J. Math. Sci. Univ. Tokyo
29 (2022), 115–148.

Lie Symmetries for Implicit Planar Webs

By Alain Hénaut

Abstract. Let $F(x,y,y')=0$ be an analytic or algebraic differential equation with y'-degree d. We deal with the qualitative study of such equation through the geometry of the planar d-web generated by the integral curves. Using meromorphic connection methods associated with the analytic class of F, Lie or infinitesimal symmetries of these configurations are studied for essentially $d \geq 3$ in the nonsingular case and from the viewpoint of their singularities. Maximal rank problems related to Abel's addition theorem are also discussed. Basic examples are given from different domains including classic algebraic geometry and Frobenius 3-manifolds or WDVV-equations.

1. Introduction

In the complex setting, we study the geometry of integral curves of an analytic or algebraic differential equation of the first order with degree $d \geq 1$

$$F(x,y,y') = a_0(x,y)\,.\,(y')^d + \cdots + a_{d-1}(x,y)\,.\,y' + a_d(x,y) = 0.$$

Using the germ language on $(\mathbb{C}^2,0)$, we have $F \in \mathcal{O}[p]$ with $p = \dfrac{dy}{dx}$ where $\mathcal{O} := \mathbb{C}\{x,y\}$ is the ring of convergent power series in two variables or globally in $\mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$, we have $F \in \mathbb{C}[x,y,p]$ through the changes of standard affine charts. To be more precise, we suppose that the p-resultant of F satisfies $R_F := \mathrm{Result}(F, \partial_p(F)) = (-1)^{\frac{d(d-1)}{2}} a_0\,.\,\Delta \neq 0$, where $\Delta \in \mathcal{O}$ is the p-discriminant of F and, to avoid parasit solutions, that all the a_i are relatively prime. If $R_F(0) \neq 0$, we have $F = a_0 \displaystyle\prod_{i=1}^{d}(p-p_i)$ and we get a (germ of a) *nonsingular d-web* $\mathcal{W}(d) = \mathcal{W}(F_1,\ldots,F_d)$ in the classic form in $(\mathbb{C}^2,0)$, that is d foliations by curves in general position. Their leaves are level sets of elements $F_i \in \mathcal{O}$ where $F_i(0)=0$ and $X_i(F_i)=0$ with analytic

2020 *Mathematics Subject Classification.* 14C21, 53A60, 32S65.
Key words: Web geometry, Lie algebra of symmetries, abelian relations, Frobenius 3-manifolds or WDVV-equations.

vector fields $X_i = \partial_x + p_i\,\partial_y$ or analytic 1-forms $dy - p_i dx$ corresponding to the different slopes $p_i = p_i(x,y)$ of $F(x,y,y') = 0$. Conversely and up to a linear transformation to avoid the "vertical" slopes, every such a web $\mathcal{W}(F_1,\ldots,F_d)$ gives rise to a differential equation $F(x,y,y') = 0$ as above by using the product of the $\partial_y(F_i)\,.\,p + \partial_x(F_i)$.

Planar web geometry studies the previous configurations $\mathcal{W}(d)$, up to local isomorphisms. As a proper field of geometry, this study has been initiated in the 1930's by Wilhelm Blaschke and his coworkers or students including Gerhard Thomsen, Gerrit Bol and Shiing-Shen Chern. The historical reference is [BB-1938] and for example [PP-2015], which deals with nonsingular webs of codimension one in general, may be consulted. The elements F and $g\,.\,F$ define, for $g \in \mathcal{O}^*$, the same analytic class of differential equations and represent the same web $\mathcal{W}(d)$. One of the main goal is to study the geometry of the class defined by $F(x,y,y') = 0$ and its singularities by using the associated web $\mathcal{W}(d)$. In the implicit setting the collection of integral curves of $F(x,y,y') = 0$, like the roots of a polynomial, will be investigated in the *mutual relations* of all. It is a reminiscence from the Galois viewpoint. This subject can also be viewed as a part of the *qualitative study* of differential equations, especially the binary differential equations of degree d.

In the following we deal with Lie or infinitesimal symmetries for such a web $\mathcal{W}(d)$. These are also called infinitesimal automorphisms of the $\mathcal{W}(d)$ at stake. With the previous notation and to be short, a *symmetry* is a vector field $X = \alpha\partial_x + \beta\partial_y$ such that the local flow $(x,y\,;t) \longmapsto \exp(tX)(x,y)$ generated by X preserves all the leaves of $\mathcal{W}(d)$. This means generically that $X_i(X(F_i)) = 0$ or equivalently $\mathcal{L}_X(\omega_i)\wedge\omega_i = 0$ for $1 \leq i \leq d$ where the 1-forms ω_i define $\mathcal{W}(d)$. Here $\mathcal{L}_X = i_X \circ d + d \circ i_X : \Omega^\bullet \longrightarrow \Omega^\bullet$ is the Lie derivative associated with X and $i_X : \Omega^\bullet \longrightarrow \Omega^{\bullet-1}$ is its interior product. The symmetries $\mathfrak{Lie}\,\mathcal{W}(d)$ of a web $\mathcal{W}(d)$ form a sheaf such that each fiber, equipped with the Lie bracket, is a Lie algebra. For singular foliations on surfaces, corresponding to planar 1-webs, these symmetries generate a pseudogroup which is the starting object of the differential Galois theory developed by Bernard Malgrange for foliations (*cf.* for example [M-2002]).

For special linear webs in $\mathbb{P}^2$ associated by duality with plane reduced algebraic curves, the study of the associated symmetries is directly related to W-curves first investigated by Felix Klein and Sophus Lie in [KL-1871] as an important step in the emergence of the theory of Lie groups. As an illustration of the equivalence problem, Élie Cartan in [C-1908] studies the generic case of a planar 3-web $\mathcal{W}(x, y, F_3)$ and its associated Lie symmetries. He pointed out that the dimension $\dim \mathfrak{Lie}\,\mathcal{W}(x, y, F_3)$, as $\mathbb{C}$-vector space, is 0, 1 or 3. In particular the basic hexagonal 3-web $\mathcal{W}(x, y, x+y)$ corresponds to maximum dimension 3. Lie symmetries do not appear in the pioneer work [BB-1938] indicated above. It is also the case in the basic revival of interest in web geometry due to Shiing-Shen Chern and Phillip A. Griffiths which essentially focus on abelian relations for webs and algebraization problems (*cf.* [Ch-1982] and references therein). Some results on these symmetries for a 3-web of codimension r in a $2r$-dimensional manifold can be found in the contributions of Maks A. Akivis with Alexander M. Shelekhov [AS-1992] or Vladislav V. Goldberg [AG-2000] (this survey contains a detailed report of the publications on this subject in Russian by N. V. Gvozdovich in the 1980's). In [GL-2006], Goldberg and Valentin V. Lychagin give a criterion for a 3-web to admit exactly one-dimensional Lie symmetry algebra. David Marín, Jorge Vitório Pereira and Luc Pirio use these symmetries coupled with plane algebraic W-curves in [MPP-2006] to provide a large class of remarkable planar webs $\mathcal{E}(d)$ for $d \geq 5$. These webs are not polylogarithmic as Bol's web $\mathcal{B}(5)$ which for example have no symmetry in the previous sense. Sergey I. Agafonov by precising results of Eugene V. Ferapontov (*cf.* [F-2004] and references therein) also studies these symmetries for particular planar 3-webs related to the geometry of Frobenius manifolds introduced by Boris Dubrovin through WDVV-equations (*cf.* [Ag-2012] for example). Some part of these previous contributions will be specified or even revisited in the following.

It is proved below that the symmetries of a web $\mathcal{W}(d)$ presented by F with $d \geq 3$ form a *local system* with rank 0, 1 or 3, outside its discriminant locus $|\Delta|$ defined by the reduced divisor associated with Δ. For $d = 3$, the local system $\mathfrak{Lie}\,\mathcal{W}(3)$ is incarnated as horizontal sections of a *connection of symmetries* (E, ∇) with rank 3 which is meromorphic on $|\Delta|$ and integrable (or flat depending on the terminology) if, and only if, $\mathcal{W}(3)$ is hexagonal. A complete discussion on $\dim \mathfrak{Lie}\,\mathcal{W}(3)$ is given with *effective methods* depending only on the coefficients of F. Generalizations for study $\mathfrak{Lie}\,\mathcal{W}(d)$ with $d \geq 3$ are also presented. The implicit viewpoint where no leaf of the web at stake is preferred plays its role, in the nonsingular case and through the singularities as well. New invariants of the class of the differential equations defined by $F(x, y, y') = 0$ which extend the classic curvature of a planar 3-web are introduced by connection methods already used in [Hé-2004]. Links between symmetries and rank problems related to abelian relations of such a $\mathcal{W}(d)$ are also studied, especially with regards to remarkable planar webs. A generalization for a planar web of Lie's equivalence for a 1-form between existence of an integrating factor and a transverse symmetry is presented. Detailed examples, discussions on singularities of $\mathfrak{Lie}\,\mathcal{W}(d)$ including regularity aspects of the meromorphic connection (E, ∇), presence of weighted Euler symmetry and some perspectives complete this article.

The author would like to thank Daniel Lehmann and Sergey I. Agafonov for general comments on the first drafts of the present text and also David Marín who moreover make available an illuminating family of hexagonal webs $\mathcal{M}_{(m,\lambda)}(3)$ in $\mathbb{P}^2$ with in particular a connection of symmetries (E, ∇) not always regular singular (*cf.* Example 6, below).

2. Description of $\mathfrak{Lie}\,\mathcal{W}(d)$ as a Local System for $d \geq 3$ and Lie's Integrating Factor for an Implicit Planar Web

In this section, we suppose that $R_F(0) \neq 0$ where $R_F = \mathrm{Result}(F, \partial_p(F))$.

By definition and from a calculus using the forms $\omega_i = dy - p_i dx$ or the vector fields $X_i = \partial_x + p_i \partial_y$ which define the nonsingular d-web $\mathcal{W}(d)$, we get that an analytic vector field $X = \alpha\partial_x + \beta\partial_y$ is a symmetry for $\mathcal{W}(d)$ that is $X \in \mathfrak{Lie}\,\mathcal{W}(d)$ if, and only if, (α, β) is an analytic solution of the following (homogeneous) linear differential system with d equations:

$$(LS)\quad -\partial_x(\beta) + (\partial_x(\alpha) - \partial_y(\beta))\,.\,p_i + \partial_y(\alpha)\,.\,p_i^2 + \partial_x(p_i)\,\alpha + \partial_y(p_i)\,\beta = 0 \qquad \text{for } 1 \leq i \leq d.$$

For $d = 1$ or $d = 2$, the $\mathbb{C}$-vector space $\mathfrak{Lie}\,\mathcal{W}(d)$ is not finite dimensional. Indeed, in these cases we may suppose $p_1 = 0$, or $p_1 = 0$ with $p_2 = 1$. Hence, every $X = \beta(y)\partial_y$ with $\beta \in \mathbb{C}\{y\}$ is a solution in the first case while every $X = \beta(y)(\partial_x + \partial_y)$ is a solution in the second case.

For $d = 3$, the system (LS) of *Lie symmetries* is equivalent to a system which looks like the linear differential system $\mathcal{M}(4)$ with 3 equations associated with the presentation of a 4-planar web in [Hé-2004]. An additional part formed by $d-3$ linear equations appears for $d \geq 3$. More precisely for $d \geq 3$, that we suppose from now on, we have the following equivalence:

$X = \alpha\partial_x + \beta\partial_y \in \mathfrak{Lie}\,\mathcal{W}(d)$ if, and only if, (α, β) is an analytic solution of the following linear differential system:

$$(LS)\quad \left\{\begin{array}{ccccc} -\,\partial_x(\beta) & + & g_d\alpha + h_d\beta & = & 0 \\ \partial_x(\alpha) - \partial_y(\beta) & + & g_{d-1}\alpha + h_{d-1}\beta & = & 0 \\ \partial_y(\alpha) & + & g_{d-2}\alpha + h_{d-2}\beta & = & 0 \\ & & g_{d-3}\alpha + h_{d-3}\beta & = & 0 \\ & & & \vdots & \\ & & g_1\alpha + h_1\beta & = & 0. \end{array}\right.$$

The previous $(d\times 2)$-matrix $(S_{ij}) := \begin{pmatrix} g_d & h_d \\ \vdots & \vdots \\ g_1 & h_1 \end{pmatrix}$ is called the *matrix of symmetries* of the presentation F of $\mathcal{W}(d)$. It comes from the two polynomials

$$G := g_1\,.\,p^{d-1} + \cdots + g_{d-1}\,.\,p + g_d \quad \text{and} \quad H := h_1\,.\,p^{d-1} + \cdots + h_{d-1}\,.\,p + h_d$$

which are the unique elements in $\mathcal{O}[p]$ with $\deg G \leq d-1$ and $\deg H \leq d-1$ such that $G(x, y, p_i) = \partial_x(p_i)$ and $H(x, y, p_i) = \partial_y(p_i)$ for $1 \leq i \leq d$. Indeed, the initial description of (LS) can be also written as the following system:

$$\begin{aligned} -\partial_x(\beta) + g_d\alpha + h_d\beta & \\ +(\partial_x(\alpha) - \partial_y(\beta) + g_{d-1}\alpha + h_{d-1}\beta)\,.\,p_i & \\ +(\partial_y(\alpha) + g_{d-2}\alpha + h_{d-2}\beta)\,.\,p_i^2 & \\ +(g_{d-3}\alpha + h_{d-3}\beta)\,.\,p_i^3 & \\ + \cdots + (g_1\alpha + h_1\beta)\,.\,p_i^{d-1} &= 0 \end{aligned}$$

for $1 \leq i \leq d$. Then, using the Vandermonde d-determinant obtained from the distinct p_i, we get the equivalence below. In the above differential system (LS), the $d-3$ conditions which do not involve derivatives can be viewed as additional conditions on the 3-upper differential part.

The p-resultant $R_F = \mathrm{Result}(F, \partial_p(F)) \in \mathcal{O}$ is classically interpreted as a $(2d-1)$-determinant. This proves that the previous G and H can be also viewed as part of the unique ordered pair of polynomials (V, G) and (W, H) in $\mathcal{O}[p]_{<d-1} \times \mathcal{O}[p]_{<d}$ such that

$$V\,.\,F - G\,.\,\partial_p(F) = \partial_x(F) \quad \text{and} \quad W\,.\,F - H\,.\,\partial_p(F) = \partial_y(F)$$

since $F(x,y,p_i)=0$ for $1\leq i\leq d$ by definition. Here $\mathcal{O}[p]_{<\ell}$ denotes the elements in $\mathcal{O}[p]$ with a p-degree strictly less than ℓ.

Therefore to be concrete and for $d=3$, we have the following Cramer-Sylvester systems:

$$\begin{pmatrix} a_0 & 0 & 3a_0 & 0 & 0 \\ a_1 & a_0 & 2a_1 & 3a_0 & 0 \\ a_2 & a_1 & a_2 & 2a_1 & 3a_0 \\ a_3 & a_2 & 0 & a_2 & 2a_1 \\ 0 & a_3 & 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} * \\ * \\ -g_1 \\ -g_2 \\ -g_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_x(a_0) \\ \partial_x(a_1) \\ \partial_x(a_2) \\ \partial_x(a_3) \end{pmatrix}$$

and

$$\begin{pmatrix} a_0 & 0 & 3a_0 & 0 & 0 \\ a_1 & a_0 & 2a_1 & 3a_0 & 0 \\ a_2 & a_1 & a_2 & 2a_1 & 3a_0 \\ a_3 & a_2 & 0 & a_2 & 2a_1 \\ 0 & a_3 & 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} * \\ * \\ -h_1 \\ -h_2 \\ -h_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_y(a_0) \\ \partial_y(a_1) \\ \partial_y(a_2) \\ \partial_y(a_3) \end{pmatrix}$$

where $G=g_1\,.\,p^2+g_2\,.\,p+g_3$ and $H=h_1\,.\,p^2+h_2\,.\,p+h_3$. For $d=4$, we get

$$\begin{pmatrix} a_0 & 0 & 0 & 4a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 3a_1 & 4a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 2a_2 & 3a_1 & 4a_0 & 0 \\ a_3 & a_2 & a_1 & a_3 & 2a_2 & 3a_1 & 4a_0 \\ a_4 & a_3 & a_2 & 0 & a_3 & 2a_2 & 3a_1 \\ 0 & a_4 & a_3 & 0 & 0 & a_3 & 2a_2 \\ 0 & 0 & a_4 & 0 & 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ -g_1 \\ -g_2 \\ -g_3 \\ -g_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \partial_x(a_0) \\ \partial_x(a_1) \\ \partial_x(a_2) \\ \partial_x(a_3) \\ \partial_x(a_4) \end{pmatrix}$$

and the analogue for H, and so on for $d\geq 5$.

It can be noted that the coefficients S_{ij} of the matrix of symmetries depend only on a_i, $\partial_x(a_i)$ and $\partial_y(a_i)$. From the singularity viewpoint these are in fact in $\mathcal{O}[1/\delta]$, that is with poles on the *discriminant locus* $|\Delta|$ of the d-web $\mathcal{W}(d)$ defined by the reduced divisor $\delta := \prod_q \Delta_q$ associated with the irreducible decomposition $\Delta = u\,.\prod_q \Delta_q^{m_q}$ of the p-discriminant of F where $u\in\mathcal{O}^*$ and every Δ_q defines an irreducible analytic germ in $(\mathbb{C}^2,0)$ with $m_q\geq 1$.

Remark 1. The geometric definition of a symmetry $X \in \mathfrak{Lie}\,\mathcal{W}(d)$ through its generated flow does not depend on the local coordinates where the d-web $\mathcal{W}(d)$ is presented. The polynomials G and H depend on the coordinates, but the presentations F and $g\,.\,F$ of the same $\mathcal{W}(d)$ for $g \in \mathcal{O}^*$ give the same G and H. These properties can be viewed also as a direct consequence of the Lagrange interpolation formula. In other words, the polynomials G and H are *invariants* of the class of the differential equations defined by $F(x, y, y') = 0$.

Remark 2 (*Linear webs and web in* $\mathbb{P}^2$ *associated by duality with a reduced algebraic curve*). If $\mathcal{W}(d) = \mathcal{L}(d)$ is a *linear* d-*web*, that is all leaves are straight lines, we have equivalently

$$X_i(p_i) = \partial_x(p_i) + p_i\partial_y(p_i) = 0 \quad \text{for} \quad 1 \leq i \leq d$$

by using the local flow generated by the vector field X_i. Therefore in this case we obtain

$$a_0 g_d = a_d h_1 \ , \ a_0(g_{d-1} + h_d) = a_{d-1} h_1 \ , \ \dots \ , \ a_0(g_1 + h_2) = a_1 h_1.$$

Indeed, if $h_1 = 0$, these equalities are coming from the Vandermonde d-determinant already used by definition of G and H. If $h_1 \neq 0$, that is $\deg H = d - 1$, then the polynomial $G + p\,.\,H \in \mathcal{O}[p]$ has the same roots p_i that the presentation F of $\mathcal{L}(d)$ and the previous equalities follow. Let $C \subset \check{\mathbb{P}}^2 := G(1, \mathbb{P}^2)$ be a reduced algebraic curve with degree d in the space of lines of $\mathbb{P}^2$. By duality, we get a special linear d-web $\mathcal{L}_C(d) \subset \mathbb{P}^2$ generically nonsingular, called *the* d-*web associated with* C. It is locally presented by

$$F(x, y, p) = P(y - px, p)$$

if $P(q, p) = 0$ is an affine equation of C. Here $F(x, y, y') = 0$ corresponds essentially to d classic Clairaut's equations $y = xy' + f_i(y')$. If C contains no lines, the leaves of $\mathcal{L}_C(d)$ are generically the tangents of the *dual curve* $\check{C} \subset \mathbb{P}^2$ of C. This one is locally defined by $\check{P}(x, y) = 0$ where $\check{P}$ is a factor of the p-resultant R_F, and the others factors are products of linear forms related to the singular points of C. If otherwise C contains lines, then the corresponding points in $\mathbb{P}^2$ give rise to pencils of lines for $\mathcal{L}_C(d)$ through these points.

Example 1. Parallel planar d-web.

We consider a particular d-web $\mathcal{SP}(d) \subset \mathbb{P}^2$ generated by the *special pencils* of lines passing through d distinct points which belong to a line $L \subset \mathbb{P}^2$. By duality $\mathcal{SP}(d) = \mathcal{L}_C(d)$ where $C \subset \check{\mathbb{P}}^2$ is the algebraic curve given by the union of d distinct lines passing through the point $\check{L} \in \check{\mathbb{P}}^2$, that is a central arrangement of d lines. With notations of Remark 2, we may suppose that C is given by $P(q,p) = \prod_{i=1}^{d}(p - p_i) = 0$ where the p_i are d distinct complex numbers and $\check{L} = [1,0,0]$ or $L = \{[X_0, X_1, X_2]\,;\, X_0 = 0\}$. From the previous definitions, the following properties are equivalent:

i) $\mathcal{W}(d) = \mathcal{SP}(d)$ is a parallel planar d-web, that is presented by $F \in \mathbb{C}[p]$;

ii) The slopes p_i of the d-web $\mathcal{W}(d)$ are d distinct constants;

iii) The $(d\times 2)$-matrix of symmetries of F is $(S_{ij}) = 0$, that is $G = H = 0$.

In this case, we get $\mathfrak{Lie}\,\mathcal{SP}(d) = \{\partial_x,\, \partial_y,\, x\partial_x + y\partial_y\}$ since the system (LS) is reduced to the following:

$$(LS) \quad \begin{cases} -\,\partial_x(\beta) & = & 0 \\ \partial_x(\alpha) - \partial_y(\beta) & = & 0 \\ \partial_y(\alpha) & = & 0. \end{cases}$$

Since in this part $R_F(0) \neq 0$, we have a nonsingular surface S defined by F with its usual de Rham complex $(\Omega^\bullet_S, d)$ where $\Omega^\bullet_S = \Omega^\bullet_{\mathbb{C}^3}/(dF \wedge \Omega^{\bullet-1}_{\mathbb{C}^3}, F\,\Omega^\bullet_{\mathbb{C}^3})$. Moreover the first projection induces a d-covering map $\pi : S \longrightarrow (\mathbb{C}^2, 0)$ with local branches $\pi_i(x,y) = (x, y, p_i(x,y))$.

The following result summarizes the previous observations. It also generalizes classical Lie's link for a 1-form between existence of an integrating factor and a transverse symmetry, to an implicit planar d-web. Agafonov gives another generalization for 3-webs in [Ag-2015] (*cf.* Theorem 1 therein). The classic link is precisely the equivalence $v) \iff ii)$ proved below for a given ω_i.

Theorem 1. *With the previous notations, in particular $\omega_i = dy - p_i dx$ and $X = \alpha\partial_x + \beta\partial_y$, the following conditions are equivalent for a d-web $\mathcal{W}(d)$ with $d \geq 3$:*

i) $X \in \mathfrak{Lie}\,\mathcal{W}(d)$, *that is* X *is a symmetry for* $\mathcal{W}(d)$*;*

ii) $\mathcal{L}_X(\omega_i) \wedge \omega_i = 0$ *for* $1 \le i \le d$*;*

iii) (α, β) *is an analytic solution of the linear differential system* (LS)*.*

Moreover if $i_X(\omega_i) \neq 0$ *for* $1 \le i \le d$*, that is* X *is transverse to* $\mathcal{W}(d)$*, then the previous conditions are equivalent to the following:*

iv) The meromorphic 1*-form* $\omega := \dfrac{dy - pdx}{\beta - \alpha p}$ *on* S *is closed, that is* $\dfrac{1}{\beta - \alpha p}$ *is an integrating factor for* $dy - pdx$ *on the surface* S*;*

v) $\dfrac{1}{i_X(\omega_i)}$ *is an integrating factor for* ω_i *for* $1 \le i \le d$*.*

Proof. The previous observations prove that $i) \iff ii) \iff iii)$.
$iii) \Longrightarrow iv)$. By hypothesis, $i_X(\omega_i) = \omega_i(X) = \beta - \alpha p_i \neq 0$ on S. In Ω^2_S, we have $\dfrac{dx \wedge dy}{\partial_p(F)} = \dfrac{dy \wedge dp}{\partial_x(F)} = \dfrac{dp \wedge dx}{\partial_y(F)}$ since $\partial_x(F)dx + \partial_y(F)dy + \partial_p(F)dp = 0$ in Ω^1_S. Hence a calculus gives

$$\begin{aligned} d\omega &= \Big(p^2\partial_y(\alpha) + p(\partial_x(\alpha) - \partial_y(\beta)) - \partial_x(\beta) \\ &\qquad - \alpha\,\frac{\partial_x(F)}{\partial_p(F)} - \beta\,\frac{\partial_y(F)}{\partial_p(F)}\Big)\frac{dx \wedge dy}{(\beta - \alpha p)^2} \\ &= \Big(p^2\partial_y(\alpha) + p(\partial_x(\alpha) - \partial_y(\beta)) - \partial_x(\beta) + \alpha \cdot G + \beta \cdot H\Big)\frac{dx \wedge dy}{(\beta - \alpha p)^2} = 0 \end{aligned}$$

on the surface S by definition of G and H through the expressions above.
$iv) \Longrightarrow v)$. For $1 \le i \le d$, we have $\pi_i^*(\omega) = \dfrac{\omega_i}{i_X(\omega_i)}$ by definition and this form is closed by hypothesis since d and π_i^* commute.
$v) \Longrightarrow ii)$. For $1 \le i \le d$, we have $0 = i_X(\omega_i \wedge d\omega_i) = i_X(\omega_i)d\omega_i - \omega_i \wedge i_X(d\omega_i)$ and by hypothesis $d(\dfrac{\omega_i}{i_X(\omega_i)}) = \dfrac{d\omega_i}{i_X(\omega_i)} - \dfrac{d(i_X(\omega_i))}{i_X(\omega_i)^2} \wedge \omega_i = 0$. Therefore, we get

$$\mathcal{L}_X(\omega_i) \wedge \omega_i = (i_X(d\omega_i) + d(i_X(\omega_i))) \wedge \omega_i = -i_X(\omega_i)d\omega_i + d(i_X(\omega_i)) \wedge \omega_i = 0. \ \square$$

Discussions on $\dim \mathfrak{Lie}\,\mathcal{W}(d)$ will be given in the next paragraphs. However, we begin with the following result:

Proposition 1. *Let $\mathcal{W}(d)$ be an implicit planar d-web presented by F with a discriminant locus $|\Delta|$. Then the sheaf $\mathfrak{Lie}\,\mathcal{W}(d)$ of symmetries of $\mathcal{W}(d)$ with $d \geq 3$ is a local system outside $|\Delta|$, that is a locally constant sheaf of $\mathbb{C}$-vector spaces with finite dimensional fibers* $\dim \mathfrak{Lie}\,\mathcal{W}(d)$. *Moreover* $\dim \mathfrak{Lie}\,\mathcal{W}(d)$ *takes values in* $\{0, 1, 3\}$.

Proof. With usual notation and basic results in $\mathcal{D}$-modules approach where $\mathcal{D}$ is the ring of linear differential operators with coefficients in $\mathcal{O}$ (*cf.* for example [GM-1993]) and after transposition, the symbol matrix of the differential system in the 3-upper part of (LS) is

$$\begin{pmatrix} 0 & \xi & \eta \\ -\xi & -\eta & 0 \end{pmatrix}.$$

Here the left $\mathcal{D}$-module at stake is the cokernel of ${}^t\rho : \mathcal{D}^3 \longrightarrow \mathcal{D}^2$ where ${}^t\rho(Q_1, Q_2, Q_3) = (Q_1 g_d + Q_2(\partial_x + g_{d-1}) + Q_3(\partial_y + g_{d-2}), Q_1(-\partial_x + h_d) + Q_2(-\partial_y + h_{d-1}) + Q_3 h_{d-2})$. Therefore the 0-th Fitting ideal associated, that is the ideal of 2×2-minors generated by ${}^t\rho$ in $\operatorname{gr}\mathcal{D} = \mathcal{O}[\xi, \eta]$ is $(\xi^2, \xi\eta, \eta^2)$. Hence as left $\mathcal{D}$-modules, we have $\operatorname{Coker}{}^t\rho = \mathcal{O}^m$ with $m \leq 3 = \operatorname{mult}\mathcal{O}[\xi, \eta]/(\xi^2, \xi\eta, \eta^2)$. Therefore $\mathfrak{Lie}\,\mathcal{W}(d)$ or the analytic solutions of the system (LS) is a local system outside the zero locus of Δ and $\dim \mathfrak{Lie}\,\mathcal{W}(d)$ is bounded by 3. If $\mathfrak{Lie}\,\mathcal{W}(d)$ has dimension 2, we may suppose $\mathfrak{Lie}\,\mathcal{W}(d) = \{\partial_x, \partial_y\}$ with $\mathcal{L}_{\partial_x}(\omega_i) \wedge \omega_i = 0 = \mathcal{L}_{\partial_y}(\omega_i) \wedge \omega_i$ for $1 \leq i \leq d$, hence we obtain that $G = H = 0$ by definition since the previous conditions are equivalent with $\partial_x(p_i) = 0 = \partial_y(p_i)$ for $1 \leq i \leq d$. In other words, $\mathcal{W}(d)$ is parallelizable (*cf.* Example 1). Therefore, we get also $x\partial_x + y\partial_y \in \mathfrak{Lie}\,\mathcal{W}(d)$ by using the system (LS), which is a contradiction. $\square$

Symmetries X for a singular d-web $\mathcal{W}(d)$ in $(\mathbb{C}^2, 0)$ are multivalued in general if there exist and the algebraic analysis of these merits to be undertaken, especially regularity and monodromy aspects, as some examples below show. Those complex analytic or holomorphic, that is with $X \in \Theta$ are interesting in regard to the singularities of $\mathcal{W}(d)$ (*cf.* [Ag-2015] for a particular case) and their different types of indices attached. This vein will be subsequently explored and related to Kyoji Saito's logarithmic vector fields, that is the free $\mathcal{O}$-module with rank 2 defined by $\operatorname{Der}(\log|\Delta|) = \{X \in \Theta\,;\, X(\delta) \in (\delta)\}$ in [S-1980]. In particular we mention the general invariance property which follows.

Proposition 2. *Let $\mathcal{W}(d)$ be an implicit planar d-web presented by $F = a_0 \,.\, p^d + a_1 \,.\, p^{d-1} + \cdots + a_d$ which admits a symmetry $X = \alpha\partial_x + \beta\partial_y$. Then the p-discriminant $\Delta = u \,.\, \prod_q \Delta_q^{m_q}$ of F is invariant by X. More precisely, we have*

$$X(\Delta) = \lambda_X \,.\, \Delta$$

with $\lambda_X = (d-1) \,.\, \Big(\dfrac{2X(a_0)}{a_0} - d \,.\, (\partial_x(\alpha) - \partial_y(\beta)) + 2\partial_y(\alpha)\,\dfrac{a_1}{a_0}\Big)$. In particular outside $|\Delta|$ with $u \equiv 1$, there exist $\lambda_{X,q} \in \mathcal{O}$ such that $X(\Delta_q) = \lambda_{X,q} \,.\, \Delta_q$, that is every irreducible component Δ_q of Δ is invariant by X with $\lambda_X = \sum_q m_q \,.\, \lambda_{X,q}$.

Proof. With the previous notations, we have classically $\Delta = a_0^{2d-2} \,.\, \prod_{1\le i<j\le d} (p_i - p_j)^2$. Since X is a symmetry, we have $X(p_i - p_j) = -(\partial_x(\alpha) - \partial_y(\beta))(p_i - p_j) - \partial_y(\alpha)(p_i^2 - p_j^2)$ from the initial presentation of the linear differential system (LS). Then the invariance formula follows by applying X on the product above and using the classical relation on the sum of the roots p_i of F with its coefficients. In particular for $u \equiv 1$, we obtain the equality

$$\sum_q m_q \,.\, X(\Delta_q)\Delta_1 \ldots \Delta_{q-1}\,\widehat{\Delta_q}\,\Delta_{q+1} \ldots \Delta_r = \lambda_X \,.\, \prod_q \Delta_q$$

outside $|\Delta|$. Hence for example $X(\Delta_1)\Delta_2 \ldots \Delta_r$ divides Δ_1 and, by successively using the Δ_q for $q \neq 1$, $X(\Delta_1)$ divides Δ_1 since the Δ_q are pairwise coprime. So we get the $\lambda_{X,q}$ as claimed. □

3. Basic Results on $\mathfrak{Lie}\,\mathcal{W}(3)$

We first recall some results for an implicit planar 3-web $\mathcal{W}(3)$. A relation $\xi_1(F_1)dF_1 + \xi_2(F_2)dF_2 + \xi_3(F_3)dF_3 = 0$ with $\xi_i \in \mathbb{C}\{t\}$ is called an *abelian relation* of a nonsingular 3-web $\mathcal{W}(3) = \mathcal{W}(F_1, F_2, F_3)$. These relations, between the normals of $\mathcal{W}(3)$, viewed as special 3-uple $(\xi_1(F_1), \xi_2(F_2), \xi_3(F_3)) \in \mathcal{O}^3$ form a local system $\mathcal{A}(3)$ such its rank, noted $\operatorname{rank} \mathcal{W}(3)$, is equal to 0 or 1 and does not depend on the choice of the F_i. The main invariant of such a nonsingular 3-web $\mathcal{W}(3)$, up to the equivalence

$\sim$ induces as pullback by an analytic isomorphism of $(\mathbb{C}^2, 0)$, is its *curvature* 2-form $\mathbf{k}$ also called its Blaschke-Dubourdieu curvature. The birth certificate of planar web geometry may be summarized as follows. (*cf.* for example [BB-1938] or [PP-2015]).

Fact. *For a nonsingular* 3*-web* $\mathcal{W}(3)$ *in* $(\mathbb{C}^2, 0)$*, the following assertions are equivalent:*

i) $\operatorname{rank} \mathcal{W}(3) = 1$;

ii) $\mathcal{W}(3) \sim \mathcal{W}(x, y, x+y)$;

iii) $\mathcal{W}(3)$ *is hexagonal* (or satisfies Thomsen's closure, 1927)*;*

iv) $\mathbf{k} = 0$.

For a planar 3-web $\mathcal{W}(3)$ presented by $F = a_0 \,.\, p^3 + a_1 \,.\, p^2 + a_2 \,.\, p + a_3$ there exists an explicit *meromorphic* 1*-form* $\gamma = Adx + Bdy$, with poles on $|\Delta|$, constructed only from the coefficients a_i, $\partial_x(a_i)$ and $\partial_y(a_i)$ such that $\mathbf{k} = d\gamma$. Description and basic properties of this 1-form γ defined in [H-2000] (*cf.* also [H-2004]) are recalled below.

With the previous notation if $R_F(0) \neq 0$, the 1-form $\gamma \in \Omega^1$ is defined by using an explicit description of $d\omega$ on the nonsingular surface S defined by F for $\omega = \dfrac{r_x dy - r_y dy}{\partial_p(F)}\cdot$ Indeed for the normalized contact 1-form $\nu := \dfrac{dy - pdx}{\partial_p(F)}$ on S, we get $d\nu = (A + B \,.\, p)\, \dfrac{dx \wedge dy}{\partial_p(F)}$ where $A := -u_3 + l_2$ and $B := -u_2 + 2l_1$ are coming from the following Cramer-Sylvester system:

$$\begin{pmatrix} a_0 & 0 & 3a_0 & 0 & 0 \\ a_1 & a_0 & 2a_1 & 3a_0 & 0 \\ a_2 & a_1 & a_2 & 2a_1 & 3a_0 \\ a_3 & a_2 & 0 & a_2 & 2a_1 \\ 0 & a_3 & 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ -l_1 \\ -l_2 \\ -l_3 \end{pmatrix} = \begin{pmatrix} \partial_y(a_0) \\ \partial_x(a_0) + \partial_y(a_1) \\ \partial_x(a_1) + \partial_y(a_2) \\ \partial_x(a_2) + \partial_y(a_3) \\ \partial_x(a_3) \end{pmatrix}.$$

The 1-forms $\nu_i := \pi_i^*(\nu) = \dfrac{dy - p_i dx}{\partial_p(F)(x, y, p_i)}$ define the 3-web $\mathcal{W}(3)$, therefore $\mathbf{k} = d\gamma$ is by definition its curvature since $\nu_1 + \nu_2 + \nu_3 = 0$ from the Lagrange interpolation formula and $d\nu_i = \gamma \wedge \nu_i$ for $1 \leq i \leq 3$. Moreover, we have the relation

$$(*) \qquad \partial_x(F) + p\, \partial_y(F) + P_0\, \partial_p(F) = (\partial_p(P_0) - A - B \,.\, p)\, F$$

where $P_0 := l_1 \,.\, p^2 + l_2 \,.\, p + l_3$ is the unique element in $\mathcal{O}[p]$ with $\deg P_0 \leq 2$ such that $P_0(x, y, p_i) = X_i(p_i)$ for $1 \leq i \leq 3$. The polynomial P_0 depends *only* on the class of $F(x, y, y') = 0$ and the leaves of its associated $\mathcal{W}(3)$ verify $y'' = P_0(x, y, y')$.

The coefficients l_j and u_i defined from the Cramer-Sylvester system above belong to $\mathcal{O}[1/\mathrm{red}\, R_F]$, that is have poles on the p-resultant of F. However the poles of A and B are only on the discriminant locus $|\Delta|$ with moreover the following detailed expressions:

$$\begin{aligned}
R_F :&= \mathrm{Result}(F, \partial_p(F)) = -a_0 \,.\, \Delta \\
\Delta &= -4a_0a_2^3 + 18a_0a_1a_2a_3 - 27a_0^2a_3^2 + a_1^2a_2^2 - 4a_1^3a_3 \\
\Delta \,.\, A &= a_3(2a_2^2 - 6a_1a_3) \,.\, \partial_y(a_0) + a_3(9a_0a_3 - a_1a_2) \,.\, (\partial_x(a_0) + \partial_y(a_1)) \\
&\quad + a_3(2a_1^2 - 6a_0a_2) \,.\, (\partial_x(a_1) + \partial_y(a_2)) \\
&\quad + (4a_0a_2^2 - a_1^2a_2 - 3a_0a_1a_3) \,.\, (\partial_x(a_2) + \partial_y(a_3)) \\
&\quad + (18a_0^2a_3 - 8a_0a_1a_2 + 2a_1^3) \,.\, \partial_x(a_3) \\
\Delta \,.\, B &= (18a_0a_3^2 - 8a_1a_2a_3 + 2a_2^3) \,.\, \partial_y(a_0) \\
&\quad + (4a_1^2a_3 - a_1a_2^2 - 3a_0a_2a_3) \,.\, (\partial_x(a_0) + \partial_y(a_1)) \\
&\quad + a_0(2a_2^2 - 6a_1a_3) \,.\, (\partial_x(a_1) + \partial_y(a_2)) \\
&\quad + a_0(9a_0a_3 - a_1a_2) \,.\, (\partial_x(a_2) + \partial_y(a_3)) + a_0(2a_1^2 - 6a_0a_2) \,.\, \partial_x(a_3).
\end{aligned}$$

It can be noted that for $g \in \mathcal{O}^*$, the presentation $g \,.\, F$ gives a 1-form ${}^g\gamma$ where

$$ {}^g\gamma = \gamma - \frac{dg}{g} $$

since we verify that $A = -\dfrac{\partial_x(a_0)}{a_0} - \partial_y(\dfrac{a_1}{a_0}) + \dfrac{a_1}{a_0}\, l_1 - 2l_2$ and $B = -\dfrac{\partial_y(a_0)}{a_0} - l_1$. Hence the 2-form $\mathbf{k} = d\gamma$ depends *only* on the class of the differential equation defined by $F(x, y, y') = 0$.

Remark 3. The previous 1-form γ such that $\mathbf{k} = d\gamma$ is "normalized" through the case of a $\mathcal{L}_C(3) \subset \mathbb{P}^2$ associated by duality with a reduced cubic $C \subset \check{\mathbb{P}}^2$, as in Remark 2. Indeed, we get $\gamma = 0$ in this case. This result can be proved by using Abel's addition theorem and is also a consequence of the relation $(*)$. Agafonov introduced an another 1-form γ_{Ag} for a 3-web $\mathcal{W}(3)$ presented by F, such that $d\gamma_{Ag} = \mathbf{k}$ (*cf.* for example [Ag-2012]). A calculus proved that the previous γ and γ_{Ag} are related by the following formula : $\gamma = \gamma_{Ag} - \frac{1}{2} \cdot \frac{d\Delta}{\Delta}$ where here $\frac{d\Delta}{\Delta} = \frac{du}{u} + \sum_q m_q \frac{d\Delta_q}{\Delta_q}\cdot$

• *Dimension of the local system* $\mathfrak{Lie}\,\mathcal{W}(3)$

For $d = 3$, the system (LS) which gives the symmetries of $\mathcal{W}(3)$ is the following:

$$(LS) \quad \left\{ \begin{array}{ccc} -\,\partial_x(\beta) & + & g_3\alpha + h_3\beta = 0 \\ \partial_x(\alpha) - \partial_y(\beta) & + & g_2\alpha + h_2\beta = 0 \\ \partial_y(\alpha) & + & g_1\alpha + h_1\beta = 0. \end{array} \right.$$

Here the matrix of symmetries of the presentation F is $(S_{ij}) = \begin{pmatrix} g_3 & h_3 \\ g_2 & h_2 \\ g_1 & h_1 \end{pmatrix}$ and there are no additional conditions, contrary to the case $d \geq 4$.

By using methods “à la Cartan-Spencer” detailed in [Hé-2004], we incarnate the local system $\mathfrak{Lie}\,\mathcal{W}(3)$ as the horizontal sections of a non necessary integrable connection (E, ∇) with rank 3, which in fact is meromorphic on the discriminant locus $|\Delta|$. This connection (E, ∇), that is equivalently a free $\mathcal{O}$-module of rank 3 endowed with a connection $\nabla : E \longrightarrow \Omega^1 \otimes_{\mathcal{O}} E$ is called the *connection of symmetries* associated with F. It is constructed by using Cartan prolongations of the linear differential system (LS) and the first Spencer complex on suitable jets. Here ∇ is represented in an *adapted basis* (e_ℓ) of E by

$$\Gamma = \begin{pmatrix} -h_3dx + g_1dy & \xi_{11}dx + \xi_{12}dy & \xi_{21}dx + \xi_{22}dy \\ -dx & g_2dx + g_1dy & h_1dy \\ -dy & -g_3dx & -h_3dx - h_2dy \end{pmatrix}$$

with explicitly

$$\begin{aligned} \xi_{11} &= (g_1 + h_2)g_3 - \partial_y(g_3) \\ \xi_{12} &= g_3h_1 + \partial_x(g_1) - \partial_y(g_2) \\ \xi_{21} &= g_3h_1 + \partial_x(h_2) - \partial_y(h_3) \\ \xi_{22} &= (g_2 + h_3)h_1 + \partial_x(h_1) \end{aligned}$$

where $E := \operatorname{Ker} j_1 = (e_1, e_2, e_3) \subset J_2(\mathcal{O}^2)$, as jets. Without all details, we have

$$e_1 = \begin{pmatrix} 0 & -1 & 0 & g_2 - h_3 & g_1 & h_1 \\ 0 & 0 & 1 & g_3 & h_3 & h_2 - g_1 \end{pmatrix},$$

$$e_2 = \begin{pmatrix} -1 & g_2 & g_1 & \cdots \\ 0 & g_3 & 0 & \cdots \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & 0 & h_1 & \cdots \\ 1 & h_3 & h_2 & \cdots \end{pmatrix}.$$

Here $j_1 : J_2(\mathcal{O}^2) \longrightarrow J_1(\mathcal{O}^3)$ is the prolongation of the initial $j_0 : J_1(\mathcal{O}^2) \longrightarrow \mathcal{O}^3$ corresponding to the linear differential operator $\rho : \mathcal{O}^2 \longrightarrow \mathcal{O}^3$ associated with the system $(\mathcal{S})$, that is $j_0 \begin{pmatrix} z_1 & p_1 & q_1 \\ z_2 & p_2 & q_2 \end{pmatrix} = \begin{pmatrix} -p_2 + g_3 z_1 + h_3 z_2 \\ p_1 - q_2 + g_2 z_1 + h_2 z_2 \\ q_1 + g_1 z_1 + h_1 z_2 \end{pmatrix}$ with Monge's notation. A *horizontal section* $(f) = {}^t(f_1, f_2, f_3) \in \operatorname{Ker} \nabla$ verify $df + \Gamma \,.\, f = 0$ in matrix form where

$$\begin{cases} f_1 = \partial_x(\alpha) + g_2\alpha = \partial_y(\beta) - h_2\beta \\ f_2 = \alpha \\ f_3 = \beta \end{cases}$$

corresponds by construction to $X = \alpha\partial_x + \beta\partial_y \in \mathfrak{Lie}\,\mathcal{W}(3)$. Moreover in the basis (e_ℓ), the curvature K of (E, ∇) has a *convenient* curvature matrix $d\Gamma + \Gamma \wedge \Gamma = \begin{pmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} dx \wedge dy$ where we have explicitly

$$\begin{aligned} k_1 &= 2\partial_x(g_1) - \partial_x(h_2) - \partial_y(g_2) + 2\partial_y(h_3) \\ k_2 &= \partial_x(\xi_{12}) - \partial_y(\xi_{11}) - (g_2 + h_3)\xi_{12} + g_3\xi_{22} \\ &= \partial_x^2(g_1) - \partial_x\partial_y(g_2) + \partial_y^2(g_3) - (g_2 + h_3)(\partial_x(g_1) - \partial_y(g_2)) \\ &\quad -\partial_y((g_1 + h_2)g_3) + \partial_x(g_3 h_1) + g_3\partial_x(h_1) \\ k_3 &= \partial_x(\xi_{22}) - \partial_y(\xi_{21}) - (g_1 + h_2)\xi_{21} + h_1\xi_{11} \\ &= \partial_x^2(h_1) - \partial_x\partial_y(h_2) + \partial_y^2(h_3) - (g_1 + h_2)(\partial_x(h_2) - \partial_y(h_3)) \\ &\quad +\partial_x((g_2 + h_3)h_1) - \partial_y(g_3 h_1) - h_1\partial_y(g_3). \end{aligned}$$

By using the explicit expressions given above, we verify by a direct calculus that we have $h_2 = 2g_1 + 2B + \dfrac{\partial_y(\Delta)}{2\Delta}$ and $h_3 = \dfrac{g_2}{2} + A + \dfrac{\partial_x(\Delta)}{4\Delta}\cdot$ Hence in the adapted basis (e_ℓ), we obtain the *trace relation*

$$\operatorname{tr}(\Gamma) = -2\gamma - \frac{d\Delta}{2\Delta}$$

where $\gamma = Adx + Bdy$. In particular, we get $\operatorname{tr}(K) = d\operatorname{tr}(\Gamma) = k_1 dx \wedge dy = -2\mathbf{k}$ where $\mathbf{k} = d\gamma$ is the curvature of the 3-web $\mathcal{W}(3)$ presented by F.

THEOREM 2. *Let $\mathcal{W}(3)$ be an implicit planar 3-web presented by F with discriminant locus $|\Delta|$ and local system of symmetries $\mathfrak{Lie}\,\mathcal{W}(3)$. The following properties hold:*

a) There exists a connection (E, ∇) with rank 3, meromorphic on $|\Delta|$, with Ker$\nabla = \mathfrak{Lie}\,\mathcal{W}(3)$ outside $|\Delta|$ and such that its curvature K vanishes if, and only if, $\dim \mathfrak{Lie}\,\mathcal{W}(3) = 3$;

b) There exists an explicit 3×3-matrix $(k_{m\ell})$ which depends only on the class of the differential equations defined by $F(x, y, y') = 0$ such that $\dim \mathfrak{Lie}\,\mathcal{W}(3) = \operatorname{corank}(k_{m\ell})$. *In particular* $\dim \mathfrak{Lie}\,\mathcal{W}(3) \geq 1 \Longleftrightarrow \det(k_{m\ell}) = 0$, *hence a general $\mathcal{W}(3)$ has no symmetry. Moreover* $\dim \mathfrak{Lie}\,\mathcal{W}(3) = 3 \Longleftrightarrow (k_{m\ell}) = 0 \Longleftrightarrow \mathcal{W}(3)$ *is hexagonal.*

PROOF. a) With the help of Cauchy-Kowalevskaya theorem, this part summarizes the previous results on the local system $\mathfrak{Lie}\,\mathcal{W}(3)$.
b) From the above relation on trace, $\mathcal{W}(3)$ is hexagonal if $\dim \mathfrak{Lie}\,\mathcal{W}(3) = 3$. Moreover with the expressions

$$\begin{aligned}
\widetilde{\Delta_x} &= (9a_0a_3^2 - 7a_1a_2a_3 + 2a_2^3)\partial_x(a_0) + (3a_0a_2a_3 - a_1a_2^2 + 2a_1^2a_3)\partial_x(a_1) \\
&\quad + (-3a_0a_1a_3 - 2a_0a_2^2 + a_1^2a_2)\partial_x(a_2) \\
&\quad + (7a_0a_1a_2 - 9a_0^2a_3 - 2a_1^3)\partial_x(a_3) \\
\widetilde{\Delta_y} &= (9a_0a_3^2 - 7a_1a_2a_3 + 2a_2^3)\partial_y(a_0) + (3a_0a_2a_3 - a_1a_2^2 + 2a_1^2a_3)\partial_y(a_1) \\
&\quad + (-3a_0a_1a_3 - 2a_0a_2^2 + a_1^2a_2)\partial_y(a_2) \\
&\quad + (7a_0a_1a_2 - 9a_0^2a_3 - 2a_1^3)\partial_y(a_3),
\end{aligned}$$

we verify that we have $k_2 = \dfrac{1}{2}\,\partial_x(k_1) - \dfrac{\widetilde{\Delta_x}}{2\Delta}\,k_1$ and $k_3 = \dfrac{1}{2}\,\partial_y(k_1) + \dfrac{\widetilde{\Delta_y}}{2\Delta}\,k_1$. Hence $\dim \mathfrak{Lie}\,\mathcal{W}(3) = 3$ if, and only if, $\mathcal{W}(3)$ is hexagonal.

We also give in this part a process to describe $\dim \mathfrak{Lie}\,\mathcal{W}(3)$ as the corank of an effective matrix $(k_{m\ell}) : \mathcal{O}^3 \longrightarrow \mathcal{O}^3$ by a result proved by Olivier Ripoll in his Thesis [R-2005] (*cf.* also [R-2005bis]). It uses basically that $\mathfrak{Lie}\,\mathcal{W}(3)$ is a local system and the curvature matrix above coupled with Nakayama's lemma. The result goes as follows. For $(f) = {}^t(f_1, f_2, f_3) \in \operatorname{Ker}\nabla$ we have $K\,.\,f = 0$, that is only $k_1f_1 + k_2f_2 + k_3f_3 = 0$. Therefore, by using ∂_x and ∂_y with substitutions we get a matrix

$$(k_{m\ell}) := \begin{pmatrix} k_1 & k_2 & k_3 \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}$$

with two other rows given by

$$\begin{cases} k_{21} = \partial_x(k_1) + h_3k_1 + k_2 \\ k_{22} = \partial_x(k_2) - \xi_{11}k_1 - g_2k_2 + g_3k_3 \\ k_{23} = \partial_x(k_3) - \xi_{21}k_1 + h_3k_3 \end{cases} \quad \text{and}$$

$$\begin{cases} k_{31} = \partial_y(k_1) - g_1k_1 + k_3 \\ k_{32} = \partial_y(k_2) - \xi_{12}k_1 - g_1k_2 \\ k_{33} = \partial_y(k_3) - \xi_{22}k_1 - h_1k_2 + h_2k_3 \end{cases}$$

such that

$$\dim \mathfrak{Lie}\,\mathcal{W}(3) = \operatorname{corank}(k_{m\ell}).$$

The presentations F and $g\,.\,F$ of the $\mathcal{W}(3)$ at stake give the same line (k_1, k_2, k_3) for $g \in \mathcal{O}^*$ or the same convenient curvature matrix $d\Gamma + \Gamma \wedge \Gamma$ of (E, ∇) since essentially it is true for G and H. Hence by construction the matrix $(k_{m\ell})$ depends only on the class of the differential equations defined by $F(x, y, y') = 0$. Which ends the proof of b). $\square$

Remark 4. The previous result can be view as a complement of the statement Fact. We recover in part b) a result obtained by Élie Cartan in the generic case in [C-1908]. The above matrix $(k_{m\ell})$ provides a new invariant associated with $F(x, y, y') = 0$ and an effective method to find $\dim \mathfrak{Lie}\,\mathcal{W}(3)$.

Remark 5. The symmetries $\mathfrak{Lie}\,\mathcal{W}(3)$ of a nonsingular hexagonal planar 3-web $\mathcal{W}(3)$ is a local system of Lie algebras isomorphic to $\mathfrak{b} = \{\partial_x, \partial_y, x\partial_x + y\partial_y\}$, that is of type V in the Bianchi classification from the statement Fact and Example 1. In particular $\mathfrak{b}$ is solvable with the derived series $\{0\} \subset [\mathfrak{b}, \mathfrak{b}] = \{\partial_x, \partial_y\} \subset \mathfrak{b}$ where $\dim[\mathfrak{b}, \mathfrak{b}] = 2$. Elements in $\mathfrak{b}$ can be considered in $\mathfrak{gl}(3, \mathbb{C})$ as a matrix $\operatorname{ad}(a\partial_x + b\partial_y + c(x\partial_x + y\partial_y)) = \begin{pmatrix} -c & 0 & a \\ 0 & -c & b \\ 0 & 0 & 0 \end{pmatrix}$. Hence the Lie group of $\mathfrak{b}$ is $B = \{ \begin{pmatrix} e^t & 0 & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix}$; $(t, u, v) \in \mathbb{C}^3\} \subset \mathrm{GL}(3, \mathbb{C})$ and can be also viewed in the affine group of $\mathbb{C}^2$ generated by $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$ where $M \in \mathrm{GL}(2, \mathbb{C})$ and $(u, v) \in \mathbb{C}^2$. These observations are exploited in a work in progress on monodromy properties of (E, ∇).

Detailed Examples (continued).

2. *The* 3*-web formed by the pencils of lines through* 3 *generic points.*

It is the basic singular hexagonal web $\mathcal{W}(x, y, \frac{y}{x}) \subset \mathbb{P}^2$, that is with slopes ∞, 0 and $\frac{y}{x}$ which is related to the relation with 3 terms given by the logarithm through the equality $x \cdot \frac{1}{y} \cdot \frac{y}{x} = 1$. In the implicit setting, this web corresponds to

$$\mathcal{W}(y - x, y + x, \frac{y}{x}) = \mathcal{L}_C(3) \subset \mathbb{P}^2$$

with slopes ± 1 and $\frac{y}{x}$ or equivalently with $C \subset \check{\mathbb{P}}^2$ locally defined by $P(q, p) = (p^2 - 1)q = 0$, hence $F = (p^2 - 1)(y - px)$. Here we have $\Delta = 4(x^2 - y^2)^2$ with $\delta = x^2 - y^2$ and the singular locus of this $\mathcal{L}_C(3)$ is globally defined in homogeneous coordinates by $X_0(X_1 - X_2)(X_1 + X_2) = 0$. Here $\gamma = 0$ from Remark 3, hence $\dim \mathfrak{Lie}\, \mathcal{L}_C(3) = 3$. The associated matrix of symmetries is $(S_{ij}) = \frac{1}{\delta} \begin{pmatrix} -y & x \\ 0 & 0 \\ y & -x \end{pmatrix}$. We verify that a basis of $\mathfrak{Lie}\, \mathcal{L}_C(3)$ is given by 2 polynomial vector fields $x\partial_x + y\partial_y$ and $y\partial_x + x\partial_y$ completed by the multivalued vector field

$$\Big(x \log(y^2 - x^2) + y \log \frac{y + x}{y - x}\Big)\partial_x + \Big(y \log(y^2 - x^2) + x \log \frac{y + x}{y - x}\Big)\partial_y.$$

3. *A family* $\mathcal{Z}_{(m,n)}(3)$ *of examples "à la Zariski" with* $\dim \mathfrak{Lie}\, \mathcal{Z}_{(m,n)}(3) = 3$.

For $(m, n) \in \mathbb{N}^2$, we consider the family of 3-webs presented by $F = p^3 + x^m y^n$. We have $\Delta = -27x^{2m}y^{2n}$ with $\gamma = -\frac{2m}{3x}\, dx - \frac{n}{3y}\, dy$ and $\mathbf{k} = d\gamma = 0$. With previous notation $(S_{ij}) = \begin{pmatrix} 0 & 0 \\ \frac{m}{3x} & \frac{n}{3y} \\ 0 & 0 \end{pmatrix}$ and $\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ -dx & \frac{m}{3x}dx & 0 \\ -dy & 0 & -\frac{n}{3y}dy \end{pmatrix}$ with $\mathrm{tr}(\Gamma) = -2\gamma - \frac{d\Delta}{2\Delta}$ and $K = 0$. Moreover, we have the following partially multivalued basis for $n \neq 3$

$$\mathfrak{Lie}\,\mathcal{Z}_{(m,n)}(3) = \{x^{-\frac{m}{3}}\partial_x,\, y^{\frac{n}{3}}\partial_y,\, (3-n)x\partial_x + (3+m)y\partial_y\}$$

and $\mathfrak{Lie}\,\mathcal{Z}_{(m,3)}(3) = \{x^{-\frac{m}{3}}\partial_x,\, y\partial_y,\, \dfrac{3x}{m+3}\,\partial_x + y\log y\,\partial_y\}$.

4. *An example* $\mathcal{W}_u(3)$ *which is equivalent to the normal form in case of dimension* 1 *given by Élie Cartan in* [C-1908].

It is the 3-web presented by $F = (p^2-1)(p-u)$ with $u := u(x)$. We have $\Delta = 4(u^2-1)^2$ with $\mathbf{k} = d\gamma = \dfrac{u'' - u^2u'' + 2u(u')^2}{(u^2-1)^2} dx \wedge dy \neq 0$ for a generic u. It associated matrix of symmetries is $(S_{ij}) = \dfrac{u'}{u^2-1}\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have $\det(k_{m\ell}) = 0$ and we verify directly that $\mathfrak{Lie}\,\mathcal{W}_u(3) = \{\partial_y\}$.

5. *Another example with* $\dim \mathfrak{Lie}\,\mathcal{W}(3) = 1$.

We consider the 3-web $\mathcal{W}(3)$ presented by $F = p^3 + x\,.\,p + y$. We have $\Delta = -4x^3 - 27y^2$ with $\gamma = -\dfrac{8x^2dx + 18ydy}{4x^3+27y^2}$ and $\mathbf{k} = d\gamma \neq 0$. It associated matrix of symmetries is $(S_{ij}) = \dfrac{1}{4x^3+27y^2}\begin{pmatrix} 6xy & -4x^2 \\ 2x^2 & 9y \\ 9y & -6x \end{pmatrix}$. With the previous notation, we verify that $\det(k_{m\ell}) = 0$ and we get $\mathfrak{Lie}\,\mathcal{W}(3) = \{\mathfrak{X}_1 = 2x\partial_x + 3y\partial_y\}$ which can be viewed in $\mathrm{Der}(\log|\Delta|) = \{\mathfrak{X}_1, \mathfrak{X}_2 = 9y\partial_x - 2x^2\partial_y\}$ from Proposition 2.

4. Symmetries for d-Planar Webs with $d \geq 3$ and Rank Problems

Let $\mathcal{W}(d)$ be a d-web in $(\mathbb{C}^2, 0)$ implicitly presented by $F \in \mathcal{O}[p]$ with $d \geq 3$. For the 3-upper part of (LS), that is the differential system at stake and from the previous paragraph, we get a connection denote again by (E, ∇). Its horizontal sections describe, up to the additional conditions above, the vector fields $X = \alpha\partial_x + \beta\partial_y \in \mathfrak{Lie}\,\mathcal{W}(d)$. In an adapted basis, this connection (E, ∇) is represented by

$$\Gamma = \begin{pmatrix} -h_d dx + g_{d-2}dy & \xi_{11}dx + \xi_{12}dy & \xi_{21}dx + \xi_{22}dy \\ -dx & g_{d-1}dx + g_{d-2}dy & h_{d-2}dy \\ -dy & -g_d dx & -h_d dx - h_{d-1}dy \end{pmatrix}$$

where $\xi_{11} = (g_{d-2} + h_{d-1})g_d - \partial_y(g_d)$ and so on, with a curvature $\begin{pmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} dx\wedge dy$ where we have explicitly $k_1 = 2\partial_x(g_{d-2}) - \partial_x(h_{d-1}) - \partial_y(g_{d-1}) + 2\partial_y(h_d)$ and so on. Moreover Γ gives rise to a matrix $(k_{m\ell}) = \begin{pmatrix} k_1 & k_2 & k_3 \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}$ such that corank $(k_{m\ell})$ is the $\mathbb{C}$-dimension of the solution of the 3-upper part of the system (LS).

In general, the $d-3$ additional conditions which appear in the system (LS) can be written

$$\frac{g_1}{h_1} = \frac{g_2}{h_2} = \cdots = \frac{g_{d-3}}{h_{d-3}} = -\frac{\beta}{\alpha}\cdot$$

Therefore in order to have $\dim \mathfrak{Lie}\,\mathcal{W}(d) \geq 1$, all the previous conditions induce specific *constraints* on the ordered pair (G, H) associated with the implicit presentation F.

- *Weighted Euler symmetry*

For a d-web $\mathcal{W}(d)$ with $\dim \mathfrak{Lie}\,\mathcal{W}(d) \geq 1$, it may happen as the previous examples show that there exists a weighted Euler symmetry $X_w \in \mathfrak{Lie}\,\mathcal{W}(d)$, that is a vector field

$$X_w := w_x x\partial_x + w_y y\partial_y \;\text{ with weight }\; (w_x, w_y) \in \mathbb{C}^2 - \{0\}.$$

Using the differential system (LS), such a weighted Euler symmetry X_w exists for a $\mathcal{W}(d)$ presented by F if, and only if, the system

$$(wE) \quad \begin{cases} w_x x g_d + w_y y h_d & = & 0 \\ w_x x g_{d-1} + w_y y h_{d-1} & = & w_y - w_x \\ w_x x g_{d-2} + w_y y h_{d-2} & = & 0 \\ & \vdots & \\ w_x x g_1 + w_y y h_1 & = & 0 \end{cases}$$

has a nonzero solution $(w_x, w_y) \in \mathbb{C}^2$, hence necessarily $\operatorname{rank}\begin{pmatrix} g_1 & \cdots & g_{d-2} & g_d \\ h_1 & \ldots & h_{d-2} & h_d \end{pmatrix} \leq 1$. Particularly useful to detect the *radial symmetry* $X_r := x\partial_x + y\partial_y$, we note that $X_w \in \mathfrak{Lie}\,\mathcal{W}(d)$ if, and only if,

$X_w(p_i) = (w_y - w_x)p_i$ for $1 \le i \le d$ which is equivalent by using Girard-Newton formulas to the equalities

$$X_w(\frac{a_i}{a_0}) = i(w_y - w_x)\frac{a_i}{a_0} \quad \text{for} \quad 1 \le i \le d.$$

From previous observations, the symmetries $\mathfrak{Lie}\,\mathcal{W}(d)$ contains at most two $\mathbb{C}$-independent weighted Euler symmetries. For a sharp example suggested by Marín, take $F = \prod_{i=1}^{d}(xp - \lambda_i y)$ with complex numbers $\lambda_i \ne \lambda_j$. Indeed, we verify here that $\mathfrak{Lie}\,\mathcal{W}(d) = \{x\partial_x, y\partial_y, x\log x\,\partial_x + y\log y\,\partial_y\}$ since $G = -\dfrac{p}{x}$ and $H = \dfrac{p}{y}\cdot$

Remark 6. Let $C \subset \check{\mathbb{P}}^2$ be a smooth cubic, then its associated 3-web $\mathcal{L}_C(3) \subset \mathbb{P}^2$ has symmetries with $\dim \mathfrak{Lie}\,\mathcal{L}_C(3) = 3$. But there is *no weighted Euler symmetry.* Indeed, we may suppose C defined by $p^3 + a\,.\,p + b - q^2 = 0$ with $4a^3 + 27b^2 \ne 0$ and we verify here that $g_1h_3 - g_3h_1 = 0$ if, and only if, $a = b = 0$. In the hexagonal case, there is a classification in [Ag-2015] of 3-webs admitting a weighted Euler symmetry.

We recall that a W-curve is invariant under a 1-parameter subgroup of $\mathrm{PGL}(3,\mathbb{C})$. It is proved in [KL-1871] that these not necessarily algebraic curves $C \subset \check{\mathbb{P}}^2$, also called anharmonic by Georges-Henri Halphen, are generated by homogeneous equations

$$X_0^{\rho_0}X_1^{\rho_1}X_2^{\rho_2} = \lambda$$

where $\lambda \in \mathbb{C}$ and $\rho_0 + \rho_1 + \rho_2 = 0$ with $\rho_j \in \mathbb{C}$. For example, let $C \subset \check{\mathbb{P}}^2$ be the algebraic W-curve of degree $d \ge 3$ with affine equation

$$P(q,p) = p^d - \lambda q^a = 0 \quad \text{where} \quad 0 \le a \le d \quad \text{and} \quad \lambda \in \mathbb{C}^*.$$

It gives rise to the d-web $\mathcal{L}_C(d) \subset \mathbb{P}^2$ endowed with a nonzero $\mathfrak{Lie}\,\mathcal{L}_C(d)$ which contains the weighted Euler symmetry

$$X = (d-a)\,.\,x\partial_x + d\,.\,y\partial_y.$$

Indeed, we have $\mathfrak{Lie}\,\mathcal{L}_C(d) = \{\partial_x,\, \partial_y,\, x\partial_x + y\partial_y\}$ for $a = 0$ from Example 1. Otherwise, the dual curve $\check{C}$ is defined by a factor $\check{P} = \mu\lambda x^d + \nu y^{d-a}$ of the

p-discriminant of $F(x,y,p) = P(y-px,p)$. Hence $d(d-a)\check{P} = X(\check{P})$ for the vector field X above and $X \in \mathfrak{Lie}\,\mathcal{L}_C(d)$ by construction of $\mathcal{L}_C(d)$. We may also use the previous results since X is weighted Euler. Moreover we may verify, for $a = d$ and at least $3 \leq d \leq 6$ that $\mathfrak{Lie}\,\mathcal{L}_C(d) = \{y\partial_x,\, y\partial_y,\, y(x\partial_x + y\partial_y)\}$, and for $0 < a < d$ and at least $4 \leq d \leq 6$ that $\dim \mathfrak{Lie}\,\mathcal{L}_C(d) = 1$.

• *Rank and remarkable planar webs*

Let $\mathcal{W}(d) = \mathcal{W}(F_1, \ldots, F_d)$ be a nonsingular d-web in $(\mathbb{C}^2, 0)$ with $d \geq 3$, implicitly presented by F. It is defined by the 1-forms $\omega_i = dy - p_i dx$ or the vector fields $X_i = \partial_x + p_i\partial_y$ where $F(x,y,p_i) = 0$. A relation $\xi_1(F_1)dF_1 + \cdots + \xi_d(F_d)dF_d = 0$ with $\xi_i \in \mathbb{C}\{t\}$ is called an *abelian relation* of $\mathcal{W}(d)$. In fact these relations viewed as special d-uple $(\xi_1(F_1), \ldots, \xi_d(F_d)) \in \mathcal{O}^d$ form a local system $\mathcal{A}(d)$ such its *rank*, noted $\operatorname{rank}\mathcal{W}(d)$ depends only on $\mathcal{W}(d)$ and is bounded by $\pi_d = \frac{1}{2}(d-1)(d-2)$. With the same notation, the rank of a planar d-web $\mathcal{W}(d)$ is defined by using its generic nonsingular web. The previous bound is optimal. Indeed, let $\mathcal{L}_C(d) \subset \mathbb{P}^2$ be the d-web associated by duality with a reduced algebraic curve $C \subset \check{\mathbb{P}}^2$ with degree d which is introduced in Remark 2. All the leaves of $\mathcal{L}_C(d)$ are straight lines and from Abel's addition theorem, we have $\operatorname{rank}\mathcal{L}_C(d) = \dim_{\mathbb{C}} H^0(C, \omega_c) = \pi_d$.

For any d with necessary $d \geq 5$, Marín, Pereira and Pirio prove in [MPP-2006] that there exist *remarkable* webs $\mathcal{E}(d)$ in $(\mathbb{C}^2, 0)$. Such a d-web $\mathcal{E}(d)$ has maximum rank π_d and is not linearizable. In particular, by a converse of Abel's addition theorem, $\mathcal{E}(d)$ does not come from an algebraic curve in $\check{\mathbb{P}}^2$. The classification of these $\mathcal{E}(d)$, even for $d = 5$ is widely open.

If $Z = \alpha\partial_x + \beta\partial_y$ is transverse to a d-web $\mathcal{W}(d)$ in $(\mathbb{C}^2, 0)$ presented by F, that is $\beta - \alpha p_i \neq 0$ for $1 \leq i \leq d$ where $F(x,y,p_i) = 0$, we may consider the planar $(d+1)$-web $\mathcal{W}(d) \sqcup Z$ presented by

$$F_Z := (\alpha \,.\, p - \beta) \,.\, F$$

if in addition $\alpha \neq 0$. In this case $\mathcal{W}(d)$ appears as a sub-d-web of $\mathcal{W}(d) \sqcup Z$.

If a d-web $\mathcal{W}(d)$ is endowed with a *transverse symmetry* $X = \alpha\partial_x + \beta\partial_y$. Then from Theorem 1, $u_i(x,y) := \displaystyle\int_z^{(x,y)} \frac{\omega_i}{i_X(\omega_i)}$ is generically well defined locally since $\dfrac{\omega_i}{i_X(\omega_i)}$ is closed. Moreover $du_i = \dfrac{\omega_i}{i_X(\omega_i)} = \dfrac{dy - p_i dx}{\beta - \alpha p_i}$ and $u_i(z) = 0$ with $\mathcal{L}_X(u_i) = i_X(du_i) = X(u_i) = 1$. Hence, if $\displaystyle\sum_{i=1}^{d} \xi_i(u_i)du_i = 0$

we have $\mathcal{L}_X(\sum_{i=1}^{d}\xi_i(u_i)du_i) = \sum_{i=1}^{d}\xi_i'(u_i)du_i = 0$. Therefore $\mathcal{L}_X$ gives rise to a $\mathbb{C}$-linear map

$$\mathcal{L}_X : \mathcal{A}(d) \longrightarrow \mathcal{A}(d)$$

defined by $(\xi_i(u_i))_i \longmapsto (\xi_i'(u_i))_i$ on the $\mathbb{C}$-vector space $\mathcal{A}(d)$ of abelian relations of $\mathcal{W}(d)$.

An efficient method introduced in [MPP-2006] to construct families of remarkable webs $\mathcal{E}(d)$, from algebraic W-curves $C \subset \check{\mathbb{P}}^2$, is based on the following result which used the linear map $\mathcal{L}_X$ through their eigenvalues and eigenspaces:

Theorem MPP (2006). *Let $\mathcal{W}(d)$ be a planar d-web with $d \geq 3$ which admits a transverse symmetry $X = \alpha\partial_x + \beta\partial_y$ with $\alpha \neq 0$. Then we have*

$$\operatorname{rank}(\mathcal{W}(d) \sqcup X) = \operatorname{rank}\mathcal{W}(d) + d - 1.$$

In particular $\mathcal{W}(d)$ is of maximum rank $\pi_d = \frac{1}{2}(d-1)(d-2)$ if, and only if, $\mathcal{W}(d) \sqcup X$ is of maximum rank π_{d+1}.

Examples and Questions.

1. The polynomial $P(q,p) = p(p^2-1)q$ gives rise to a $\mathcal{L}_C(4) = \mathcal{W}(y, y-x, y+x, \frac{y}{x})$, that is with slopes 0, ± 1 and $\frac{y}{x}\cdot$ It corresponds to the linear 4-web generated by the pencils of lines through 4 distinct points such that exactly 3 of them are aligned. Here we have $F = p(p^2-1)(y-px)$, hence $R_F = -4xy^2(x-y)^2(x+y)^2$ and $\Delta = 4y^2(x-y)^2(x+y)^2$. We verify that its $\mathfrak{Lie}\,\mathcal{L}_C(4)$ is only generated by the radial symmetry X_r, which moreover is *not* transverse to the the $\mathcal{L}_C(4)$ at stake.

2. For the W-curve C defined by $P(q,p) = p^4 - q$ and by using the previous methods, we verify that its $\mathfrak{Lie}\,\mathcal{L}_C(4)$ is only generated by $X = 3x\partial_x + 4y\partial_y$ which moreover is transverse. Indeed, here $\Delta := -27x^4 - 256y^3$ and we have $(S_{ij}) = \dfrac{1}{\Delta}\begin{pmatrix} 36x^2y & -27x^3 \\ -9x^3 & -64y^2 \\ 64y^2 & -48xy \\ 48xy & -36x^2 \end{pmatrix}$. For the 3-upper part of (LS), we get $k_1 \neq 0$ and $\det(k_{m\ell}) = 0$. Hence there exists a unique solution (α, β) for this differential system, up to a complex number. In fact $(3x, 4y)$ verify

the system (LS). Then, we may prove directly by connection methods that $\mathcal{L}_C(4) \sqcup X$, presented by $F_X := (3x\,.\,p - 4y)(p^4 + x\,.\,p - y)$, is remarkable. Let us note that it is not linearizable since its linearization polynomial is $P_0 = -\dfrac{36x^2}{\Delta} \cdot p^4 + \cdots$ and therefore $\deg P_0 > 3$ (*cf.* for example [Hé-2014] for details and references).

3. *From an example suggested by Gilles Robert.* We start with the 4-web presented by $F = (p^2 - px + y)(p^2 - px + y + 1)$. It is a $\mathcal{L}_C(4)$ defined by $P = (p^2+q)(p^2+q+1)$ with $\Delta = (x^2-4y-4)(x^2-4y) := \Delta_1\,.\,\Delta_2$. We verify that its $\mathfrak{Lie}\,\mathcal{L}_C(4)$ is only generated by the symmetry $X = 2\partial_x + x\partial_y$ which is not weighted Euler, but with $X(\Delta_j) = 0$ for $1 \le j \le 2$. Moreover, we may check directly by connection methods that $F_X := (2p - x)\,.\,F$ provides a *not* completely decomposable remarkable 5-web $\mathcal{E}(5)$.

4. Let $\mathcal{E}(d)$ be a remarkable planar d-web, then necessarily $d \ge 5$ from a classic result. Moreover the following *dichotomy* holds: $\dim \mathfrak{Lie}\,\mathcal{E}(d)$ *is* 0 *or* 1, otherwise $\mathcal{E}(d)$ would be parallelizable from Proposition 1.

The pre-Bol has no symmetry for example, hence Bol's web $\mathcal{B}(5)$ too. More precisely, the algebraic 4-web $\mathcal{GP}(4) \subset \mathbb{P}^2$ formed by the pencils of lines through 4 generic points has no symmetry. Indeed, we may choose the 4 vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, hence with slopes

$$p_1 = \frac{y}{x},\ p_2 = \frac{y}{x-1},\ p_3 = \frac{y-1}{x-1},\ p_4 = \frac{y-1}{x}\cdot$$

Then we verify by a calculus that its presentation F gives $\det(k_{m\ell}) \neq 0$ for the 3-upper part of its associated system (LS). Here we recall that Bol's example $\mathcal{B}(5) := \mathcal{GP}(4) \sqcup Z$ given in [B-1936] where $Z = x(1-x)(1-2y)\partial_x + y(1-y)(1-2x)\partial_y$ is related to the five-term relation of the dilogarithm and is the first $\mathcal{E}(5)$ discovered.

There exist several examples of 5-webs $\mathcal{E}(5) := \mathcal{L}_C(4) \sqcup Z$ endowed with a transverse symmetry X. It is the case of a Terracini's example initially introduced in [Te-1937] and a Pirio's example (*cf.* for instance [MPP-2006]) implicitly presented by $F_Z := (yp-x)(p^4-1)$ (resp. $(xp+y)(p^4-1)$) with the radial symmetry X_r coming from the W-curve with equation $p^4 - 1 = 0$. In fact from the previous results and for a meromorphic germ $\xi = \xi(x,y)$, the planar 5-web presented by $F_\xi := (p - \xi)(p^4 - 1)$, hence with $\xi^4 \neq 1$, admits the transverse radial symmetry X_r if, and only if, we have $x\partial_x(\xi) + y\partial_y(\xi) =$

0 with $\xi \neq \dfrac{y}{x}\cdot$ Hence to get all the $\mathcal{E}(5)$ presented by F_ξ, from the partial differential equation viewpoint, it is sufficient to use the explicit 6 differential conditions on ξ given by connection methods in [R-2005] and the additional condition $\partial_x(\xi) + \xi\partial_y(\xi) \neq 0$ by using its linearization polynomial. For example the first condition, corresponding to the vanishing of its generalized curvature, is $(\xi^4 - 1)(\xi^2\partial_x^2(\xi) - \partial_y^2(\xi)) - 2\xi(\xi^4+1)\partial_x(\xi)^2 + 4\xi^3\partial_y(\xi)^2 = 0$.

With the terminology of [Hé-2014]), we consider the map $\mathfrak{u} : (\mathbb{C}^2, 0) \longrightarrow \check{\mathbb{P}}^{\pi_d - 1}$ classically associated "à la Poincaré-Blaschke" with a remarkable d-web $\mathcal{E}(d)$ or more generally a planar Bompiani d-web $\mathcal{W}_f(d)$ associated with a special analytic map $f : (\mathbb{C}^2, 0) \longrightarrow \check{\mathbb{P}}^{\pi_d - 1}$. In both cases, how to use the " additional property " for study the geometry of $\mathfrak{u}$ or f, if the web at stake is endowed with a symmetry? For $d = 5$, an analogous question appears already in [MPP-2006] for $\mathfrak{u}$. A particular global interest can be noted if moreover $\mathfrak{u}$ or f is rational, since such examples exist.

In order to precise these problems at least for $d = 5$ and with projective differential methods initiated by Alessandro Terracini [Te-1937] and following [Hé-2014], let $f : (\mathbb{C}^2, 0) \longrightarrow \check{\mathbb{P}}^5$ be an analytic map with maximum 2-osculation and associated 4×3-matrix (α_{ij}). We assume that f gives rise to a planar Bompiani 5-web $\mathcal{W}_f(\infty, 0\,; F)$ presented in normal form. This means in particular that this planar 5-web contains the canonical "warp and weft" 2-web $\mathcal{W}(x, y)$ and all its leaves are 1-principal curves associated with f. From the projective definition of f, we may always suppose

$$(\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & \alpha_{32} & 0 \\ 0 & \alpha_{42} & \alpha_{43} \end{pmatrix}$$

with $F := \alpha_{42}\,.\,p^3 - (3\alpha_{21} - 3\alpha_{32} + \alpha_{43})\,.\,p^2 - (\alpha_{11} - 3\alpha_{22})\,.\,p + \alpha_{12}$ such its associated polynomial $P_0 = l_1\,.\,p^2 + l_2\,.\,p + l_3$ presented above verifies the following *three* explicit differential conditions of the first order on the α_{ij}:

$$l_1 = 2\alpha_{21} - \alpha_{32},\ l_2 = \alpha_{11} - 2\alpha_{22},\ l_3 = -\alpha_{12}.$$

Moreover this 5-web $\mathcal{W}_f(\infty, 0\,; F)$ corresponds to a $\mathcal{E}(5)$ if, and only if, in addition only its generalized curvature verifies $k_{\mathcal{W}_f(\infty,0\,;F)} = 0$, that is *one* explicit differential condition of the second order on the α_{ij}. By using a previous result, $\mathcal{W}_f(\infty, 0\,; F)$ admits for example the radial symmetry X_r if, and only if, we have *three* explicit differential conditions of the first order on the α_{ij}. However, even with all these hypothesis, the geometric description of such f *modulo* $\mathrm{PGL}(6, \mathbb{C})$ seems open.

5. Singularities for Symmetries and Abelian Relations of a Planar 3-Web

Let (E, ∇) be the meromorphic connection of symmetries associated with the presentation F of an implicit 3-web $\mathcal{W}(3)$. It has rank 3 and it is meromorphic on the zero locus $|\Delta|$ of the p-discriminant Δ of F. Let $(\det E, \det \nabla)$ be the *determinant connection* associated with (E, ∇). By definition $\det E$ is the line bundle $\wedge^3 E$ and its associated connection is presented by $\mathrm{tr}(\Gamma) \in \Omega^1$ in the basis $e_1 \wedge e_2 \wedge e_3$ if (e_ℓ) is a basis of E. Hence $(\det E, \det \nabla)$ is meromorphic on $|\Delta|$.

The meromorphic connection of symmetries (E, ∇) is *regular singular* along $|\Delta|$ if for any transversal morphism $u : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, z)$ at a smooth point $z \in |\Delta|$, the pullback connection $(u^*E, u^*\nabla)$, which is always integrable, has a regular singularity at 0 in the usual sense in dimension one. This means for example that after a possible meromorphic change of basis of u^*E, the connection $u^*\nabla$ is presented by $\mathbb{B}(t)\dfrac{dt}{t}$ with an analytic 3×3-matrix $\mathbb{B}$ or equivalently there exists a cyclic element for $(u^*E, u^*\nabla)$ with the following matrix presentation:

$$\begin{pmatrix} 0 & 0 & -\phi_3 \\ 1 & 0 & -\phi_2 \\ 0 & 1 & -\phi_1 \end{pmatrix} dt$$

such that the $\phi_\ell \in \mathbb{C}\{t\}[1/t]$ verify *Fuchs' conditions*, namely $t^\ell \phi_\ell \in \mathbb{C}\{t\}$ for $1 \leq \ell \leq 3$. It can be verified with the help of $\xi \longmapsto {}^t(\xi, \xi', \xi'')$ that solutions of the linear differential equation

$$\xi'''(t) + \phi_1(t)\,.\,\xi''(t) + \phi_2(t)\,.\,\xi'(t) + \phi_3(t)\,.\,\xi(t) = 0$$

can be viewed as the horizontal sections of the dual connection associated with $(u^*E, u^*\nabla)$. In classic study of families of projective varieties parametrized by $\mathbb{P}^1$ for example, the corresponding linear differential equation is the so-called *Picard-Fuchs equation* associated with the *Gauss-Manin* *connection* of the family at stake. We indicate also that for an exact sequence of integrable meromorphic connections

$$0 \longrightarrow (E_1, \nabla) \longrightarrow (E_2, \nabla) \longrightarrow (E_3, \nabla) \longrightarrow 0\,,$$

then (E_2, ∇) is regular singular if, and only if, (E_1, ∇) and (E_3, ∇) are regular singular. Moreover the connection determinant $(\det E, \det \nabla)$ associated with (E, ∇) is regular singular if (E, ∇) is regular singular.

Let (E_a, ∇_a) be the meromorphic connection, with poles on $|\Delta|$, associated with the abelian relations $\mathcal{A}(3)$ of $\mathcal{W}(3)$ through F. We have $E_a = \operatorname{Ker} p_0$ where $p_0 : J_1(\mathcal{O}) \longrightarrow \mathcal{O}^2$ is given by $p_0(z,p,q) = (p+Az, q+Bz)$ and $\nabla_a : E_a \longrightarrow \Omega^1 \otimes_{\mathcal{O}} E_a$ is presented by $\gamma = Adx + Bdy$ in the basis $\varepsilon = (1, -A, -B)$. We recall that the horizontal sections of (E_a, ∇_a) are identified with $\mathcal{A}(3)$. In this approach related to Abel's addition theorem, any abelian relation of $\mathcal{W}(3)$ is interpreted as the vanishing trace associated with π through the π_i of a closed 1-form $a(x,y) \cdot \dfrac{dy - pdx}{\partial_p(F)} = a\,.\,\nu$ on the surface S defined by F, that is such a is an analytic solution of the linear differential system

$$\mathcal{M}(3) \qquad \begin{cases} \partial_x(a) + A\,a = 0 \\ \partial_y(a) + B\,a = 0. \end{cases}$$

In fact, we have a basic isomorphism $\mathcal{A}(3) \xrightarrow{\sim} \mathfrak{a}_F$ where $\mathfrak{a}_F = \{\omega = a\,.\,\nu\,;\, d\omega = 0\}$ such the element $(\xi_i(F_i))_i \in \mathcal{A}(3)$ corresponds to $a\,.\,\nu$ where $a = F \cdot (\displaystyle\sum_{i=1}^{3} \frac{\xi_i(F_i)\partial_y(F_i)}{p - p_i})$.

In the hexagonal case and with the trace relation of the previous paragraph, we obtain from the properties above that (E_a, ∇_a) *is regular singular along* $|\Delta|$ *if it is the case for* (E, ∇).

Let $\mathcal{H}(3)$ be a *singular hexagonal* planar 3-web presented by F, that is $\mathbf{k} = d\gamma = 0$ and $\Delta(0) = 0$. For such a $\mathcal{H}(3)$, we have $\dim \mathfrak{Lie}\,\mathcal{H}(3) = 3$ and we may consider three symmetries $X = \alpha_1\partial_x + \beta_1\partial_y$, $Y = \alpha_2\partial_x + \beta_2\partial_y$ and $Z = \alpha_3\partial_x + \beta_3\partial_y$ in $\mathfrak{Lie}\,\mathcal{H}(3)$. With the identification of $\bigwedge^3 \mathbb{C}^3$ and $\mathbb{C}$, we set

$$\begin{aligned} \theta := \det(X,Y,Z) &= \begin{vmatrix} \partial_y(\beta_1) - h_2\beta_1 & \partial_y(\beta_2) - h_2\beta_2 & \partial_y(\beta_3) - h_2\beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} \\ &= \alpha \wedge \beta \wedge \partial_y(\beta) \end{aligned}$$

for the *determinant* of the corresponding horizontal sections of (E,∇). It is multivalued in general. Here we have $\alpha = (\alpha_\ell)$ and $\beta = (\beta_\ell)$ as column vectors, and we get the following characterization:

$$\mathfrak{Lie}\,\mathcal{H}(3) = \{X, Y, Z\} \textit{ if, and only if, } \theta \text{ is nonzero outside the singular locus } |\Delta|.$$

In this case and with the previous notation and results, θ is a horizontal section of $(\det E, \det\nabla)$, that is $0 = d\theta + \mathrm{tr}(\Gamma)\,.\,\theta = d\theta - \theta(2\gamma + \dfrac{d\Delta}{2\Delta})$ and we have $\gamma = \dfrac{d\theta}{2\theta} - \dfrac{d\Delta}{4\Delta}\cdot$ Therefore $\mathbf{k} = d\gamma = 0$ hence there exists, locally and outside $|\Delta|$, an analytic solution a of the previous linear differential system $\mathcal{M}(3)$. In particular, the closed 1-form $a \cdot \dfrac{dy - p dx}{\partial_p(F)}$ corresponds to an abelian relation of $\mathcal{H}(3)$ with $\dfrac{da}{a} = \dfrac{d\Delta}{4\Delta} - \dfrac{d\theta}{2\theta}\cdot$

For a singular hexagonal 3-web $\mathcal{H}(3)$ presented by F with a reduced divisor $\delta = \prod_q \Delta_q$ associated with the p-discriminant Δ, it may happen that $\gamma \in \Omega^1(\log|\Delta|)$ in the sense of Kyoji Saito (*cf.* [S-1980]). This means here that only $\delta\gamma \in \Omega^1$, since $\delta d\gamma = 0$ by hypothesis. In this case the integrable meromorphic connection (E_a, ∇_a) (resp. $(\det E, \det\nabla)$) is regular singular along $|\Delta|$. Moreover by using a property of the 1-form γ recall in Paragraph 3 (above remark 3) and the exact sequence of K. Saito-Aleksandrov, we may suppose up to a change of presentation, that we have

$$\gamma = \sum_q r_q \cdot \frac{d\Delta_q}{\Delta_q}$$

with *residues* $r_q := \mathrm{res}_{\Delta_q}(\gamma)$ of γ, in the sense of Poincaré-Leray, along the irreducible components Δ_q of $|\Delta|$. These are *complex numbers* which depend *only* on the class of $F(x,y,y') = 0$ (*cf.* [Hé-2006]). The previous formula, also called the *determinant formula*, is a way to encode informations related to the singularities of the local system $\mathfrak{a}_F$ corresponding to the abelian relations of $\mathcal{H}(3)$.

In regard to regular singularities for symmetries, a large class of planar hexagonal 3-webs with *nonpositive rational residues* r_q is expected. For $X \in \mathfrak{Lie}\,\mathcal{H}(3)$ and with the notation of Proposition 2, we remark that in this case we get $\gamma(X) = \sum_q r_q \,.\, \lambda_{X,q}$ through the invariance property. More generally, a natural problem is to characterize singular hexagonal $\mathcal{H}(3)$ with a *regular singular* connection (E, ∇) of symmetries such the *transversal local monodromy theorem* holds. This means that through a generic transversal pullback of (E, ∇) to the smooth part of $|\Delta|$, all the eigenvalues of the local monodromy obtained are roots of unity.

Example 2 (continued). For the $\mathcal{L}_C(3)$ presented by $F = (p^2-1)(y - px)$, we have $\Delta = 4(x^2 - y^2)^2$. With the basis of $\mathfrak{Lie}\,\mathcal{L}_C(3)$ given above, we get $\theta = 2(x^2 - y^2)$ and $\gamma = 0$. Here $a = 1$ corresponds to an abelian relation of $\mathcal{L}_C(3)$, which is a general fact from Abel's addition theorem.

Example 3 (continued). For the hexagonal web $\mathcal{Z}_{(m,n)}(3)$ presented by $F = p^3 + x^m y^n$, we have $\Delta = -27x^{2m}y^{2n}$ and with $\Delta_1 = x$ and $\Delta_2 = y$, we get with corresponding residues

$$\gamma = -\frac{2m}{3} \cdot \frac{d\Delta_1}{\Delta_1} - \frac{n}{3} \cdot \frac{d\Delta_2}{\Delta_2}.$$

Here the connection of symmetries (E, ∇) is *logarithmic* along $|\Delta|$ since there exists a basis of E such that ∇ is represented by a connection matrix $\Gamma \in \Omega^1(\log|\Delta|)$. For $n \neq 3$, the basis of symmetries $\mathfrak{Lie}\,\mathcal{Z}_{(m,n)}(3) = \{x^{-\frac{m}{3}}\,\partial_x,\ y^{\frac{n}{3}}\,\partial_y,\ (3-n)x\partial_x + (3+m)y\partial_y\}$ gives $\theta = -\dfrac{(m+3)(n-3)}{3}\, x^{-\frac{m}{3}} y^{\frac{n}{3}}$ and, up to a complex number, $a = x^{\frac{2m}{3}} y^{\frac{n}{3}}$ corresponds to an abelian relation. It can be noted, contrary to the other symmetries of the basis above, that only the weighted Euler symmetry $X = (3-n)\,x\partial_x + (3+m)\,y\partial_y$ gives a complex number $\gamma(X) = -2m - n + \frac{1}{3}mn$. In case $n = 3$, it is also true as before only for $X = y\partial_y$ with $\gamma(X) = -1$.

Example 6 (Marín's family $\mathcal{M}_{(m,\lambda)}(3)$ with parameters $(m, \lambda) \in \mathbb{N} \times \mathbb{C}^*$). It is the family of 3-webs presented by

$$F = p(\lambda xp - y^m)(\lambda x(1-x)p - y^m) = 0.$$

Here $\Delta = \lambda^2 x^4 y^{6m}$ and $\gamma = -(\dfrac{\lambda}{y^m} + \dfrac{m}{y})dy$. Hence we get a (m, λ)-family of hexagonal 3-webs in $\mathbb{P}^2$ with a pole of order m for γ and residue $-(\lambda+1) \in \mathbb{C}$

along $\{y=0\}$ if $m=1$. For $m\neq 1$ (resp. $m=1$) a basis of $\mathfrak{Lie}\,\mathcal{M}_{(m,\lambda)}(3)$ is available with $\theta=-\lambda x^2y^me^{-2\lambda\frac{y^{1-m}}{1-m}}$ (resp. $\theta=-\lambda x^2y^{-2\lambda+1}$) which corresponds to a generator $a=y^me^{\lambda\frac{y^{1-m}}{1-m}}$ (resp. $a=y^{\lambda+1}$) for the abelian relations of the corresponding 3-web. In particular the connection of symmetries (E,∇) is not regular singular for $m\geq 2$. However it is regular singular for $m=1$, with a rational residue if, and only if, $\lambda\in\mathbb{Q}$.

Remark 7. Suppose Δ is reduced. It is for example the case for a $\mathcal{L}_C(3)\subset\mathbb{P}^2$ associated with a smooth cubic $C\subset\check{\mathbb{P}}^2$. Then according to the order of the poles in the initial presentation $\Gamma=(\Gamma_{ij})$ of (E,∇) in Paragraph 3, the meromorphic change of basis P defined by $(\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3)=(e_1,e_2,e_3)P$ where $P=\begin{pmatrix}\delta^{-1}&0&0\\0&1&0\\0&0&1\end{pmatrix}$ gives rise to a presentation $\widetilde{\Gamma}=P^{-1}\Gamma P+P^{-1}dP=\begin{pmatrix}\Gamma_{11}-\dfrac{d\delta}{\delta}&\delta\,\Gamma_{12}&\delta\,\Gamma_{13}\\-\dfrac{dx}{\delta}&\Gamma_{22}&\Gamma_{23}\\-\dfrac{dy}{\delta}&\Gamma_{32}&\Gamma_{33}\end{pmatrix}$ which has at worst simple poles on $|\Delta|$. This proves in this case that the connection (E,∇) and therefore also (E_a,∇_a) is regular singular along $|\Delta|$.

• *Singular hexagonal 3-webs related to WDVV-equations*

From geometry of 3-dimensional Frobenius manifolds only two non-equivalent normal forms appear as corresponding associativity equation of order three for $f\in\mathcal{O}$. For the basic two types these nonlinear partial differential WDVV-equations, with standard notation in the Frobenius world for partial derivatives, are

$$\begin{aligned}E_1 &:= f_{yyy}+f_{xxx}\cdot f_{xyy}-f_{xxy}^2=0\\ E_2 &:= f_{xxx}\cdot f_{yyy}-f_{xxy}\cdot f_{xyy}-1=0.\end{aligned}$$

They give rise to special planar webs called *characteristic* 3-*webs* by Ferapontov in [F-2004] and *booklet* 3-*webs* by Agafonov in [Ag-2012]. Corresponding to $E_j=0$ and denoted by $\mathcal{W}_j(3)$, these are respectively presented by

$$\begin{aligned}W_1 &:= f_{xyy}\,.\,p^3+2f_{xxy}\,.\,p^2+f_{xxx}\,.\,p-1\\ W_2 &:= f_{yyy}\,.\,p^3+f_{xyy}\,.\,p^2-f_{xxy}\,.\,p-f_{xxx}.\end{aligned}$$

With the previous notation and by calculus, the following identities hold for W_1:

$$\begin{aligned} 2\Delta\,.\,A+\partial_x(\Delta) &= -4\partial_x(E_1)(6f_{xxy}+f_{xxx}^2) \\ 2\Delta\,.\,B+\partial_y(\Delta) &= -2\partial_x(E_1)(9f_{xyy}+2f_{xxx}\cdot f_{xxy}) \end{aligned}$$

and for W_2:

$$\begin{aligned} 2\Delta\,.\,A+\partial_x(\Delta) &= -2\partial_x(E_2)(9f_{xxx}\cdot f_{yyy}-f_{xxy}\cdot f_{xyy}) \\ &\quad -4\partial_y(E_2)(3f_{xxx}\cdot f_{xyy}+f_{xxy}^2) \\ 2\Delta\,.\,B+\partial_y(\Delta) &= -4\partial_x(E_2)(3f_{xxx}\cdot f_{xyy}+f_{xxy}^2) \\ &\quad -2\partial_y(E_2)(9f_{xxx}\cdot f_{yyy}-f_{xxy}\cdot f_{xyy}). \end{aligned}$$

Hence from the above identities and the irreducible decomposition of the p-discriminant $\Delta = u\,.\prod_q \Delta_q^{m_q}$ of the presentation W_1 (resp. W_2), we get the following result:

Proposition 3. *The* 3*-webs* $\mathcal{W}_1(3)$ *and* $\mathcal{W}_2(3)$ *are hexagonal with* $\gamma = -\frac{1}{2}\cdot\frac{d\Delta}{\Delta}\cdot$ *In particular, for the two types, these hexagonal* 3*-webs are regular singular from abelian relations viewpoint and the corresponding residues along* Δ_q *are nonpositive rational numbers equal to* $-\frac{m_q}{2}\cdot$

Before to give some examples with the E_1-type, the analogy in the previous result for the two types should be put into perspective with the Ferapontov-Mokhov transformation indicated in [F-2004] between these two types.

Examples WDVV-1. In addition to $E_j = 0$, solutions of WDVV-equations have quasihomogeneity constraints. By using Dubrovin's normal forms found in [D-1996] we get the following examples of 3-webs $\mathcal{W}_1(3)$ through some explicit $f\in\mathcal{O}$, with rational residues from above. The vector field $E = t\partial_t + [(1-q_1)x+r_1]\partial_x + [(1-q_2)y+r_2]\partial_y$, with $(q_i, r_i)\in\mathbb{C}^2$ where $r_i\neq 0$ only if $q_i = 1$, attached to the Frobenius structure at stake provides the indicated E-symmetry:

1. $f_1 = \dfrac{x^2y^2}{4}+\dfrac{y^5}{60}$ with $E = \dfrac{1}{4}(4t\partial_t + 3x\partial_x + 2y\partial_y)$ gives rise to a $\mathcal{W}_1(3)$ presented by $F_{f_1} = x\,.\,p^3 + 2y\,.\,p^2 - 1$ where $\Delta = -27x^2 + 32y^3$ and the

E-symmetry $3x\partial_x + 2y\partial_y$. A variant of this hexagonal web with a cusp as discriminant locus has been given by Alcides Lins Neto and Isao Nakai. Under the so-called regularity condition, it is the unique companion of the 3-web $\mathcal{L}_C(3) \subset \mathbb{P}^2$ associated with $P(q,p) = p^3 - q = 0$ (*cf.* [N-2014] for details and references).

2. $f_2 = \dfrac{x^3y}{6} + \dfrac{x^2y^3}{6} + \dfrac{y^7}{210}$ with $E = \dfrac{1}{6}(6t\partial_t + 4x\partial_x + 2y\partial_y)$ gives the presentation $F_{f_2} = 2xy \,.\, p^3 + 2(x+y^2) \,.\, p^2 + y \,.\, p - 1$ where $\Delta = 4(2x + y^2)(2x - 3y^2)^2$ and the E-symmetry $2x\partial_x + y\partial_y$.

3. $f_3 = \dfrac{x^3y^2}{6} + \dfrac{x^2y^5}{20} + \dfrac{y^{11}}{3960}$ with $E = \dfrac{1}{10}(10t\partial_t + 6x\partial_x + 2y\partial_y)$ gives the presentation $F_{f_3} = x(x+2y^3) \,.\, p^3 + y(4x+y^3) \,.\, p^2 + y^2 \,.\, p - 1$ where $\Delta = -(27x + 5y^3)(x - y^3)^3$ and the E-symmetry $3x\partial_x + y\partial_y$.

4. $f_4 = -\dfrac{x^4}{24} + xe^y$ with $E = \dfrac{1}{2}(2t\partial_t + x\partial_x + 3\partial_y)$ gives the presentation $F_{f_4} = e^y \,.\, p^3 - x \,.\, p - 1$ where $\Delta = e^y(4x^3 - 27e^y)$ and the E-symmetry $x\partial_x + 3\partial_y$.

References

[Ag-2012] Agafonov, S. I., Flat 3-webs via semi-simple Frobenius 3-manifolds, J. Geom. Phys. **62** (2012), 361–367.

[Ag-2015] Agafonov, S. I., Local classification of singular hexagonal 3-webs with holomorphic Chern connection form and infinitesimal symmetries, Geom. Dedicata **176** (2015), 87–115.

[AG-2000] Akivis, M. A. and V. V. Goldberg, *Differential Geometry of Webs*, in Handbook of Differential Geometry, Vol. 1, Ed. F. J. E. Dillen and L. C. A. Verstraelen, Elsevier Science, Amsterdam, 2000, 1–152.

[AS-1992] Akivis, M. A. and A. M. Shelekhov, *Geometry and Algebra of Multidimensional Three-Webs*, Kluwer Academic Publishers, Dordrecht, 1992.

[BB-1938] Blaschke, W. und G. Bol, *Geometrie der Gewebe*, Springer, Berlin, 1938.

[B-1936] Bol, G., Über ein bemerkenswertes Fünfgewebe in der Ebene, Abh. Hamburg **11** (1936), 387–393.

[C-1908] Cartan, É., Les sous-groupes des groupes continus de transformations, Ann. Sci. École Norm. Sup. **25** (1908), 57–194.

[Ch-1982] Chern, S. S., Web Geometry, Bull. Amer. Math. Soc. **6** (1982), 1–8.

[D-1996] Dubrovin, B., Geometry of $2D$ topological field theories, Springer

Lecture Notes in Math. **1620** (1996), 120–348.

[F-2004] Ferapontov, E. V., Hypersurfaces with Flat Centroaffine Metric and Equations of Associativity, Geometriae Dedicata **103** (2004), 33–49.

[GL-2006] Goldberg, V. V. and V. V. Lychagin, On the Blaschke conjecture for 3-webs, J. Geom. Anal. **16** (2006), 69–115.

[GM-1993] Granger, M. and P. Maisonobe, A basic course on differential modules, in *$\mathcal{D}$-modules cohérents et holonomes*, Travaux en cours **45**, Hermann, Paris, 1993, 103–168.

[Hé-2000] Hénaut, A., Sur la courbure de Blaschke et le rang des tissus de $\mathbb{C}^2$, Natural Science Report, Ochanomizu Uni. **51** (2000), 11–25.

[Hé-2004] Hénaut, A., On planar web geometry through abelian relations and connections, Ann. of Math. **159** (2004), 425–445.

[Hé-2006] Hénaut, A., Planar web geometry through abelian relations and singularities, in *Inspired by S. S. Chern, Nankai Tracts in Math* **11**, Ed. P. A. Griffiths, World Scientific, Singapore, 2006, 269–295.

[Hé-2014] Hénaut, A., Planar webs of maximum rank and analytic projectives surfaces, Math. Z. **278** (2014), 1133–1152.

[KL-1871] Klein, F. und S. Lie, Über diejenigen ebenen Kurven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen, Math. Ann. **4** (1871), 50–84.

[M-2002] Malgrange, B., On nonlinear differential Galois theory, Chinese Ann. Math. **23** (2002), 219–226.

[MPP-2006] Marín, D., Pereira, J. V. and L. Pirio, On planar webs with infinitesimal automorphisms, in *Inspired by S. S. Chern, Nankai Tracts in Math.* **11**, Ed. P. A. Griffiths, World Scientific, Singapore, 2006, 351–364.

[N-2014] Nakai, I., Webs and singularities, Journal of Singularities **9** (2014), 151–167.

[PP-2015] Pereira, J. V. and L. Pirio, *An Invitation to Web Geometry*, IMPA Monographs, Vol. 2, Springer, 2015.

[R-2005] Ripoll, O., Géométrie des tissus du plan et équations différentielles, Thèse de doctorat, Université Bordeaux 1, 2005.

[R-2005bis] Ripoll, O., Détermination du rang des tissus du plan et autres invariants géométriques, C. R. Acad. Sci. Paris **341** (2005), 247–252.

[S-1980] Saito, K., Theory of logarithmic differential forms and logarithmic vector fields, J. Fac. Sci. Univ. Tokyo **27** (1980), 265–291.

[Te-1937] Terracini, A., Su una possibile particolarità delle linee principali di una superficie, Note I e II, Rend. della R. Acc. dei Lincei **26** (1937), 84–91 & 153–158.

(Received November 25, 2021)
(Revised January 12, 2022)

Institut de Mathématiques de Bordeaux
Université de Bordeaux et CNRS (UMR 5251)
F-33400 Talence, France
E-mail: Alain.Henaut@math.u-bordeaux.fr