $i = 0,1$. Since the condition $w|_{\Gamma_\varepsilon^i} \cdot n_\varepsilon = 0$ on Γ_ε^i is equivalent to (E.10) and $w \in \mathcal{R}_0 \cap \mathcal{R}_1 \subset \mathcal{R}$ satisfies $w|_\Gamma \cdot n = 0$ on Γ, it is sufficient for (a) to show that

$$w(y) \cdot \tau_\varepsilon^i(y) = 0, \quad \{a \times n(y)\} \cdot \tau_\varepsilon^i(y) = 0 \quad \text{for all} \quad y \in \Gamma. \tag{E.11}$$

Hereafter we fix and suppress the argument y. If $w = 0$, then $a \times n = 0$ by (E.1) and (E.11) follows (note that we can apply Lemma E.2 by $w \not\equiv 0$). Suppose $w \neq 0$. Then w is the eigenvector of W corresponding to the eigenvalue λ by (E.1). Since W has the eigenvalues κ_1, κ_2, and zero with $Wn = 0$ and $w \neq n$ by $w \cdot n = 0$, we have $\lambda = \kappa_1$ or $\lambda = \kappa_2$. Without loss of generality, we may assume $\lambda = \kappa_1$, i.e. $Ww = \kappa_1 w$. Then since $1 - \varepsilon g_i \kappa_1 > 0$ and $I_3 - \varepsilon g_i W$ is invertible by $|g_i| < \delta$ on Γ, (3.2), and Lemma 3.3, we have

$$(I_3 - \varepsilon g_i W)^{-1} w = (1 - \varepsilon g_i \kappa_1)^{-1} w. \tag{E.12}$$

We use (3.26), $W^T = W$, (E.12), and $w \cdot \nabla_\Gamma g_i = 0$ by $w \in \mathcal{R}_i$ to get

$$w \cdot \tau_\varepsilon^i = (I_3 - \varepsilon g_i W)^{-1} w \cdot \nabla_\Gamma g_i = 0. \tag{E.13}$$

Moreover, by (E.1) with $\lambda = \kappa_1$ and (E.12),

$$(I_3 - \varepsilon g_i W)^{-1}(a \times n) = -\kappa_1 (I_3 - \varepsilon g_i W)^{-1} w = -\kappa_1 (1 - \varepsilon g_i \kappa_1)^{-1} w.$$

Using this equality we get $(a \times n) \cdot \tau_\varepsilon^i = 0$ as in (E.13). Hence (E.11) holds and we have $w|_{\Gamma_\varepsilon^i} \cdot n_\varepsilon = 0$ on Γ_ε^i for all $\varepsilon \in (0,1]$ and $i = 0, 1$, i.e. (a) is valid. $\square$

By Lemmas E.1 and E.5, we observe that the nontriviality of $\mathcal{R}_0 \cap \mathcal{R}_1$ implies the uniform axial symmetry of Ω_ε.

Lemma E.6. *If there exists a vector field $w(x) = a \times x + b \in \mathcal{R}_0 \cap \mathcal{R}_1$ such that $w \not\equiv 0$, then $a \neq 0$, $a \cdot b = 0$, and Ω_ε is axially symmetric around the line parallel to a and passing through $b_a = |a|^{-2}(a \times b)$ for all $\varepsilon \in (0,1]$.*

Proof. Let $w(x) = a \times x + b \in \mathcal{R}_0 \cap \mathcal{R}_1$. Then $w|_{\Gamma_\varepsilon^i} \cdot n_\varepsilon = 0$ on Γ_ε^i for each $\varepsilon \in (0,1]$ and $i = 0, 1$ by Lemma E.5. Hence if $w \not\equiv 0$ then Lemma E.1 implies that $a \neq 0$, $a \cdot b = 0$, and both Γ_ε^0 and Γ_ε^1 are axially symmetric around the line parallel to a and passing through b_a, which yields the same axial symmetry of Ω_ε. $\square$

Also, the triviality of $\mathcal{R}_g$ yields the axial asymmetry of Ω_ε.

Lemma E.7. *If $\mathcal{R}_g = \{0\}$, then there exists a constant $\tilde{\varepsilon} \in (0,1]$ such that Ω_ε is not axially symmetric around any line for all $\varepsilon \in (0, \tilde{\varepsilon}]$.*

Proof. We prove the contrapositive statement: if there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty$ convergent to zero such that Ω_{ε_k} is (and thus $\Gamma_{\varepsilon_k}^0$ and $\Gamma_{\varepsilon_k}^1$ are) axially symmetric around some line l_k for each $k \in \mathbb{N}$, then $\mathcal{R}_g \neq \{0\}$.

Suppose that such a sequence $\{\varepsilon_k\}_{k=1}^\infty$ exists and that for each $k \in \mathbb{N}$ the line l_k is of the form $l_k = \{sa_k + b_k \mid s \in \mathbb{R}\}$ with $a_k, b_k \in \mathbb{R}^3$, $a_k \neq 0$, i.e. l_k is parallel to a_k and passing through b_k. Replacing a_k with $a_k/|a_k|$ we may assume $a_k \in S^2$ for all $k \in \mathbb{N}$ without changing l_k. Since Ω_ε is contained in the bounded set N for all $\varepsilon \in (0,1]$, there exists an open ball B_R centered at the origin of radius $R > 0$ such that $\Omega_{\varepsilon_k} \subset B_R$ for all $k \in \mathbb{N}$. Then, by the axial symmetry of Ω_{ε_k} around the line l_k, the intersection $l_k \cap B_R$ is not empty: otherwise the ball generated by the rotation of B_R through the angle π around l_k does not intersect with B_R and thus $\Omega_{\varepsilon_k} \subset B_R$ is not axially symmetric around l_k. Hence we may assume $b_k \in l_k \cap B_R$ for all $k \in \mathbb{N}$ by replacing b_k with $b_k - sa_k$ for an appropriate $s \in \mathbb{R}$. Now $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ are bounded in $\mathbb{R}^3$ and thus converge (up to subsequences) to some $a \in S^2$ and $b \in \mathbb{R}^3$, respectively.

Let us prove $w(x) := a \times (x - b) \in \mathcal{R}_g$. For $k \in \mathbb{N}$ and $i = 0, 1$ let $\tau_{\varepsilon_k}^i$ be the vector field on Γ given by (3.26) and $w_k(x) := a_k \times (x - b_k)$, $x \in \mathbb{R}^3$. Then, by (3.29) and $a_k \to a$, $b_k \to b$ as $k \to \infty$,

$$\lim_{k\to\infty} \tau_{\varepsilon_k}^i(y) = \nabla_\Gamma g_i(y), \quad \lim_{k\to\infty} w_k(y) = w(y), \quad y \in \Gamma. \tag{E.14}$$

For each $k \in \mathbb{N}$ and $i = 0, 1$, since $\Gamma_{\varepsilon_k}^i$ is axially symmetric around the line l_k, Lemma E.1 implies that $w_k|_{\Gamma_{\varepsilon_k}^i} \cdot n_{\varepsilon_k} = 0$ on $\Gamma_{\varepsilon_k}^i$. By the proof of Lemma E.5 (see (E.10)) this condition is equivalent to

$$w_k|_\Gamma \cdot n - \varepsilon_k w_k|_\Gamma \cdot \tau_{\varepsilon_k}^i - \varepsilon_k^2 g_i(a_k \times n) \cdot \tau_{\varepsilon_k}^i = 0 \quad \text{on} \quad \Gamma. \tag{E.15}$$

We send $k \to \infty$ in (E.15) to get $w|_\Gamma \cdot n = 0$ on Γ by (E.14) and $a_k \to a$. Thus $w \in \mathcal{R}$. Next we subtract (E.15) for $i = 1$ from that for $i = 0$ and divide the resulting equality by ε_k. Then since $w_k|_\Gamma \cdot n$ does not depend on i, we have

$$w_k|_\Gamma \cdot (\tau_{\varepsilon_k}^1 - \tau_{\varepsilon_k}^0) + \varepsilon_k (a_k \times n) \cdot (g_1 \tau_{\varepsilon_k}^1 - g_0 \tau_{\varepsilon_k}^0) = 0 \quad \text{on} \quad \Gamma.$$

We send $k \to \infty$ in this equality and use (E.14) and $a_k \to a$ to get $w|_\Gamma \cdot \nabla_\Gamma g = 0$ on Γ by $g = g_1 - g_0$. Hence $w \in \mathcal{R}_g$. Since $w \not\equiv 0$ by $a \in S^2$, we obtain $\mathcal{R}_g \neq \{0\}$. □

Finally, we give a relation between $\mathcal{R}_0 \cap \mathcal{R}_1$ and $L^2_\sigma(\Omega_\varepsilon)$.

LEMMA E.8. *We have $\mathcal{R}_0 \cap \mathcal{R}_1 \subset L^2_\sigma(\Omega_\varepsilon)$ for all $\varepsilon \in (0,1]$.*

PROOF. If $w(x) = a \times x + b \in \mathcal{R}_0 \cap \mathcal{R}_1$, then $\operatorname{div} w = 0$ in $\mathbb{R}^3$ by direct calculations and $w|_{\Gamma_\varepsilon} \cdot n_\varepsilon = 0$ on Γ_ε by Lemma E.5, i.e. $w \in L^2_\sigma(\Omega_\varepsilon)$. $\square$

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Department of Mathematics
Kyoto University
Kitashirakawa Oiwake-cho, Sakyo-ku
Kyoto 606-8502, Japan
E-mail: t.miura@math.kyoto-u.ac.jp

Present address
Graduate School of Science and Technology
Hirosaki University
3, Bunkyo-cho, Hirosaki-shi
Aomori 036-8561, Japan
E-mail: thmiura623@hirosaki-u.ac.jp