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Construction à la Ibukiyama of Symmetry Breaking Differential Operators

By Jean-Louis Clerc

Abstract. The construction of symmetry breaking differential operators, using invariant pluriharmonic polynomials, due to T. Ibukiyama in the context of the Siegel upper half space, is extended for scalar representations to general Hermitian symmetric spaces of tube-type. The new context is described in terms of Euclidean Jordan algebras and their representations. As an example, new and explicit differential operators are obtained for the restriction from the tube domain over the light cone to the product of two upper half-planes.

Introduction

In his seminal paper [6] T. Ibukiyama introduced a construction of holomorphic differential operators, in the geometric context of the Siegel upper half space, its group of holomorphic diffeomorphisms (the symplectic group) and the holomorphic series of representations. The differential operators he constructs are examples of symmetry breaking differential operators in the sense of T. Kobayashi (see [9]). They can also be viewed as generalizations of the classical Rankin-Cohen brackets, and they play an important rôle in the theory of Siegel modular forms. For more on the subject, see [7] and the bibliography therein.

In the present paper, a broader geometric context is considered, namely Hermitian symmetric spaces of tube-type. The theory of Euclidean Jordan algebras is very useful to handle these situation and the (lesser known) notion of *representation* of a Euclidean Jordan algebra, as introduced in [5] and further studied in [1, 2, 4] is central to the present article.

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The main theorem of [6] is rephrased and proved in this broader context. However, we only explore the situation called (I) in [6], and we only consider the case of *scalar* holomorphic representations.

As a test of the efficiency of the process to produce new and explicit examples, some symmetry breaking differential operators are obtained, in relation with the restriction from the tube-domain over the light cone to the product of two upper-half planes (or equivalently from the Lie ball to the bi-disc).

CONTENTS

To complete the introduction, here is a more precise description of this paper. Let D be a Hermitian symmetric space, and let $\mathrm{Aut}(D)$ be its group of holomorphic diffeomorphisms. Let D' be a Hermitian symmetric subspace of D. Let G be the subgroup of $\mathrm{Aut}(D)$ which preserves the smaller domain D'. Let π a representation of $\mathrm{Aut}(D)$ realized on the space $\mathcal{O}(D)$ of holomorphic functions on D, and let π' be a representation of G which is realized on the space $\mathcal{O}(D')$. Finally let $\mathrm{res} : \mathcal{O}(D) \longrightarrow \mathcal{O}(D')$ be the restriction map. In this context, a *symmetry breaking differential operator* (SBDO for short) is a holomorphic differential operator $\mathcal{D}$ on D such that, for any $g \in G$

$$(\mathrm{res} \circ \mathcal{D}) \circ \pi(g) = \pi'(g)(\mathrm{res} \circ \mathcal{D}) \ .$$

Among the Hermitian symmetric domains, there is the subclass of domains of tube-type, those which can be realized as Siegel domains of type I, i.e. complex tubes over convex symmetric cones in a Euclidean space. In turn, symmetric cones are related to Euclidean Jordan algebras. More precisely to any Euclidean Jordan algebra J is associated a convex symmetric cone Ω and a Hermitian symmetric space of tube-type T_Ω, and vice versa, in a very functorial way (see [11] Ch. I Section 9). The notion of *complete system of orthogonal idempotents* (for short CSOI) allows to construct specific Jordan subalgebras J' such that the associated tube-type domain $T_{\Omega'}$ is a Hermitian subdomain of T_Ω, both of the same rank. An example is the situation studied by Ibukiyama, where $J = \mathrm{Symm}(r)$ is the Jordan algebra of real symmetric matrices of size r, $J' = \mathrm{Symm}(r_1) \oplus \cdots \oplus \mathrm{Symm}(r_k)$ and $r_1 + r_2 \cdots + r_k = r$. The corresponding tube-type domains are the Siegel upper half-space $T_\Omega = \mathbb{H}_r$ and $T_{\Omega'} = \mathbb{H}_{r_1} \oplus \cdots \oplus \mathbb{H}_{r_k}$.

Section 1 is devoted to study this situation for a general Euclidean Jordan algebra and a general CSOI. The structure of the subgroup of $\mathrm{Aut}(T_\Omega)$ which preserves the smaller tube-type domain $T_{\Omega'}$ is precisely described.

The notion of *representation of a Euclidean Jordan algebra*, systematically introduced by J. Faraut and A. Korányi (see [5]) offers a nice framework to reinterpret and generalize Ibukiyama's construction. The notion is recalled in Section 2, and further developped in the context of Section 1. The notion of pluriharmonic polynomials is introduced in Section 3, and the classical Hecke formula (already extended in [8]) is further extended to the present situation.

Section 4 recalls the construction of the (scalar) holomorphic series of representations for the group $\mathrm{Aut}(T_\Omega)$, in fact for a twofold covering of its neutral component.

Section 5 contains the main result, and reduces the analytic problem to an algebraic problem about a class of polynomials on J. The problem is hard to solve in general, but many cases can be investigated, using in particular the classical theory of compact simple Lie groups and theory of invariants (see [7]).

Section 6 is devoted to an example, corresponding to the case where the Jordan algebra J is of rank 2. The domain T_Ω is the tube-domain over the light cone, the subdomain $T_{\Omega'}$ is a product of two upper half-planes. The representations of J involved are interpreted as *Clifford modules.* In this case, we investigate the algebraic problem and give explicit solutions, producing new pluriharmonic polynomials and new SBDO.

1. The Algebraic/Geometric Setting

Let J be a Euclidean Jordan algebra. The main reference for results and notation is [5]. See also [11].

1.1. Complete system of orthogonal idempotents and the associated subalgebra

Definition 1.1. A *complete system of orthogonal idempotents* (CSOI for short) of J is a family $\mathbf{c} = (c_1, c_2, \dots, c_k)$ of mutually orthogonal idempotents of J such that $e = c_1 + c_2 + \cdots + c_k$.

Let $\mathbf{c} = (c_1, c_2, \dots, c_k)$ be a CSOI of J. For each $j, 1 \leq j \leq k$, let J_j be the Euclidean Jordan subalgebra defined by

$$J_j = J(c_j, 1) = \{x \in J, c_j x = x\} \ .$$

Each J_j is a subalgebra of J, and for any i, j such that $1 \leq i \neq j \leq k$, $J_i \cap J_j = \{0\}$ and $J_i J_j = 0$. Define

$$J(\mathbf{c}) = \bigoplus_{j=1}^{k} J_j \ .$$

Then $J(\mathbf{c})$ is a Euclidean Jordan subalgebra of J, and we refer to it as the *subalgebra associated to the CSOI* $\mathbf{c}$.

1.2. The group $L(\mathbf{c})$

Let $\mathrm{Str}(J)$ be the structure group of J. Its elements may be characterized as follows : an element $\ell \in GL(J)$ belongs to $\mathrm{Str}(J)$ if and only if ℓ preserves $J^\times$ (the open set of invertible elements in J) and there exists $h \in GL(J)$ such that for any $x \in J^\times$,

$$(\ell x)^{-1} = hx^{-1} \ .$$

Moreover, the element h is unique and equal to ℓ^{t-1}.

Consequently, the structure group $\mathrm{Str}(J)$ is a closed Lie subgroup of $GL(J)$, stable by the Cartan involution $\ell \longmapsto \ell^{t-1}$.

Let Ω be the symmetric cone in J which can be defined as the set of squares of invertible elements. The group $G(\Omega)$ of all linear transformations of J which preserve Ω is closely connected to $\mathrm{Str}(J)$. In fact, both groups have the same neutral component, henceforth denoted by L. The next result is introduced (with proof) because of lack of reference.

Proposition 1.1. *Let J be a Euclidean Jordan algebra. The closed subgroup of $GL(J)$ generated by $\{P(x), x \in \Omega\}$ is equal to L.*

Proof. Let L_1 be the closed subgroup generated by $\{P(x), x \in \Omega\}$. For $x \in \Omega$, $P(x)$ belongs to L, so that $L_1 \subset L$.

Let $\mathfrak{l} = \mathrm{Lie}(L) = \mathfrak{str}(J)$ and $\mathfrak{l}_1 = \mathrm{Lie}(L_1)$. Let $\mathfrak{p} = \{L(x), x \in J\}$. Recall that for any $x \in J$, $P(\exp x) = \exp 2L(x)$. Moreover, $\Omega = \exp J$, so that $\mathfrak{l}_1 \supset \mathfrak{p}$, and hence

$$\mathfrak{l}_1 \supset [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p} \ .$$

On the other hand,

$$\mathfrak{l} = \mathrm{Der}(J) \oplus \mathfrak{p}$$

where $\mathrm{Der}(J)$ is the space of derivations of J, which is known to be equal to $[\mathfrak{p}, \mathfrak{p}]$, so that

$$\mathfrak{l} = [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p} \ .$$

and hence $\mathfrak{l}_1 = \mathfrak{l}$. As both L and L_1 are connected and $L_1 \subset L$, the conclusion follows. □

Let J be a Euclidean Jordan algebra, let $\mathbf{c} = (c_1, c_2, \dots, c_k)$ be a CSOI and let $J(\mathbf{c})$ be the associated subalgebra. Define

$$\mathrm{Str}(\mathbf{c}) = \{\ell \in \mathrm{Str}(J), \ell\big(J(\mathbf{c})\big) = J(\mathbf{c})\} \ .$$

The group $\mathrm{Str}(\mathbf{c})$ is a Lie subgroup and its Lie algebra $\mathfrak{str}(\mathbf{c})$ is given by

$$\mathfrak{str}(\mathbf{c}) = \{T \in \mathfrak{str}(J), T\big(J(\mathbf{c})\big) \subset J(\mathbf{c})\} \ .$$

Proposition 1.2. *The group* $\mathrm{Str}(\mathbf{c})$ *is stable by the Cartan involution.*

Proof. Let $J^\times$ be the open subset of invertible elements in J, and let $J(\mathbf{c})^\times$ the open subset of invertible elements of $J(\mathbf{c})$. Then

$$J(\mathbf{c})^\times = J(\mathbf{c}) \cap J^\times \ . \tag{1}$$

In fact, if $x \in J(\mathbf{c})$ is invertible in J, its inverse belongs to $\mathbb{R}[x]$. As e and x belong to $J(\mathbf{c})$, $\mathbb{R}[x] \subset J(\mathbf{c})$, so that x is invertible in $J(\mathbf{c})$. Hence $J(\mathbf{c}) \cap J^\times \subset J(\mathbf{c})^\times$. The opposite inclusion $J(\mathbf{c})^\times \subset J^\times$ is trivial.

Now assume that ℓ belongs to $\mathrm{Str}(\mathbf{c})$. Let $x \in J(c) \cap J^\times$. As ℓ^{t-1} belongs to $\mathrm{Str}(J)$,

$$\ell^{t-1}x = (\ell x)^{-1} \in J^\times \cap J(\mathbf{c})$$

and (1) implies that ℓ^{t-1} maps $J(\mathbf{c})^\times$ into itself. As $J(\mathbf{c})^\times$ is dense in $J(\mathbf{c})$, ℓ^{t-1} maps $J(\mathbf{c})$ into itself. Hence the group $\mathrm{Str}(\mathbf{c})$ is stable by the involution $\ell \longmapsto \ell^{t-1}$. $\square$

Notice that, as a consequence of Proposition 1.2, the Lie algebra $\mathfrak{s}tr(\mathbf{c})$ is stable by the Cartan involution $T \longmapsto -T^t$.

Proposition 1.3. *Let* $T \in \mathfrak{str}(\mathbf{c})$*. Then* T *maps each* J_j *into itself.*

Proof. If $T \in \mathfrak{l}(\mathbf{c})$, then both $\frac{1}{2}(T - T^t)$ and $\frac{1}{2}(T + T^t)$ belong to $\mathfrak{l}(\mathbf{c})$, so that it suffices to prove Proposition 1.3 separately for $T = D \in \mathfrak{l}(\mathbf{c}) \cap \mathrm{Der}(J)$ and for those $T = L(v), v \in J$ which belong to $\mathfrak{l}(\mathbf{c})$.

So let D be a derivation of J which maps $J(\mathbf{c})$ into itself. For $j, 1 \le j \le k$, $Dc_j = Dc_j^2 = 2c_jDc_j$ so that $Dc_j \in J(c_j, \frac{1}{2})$. As $J(c_j, \frac{1}{2}) \cap J(\mathbf{c}) = \{0\}$,

$Dc_j = 0$. Now for $x \in J_j$, $Dx = D(c_jx) = c_jDx$ and hence $Dx \in J_j$. As this is true for any j, the conclusion follows in this case.

Next, let $T = L(v)$ for some $v \in J$ and assume that $L(v)$ maps $J(\mathbf{c})$ into itself. Let $j, 1 \le j \le k$ and let $v = v_1 + v_{\frac{1}{2}} + v_0$ be its decomposition relative to the idempotent c_j. Then

$$L(v)c_j = L(c_j)v = v_1 \ + \ \frac{1}{2}\, v_{\frac{1}{2}}\ .$$

As $J(c_j, \frac{1}{2}) \cap J(\mathbf{c}) = \{0\}$, this forces $L(v)c_j = v_1 \in J_j$. So, for any $j, 1 \le j \le k$, $L(v)c_j \in J_j$. Hence

$$v = L(v)e = L(v)(c_1 + \cdots + c_k) = L(v)c_1 + \cdots + L(v)c_k$$

belongs to $J(\mathbf{c})$. But now for $v \in J(\mathbf{c})$, $L(v)$ maps each J_j into itself and the conclusion follows. □

Let $L(\mathbf{c})$ be the neutral component of $\mathrm{Str}(\mathbf{c})$.

PROPOSITION 1.4. *Let $\ell \in L(\mathbf{c})$. Then ℓ maps J_j into itself for any $j, 1 \le j \le k$. For each $j, 1 \le j \le k$, the induced map $\ell_j : J_j \longrightarrow J_j$ belongs to L_j, the connected component of* $\mathrm{Str}(J_j)$*.*

PROOF. Let $\ell_t = \exp tT$ be a one-parameter subgroup in $L(\mathbf{c})$. Then by differentiation at $t = 0$, X belongs to $\mathfrak{l}(\mathbf{c})$, which by Proposition 1.3 implies $X(J_j) \subset J_j$ for any $j, 1 \le j \le k$. Hence $\ell_t(J_j) \subset J_j$ for any $t \in \mathbb{R}$. So the proposition is satisfied for all elements of $L(\mathbf{c})$ sufficiently closed to the identity. As $L(\mathbf{c})$ is connected, the property $\ell(J_j) \subset J_j$ is valid for all $\ell \in L(\mathbf{c})$.

For the second part of the proposition, recall that for $\ell \in \mathrm{Str}(J)$ and $x \in J$,

$$P(\ell x) = \ell P(x) \ell^t \tag{2}$$

Let $\ell \in L(\mathbf{c})$ and let $x \in J_j$, so that $\ell x_j \in J_j$. The operators $\ell, \ell^t, P(x)$ map J_j into itself, so that by (2), $P(\ell x)$ also maps J_j into itself. By restriction to J_j, (2) implies

$$P_j(\ell_j x) = \ell_j P_j(x)(\ell^t)_j$$

By [5] Lemma VIII.2.3 applied to J_j, this implies $\ell_j \in \mathrm{Str}(J_j)$ and $(\ell^t)_j = \ell_j^t$. As the restriction map $\ell \longmapsto \ell_j$ is continuous, it follows that ℓ_j belongs to L_j. □

The last proposition allows to define the *restriction map*

$$L(\mathbf{c}) \ni \ell \longmapsto (\ell_1, \ell_2, \dots, \ell_k) \in L_1 \times L_2 \times \cdots \times L_k \ .$$

For each $j, 1 \le j \le k$ let Ω_j be the symmetric cone of J_j.

Proposition 1.5. *Let $x_1 \in J_1, \dots, x_j \in J_j, \dots, x_k \in J_k$ and let $x = x_1 + x_2 + \cdots + x_k \in J(\mathbf{c})$. Then x belongs to Ω iff x_j belongs to Ω_j for $1 \le j \le k$.*

Proof. For any $j, 1 \le j \le k$ let r_j be the rank of J_j. There exists a Jordan frame $(e_j^{(1)}, \dots, e_j^{(r_j)})$ of J_j such that

$$x_j = a_j^{(1)} e_j^{(1)} + \cdots + a_j^{(r_j)} e_j^{(r_j)} \ .$$

for some $a_j^{(i)} \in \mathbb{R}, 1 \le i \le r_j$. The collection $\left\{e_1^{(1)}, \dots, e_1^{(r_1)}, \dots, e_k^{(1)}, \dots, e_k^{(r_k)}\right\}$ is a Jordan frame of J and

$$x = x_1 + x_2 + \cdots + x_k = \sum_{j=1}^{k} \sum_{i=1}^{r_j} a_j^{(i)} e_j^{(i)} \ .$$

Now x_j belongs to Ω_j if and only if $a_j^{(i)} > 0$ for $1 \le i \le r_j$, and x belongs to Ω if and only if $a_j^{(i)} > 0$ for $1 \le j \le k$ and $1 \le i \le r_j$. The proof of the equivalence of the two properties follows easily. □

Consequently, let

$$\Omega(\mathbf{c}) = J(\mathbf{c}) \cap \Omega = \Omega_1 + \cdots + \Omega_j + \cdots + \Omega_k.$$

Proposition 1.6. *Let c an idempotent of J and let $x \in J(c,1)$, $y \in J(c,0)$. Then $P(x+y)$ maps $J(c,\lambda)$ into itself ($\lambda = 1, \frac{1}{2}, 0$) and is equal to*

$$\begin{array}{lcl} i) & P_1(x) & \text{on } J(c,1)) \\ ii) & 2\big(L(x)L(y) + L(y)L(x)\big) & \text{on } J\big(c, \tfrac{1}{2}\big) \\ iii) & P_0(y) & \text{on } J_0(c) \end{array} \ .$$

Proof. As $P(y) = 2L(y)^2 - L(y^2)$ we get

$$P(x+y) =$$

$$2L(x+y)^2 - L\big((x+y)^2\big)$$

$$= 2L(x)^2 + 2L(y)^2 + 2\big(L(x)L(y) + L(y)L(x)\big) - L(x^2) - L(y^2) - 2L(xy)$$

$$= P(x) + 2\big(L(x)L(y) + L(y)L(x)\big) + P(y)\ .$$

Now, as $P(c)x = x$, $P(x) = P\big(P(c)x\big) = P(c)P(x)P(c)$, $P(x)$ maps $J(c,1)$ on itself and is 0 on $J(c,\frac{1}{2}) \oplus J_0(c)$. Permuting the role of c (resp. x) and $(e-c)$ (resp. y), $P(y)$ maps $J(c,0)$ on itself and is 0 on $J(c,1)\oplus J(c,\frac{1}{2})$. Next, $L(x)$ is 0 on $J(c,1)$ and $L(y)$ is 0 on $J(c,1)$, so that $2(L(x)L(y)+L(y)L(x))$ is 0 on $J(c,1)\oplus J(c,0)$, and finally both $L(x)$ and $L(y)$ maps $J(c,\frac{1}{2})$ into itself, so that $2(L(x)L(y)+L(y)L(x))$ maps $J(c,\frac{1}{2})$ into itself. This completes the proof of Proposition 1.6. □

Proposition 1.7. *Let $x_1 \in \Omega_1, \dots, x_k \in \Omega_k$ and set $x = x_1 + x_2 + \cdots + x_k$. Then $P(x)$ belongs to $L(\mathbf{c})$ and its image by the restriction map is equal to*

$$\big(P_1(x_1), P_2(x_2), \dots, P_k(x_k)\big)\ .$$

Proof. As x belongs to Ω, $P(x)$ belongs to L. Let $j, 1 \le j \le k$ and set $y_j = x - x_j$. Then $x_j \in J(c_j,1)$ and $y_j \in J(c_j,0)$. Hence, by Proposition 1.6, $P(x) = P(x_j + y_j)$ maps $J(c_j,1) = J_j$ into itself and the induced restriction on $J(c_j,1) = J_j$ is equal to $P_j(x_j)$. □

Let K be the neutral component of the group of automorphisms (for the Jordan structure) of J. Recall that K is a maximal compact subgroup of L and its Lie algebra $\mathfrak{k}$ is equal to $\mathrm{Der}(J)$. It is also the stabilizer of the neutral element e in L. For each $j, 1 \le j \le k$, the definition of K_j is the same, adapted to the Jordan algebra J_j.

Proposition 1.8. *The image of $K \cap L(\mathbf{c})$ by the restriction map is equal to $K_1 \times K_2 \times \cdots \times K_k$.*

Proof. Let $k \in K \cap L(\mathbf{c})$ and let $(k_1, k_2, \ldots, k_k)$ be its image by the restriction map. Then $ke = e$, so that, for any j, $k_j c_j = c_j$ and hence $k_j \in K_j$. Next, let u_j, v_j be in J_j. Then $[L(u_j), L(v_j)]$ is a derivation of J which preserves $J(\mathbf{c})$ and induces

$$(0, \ldots, 0, [L_j(u_j), L_j(v_j)], 0, \ldots, 0)$$

on $J(\mathbf{c})$. As elements of the form $[L(u_j), L(v_j)]$ generate $\mathrm{Der}(J_j)$, it follows that $\mathrm{Der}(J_1) \times \mathrm{Der}(J_2) \times \cdots \times \mathrm{Der}(J_k)$ is contained in the image by the restriction map of $\mathrm{Der}(J) \cap L(\mathbf{c})$. Hence the image by the restriction map of $K \cap L(\mathbf{c})$ contains a neigbourhood of the neutral element in $K_1 \times K_2 \times \cdots \times K_k$. As the image of $K \cap L(\mathbf{c})$ is compact and as $K_1 \times K_2 \times \cdots \times K_k$ is connected, the image is equal to $K_1 \times K_2 \times \cdots \times K_k$. □

Let

$$M(\mathbf{c}) = \{\ell \in L(\mathbf{c}), \ell x = x \text{ for all } x \in J(\mathbf{c})\} .$$

Observe that $M(\mathbf{c})$ is a closed normal subgroup of $L(\mathbf{c})$, which is contained in $\mathrm{Aut}(J)$. Let $\mathrm{Lie}\big(M(\mathbf{c})\big) = \mathfrak{m}(\mathbf{c})$.

Proposition 1.9.

i) The restriction map

$$L(\mathbf{c}) \ni \ell \longmapsto (\ell_1, \ldots, \ell_k) \in L_1 \times \cdots \times L_k$$

is a surjective homomorphism.

ii) The kernel of the restriction map is equal to $M(\mathbf{c})$.

iii) $L(\mathbf{c})$ *is the closed subgroup of* L *generated by* $\{P(x),\ x \in \Omega(\mathbf{c})\}$ *and* $M(\mathbf{c})$.

Proof. Any element of L_j can be written as $k_j P(x_j)$ for some $k \in K_j$ and some $x_j \in \Omega_j$. So *i)* follows easily from Proposition 1.7 and Proposition 1.8. Further, *ii)* is a merely a rephrasing of the definition of $M(\mathbf{c})$.

Finally, let $x_j \in \Omega_j$ and set $\widetilde{x}_j = c_1 + \cdots + c_{j-1} + x_j + \cdots + c_k$. The image of $P(\widetilde{x}_j)$ by the restriction map is equal to

$$(\mathrm{Id}_1, \ldots, \mathrm{Id}_{j-1}, P_j(x_j), \mathrm{Id}_{j+1}, \ldots, \mathrm{Id}_k) .$$

As L_j is the closed subgroup generated by $\{P_j(x_j), x_j \in \Omega_j\}$, the image by the restriction map of the closed subgroup generated by $\{P(\widetilde{x}_j), \widetilde{x}_j \in \Omega_j\}$ is equal to $\{\mathrm{Id}_1\} \times \cdots \times \{\mathrm{Id}_{j-1}\} \times L_j \times \cdots \times \{\mathrm{Id}_k\}$ and *iii*) follows by repeating the argument for all $j, 1 \leq j \leq k$. □

1.3. The group $G(\mathbf{c})$

Let $\mathbb{J}$ be the complexification of J, and let

$$T_\Omega = J + i\Omega \subset \mathbb{J}$$

be the tube over the cone Ω, which is a Hermitian symmetric domain. Let G be the neutral component of the group $\mathrm{Aut}(T_\Omega)$ of all bi-holomorphic automorphisms of T_Ω.

Let $\mathbb{J}_j$ (resp. $\mathbb{J}(\mathbf{c})$) be the complexification of J_j (resp. $J(\mathbf{c})$). For $j, 1 \leq j \leq k$ consider the tube-type domain $T_{\Omega_j} \subset \mathbb{J}_j$. As a consequence of Proposition 1.5,

$$T_\Omega \cap \mathbb{J}(\mathbf{c}) = T_{\Omega_1} \oplus \cdots \oplus T_{\Omega_j} \oplus \cdots \oplus T_{\Omega_k},$$

and this domain will be denoted by $T_{\Omega(\mathbf{c})}$.

Let $G(\mathbf{c})$ be the neutral component of the subgroup of G defined by

$$\{g \in G, \quad g(T_{\Omega(\mathbf{c})}) \subset T_{\Omega(\mathbf{c})}\} \ .$$

The subgroup $G(\mathbf{c})$ is a closed Lie subgroup of G. We first determine the Lie algebra $\mathfrak{g}(\mathbf{c})$ of $G(\mathbf{c})$. Recall that the Lie algebra of $\mathfrak{g}$ is given by

$$\mathfrak{g} = J \oplus \mathfrak{l} \oplus J = \{X = (u, T, v), \quad u \in J, \quad T \in \mathfrak{l}, \quad v \in J\}$$

with Lie bracket

$$[(u_1, T_1, v_1), (u_2, T_2, v_2)] = (u, T, v)$$

$$\begin{aligned} u =& T_1 u_2 - T_2 u_1, \\ T =& [T_1, T_2] + 2u_1 \Box v_2 - 2u_2 \Box v_1, \\ v =& -T_1^t v_2 + T_2^t v_1 \ . \end{aligned} \tag{3}$$

For $X = (u, T, v) \in \mathfrak{g}$, the vector field induced on T_Ω by the one-parameter subgroup $\exp tX$ is equal to

$$X(z) = u + Tz - P(z)v \ , \qquad z \in T_\Omega \ .$$

See [5] X.5 page 211.

Proposition 1.10. *The Lie algebra of $G(\mathbf{c})$ is given by*

$$\mathfrak{g}(\mathbf{c}) = \{X = (u, T, v), \qquad u \in J(\mathbf{c}), \quad T \in \mathfrak{l}(\mathbf{c}), \quad v \in J(\mathbf{c})\}.$$

Proof. Assume that $X = (u, T, v)$ with $u \in J(\mathbf{c}), T \in \mathfrak{l}(\mathbf{c}), v \in J(c)$. Let $z \in T_{\Omega(c)}$. Then, by an elementary calculation $u + Tz - P(z)v$ belongs to $\mathbb{J}(\mathbf{c})$ and hence, by exponentiation, $\exp tX$ preserves $T_{\Omega(\mathbf{c})}$ so that $X \in \mathfrak{g}(\mathbf{c})$.

Conversely, suppose that the vector field $X(z), z \in T_\Omega$ is associated to a one-parameter group of $G(\mathbf{c})$. Then $X(z) = u + Tz - P(z)v$ must belong to $\mathbb{J}(\mathbf{c})$ whenever z belongs to $T_{\Omega(\mathbf{c})}$. As $X(z)$ is polynomial in z, this property has to be valid for any $z \in \mathbb{J}(\mathbf{c})$. Set $z = 0$ to get $u \in J(\mathbf{c})$. Next, for any $t \in \mathbb{C}^\times$ and $z \in \mathbb{J}(\mathbf{c})$

$$\frac{1}{t}\big(X(tz) - u\big) = Tz - tP(z)v \in \mathbb{J}(\mathbf{c}) \ .$$

As this has to be valid for any $t \in \mathbb{C}^\times$, necessarily, $Tz \in \mathbb{J}(\mathbf{c})$ and $P(z)v \in \mathbb{J}(\mathbf{c})$ for any $z \in \mathbb{J}(\mathbf{c})$. This implies that T maps $J(\mathbf{c})$ into itself, hence belongs to $\mathfrak{l}(\mathbf{c})$. Finally, set $z = e \in \mathbb{J}(\mathbf{c})$ to get $P(e)v = v \in J(\mathbf{c})$. □

Proposition 1.11. *Let $g \in G(\mathbf{c})$. Then there exists $g_1 \in G_1, \dots, g_k \in G_k$ such that for any $z = z_1 + z_2 + \dots + z_k \in T_{\Omega(\mathbf{c})}$,*

$$g(z) = g_1(z_1) + g_2(z_2) + \dots + g_k(z_k) \ . \tag{4}$$

Proof. Let first assume that $g = \exp X$, where $X \in \mathfrak{g}(\mathbf{c})$. Then the vector field generated by the one parameter group $g_t = \exp tX$ is given by

$$X(z) = u + Tz - P(z)v \ , u, v \in J(\mathbf{c}), T \in \mathfrak{l}(\mathbf{c}) \ .$$

More explicitly, let

$$u = u_1 + u_2 + \dots + u_k, \qquad v = v_1 + v_2 + \dots + v_k,$$

where for any $j, 1 \leq j \leq k$, $u_j, v_j \in J_j$. By Proposition 1.3, T maps each J_j into itself and induces for each $j, 1 \leq j \leq k$ an endomorphism T_j, which belongs to $\mathfrak{l}_j$. As a consequence, at any point $z = z_1 + z_2 + \cdots + z_k \in T_{\Omega(\mathbf{c})}$,

$$X(z) = X_1(z_1) + X_2(z_2) + \cdots + X_k(z_k)$$

where $X_j(z_j) = u_j + T_j z_j - P(z_j)v_j$. Now let $X_j = X(u_j, T_j, v_j) \in \mathfrak{g}_j$ and, for $t \in \mathbb{R}$, let $g_{j,t} = \exp tX_j \in G_j$. Then by integration

$$g_t(z_1 + z_2 + \cdots + z_k) = g_{1,t}(z_1) + g_{2,t}(z_2) + \cdots + g_{k,t}(z_k)$$

for any $z = z_1 + z_2 + \ldots z_k \in T_{\Omega(\mathbf{c})}$. So, the existence of a family $(g_1, \ldots, g_k)$ which satisfies (4) is proven for any g in the image of the exponential map, in particular in some neighborhood of the identity in $G(\mathbf{c})$. Now property (4) is stable by composition, i.e. if satisfied by two elements g and h of $G(\mathbf{c})$, it is satisfied for gh. As $G(\mathbf{c})$ is connected, the property (4) is satisfied for all elements of $G(\mathbf{c})$. This finishes the proof of the proposition. □

The last proposition allows to define the *restriction map*

$$G(\mathbf{c}) \ni g \longmapsto (g_1, g_2, \ldots, g_k) \in G_1 \times G_2 \times \cdots \times G_k \ .$$

THEOREM 1.1. *The restriction map is surjective. Its kernel is equal to $M(\mathbf{c})$.*

PROOF. The restriction map induces an homomorphism from $\mathfrak{g}(c)$ into $\mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_k$. From the proof of Proposotion 1.11, this homomorphism is given by

$$X = (u, T, v) \longmapsto \big((u_1, T_1, v_1), \ldots, (u_j, T_j, v_j), \ldots, (u_k, T_k, v_k)\big)$$

where $u = u_1 + u_2 + \cdots + u_k, u_j \in J_j$, $v = v_1 + \cdots + v_k, v_j \in J_j$ and $T_j = T_{|J_j}$. By (the Lie algebra version of) Proposition 1.9, this homomorphism is surjective. Hence the image of the restriction map contains a neighborhood of the neutral element in $G_1 \times \cdots \times G_k$. As for any $j, 1 \leq j \leq k$, G_j is connected, this implies the surjectivity.

Now let g be an element of $G(\mathbf{c})$ which is in the kernel of the restriction map. Then, as $c_1 + c_2 + \ldots c_k = e$, the point ie belongs to $T_{\Omega(\mathbf{c})}$ and hence by assumption, $g(ie) = ie$. The differential $Dg(ie)$ of g at ie belongs to L. The tangent space to $T(\Omega(\mathbf{c}))$ at ie is equal to $\mathbb{J}(\mathbf{c})$, and $Dg(ie)$ induces the identity on it, hence belongs to $M(\mathbf{c})$, say $Dg(ie) = m$ with $m \in M(\mathbf{c})$. Now consider $g' = g \circ m^{-1}$. Then g' is an isometry of the tube T_Ω for the Bergman metric which fixes the point ie and has differential $Dg'(ie) = \mathrm{id}_{\mathbb{J}}$. Hence $g' = \mathrm{id}$ and so $g \in M(\mathbf{c})$. □

1.4. c-homogeneous polynomials and $L(\mathbf{c})$-covariant differential operators

In this section we assume that J is a *simple* Euclidean Jordan algebra.

Proposition 1.12. *Let J be a simple Euclidean Jordan algebra and let c be an idempotent of J. Then $J(c,1)$ is a simple Jordan algebra.*

Proof. Let $c = d_1 + d_2 + \cdots + d_m$ be a decomposition of c as a sum of primitive idempotents of J. For any $1 \le j \le m$, as $cd_j = d_j$, d_j belongs to $J(c,1)$. This family of primitive orthogonal idempotents can be completed to obtain a Jordan frame of J, namely $\{d_1, d_2, \ldots, d_m, d_{m+1}, \ldots, d_r\}$. The corresponding Peirce decomposition yields

$$J = \bigoplus_{1 \le i \le j \le r} J_{ij}.$$

Now by an elementary check,

$$J_{ij} \cap J(c,1) \neq \{0\} \text{ if and only if } 1 \le i \le j \le m \ .$$

Hence $J(c) = \bigoplus_{1 \le i \le j \le k} J_{ij}$. Now assume that $J(c,1)$ could be decomposed as a sum of two non trivial Jordan subalgebras $J(c,1) = J_1 \oplus J_2$. Let e_1 (resp. e_2) be the neutral element of J_1 (resp. J_2). Then $c = e_1 + e_2$ is a decomposition of c as a sum of two orthogonal idempotents, and there is a Jordan frame of $J(c,1)$ say $\{d_1, d_2, \ldots, d_m\}$ such that

$$e_1 = d_1 + \cdots + d_p, \qquad e_2 = d_{p+1} + \cdots + d_m \ .$$

Notice that $d_1, \ldots, d_p \in J_1, d_{p+1}, \ldots, d_m \in J_2$. The space $J_{1,p+1}$ is not reduced to $\{0\}$ (a consequence of the fact that J is assumed to be simple). Let $w \in J_{1,p+1}, w \neq 0$. Then $w = w_1 + w_2, w_1 \in J_1, w_2 \in J_2$. Now

$d_1 w_1 = d_1 w = \frac{1}{2} w$, hence $w \in J_1$. But similarly $d_{p+1} w_2 = \frac{1}{2} w$ and hence $w \in J_2$. As $w \neq 0$ whereas $J_1 \cap J_2 = \{0\}$, this yield a contradiction. □

Let $\mathbf{c} = (c_1, c_2, \dots, c_k)$ be a complete system of idempotents and let $J(\mathbf{c})$ be the associated algebra. As a consequence of Proposition 1.12, $J_j = J(c_j, 1)$ is simple for any $j, 1 \leq j \leq k$.

There exists a character χ on L, such that

$$\text{for } \ell \in L, \quad x \in J, \qquad \det(\ell x) = \chi(\ell) \det x \ , \tag{5}$$

and similarly for any $j, 1 \leq j \leq k$

$$\text{for } \ell_j \in L_j, \quad x \in J_j, \qquad \det_j(\ell_j x_j) = \chi_j(\ell_j) \det_j x_j \ .$$

PROPOSITION 1.13. *Let $\ell \in L(\mathbf{c})$ and let $(\ell_1, \ell_2, \dots, \ell_k)$ be its restriction to $J(\mathbf{c})$. Then*

$$\chi(\ell) = \chi_1(\ell_1)\chi_2(\ell_2) \dots \chi_k(\ell_k) \ .$$

PROOF. Let $x = x_1 + x_2 + \cdots + x_k$ be in $\in J(\mathbf{c})$. Then

$$\det x = \det_1 x_1 \det_2 x_2 \ \dots \ \det_k x_k \ .$$

Similarly, $\ell x = \ell_1 x_1 + \ell_2 x_2 + \cdots + \ell_k x_k$ and

$$\det \ell x = \det_1 (\ell_1 x_1) \det_2 (\ell_2 x_2) \dots \det_k (\ell_k x_k) \ .$$

and the statement follows easily by choosing x such that $\det x \neq 0$. □

DEFINITION 1.2. A polynomial q on J is said to be $\mathbf{c}$-*homogenous of degree* $(p_1, p_2, \dots, p_k)$, if for any $\ell \in L(\mathbf{c})$, for any $x \in J$

$$q(\ell x) = \chi_1(\ell_1)^{p_1} \dots \chi_k(\ell_k)^{p_k} q(x) \ , \tag{6}$$

where $(\ell_1, \ell_2, \dots, \ell_k)$ is the restriction of ℓ to $J(\mathbf{c})$.

Let q be a polynomial on J, which we extend as a holomorphic polynomial on $\mathbb{J}$. Let D_q be the unique constant coefficients holomorphic differential operator on $\mathbb{J}$ such that for any $v \in \mathbb{J}$

$$D_q\, e^{(z,v)} = q(v)\, e^{(z,v)} \ . \tag{7}$$

The polynomial q is said to be the *algebraic symbol* of D_q.

PROPOSITION 1.14. *Let q be a polynomial on J and assume that q is $\mathbf{c}$-homogeneous of multidegree $(p_1, p_2, \dots, p_k)$. Then the differential operator D_q satisfies for all $f \in C^\infty(\mathbb{J})$*

$$D_q(f \circ \ell^{-1}) = \prod_{j=1}^{k} \chi_j(\ell_j)^{-p_j} (D_q f) \circ \ell^{-1} \ . \tag{8}$$

where $\ell \in L(\mathbf{c})$ and $(\ell_1, \ell_2, \dots, \ell_k)$ is its restriction to $J(\mathbf{c})$.

PROOF. For $v \in \mathbb{J}$ let f_v be the function defined on $\mathbb{J}$ by

$$f_v(z) = e^{(z,v)}.$$

It is enough to prove (8) for the family of functions $(f_v), v \in \mathbb{J}$. Now $f_v \circ \ell^{-1} = f_{\ell^{-1^t} v}$, so that

$$D_q(f_v \circ \ell^{-1}) = D_q f_{\ell^{-1^t} v} = q(\ell^{-1^t} v) f_{\ell^{-1^t} v}$$

whereas

$$D_q f_v = q(v) f_v, \quad D_q f_v \circ \ell^{-1} = q(v) f_{\ell^{-1^t} v} \ .$$

Use the covariance property (6) of q to conclude. □

2. Jordan Algebra Representations

2.1. Generalities

Recall that a *representation* of J is a triple $(E, \langle\, .\, ,\, .\, \rangle, \Phi)$ where $(E, \langle\, .\, ,\, .\, \rangle)$ is a finite dimensional Euclidean vector space and $\Phi : J \to \mathrm{Symm}(E)$ a unital Jordan algebra morphism, i.e. Φ satisfies

i) Φ is linear

ii) $\Phi(xy) = \frac{1}{2}\big(\Phi(x)\Phi(y) + \Phi(y)\Phi(x)\big)$, for all $x, y \in J$

iii) $\Phi(e) = \mathrm{id}_E$

iv) $\langle \Phi(x)\xi, \eta\rangle = \langle \xi, \Phi(x)\eta\rangle$, for all $x \in J$, $\xi, \eta \in E$.

For general results on the subject, see [5], IV.4 and ch. XVI, and [1].

The assumptions imply that

(9) $\Phi(x)$ is invertible if x is invertible and then $\Phi(x)^{-1} = \Phi(x^{-1})$,

(10) $$\Phi(P(x)y) = \Phi(x)\Phi(y)\Phi(x), \text{ for all } x, y \in J \ .$$

To a representation E of J is associated a symmetric bilinear map $H : E \times E \to J$ define by

$$\text{for all } x \in J, \qquad \big(H(\xi, \eta), x\big) = \langle \Phi(x)\xi, \eta\rangle \ .$$

Denote by $Q : E \to J$ the associated quadratic map, defined by

$$Q(\xi) = H(\xi, \xi) \ .$$

Let us recall some results which will be necessary later. For proofs, see [1].

Proposition 2.1. *An element $\xi \in E$ is said to be* regular *if one of the four equivalent propositions is satisfied :*

i) $x \in J, \Phi(x)\xi = 0 \Longrightarrow x = 0$

ii) $Q(\xi) \in \Omega$

iii) $\det Q(\xi) \neq 0$

iv) the differential $d_\xi Q$ of Q at ξ is surjective.

Proposition 2.2. *A representation Φ on E is said to be* regular *if one of the three equivalent propositions is verified :*

i) $\exists\, \xi \in E,\ Q(\xi) = e$

ii) $\exists\, \xi \in E,\ \xi$ *regular*

iii) $Q(E) \supset \Omega$.

Proposition 2.3. *Let J be a simple Euclidean Jordan algebra of rank r, and let (E, Φ) be a representation of J of dimension N.*

i) for a primitive idempotent d of J, $q := \operatorname{rank}(\Phi(d))$ is independent of d and $N = rq$.

ii) for any idempotent c of rank r_c, $\operatorname{rank}(\Phi(c)) = r_c q$.

iii) for $x \in J$, $\qquad \operatorname{Det}\Phi(x) = (\det x)^{\frac{N}{r}}$.

Proof. For *i)* and *iii)*, see [5]. For *ii)*, let $c = d_1 + \cdots + d_{r_c}$ be a Peirce decomposition of c as a sum of mutually orthogonal idempotents. Then $\Phi(c) = \Phi(d_1) + \cdots + \Phi(d_{r_c})$ is a sum of mutually orthogonal projectors, each of rank q. Hence $\operatorname{rank}\big(\Phi(c)\big) = r_c q$. □

Let us give two typical examples of representations.

Let $J = \operatorname{Symm}(r)$ the space of real symmetric matrices of size r, with the Jordan product $x.y = \frac{1}{2}(xy + yx)$. Let q be an integer, $q \geq 1$ and let $E = \operatorname{Mat}(r, q)$ be the space of real matrices of size (r, q), equipped with its standard inner product $\langle \xi, \eta \rangle = \operatorname{tr}(\xi \eta^t)$. For $x \in J$ and $\xi \in E$, define

$$\Phi(x)\xi = x\xi \ .$$

Then Φ is a representation of J on E. The representation is regular if and only if $q \geq r$. This example plays a fundamental rôle in Ibukiyama's work.

The second example will be studied in Section 7. Let J be the quadratic algebra of dimension n, i.e. $J = \mathbb{R} \oplus \mathbb{R}^{n-1}$, with the Jordan product

$$(s, x)(t, y) = (st + x.y, sy + tx), \qquad s, t \in \mathbb{R}, x, y \in \mathbb{R}^{n-1} \ ,$$

where $x.y = x_1 y_1 + \ldots x_{n-1} y_{n-1}$.

Let $Cl(\mathbb{R}^{n-1})$ be the Clifford algebra of $\mathbb{R}^{n-1}$, generated by $\mathbb{R}^{n-1}$ and the relations[1]

$$x, y \in \mathbb{R}^{n-1}, \qquad xy + yx = 2\ x.y \ .$$

The representations of J coincide with the *Clifford modules* for the algebra $Cl(\mathbb{R}^{n-1})$. In fact, If E is a Clifford module, then for $x \in J$ and $\xi \in E$,

$$\Phi(s, x)\,\xi = s\,\xi + x\,\xi$$

defines a representation of J on E, and vice versa. The regularity of these representations is a rather subtle question, see [1].

[1] Notice that it differs from the usual convention by the absence of a sign $-$.

2.2. Restriction to $J(\mathbf{c})$

We now consider the situation studied in Section 1. So let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a CSOI of J.

Proposition 2.4. *The operators $\{\Phi(c_j), 1 \le j \le k\}$ form a complete family of orthogonal projectors on E.*

The proof is similar to the proof given for the case of a Jordan frame of J in [5] Section IV.4 or in [1].

Let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ be the corresponding orthogonal decomposition of the space E.

Proposition 2.5. *Let $j, 1 \le j \le k$, and let $x \in J_j$. Then*

i) for $i \ne j$, $\Phi(x)E_i = 0$

ii) $\Phi(x)E_j \subset E_j$

iii) The map $\Phi_j : J_j \longrightarrow \mathrm{End}(E_j)$ given by $\Phi_j(x) = \Phi(x)_{|E_j}$ yields a representation of the Jordan algebra J_j.

Proof. *i)* Let $i \ne j$ and let $\xi_i \in E_i$. Then

$$\Phi(x)\xi_i = \Phi(c_j x)\xi_i = \frac{1}{2}\big(\Phi(c_j)\Phi(x) + \Phi(x)\Phi(c_j)\big)\xi_i = \frac{1}{2}\Phi(c_j)\Phi(x)\xi_i \ .$$

But 2 is not an eigenvalue of $\Phi(c_j)$, so that $\Phi(x)\xi_i = 0$.

ii) Let $\xi \in E_j$. Then

$$\begin{aligned}\Phi(x)\xi = \Phi(c_j x)\xi &= \frac{1}{2}\big(\Phi(c_j)\Phi(x) + \Phi(x)\Phi(c_j)\big)\xi \\ &= \frac{1}{2}\Phi(c_j)\Phi(x)\xi + \frac{1}{2}\Phi(x)\xi \ ,\end{aligned}$$

so that

$$\Phi(x)\xi \ = \Phi(c_j)\Phi(x)\xi$$

and hence $\Phi(x)\,\xi$ belongs to E_j.

iii) The verification of this result is elementary. □

For each $j, 1 \le j \le k$, let $Q_j : E_j \longrightarrow J_j$ be the quadratic map associated to the representation (E_j, Φ_j).

Let proj($\mathbf{c}$) be the orthogonal projection from J onto $J(\mathbf{c})$.

Proposition 2.6. *For any $\xi = \xi_1 + \xi_2 + \cdots + \xi_k \in E$,*

$$\operatorname{proj}(\mathbf{c})Q(\xi) = \sum_{j=1}^{k} Q_j(\xi_j) \ . \tag{11}$$

Proof. Let $x = x_1 + \cdots + x_k \in J(\mathbf{c})$ and let $\xi \in E$. Then

$$\begin{aligned}\big(x, Q(\xi)\big) &= \sum_{j=1}^{k} \big(x_j, Q(\xi)\big) = \sum_{j=1}^{k} \langle \Phi(x_j)\xi, \xi\rangle \\ &= \sum_{j=1}^{k} \langle \Phi_j(x_j)\xi_j, \xi_j\rangle = \sum_{j=1}^{k} \big(x_j, Q_j(\xi_j)\big) \ .\end{aligned}$$

As $\sum_{j=1}^{k} Q_j(\xi_j)$ belongs to $J(\mathbf{c})$, the statement follows. □

Proposition 2.7. *If the representation (E, Φ) is regular, then for any $j, 1 \le j \le k$ the representation (E_j, Φ_j) of J_j is regular.*

Proof. By assumption, there exists $\xi^0 \in E$ such that $Q(\xi_0) = e$. Now, use (11) to obtain

$$e = \operatorname{proj}(\mathbf{c})e = \operatorname{proj}(\mathbf{c})Q(\xi^0) = \sum_{j=1}^{k} Q_j(\xi_j^0),$$

and hence for each $j, 1 \le j \le k$

$$Q_j(\xi_j^0) = c_j \ ,$$

so that the representation (E_j, Φ_j) is regular. □

Proposition 2.8. *Let J be a* simple *Euclidean Jordan algebra, and let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a CSOI, and let r_j be the rank of J_j. Let (E, Φ) be a representation of J, and let $N = rq$ be its dimension. Then, for any $j, 1 \le j \le k$,*

$$\dim(E_j) = N_j = r_j q \ . \tag{12}$$

Proof. This is a direct application of Proposition 2.3. □

Proposition 2.9. *Let q be a polynomial on J which is $\mathbf{c}$-homogeneous of multidegree $(p_1, \dots, p_k)$. Let p be the polynomial on E defined by $p = q \circ Q$. Then p satisfies for any $x = x_1 + x_2 + \cdots + x_k \in J(\mathbf{c})$,*

$$p(\Phi(x)\xi) = \prod_{j=1}^{k} \det_j(x_j)^{2p_j} p(\xi) \ . \tag{13}$$

Proof. Let $x = x_1 + x_2 + \cdots + x_k \in \Omega(\mathbf{c})$. Then $P(x)$ belongs to $L(\mathbf{c})$ and its restriction to E_j is equal to $P_j(x_j)$. As $\chi_j((P_j(x_j)) = \det_j(x_j)^2$ follows

$$p(\Phi(x)\xi) = q\big(Q(\Phi(x)\xi)\big) = q\big(P(x)Q(\xi)\big) = \prod_{j=1}^{k} \det_j(x_j)^{2p_j} q(Q(\xi)) \ .$$

As both sides of (13) are polynomial in x, and $\Omega(\mathbf{c})$ is an open set of of $J(\mathbf{c})$, the identity is valid on $J(\mathbf{c})$. □

3. Pluriharmonic Polynomials

Let J be a Euclidean Jordan algebra, (E, Φ) a representation of J, and let $Q : E \times E \longrightarrow J$ be the associated quadratic map.

Definition 3.1. A polynomial p on E is said to be *pluriharmonic* if for all $x \in J$

$$\Delta_E(p \circ \Phi(x)) = 0 \ , \tag{14}$$

where Δ_E is the Laplacian on E.

3.1. The Hecke formula for pluriharmonic polynomials

A key result to be used later is a generalization of the classical Hecke formula, which we recall now, with a proof. For earlier occurrences of such results, see [8, 6]. Let us also mention the paper [3], where a similar result in an even more general context is proved.

Let J be a simple Euclidean Jordan algebra, and assume that (E, Φ) is a regular representation of J of dimension N.

PROPOSITION 3.1. *Let p be a pluriharmonic polynomial on E. Then for $x \in \Omega$,*

(15)
$$\int_E e^{i\langle\xi,\eta\rangle} e^{-\frac{1}{2}(x,Q(\xi))}\, p(\xi)\, d\xi = (2\pi)^{\frac{N}{2}}\, (\det x)^{-\frac{N}{2r}}\, e^{-\frac{1}{2}(x^{-1},\, Q(\eta))}\, p\big(i\phi(x^{-1})\eta)\big)$$

PROOF. First notice that for $x \in \Omega$, the quadratic form $(x, Q(\xi)) = \langle\Phi(x)\xi, \xi\rangle = \langle\Phi(x^{\frac{1}{2}})\xi, \Phi(x^{\frac{1}{2}})\xi\rangle$ is positive definite on E. Hence the integral on the left hand side converges. Next, recall that for any harmonic polynomial q on a Euclidean space F,

$$\int_F e^{i\langle\xi,\eta\rangle}\, e^{-\frac{1}{2}\langle\xi,\xi\rangle}\, q(\xi) d\xi = (2\pi)^{\frac{N}{2}}\, e^{-\frac{1}{2}\langle\eta,\eta\rangle}\, q(i\eta)\ . \tag{16}$$

Let $\xi' = \Phi(x^{\frac{1}{2}})\xi$. Then

$$\int_E e^{i\langle\xi,\eta\rangle} e^{-\frac{1}{2}(x,Q(\xi))}\, p(\xi)\, d\xi =$$

$$\int_E e^{i\langle\xi',\Phi(x^{-\frac{1}{2}}\eta)\rangle} e^{-\frac{1}{2}\langle\xi',\xi'\rangle} p(\Phi(x^{-\frac{1}{2}})\xi')\, \mathrm{Det}\,\Phi(x^{-\frac{1}{2}})\, d\xi'\ .$$

$$= (2\pi)^{\frac{N}{2}} (\det x)^{-\frac{N}{2r}} e^{-\frac{1}{2}(x^{-1},\, Q(\eta))}\, p\big(i\phi(x^{-1})\eta\big)\ ,$$

where we apply (16) to $q = p \circ \Phi(x^{-\frac{1}{2}}), \eta' = \Phi(x^{-\frac{1}{2}})\eta$ and notice that by Proposition 2.3 *iii*), $\mathrm{Det}\,\Phi(x^{-\frac{1}{2}}) = (\det x)^{-\frac{N}{2r}}$. $\square$

Let $\mathbb{J}$ be the complexified Jordan algebra, $\mathbb{E}$ the complexification of E, and extend Φ $\mathbb{C}$-linearly to $\mathbb{J}$. The previous formula has an extension to this complex setting.

PROPOSITION 3.2. *Let p be a pluriharmonic polynomial on E and extend it holomorphically to $\mathbb{E}$. Then for any $z \in T_\Omega$,*

$$\begin{aligned} &\int_E e^{i\langle\xi,\eta\rangle} e^{\frac{i}{2}(z,Q(\xi))}\, p(\xi)\, d\xi \\ &= (2\pi)^{\frac{N}{2}}\, \big(\det(\frac{z}{i})\big)^{-\frac{N}{2r}}\, e^{\frac{i}{2}(-z^{-1},\, Q(\eta))}\, p\big(\phi(-z^{-1})\eta)\big)\ , \end{aligned} \tag{17}$$

where $\left(\det(\frac{z}{i})\right)^{-\frac{N}{2r}}$ *is computed using the determination which is equal to* $(\det y)^{-\frac{N}{2r}}$ *for* $z = iy, y \in \Omega$.

PROOF. The left hand side of (17) is a convergent integral depending holomorphically on the parameter $z \in T_\Omega$. The right handside is also holomorphic and both sides coincide on $i\Omega \subset T_\Omega$ by (16). The conclusion follows. □

3.2. c-pluriharmonic polynomials

Let J be a simple Euclidean Jordan algebra, let $\mathbf{c}$ be a CSOI of J, and let (E, Φ) a representation of J.

DEFINITION 3.2. A polynomial p on E is said to be $\mathbf{c}$-*pluriharmonic* (with respect to Φ) if for any $x \in J(\mathbf{c})$

$$\text{for any } x \in J(\mathbf{c}), \qquad \Delta_E\big(p \circ \Phi(x)\big) = 0\ . \tag{18}$$

Equivalently,

for any $\xi_1 \in E_1, \dots, \xi_{j-1} \in E_{j-1}, \xi_{j+1} \in E_{j+1}, \dots, \xi_k \in E_k$,

$$\begin{aligned} &\text{the polynomial } \xi_j \longmapsto p(\xi_1, \xi_2, \dots, \xi_{j-1}, \xi_j, \xi_{j+1}, \dots, \xi_k) \\ &\qquad\qquad\qquad \text{is pluriharmonic on } E_j \text{ (w.r.t. } \Phi_j). \end{aligned} \tag{19}$$

In fact, assume that p is $\mathbf{c}$-pluriharmonic. Let $x_j \in J_j$. For $\xi = \xi_1 + \dots + \xi_k \in E$,

$$\begin{aligned} &\Phi(c_1 + \dots + c_{j-1} + x_j + c_{j+1} + \dots + c_k)(\xi) \\ &= \xi_1 + \dots + \xi_{j-1} + \Phi_j(x_j)\xi_j + \xi_{j+1} + \dots + \xi_k, \end{aligned}$$

and hence the equivalent definition is satisfied.

Conversely, assume p is a polynomial on E which satisfies the conditions (19). Let $x = x_1 + \dots + x_k \in J(\mathbf{c})$. For $\xi = \xi_1 + \dots + \xi_k \in E$,

$$p(\Phi(x)\xi) = p\big(\Phi_1(x_1)\xi_1, \dots, \Phi(x_k)\xi_k\big)\ .$$

So, if p satisfies the condition (19), $\Delta_{E_j}(p \circ \Phi(x)) = 0$ for any $j, 1 \le j \le k$ and hence $\Delta_E\big(p \circ \Phi(x)\big) = 0$, as $\Delta_E = \Delta_{E_1} + \dots + \Delta_{E_k}$.

4. Holomorphic Representations

Let J be a simple Euclidean Jordan algebra, denote by $\mathbb{J}$ its complexification. Form the corresponding *tube-type domain* $T_\Omega = J + i\Omega \subset \mathbb{J}$. Let $G(T_\Omega)$ be the group of bi-holomorphic automorphisms of T_Ω, which turns out to be a Lie group. Its neutral component $G(T_\Omega)^0$ is generated by

- the translations $t_u : z \mapsto z + u, u \in J$
- the group $L = \mathrm{Str}(J)^0 = G(\Omega)^0$
- the inversion $\iota : z \mapsto -z^{-1}$

(see [5] Ch. X).

It turns out to be wise to work with a two-fold covering of the group $G = G(T_\Omega)^0$. Notice that already for the upper-half plane $\mathcal{H} = \mathbb{R} + i\mathbb{R}^+$ in $\mathbb{C}$, the relevant group is $SL(2,\mathbb{R})$, whereas the group of holomorphic diffeomorphisms of $\mathcal{H}$ is $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\,\mathrm{id}\}$.

Viewing $\mathbb{J}$ as a complex Jordan algebra, there is a corresponding structure group $\mathrm{Str}(\mathbb{J})$ which is a complexification of $\mathrm{Str}(J)$. Let $\mathbb{L}$ be its neutral connected component, which can be called *the* complexification of L. The character χ, being defined by an algebraic condition (5), has a natural complex extension to $\mathbb{L}$.

We now recall the construction of a two-fold covering group of $G(T_\Omega)^0$. For $g \in G(T_\Omega)^0$ and $z \in T_\Omega$ denote by $D(g,z)$ the differential of g at z. A remarkable fact is that $Dg(z)$ belongs to $\mathbb{L}$. This is obtained by verifying the property for the generators of G and extending to the full group by the chain rule.

Let $\chi(g,z) = \chi\big(Dg(z)\big)$. This is a smooth cocycle, and the covering is defined using a square root of this cocycle. Let

(20)
$$\widetilde{G} = \{(g,\psi_g), g \in G(T_\Omega)^0, \psi_g : T_\Omega \to \mathbb{C} \text{ holomorphic}, \psi_g(z)^2 = \chi(g,z)\},$$

with the group law

$$(g,\psi_g)(h,\psi_h) = (gh, (\psi_g \circ h)\,\psi_h)\ .$$

Then $\widetilde{G}$ has a natural structure of Lie group, and the projection $(g,\psi_g) \longmapsto g$ is a twofold covering of G. Elements of $\widetilde{G}$ will denoted simply by g and we let $\psi(g,z) = \psi_g(z)$ be the corresponding choice of a square root of $\chi(g,z)$.

The group $\widetilde{G}$ has generators very similar to those for G.

• For $u \in$, $\chi(t_u, z) \equiv 1$, so that the element $(t_u, 1)$ belongs to $\widetilde{G}$ and is (with some abuse of notation) still denoted by t_u.

• For the inversion ι, $\chi(\iota, z) = (\det z)^{-2}$, so $(\iota, (\det z)^{-1})$ belongs to $\widetilde{G}$ and is (again with some abuse of notation) denoted by ι.

• Finally, let $\widetilde{L} = \{(\ell, \psi_\ell = \pm\chi(\ell)^{1/2}), \ell \in L = \mathrm{Str}(J)^0\}$, a twofold covering of L.

The group $\widetilde{G}$ is generated by the translations $\{t_u, u \in J\}$, the group $\widetilde{L}$ and the inversion ι.

It turns out to be necessary to consider also the twofold covering $\widetilde{\mathbb{L}}$ of $\mathbb{L} = \mathrm{Str}(\mathbb{J})^0$, constructed exactly as the twofold covering of L, using a square root of the complex-valued character χ of $\mathrm{Str}(\mathbb{J})$. Denote by ψ the corresponding (well-defined) character on $\widetilde{\mathbb{L}}$. It is worthwile to notice that for any $g \in \widetilde{G}$, the function $z \longmapsto \psi(g, z)^{-1}$ is (the restriction to T_Ω of) a polynomial function on $\mathbb{J}$.

Let m be an integer. For any holomorphic function F on T_Ω and any element $g \in \widetilde{G}$, let

$$\pi_m(g)F(z) = \psi(g^{-1}, z)^m F\big(g^{-1}(z)\big) \ . \tag{21}$$

This defines a smooth representation of $\widetilde{G}$ on the space $\mathcal{O}(T_\Omega)$ of holomorphic functions on T_Ω, equipped with the Montel topology.

Let us write the expression of the representation π_m for generators of the group $\widetilde{G}$.

For translations $t_v, v \in J$,

$$\big(\pi_m(t_v)F\big)(z) = F(z - v) \ , \tag{22}$$

for an element $\ell \in \widetilde{L}$

$$\big(\pi_m(\ell)F\big)(z) = \psi(\ell)^{-m} F(\ell^{-1} z) \ , \tag{23}$$

for the inversion ι,

$$\big(\pi_m(\iota)F\big)(z) = (\det z)^{-m} F(-z^{-1}) \ . \tag{24}$$

This construction of a twofold covering can be done in the context of the algebra $J(\mathbf{c})$ for $\mathbf{c}$ being a complete system of mutually orthogonal idempotents of J. We skip details, but just mention that all the results proved in

Section 1 (and particularly Theorem 1.1) have an almost trivial extension to the groups $\widetilde{L}, \widetilde{G}, \widetilde{M_{\mathbf{c}}}$ and $\widetilde{L}_j, \widetilde{G}_j$.

Similarly, for each $j, 1 \leq j \leq k$, we may define representations of the group $\widetilde{G}_j$ on the space of holomorphic $\mathcal{O}(T_{\Omega_j})$ by

$$\pi_{m_j}^{(j)}(g)F(z_j) = \psi_j(g^{-1}, z_j)^{m_j} F\big(g^{-1}(z_j)\big) \ .$$

Given $\mathbf{m} = (m_1, m_2, \dots, m_k)$, consider the tensor product representation $\pi_{m_1}^{(1)} \otimes \pi_{m_2}^{(2)} \otimes \cdots \otimes \pi_{m_k}^{(k)}$ of $\widetilde{G}_1 \times \cdots \times \widetilde{G}_k$ defined on $\mathcal{O}(T_{\Omega_{\mathbf{c}}})$ by

$$\pi_{m_1}^{(1)}(g_1) \otimes \pi_{m_2}(g_2) \otimes \cdots \otimes \pi_{m_k}(g_k)F(z_1, z_2, \dots z_k)$$

$$= \prod_{j=1}^{k} \psi_j(g_j^{-1}, z_j)^{m_j} \ F\big(g_1^{-1}(z_1), g_2^{-1}(z_2), \dots, g_k^{-1}(z_k)\big) \ .$$

Proposition 4.1. *The restriction map* res : $F \longmapsto F_{|T_{\Omega_{\mathbf{c}}}}$ *intertwines the restriction of* π_m *to* $\widetilde{G}(\mathbf{c})$ *and the tensor representation* $\pi_m^{(1)} \otimes \pi_m^{(2)} \otimes \cdots \otimes \pi_m^{(k)}$ *of* $\widetilde{G}_1 \times \cdots \times \widetilde{G}_k$.

Proof. First notice that for $g \in \widetilde{G}(\mathbf{c})$ and $z \in T_{\Omega_{\mathbf{c}}}$, $\pi_m(g)F(z)$ only depends on $g \bmod \widetilde{M_{\mathbf{c}}}$. Next use Proposition 1.13 (more exactly its analog for the characters ψ and ψ_j) and the conclusion follows. □

5. The Main Theorem

In this section, we want to construct a differential operator D on T_Ω such that $res(\mathbf{c}) \circ D$ intertwines the restriction of π_m to $\widetilde{G}(\mathbf{c})$ and $\bigotimes_{j=1}^k \pi_{m_j}^{(j)}$ for an appropriate choice of $\mathbf{m} = (m_1, m_2, \dots, m_k)$.

5.1. The data and the statement of the main theorem

Theorem 5.1. *Let* (E, Φ) *be a regular representation of* J *and assume that* $N = \dim E = 2rm$ *for some* $m \in \mathbb{N}$.

Let q *be a polynomial on* J *which satisfies*

(25)
i) q *is* $\mathbf{c}$*-homogeneous of multidegree* $\mathbf{p} = (p_1, p_2, \dots, p_k)$
ii) $p = q \circ Q$ *is* $\mathbf{c}$*–pluriharmonic on* E .

Let

$$m_1 = m + 2p_1, \quad m_2 = m + 2p_2, \quad \ldots, \quad m_k = m + 2p_k \ .$$

Let D_q *be the holomorphic constant coefficients differential operator on* $\mathbb{J}$ *with algebraic symbol* q. *Then* D_q *satisfies, for any* $g \in \widetilde{G}(\mathbf{c})$ *whose restriction to* $J(\mathbf{c})$ *is equal to* $(g_1, g_2, \ldots, g_k)$

$$\text{(26)} \qquad \operatorname{res} \circ D_q \circ \pi_m(g) = (\pi^{(1)}_{m_1}(g_1) \otimes \cdots \otimes \pi^{(k)}_{m_k}(g_k)) \circ \operatorname{res} \circ D_q \ .$$

5.2. The proof of the main theorem

It suffices to verify (26) for generators of $\widetilde{G}(\mathbf{c})$. First, the operator D has constant coefficients, hence commutes to translations and (26) follows for $g = t_v, v \in J(\mathbf{c})$.

When g is in $\widetilde{L}(\mathbf{c})$, the proof to get (26) is longer and we state it as a lemma. Notice however that the intertwining property (26) is trivially satsified for $m \in \widetilde{M}(\mathbf{c})$.

Lemma 5.1. *For* $\ell \in \widetilde{L}(\mathbf{c})$ *and for* $z \in T_{\Omega(\mathbf{c})}$,

$$\text{(27)} \qquad D \circ \pi_m(\ell) F(z) = (\pi^{(1)}_{m_1}(\ell_1) \otimes \cdots \otimes \pi^{(k)}_{m_k}(\ell_k))(DF_{|T_{\Omega(\mathbf{c})}})(z) \ .$$

Proof. As D is a constant coefficients differential operator, it suffices to prove (27) for the family of functions $f_v, v \in \mathbb{J}$, where $f_v(z) = e^{(z,v)}$.

Let $\ell \in \widetilde{L}(\mathbf{c})$. For any $z \in T_\Omega$,

$$\pi_m(\ell) f_v(z) = \psi(\ell)^{-m} f_{\ell^{-1 t} v}(z) \ .$$

As $Df_v = q(v) f_v$,

$$D \circ \pi_m(\ell) f_v(z) = \psi(\ell)^{-m} q(\ell^{-1 t} v) f_{\ell^{-1 t} u}(z) \ ,$$

so that, using (6)

$$\text{(28)} \qquad D \circ \pi_m(\ell) f_v(z) = \prod_{j=1}^{k} \psi_j(\ell_j)^{-m-2p_j} q(v) f_{\ell^{-1 t} v}(z) \ .$$

Let further $\operatorname{proj}(\mathbf{c})v = v_1 + v_2 + \dots v_k$, where $v_j \in J_j$. For $z = z_1 + z_2 + \dots + z_k \in T_{\Omega(\mathbf{c})}$,

$$f_v(z) = \prod_{j=1}^{k} e^{i(z_j, v_j)} \ ,$$

and hence

$$DF_{|T_{\Omega(\mathbf{c})}}(z) = q(v) \prod_{j=1}^{k} e^{i(z_j, v_j)} \ ,$$

so that

(29)

$$(\pi^{(1)}_{m_1}(\ell_1) \otimes \dots \otimes \pi^{(k)}_{m_k}(\ell_k))(DF_{|T_{\Omega(\mathbf{c})}})(z) = q(v) \prod_{j=1}^{k} \psi_j(\ell_j)^{-m_j} \prod e^{(z_j, \ell_j^{-1\,t} u_j)} \ .$$

Now compare (28) and (29) to finish the proof of Lemma 5.1. □

It remains to prove (26) for $g = \iota$, which we also state as a lemma.

LEMMA 5.2. *Let $z \in T_{\Omega(\mathbf{c})}$. Then*

$$D \circ \pi_m(\iota) F(z) = (\pi^{(1)}_{m_1}(\iota_1) \otimes \dots \otimes \pi^{(k)}_{m_k}(\iota_k))(DF_{|T_{\Omega(\mathbf{c})}})(z) \ . \tag{30}$$

PROOF. For $\xi \in E$, let

$$F_\xi(z) = e^{\frac{i}{2}\left(z, Q(\xi)\right)} \ .$$

As the representation (E, Φ) is assumed to be regular, it is enough to prove (30) for the family of functions $F_\xi, \xi \in E$.

For $z \in T_\Omega$

$$\pi_m(\iota) F_\xi(z) = (\det z)^{-m} e^{\frac{i}{2}\left(-z^{-1}, Q(\xi)\right)}.$$

Notice that for $z = x + iy \in T_\Omega$ the quadratic form

$$\big(y, Q(\xi)\big) = \langle \Phi(y^{\frac12})\xi, \Phi(y^{\frac12})\xi \rangle$$

is positive-definite, and hence, $\xi \longmapsto F_\xi(z)$ is in the Schwartz class $\mathcal{S}(E)$. Use (3.2) for $p \equiv 1$ on E and notice that $m = \frac{N}{2r}$ to get

$$\pi_m(\iota)F_\xi(z) = (2\pi)^{-\frac{N}{2}}(-i)^{rm}\int_E e^{i\langle\eta,\xi\rangle}e^{\frac{i}{2}(z,Q(\eta))}d\eta\ .$$

Now use (7) to get

$$D\pi_m(\iota)F_\xi(z) = (2\pi)^{-\frac{N}{2}}(-i)^{rm}2^{-\mathbf{r}.\mathbf{p}}\int_E q\big(Q(\eta)\big)e^{i\langle\xi,\eta\rangle}e^{\frac{i}{2}\left(z,Q(\eta)\right)}d\eta\ ,$$

where $\mathbf{r}.\mathbf{p} = \sum_{j=1}^k r_j p_j$.

Next, assume that $z = z_1 + z_2 + \cdots + z_k \in T_{\Omega(\mathbf{c})}$. Then, as $\mathrm{proj}_{\mathbf{c}}(Q(\eta)) = \sum_{j=1} Q_j(\eta_j)$,

$$D\pi_m(\iota)F_\xi(z) = (2\pi)^{-\frac{N}{2}}(-i)^{rm}2^{-\mathbf{r}.\mathbf{p}}$$

$$\int_{E_1}\cdots\int_{E_k} p(\eta_1,\dots,\eta_k)\prod_{j=1}^k e^{i\langle\xi_j,\eta_j\rangle}e^{\frac{i}{2}\left(z_j,Q_j(\eta_j)\right)}d\eta_1\dots d\eta_k\ .$$

Use Fubini theorem and (3.2) repeatedly for $j = 1,\dots,k$, take into account that $N_j = 2r_j m$ (see (12)) to get

$$D\pi_m(\iota)F_\xi(z) = 2^{-\mathbf{r}.\mathbf{p}}\prod_{j=1}^k \det(\frac{z_j}{i})^{-m}e^{\frac{i}{2}(-z_j^{-1},Q_j(\xi_j)}p(\dots,\Phi(-z_j^{-1})\xi_j,\dots).$$

Now use (12) and (13) to obtain

$$D\pi_m(\iota)F_\xi(z) = 2^{-\mathbf{r}.\mathbf{p}}\prod_{j=1}^k \det(\frac{z_j}{i})^{-m-2p_j}e^{\frac{i}{2}\left(-z_j^{-1},Q_j(\xi_j)\right)}p(\xi_1,\dots,\xi_k)\ . \tag{31}$$

On the other hand, for $z \in T_\Omega$,

$$DF_\xi(z) = q(\frac{1}{2}Q(\xi))\,e^{\frac{i}{2}\left(z,Q(\xi)\right)} = 2^{-\mathbf{r}.\mathbf{p}}p(\xi)e^{\frac{i}{2}\left(z,Q(\xi)\right)}\ .$$

Hence for $z = z_1 + z_2 + \cdots + z_k \in T_{\Omega(c)}$

$$DF_\xi(z) = 2^{-\mathbf{r}.\mathbf{p}}p(\xi)\prod_{j=1}^k e^{\frac{i}{2}\left(z_j,Q(\xi_j)\right)}$$

so that

$$\begin{aligned}(32)\qquad &\big(\pi^{(1)}_{m_1}(\iota_1)\otimes\cdots\otimes\pi^{(k)}_{m_k}(\iota_k)\big)DF_{\xi_{|T_{\Omega(\mathbf{c})}}}(z_1,z_2,\dots,z_k)\\ &\qquad = 2^{-\mathbf{r}.\mathbf{p}}p(\xi)\prod_{j=1}^{k}\det{}_j(\frac{z_j}{i})^{-m_j}\, e^{\frac{i}{2}\big(z_j,Q(\xi_j)\big)}\ .\end{aligned}$$

Compare (31) and (32) to conclude. □

This achieves the proof of the main theorem.

6. Examples in Rank 2

In this section, let $J=J_n$ be the simple Euclidean Jordan algebra of rank 2 and dimension n, i.e. $J_n=\mathbb{R}\oplus\mathbb{R}^{n-1}$, with the Jordan product

$$(s,x_1,\dots,x_{n-1})(t,y_1,\dots,y_{n-1})=(st+x.y,sy_1+tx_1,\dots,sy_{n-1}+tx_{n-1})\ ,$$

where $x.y=x_1y_1+\cdots+x_{n-1}y_{n-1}$.

We will assume that $n\geq 4$. In fact, for $n=2$, the algebra J_2 is not simple. For $n=3$, J is isomorphic to $Symm(2,\mathbb{R})$ and this case differs from the general case. See next footnote and final remark.

The trace and the determinant are given by

$$\mathrm{tr}(s,x)=2s,\qquad \det(s,x)=s^2-|x|^2\ .$$

The structure group of J is the product $\mathbb{R}^*\times O(1,n-1)$ and the group denoted by $\widetilde{L}$ in the general case is equal to $\mathbb{R}^+\times Spin_0(1,n-1)$.

The cone $\Omega=\Omega_n$ is the *forward cone*

$$\Omega=\{(s,x)\in\mathbb{R}\times\mathbb{R}^{n-1},\quad s^2-|x|^2>0,\ s>0\}\ .$$

The tube-type domain is

$$T_\Omega=\{Z=(z_0,z_1,\dots,z_{n-1})\in\mathbb{C}^n,\ \mathrm{Im}(Z)\in\Omega\}\ ,$$

the group of biholomorphic automorphisms is isomorphic to $O(2,n)/\{\pm\,\mathrm{id}\}$, and the group $\widetilde{G}$ is isomorphic to $Spin_0(2,n)$, see [11]. The domain has a bounded realization known as the *Lie ball.*

It will be convenient to use for J the standard inner product on a Euclidean Jordan algebra, which in this case is given by

$$\big((s,x),(t,y)\big) = 2(st + x.y)\ .$$

An idempotent of rank 1 is of the form $(\frac{1}{2}, x)$, with $|x| = \frac{1}{2}$. Up to an isomorphism of J, there is only one (non-trivial) CSOI, namely

$$\mathbf{c} = (c_1, c_2), \qquad c_1 = \big(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0\big), \quad c_2 = \big(\frac{1}{2}, -\frac{1}{2}, 0, \dots, 0\big).$$

Consequently,

$$J(\mathbf{c}) = \mathbb{R}c_1 \oplus \mathbb{R}c_2.$$

The corresponding Peirce decomposition is given by

$$J = \mathbb{R}c_1 \oplus \mathbb{R}c_2 \oplus J_{\frac{1}{2}}$$

where

$$J_{\frac{1}{2}} = \big\{(s,x), s = 0, x_1 = 0\big\} \simeq \mathbb{R}^{n-2}\ .$$

A well-adapted orthonormal basis of J is given by

$$f_1 = c_1, \quad f_2 = c_2, \quad \text{and for } j \geq 3 \quad f_j = (0, \dots, 0, \frac{1}{\sqrt{2}}, 0, \dots, 0)$$

where $\frac{1}{\sqrt{2}}$ is in the j-th place. A generic element of J will be denoted by

$$y = (y_1, y_2, \dots, y_j, \dots, y_n) = (y_1, y_2, y')$$

where y_j is the j-th coordinate of the element in the new basis $\{f_1, \dots, f_n\}$. The formula for the base change is given by

$$y_1 = s + x_1, \quad y_2 = s - x_1, \quad y_j = \sqrt{2}\, x_{j-1},\ 3 \leq j \leq n\ .$$

The group $L(\mathbf{c})$ preserves $J(\mathbf{c})$ and hence also its orthogonal $J_{\frac{1}{2}}$. In the basis $\{f_1, \dots, f_n\}$, an element of $L(\mathbf{c})$ is represented by

$$\ell(u,v,m) = \left\{ \begin{pmatrix} \begin{matrix} ue^{v} & 0 \\ 0 & ue^{-v} \end{matrix} & 0 \\ 0 & u\,m \end{pmatrix}, \quad \left\{ \begin{matrix} u \in \mathbb{R}_+,\ v \in \mathbb{R} \\ m \in Spin(n-2) \end{matrix} \right. \right\}.$$

L_1 and L_2 are both isomorphic to $\mathbb{R}_+$, and the restriction map is given by

$$L \ni \ell(u,v,m) \quad \longmapsto \quad (ue^v, ue^{-v}) \in L_1 \times L_2 \ .$$

The group $\widetilde{M}(\mathbf{c})$ is isomorphic to $Spin(n-2)$.

Lemma 6.1. *A polynomial q on J is $\mathbf{c}$-homogeneous of multi-degree (p_1,p_2) if and only if q is of the form*

$$q(y) = \sum_{j=0}^{\inf(p_1,p_2)} a_j\, y_1^{p_1-j} y_2^{p_2-j} |y'|^{2j} \tag{33}$$

for some $a_j \in \mathbb{C}$.

Proof. The group $M(\mathbf{c})$ is isomorphic to $Spin(n-2)$, fixes the subalgebra $J(\mathbf{c})$ and acts as $SO(n-2)$ on $J_{\frac{1}{2}}$. Hence[2] a $\mathbf{c}$-homogeneous polynomial can be written as a polynomial in y_1, y_2 and $|y'|^2$.

Now consider the elementary polynomial $y_1^{m_1} y_2^{m_2} |y'|^{2m_3}$. It satisfies the required conditions for $\mathbf{c}$-homogeneity if and only if m_1, m_2, m_3 satisfy

$$\begin{aligned} m_1 + m_2 + m_3 &= p_1 + p_2 \\ m_1 - m_2 &= p_1 - p_2 \ . \end{aligned} \tag{34}$$

Hence $m_1 = p_1 - j, m_2 = p_2 - j, m_3 = j$ for some $j, 0 \le j \le p_1, p_2$ and the conclusion follows. □

The cone $\Omega(c)$ is equal to $\{a_1c_1 + a_2c_2, a_1, a_2 \in \mathbb{R}^+\}$ and is a product of two positive half-lines, the tube-domain $T_{\Omega(\mathbf{c})}$ is equal to

$$\{(z_0, z_1, 0, \dots, 0), \mathrm{Im}(z_0 + z_1) > 0, \mathrm{Im}(z_0 - z_1) > 0\}$$

and is a product of two complex upper half-planes, the groups $\widetilde{G}_1$ and $\widetilde{G}_2$ are isomorphic to $SL(2,\mathbb{R})$.

Let (E, Φ) be a representation of J. For $v, w \in \mathbb{R}^{n-1}$

$$(0,v)(0,w) = (v.w, 0, \dots, 0) \ , \qquad v.w = \sum_{j=1}^{n-1} v_j w_j \ ,$$

[2] Here is the reason to exclude the case $n = 3$, as $SO(1) = \{\mathrm{id}\}$ and the invariance by $M(\mathbf{c})$ imposes no condition in this case.

so that

$$\Phi\big((0,v)\big)\Phi\big((0,w)\big) + \Phi\big((0,w)\big)\Phi\big((0,v)\big) = 2\,v.w\,\mathrm{Id}_E \ .$$

Hence E is a *Clifford module* for the Clifford algebra $Cl(n-1)$ generated by $\mathbb{R}^{n-1}$ with the relation (beware of the absence of sign $-$)

$$vw + wv = 2\,v.w \ .$$

Conversely, if E is a Clifford module for $Cl(n-1)$, then set

$$(s,x) \in J,\, \xi \in E, \qquad \Phi\big((s,x)\big)\xi = s\xi + x\xi$$

to obtain a representation of J. For a deeper study of these representations, see [1].

Let $E = E_1 \oplus E_2$ the decomposition of E with respect to the CSOI $\mathbf{c} = (c_1, c_2)$. For $v \in \mathbb{R}^{n-2}$, $\ c_1(0,0,v) = \frac{1}{2}(0,0,v)$ and hence $\Phi\big((0,0,v)\big) = 2\Phi\big(c_1(0,0,v)\big) = \Phi(c_1)\Phi\big((0,0,v)\big) + \Phi\big((0,0,v)\big)\Phi(c_1)$, so that

$$\Phi\big((0,0,v)\big)\Phi(c_2) = \Phi(c_1)\Phi\big((0,0,v)\big) \ .$$

Hence $\Phi\big((0,0,v)\big)$ permutes E_1 and E_2.

The quadratic map Q is given in the original basis of J by

$$Q(\xi_1+\xi_2) = \Big(\frac{1}{2}(|\xi_1|^2) + |\xi_2|^2), \frac{1}{2}(|\xi_1|^2 - |\xi_2|^2), \dots, \langle \Phi(e_j)\xi_1, \xi_2\rangle, \dots\Big) \ .$$

and hence in the basis $\{f_1, f_2, \dots, f_n\}$ by

$$Q(\xi_1+\xi_2) = \big(|\xi_1|^2, |\xi_2|^2, \dots, 2\langle \Phi(f_j)\xi_1, \xi_2\rangle, \dots\big) \ .$$

Let $\{\epsilon_k, 1 \leq k \leq N_1\}$ be an orthogonal basis of E_1 and denote by $(\xi_{1,k})_{1\leq k\leq N_1}$ the corresponding coordinates of a generic element $\xi_1 \in E_1$. Finally, denote by Δ_1 the partial Laplacian on E related to E_1, i.e.

$$\Delta_1 = \sum_{k=1}^{N_1} \frac{\partial^2}{\partial \xi_{1,k}^2} \ .$$

For the next statements, let $\partial_j = \dfrac{\partial}{\partial y_j}$ be the partial derivative (on J) with respect to the coordinate y_j.

PROPOSITION 6.1. *Let q be a polynomial on J and let p be the polynomial on E defined by $p = q \circ Q$. Then for $\xi = (\xi_1, \xi_2)$*

$$\Delta_1 p(\xi_1,\xi_2) = (\delta_1 q)\big(Q(\xi_1,\xi_2)\big)$$

where

$$\delta_1 = 2N_1\partial_1 + 4\, y_1\, \partial_1^2 + 4\sum_{j=3}^{n} y_j\, \partial_1\partial_j + 2y_2 \sum_{j=3}^{n} \partial_j^2 \ . \tag{35}$$

PROOF. Consider p as a polynomial on $E_1 \oplus E_2$, so that

$$p(\xi_1,\xi_2) = q\big(|\xi_1|^2, |\xi_2|^2, \dots, 2\langle \xi_1, \Phi(f_j)\xi_2\rangle, \dots\big).$$

Then, for any $k, 1 \le k \le N_1$

$$\frac{\partial}{\partial \xi_{1,k}}\, p\,(\xi_1,\xi_2) = 2\,\xi_{1,k}\,\partial_1 q + 2\sum_{j=3}^{n} \langle \epsilon_k, \Phi(f_j)\xi_2\rangle\, \partial_j q$$

$$\frac{\partial^2}{\partial \xi_{1,k}^2}\, p\,(\xi_1,\xi_2) = 2\,\partial_1 q + 4\,\xi_{1,k}^2\, \partial_1^2 q + 8\,\xi_{1,k} \sum_{j=3}^{n} \langle \epsilon_k, \Phi(f_j)\xi_2\rangle\, \partial_1\partial_j q$$
$$+4\sum_{j=3}^{n}\sum_{i=3}^{n} \langle \epsilon_k, \Phi(f_j)\xi_2\rangle\ \langle \epsilon_k, \Phi(f_i)\xi_2\rangle\, \partial_i\partial_j q\ .$$

Sum over $k, 1 \le k \le N_1$ and use the formula

$$\sum_{k=1}^{N_1} \langle \epsilon_k, \Phi(f_j)\xi_2\rangle \langle \epsilon_k, \Phi(f_i)\xi_2\rangle = \langle \Phi(f_j)\xi_2, \Phi(f_i\xi_2)\rangle\ ,$$

to get

$$\Delta_1 p(\xi_1,\xi_2) = 2\,N_1\partial_1 q + 4|\xi_1|^2\partial_1^2 q + 8\sum_{j=3}^{n} \langle \xi_1, \Phi(f_j)\xi_2\rangle \partial_1\partial_j q$$
$$+4\,\sum_{i=3}^{n}\sum_{j=3}^{n} \langle \Phi(f_j)\xi_2, \Phi(f_i)\xi_2\rangle\, \partial_i\partial_j\, q\ .$$

For $3 \leq i \neq j \leq n$, $f_i f_j = 0$, so that $\Phi(f_i)\Phi(f_j) = -\Phi(f_j)\Phi(f_i)$, whereas $f_j^2 = \frac{1}{2}e$ and hence $\langle \Phi(f_j)\xi_2, \Phi(f_j)\xi_2 \rangle = \frac{1}{2}|\xi_2|^2$. The formula follows from these observations. □

LEMMA 6.2. *Let $p_1, p_2 \in \mathbb{N}$ and let $k \in \mathbb{N}$, $0 \leq k \leq \inf(p_1, p_2)$. Then*

$$\begin{aligned} &\delta_1\big(y_1^{p_1-k} y_2^{p_2-k} |y'|^{2k}\big) = \\ &(p_1 - k)\Big(4k + 2\,N_1 + 4p_1 - 4\Big) y_1^{p_1-k-1} y_2^{p_2-k} |y'|^{2k} \\ &+ 2k(2k + n - 3) y_1^{p_1-k} y_2^{p_2-k+1} |y'|^{2k-2} \ . \end{aligned} \tag{36}$$

The verification is elementary and left to the reader.

PROPOSITION 6.2. *Let q be the polynomial defined by*

$$q(y) = \sum_{j=0}^{\inf(p_1,p_2)} a_j\, y_1^{p_1-j} y_2^{p_2-j} |y'|^{2j} \ , \tag{37}$$

and assume that $q \not\equiv 0$. Then $p(\xi) = q(Q(\xi))$ is $\mathbf{c}$-pluriharmonic if and only if $p_1 = p_2 = p$ and the coefficients a_j satisfy the relation

$$(j+1)\big(j + \frac{n-1}{2}\big) a_{j+1} + (p - j)(j + \frac{N_1}{2} + p - 1) a_j = 0 \ . \tag{38}$$

PROOF. The polynomial $p = q \circ Q$ is $\mathbf{c}$-pluriharmonic if and only if

$$\delta_1 q = 0, \qquad \delta_2 q = 0 \ , \tag{39}$$

where

$$\delta_2 = 2N_1 \partial_2 + 4y_2 \partial_2^2 + 4 \sum_{j=3} y_j \partial_2 \partial_j + 2y_1 \sum_{j=3}^{n} \partial_j^2 \ .$$

Use (36) to compute $\delta_1 q$ and simlarly for $\delta_2 q$. The conditions (39) are satisfied if and only if, for any $k, 0 \leq k \leq \inf(p_1, p_2) - 1$

$$\begin{aligned} &(k+1)(k + \frac{n-1}{2}) a_{k+1} + (p_1 - k)\big(k + \frac{N_1}{2} + p_1 - 1\big) a_k = 0 \\ &(k+1)(k + \frac{n-1}{2}) a_{k+1} + (p_2 - 1)\big(k + \frac{N_1}{2} + p_2 - 1\big) a_k = 0 \ . \end{aligned} \tag{40}$$

Hence $p_1 = p_2$ and (38) follows. □

From Proposition 6.2 follows

$$a_j = -\,\frac{(p+1-j)\big(j+\frac{N_1}{2}+p-2\big)}{j\,\big(j+\frac{n-3}{2}\big)}a_{j-1} \tag{41}$$

and hence

$$a_j = (-1)^j \frac{p\dots(p-(j-1))\,\left(\frac{N_1}{2}+p-1\right)\dots\left(\frac{N_1}{2}+p-1+(j-1)\right)}{1\,2\dots j\,\left(\frac{n-1}{2}\right)\dots\big(\frac{n-1}{2}+(j-1)\big)}\;a_0 \tag{42}$$

For $n,p,m\in\mathbb{N}$, let for $j\in\mathbb{N},\ 0\le j\le p$

$$a_j^{n,p,m} = (-1)^j \frac{p!}{j!(p-j)!}\,\frac{(m+p-1)\dots\big(m+p-1+(j-1)\big)}{\left(\frac{n-1}{2}\right)\dots\left(\frac{n-1}{2}+j-1\right)}\ .$$

Theorem 6.1. *Let $p,m\in\mathbb{N}$, and let D be the holomorphic diffferential operator defined on T_Ω by*

$$D = D_{n,p,m} = \sum_{j=0}^{p} a_j^{n,p,m}\left(\frac{\partial^2}{\partial z_0^2}-\frac{\partial^2}{\partial z_1^2}\right)^{p-j}\Delta_{n-2}^j\ . \tag{43}$$

For any $g\in\widetilde{G}(\mathbf{c})$, whose restriction to $T_{\Omega(\mathbf{c})}$ is equal to (g_1,g_2), the operator D satisfies

$$(\mathrm{res}\circ D)\circ\pi_m(g) = \big(\pi^{(1)}_{m+2p}(g_1)\otimes\pi^{(2)}_{m+2p}(g_2)\big)\circ(\mathrm{res}\circ D)\ . \tag{44}$$

Proof. From the study fo Clifford modules for $Cl(n-1)$, recalled for instance in [1], it is easily seen that there exists a regular representation E_{m_0} of J whose dimension $N = 4m_0$ is a multiple of 4. Then $\frac{N_1}{2} = \frac{1}{4}\dim(E) = m_0$ and the assumptions for Theorem 5.1 (adapted to the present context) are satisfied for E_{m_0}. Now consider the Clifford module $E_{m_0q} = \bigoplus_{j=1}^{q} E_j$, where each E_j is a copy of E_{m_0}. The corresponding representation of J satisfies all requirements needed for Theorem 5.1 to be valid. Hence Theorem 6.1 is valid for any $m = m_0 q, q \in \mathbb{N}$. Now the differential operator $D_{n,p,m}$ has coefficients which are polynomial in m. A simple argument, based on the fact that two polynomials of one variable which coincide on an infinite subset have to be equal, allows to conclude that Theorem 6.1 is valid for *all* values of m. Explicitly, as $\widetilde{G}(\mathbf{c})$ is connected, (44) is equivalent to its infinitesimal version

(45)
$$(\mathrm{res} \circ D) \circ d\pi_m(X) = \big(d\pi^{(1)}_{m+2p}(X_1) \otimes \mathrm{id} + \mathrm{id} \otimes d\pi^{(2)}_{m+2p}(X_2)\big) \circ (\mathrm{res} \circ D),$$

where $X \in \mathfrak{g}(\mathbf{c})$ and X_1 (resp. X_2) is the restriction of X to $\mathbb{J}_1$ (resp. $\mathbb{J}_2$). It is easily seen by differentiation of (21) that $d\pi_m(X)$ (resp. $d\pi^{(1)}_{m+2p}(X_1)$, $d\pi^{(2)}_{m+2p}(X_2)$) is expressed by a differential operator on $\mathbb{J}$ (resp. $\mathbb{J}^{(1)}, \mathbb{J}^{(2)}$) which depends polynomially (of degree 1) on m. Hence both sides of (45) are polynomial in m and the conclusion follows. □

Notice that the definition of $a^{n,p,m}$ makes sense for any real number m. This could be used to further extend our construction to the case of the *universal covering* of $G(\mathbf{c})$.

Remark. When $n = 3$, as $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$, the pair $\big(\mathfrak{g}, \mathfrak{g}(\mathbf{c})\big)$ is isomorphic to $\big(\mathfrak{so}(3,2), \mathfrak{so}(2,2)\big)$ and this case has been thoroughly studied in [10].

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Université de Lorraine
CNRS, IECL, F-54000 Nancy
France
E-mail: jean-louis.clerc@univ-lorraine.fr