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Cellular Chain Complexes of Universal Covers of Some 3-Manifolds

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Abstract. For a closed 3-manifold M in a certain class, we give a presentation of the cellular chain complex of the universal cover of M. The class includes all surface bundles, some surgeries of knots in S^3, some cyclic branched cover of S^3, and some Seifert manifolds. In application, we establish a formula for calculating the linking form of a cyclic branched cover of S^3, and develop procedures of computing some Dijkgraaf-Witten invariants.

1. Introduction

In order to investigate a connected CW-complex X with a non-trivial fundamental group $\pi_1(X)$, it is important to give a concrete presentation of the cellular chain complex, $C_*(\widetilde{X};\mathbb{Z})$, and the cup-products of the universal cover $\widetilde{X}$. In fact, the homology of X with local coefficients and the (twisted) Reidemeister torsion of X are defined from $C_*(\widetilde{X};\mathbb{Z})$. If X is a $K(\pi,1)$-space, the chain complex means a projective resolution of the group ring $\mathbb{Z}[\pi_1(X)]$. Thus, it is also of use for computing many invariants to concretely present $C_*(\widetilde{X};\mathbb{Z})$.

This paper focuses on a class of closed 3-manifolds satisfying the following condition:

Assumption (†). *A closed oriented 3-manifold M satisfies that any closed 3-manifold M' with a group isomorphism $\pi_1(M)\cong\pi_1(M')$ admits a homotopy equivalence $M\simeq M'$.*

For example, M satisfies this assumption if M is an Eilenberg-MacLane space of type $(\pi_1(M),1)$, which is equivalent to that M is irreducible and has an infinite fundamental group. In Section 3, we examine many 3-manifolds,

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including all surface bundles, some surgeries of knots in S^3, spliced sums, cyclic branched covers of S^3 with Assumption (†), and some Seifert manifolds. For when M is one of these, we describe presentations of the complex $C_*(\widetilde{M};\mathbb{Z})$ and of the cup-product $H^1(M;N)\otimes H^2(M;N') \to H^3(M;N\otimes N')$ for any local coefficient modules N, N'. The procedure for obtaining such descriptions essentially follows from the work of [Sie, Tro] in terms of "identity", which we review in Section 2. This procedure can also be used to describe the fundamental homology 3-class, $[M]$ of M; see Remark 2.4.

In application, we give a formula for the linking forms of cyclic branched covers of S^3 with Assumption (†) (see Propositions 4.1). Furthermore, we develop procedures of computing some Dijkgraaf-Witten invariants from the above descriptions; see §5. In addition, such descriptions of identities are used for computing knot concordance groups, Reidemeister torsions, and Casson invariants; see [MP, No1, Waki]. There might be other applications from the above presentations of the complexes $C_*(\widetilde{M};\mathbb{Z})$

Conventional notation. In this paper, every manifold is understood to be smooth, connected, and orientable. By M, we mean a closed 3-manifold with orientation $[M]$.

2. Taut Identities and Cup-Products

2.1. Review: identities and cup-products

Let us recall the procedure of obtaining cellular chain complexes of some universal covers, as described in the papers [Sie] and [Tro]. There is nothing new in this section.

We will start by reviewing identities. Take a finitely presented group $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_m\rangle$ of deficiency zero. Setting up the free groups $F := \langle x_1, \ldots, x_m \mid\rangle$ and $P := \langle \rho_1, \ldots, \rho_m \mid\rangle$, let us consider the homomorphism,

$$\psi : P * F \longrightarrow F \quad \text{defined by} \quad \psi(\rho_j) = r_j, \quad \psi(x_i) = x_i.$$

An element $s \in P * F$ is *an identity* if $s \in \mathrm{Ker}(\psi)$ and s can be written as $\prod_{k=1}^{n} \omega_k \rho_{j_k}^{\epsilon_k} \omega_k^{-1}$ for some $w_k \in F$, $\epsilon_k \in \{\pm 1\}$ and indices j_k's.

Given a closed 3-manifold M with a genus-m Heegaard splitting, let us review the cellular complex of the universal cover, $\widetilde{M}$, of M. A CW-complex structure of M induced by the splitting consists of a single zero-cell, m one-handles, m two-handles, and a single three-handle. Therefore, $\pi_1(M)$ has a group presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle$, and the cellular complex of $\widetilde{M}$ is described as

$$C_*(\widetilde{M}; \mathbb{Z}) : 0 \to \mathbb{Z}[\pi_1(M)] \xrightarrow{\partial_3} \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(M)]^m \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(M)] \to 0. \tag{1}$$

Here, $\mathbb{Z}[\pi_1(M)]$ is the group ring of $\pi_1(M)$. We will explain the boundary maps ∂_* in detail. Let $\{a_1, \ldots, a_m\}$, $\{b_1, \ldots, b_m\}$, and $\{c\}$ denote the canonical bases of $C_1(\widetilde{M}; \mathbb{Z})$, $C_2(\widetilde{M}; \mathbb{Z})$, and $C_3(\widetilde{M}; \mathbb{Z})$ as left $\mathbb{Z}[\pi_1(M)]$-modules, respectively. Then, as is shown in [Lyn], $\partial_1(a_i) = 1 - x_i$, and $\partial_2(b_i) = \sum_{k=1}^m [\frac{\partial r_i}{\partial x_k}] a_k$, where $\frac{\partial r_i}{\partial x_k}$ is the Fox derivative. Moreover, the main result in [Sie] is that there exists an identity s such that $\partial_3(c) = \sum_k [\psi(\frac{\partial s}{\partial \rho_k})] b_k$.

Next, we will briefly give a formula for the cup-product in terms of the identity, which is a result of [Tro, §2.4]. Let N and N' be left $\mathbb{Z}[\pi_1(M)]$-modules. We can define the cochain complex on $C^*(M; N) := \mathrm{Hom}_{\mathbb{Z}[\pi_1(M)]}(C_*(\widetilde{M}; \mathbb{Z}), N)$ with local coefficients. Recalling the definition of the identity $s = \prod_{k=1}^n \omega_k \rho_{j_k}^{\epsilon_k} \omega_k^{-1}$, define

$$D^\sharp(c) = \sum_{k=1}^n \epsilon_k \Big(\sum_{\ell=1}^m [\frac{\partial \omega_k}{\partial x_\ell}] a_\ell \otimes \omega_k b_{j_k} \Big) \in C_1(\widetilde{M}; \mathbb{Z}) \otimes C_2(\widetilde{M}; \mathbb{Z}). \tag{2}$$

Then, for cochains $p \in C^1(M; N)$ and $q \in C^2(M; N')$, we define a 3-cochain $p \smile q$ by

$$p \smile q(uc) := (p \otimes q)(u D^\sharp(c)) \in N \otimes_{\mathbb{Z}} N'.$$

Here, $u \in \mathbb{Z}[\pi_1(M)]$. Then, the map

$$\smile : C^1(M; N) \otimes C^2(M; N') \to C^3(M; N \otimes_{\mathbb{Z}} N'); \quad (p, q) \mapsto p \smile q,$$

induces the bilinear map on cohomology, which is known to be equal to the usual cup-product. Here, notice that, since the third $\partial_3 \otimes_{\mathbb{Z}[\pi_1(M)]} \mathrm{id}_{\mathbb{Z}}$ is zero, the 3-class $s \otimes 1 \in C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}$ is a generator of $H_3(C_*(\widetilde{M}) \otimes \mathbb{Z}) \cong H_3(M; \mathbb{Z}) \cong \mathbb{Z}$, which represents the fundamental 3-class $[M]$; thus, given a $\pi_1(M)$-invariant bilinear map $\psi : N \otimes N' \to A$ for some abelian group A, we have the following equality on the pairing of $[M]$:

$$\psi \circ \smile (p, q) = \psi \langle p \smile q, [M] \rangle \in A, \tag{3}$$

for any cochains $p \in C^1(M; N)$ and $q \in C^2(M; N')$.

In summary, for a description of the complex $C_*(\widetilde{M})$ and the cup-product, it is important to describe an identity from M.

2.2. Taut identities

In order to find such identities giving the complex (1), we review tautness from [Sie]; see also [Waki, Appendix] for a brief explanation. Fix a finite presentation $\langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$. Let $s = \prod_{k=1}^{2m} w_k \rho_{j_k}^{\epsilon_k} w_k^{-1} \in P * F$ be an identity, where ρ_{j_k} and w_k can be written in

$$\rho_{j_k} = a_{k,1}^{\epsilon_{k,1}} \cdots a_{k,\ell_k}^{\epsilon_{k,\ell_k}}, \qquad w_k = b_{k,1}^{\eta_{k,1}} \cdots b_{k,n_k}^{\eta_{k,n_k}}, \qquad (\epsilon_{i,j}, \eta_{i,j} \in \{\pm 1\}).$$

Here, $a_{k,\ell}$ and $b_{k,\ell}$ lie in $\{x_1, \dots, x_m\}$. For each $w_k \rho_{j_k}^{\epsilon_k} w_k^{-1}$, take the ℓ_k-gon D_{j_k} whose i-th edge is labeled by $a_{k,i}^{\epsilon_{k,i}}$, and the segment $I_k = [0, n_k]$ such that $[i-1, i]$ is labeled by $b_{k,i}^{\eta_{k,i}}$.

DEFINITION 2.1 ([Sie]).

(1) A self-bijection

$$\mathcal{I} : \cup_{k=1}^{2m} \{(k,1), \dots, (k, \ell_k)\} \to \cup_{k=1}^{2m} \{(k,1), \dots, (k, \ell_k)\}$$

is called a *syllable* if $a_{\mathcal{I}(i,j)} = a_{i,j} \in F$ and $\epsilon_{i,j} = -\epsilon_{\mathcal{I}(i,j)} \in \{\pm 1\}$.

(2) For a syllable $\mathcal{I}$, consider the following equivalence on the disjoint union $\sqcup_{i=1}^{2m} D_{r_i}$: the interval with labeling $a_{i,j}$ is identified with those with labeling $a_{\mathcal{I}(i,j)}$.

(3) An identity s is said to be *taut* if there is a syllable $\mathcal{I}$ such that the quotient space $\sqcup_{i=1}^{2m} D_{r_i} / \sim$ of $\sqcup_{i=1}^{2m} D_{r_i}$ subject to the above equivalence $\sim$ is homeomorphic to S^2, and if there are injective continuous maps

$$\kappa_k : I_k = [0, n_k] \to \sqcup_{i=1}^{2m} \partial D_{r_i} / \sim, \qquad \lambda_k : [0, \ell_k) \to \partial D_{r_k} / \sim$$

satisfying the following condition (*).

(*) For each k, the image $\kappa_k([i-1, i])$ coincides with an edge labeled by $b_{k,i}$ compatible with the orientations, and $\lambda_k([j-1, j])$ coincides with the j-th edge of D_{r_k} compatible with the orientations. Furthermore, $\kappa_k(n_k) = \lambda_k(0) = \lambda_k(\ell_k)$.

This paper is mainly based on the following theorem of Sieradski:

THEOREM 2.2 ([Sie]). *Given a group presentation $\langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$ with a taut identity s, there exists a closed 3-manifold M with* *a genus-m Heegaard splitting such that the complex* $C_*(\widetilde{M};\mathbb{Z})$ *is isomorphic to the complex* (1)*.*

In a concrete situation where an identity s is explicitly described, it is not so hard to find such a $\mathcal{I}$ and show the tautness of s (in fact, this check is to construct a 2-sphere from the disjoint union $\sqcup_{i=1}^{n} D_{r_i}$ as a naive pasting). In all the statements in §3, we will claim that some identities satisfy the taut condition; however, we will also omit the check by elementary complexity, as in other papers on taut identities [BH, Sie, Tro].

Example 2.3. As an easy example of the pasting, we focus on the 3-dimensional torus $M = (S^1)^3$ with presentation $\pi_1(M) = \langle x, y, z \mid r, s, u \rangle$, where $r = [x, y], s = [y, z], u = [z, x]$. As in [Sie], consider the following identity.

$$W_{(S^1)^3} = r(y^{-1}u^{-1}y)s(z^{-1}rz)u(x^{-1}s^{-1}x).$$

Then, Figure 1 gives a self-bijection and λ_m, κ_m satisfy the tautness. Moreover, if we attach a 3-ball in the right hand side in the figure along the boundary of the 3-cube, the resulting space is equal to $(S^1)^3$.

REMARK 2.4. Suppose that we find a taut identity s from $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle$, and the resulting 3-manifold M satisfies Assumption (†). Then, by Assumption (†), the resulting 3-manifold up to homotopy does not depend on the choice of s. In particular, we emphasize that, if M satisfies Assumption (†) and we find a taut identity from $\pi_1(M) = \langle x_1, \ldots, x_m \mid$

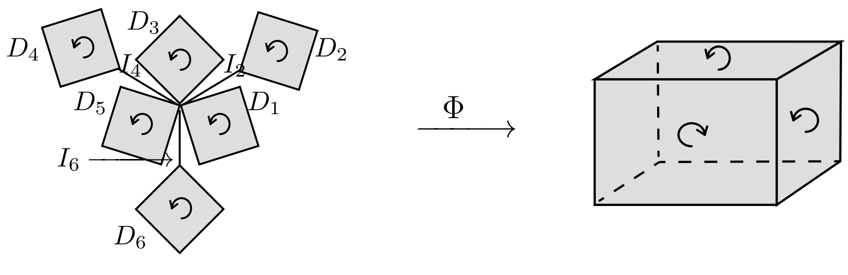

Fig. 1. The tautness of $(S^1)^3$. The right side means the 2-sphere obtained as the quotient $\sqcup_{i=1}^{6} D_i/\sim$. Here, the restriction map on I_i of Φ means λ_i, and the restriction map on ∂D_i of Φ means κ_i.

$r_1, \ldots, r_m\rangle$, then the third ∂_3 and the cup-product are uniquely determined, up to homotopy, by the identity. In fact, if we have another identity ω' and consider the associated $C_*(\widetilde{M})'$, Assumption (†) ensures a chain map $C_*(\widetilde{M}) \to C_*(\widetilde{M})'$, which induces a homotopy equivalence.

3. Descriptions of Taut Identities of Various 3-Manifolds

In this section, we give several examples of identities from some classes of 3-manifolds. We will describe the cellular complexes of some universal covers.

3.1. Fibered 3-manifolds with surface fibers over the circle

First, we will focus on surface bundles over S^1. Let Σ_g be an oriented closed surface of genus g and $f : \Sigma_g \to \Sigma_g$ an orientation-preserving diffeomorphism. *The mapping torus*, T_f, is the quotient space of $\Sigma_g \times [0,1]$ subject to the relation $(y,0) \sim (f(y),1)$ for any $y \in \Sigma_g$. The homeomorphism type of T_f depends on the mapping class of f. Conversely, if a closed 3-manifold M is a fibered space over S^1, then M is homeomorphic to T_f for some f. Since T_f is a Σ_g-bundle over S^1, it is a $K(\pi,1)$-space and therefore satisfies Assumption (†).

We will construct an identity. Choose a generating set $\{x_1, \ldots, x_{2g}\}$ of $\pi_1(\Sigma_g)$, which gives the isomorphism $\pi_1(\Sigma_g) \cong \langle x_1, \ldots, x_{2g} \mid [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \ \rangle$. Following a van Kampen argument, we can verify the presentation of $\pi_1(T_f)$ as

$$\text{(4)} \quad \langle\ x_1, \ldots, x_{2g}, \gamma \mid r_i := \gamma f_*(x_i)\gamma^{-1}x_i^{-1}, \quad (i \leq 2g), \\ r_{2g+1} := [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]\rangle.$$

Here, γ represents a generator of $\pi_1(S^1)$. For $i \leq 2g$, define $w_i = \prod_{j=1}^{i}[x_{2j-1}, x_{2j}] \in F$, and

$$W_i := w_{i-1}\rho_{2i-1}w_{i-1}^{-1} \cdot (w_{i-1}x_{2i-1})\rho_{2i}(w_{i-1}x_{2i-1})^{-1} \\ \cdot (w_i x_{2i})\rho_{2i-1}^{-1}(w_i x_{2i})^{-1} \cdot w_i \rho_{2i}^{-1} w_i^{-1}.$$

Since f can be isotoped so as to preserve a point $z \in \Sigma_g$, we regard the induced map f_* as a homomorphism : $\pi_1(\Sigma_g \setminus \{z\}) \to \pi_1(\Sigma_g \setminus \{z\})$. Since

f_* is a group isomorphism, there exists a unique element $q_f \in \langle x_1, \ldots, x_{2g} | \rangle$ satisfying

$$f_*([x_1, x_2] \cdots [x_{2g-1}, x_{2g}]) = q_f([x_1, x_2] \cdots [x_{2g-1}, x_{2g}]) q_f^{-1} \in \langle x_1, \ldots, x_{2g} | \rangle.$$

THEOREM 3.1. *Let W be $(\Pi_{i=1}^g W_i)\rho_{2g+1}(\gamma q_f \rho_{2g+1}^{-1} q_f^{-1} \gamma^{-1}) \in F * P$. Then, W is an identity.*

PROOF. Direct calculation gives $\psi(W_i) = w_{i-1}\gamma[f_*(x_{2i-1}), f_*(x_{2i})]\gamma^{-1} w_i^{-1}$, which implies

$$\begin{aligned} \psi(\Pi_{i=1}^g W_i) &= \gamma(\Pi_{i=1}^g [f_*(x_{2i-1}), f_*(x_{2i})])\gamma^{-1} w_g^{-1} \\ &= \gamma \Pi_{i=1}^g [f_*(x_{2i-1}), f_*(x_{2i})]\gamma^{-1} (\Pi_{i=1}^g [x_{2i-1}, x_{2i}])^{-1}. \end{aligned}$$

Hence, $\psi(W) = 1$ by definition; that is, W turns out to be an identity. □

Furthermore, we can verify that W is taut by the definition of W. Hence, from the discussion in §2, we can readily prove the following corollary.

COROLLARY 3.2. *Under the above terminology, the cellular chain complex of $\widetilde{T_f}$ is given by*

$$\begin{aligned} C_*(\widetilde{T_f}; \mathbb{Z}) : 0 \to \mathbb{Z}[\pi_1(T_f)] &\xrightarrow{\partial_3} \mathbb{Z}[\pi_1(T_f)]^{2g+1} \\ &\xrightarrow{\partial_2} \mathbb{Z}[\pi_1(T_f)]^{2g+1} \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(T_f)] \to 0. \end{aligned}$$

Here, $\partial_1(a_i) = 1 - x_i$, $\partial_1(\gamma) = 1 - \gamma$, and ∂_2 and ∂_3 have the matrix presentations,

$$\begin{pmatrix} \left\{\gamma \frac{\partial f_*(x_i)}{\partial x_j} - \delta_{ij}\right\}_{1 \le i,j \le 2g} & \left\{1 - x_i\right\}_{1 \le i \le 2g}^{\text{transpose}} \\ \left\{\frac{\partial r_{2g+1}}{\partial x_j}\right\}_{1 \le j \le 2g} & 0 \end{pmatrix},$$

$$\left(\left\{w_{j-1} - w_j x_{2j}, w_{j-1}x_{2j-1} - w_j\right\}_{1 \le j \le g}, \; 1 - \gamma q_f \right).$$

Furthermore, the diagonal map $D^\sharp(c)$ is represented by

$$\Big(\sum_{i=1}^{g}\sum_{k=1}^{2g}\frac{\partial w_{i-1}}{\partial x_k}a_k\otimes w_{i-1}b_{2i-1}-\frac{\partial(w_ix_{2i-1})}{\partial x_k}a_k\otimes w_ix_{2i}b_{2i-1}$$
$$-\frac{\partial(w_{i-1}x_{2i-1})}{\partial x_k}a_k\otimes w_{-1i}x_{2i-1}b_{2i}+\frac{\partial w_i}{\partial x_k}a_k\otimes w_ib_{2i}\Big)$$
$$-\Big(\sum_{k=1}^{2g}\frac{\partial(\gamma q_f)}{\partial x_k}a_k\otimes\gamma q_fb_{2g+1}\Big)+a_{2g+1}\otimes(1-\gamma q_f)b_{2g+1}.$$

REMARK 3.3. Corollary 3.2 for every g is a generalization of the result of [Mar]; the paper gives the cellular complexes of $\widetilde{T_f}$ only in the case $g=1$. We can verify that Corollary 3.2 with $g=1$ coincides with the results in [Mar].

Finally, we mention the virtually fibered conjecture, which was eventually proven by Wise; see, e.g., [Wise]. This conjecture states that every closed, irreducible, atoroidal 3-manifold M with an infinite fundamental group has a finite cover, which is homeomorphic to T_f for some f. Let $d\in\mathbb{N}$ be the degree of the covering. Then, if we can find such a cover $p:T_f\to M$, the pushforward of the above identity W gives an algebraic presentation of $d[M]$.

3.2. Spliced sums and $(p/1)$- and $(1/q)$-surgeries of S^3 along knots

We will focus on spliced sums and some surgeries of S^3 along knots and construct taut identities. This section supposes that the reader has basic knowledge of knot theory, as in [Lic, Chapters 1–11].

Let us review spliced sums. Take two knots $K,K'\subset S^3$ and an orientation-reversing homeomorphism $h:\partial(S^3\setminus\nu K)\to\partial(S^3\setminus\nu K')$, where νK means an open tubular neighborhood of K. Then, we can define a closed 3-manifold, $\Sigma_h(K,K')$, as the attaching space $(S^3\setminus\nu K)\cup_h(S^3\setminus\nu K')$ with $\partial(S^3\setminus\nu K)$ glued to $\partial(S^3\setminus\nu K')$ by h. This space is commonly referred to as *the spliced sum of* (K,K') *via* h. Spliced sums sometimes appear in discussions on additivity of topological invariants; see, e.g., [BC]. Further, choose the preferred meridian-longitude pair $(\mathfrak{m},\mathfrak{l})$ (resp. $(\mathfrak{m}',\mathfrak{l}')$) as a generating set of $\pi_1\partial(S^3\setminus\nu K)$ (resp. of $\partial(S^3\setminus\nu K')$). If $h_*:\pi_1\partial(S^3\setminus\nu K)\to$

$\pi_1\partial(S^3 \setminus \nu K')$ is represented by $\begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ q & -1 \end{pmatrix}$) for some $p, q \in \mathbb{Z}$, we denote $\Sigma_h(K, K')$ by $\Sigma_{p/1}(K, K')$ (resp. $\Sigma_{1/q}(K, K')$). In particular, if K' is the unknot, then $\Sigma_{p/1}(K, K')$ and $\Sigma_{1/q}(K, K')$ are the closed 3-manifolds obtained by $(p/1)$- and $(1/q)$-Dehn surgery on K in S^3, respectively.

Since the identities of $\Sigma_{p/1}(K, K')$ and $\Sigma_{1/q}(K, K')$ will be constructed in an analogous way to [Tro, Page 481], let us review the terminology in [Tro]. Choose a Seifert surface Σ of genus g and a bouquet of circles $W \subset \Sigma$ such that W is a deformation retract of Σ and $\pi_1(S^3 \setminus \Sigma)$ is a free group. 21. Page. For example, any Seifert surface obtained by a Seifert algorithm admits such a bouquet. Choose a bicollar $\Sigma \times [-1, 1]$ of Σ such that $\Sigma \times \{0\} = \Sigma$. Let $\iota_\pm : \Sigma \to S^3 \setminus \Sigma$ be embeddings whose images are $\Sigma \times \{\pm 1\}$. Take generating sets $\{v_1, \ldots, v_{2g}\}$ of $\pi_1\Sigma$ and $\{x_1, \ldots, x_{2g}\}$ of $\pi_1(S^3 \setminus \Sigma)$, and set $u_i^\sharp := (\iota_+)_*(v_i)$ and $u_i^\flat := (\iota_-)_*(v_i)$, where we may suppose that $[v_1, v_2] \cdots [v_{2g-1}, v_{2g}]$ represents a loop of $\pi_1\partial\Sigma$; a van Kampen argument yields a presentation

$$\langle\, x_1, \ldots, x_{2g}, \mathfrak{m} \mid r_i := \mathfrak{m} u_i^\sharp \mathfrak{m}^{-1} (u_i^\flat)^{-1} \quad (1 \le i \le 2g) \,\rangle \tag{5}$$

of $\pi_1(S^3 \setminus K)$. Here, $\mathfrak{m}$ is a representative of a meridian in $\pi_1(S^3 \setminus K)$, and the x_i's lie in the commutator subgroup of $\pi_1(S^3 \setminus K)$. Since the boundary loops of $\pi_1\Sigma$ and $\pi_1(S^3 \setminus \Sigma)$ are equal by definition, we should notice

$$[u_1^\flat, u_2^\flat] \cdots [u_{2g-1}^\flat, u_{2g}^\flat] = [u_1^\sharp, u_2^\sharp] \cdots [u_{2g-1}^\sharp, u_{2g}^\sharp] \in \pi_1(S^3 \setminus \Sigma), \tag{6}$$

which we denote by $\mathfrak{l}$. In other words, $\mathfrak{l}$ means a preferred longitude of K.

In a parallel way, concerning the other K', we have a generating set $\{x'_1, \ldots, x'_{2g'}\}$ of $\pi_1(S^3 \setminus \Sigma')$ and can define appropriate words $\mathfrak{u}_i^\sharp$ and $\mathfrak{u}_i^\flat$ such that

$$\pi_1(S^3 \setminus K') \cong \langle\, x'_1, \ldots, x'_{2g'}, \mathfrak{m}' \mid r'_i := \mathfrak{m}' \mathfrak{u}_i^\sharp (\mathfrak{m}')^{-1} (\mathfrak{u}_i^\flat)^{-1} \quad (1 \le i \le 2g') \,\rangle.$$

We also redefine $\mathfrak{l}'$ by $[\mathfrak{u}_1^\flat, \mathfrak{u}_2^\flat] \cdots [\mathfrak{u}_{2g-1}^\flat, \mathfrak{u}_{2g}^\flat]$.

Before we state Theorem 3.4, we should notice from the van Kampen theorem that the fundamental groups $\pi_1(\Sigma_{p/1}(K, K'))$ and $\pi_1(\Sigma_{1/q}(K, K'))$ are presented by

$$\begin{aligned} \langle\, x_1, \ldots, x_{2g}, \mathfrak{m}\ x'_1, \ldots, x'_{2g'} \mid r_1, r_2, \ldots, r_{2g}, r'_1, \ldots, r'_{2g'}, \\ r_{2g+1} := \mathfrak{l}\mathfrak{m}^p (\mathfrak{l}')^{-1} \,\rangle, \end{aligned} \tag{7}$$

$$\langle\ x_1,\dots,x_{2g},\mathfrak{m}\ x'_1,\dots,x'_{2g'},\mathfrak{m}' \mid r_1,r_2,\dots,r_{2g},r'_1,\dots,r'_{2g'},$$
$$r_\dagger := \mathfrak{m}\mathfrak{l}^q(\mathfrak{l}')^{-1}, r_\star := \mathfrak{m}'\mathfrak{l}^{-1}\ \rangle.$$

Here, in (7), we identify $\mathfrak{m}$ with $\mathfrak{m}'$. Define w_i to be $\prod_{j=1}^{i}[u^\flat_{2j-1},u^\flat_{2j}]$, and

$$W_i := w_{i-1}\rho_{2i-1}w_{i-1}^{-1}\cdot(w_{i-1}u^\flat_{2i-1})\rho_{2i}(w_{i-1}u^\flat_{2i-1})^{-1}\cdot$$
$$(w_iu^\flat_{2i})\rho_{2i-1}^{-1}(w_iu^\flat_{2i})^{-1}\cdot w_i\rho_{2i}^{-1}w_i^{-1}.$$

Likewise, we also define words w'_i and W'_i. We consider the two words,

$$W^{K,K'}_{p/1} := (\Pi^g_{i=1}W_i)\cdot\rho_{2g+1}\cdot(\Pi^{g'}_{i=1}W'_i)^{-1}(\mathfrak{m}\rho^{-1}_{2g+1}\mathfrak{m}^{-1}),$$
$$W^{K,K'}_{1/q} := (\Pi^g_{i=1}W_i)\cdot\rho_\star^{-1}\cdot(\mathfrak{m}'\rho_\dagger(\mathfrak{m}')^{-1})\cdot(\Pi^{g'}_{i=1}W'_i)\cdot(\mathfrak{l}'\rho_\star(\mathfrak{l}')^{-1})\cdot\rho_\dagger^{-1}.$$

THEOREM 3.4. *Then, $W^{K,K'}_{p/1}$ and $W^{K,K'}_{1/q}$ are taut identities with respect to the presentations* (7) *of $\pi_1(\Sigma_{p/1}(K,K'))$ and $\pi_1(\Sigma_{1/q}(K,K'))$, respectively.*

PROOF. An immediate computation gives $\psi(W_i) = w_{i-1}\mathfrak{m}[u^\sharp_{2i-1}, u^\sharp_{2i}]\mathfrak{m}^{-1}w_i^{-1}$, so that $\psi(\Pi^g_{i=1}W_i) = \mathfrak{m}\Pi^g_{i=1}[u^\sharp_{2i-1},u^\sharp_{2i}]\mathfrak{m}^{-1}w_g^{-1}$. Then, $W^{K,K'}_{p/1}$ and $W^{K,K'}_{1/q}$ turn out to be identities by (7). Furthermore, by the definition of $W^{K,K'}_\bullet$, we verify that $W^{K,K'}_\bullet$ are taut. $\square$

As a corollary, if K' is the unknot, we have the complex $C_*(\widetilde{M};\mathbb{Z})$, where M is the 3-manifold, $M_{p/1}(K)$, obtained by $p/1$-surgery of S^3 along K:

COROLLARY 3.5. *If $M := M_{p/1}(K)$ satisfies Assumption (†), then the boundary maps ∂_2 and ∂_3 in the associated complex $C_*(\widetilde{M};\mathbb{Z})$ in* (1) *are given by the following matrix presentations:*

$$\begin{pmatrix} \left\{\mathfrak{m}\frac{\partial u^\sharp_i}{\partial x_j}-\frac{\partial u^\flat_i}{\partial x_j}\right\}_{1\le i,j\le 2g} & \left\{1-\frac{\partial\mathfrak{l}}{\partial x_j}\right\}^{\text{transpose}}_{1\le j\le 2g} \\ \left\{\frac{\partial\mathfrak{l}}{\partial x_j}\mathfrak{m}^p\right\}_{1\le j\le 2g} & \mathfrak{l}\frac{\partial\mathfrak{m}^p}{\partial\mathfrak{m}} \end{pmatrix},$$

$$\partial_3(s) = (1-\mathfrak{m})b_{2g+1} + \sum_{i=1}^{g}(w_{i-1}-w_iu^\flat_{2i})b_{2i-1} + (w_{i-1}u^\flat_{2i-1}-w_i)b_{2i}.$$

REMARK 3.6. We give a comparison to Theorem 3.9 in [MP]. The authors give an expression of the chain complex $C_*(\widetilde{M};\mathbb{Z})$, where $M = M_{0/1}(K)$. However, the numbers of basis of C_3, C_2, C_1 are $2, c+1, c$, respectively, where c is the crossing number of K, while those in Corollary 3.5 are fewer.

Let us recall the cabling conjecture, which predicts that if K is not a cabling knot, then $M_0(K)$ is irreducible; this conjecture has been proven for some classes of knots. Since $\pi_1(M_0(K))$ is of infinite order, it is fair to say that most $M_0(K)$ satisfy Assumption (†). Incidentally, it is a problem for the future to clarify a taut identity for the (p/q)-surgery for any $p/q \in \mathbb{Q}$.

3.3. Branched covering spaces of S^3 branched over a knot

Take a knot K in S^3, and $d \in \mathbb{N}$. In this subsection, we will give a taut identity of $\pi_1(B_K^d)$, where we mean by B_K^d the d-fold cyclic covering space of S^3 branched over K. We should remark the fact that, if K is a prime knot and $\pi_1(B_K^d)$ is of infinite order, then B_K^d is aspherical and therefore admits Assumption (†). Let $p : E_K^d \to S^3 \setminus K$ be the d-fold cyclic covering. For $k \in \mathbb{Z}/d$, let $x_i^{(k)}$ be a copy of x_i and $u_{i,k}^\sharp$ be the word obtained by replacing x_i with $x_i^{(k)}$ in the word $u_i^\sharp$. We similarly define the word $u_{i,k}^\flat$. Then, by using the Reidemeister-Schreier method (see, e.g., [Kab, Proposition 3.1]), it follows from presentation (5) that $\pi_1(E_K^d)$ is presented by

$$(8) \quad \langle\, x_1^{(k)}, \dots, x_{2g}^{(k)}, \overline{\mathfrak{m}} \quad (k \in \mathbb{Z}/d) \mid \overline{\mathfrak{m}} u_{i,k}^\sharp \overline{\mathfrak{m}}^{-1} \ (u_{i,k+1}^\flat)^{-1} \quad (1 \le i \le 2g,\ k \in \mathbb{Z}/d) \ \rangle.$$

Since B_K^d is obtained from E_K^d by attaching a solid torus which annihilates the meridian $\overline{\mathfrak{m}}$, $\pi_1(B_K^d)$ is presented by the quotient of $\pi_1(E_K^d)$ subject to $\overline{\mathfrak{m}} = 1$; that is,

$$(9) \quad \pi_1(B_K^d) \cong \langle\, x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d) \mid r_{i,k} := u_{i,k}^\sharp (u_{i,k+1}^\flat)^{-1} \quad (1 \le i \le 2g,\ k \in \mathbb{Z}/d) \ \rangle.$$

Let F be the free group $\langle\, x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d) \mid \rangle$. From (6), we should notice that $[u_{1,k}^\flat, u_{2,k}^\flat] \cdots [u_{2g-1,k}^\flat, u_{2g,k}^\flat] = [u_{1,k}^\sharp, u_{2,k}^\sharp] \cdots [u_{2g-1,k}^\sharp, u_{2g,k}^\sharp] \in F$ for any $k \in \mathbb{Z}/d$.

Similarly to §3.2., we will give an identity with respect to the presentation (9). For $1 \leq i \leq g, 1 \leq k \leq d$, define $w_{i,k} = \prod_{j=1}^{i}[u^{\flat}_{2j-1,k+1}, u^{\flat}_{2j,k+1}]$, and

$$W_{i,k} = w_{i-1,k}\rho_{2i-1,k}w_{i-1,k}^{-1} \cdot (w_{i-1,k}u^{\flat}_{2i-1,k+1})\rho_{2i,k}(w_{i-1,k}u^{\flat}_{2i-1,k+1})^{-1} \cdot \\ (w_{i,k}u^{\flat}_{2i,k+1})\rho_{2i-1,k}^{-1}(w_{i,k}u^{\flat}_{2i,k+1})^{-1} \cdot w_{i,k}\rho_{2i,k}^{-1}w_{i,k}^{-1}.$$

Proposition 3.7. *Define W to be $\Pi_{k=1}^{d}W_{1,k}W_{2,k}\cdots W_{g,k}$, by the above equality in F. Then, W is a taut identity. In particular, if B_K^d satisfies Assumption (†), the associated complex in* (1) *is isomorphic to the cellular chain complex of the universal cover of B_K^d.*

Proof. Direct calculation gives $\psi(W_{i,k}) = w_{i-1,k}[u^{\sharp}_{2i-1,k}, u^{\sharp}_{2i,k}]w_{i,k}^{-1}$, which deduces

$$\begin{aligned}\psi(\Pi_{i=1}^{g}W_{i,k}) &= (\Pi_{i=1}^{g}[u^{\sharp}_{2i-1,k}, u^{\sharp}_{2i,k}])w_{g,k}^{-1} \\ &= \Pi_{i=1}^{g}[u^{\sharp}_{2i-1,k}, u^{\sharp}_{2i,k}](\Pi_{i=1}^{g}[u^{\flat}_{2i-1,k+1}, u^{\flat}_{2i,k+1}])^{-1}.\end{aligned}$$

Thus, W turns out to be an identity. Furthermore, since we can verify that W is taut by the definition of W, Remark 2.2 readily leads to the latter part. □

Example 3.8. Let K be the figure-eight knot. It can be verified that the presentation (5) can be written as

$$\langle\ x_1, x_2, \mathfrak{m} \mid \mathfrak{m}x_1x_2\mathfrak{m}^{-1} = x_1,\ \mathfrak{m}x_2x_1x_2\mathfrak{m}^{-1} = x_2\ \ \rangle.$$

Thus, by (9), we have

$$\pi_1(B_K^d) \cong \langle\ x_1^{(i)}, x_2^{(i)}\ \ (1 \leq i \leq d) \mid x_1^{(i)}x_2^{(i)} = x_1^{(i+1)}, \\ x_2^{(i)}x_1^{(i)}x_2^{(i)} = x_2^{(i+1)}\ \ (1 \leq i \leq d)\ \ \rangle.$$

Annihilating $x_2^{(i)}$ by using the relation $x_1^{(i)}x_2^{(i)} = x_1^{(i+1)}$, we have

$$\pi_1(B_K^d) \cong \langle\ x_1^{(1)}, \ldots, x_1^{(d)} \mid (x_1^{(i)})^{-1}(x_1^{(i+1)})^2(x_1^{(i+2)})^{-1}x_1^{(i+1)}\ \ (1 \leq i \leq d)\ \rangle.$$

This isomorphism coincides exactly with the result in [KKV, Page 963].

Likewise, we can verify that some groups, called "cyclically presented groups" in [KKV] and references therein, are isomorphic to $\pi_1(B_K^d)$ for some K and d.

Remark 3.9. As the referee points out, it is reasonable to hope that Proposition 3.7 is true without Assumption (†). In fact, as seen in [Sie], given a Heegaard diagram, we can construct a "squashing map" and a taut identity compatible with the complex (1). Thus, it is a conjecture that we can find an appropriate Heegaard diagram of B_K^d such that the associated taut identity is equal to the above W.

3.4. 0-Surgery-like spaces from branched covering spaces of S^3

Using the notation in the preceding subsection, we can examine the 3-manifold obtained by the 0-surgery on the knot $p^{-1}(K) \subset B_K^d$. The 0-surgery appears in the topic of the concordance group including the Casson-Gordon invariant [CG]. More precisely, regarding the boundary of E_K^d as a knot in B_K^d, we consider the 3-manifold obtained by 0-surgery on the knot in B_K^d. Notice from (8) that the fundamental group canonically has a group presentation

$$(10) \quad \langle\, x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d),\ \overline{\mathfrak{m}} \mid r_i^{(k)} \quad (i \leq 2g,\ k \in \mathbb{Z}/d), \\ \Pi_{i=1}^{g} [u^\flat_{2i-1,1}, u^\flat_{2i,1}] \rangle.$$

Let $\mathfrak{l}^{(k)} := \Pi_{i=1}^{g} [u^\flat_{2i-1,k}, u^\flat_{2i,k}]$, and consider an analogous presentation

$$(11) \quad \langle\, x_1^{(k)}, \dots, x_{2g}^{(k)} \quad (k \in \mathbb{Z}/d),\ \overline{\mathfrak{m}} \mid r_i^{(k)} \quad (i \leq 2g,\ k \in \mathbb{Z}/d), \\ r_\ell := \mathfrak{l}^{(1)}\mathfrak{l}^{(2)} \cdots \mathfrak{l}^{(d)} \rangle.$$

Similarly to §3.3, we can construct an identity. For $i \leq 2g, k \leq d$, define $z_k = \mathfrak{l}^{(1)}\mathfrak{l}^{(2)} \cdots \mathfrak{l}^{(k)}$ and

$$W_{i,k} = z_k w_{i-1,k} \rho_{2i-1,k} w_{i-1,k}^{-1} z_k^{-1} \cdot \\ (z_k w_{i-1,k} u^\flat_{2i-1,k+1}) \rho_{2i,k} (z_k w_{i-1,k} u^\flat_{2i-1,k+1})^{-1} \cdot \\ (z_k w_{i,k} u^\flat_{2i,k+1}) \rho_{2i-1,k}^{-1} (z_k w_{i,k} u^\flat_{2i,k+1})^{-1} \cdot z_k w_{i,k} \rho_{2i,k}^{-1} w_{i,k}^{-1} z_k^{-1}.$$

In the usual way, we can easily show the following:

PROPOSITION 3.10. *Define W to be $(\Pi_{k=1}^{d}\Pi_{i=1}^{g}W_{i,k})\cdot\rho_\ell\cdot(\mathfrak{m}\rho_\ell^{-1}\mathfrak{m}^{-1})$. Then, W is a taut identity. In particular, Remark 2.2 ensures that if the fundamental group of a closed 3-manifold satisfying Assumption* (†) *is isomorphic to* (11)*, then the cellular chain complex of the universal cover is isomorphic to the complex* (1).

3.5. Some Seifert fibered spaces over S^2

In the last subsection, we will discuss some of the Seifert fibered spaces and Brieskorn manifolds. The theorem of Scott [Sc] shows that the homeomorphism types of such spaces with infinite π_1 can be detected by the fundamental groups; thus, the spaces satisfy Assumption (†).

Let us state Proposition 3.11. Take integers $a_1,\dots,a_{n+1}$ with $a_i\geq 2$, and $\epsilon_1,\dots,\epsilon_n\in\{\pm 1\}$. Let M be a Seifert fibered space of the form

$$\Sigma(0;(0,1),(a_1,\epsilon_1),(a_2,\epsilon_2),\dots,(a_n,\epsilon_n),(a_{n+1},1)).$$

Then, as is classically known, the fundamental group has the presentation

$$\langle\ x_1,\dots,x_{n+1},h\mid hx_ih^{-1}x_i^{-1},\ \ x_i^{a_i}h^{\epsilon_i}\ \ (i\leq n)\quad ,x_{n+1}^{a_{n+1}}h,\ \ x_1\cdots x_{n+1}\ \ \rangle.$$

Furthermore, let us consider a group G with the presentation

$$\langle\ x_1,\dots,x_n\mid r_i:=(x_ix_{i+1}\cdots x_nx_1\cdots x_{i-1})^{-a_{n+1}}x_i^{\epsilon_ia_i}\qquad(i\leq n)\ \ \rangle. \tag{12}$$

We can easily check that the correspondence $x_i\mapsto x_i,\ \ x_{n+1}\mapsto(x_1\cdots x_n)^{-1}$, $h\mapsto x_1^{\epsilon_1a_1}$ gives rise to a group isomorphism $\pi_1(M)\cong G$. Therefore, we shall define a taut identity on the presentation (12):

PROPOSITION 3.11. *Suppose that $\pi_1(M)$ is of infinite order. Define W to be*

$$\rho_1(x_1^{-1}\rho_1^{-1}x_1)\rho_2(x_2^{-1}\rho_2^{-1}x_2)\cdots\rho_n(x_n^{-1}\rho_n^{-1}x_n).$$

Then, W is a taut identity of the presentation (12).

The proof is similar to the ones above, so we will omit the details.

REMARK 3.12. The taut identity when $n=2$ is presented in [Sie, p. 127]. The paper does not mention the homeomorphism type of the associated 3-manifold; however, Proposition 3.11 implies that the homeomorphism type can be detected by a Seifert structure.

Finally, let us turn to the topic of Brieskorn 3-manifolds. Choose integers $a, b, p, q, m \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$ satisfying $ap + bq = 1$ and $p, q, m > 1$. We will focus on the Brieskorn 3-manifold of the form,

$$\begin{aligned} M &:= \Sigma(p, q, mpq + \varepsilon) \\ &:= \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^{mpq+\varepsilon} = 0, \quad |x|^2 + |y|^2 + |z|^2 = 1 \ \}, \end{aligned}$$

which is an Eilenberg-MacLane space if $1/p + 1/q + 1/(mpq + \varepsilon) < 1$. The manifold is known to be homeomorphic to a 3-manifold obtained from (ε/m)-surgery on the (p, q)-torus knot $T_{p,q}$. Recall the presentation of $\pi_1(S^3 \setminus T_{p,q})$ as $\pi_1(S^3 \setminus T_{p,q}) \cong \langle\ x, y \mid x^q = y^p\ \rangle$, and that the meridian $\mathfrak{m}$ and the preferred longitude $\mathfrak{l}$ are identified with $x^a y^b$ and $(x^a y^b)^{-pq} x^q$, respectively. Therefore, $\pi_1(M)$ admits a genus-two Heegaard decomposition and has the group presentation,

$$\pi_1(M) \cong \langle\ x, y \mid r_1 := x^{qm}(x^a y^b)^{-mpq-\varepsilon}, \quad r_2 := (x^a y^b)^{mpq+\varepsilon} y^{-p} x^{-qm-q} \ \rangle. \tag{13}$$

Likewise, we can show the following result:

Proposition 3.13. *Suppose* $1/p + 1/q + 1/(mpq + \varepsilon) < 1$ *as above. Then the following word is a taut identity of the presentation* (13).

$$\rho_1 \rho_2^{-1} \rho_1^{-1} (x^{qm} y^{-p} x^{-qm-q} \rho_2 x^{qm+q} y^p x^{-qm}).$$

4. First Application to the Linking Forms of Branched Covers

4.1. Review of the linking form and a theorem

Here, we will review the linking form of M for a closed 3-manifold M with $H_*(M; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q})$. Considering the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{14}$$

we can easily check that the Bockstein maps

$$\beta : H_2(M; \mathbb{Q}/\mathbb{Z}) \cong H_1(M; \mathbb{Z}), \qquad \beta : H^1(M; \mathbb{Q}/\mathbb{Z}) \cong H^2(M; \mathbb{Z}),$$

are isomorphisms from the long exact homology sequences. Let $\mathrm{PD}^{\mathbb{Z}}_M$ be the Poincaré duality on the integral (co)-homology. We denote by Ω the composite map defined by setting

$$H_1(M;\mathbb{Z}) \xrightarrow{\mathrm{PD}^{\mathbb{Z}}_M} H^2(M;\mathbb{Z}) \xrightarrow{\beta^{-1}} H^1(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_1(M;\mathbb{Z});\mathbb{Q}/\mathbb{Z}),$$

where the last map is the Kronecker evaluation map. Then, *the linking form of M* is

$$\lambda_M : H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined by $\lambda_M(a,b) = \Omega(a)(b)$. This bilinear map is known to be symmetric and non-singular. This definition goes back to Seifert [Sei], and the form has sometimes appeared in the study of algebraic surgery theory (see, e.g., [Wall]) and the concordance groups of knots [CG]. Recently, the linking form of M can be computed in terms of Heegaard splittings [CFH].

Of particular interest to us is an application to the Casson-Gordon invariant [CG] and a procedure for computing λ_M in another way. In what follows, let B^d_K be the d-fold cyclic covering space of S^3 branched over a knot K. In the context of the invariant, the linking form of B^d_K plays an important role: more precisely, it is important to calculate metabolizers of the form; see, e.g., [CG].

Now let us give a matrix presentation of the homology $H_1(B^d_K;\mathbb{Z})$ and state the main theorem. Choose a Seifert surface Σ of K whose genus is g, as in §3.2.. Then, we have the Seifert form $\alpha : H_1(\Sigma;\mathbb{Z}) \otimes H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$; see [Lic, Chapter 6] for the definition. Let J be the inverse matrix $(V - {}^tV)^{-1}$, where $\det(V - {}^tV) = 1$ is known (see [Lic, Theorem 6.10]). The matrix presentation is often written as $V \in \mathrm{Mat}(2g \times 2g;\mathbb{Z})$ and is called *the Seifert matrix*. Consider the following matrices of size $(2gd \times 2gd)$:

$$A := \begin{pmatrix} -V & 0 & \cdots & 0 & {}^tV \\ {}^tV & -V & \cdots & 0 & 0 \\ 0 & {}^tV & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & {}^tV & -V \end{pmatrix},$$

$$B := \begin{pmatrix} 0 & 0 & \cdots & 0 & J^tV \\ J^tV & 0 & \cdots & 0 & 0 \\ 0 & J^tV & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & J^tV & 0 \end{pmatrix},$$

which appear in [Tro, Page 494]. As is known (see [Sei, Tro] or [Lic, Theorem 9.7]), the first homology $H_1(B^d_K;\mathbb{Z})$ is isomorphic to the cokernel of A, i.e., $H_1(B^d_K;\mathbb{Z}) \cong \mathbb{Z}^{2gd}/\,{}^tA\mathbb{Z}^{2gd}$. In particular, $\det(A) \neq 0$ if and only if $H_1(B^d_K;\mathbb{Q}) \cong 0$. The linking formula of B^d_K can be algebraically formulated in the above notation as follows:

THEOREM 4.1. *Suppose that B^d_K satisfies Assumption (†) and $H_1(B^d_K;\mathbb{Q}) \cong 0$. Then, the matrix multiplication $B : \mathbb{Z}^{2gd} \to \mathbb{Z}^{2gd}$ induces an isomorphism $\mathcal{B} : \mathbb{Z}^{2gd}/\,{}^tA\mathbb{Z}^{2gd} \to \mathbb{Z}^{2gd}/\,{}^tA\mathbb{Z}^{2gd}$ and the linking form $\lambda_{B^d_K}$ of B^d_K is equal to the form,*

$$\begin{aligned} \mathbb{Z}^{2gd}/\,{}^tA\mathbb{Z}^{2gd} \times \mathbb{Z}^{2gd}/\,{}^tA\mathbb{Z}^{2gd} &\longrightarrow \mathbb{Q}/\mathbb{Z}; \\ (v,w) &\longmapsto {}^tv\mathrm{adj}(A)^t\mathcal{B}^{-1}w/\Delta. \end{aligned} \tag{15}$$

Here, $\mathrm{adj}(A)$ *is the adjugate matrix of A, and Δ is the order $|H_1(B^d_K;\mathbb{Z})| \in \mathbb{N}$.*

This statement is implicitly connoted in [Sei, Satz I] and [Tro, p. 496][1]; however, there is no complete proof for this statement in the literature.

Here, let us make a few remarks. Whereas the matrix $\mathrm{adj}(A)^t\mathcal{B}^{-1}$ is not always symmetric, the quotient on $\mathbb{Z}^{2gd}/\,{}^tA\mathbb{Z}^{2gd}$ is symmetric. Next, the second condition of $H_1(B^d_K;\mathbb{Q}) \cong 0$ is not so strong: indeed, according to [Lic, Corollary 9.8], if any d-th root of unity is not a zero point of the Alexander polynomial of K (e.g., the case d is a prime power), then $H_1(B^d_K;\mathbb{Q}) \cong 0$. Furthermore, as the proof and Remark 3.9 imply, one may hope that the theorem is true even if we drop the condition of Assumption (†).

[1] To be precise, the original statements implicitly claim that the linking form $\lambda_{B^d_K}$ is equal to the matrix presentation $B\mathrm{adj}(A)$ up to isomorphisms. However, for applications to the Casson-Gordon invariants, we should describe the linking form from a basis of $H_1(B^d_K;\mathbb{Z})$.

PROOF OF THEOREM 4.1. It is known [CFH, Lemma 2.5] that the linking form can be formulated in the terminology of cohomology as

$$\lambda_M(a,b) = \langle (\beta^{-1} \circ \mathrm{PD}_M^{\mathbb{Z}})(a) \smile \mathrm{PD}_M^{\mathbb{Z}}(b), [M] \rangle. \tag{16}$$

Here, $\smile$ is the cup-product $H^1(M;\mathbb{Q}/\mathbb{Z}) \otimes H^2(M;\mathbb{Z}) \to H^3(M;\mathbb{Q}/\mathbb{Z})$.

Let M be B_K^d, and let R be one of $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ as trivial coefficients. Let $\varepsilon : \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}$ be the augmentation map. Then, as is known (see [Tro, Proposition 4.1]), by choosing a Seifert surface, the integral matrices $\{\varepsilon(\frac{\partial u_i^\sharp}{\partial x_j})\}_{1 \leq i,j \leq 2g}$ and $\{\varepsilon(\frac{\partial u_i^\flat}{\partial x_j})\}_{1 \leq i,j \leq 2g}$ are equal to V and ${}^tV$, respectively. Let us identify the complex $C^*(M;R)$ in the coefficients R with $C^*(\widetilde{M};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} R$ via ε. Then, by presentation (9), the complex $C^*(M;R)$ reduces to

$$0 \to C^0(M;\mathbb{Z}) \xrightarrow{0} C^1(M;R) \xrightarrow{A} C^2(M;R) \xrightarrow{0} C^3(M;R) \to 0. \tag{17}$$

If $R = \mathbb{Q}$, the matrix A is an isomorphic because of $H^*(M;\mathbb{Q}) \cong H^*(S^3;\mathbb{Q})$. Therefore, from the definition of the Bockstein inverse map β^{-1} : $C^2(M;\mathbb{Z}) \to C^1(M;\mathbb{Q}/\mathbb{Z})$ is identified with $\mathbb{Z}^{2gd} \to (\mathbb{Q}/\mathbb{Z})^{2gd}; v \mapsto \mathrm{adj}(A)v/\Delta$.

Meanwhile, from the formula for the identity W in Proposition 3.7 and the formula (3), the cup-product $\smile$: $C^1(M;\mathbb{Q}/\mathbb{Z}) \times C^2(M;\mathbb{Z}) \to C^3(M;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ is considered to be $(\mathbb{Q}/\mathbb{Z})^{2gd} \times \mathbb{Z}^{2gd} \to \mathbb{Q}/\mathbb{Z}; (v,w) \mapsto {}^t vBw$. The Poincaré duality ensures the non-degeneracy of the cup product on cohomology. In particular, the desired induced map $\mathcal{B}$ is an isomorphism, and is identified with the duality $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$, where $H^2(M;\mathbb{Z})$ is canonically regarded as $\mathrm{Coker}(A) = \mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd}$ by (17). Hence, upon the identification $H^2(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) \cong \mathbb{Z}^{2gd}/{}^tA\mathbb{Z}^{2gd}$, the formula (16) immediately implies that the linking form is equal to the required (15). □

4.2. Example computations

It is easier to quantitatively compute kernels rather than cokernels. Let us examine Corollary 4.2 below. Let $\mathrm{Ker}(A)_{\mathbb{Z}/\Delta}$ be $\{v \in (\mathbb{Z}/\Delta\mathbb{Z})^{2gd} \mid Av = 0 \in (\mathbb{Z}/\Delta\mathbb{Z})^{2gd}\}$. Consider the linear map

$$\mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd} \longrightarrow \mathrm{Ker}(A)_{\mathbb{Z}/\Delta}; \quad v \longmapsto \mathrm{adj}(A)v.$$

This map is an isomorphism if $|\Delta| \neq 0$: in fact, with a choice of the section $\mathfrak{s} : \mathbb{Z}^{2gd}/A\mathbb{Z}^{2gd} \to \mathbb{Z}^{2gd}$, the inverse map is defined by $w \mapsto (A\mathfrak{s}(w))/\Delta$. In summary, from Theorem 4.1, we immediately have the following:

COROLLARY 4.2. *Let Δ, A, B and $\mathrm{adj}(A)$ be as in Theorem 4.1. Under the supposition in Theorem 4.1, the linking form $\lambda_{B_K^d}$ of B_K^d is isomorphic to the bilinear form*

$$\mathrm{Ker}(A)_{\mathbb{Z}/\Delta} \times \mathrm{Ker}(A)_{\mathbb{Z}/\Delta} \longrightarrow \mathbb{Q}/\mathbb{Z}; \qquad (v, w) \longmapsto {}^t\mathfrak{s}(v)^t ABw/\Delta^2.$$

Example 4.3. Let $p, q, r \in \mathbb{Z}$ be odd numbers. Let K be the Pretzel knot $P(p, q, r)$. When $d = 2$, the branched cover B_K^2 is known to be a Seifert fibered space of type $\Sigma(p, q, r)$ over S^2. Furthermore, we can choose a Seifert matrix of the form $V = \frac{1}{2}\begin{pmatrix} p+q & q+1 \\ q-1 & q+r \end{pmatrix}$, and $\Delta = pq+qr+rp$; see [Lic, Example 6.9].

First, consider the case where p, q, r are relatively prime. Then, $\mathrm{Ker}(A)$ is generated by $(-r-q, q, -r-q, q)$, and we can easily verify that the linking form equal to $2(q + r)/\Delta$.

However, if p, q, r are not relatively prime, $\mathrm{Ker}(A)$ and the linking form are complicated. For example, if $(p, q, r) = (p, -p, p)$, then $\mathrm{Ker}(A) \cong (\mathbb{Z}/p)^2$ is generated by $(0, p, 0, p)$ and $(p, 0, p, 0)$; the linking matrix is equal to $\frac{2}{p}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Meanwhile, if $(p, q, r) = (p, p, p)$ and p is not divisible by 3, then $\mathrm{Ker}(A) \cong \mathbb{Z}/p \oplus \mathbb{Z}/3p$ possesses a basis, $v = (0, 3p, 0, 3p)$, $w = (3p + p^2, p^2, 3p + p^2, p^2)$. Hence, $\begin{pmatrix} \mathrm{lk}(v,v) & \mathrm{lk}(v,w) \\ \mathrm{lk}(w,v) & \mathrm{lk}(w,w) \end{pmatrix}$ can be computed as $\frac{2}{p}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

In a similar way, we can compute many linking forms of d-fold branched covering spaces for small d with the help of a computer program.

5. Second Application to Dijkgraaf-Witten Invariants

As another application, we develop procedures of computing some Dijkgraaf-Witten invariants in terms of identities.

We start by reviewing the Dijkgraaf-Witten invariant [DW]. Let G be a finite group, A a commutative ring, and ψ a group 3-cocycle of G. Denoting by BG an Eilenberg-MacLane space of type $(G, 1)$, we have a classifying map $\iota : M \hookrightarrow B\pi_1(M)$ uniquely up to homotopy. Then, as is known, ψ can be regarded as a 3-cocycle of $H^3(BG;A)$, and any group homomorphism $f: M \to G$ canonically gives rise to the composite

$$\iota^* \circ f^* : H^*(BG;A) \to H^*(B\pi_1(M);A) = H^*(\pi_1(M);A) \to H^*(M;A).$$

Then, the *Dijkgraaf-Witten invariant of M* is defined as a formal sum in the group ring $\mathbb{Z}[A]$ by setting

$$\mathrm{DW}_\psi(M) := \sum_{f \in \mathrm{Hom}(\pi_1(M),G)} \langle \iota^* \circ f^*(\psi), [M] \rangle \in \mathbb{Z}[A].$$

Although the definition seems rather simple or direct, it is not easy to compute $\mathrm{DW}_\psi(M)$ except in the case where G is abelian, because it is not trivial to explicitly express $[M]$ and f^* (however, see [DW, Wakui] for the abelian case and [No2] for a partially non-abelian case). To the knowledge of the author, there are few examples of Dijkgraaf-Witten invariants when G is non-abelian.

This section develops a method for computing the invariants, and gives non-abelian examples. First, for simplicity, we now restrict on the case $\psi = \gamma \smile \delta$ for some $\gamma \in H^1(G;A)$ and $\delta \in H^2(G;A)$. Take a group homomorphism $f: \pi_1(M) \to G$ and a group presentation $G = \langle y_1, \ldots, y_n \mid s_1, \ldots, s_\ell \rangle$. Then, as in (1), we have a commutative diagram:

$$\begin{tikzcd}
C_*(M;A): \arrow[r, "\partial_3"] & A \otimes \mathbb{Z}[\pi_1(M)]^m \arrow[r, "\partial_2"] \arrow[d, "f_*"'] & A \otimes \mathbb{Z}[\pi_1(M)]^m \arrow[r, "0"] \arrow[d, "f_*"'] & A \otimes \mathbb{Z}[\pi_1(M)] \arrow[d, "f_*"'] \\
C_*(G;A): \arrow[r, "\partial'_3"] & A \otimes \mathbb{Z}[G]^\ell \arrow[r, "\partial'_2"] & A \otimes \mathbb{Z}[G]^n \arrow[r, "\partial'_1=0"] & A \otimes \mathbb{Z}[G].
\end{tikzcd}$$

Here, the tensors are over $\mathbb{Z}[G]$, and $\partial'_2(b'_i) = \sum_{k=1}^n [\frac{\partial s_i}{\partial y_k}] a'_k$.

Example 5.1. Suppose $p, q \in \mathbb{N}$ such that $(p,q) = 1$. Let $A = G = \mathbb{Z}/p$, and M be the lens space $L(p,q)$. Then, as is known, $H^*(G;A) \cong \mathbb{Z}/p$, and we can choose appropriate generators $\alpha_i \in H^i(G;A) \cong \mathbb{Z}/p$ such that $\alpha_3 = \alpha_1 \smile \alpha_2$. We fix a presentation $G = \pi_1(M) = \langle x \mid \ |s := x^p \rangle$. Then, the taut identity of $L(p,q)$ is known to be $W_{p,q} = sx^{-q}s^{-1}x^q$; see [Sie]. Then, for $i \leq 3$, we can regard α_i as a map $\mathbb{Z}/p = C_*(M;\mathbb{Z}/p) \to \mathbb{Z}/p$ that sends a generator to 1. Then it follows from (2) that the cup product $\smile: H^1(L(p,q);\mathbb{Z}/p) \times H^2(L(p,q);\mathbb{Z}/p) \to \mathbb{Z}/p$ is computed as $(a,b) \mapsto qab$.

Moreover, for $a \in \mathbb{Z}/p$, if we define $f_a : \pi_1(M) \to G$ by setting $x \mapsto a$, then $\mathrm{Hom}(\pi_1(M), G)$ is equal to $\{f_a | a \in \mathbb{Z}/p\}$, and we can compute

$$\langle f_a^*(\alpha_3), \iota_*[M]\rangle = \langle f_a^*(\alpha_1 \smile \alpha_2), \iota_*[M]\rangle = \langle a\alpha_1 \smile a\alpha_2, \iota_*[M]\rangle = qa^2.$$

In conclusion,

$$\mathrm{DW}_{\alpha_3}(L(p,q)) = \sum_{a\in\mathbb{Z}/p} 1\{qa^2\} \in \mathbb{Z}[\mathbb{Z}/p].$$

In a similar way, if M is another manifold such that the cohomology ring is known, we can compute $\mathrm{DW}_{\alpha_3}(M)$ for $G = \mathbb{Z}/p$. Comparing with [DW, Wakui] as original computations, the above computation seems easier.

Example 5.2. Let m, n be natural numbers such that m is relatively prime to $6n$. Let G be the non-abelian group of order m^3 which has a group presentation

$$\langle x, y, z \mid x^m, y^m, z^m, s := xzx^{-1}z^{-1}, t := yzy^{-1}z^{-1}, u := zyxy^{-1}x^{-1}\rangle. \tag{18}$$

The (co)-homology of G is known (see, e.g., [Lea]). As a result, $H_1(G;\mathbb{Z}) \cong (\mathbb{Z}/m)^2$. Dually, the first cohomology $H^1(G;\mathbb{Z}/m) \cong (\mathbb{Z}/m)^2$ is generated by the maps α and β defined by $\alpha(x) = \beta(y) = 1$ and $\alpha(y) = \beta(x) = 0$. Furthermore, the Massey product $\langle\alpha, \beta, \alpha\rangle$ and the product $\psi := \beta \smile \langle\alpha, \beta, \alpha\rangle$ are known to be non-trivial. The equality $\psi = -\alpha \smile \langle\beta, \alpha, \beta\rangle$ is also known. Since the cup product $C^1 \otimes C^1 \to C^2$ is well described in [Tro, §2.4], the Massey product $\langle\alpha, \beta, \alpha\rangle$ can be, by definition, regarded as the map $C_2(G;\mathbb{Z}/m) \to \mathbb{Z}/m$ by setting

$$x^m \mapsto 0, \quad y^m \mapsto 0, \quad z^m \mapsto 0, \quad s \mapsto 0, \quad t \mapsto 0, \quad u \mapsto 2. \tag{19}$$

On the other hand, for simplicity, we specialize to the Seifert manifolds of type $M_{m,n} := \Sigma(0,(1,0),(m,1),(m,-1),(n,-1))$ over S^2, whose fundamental groups are presented by

$$\pi_1(M_m) = \langle x_1, x_2 \mid r_1 := x_1^m(x_1^{-1}x_2^{-1})^n,\ r_2 := x_2^m(x_2^{-1}x_1^{-1})^n\rangle.$$

By Proposition 3.11, the identity is $W := r_2x_2r_2^{-1}x_2^{-1}r_1x_1r_1^{-1}x_1^{-1}$. We further analyze the set $\mathrm{Hom}(\pi_1(M_{m,n}), G)$. For $a, b, c \in \mathbb{Z}/m$, consider the homomorphism $f_{a,b,c} : \pi_1(M_{m,n}) \to G$ defined by

$$f_{a,b,c}(x_1) := x^a y^b z^c, \quad f_{a,b,c}(x_2) := x^{-a}y^{-b}z^{-c+ab}.$$

It is not so hard to check the bijectivity of $(\mathbb{Z}/m)^3 \leftrightarrow \mathrm{Hom}(\pi_1(M_{m,n}), G)$ which sends (a, b, c) to $f_{a,b,c}$. Then, the conclusion is as follows:

PROPOSITION 5.3. *Let ψ be $\beta \smile \langle \alpha, \beta, \alpha \rangle \in H^3(G, \mathbb{Z}/m)$. Let $m \in \mathbb{Z}$ be relatively prime to $6n$. Then, upon the identification $(\mathbb{Z}/m)^3 \leftrightarrow \mathrm{Hom}(\pi_1(M_{m,n}), G)$, the Dijkgraaf-Witten invariant is equal to*

$$\mathrm{DW}_\psi(M_{m,n}) = \sum_{(a,b,c)\in(\mathbb{Z}/m)^3} 1\{n\big(2abc - a(a-1)b(b-1)\big)\} \in \mathbb{Z}[\mathbb{Z}/m].$$

PROOF. Recall from (1) that the basis of $C_2(\widetilde{M_{m,n}}) \cong \mathbb{Z}[\pi_1(M_m)]^2$ is denoted by b_1, b_2, where b_i corresponds to the relator r_i. We now analyse $(f_{a,b,c})_*(b_1) \in C_2(G; \mathbb{Z}/m)$. We can easily check that $f_{a,b,c}(r_i)$ is transformed to $x^{am}y^{bm}z^{cm(m+1)/2}$ by the above relators s, t, u. Let us define $N_{b_i} \in \mathbb{Z}$ to be the numbers of applying u when we transform $(f_{a,b,c})(r_i)$ by $x^{ma}y^{bm}z^{cm(m+1)/2}$. Then, by (19), the pairing $\langle \langle \alpha, \beta, \alpha \rangle,\ (f_{a,b,c})_*(b_1) \rangle$ is equal to $2N_{b_1}$. From the definition of N_{b_1}, a little complicated computation can lead to

$$N_{b_1} = \frac{m(m+1)ac}{2} + (\sum_{i=1}^{m-1} \frac{ia(ia-1)}{2}) + nac - \frac{na(a-1)(b-1)}{2} \in \mathbb{Z}.$$

Since m is relatively prime to $6n$, we can easily check the first and second terms to be zero modulo m. Hence, using the above description of W and the formula (2), we have

$$\begin{aligned} \langle \psi, (f_{a,b,c})_*[M_{m,n}] \rangle &= 0 + b \cdot 2N_{b_1} - 0 \cdot N_{b_2} + 0 \\ &= b(2nac - na(a-1)(b-1)) \in \mathbb{Z}/m, \end{aligned}$$

which immediately leads to the conclusion. □

The above computation is relatively simple, since so are the presentations of $\pi_1(M)$ and G; however, a similar computation seems to be harder if $\pi_1(M)$ is complicated.

In contrast, we conclude this paper by suggesting another procedure of computing $\mathrm{DW}_\psi(M)$, which is implicitly discussed in [No1, §4]. Hereafter $\psi \in H^3(G; A)$ may be an arbitrary 3-cocycle.

Let $C_*^{\mathrm{nh}}(G;\mathbb{Z})$ be the normalized homogenous complex of G, which is defined as the quotient $\mathbb{Z}$-free module of $\mathbb{Z}[G^{n+1}]$ subject to the relation $(g_0,\dots,g_n)\sim 0$ if $g_i=g_{i+1}$ for some i; see [Bro, 19 page]. Assume that we know an explicit expression of $\psi: G^4\to A$ as an element of $C^3_{\mathrm{nh}}(G,A)$. When $*\le 3$, we now define a chain map $c_*: C_n(\widetilde{M};\mathbb{Z})\to C_n^{\mathrm{nh}}(\pi_1(M);\mathbb{Z})$ as follows. Let c_0 be the identity map. Let $A\in\mathbb{Z}[\pi_1(M)]$ be any element. Define $c_1(Ax_i):=(A,Ax_i)$. If r_i is expanded as $x_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2}\cdots x_{i_n}^{\epsilon_n}$ for some $\epsilon_k\in\{\pm1\}$, we define

$$c_2(Ar_i)=\sum_{m:1\le m\le n}\epsilon_m(A, Ax_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2}\cdots x_{i_{m-1}}^{\epsilon_{m-1}}x_{i_m}^{(\epsilon_m-1)/2},$$
$$Ax_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2}\cdots x_{i_{m-1}}^{\epsilon_{m-1}}x_{i_m}^{(\epsilon_m+1)/2})\in C_2^{\mathrm{nh}}(\pi_1(M);\mathbb{Z}).$$

Then, we can easily verify $\partial_1^\Delta\circ c_1=c_0\circ\partial_1$ and $\partial_2^\Delta\circ c_2=c_1\circ\partial_2$. Let $\mathcal{O}_M\in C_3(\widetilde{M};\mathbb{Z})$ be the basis. Notice that $\partial_2^\Delta\circ c_2\circ\partial_3(\mathcal{O}_M)=c_1\circ\partial_2\circ\partial_3(\mathcal{O}_M)=0$, that is, $c_2\circ\partial_3(\mathcal{O}_M)$ is a 2-cycle. If we expand $c_2\circ\partial_3(\mathcal{O}_M)$ as $\sum n_i(g_0^i,g_1^i,g_2^i)$ for some $n_i\in\mathbb{Z}, g_j^i\in G$, then $\mathcal{O}'_M:=-\sum n_i(1,g_0^i,g_1^i,g_2^i)$ satisfies $\partial_3^\Delta(\mathcal{O}'_M)=c_2\circ\partial_3(\mathcal{O}_M)$. Therefore, the correspondence $\mathcal{O}_M\mapsto\mathcal{O}'_M$ gives rise to a chain map $c_3: C_*(\widetilde{M})\to C_*^{\mathrm{Nor}}(\pi_1(M);\mathbb{Z})$, as desired. In conclusion, the above discussion can be summarized as follows:

PROPOSITION 5.4. *For any homomorphism $f:\pi_1(M)\to G$, the push-forward $f_*\circ\iota_*[M]$ is equal to $1\otimes_{\pi_1(M)}f_*\circ c_3(\mathcal{O}_M)$ in $H_3^{\mathrm{nh}}(G;\mathbb{Z})$.*

To conclude, if we know an explicit presentation of $\pi_1(M)$ and a representative of the 3-cocycle $\psi: G^4\to A$, in principle, we can compute $\mathrm{DW}_\psi(M)$ in terms of the chain map c_* (with the help of computer program).

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