

## *Polystable Log Calabi-Yau Varieties and Gravitational Instantons*

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**Abstract.** Open Calabi-Yau manifolds and log Calabi-Yau varieties have been broadly studied over decades. Regarding them as “semistable” objects, we propose to consider their good proper subclass, which we regard as certain *polystable* ones, morally corresponding to *semistable with closed (minimal) orbits* as the classical analogue of GIT.

We partially confirm that the new polystability seems equivalent to the existence of non-compact complete Ricci-flat Kähler metrics with small volume growths, notably many examples of gravitational instantons. Also, we prove some compactness or polystable reduction type results, partially motivated by bubbles of compact Ricci-flat metrics.

### 1. Introduction

#### 1.1. History and motivation

Since the celebrated existence theorem of compact Ricci-flat Kähler manifolds [Yau78a], sometimes under the name as the Calabi conjecture, there has been also plenty of nice works of constructing *non-compact complete* Ricci-flat Kähler metrics. Another origin is the group of various gravitational instantons in real four dimensions with rapid curvature decays, while their signature being Lorenzian [Hawk77, GH78, EH78], in the context of general relativity. We are not able to make an exhaustive list of references right here due to its numbers, but our discussion to follow necessarily includes many important examples. The first age general existence theorems in Kähler setting seem to be due to the papers [BK87, TY90, TY91, BK90] (cf., also [Yau78b]) outside *smooth* complement divisors in smooth Kähler manifolds.

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This paper means to be the first part of our attempt to lay an algebro-geometric foundation for these spaces, especially to allow the complement of *singular* divisors which are much subtler. Therefore, although we basically work over complex numbers  $\mathbb{C}$ , all algebro-geometric arguments work over an arbitrary algebraic closed field of characteristic 0 at least. Such hope for presence of such algebro-geometric treatment could have been in the line along the conjecture in [Yau78b], which predicted that any complete Ricci-flat Kähler metrics on  $X^\circ$  may be compactifiable to a compact Kähler manifold  $X$  whose complement is the support of an anticanonical divisor  $D$ . However, the conjecture of *loc.cit* is not necessarily true as discovered counterexamples by [AKL89, Goto94, Goto98, Hat11] etc, hence this paper restricts our attention to *compactifiable case* i.e., when  $X^\circ$  can be written as complement of analytic closed subset of projective normal variety  $X$ .

Combined with the Hironaka's log resolution in such compactifiable case, there should be a log pair  $(X, D)$  such that  $X$  is smooth,  $D$  is simple normal crossing such that  $X^\circ = X \setminus \text{Supp}(D)$ . We call such pair *log smooth* in this paper as we expect no confusion, taking our context to account. Also, just in this introduction, we suppose  $D$  is a reduced ( $\mathbb{Z}$ -)divisor for simplicity of exposition, while we discuss with  $\mathbb{Q}$ -divisor  $D$  from the next section.

The notion of *dlt (divisorially log terminal)* pair of a variety  $X$  and a boundary divisor  $D$ , introduced by Shokurov [Sho93] and more clarified in [Sza95], [KM98] (cf., also [KMM87], [Fjn07]), slightly extends the ubiquitous log smooth setup. We propose to use this notion of dlt pairs more systematically, in much wider context of studying limiting behaviour of canonical Kähler metrics, their singularities, or its non-compact versions among others. For the details of the notion, we refer to [KM98], [Fjn18], [Kol13] for instance. As a technical important tool, extension of the minimal model program for only klt singular setting (as the groundbreaking [BCHM10]) to even more singular setting will be effectively used as in this paper; allowing dlt, log-canonical or even semi-log-canonical singularities. See e.g., [Fjn00, Amb03, KK10, Gon11a, Gon11b, FG14, HX13, Fjn17, Has16, HH20] as foundational results and recent developments in such direction.

We start with raising the following conjecture, partially to set the scene.

**CONJECTURE 1.1 (Asymptotics).** *Suppose  $X^\circ$  is a non-proper but separated (Hausdorff) variety which only has kawamata-log-terminal singular-*

ities (assume smoothness for simplicity, if not familiar) over  $\mathbb{C}$  and admits a complete Ricci-flat weak Kähler metric  $g$  in which we fix a base point  $p$ . We suppose that

$$\text{vol}(B(p, r)) \sim cr^d,$$

for  $r \rightarrow +\infty$  where  $B(p, r)$  denotes the geodesic balls of radius  $r > 0$  with the center  $p$ , and call this  $d$  as the volume growth dimension  $\text{vg}(X^o, g)$ .

- (i) if  $\text{vg}(X^o, g) > \dim_{\mathbb{C}}(X)$  then the logarithmic Iitaka dimension ([Iit77]<sup>1</sup>) satisfies  $\bar{\kappa}(X^o) = -\infty$ . In particular, any compactification  $(X^o \subset) X$  is covered by rational curves i.e., uniruled<sup>2</sup>.
- (ii) If  $X^o$  is compactified to be a log pair  $(X, D)$  of smooth projective variety  $X$  and its simple normal crossing divisor  $D(\in | -K_X|)$  such that  $X \setminus \text{Supp}(D) = X^o$ , then the following inequalities hold;

$$\dim_{\mathbb{R}} \tilde{\Delta}(D) = \dim(X^o)^{\text{an}} \leq \text{vg}(X^o, g) \leq \dim_{\mathbb{R}}(X) = 2 \dim_{\mathbb{C}}(X),$$

where  $\tilde{\Delta}(D)$  means the dual intersection cone complex<sup>3</sup> of  $D$  and  $(X^o)^{\text{an}}$  is the Berkovich analytification of  $X^o$  for trivial valuation (cf., [Ber90]).

Note that, the volume growth dimension  $\text{vg}(X^o, g)$  is at most  $2\dim_{\mathbb{C}}(X^o) = \dim_{\mathbb{R}}(X)$  by the Bishop-Gromov comparison and it is not necessarily integer. Indeed, in the classical examples of [TY90],  $\frac{2 \dim(X)}{\dim(X)+1}$  is attained. Also, although we excluded in the above situation, compact Ricci-flat Kähler varieties can be also regarded as those with volume growth dimension 0.

REMARK 1.2. For above (i) case, the converse does *not* hold in general. For instance, appropriate products of known gravitational instantons easily give examples. Also, even an indecomposable (surface) example exists by

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<sup>1</sup>contains an interesting metaphor from the 70s by its author as follows in *loc.cit* §11.16 “ $V = \mathbb{P}^2 - \bar{D}$  is algebraic geometry for dimension 1.5” (translated from Japanese version)

<sup>2</sup>For instance, if  $X^o$  is a smooth surface, more strongly it holds that  $\bar{\kappa}(X^o) = -\infty$  is equivalent to that  $X^o$  is dominated by images of  $\mathbb{A}_{\mathbb{C}}^1 = \mathbb{C}$ . This is due to the works of [MT84a, MT84b, KeMc99].

<sup>3</sup>Note that the “common” dual intersection complex is  $\Delta(D) = (\tilde{\Delta}(D) \setminus \vec{0})/\mathbb{R}_{>0}$

the thesis of Hein [Hein12]. Indeed, the complement of type II, III, IV degenerations (resp., type  $I^*$  degenerations) in rational elliptic surfaces with Tian-Yau metric [TY90] are shown to be such examples, in [Hein12] (resp., [Hein12]).

More details are as follows. For any of those fibers of type II, III, IV, after passing to a log resolution, the log crepant boundary divisor supported on the fiber becomes “combinatorially same” i.e., they all become simple normal crossing with 4 components with just one non-reduced component of multiplicity 2 penetrating the others. On the other hand, we remark that the conical angle of the ALG type bubble in that case ([Hein12], [CVZ20]) is the inverse of the ramification index for the necessary *semistable reduction*. Also, Type  $I^*$  (minimal) degenerations are originally non-reduced.

This paper mainly focuses on the case when they are non-compact and also the volume growth dimension should not exceed the original complex dimension of  $X$ . We wish to discuss the remained case in near future, and mean this paper to be the first for such series (We would like to call the property of complete Ricci-flat Kähler manifold being  $\text{vg}(X^o, g) > \dim_{\mathbb{C}}(X)$  *very open-ness*.) Anyhow, for various examples of open Calabi-Yau manifolds, by which we mean the open part  $X^o = X \setminus \text{Supp}(D)$  of log Calabi-Yau manifolds  $(X, D)$ , it is not known if  $X^o$  admits any complete Ricci-flat Kähler metric. Recently, S. Sun asked a question if possible obstruction is detectable in a systematic algebro-geometric framework.

Before defining our stability conditions to answer his question, we recall further background. The K-stability of polarized variety [Tia97, Don02] has a version for log pairs introduced by [Don12] which aims to understand existence of *edge-conical singular* (weak) Kähler-Einstein metrics with cone angle  $2\pi\beta$ , when the coefficient  $1 - \beta$  of the divisors are *strictly less* than 1 (i.e.,  $\beta > 0$ ). There is also the framework via intersection numbers [OS15]. One may think it would be natural to use this log K-stability notion, even in the case the boundary divisors have coefficients 1, so we recall the following result, formally applying the definition of log stabilities [Don12, OS15] even for “conical angle 0” which do not a priori make sense for metrics.

**THEOREM 1.3** (cf., [OS15]). *Assume  $X$  is a smooth projective variety,  $(X, D)$  is a log smooth Calabi-Yau pair, i.e.,  $K_X + D$  is numerically equiv-*

alent to zero divisor with a polarization  $L$ .<sup>4</sup> Then,  $((X, D), L)$  is logarithmically  $K$ -semistable if and only if coefficients of  $D$  are at most 1 but it cannot be log  $K$ -polystable unless  $\lfloor D \rfloor = 0$ .

More generally, semi-log-canonical Calabi-Yau polarized pair  $((X, D), L)$  is log  $K$ -semistable but it is log  $K$ -polystable if and only if the pair has only klt singularities.

Therefore, the main issue is to refine the log stability notion, which will be our stability notions. Another motivation of the introduction of the notion for the author is for the problem of compactifying the moduli with its relation with bubbles of Ricci-flat Kähler metrics, as we indeed prove some “valuative criteria of properness” type results in §4. We leave the details of the actual stability conditions to section 2 while we give some hints here. We introduce following three variants with slightly different purposes, reflecting the subtlety of setup of discussing complete Ricci-flat Kähler metrics, and for later each usefulness.

- (Definition 2.3) We first introduce *weakly open  $K$ -polystability* of open Calabi-Yau pair, which concerns log test configurations with only irreducible degeneration fibers. As we show, one of our main points of the notion is it can be studied fairly systematically and easily (unlike the usual  $K$ -stability!) in terms of log canonical valuations and log canonical centers. In particular, the set of test configurations for this are only *countably* infinite even over  $\mathbb{C}$ .

Another point is that the definition also somewhat resembles the original Futaki’s obstruction [Fut83].

- (Definition 2.11) A somewhat stronger notion *open  $K$ -polystability*, stronger than weakly open  $K$ -polystability, means to detect the existence of complete Ricci-flat Kähler metric.
- (Definition 2.13) The *strongly open  $K$ -polystability*, stronger than open  $K$ -polystability, means to detect the existence of complete Ricci-flat Kähler metric which is limit of conical singular canonical metric with angle converging to 0. The definition involves log  $K$ -polystability ([Don12, OS15]).

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<sup>4</sup>We remark that “polarization” in our context refers to an ample line bundle unless otherwise stated.

- We show some stable reduction type results (compactness theorem) in §4, partially motivated by minimal non-collapsing pointed Gromov-Hausdorff limits of compact Ricci-flat Kähler metrics for special maximal degenerating holomorphic families.

## 1.2. Conjectures on metrics

As mentioned above, one side of the main motivations for our attempt to introduce stability notions is following expectation on metric existence.

CONJECTURE 1.4 (Non-compact Yau-Tian-Donaldson conjecture). *Take an arbitrary a polarized dlt log Calabi-Yau pair  $((X, D), L)$  with  $[D] = \sum_i D_i$ ,  $K_X + D \equiv 0$  and  $L$  being ample on  $X^\circ := X \setminus \text{Supp}([D])$ .*

- (i) *Then, if there is a complete Ricci-flat Kähler metric  $g$  on  $X^\circ$  of the volume growth dimension at most  $\dim_{\mathbb{R}}(X) = 2 \dim_{\mathbb{C}}(X)$ , whose Kähler class is  $c_1(L)|_{X^\circ}$ ,  $(X^\circ, L^\circ)$  is weakly open  $K$ -polystable.*
- (ii) *Furthermore, such a complete Ricci-flat Kähler metric with any base point is the pointed Gromov-Hausdorff limit of the conical singular weak<sup>5</sup> cscK metric on some polarized dlt log Calabi-Yau compactification  $((X, (1 - \epsilon)D), L_\epsilon)$  for  $\epsilon \rightarrow 0$  with appropriate base points with  $c_1(L_\epsilon|_{X^\circ}) \rightarrow c_1(L|_{X^\circ})$  for  $\epsilon \rightarrow 0$ , if and only if  $(X^\circ, L^\circ)$  is strongly open  $K$ -polystable.*

This obviously reflects the following question.

QUESTION 1 (Approximability / Metric compactifiability). For an arbitrary open complete Ricci-flat Kähler manifold  $(X^\circ, g)$  with the volume growth dimension at most  $\dim_{\mathbb{C}}(X)$ , is there a compactification  $X^\circ \subset X$  such that  $(X, X \setminus X^\circ =: D)$  is dlt, where  $D$  is a reduced integral divisor,  $g$  is the limit of a sequence of conical singular weak cscK metrics (cf., e.g., [KZ18, LWZ20])  $g_i (i = 1, 2, \dots)$  on  $X$  whose singularities are supported

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<sup>5</sup>Note that this notion of conical singular weak cscK metric makes sense in the same way as weak Kähler-Einstein metric on klt pairs (see e.g. [BBEGZ19, Ber16, KZ18, LT19, LWZ20]), since the concerned log pairs  $(X, (1 - \epsilon)D)$  are klt for  $\epsilon > 0$  in our situation. More precisely, it refers to the Kähler current of conical singular cscK metric (cf., [KZ18, LWZ20]) on the log smooth locus which defines a current on whole  $X$  whose volumes are right intersection numbers.

in  $D$ , whose conical angles  $\beta_i$  converge to 0? If not true in general, what conditions on  $g$  ensures such approximability?

For a special case of the above question for Tian-Yau metric [TY90], especially when  $X$  is a Fano manifold and  $D$  is smooth (cf., Theorem 3.1 (i), (iii) later) the answer is expected to be affirmative in [Don12]. Also, a partial confirmation of Conjecture 1.4 below follows from a result of Berman [Ber16] straightforwardly.

**COROLLARY 1.5** (of [Ber16]). *In the setting of Conjecture 1.4, under further assumption that  $L_\epsilon \equiv -(K_X + (1 - \epsilon)D)$ , existence of a sequence of conical singular weak Kähler-Einstein metrics for  $((X, (1 - \epsilon)D), L_\epsilon)$  with  $\epsilon \rightarrow 0$  (which is expected to automatically converge to a complete Ricci-flat weak Kähler metric on  $X^\circ$ ) implies strongly open K-polystability of  $(X^\circ, L^\circ)$ .*

In general, there are many difficulties to work on Kähler geometry for open manifolds compared with compact case. For instance, even for relatively simple complete Kähler manifolds,  $\partial\bar{\partial}$  lemma does *not* holds. (Nevertheless, some affirmative results are in [Del90], [CH13], for instance.) We also remark there is another systematically studied class of non-compact complete canonical Kähler metrics; the Kähler-Einstein metrics with *negative Ricci curvature* i.e., hyperbolic metrics on smooth locus of Deligne-Mumford stable curves and its higher dimensional extension to KSBA varieties (semi-log-canonical models), whose existence ([BG14]) matches to corresponding algebro-geometric K-stability results ([Od13a, Od12, OS15]).

This paper focuses on the Ricci-flat case and more precisely on concrete analysis of our new stability notions which provide many evidences to the above conjecture 1.4 on the existence of complete Ricci-flat Kähler metrics, and also explore an application to the non-collapsing limits of families of compact Ricci-flat Kähler metrics, and the moduli compactification problem: see our next subsection and §4. The supporting evidences for the above conjecture 1.4 are obtained from some results of concrete analysis of the stability notion below.

**THEOREM 1.6** (A weaker version of Theorem3.1). *Suppose  $(X^\circ, L^\circ)$  is an open  $n$ -dimensional polarized Calabi-Yau pair, and its compactifying polarized smooth log Calabi-Yau pair is denoted as  $((X, D), L)$ . In this introduction, we assume  $X$  is smooth and  $D$  is a reduced (coefficients 1) integral*

simple normal crossing divisor, for simplicity just in this introduction. (See more singular extensions in Theorem 3.1.)

- (i) If  $D$  is smooth, then  $(X^\circ, L^\circ)$  is weakly open  $K$ -polystable for any  $L$ . If  $X$  is a Fano manifold, then it is strongly open  $K$ -polystable. (We extend to slightly more singular case in Theorem 3.1 ii).
- (ii) If  $X^\circ$  is algebraic torus, then strongly open  $K$ -polystable.
- (iii) Moreover, if  $X^\circ$  is semi-abelian variety, then strongly open  $K$ -polystable.
- (iv) If  $G := \text{Aut}^\circ(X)$  is reductive and  $((X, D), L)$  is weakly open  $K$ -polystable, then  $D$  is GIT polystable with respect to the  $G$ -action.
- (v) cluster log surface (cf e.g. [GHK]) is not even weakly open  $K$ -polystable with respect to any polarization.
- (vi) rational elliptic surface  $X$  with  $D$  a nodal reduced fiber of  $I_\nu (\nu \geq 1)$  Kodaira type, then  $(X^\circ, L^\circ)$  is strongly open  $K$ -polystable at least for some  $L^\circ$ .
- (vii) If  $((X, D), L)$  is weakly open  $K$ -polystable and some irreducible component  $D_i$  of  $D$  satisfies that

- $L|_{X \setminus D_i} \sim_{\mathbb{Q}} 0$  and
- $D_i$  is ample (e.g. when  $\rho(X) = 1$ ),

then  $((X, D), L)|_{X \setminus D_i}$  is the affine cone of a certain  $(n - 1)$ -dimensional dlt log Calabi-Yau pair  $((X', D'), L') := ((D_i, \cup_{j \neq i} D_j \cap D_i), N_{D_i/X})$ .

The above results match with known existence of gravitational instantons such as: [TY90, BK90], [Hein12] (cf., also [CJL21]), [GCh17, CC15, CC21], [CHNP13, HHN15] for example.

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## 2. The Stability Conditions

This section introduces various stability notions for “polarized Calabi-Yau varieties” in somewhat generalized senses which are allowed to be either non-proper, or “log pairs” i.e.,  $(X, D)$  with  $K_X + D \equiv 0$ , or with non-normal reducible  $X$ . The author believes the number of variants for these stability notions below indirectly reflect the subtlety of handling general complete Ricci-flat Kähler metrics. Indeed, we use most of the notions introduced here in later sections.

### 2.1. Irreducible case

Firstly, we make the following usual setup.

DEFINITION 2.1.

(i) A polarized (dlt) log Calabi-Yau pair  $((X, D), L)$  in this paper means

- $K_X$  is  $\mathbb{Q}$ -Cartier (“ $\mathbb{Q}$ -Gorenstein” condition),<sup>6</sup>
- $D$  is  $\mathbb{Q}$ -divisor such that the pair  $(X, D)$  is dlt,  $K_X + D \equiv 0$  (which is equivalent to  $K_X + D \sim_{\mathbb{Q}} 0$  by [Fjn00, Gon11a]),
- the rounddown part of  $D$  which we write as  $\lfloor D \rfloor$ , and its fractional part  $\{D\}(= D - \lfloor D \rfloor)$  are both  $\mathbb{Q}$ -Cartier
- $L$  is ample .

We denote  $X^\circ := X \setminus \text{Supp}[D] = X^{\text{klt}}$  as its klt locus, a Zariski open subset, and call it *open locus* sometimes in this paper. We also set  $L^\circ := L|_{X^\circ}$ . Any polarized log Calabi-Yau pair is log K-semistable as reviewed as Theorem 1.3 ([OS15]) above. If further  $L = -K_{(X, \{D\})} := -(K_X + \{D\})$ , we call  $((X, D), L)$  is *anti-log-canonically polarized*.

Also, if we consider  $((X, D), L)$  which satisfies all above condition except for the dlt property while  $(X, D)$  is still log canonical (resp., semi-log-canonical), each time we write so as *log-canonical (resp., semi-log-canonical) polarized log Calabi-Yau pair*.

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<sup>6</sup>We put it here for simplicity. It is not necessary for various part of the following discussions.

- (ii) A non-proper polarized log pair  $(Y, M)$  is said to be *open polarized Calabi-Yau pair* in this notes if it is compactifiable to a polarized log Calabi-Yau pair  $((X, D), L)$  which satisfies  $[D] \neq 0$ , associated with a fixed isomorphism  $(X^\circ, L^\circ) \simeq (Y, M)$ .

REMARK 2.2. Note that the above compactifications are not unique for a fixed open polarized Calabi-Yau pair as well-known. Nevertheless, as an easy obvious case of the Shokurov-Kollár connectedness principle, between different such compactifications, still the set of the connected components of  $[D]$  are canonically bijective to each other. They are at most two connected components of  $[D]$  (cf., e.g., [Fjn00], [Gon11a]).

Now we define weakly open polystability notion and prepare its foundation.

DEFINITION 2.3 (Weakly open polystability).

- (i) For a polarized log Calabi-Yau pair  $((X, D), L)$ , a log test configuration (see [Don12], [OS15] for the definition)  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  is called *plt-type* in this paper, if  $(\mathcal{X}, \mathcal{X}_0)$  is plt.<sup>7</sup>
- (ii) Take an arbitrary log Calabi-Yau pair  $(X, D)$  which is dlt so that for its any polarization  $L$  log K-semistability holds by Theorem 1.3 above. We denote the restricted polarization as  $L^\circ := L|_{X^\circ}$ . A pair  $(X^\circ, L^\circ)$  or a triple  $((X, D), L^\circ)$  is said to be *weakly open K-polystable* if any plt-type test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  of  $((X, D), L)$  whose log Donaldson-Futaki invariant vanishes  $\text{DF}((\mathcal{X}, \mathcal{D}), \mathcal{L}) = 0$  ([Don12], [OS15]) is product log test configuration i.e.,  $((X, D), L)$ -fiber bundle over  $\mathbb{P}^1$ . We will see in Proposition 2.8 that the notion only depends on  $(X^\circ, L^\circ)$  to  $L$ , hence the terminology.

Furthermore, if the only product log test configuration is trivial one, i.e., when  $\text{Aut}((X, D), L)$  is finite, then we call the triple  $((X, D), L^\circ)$  *weakly open K-stable*.

The plt-type condition together with the automatic Cartierness property of  $\mathcal{X}_0$  imply that  $\mathcal{X}_0$  is klt and vice versa ([KM98]). If  $L = -K_X$

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<sup>7</sup>plt means purely log terminal. See [KM98] for details if not familiar. This condition in particular implies  $\mathcal{X}_0$  is klt and there is associated dreamy valuation for  $K(X \times \mathbb{A}^1)$ .

modulo  $\mathbb{Q}D$ , then  $\mathcal{L} \equiv -K_{\mathcal{X}/\mathbb{P}^1}$  modulo  $\mathbb{Q}D$  (cf., [OSS16]). The notion may remind the experts of the notion of “special test configurations” for anticanonically polarized  $\mathbb{Q}$ -Fano varieties ([DT92, Tia97, LX14]) as examples, but note that we do not require  $\mathcal{L}$  is anticanonical which will be important later on. Therefore, our notion strictly extends the notion of special test configuration of Fano varieties. Also, it is easy to see that if we would allow *log canonical* Calabi-Yau variety as  $\mathcal{X}_0$ , then [TY90] examples do not satisfy the corresponding polystability notion e.g.  $\mathbb{P}^2$  degenerating into the projective cone over elliptic curve.

We make following remark about the *product-ness* of log test configuration.

**PROPOSITION 2.4.** *The automorphism groups  $\text{Aut}^o((X, D), L)$  and  $\text{Aut}^o(X^o, L^o)$  are different for general triple  $((X, D), L)$ , while they coincide when  $K_X + D = 0$  (assuming the base field  $k$  has uncountable order).  $\text{Aut}^o(-)$  mean the identity connected component of the automorphism groups here and henceforth.*

**PROOF OF PROPOSITION 2.4.** As an example, of which the above two automorphism groups are different, simply we can take the affine space  $X^o = \mathbb{A}^n$  so that  $(X^o, L|_{X^o})$  includes the whole polynomial ring  $k[X_2, \dots, X_n](\ni f)$  as

$$(a_1, \dots, a_n) \mapsto (a_1 + f(a_2, \dots, a_n), a_2, \dots, a_n),$$

hence far from being an algebraic group, as it would be of course finite dimensional if so.

**LEMMA 2.5.** *For any fixed dlt log Calabi-Yau pair  $(X, D)$ , there are only at most countably many log crepant  $(X', D')$  with  $X \setminus \text{Supp}[D] = X' \setminus \text{Supp}[D']$ , modulo the isomorphisms.*

Note that for instance, there are countable infinite order of toric pairs for each fixed dimension more than 1.

**PROOF OF LEMMA 2.5.** Since  $(X', D')$  is log crepant to  $(X, D)$  for the obvious reason, all the irreducible components of  $D'$  are log canonical places for the log pair  $(X, D)$ . From Zariski lemma (cf., e.g., [KM98]) and the dlt

property of  $(X, D)$ , there are only countably many such log canonical places hence we conclude the proof.  $\square$

We now return to continue the proof of Proposition 2.4. We take a connected algebraic subgroup of  $\text{Aut}^o(X^o, L|_{X^o})$  which we denote as  $G^o$ . Then, thanks to the cute lemma above 2.5,  $((X, D), L)$  also admit natural  $G^o$ -action so that  $G^o \subset \text{Aut}^o((X, D), L)$ . Then, ranging  $G^o$  finishes the proof.  $\square$

**COROLLARY 2.6** (Product-ness). *Any plt-type log test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  of  $((X, D), L)$  which contains a product test configuration of  $(X^o, L|_{X^o})$  as the complement of the closure of  $D \times (\mathbb{P}^1 \setminus \{0\})$  is itself a product log test configuration of  $((X, D), L)$ . Further, it is determined by the open product test configuration associated in that way.*

**PROOF OF COROLLARY 2.6.** Thanks to Proposition 2.4, what remains is to show there is no plt-type testconfiguration  $((\mathcal{X}', \mathcal{D}'), \mathcal{L}')$  which coincides outside  $\text{Supp}(\mathcal{D})$  and  $\text{Supp}(\mathcal{D}')$  but it follows from the presence of polarizations since the two spaces are isomorphic in codimension 1 with same restriction of the polarizations in the common open locus (cf., [MM64]).  $\square$

Next we characterize the plt-type log test configuration with vanishing log Donaldson-Futaki invariant, which is one of the keys for our later analysis of the stability notion.

**PROPOSITION 2.7** (Test configuration as log canonical valuations). *Plt-type test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  for a polarized log Calabi-Yau variety  $((X, D), L)$  has vanishing log Donaldson-Futaki invariant if and only if the discrepancy vanishes*

$$(1) \quad a_{((X, D) \times \mathbb{P}^1)}(\mathcal{X}_0) = 0.$$

*Note that this in particular implies the central fiber  $\mathcal{X}_0$  is log canonical place of  $((X \times \mathbb{P}^1, D \times \mathbb{P}^1 + X \times \{0\})$  i.e., the log discrepancy over  $((X \times \mathbb{P}^1, D \times \mathbb{P}^1 + X \times \{0\})$  vanishes. Furthermore, such log test configuration is determined by the log canonical place as valuation  $v_{\mathcal{X}_0}$ .*

**PROOF.** The former claim follows straightforward from the intersection number formula of (log) Donaldson-Futaki invariant [OS15]. The latter

claim follows from that the plt type test configuration has irreducible central fiber so that  $v_{\mathcal{X}_0}$  determines  $\mathcal{X}$  up to isomorphism in codimension 1 and the same argument as previous Corollary 2.6 ([MM64]) using the polarization.  $\square$

The following supports the terminology.

**PROPOSITION 2.8.** *Weakly open K-polystability (resp., weakly open K-stability) of polarized log Calabi-Yau pair  $((X, D), L)$  only depends on the open locus  $(X^\circ, L^\circ)$ , not on the choice of compactification  $((X, D), L)$ .*

**PROOF.** For  $X^\circ$ , suppose there is a triple  $((X_1, D_1), L_1)$  which satisfies the weakly open K-polystability condition as Definition 2.3. It suffices to show the following; take another arbitrary polarized dlt Calabi-Yau compactification  $((X_2, D_2), L_2)$  and we prove it also satisfies the condition of 2.3. Consider a log test configuration of  $((X_2, D_2), L_2)$  of plt type, with vanishing log Donaldson-Futaki invariant as 2.3, whose total space is denoted as  $((\mathcal{X}_2, \mathcal{D}_2), \mathcal{L}_2)$  and its open subset  $\mathcal{X}_2^\circ := \mathcal{X}_2 \setminus \text{Supp}[\mathcal{D}_2]$ . Take an a priori smaller open subset  $\mathcal{X}_2^{\circ\circ}(\mathcal{X}_2^\circ)$  from which the birational map to  $X^\circ \times \mathbb{P}^1$  is a morphism. We take  $(\mathcal{X}'_1, \mathcal{D}'_1) := ((X_1, D_1) \times (\mathbb{P}^1 \setminus \{0\})) \cup \mathcal{X}_2^{\circ\circ}$  on which there is also a naturally glued polarization  $\mathcal{L}'_1$ . We compactify the triple  $((\mathcal{X}'_1, \mathcal{D}'_1), \mathcal{L}'_1)$  to a log test configuration  $((\mathcal{X}''_1, \mathcal{D}''_1), \mathcal{L}''_1)$  so that  $(\mathcal{X}''_1, \mathcal{D}''_1)$  is  $\mathbb{Q}$ -factorial dlt pair with birational *morphism*  $f$  to  $X_1 \times \mathbb{P}^1$ , by the use of relative minimal model program over  $X_1 \times \mathbb{P}^1$ . By applying Lemma 2.7 to  $\mathcal{X}_2$ , and combining with the log canonicity of  $((X_1 \times \mathbb{P}^1, D_1 \times \mathbb{P}^1 + X_1 \times \{0\})$ , we see that there is a  $\mathbb{Q}$ -divisor  $F$  supported on the central fiber  $(\mathcal{X}''_1)|_0$  such that  $(\mathcal{X}''_1, F)$  is log crepant to  $((X_1 \times \mathbb{P}^1, D_1 \times \mathbb{P}^1 + X_1 \times \{0\})$ . We denote a birational map  $\mathcal{X}''_1 \dashrightarrow \mathcal{X}_2$  as  $g$ . We take a small enough  $0 < \epsilon \ll 1$  and run again the relative minimal model program, this time over  $\mathbb{P}^1$ , from  $(\mathcal{X}''_1, F - \epsilon g_*^{-1}(\mathcal{X}_2)|_0)$ . Then we obtain a dlt minimal model  $(\mathcal{X}'''_1, \mathcal{D}'''_1 + (\mathcal{X}''_1)|_0)$  with generic fiber  $(X_1, D_1)$ . We take a general relative section of  $|mL_1 \times \mathbb{P}^1|$  and its closure in  $\mathcal{X}'''_1$ . Take lc model of  $(\mathcal{X}'''_1, \mathcal{D}'''_1 + \epsilon' A)$  for  $0 < \epsilon' \ll 1$ , then by the negativity lemma (cf., e.g., [KM98]), we obtain the desired log test configuration  $(\mathcal{X}_1, \mathcal{D}_1)$  with the polarization extending that of  $L_1 \times (\mathbb{P}^1 \setminus \{0\})$ . Its log Donaldson-Futaki invariant vanishes due to Lemma 2.7, hence by the weakly open K-polystability (with respect to  $((X_1, D_1), L_1)$ ) assumption we see that this is obtained by one parameter subgroup of the automorphism group discussed in Proposition 2.4. By

the same proposition, this also provides log product test configuration of  $((X_2, D_2), L_2)$  which has small birational map to  $((\mathcal{X}_2, \mathcal{D}_2), \mathcal{L}_2)$ . Applying [MM64], we obtain it is isomorphism. Hence we conclude the proof.  $\square$

From the above lemma 2.7, the theory of log canonical centers ([Amb03], [Fjn18]) play an essential role for this weakly open K-polystability notion. See the arguments in the next section §3.

The following is motivated by the Shokurov connectedness principle and the above Proposition 2.7. Take an arbitrary polarized (dlt) log Calabi-Yau pair  $((X, D), L)$  (Definition 2.1 (i)). For two log test configurations of plt-type  $((\mathcal{X}_i, \mathcal{D}_i), \mathcal{L}_i) (i = 1, 2)$ , we call one is *elementary transform* of the other if there is a test configuration of  $X$  which is the blow up of a log canonical center (with reduced structure) of  $(\mathcal{X}_i, \mathcal{D}_i + \mathcal{X}_{i,0})$  for both  $i$ .

**PROPOSITION 2.9 (Connectedness).** *For any polarized (dlt) log Calabi-Yau pair  $((X, D), L)$  and any two log test configurations of plt-type  $((\mathcal{X}_i, \mathcal{D}_i), \mathcal{L}_i) (i = 1, 2)$  with  $\text{DF}((\mathcal{X}_i, \mathcal{D}_i), \mathcal{L}_i) = 0$  for  $i = 1, 2$ , is there always a finite sequence of such plt type log test configurations  $((\mathcal{X}'_i, \mathcal{D}'_i), \mathcal{L}'_i) (i = 1, 2, \dots, m)$  such that the following holds?*

*$(\mathcal{X}'_i, \mathcal{D}'_i)$  and  $(\mathcal{X}'_{i+1}, \mathcal{D}'_{i+1})$  are elementary transforms of each other for  $i = 1, \dots, m - 1$ , with*

$$(\mathcal{X}_1, \mathcal{D}_1) = (\mathcal{X}'_1, \mathcal{D}'_1),$$

$$(\mathcal{X}_2, \mathcal{D}_2) = (\mathcal{X}'_m, \mathcal{D}'_m).$$

Finally, as example, we give a systematic construction of plt-type test configurations which we use in the next section.

**LEMMA 2.10.** *If  $((X, D), L)$  is weakly open K-polystable and some irreducible component  $D_i$  of  $[D]$  satisfies that*

- $L|_{X \setminus D_i} \sim_{\mathbb{Q}} \mathcal{O}$  and
- $D_i$  is ample (e.g. when  $\rho(X) = 1$ ),

*then there is a plt-type log test configuration of  $((X, D), L)$  such that the central fiber is the projective cone of  $((D_i, \cup_{j \neq i} D_j \cap D_i), N_{D_i/X})$ .*

PROOF. We first blow up  $D_i \times \{0\} \subset X \times \mathbb{A}^1$  to obtain  $b: \mathcal{B} \rightarrow (X \times \mathbb{A}^1)$  whose central fiber is  $b_*^{-1}X \cup \mathbb{P}(N_{D_i/X} \oplus \mathcal{O}_{D_i})$ . The intersection is still  $b_*^{-1}X \cap \mathbb{P}(N_{D_i/X} \oplus \mathcal{O}) \simeq D_i$ . As the component  $\mathbb{P}(N_{D_i/X} \oplus \mathcal{O})$  is exceptional divisor, we simply denote it as  $E$ . Then from the assumption it is easy to see that  $b^*(L \times \mathbb{A}^1)(-cE)$  gives a contraction to a polarized test configuration  $\mathcal{X}$  whose central fiber  $\mathcal{X}_0$  is the contraction of  $\mathbb{P}(N_{D_i/X} \oplus \mathcal{O})$  along the 0-section, i.e., the projective cone of  $((D_i, \cup_{j \neq i} D_j \cap D_i), N_{D_i/X})$  which is a process of the usual minimal model program which preserves dlt condition (cf., [KM98]). Since the fiber  $\mathcal{X}_0$  is irreducible, dlt condition implies that the obtained polarized family is a plt-type log test configuration.  $\square$

Now we are ready to define open K-polystability and its further strengthening, following Definition 2.3.

From here, a main idea could be expressed as to study the structure of the ends of the complete metrics to expect, “virtually” considering a compact (non-canonical) model which “close the ends” with small angles, which we analyze by algebro-geometric tools.

DEFINITION 2.11 (Open polystability).  $(X^\circ, L^\circ)$  is said to be *open K-polystable* if and only if there is a polarized dlt log Calabi-Yau pair compactification  $(X^\circ, L^\circ) \subset ((X, D), L)$  in the sense of Definition 2.1 which satisfies the followings;

- (i)  $[D](\equiv -K_{(X, \{D\})} := -(K_X + \{D\}))$  is nef,
- (ii) there is a constant  $\epsilon \in (0, 1)$  such that for any  $\beta \in [0, \epsilon)$ , the log Donaldson-Futaki invariant ([Don12, OS15]) is non-negative:

$$(2) \quad \text{DF}((\mathcal{X}, \{\mathcal{D}\} + (1 - \beta)[\mathcal{D}]), \mathcal{L}(\frac{c}{\beta}[\mathcal{D}])) \geq 0$$

for *plt-type* log test configuration

$$((\mathcal{X}, \{\mathcal{D}\} + (1 - \beta)[\mathcal{D}]), \mathcal{L}(\frac{c}{\beta}[\mathcal{D}]))$$

in the sense of Definition 2.3 of  $(X, \{D\} + (1 - \beta)[D])$  and the value attains 0 if and only if  $((\mathcal{X}, \{\mathcal{D}\} + (1 - \beta)[\mathcal{D}]), \mathcal{L}(\frac{c}{\beta}[\mathcal{D}]))$  is a product log test configuration.

We call such a compactification itself  $(X^\circ, L^\circ) \subset ((X, D), L)$  *stabilizing*, or simply open K-polystable.

In many concrete situations, we only consider the cases when  $D$  is integral, i.e.,  $D = \lfloor D \rfloor$  so that the description is simpler.

For justification of the introduction of the constant  $c$  and the variation  $\frac{c}{\beta} \lfloor \mathcal{D} \rfloor$  of polarization in the above definition, see the elliptic type spherical (real) surfaces i.e., when  $X = \mathbb{P}^1$  and  $D$  consists of two points, for instance. From the definition, the lemma below immediately follows.

LEMMA 2.12. *Suppose  $(X^\circ, L^\circ)$  is open K-polystable and take a stabilizing polarized dlt log Calabi-Yau pair compactification  $(X^\circ, L^\circ) \subset ((X, D), L)$ . Then, it implies:*

- (i)  $(X^\circ, L^\circ)$  is weakly open K-polystable
- (ii) if further  $L = \lfloor D \rfloor$ , there is a polarized dlt Calabi-Yau pair compactification  $((X, D), L)$  such that for any log product log test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  satisfies  $(\lfloor \mathcal{D} \rfloor)^{\cdot(n+1)} = 0$ .

We call a compactification  $((X, D), L)$  of  $(X, L)$  satisfying the latter condition *balancing*.

The last variant definition is:

DEFINITION 2.13 (Strongly open polystability). In the setting of above Definition 2.11,  $(X^\circ, L^\circ)$  is called *strongly open K-polystable* if the chosen compactification  $((X, D), L)$  further satisfies that

- (i)  $\lfloor D \rfloor (\equiv -K_{(X, \{D\})} := -K_X - \{D\})$  is nef and
- (ii) for some fixed positive constant  $c > 0$ ,

$$((X, \{D\} + (1 - \beta)\lfloor D \rfloor, L(\frac{c}{\beta}\lfloor D \rfloor))$$

is log K-polystable and any  $0 < \beta \ll 1$ .

We call such compactification  $((X, D), L)$  in the either way: *strongly stabilizing* compactification, *stable* compactification, or simply, being strongly open K-polystable.



Then it follows straightforward from the definitions that:

**PROPOSITION 2.14.** *Strongly open K-polystable  $(X^\circ, L^\circ)$  is open K-polystable.*

On the other hand, [LX14] immediately implies the following.

**PROPOSITION 2.15.** *In the above situation as 2.13, suppose that open K-polystable  $(X^\circ, L^\circ)$  has a stabilizing compactification  $((X, D), L)$  which is anti-log-canonically polarized (see [CR15] for related notion) in the sense that there is a small enough positive real number  $\beta \ll 1$  such that  $L$  is proportional to  $-(K_X + \{D\} + (1 - \beta)[D])$ . Then, it is strongly stabilizing compactification, so that  $(X^\circ, L^\circ)$  is strongly open K-polystable in particular.*

We add a useful remark, in the spirit of [LS14].

**LEMMA 2.16.** *In the above situation as 2.13, if  $(X, L(t[D]))$  is K-polystable for any  $t \gg 1$ , then  $(X^\circ, L^\circ)$  is strongly open K-polystable.*

**PROOF.** This follows straightforward from the linearity of the log Donaldson-Futaki invariant with respect to the linear change of the coefficient of the boundary divisor  $D$  (see [OS15] and [LS14]).  $\square$

**REMARK 2.17.** In the case when there is a non-negative linear combination of  $D_i$  which is ample, so that  $(X, D)$  is asymptotically log Fano in the sense of [CR15], effective K-stability criteria have already studied much so that the above stability should be able to be studied by using them.

Finally, we remark that, specifying an action of algebraic group  $G$  on  $(X^\circ, L^\circ)$  or  $(X, L)$ , we can and do naturally define the  $G$ -equivariant version, such as  $G$ -equivariantly (weak) open K-polystable etc, of above stability notions mean that we only concern log test configurations with  $G$ -action on whole total space with  $G$ -linearization. Then we can show:

**LEMMA 2.18.** *If  $G$  is a connected algebraic group acting on open Calabi-Yau polarized variety  $(X^\circ, L^\circ)$ , then for  $G$ -equivariant weakly open K-polystability of  $(X^\circ, L^\circ)$  and weakly open K-polystability of  $(X^\circ, L^\circ)$  are equivalent.*

PROOF. This follows from Proposition 2.7 (also see Corollary 2.6) which implies all the log test configurations satisfying the condition in *loc.cit* should hold  $G$ -action automatically.  $\square$

## 2.2. Reducible setup

Now we discuss stability notions for polarized Calabi-Yau varieties  $(X, L)$  where  $X$  is non-normal. More precisely, we use the notion of semi-divisorially log terminality (sdlt for short), as a natural non-normal or demi-normal version of original divisorial log terminality. See e.g., [KM98], [Fjn00] for details.

DEFINITION 2.19. Suppose a connected projective scheme  $X$  has only semi-dlt singularities and  $K_X \equiv 0$ , hence equivalently  $K_X \sim_{\mathbb{Q}} 0$  by [Fjn00, Gon11a].

We denote its irreducible decomposition as  $X = \cup_i V_i$  with the double locus (conductor divisor) as  $D_i \subset V_i$  so that  $(V_i, D_i)$  is log Calabi-Yau dlt pair for each  $i$ . We also consider a polarization i.e., an ample line bundle on  $X$ .

- (i) Consider all log test configurations  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  of  $((X, D), L)$  which satisfy that
- $\mathcal{X}$  satisfies Serre's  $S_2$  condition,
  - its restriction to the closure of  $V_i \times (\mathbb{P}^1 \setminus \{0\})$  is of plt type in the sense of Definition 2.3 (i),
  - the log Donaldson-Futaki invariant (cf., [Don12], [OS15]) vanishes.

The polarized log Calabi-Yau pair  $((X, D), L)$  is *weakly open  $K$ -polystable* if any such above type log test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  satisfies that *the klt (open) locus of  $(\mathcal{X}, \mathcal{D})$*  is a log test configuration of product type of the open locus  $(X^\circ, L^\circ)$ . By applying Lemma 2.6 to all its components, we see that the condition of such product-ness is also equivalent to that the restriction of the log test configuration to the closure of  $V_i \times (\mathbb{P}^1 \setminus \{0\})$  are log *product* test configurations for every  $i$ .

- (ii)  $((X, D), L)$  is *open K-polystable* if it is weakly open K-polystable and furthermore for each  $i$ ,  $((V_i, D_i), L|_{V_i})$  is stabilizing in the sense of Definition 2.11.
- (iii)  $((X, D), L)$  is *strongly open K-polystable* if it is open K-polystable and furthermore for each  $i$ ,  $((V_i, D_i), L|_{V_i})$  is strongly stabilizing in the sense of Definition 2.13.

Specifying an action of algebraic group  $G$  on  $((X, D), L)$ , we can naturally introduce the  $G$ -equivariant version of above stability notions, such as  $G$ -equivariantly open K-polystable etc. They mean that we only concern log test configurations with  $G$ -action on whole  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$ .

Similarly to Lemma 2.18, the following holds.

LEMMA 2.20. *If  $G$  is a connected algebraic group acting on sdlc log Calabi-Yau polarized variety  $((X, D), L)$ , then  $G$ -equivariant weakly open K-polystability of  $((X, D), L)$  holds if and only if weakly open K-polystability of  $((X, D), L)$  holds.*

PROOF. “If” direction is obvious by definition. The “only if” direction is reduced to Lemma 2.18 as follows: suppose we take a log test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  satisfying the three conditions in Definition 2.19(i). Then, if we take the normalization of  $\mathcal{X}$ , the log Donaldson-Futaki invariant  $DF((\mathcal{X}, \mathcal{D}), \mathcal{L})$  decomposes to the contributions of the log test configurations of each closures of  $V_i \times (\mathbb{P}^1 \setminus \{0\})$  e.g., by the intersection number formula ([OS15]). Hence, the assertion follows from Lemma 2.18 or more directly from Proposition 2.7 that each components are product test configurations admitting extended  $G$ -action i.e., corresponding to a  $\mathbb{C}^*$ -action which commutes with the given  $G$ -action. By the uniqueness of gluing along conductors ([Kol13], also cf., [Fjn00]), the action also extends to the whole log test configuration.  $\square$

QUESTION 2 (Component-wise nature?). For a polarized semi-dlt Calabi-Yau variety  $(X = \cup_i V_i, L)$  to be weakly open K-polystable, is it equivalent to the weakly open K-polystability for *all*  $(V_i^o, L|_{V_i^o})$ ? Here,  $V_i^o$  denotes the open subset of  $V_i$  as the complement of the double locus  $V_i \cap (\cup_{j \neq i} V_j)$ .

We end the section by observing that at least partially this is true.

**PROPOSITION 2.21.** *For a polarized semi-dlt Calabi-Yau variety  $(X = \cup_i V_i, L)$ , weakly open  $K$ -polystabilities for all  $(V_i^\circ, L|_{V_i^\circ})$  imply that of  $(X, L)$ .*

*Furthermore, in case  $V_i$  is smooth (factoriality is enough) and the dual complex is one dimensional (i.e., there is no three distinct  $V_i, V_j, V_k$  intersecting), the converse also holds.*

**PROOF.** The former statement is obvious from the definition. We prove the latter statement by contradiction. If there is an index  $i$  such that  $(V_i^\circ, L|_{V_i^\circ})$  is not weakly open  $K$ -polystable, there is a plt type test configuration  $(\mathcal{V}_i, \mathcal{L}_i)$  of  $(V_i, L|_{V_i})$  whose log Donaldson-Futaki invariant is zero and is dominated by composite of blow up of the connected (or equivalently, irreducible) components of  $V_i \cap (\cup_{j \neq i} V_j)$ . In particular, the double loci remain isomorphic as original.

Then we can glue trivial test configurations of  $(V_k, L_k)$  for all  $k \neq i$  and  $(\mathcal{V}_i, \mathcal{L}_i)$  by [Koll13] (as varieties) and [Kol13] (polarizations). This contradicts the weakly open  $K$ -polystability of  $(X, L)$ .  $\square$

### 3. Testing Known Examples

This section shows our analysis of the stability notions 2.3, 2.11, 2.13 for various class of examples below, which match to known (and unknown) gravitational instantons as well as some phenomena observed in examples of moduli. Here is the list of the results.

**THEOREM 3.1.** *Suppose  $((X, D), L)$  is an  $n$ -dimensional polarized dlt log Calabi-Yau pair, and take its klt open locus as  $X^\circ := X^{\text{klt}}, L^\circ = L|_{X^\circ}$ . As in section 2, we do not assume  $D$  to be integral divisor but can be real divisor.*

- (i) *If  $D$  is smooth and Cartier, then  $(X^\circ, L^\circ)$  is weakly open  $K$ -polystable for any  $L$ .*
- (ii) *More generally, suppose  $(X, D)$  is purely log terminal (plt) and further  $D$  has only canonical singularities and satisfies adjunction i.e.,  $K_D = 0$  (equivalently, the different in Shokurov's sense is trivial). Then it is weakly open  $K$ -polystable for any  $L$ .*

- (iii) *Under the situation (ii) above, if further one can take  $L$  as  $-K_X$  (hence, with  $\mathbb{Q}$ -Fano  $X$ ), then  $(X^\circ, L^\circ)$  is strongly open  $K$ -polystable.*
- (iv) *Under the situation (ii) above, if we can take the compactification  $X$  as a semi-Fano manifold in the sense of [CHNP13], then  $(X^\circ, L^\circ)$  is again strongly open  $K$ -polystable.*
- (v) *If  $X^\circ$  is algebraic torus, then  $(X^\circ, L^\circ)$  is strongly open  $K$ -polystable.*
- (vi) *More generally, if  $X^\circ$  is semi-abelian variety, then  $(X^\circ, L^\circ)$  is strongly open  $K$ -polystable.*
- (vii) *If  $G$  is a reductive algebraic subgroup of  $\text{Aut}^\circ(X)$  and  $((X, D), L)$  is weakly open  $K$ -polystable, then  $D$  is GIT polystable with respect to the  $G$ -action.*
- (viii) *If  $(X, D)$  is a cluster log surface ([FoGo09, GHK]), then  $((X, D), L)$  is not even weakly open  $K$ -polystable for any  $L$ .*
- (ix) *If  $X$  is a rational elliptic surface with  $D$  a nodal fiber of  $I_\nu$  ( $\nu \geq 1$ ) type, then  $((X, D), L)$  is strongly open  $K$ -polystable at least for some  $L$ .*
- (x) *If  $((X, D), L)$  is weakly open  $K$ -polystable and some irreducible component  $D_i$  of  $[D]$  satisfies that*

- $L|_{X \setminus D_i} \sim_{\mathbb{Q}} 0$  and
- $D_i$  is ample (e.g. when  $\rho(X) = 1$ ),

*then  $((X, D), L)|_{X \setminus D_i}$  is the affine cone of a certain  $(n - 1)$ -dimensional dlt log Calabi-Yau pair  $((X', D'), L') := ((D_i, \cup_{j \neq i} D_j \cap D_i), N_{D_i/X})$ .*

- (xi) *Suppose  $(X, [D]) = D_1 \sqcup D_2$  with Cartier  $[D]$  and connected  $D_i$ , which dominates birationally  $X'$  which is a  $\mathbb{P}^1$ -bundle over a  $(n - 1)$ -dimensional projective variety, such that the images  $D'_i$  of  $D_i$  are its two sections. Then the following holds.*
  - $(D'_i, \text{Diff}_{D'_i}(0))$  ( $i = 1, 2$ ), where  $\text{Diff}(-)$  denotes the Shokurov different (cf., e.g., [Kol13]), is a klt log Calabi-Yau pair, canonically isomorphic to each other. We denote it as  $(B, D_B)$ .

- If  $((X, D), L)$  is weakly open  $K$ -polystable for a polarization  $L$ , then  $X \rightarrow X'$  is isomorphism and there is a holomorphic line bundle  $N$  on  $B$  such that  $X' = \mathbb{P}_B(\mathcal{O} \oplus N)$  and  $D'_1$  and  $D'_2$  are the natural sections, the 0-section and the infinity section with respect to the splitting.
- Suppose  $B$  is an elliptic curve.<sup>8</sup> If  $(X^o, L^o)$  is further open  $K$ -polystable as  $((X, D), L)$  is stabilizing, if and only if  $N$  is a numerically trivial line bundle.

REMARK 3.2. For (ii), recall that if  $D$  is not necessarily Cartier, then in general klt condition of  $X^o$ ,  $D$  alone are not enough to imply plt condition of  $(X, D)$  because of the failure of adjunction (yielding nontrivial Different in the sense of Shokurov).

REMARK 3.3. For (ii) again, [FA91], [Fjn18] show that  $\lfloor D \rfloor$  has at most two connected component and if there are two connected component  $D_1$  and  $D_2$ , they are birational through a  $\mathbb{P}^1$ -fibration over some  $(n-1)$ -dimensional log terminal base  $B$ , up to a log crepant birational transform. We expect that the only open  $K$ -polystable such  $(X^o, L^o)$  have a structure  $\mathbb{P}_B(\mathcal{O} \oplus M)$  where  $B \in \text{Pic}^0(B)$ .

REMARK 3.4. *Cluster log surface* in (viii) simply means the following in our paper, as a simple variant of 2-dimensional cluster varieties introduced originally in [FoGo09] (see also [GHK, MV20]).

Let us start from another log smooth Calabi-Yau surface  $(X', D')$ . Then we can blow up a smooth point  $p$  of  $\lfloor D' \rfloor$ , which we denote as  $\varphi$  as a birational morphism here, and take the strict transform of  $D'$  to obtain a new log smooth Calabi-Yau surface  $(X = \text{Bl}_p(X'), D = \varphi_*^{-1} D')$ . In general, cluster log surface in our paper means a log smooth Calabi-Yau surface obtained by applying this procedure finite times (at least once) from a toric log Calabi-Yau pair.

Also, the statements and the proof of (viii) should be easily extended to its higher dimensional analogue log Calabi-Yau varieties of type [AG20] in which much more deep analysis for the mirror symmetry is done. We wish to leave the details to some readers.

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<sup>8</sup>We expect this dimension condition would be removed if we do more refined discussion.

PROOF OF THEOREM 3.1. We provide proofs to each item above one by one.

Proof of (ii): The Iitaka dimension of  $D$  is 0 because of the canonicity of the singularity, and therefore in particular it is not uniruled. Consider plt-type testconfiguration with vanishing Donaldson-Futaki invariant  $\mathcal{X}$  then the strict transform of  $X \times \{0\}$  is log canonical place of  $(\mathcal{X}_0, \mathcal{D}_{\mathcal{X}_0})$  as Proposition 2.7 shows. On the other hand, the adjunction says  $\mathcal{X}_0$  is klt. Therefore, [HM07] implies  $D$  with  $\kappa(D) = 0$  cannot be contracted to dimension less than  $n - 1$  in  $\mathcal{X}_0$ . From the fiber connectedness of birational morphisms between normal varieties (Zariski's main theorem), the center of  $\mathcal{X}_0$  is birational to  $D$ . Consider the 1-dimensional general fiber  $F$  of the birational map  $\mathcal{X}_0 \dashrightarrow D$ , the genus  $g(F)$  must be 0 because of the Iitaka conjecture (this case of relative dimension 1 is certainly a theorem cf., e.g., [Kwmt85]).

Proof of Fano manifolds case (iii): This follows from (ii) and [OS15], [Od13c] (cf., also preceding [Ber13]), which shows log K-stability for small angle with effective bound in terms of the alpha invariant.

Proof of semi-Fano manifolds case (iv): if we apply the Kawamata basepointfree theorem to  $X$ , as in [CHNP13], then it reduces to (iii) case.

Proof of (v) and (vi): we first prove the weakly open K-polystability. As preparation, we show the following lemma which should be fundamental and may be known to some experts but the author could not find the literatures. We write for convenience of readers:

LEMMA 3.5 (Torus invariance lemma). *Consider an arbitrary toric log Calabi-Yau pair  $(V, \Delta_V)$  i.e., toric variety  $V$  with the sum  $\Delta_V$  of all torus invariant prime divisors. Its divisorial valuation  $v$  whose center exists inside  $V$  is log canonical valuation i.e.,  $A_{(V, \Delta_V)}(v) = 0$  if and only if it is torus invariant.*

PROOF OF LEMMA 3.5. Suppose  $v$  is realized as a prime divisor  $F$  inside a blow up of  $V$  as  $b: W \rightarrow V$ . If  $v$  is toric valuation i.e., torus invariant, then we can take  $W$  and  $b$  inside the category of toric varieties. If we write the sum of torus invariant prime divisor of  $W$  as  $\Delta_W$ , then obviously  $(W, \Delta_W) \rightarrow (V, \Delta_V)$  is log crepant so that  $F$ , which is a component of  $\Delta_W$ , is obviously a log canonical valuation.

Conversely, suppose  $F$  gives a log canonical valuation  $v = v_F$ . Without loss of generality, by toric log resolution again, we can and do assume  $(V, \Delta_V)$  is dlt or even log smooth. From a lemma of Zariski (cf., [KM98]), then  $W'$  (which could be a priori non-toric) can be obtained as a composition of blow up along the log canonical center of  $v$ . From [Amb03], [Fjn18], it follows that the log canonical center (of  $v$ ) in  $V$  is a torus invariant strata. Therefore, the above obtained  $W'$  and the morphism  $b': W' \rightarrow V$  is again toric and  $F \subset W'$  realizing  $v$  is toric. Hence we complete the proof of Lemma3.5.  $\square$

To show the weakly open K-polystability of an algebraic torus  $X^o = T$  (there is no ambiguity of  $L^o$  since  $\text{Pic}(X^o) = 0$ ), we use Proposition 2.7 again. We take an arbitrary toric compactification  $((X, D), L) \supset (X^o, L^o)$  and consider a plt type log test configuration  $\mathcal{X}$  such that  $\mathcal{X}_0$  gives a log canonical valuation of  $(X, D) \times \mathbb{P}^1$ . Proposition2.7 says it is enough to see that  $(\mathcal{X}, \overline{D} \times (\mathbb{P}^1 \setminus \{0\}))$  is a product test configuration. Lemma 3.5 shows  $v_{\mathcal{X}_0}$  is toric, while toric valuation is parametrized by  $\text{Hom}(\mathbb{G}_m, T \times \mathbb{G}_m) \otimes \mathbb{Q}$  i.e., there is a product test configuration  $\mathcal{X}'$  such that  $\mathcal{X}'_0$  also gives  $v_{\mathcal{X}_0}$ . From [MM64] again, we conclude that  $\mathcal{X}' \simeq \mathcal{X}$  hence the proof of weakly open K-polystability of (polarized) algebraic torus is completed. The weakly open K-polystability of polarized semiabelian variety is basically the same since it is étale locally product of an algebraic torus and smooth base.

Strong open K-polystability of  $X^o$  follows from that semi-abelian variety  $X^o$  has a unique<sup>9</sup> short exact sequence structure  $1 \rightarrow T \rightarrow X^o \rightarrow A \rightarrow 1$  where  $T$  is an algebraic torus,  $A$  is an abelian variety and if we see this as principal  $(\mathbb{C}^*)^r (\simeq T(\mathbb{C}))$ -bundle, it has flat unitary connection or equivalently it is unitary local system. Therefore, we can compactify naturally to  $X \rightarrow A$  as  $\mathbb{P}^n$ -fiber bundle with transition function locally constant. That projective bundle corresponds to polystable bundle, and hence from Lemma 2.16, we end the desired proof of (vi).

Proof of (vii): We fix  $X$  and consider the natural universal family of  $((X, D), L)$  where only  $D$  deforms as all  $D \in |-K_X|$ , which we denote as  $\mathcal{U}_X \rightarrow B_X = \mathbb{P}(H^0(-K_X))$ . Then consider log CM line bundle  $\lambda_{CM}$  on  $B_X$  (cf., e.g., [PT06], [ADL19]) which admits the natural  $G$ -linearization. Since we assume  $(X, D)$  is dlt, hence a log-canonical pair, Theorem1.3 (from

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<sup>9</sup>thanks to the algebraicity of  $A$



[OS15]) applies to conclude that it is log K-semistable. Note that the log CM line bundle is of the form  $\mathcal{O}_{\mathbb{P}(H^0(-K_X))}(c)$  but as in the argument [OSS16], [ADL19], we have  $c > 0$  and log Donaldson-Futaki invariant of  $((X, D), L)$  is proportional to the GIT weight (of  $D$ ) and the proportionality constant is positive. Therefore, from the arguments [PT06], [OSS16], [ADL19], we see that  $D$  is GIT semistable with respect to the  $G$ -action. A special case where  $X = \mathbb{P}^{n+1}$  is proved in [Lee08, Okw11].

Now, we consider log test configuration of  $((X, D), L)$  where the ambient space is a  $X$ -fiber bundle, in particular, of plt type. Since, we now know  $D$  is at least semistable with respect to  $G$ -action, weakly open K-semistability implies the  $G$ -orbit is closed in  $G$ -invariant affine open subset of  $B_X$ . Therefore, we see that weakly open K-polystability of  $((X, D), L)$  implies the GIT polystability of  $D$ .

Proof of (viii): this is easy to see since such a polarized pair  $((X, D), L)$  degenerates to a toric Calabi-Yau pair by degenerating the blow up centers to nodes in the toric boundary. Since  $X$  has vanishing irregularity, the polarization naturally preserves as well. This forms a product test configuration which is obviously plt-type. On the other hand, since toric polarized Calabi-Yau pair is semi-log-canonical by the presence of toric log resolution (cf., e.g. [Ale96]), [OS15] or Theorem 1.3 implies the Donaldson-Futaki invariant vanishes. We conclude the proof of (viii).

Proof of (ix): First we prove weak K-polystability. Consider a plt type test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  of vanishing Donaldson-Futaki invariant. From 2.7,  $\mathcal{X}_0$  is a log canonical place of  $(X \times \mathbb{A}^1, D \times \mathbb{A}^1 + X \times \{0\})$  i.e.,  $A_{(X \times \mathbb{A}^1, D \times \mathbb{A}^1 + X \times \{0\})}(\mathcal{X}_0) = 0$ . From a lemma due to Zariski (cf., [KM98]), it follows that finite time blow up of  $X \times \mathbb{A}^1$  at the log-canonical center of  $\mathcal{X}_0$  realizes  $\mathcal{X}_0$ . We concretely trace such possibility: the irreducible component  $D_i$  of the nodal degenerate fiber  $D$  has negative self-intersection as it is well-known (Hodge index theorem), hence we see that the strict transform of the irreducible component  $D_i$  in the strict transform of  $X(\times\{0\})$  is still negative. We write  $X'$  the other component than  $X(\times\{0\})$  which includes  $D_i$ . From the d-semistability result (a.k.a. the triple point formula) we easily see that  $D_i \subset X'$  has positive self intersection, hence not contractible. This would contradict unless the log canonical center of  $\mathcal{X}_0$  is  $X \times \{0\}$ , hence weak K-polystability of  $(X^o, L^o)$  follows.

Now we prove the strongly open K-polystability. We regard  $X$  as a

blow up (with the morphism  $\varphi: X \rightarrow X'$  and the exceptional  $(-1)$ -curve  $E$ ) of DelPezzo surface  $X'$  of degree 1 at the base point of elliptic anti-canonical linear system which we denote as  $\pi$ . Then, applying [AP06], it follows that  $X$  admits a polarization  $L_\epsilon := \varphi^*(-K'_X) - \epsilon E$  for  $\epsilon \ll 1$  such that  $(X, L)$  has a corresponding cscK metric hence K-polystable by [Stp09]. Therefore, from Lemma 2.16, we obtain the strongly open K-polystability of  $(X^\circ, L^\circ)$ .

Proof of (x) follows from Lemma 2.10. We would like to call this pair as being type E as in [Od20b], to which we consult the details.

Proof of (xi): The first item should be essentially known to birational geometry experts. From a classification result (cf., e.g., [Fjn00], [Gon11a]), it follows that  $X'$  is indeed a  $\mathbb{P}^1$ -bundle over a klt log Calabi-Yau pair  $(D_i, \text{Diff}_{D_i}(0))$ . Since  $(X, D)$  is a dlt log Calabi-Yau pair, as we assume always during Theorem 3.1,  $(X, D)$  is plt around  $D_i$ . From Cartier assumption of  $D_i$ , if we consider the deformation to the normal cone with respect to  $D_i \subset X$ , we can contract the strict transform of the original central fiber  $X \times \{0\}$  to get a log test configuration of  $(X, D)$  degenerating to  $\mathbb{P}_{D_i}(\mathcal{O} \oplus N_{D_i/X})$ , whose log Donaldson-Futaki invariant vanishes. Therefore, the second statement on the weakly open K-polystable case holds if we put  $N = N_{D'_i/X'}$ .

Finally, for the third statement, we suppose  $((X, D), L)$  is a stabilizing compactification. Then from [RT07] (also [AT08, AK19]), it follows that  $(X', L')$  has vanishing Futaki character if and only if  $N$  is numerically trivial.  $\square$

For the last claim on (xi), we have not been able to succeed to remove dimension assumption on  $B$  at this point, but we nevertheless expect following more general claim would be algebraically proved eventually; connecting the phenomenon observed in birational geometry ([FA91, Fjn00, Gon11a] etc) and the Cheeger-Gromoll splitting [CG71].

**CONJECTURE 3.6** (Algebraic Cheeger-Gromoll). *If  $((X, D), L)$  is open K-polystable and there are two connected components of  $\text{Supp}([D])$  (i.e., “have two ends”), then there is a klt log Calabi-Yau variety  $(B, D_B)$  and a numerically trivial line bundle  $N$  such that  $X \simeq \mathbb{P}(\mathcal{O} \oplus N)$ ,  $D$  is the union of two natural sections, and a complete Ricci-flat weak Kähler metric on  $X^\circ$*

is complex analytically locally a product of lower dimension Ricci-flat weak Kähler metric and a flat metric.

Conversely, if  $(B, D_B = 0)$  is smooth, then regarding  $X^\circ$  as a unitary local system of rank 1 modulo the natural finite base change, we see the existence of complete Ricci-flat Kähler metrics on  $X^\circ$  in the above form.

**COROLLARY 3.7** (Classification for smooth surface case). *Open  $K$ -polystable (resp., weakly open  $K$ -polystable) smooth surfaces  $X^\circ$  compactifiable in log smooth  $(X, D)$  are classified as follows:*

- (i)  $X^\circ$  is the 2-dimensional algebraic torus (i.e.,  $(X, D)$  is a toric pair), or
- (ii) the complement of an elliptic curve in  $X$  or
- (iii) rank 1 local system over elliptic curves (resp., holomorphic principle  $\mathbb{C}^*$ -bundle over elliptic curves).

**PROOF.** Recall that  $D$  is connected or otherwise consists of two isomorphic disjoint sections of a ruled surface structure (Remark 3.3). The latter case is reduced to above (ii) which is easy to show.

After log resolution, we can assume  $(X, D)$  is either of type (i), i.e.,  $D$  is smooth, or  $D$  is nodal i.e., so-called Looijenga pair first studied systematically in [Looi81]. Now, the assertion follows from Theorem 3.1 (v), (viii) since [GHK] shows that after finite times composition of blow ups of nodes of the boundary curve  $D$ , the surface admits cluster surface structure.  $\square$

From now on, we examine some known gravitational instantons and show their presence (and some non-existence) match to above Theorem 3.1.

*Example 3.8.* For (ix), this matches to the H-J.Hein’s gravitational instanton [Hein12] as well as recent result of [CJL21] which relates it with Tian-Yau metric [TY90] via hyperKähler rotation. Also recall G.Chen ([GCh17] or [CC15]) shows that ALG space  $X^\circ$  with curvature decay *faster than quadratic order* can be compactified to a rational elliptic surface  $X$  and its complement  $D$  is its singular nodal fiber. In particular, we expect our (ix) holds for more general polarization  $L$ .

*Example 3.9.* An easy typical example of (vii) is when  $X$  is the projective plane  $\mathbb{P}^2$ . Then weakly open  $K$ -polystability of  $(X, D)$  implies that  $D$  is either smooth or union of three mutually transversal lines which fits to i and v, but can not be the union of conic and line.

*Example 3.10.* [TY90] matches to above Theorem 3.1 (iii) well. Indeed, recall that smooth  $X$  with smooth  $D$  form the most typical plt pair. In the construction of Tian-Yau [TY90] of complete Ricci-flat Kähler metric on  $X^\circ$ , they assume  $D$  is (almost) ample i.e.,  $X$  is Fano, as the construction starts with positive curvature hermitian metric on  $\mathcal{O}_X(D)$  as a source of reference metric.

*Example 3.11.* [HHN15] treats the case when  $X$  admits certain kind of cyclic quotient singularity while  $D(\supset \text{Sing}(X))$  is also a cyclic finite quotient of a Calabi-Yau manifold. Since the finite cyclic group action, “ $\iota$ ” in *loc.cit.*, do not have fixing divisor,  $(X, D)$  is plt and as they show the adjunction holds in this case. Therefore, the existence result [HHN15] of complete Ricci-flat Kähler metric of asymptotically cylindrical type (“ACyl”) perfectly fits as example of above condition in Theorem 3.1 (ii).

As 3-dimensional special case, there is a work of [CHNP13] when  $X$  is certain kind of smooth weak Fano manifold, which are then applied to construction of compact  $G_2$  manifolds ([CHNP12]).

We end this subsection by algebraically showing the following analogue of Matsushima reductivity theorem ([Mat57]) by reducing to another theorem of Y. Matsushima ([Mat60]) on homogeneous spaces! However, note that the result is partial for now due to reductivity assumption of the ambient symmetry  $\text{Aut}(X, L)$ .

**COROLLARY 3.12** (Matsushima-type theorem). *Consider a weakly open  $K$ -polystable polarized log Calabi-Yau pair  $((X, D), L)$ . If we assume reductivity of  $\text{Aut}(X, L)$ , then  $\text{Aut}((X, D), L)$  is also reductive.*

**PROOF.** From the previous Theorem 3.1 (vii), we see that  $D \in | -K_X |$  is GIT polystable with respect to the natural action of  $\text{Aut}(X, L)$ . Therefore, its  $\text{Aut}(X, L)$ -orbit is closed in an affine subset hence so is affine. Thus, [Mat60] implies the isotropy of  $D$  is reductive which is nothing but  $\text{Aut}((X, D), L)$ .  $\square$

## 4. Compactness and Stable Reduction

### 4.1. Introductory remarks

An important piece of general construction of compact or proper moduli space is to establish the formal procedure of constructing canonical limits of a sequence or a punctured family of objects in concern. In algebraic geometry, it is formulated as the so-called valuative criterion of properness. The purpose of this section is to make a progress in the case of moduli of polarized Calabi-Yau varieties  $(X, L)$ s, by following similar idea to the classical Jordan-Holder filtrations. With differential geometric perspective, another purpose is to also understand the bubbling phenomena of maximally degenerating Ricci-flat Kähler metrics.

First we observe that flops can cause serious non-separatedness of moduli, even in polarized setting, if we do not put any stability assumption. This phenomenon also partially motivates our study.

*Example 4.1* (Flops cause **un**-separatedness, even with polarizations, without polystability conditions). It has been well-known that families of Calabi-Yau varieties can flop, which causes unseparatedness (non-Hausdorff properties) of moduli.

Here, we clarify and enhance the meaning by seeing such examples even under the presence of polarizations for convenience of readers, as the author could not find literature.

Consider the well-studied family in 2-dimensional case  $\mathcal{X} \rightarrow \mathbb{A}_t^1$  as  $\mathcal{X}_t = [xyzw + tF_4(x, y, z, w)] \subset \mathbb{P}_{x,y,z,w}^3$ , for general  $F_4 \in H^0(\mathbb{P}^3, \mathcal{O}(4))$ . There are four  $A_1$ -singularities (conifold points)  $p_1, \dots, p_4$  on  $x = y = t = 0$  inside  $\mathcal{X}_0$  which is intersection of the projective plane  $V_1 = (t = x = 0)$  and another projective plane  $V_2 = (t = y = 0)$ . To be precise,  $p_i$ s are zeroes of  $F_4(0, 0, z, w)$ . (More generally, the total space of generically smooth hypersurfaces degeneration  $\mathcal{X} = \{tF + G = 0\}$  is  $V(F) \cap \text{Sing}(\mathcal{X}_0)$ ). If we blow up  $V_1$  (resp.,  $V_2$ ) in  $\mathcal{X}$ , then we obtain a different polarized model  $\mathcal{X}_2$  (resp.,  $\mathcal{X}_1$ ). On the other hand, if we blow up  $p_1$ , then we get yet another model  $\tilde{\mathcal{X}}$  which dominates both  $\mathcal{X}_i$ s. The exceptional divisor of  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is  $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and since  $\tilde{\mathcal{X}}$  is smooth and no critical point generically inside  $E$ , a local section  $s(t): \Delta \rightarrow \tilde{\mathcal{X}}$  converging to a general point in  $E$ , the image section inside  $\mathcal{X}_i$  converges to  $V_i \subset \mathcal{X}_0$  as  $t \rightarrow 0$ , whose open locus is not isomorphic. In particular, if we simply consider the moduli functor of

flat families of pointed open polarized Calabi-Yau varieties (“ $(X^o, L^o)$ ”s), it is not separated.

Another motivation for the following compactness type results come from the desire to understand more local behavior of some special maximally degenerating Calabi-Yau metrics.

More precisely, we are concerned with the pointed Gromov-Hausdorff limits of Ricci-flat Kähler manifolds with minimal non-collapsing rescale in the following sense, somewhat analogous to the notions of “regularity scales” although not quite identical.

**DEFINITION 4.2** (Minimal non-collapsing rescale). A polarized flat ( $\mathbb{Q}$ -Gorenstein) punctured holomorphic family  $\pi^*: (\mathcal{X}^*, \mathcal{L}^*) \rightarrow \Delta^* := \Delta \setminus \{0\}$  of  $n$ -dimensional polarized klt projective varieties with continuous family of Kähler metrics  $g_t$  on  $\mathcal{X}_t$  whose Kähler class is  $c_1(\mathcal{L}_t)$ , take a sequence  $p_i \in \mathcal{X}_{t_i}$  ( $i = 1, 2, \dots$ ) such that  $t_i (\neq 0) \rightarrow 0$  for  $i \rightarrow \infty$ . Here,  $\mathcal{X}_t := (\pi^*)^{-1}(t)$  and  $\mathcal{L}_t := \mathcal{L}^*|_{\mathcal{X}_t}$  as usual. A sequence of real numbers  $r_i$  is called *minimal non-collapsing order* if  $(p_i \in \mathcal{X}_{t_i}, r_i g_{t_i})$  has non-collapsing<sup>10</sup> pointed Gromov-Hausdorff limit as Ricci-flat weak (klt) Kähler spaces but for any  $\epsilon_i > 0$  ( $i = 1, 2, \dots$ ) with  $\lim_i \epsilon_i = 0$ ,  $(p_i \in \mathcal{X}_{t_i}, \epsilon_i r_i g_{t_i})$  does not have such limit.

We are concerned with the natural equivalence class of the sequence  $\{r_i\}_i$  by  $\sim$  where

$$\{r_i\}_i \sim \{r'_i\}_i,$$

simply means  $\{\log \frac{r'_i}{r_i}\}_i$  is bounded from both sides.

Particularly, we are interested in the case when  $p_i$  moves along a meromorphic section over  $\Delta^*$ ,  $K_{\mathcal{X}^*/\Delta^*} \sim_{\mathbb{Q}} 0$ , and  $g_t$  are Ricci-flat Kähler. For instance, for degeneration of polarized elliptic curves with standard flat metrics, the minimal non-collapsing limit is a cylinder  $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$  with a standard flat metric. The rescaling parameter  $r_t$  simply makes the injectivity radius of  $r_t g_t$  bounded. Now, we put an expectation which is inspired by [DonSun17], which in particular gives partial confirmation for smooth case, other than the metric completeness.

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<sup>10</sup>in the sense of e.g., [DonSun17]

CONJECTURE 4.3 (Existence of minimally non-collapsing limits). *In the above situation, there is a unique equivalence class of minimal non-collapsing order. Furthermore, for sequences of points  $p_i \in \mathcal{X}_i$  with  $t_i (\neq 0) \rightarrow 0$  which are general enough in a certain sense, the pointed Gromov-Hausdorff limits are complete Ricci-flat (weak) Kähler space.*

Here, the “generality” condition should be necessary since for some sequences, we expect e.g., (multi-)Taub-NUT metrics as the occurring limits as [HSVZ22] for instance.

We further put an imprecise expectation as it actually served as one of the motivations of our open K-polystability notion.

QUESTION 3 (Special maximal degeneration case). We expect that for certain special maximally degenerating families of polarized Calabi-Yau varieties, which are dlt minimal models over a curve, the minimal non-collapsing pointed Gromov-Hausdorff limit in Conjecture 4.3 are nothing but the open strata of open polystable dlt model reductions. Characterize when this is true.

However we observe the following.

*Caution-Conjecture 4.4.* Note that minimal non-collapsing limits do not respect product so that the above expectation 3 can not be extended to general degenerations.

To provide a simple counterexample, first we take  $(\mathcal{X}^*, \mathcal{L}^*) \rightarrow \Delta^*$  which satisfies affirmatively the above Question 3. Then, if we take a fixed pointed polarized  $m$ -dimensional Calabi-Yau variety  $s' \in (S, M)$  and consider  $(s(t), s') \in (\mathcal{X} \times S, p_1^* \mathcal{L} \otimes p_2^* M) \rightarrow \Delta^*$ , then we expect the minimal non-collapsing limit of the fiber Ricci-flat Kähler metrics on the fiber  $\mathcal{X}_t \times S$  to be the isometric to the metric product of the following two metric spaces. One is the minimal non-collapsing pointed Gromov-Hausdorff limit of  $\mathcal{X}_t \ni s(t)$  with respect to the original Ricci-flat Kähler metrics, and the other is the (metric) tangent cone of  $S$  at  $s'$  rather than  $S$  itself.

Indeed, this is easily verified in the case of degenerating polarized abelian varieties as [Od19].

Furthermore, by the concrete analysis in [Od19], even some maximal degenerations of polarized abelian varieties can have negative answers to

Question 3. However, certainly good “balancing” class of maximal degenerations do exist, of which the answer to Question 3 are affirmative. We leave the details to future papers.

## 4.2. Weakly open stable reduction

We used the notion of the dlt minimal models for flat family of Calabi-Yau varieties in the formulation of above Question 3. It is obtained the relative minimal model program over the base curve (see [Fjn11b]) gives a powerful method of constructing degenerate Calabi-Yau varieties still with trivial canonical divisor, in a certain sense. A good news is that, as the name suggests, the singularities in such degeneration are controlled to be mild (semi-dlt), but the filling is rather far from unique. As we mentioned, [OS15] (Theorem 1.3) ensures their log K-semistability but never satisfies log K-polystability, hence it does not help to obtain uniqueness. This section aims to try to fix the problem by restricting the class of such occurring minimal degenerations further by imposing our our new *poly*-stability notions in §2, to obtain (unique and canonical) stable reduction type results.

We work in the setting of flat proper family over  $\Delta \ni 0$ , a germ of smooth algebraic curve. Nevertheless, we expect the same holds for analytic or formal germ if we replace the use of semistable MMP [Fjn11b] by its technical extension to formal equicharacteristic setting (cf., e.g., [HP16, NKX18]). We often denote  $\Delta \setminus \{0\}$  as  $\Delta^*$ . The superscript  $*$  denotes for punctured setting i.e., away from central fiber as boundary, while the superscript  $^o$  means outside the horizontal boundary  $\text{Supp}(\lfloor \mathcal{D} \rfloor)$ . In this section, to avoid technical difficulties of dealing with degenerations of boundary divisors (cf., e.g., [Kol]), we suppose  $\mathcal{D}$  is reduced i.e., all coefficients being 1.

**THEOREM 4.5** (Weakly open polystable reduction, Type I case). *Consider a flat  $\mathbb{Q}$ -Gorenstein family of open polarized Calabi-Yau varieties (recall Definition 2.1 (ii)) which we denote by  $(\mathcal{X}^{*,o}, \mathcal{L}^{*,o}) \rightarrow \Delta^* = \Delta \setminus \{0\}$  i.e., both horizontaliry and vertically compactifiable to  $((\mathcal{X}, \mathcal{D}), \mathcal{L}) \rightarrow \Delta$  which is a family of log dlt Calabi-Yau varieties such that*

- *All components of  $\mathcal{D}$  have coefficients 1 and are horizontal i.e., maps to  $\Delta$  dominantly,*
- $\mathcal{X}^* \setminus \text{Supp}(\lfloor \mathcal{D}^* \rfloor) = \mathcal{X}^{*,o},$



- $\mathcal{L}^*|_{\mathcal{X}^{*,o}} = \mathcal{L}^{*,o}$ ,
- and  $(\mathcal{X}, \mathcal{X}_0)$  is of plt-type i.e., the open central fiber pair  $(\mathcal{X}_0^o, \mathcal{D} - \lfloor \mathcal{D} \rfloor|_{\mathcal{X}_0^o})$  is klt (we call “Type I” condition<sup>11</sup>).

Then, possibly after a finite base change of  $\Delta \ni 0$ , we can perform a birational transform along the fiber over 0, to obtain a priori new  $((\mathcal{X}, \mathcal{D}), \mathcal{L}) \rightarrow \Delta$  such that again

- All components of  $\mathcal{D}$  have coefficients 1 and are horizontal i.e., maps to  $\Delta$  dominantly,
- $\mathcal{X}^* \setminus \text{Supp}(\lfloor \mathcal{D}^* \rfloor) = \mathcal{X}^{*,o}$ ,
- $\mathcal{L}^*|_{\mathcal{X}^{*,o}} = \mathcal{L}^{*,o}$ ,

and further that, most importantly,

- The central fiber  $((\mathcal{X}_0, \mathcal{D}|_{\mathcal{X}_0}), \mathcal{L}|_{\mathcal{X}_0})$  does not admit a plt type log test configuration which is not a product log test configuration. In particular, the open central fiber  $((\mathcal{X}_0^o, \mathcal{D} - \lfloor \mathcal{D} \rfloor|_{\mathcal{X}_0^o}), \mathcal{L}|_{\mathcal{X}_0^o})$  is weakly open K-polystable klt open Calabi-Yau polarized variety in the sense of Definition 2.19 (i).

A special consequence for when the general fiber is smooth, is as follows.

**COROLLARY 4.6** (A special case of Theorem 4.5). *If there is a flat family of polarized open Calabi-Yau manifolds  $(\mathcal{X}^o, \mathcal{L}^o) \rightarrow \Delta$ , we can replace the central fiber after finite base change to make it weakly open K-polystable.*

**PROOF OF THEOREM 4.5.** By [Fjn11b], there is at least such  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  which satisfies the conditions except for the last one i.e., non-existence of non-product plt log test configuration of the central fiber is not ensured.

*Step 1.* Suppose there is a there is a plt-type log test configuration  $((\mathcal{X}', \mathcal{D}'), \mathcal{L}')$  of the central fiber  $((\mathcal{X}_0, \mathcal{D}_0), \mathcal{L}|_{\mathcal{X}_0})$ . By the arguments of

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<sup>11</sup>after the Kulikov-Pinkham-Persson classification of log smooth minimal degenerations of K3 surfaces. This roughly means that the pair does not degenerate much.

Lemma 2.18, Lemma 2.20, we can even assume it is  $\text{Aut}^o(\mathcal{X}_0^o, \mathcal{L}|_{\mathcal{X}_0^o})$ -equivariant.

Suppose the base curve germ  $\Delta \ni 0$  is realized in a smooth algebraic curve  $C \ni p$ . Possibly after a finite base change, we want to glue “base changed enough” base curve  $C$  of  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  and the base  $\mathbb{P}^1$  of  $((\mathcal{X}', \mathcal{D}'), \mathcal{L}')$  to obtain a birational transform of  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  only along the fiber over 0 so that the new fiber over 0 becomes  $(\mathcal{X}'_0, \mathcal{L}'|_{\mathcal{X}'_0})$ . We provide its details as following Step 2, Step 3, and Step 4.

*Step 2* (Indeterminacy resolution). We take a certain component of the incidence locus of the two Hilbert schemes one of which parametrizes the fibers  $(X_t, L_t^{\otimes m})$  for uniform  $m \gg 0$ , as  $\text{Hilb}(\mathbb{P}^{N_X})$ , while the other  $\text{Hilb}(\mathbb{P}^{N_D})$  parametrizes at least  $(D_t, \mathcal{L}^{\otimes m}|_{D_t})$  for the same fixed  $m$ . We denote such incidence locus by  $H$ . Furthermore, we suppose the destabilizing log test configuration  $((\mathcal{X}', \mathcal{D}'), \mathcal{L}')$  is induced by a  $\mathbb{C}^*$ -action  $\lambda$  on  $\mathbb{P}^N$  hence on  $H$ .

This gives a flat projective family over  $\mathbb{C}^* \times C$  in a  $\mathbb{C}^*$ -equivariant manner, extending  $\mathcal{X}$  over  $\{1\} \times C$ , hence a morphism  $\mathbb{C}^* \times C \rightarrow \text{Hilb}(\mathbb{P}^N)$ . In particular, it induces a rational map

$$\mathbb{P}^1 \times C \dashrightarrow H.$$

We consider its indeterminacy locus which must be finite closed points inside  $\{0, \infty\} \times C$ . In particular, we can take  $\{p = p_1, \dots, p_l\} \subset C$  so that the indeterminacy locus is inside  $\{0, \infty\} \times \{p = p_1, \dots, p_l\}$ . Since we assume that the order of the base field  $\mathbf{k}$  is infinity, note that we can replace the subset  $\{p = p_1, \dots, p_l\}$  of  $C$  by a larger one with arbitrarily big order if we need, as we do in the next Step 3.

Now we consider a  $\mathbb{C}^*$ -equivariant resolution of indeterminacy of the rational map which we denote as  $\mathcal{B} \rightarrow H$ , which is a blow up of some close subscheme of  $\mathbb{P}^1 \times C$  which we denote as  $\Sigma$ . Then we obtain a diagram

$$\begin{array}{ccc} & \mathcal{B} = \text{Bl}_{\Sigma}(\mathbb{P}^1 \times C) & \\ & \swarrow \pi & \searrow \\ \mathbb{P}^1 \times C & & H, \end{array}$$

and our flat projective family (over  $(\mathbb{C}^* \times C)$ ) extends over  $\mathcal{B}$ . This is an easy example of so-called flattening procedure.

Now we define *depth* of the blow up  $\pi$  at a point in  $C \times \mathbb{P}^1$ . First, since  $\pi$  is a birational proper morphism from regular surface to regular surface, it can be decomposed as the composition of finite maximal ideal blow up as  $\pi = \pi^{(d)} \circ \pi^{(d-1)} \circ \dots \circ \pi^{(1)}$  where each  $\pi^{(i)}$  is a blow up with its corresponding maximal ideal defining a closed point  $c_i$ . We define  $\text{depth}(\pi, (x, y))$  as

$$\#\{i \mid c_i = (\pi^{(i-1)} \circ \dots \circ \pi^{(1)})^*(x \times C) \cap (\pi^{(i-1)} \circ \dots \circ \pi^{(1)})_*^{-1}(\tilde{C} \times y)\}.$$

Note that  $\text{depth}(\pi, (x, y)) = 0$  unless  $x$  is either 0 or  $\infty$  and  $y$  is one of  $p_i$ s. Then take a positive integer  $m$  such that

$$(3) \quad m > \max_{(x,y)} \{\text{depth}(\pi, (x, y))\},$$

*Step 3 (Cyclic covering).* If we replace  $\{p_1, \dots, p_l\}$  by a larger set if necessary, we can and do assume that there is an effective  $\mathbb{Z}$ -divisor  $D$  on  $C$  such that  $(\sum_{1 \leq i \leq l} p_i) \sim mD$ .<sup>12</sup>

We set  $L := \mathcal{O}_C(D)$  on  $C$ , and exploits the standard cyclic cover construction:

$$\tilde{C} := \text{Spec}_{\mathcal{O}_C}(\oplus_{0 \leq a < m} \mathcal{O}_C(aD)) \rightarrow C,$$

where the ring structure on the right hand side is induced by  $\mathcal{O}_C(mD) \simeq \mathcal{O}_C(\sum_i p_i) \hookrightarrow \mathcal{O}_C$ . Since  $m$  is coprime to the characteristic of  $\mathbf{k}$ , this gives an integral normal curve  $\tilde{C}$  which is cyclic cover of  $C$ . We denote the covering by  $f$  and write its graph in  $\tilde{C} \times C$  as  $\Gamma_f$ . Recall that  $\Gamma_f$  is isomorphic to  $\tilde{C}$  through the projection.

Note that  $f^*p_i = mp'_i$  with  $p'_i \in \tilde{C}$  for each  $i$ .

*Step 4 (Rational function for base change).* We take a pair of disjoint finite closed sets  $\{q_j\}_j \subset C$  and  $\{q'_j\}_j \subset C$  such that:

- (i)  $\{q_j\} = \{q'_j\} \geq 2g(\tilde{C}) + 1$ ,
- (ii)  $\{p'_i\} \cap \{q_j\} = \{p'_1\}$ ,
- (iii)  $(\{p'_i\} \cup \{q_j\}) \cap \{q'_j\} = \emptyset$ ,
- (iv)  $\sum_j q_j \sim \sum_j q'_j$ .

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<sup>12</sup>Here we used that the base field is characteristic 0 hence infinity order in particular.

From the last condition, we have a rational function  $r \in K(\tilde{C})^*$  whose zeroes are  $\{q_j\}$  with multiplicities 1 and poles  $\{q'_j\}$  with multiplicities 1. In particular,  $r$  is étale over  $\{0, \infty\} \subset \mathbb{P}^1$ . Therefore,  $\tilde{\mathcal{B}} := \mathcal{B} \times_{(\mathbb{P}^1 \times C)} (\tilde{C} \times C)$  is smooth, on which we also have a flat polarized family extending that of original  $(\mathcal{X}, \mathcal{L})$  which is isotrivial for  $\tilde{C}$ -direction. We denote the obtained morphism  $\tilde{\mathcal{B}} \rightarrow (\tilde{C} \times C)$  by  $\tilde{\pi}$ , which can be also seen as an indeterminacy resolution for the corresponding rational map to  $\text{Hilb}(\mathbb{P}^N)$ . Summarizing, we have the following diagram:

$$\begin{array}{ccc}
 & & \Gamma_f \\
 & & \downarrow \\
 \tilde{\mathcal{B}} & \xrightarrow{\tilde{\pi}} & (\tilde{C} \times C) \\
 \downarrow \widetilde{(r \times id)} & & \downarrow (r \times id) \\
 \mathcal{B} = \text{Bl}_\Sigma(\mathbb{P}^1 \times C) & \xrightarrow{\pi} & (\mathbb{P}^1 \times C).
 \end{array}$$

Then we consider  $\tilde{\pi}_*^{-1} \Gamma_f \subset \tilde{\mathcal{B}}$  which is isomorphic to  $\tilde{C}$ . The family over  $\Gamma_f \simeq \tilde{C}$ , pulled back from the incidence locus of  $\text{Hilb}(\mathbb{P}^{N_X}) \times \text{Hilb}(\mathbb{P}^{N_D})$  is our desired glued family of polarized log Calabi-Yau varieties.

*Step 5.* We denote the normalization of the new family over  $\tilde{C}$  as  $\mathcal{X}(1)$  and replace the notion  $\tilde{C}$  by  $C$  for simplicity. Note the normalization map of  $\mathcal{X}$  is bijective, which follows from fiberwise  $S_2$ -condition. Therefore, the pullback of the boundary divisor makes sense which we denote as  $\mathcal{D}(1)$ . Furthermore, we put the pullback of the polarization denoted as  $\mathcal{L}(1)$ . A priori,  $(\mathcal{X}(1), \mathcal{D}(1))$  may be only log canonical, not necessarily dlt, by the inversion of adjunction ([Kwkt07]). Therefore, we consider its dlt modification (cf., e.g., [Kol13, FG14, OX12]) to modify to a dlt minimal model over  $\Delta$ .

*Step 6.* Notice that the rank of  $\text{Aut}^o(\mathcal{X}_0^o, \mathcal{L}|_{\mathcal{X}_0^o})$  increases by the above procedure, hence the replacement either stops after finitely many times, to obtain a sequence  $((\mathcal{X}(i), \mathcal{D}(i)), \mathcal{L}(i))(i = 1, 2, \dots)$  or  $\mathcal{X}_0^o$  becomes an algebraic torus so that the assertion holds by Theorem 3.1 (v) after all.  $\square$

To state a variant for general reducible degenerations, we introduce the following a priori variant stability notion, which might remind the readers

of the notion of pointed Gromov-Hausdorff limits in metric geometry. However, we recall from Caution 4.4 that the open parts of the components of degenerate varieties which satisfy these (pointed) open polystability notion can *not* be pointed Gromov-Hausdorff limits themselves in general, especially for non-maximal degenerations, even after rescale (cf., e.g., [Od19]). Nevertheless, weaker relations can be expected in special situations and we wish to explore relations in more general context in future.

DEFINITION 4.7. We suppose a projective variety  $X$  has only sdlt singularities and  $K_X \sim_{\mathbb{Q}} 0$  and the irreducible decomposition as  $X = \cup_i V_i$  so that each  $(V_i, D_i)$  is a log Calabi-Yau dlt pair, where  $D_i$  denotes the conductor divisor.

Now we fix a reference closed point  $p$  inside the klt locus of  $\mathcal{X}_0$ . We also fix a polarization i.e., an ample line bundle on  $X$ .

For a log test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  of  $((X, D), L)$  such that

- $\mathcal{X}$  satisfies Serre's  $S_2$  condition.
- Its restriction to the closure of  $V_i \times (\mathbb{P}^1 \setminus \{0\})$  is of plt type in the sense of Definition 2.3 (i).
- The log Donaldson-Futaki invariant  $\text{DF}((\mathcal{X}, \mathcal{D}), \mathcal{L})$  (in the sense of [Don12, OS15]) vanishes.
- The limit of  $p$  i.e.,  $\overline{\mathbb{G}_m \cdot (p \times \{1\})} \cap \mathcal{X}_0$  is inside the klt locus of  $(\mathcal{X}_0, \mathcal{D}_0)$ .

$((X, D), L)$  is *weakly pointed open K-polystable* if every such log test configuration  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  satisfies that the klt locus of  $(\mathcal{X}, \mathcal{D})$  is of product type.

THEOREM 4.8 (Weak (proper) pointed stable reduction). *Take an arbitrary flat  $\mathbb{Q}$ -Gorenstein family  $((\mathcal{X}^*, \mathcal{D}^*), \mathcal{L}^*) \rightarrow \Delta^*$  of polarized projective klt log Calabi-Yau varieties over a punctured germ of smooth curve  $0 \in \Delta$ , with  $K_{\mathcal{X}^*/\Delta^*} + \mathcal{D}^* \equiv 0$ <sup>13</sup> and a meromorphic section  $s: \Delta^* \rightarrow \mathcal{X}^*$ , possibly after a finite base change of  $\Delta \ni 0$ , there is a dlt minimal model  $(\mathcal{X}, \mathcal{D} + \mathcal{X}_0) \rightarrow \Delta$  such that the following additional conditions hold:*

- (i)  $(\mathcal{X}_0, \mathcal{D}|_{\mathcal{X}_0})$  is semi-dlt and  $K_{\mathcal{X}_0} + \mathcal{D}|_{\mathcal{X}_0} \sim_{\mathbb{Q}} 0$ ,

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<sup>13</sup>again, equivalent to  $K_{\mathcal{X}^*/\Delta^*} + \mathcal{D}^* \sim_{\mathbb{Q}} 0$  by [Fjn00, Gon11a]

- (ii)  $\lim_{t \rightarrow 0} s(t)$  lies inside the klt locus  $\mathcal{X}_0^{\text{klt}}$  of the central fiber  $(\mathcal{X}_0, \mathcal{D}_0)$
- (iii)  $((\mathcal{X}_0, \mathcal{D}|_{\mathcal{X}_0}), \mathcal{L}|_{\mathcal{X}_0})$  is a weakly pointed open  $K$ -polystable Calabi-Yau polarized variety with respect to  $\lim_{t \rightarrow 0} s(t)$  in the sense of Definition 4.7.

Again, we also write brief version for a rather special i.e., log smooth case, for readers' convenience.

**COROLLARY 4.9.** *Take an arbitrary degenerating polarized family  $(\mathcal{X}, \mathcal{L}) \rightarrow \Delta$  of  $n$ -dimensional pointed smooth (projective) polarized Calabi-Yau manifolds also with a holomorphic section  $\{s(t)\}_t$ . We write the fibers as  $(\mathcal{X}_t, \mathcal{L}_t) \ni s(t)$  ( $t \neq 0$ ) and suppose the degeneration  $\mathcal{X}_0$  is a simple normal crossing Calabi-Yau variety. For instance, polarized Kulikov families of pointed  $K3$  surfaces, not of Type I, satisfy the condition.*

*Then, we can birationally modify the central fiber, possibly after a finite base change, to make it a "better" degenerate polarized Calabi-Yau varieties  $(\mathcal{X}_0, \mathcal{L}_0)$  which satisfies:*

- (i)  $\mathcal{X}_0$  is simple normal crossing away from a closed subset of dimension  $n - 2$ ,
- (ii) the limit point of  $\lim_{t \rightarrow 0} s(t)$  stays outside the double locus of  $\mathcal{X}_0$ ,
- (iii)  $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$  is weakly pointed open  $K$ -polystable with respect to the limit point of  $\lim_{t \rightarrow 0} s(t)$ .

**PROOF OF THEOREM 4.8.**

*Step 1.* By the semistable reduction theorem [KKMSD73] and the semistable minimal model program [Fjn11b], we can at least construct a dlt minimal model  $(\mathcal{X}, \mathcal{D} + \mathcal{X}_0) \rightarrow \Delta$ .

*Step 2.* The main innovation from here is the following base change trick, which more or less yields "sub-divided" dlt minimal models with much more irreducible components in the central fiber.

Using the  $N$ -th ramifying finite morphism  $b^{(N)}: (\Delta \ni 0) \rightarrow (\Delta \ni 0)$  with  $N \in \mathbb{Z}_{>0}$ , we take the base change of  $(\mathcal{X}, \mathcal{D})$  which we denote as

$(\mathcal{X}^{(N)}, \mathcal{D}^{(N)}) = \mathcal{X} \times_{\Delta, b^{(N)}} \Delta$ . Recalling the determination of log canonical centers of dlt pairs (cf., [Amb03], [Fjn07]), combined with [KM98], the lc centers of  $(\mathcal{X}^{(N)}, \mathcal{D}^{(N)} + \mathcal{X}_0^{(N)})$  are simply the preimages of those of  $(\mathcal{X}, \mathcal{D} + \mathcal{X}_0)$ . Let us consider the open subset of  $\mathcal{X}$  where  $(\mathcal{X}, \mathcal{D} + \mathcal{X}_0)$  is log smooth, which we denote as  $\mathcal{X}^{\text{lsm}}$  and think of étale local structure of  $\mathcal{X}^{\text{lsm}} \times_{\Delta, b^{(N)}} \Delta$ . From the log smoothness of  $(\mathcal{X}, \mathcal{D}) \cap \mathcal{X}^{\text{lsm}}$ , the preimage of  $\mathcal{X}^{\text{lsm}}$  in  $(\mathcal{X}^{(N)}, \mathcal{D}^{(N)} + \mathcal{X}_0^{(N)})$  are toroidal and have simple explicit combinatorial description by a rational polyhedral fan over the dual intersection complex ([KKMSD73]). We take an arbitrary regular subdivision of the corresponding fan, then it gives a crepant toric log resolution of the preimage of  $\mathcal{X}^{\text{lsm}}$  inside  $\mathcal{X}^{(N)}$ .

*Step 3.* Suppose that log resolution is given by the blow up of a coherent ideal  $I^\circ$  in the preimage of  $\mathcal{X}^{\text{lsm}}$  inside  $\mathcal{X}^{(N)}$ . We extend the coherent ideal  $I^\circ$  to a coherent ideal  $I$  of whole  $\mathcal{O}_{\mathcal{X}^{(N)}}$  whose normalized blow up gives still a log resolution (or we take the log canonical closure of [HX13]). Then we run the relative minimal model program over  $\mathcal{X}^{(N)}$ , which is now allowed by using [HX13, HH20] etc. This gives a dlt minimal model over  $\mathcal{X}^{(N)}$  which we denote by  $(\mathcal{X}^{[N]}, \mathcal{D}^{[N]})$ . By a simple phenomenon that the all log canonical centers of  $(\mathcal{X}, \mathcal{D})$  intersect with  $\mathcal{X}^{\text{lsm}}$  (cf., [Amb03], [Fjn07]), it follows that  $(\mathcal{X}^{[N]}, \mathcal{D}^{[N]})$  is also a dlt minimal model *over the base curve*  $\Delta$  as well.

From our earlier determination of log canonical centers of  $(\mathcal{X}^{(N)}, \mathcal{D}^{(N)} + \mathcal{X}_0^{(N)})$ , it easily follows that all the irreducible components of  $\mathcal{X}_0^{[N]}$  intersect with the preimage of  $\mathcal{X}^{\text{lsm}}$ .

*Step 4.* Now we consider  $\text{order}_{t=0}(s^*V_i) \in \mathbb{Q}$  and  $\text{order}_{t=0}(s^*\mathcal{D}_i) \in \mathbb{Q}$  where  $V_i$  denotes the irreducible component of  $\mathcal{X}_0$  and  $\mathcal{D}_i$  denotes irreducible components of  $\mathcal{D}$ . Comparing with those rational numbers, if we take a prime number  $N$  which do not divide neither the numerators or denominators, then our desired assertion (ii) easily follows. This fact can be shown in standard explicit blow up calculation while it more systematically follows by use of the Morgan-Shalen-Boucksom-Jonsson construction (cf., e.g., [Od19]) which we expand in details in another paper [Od20c] more. So now we finish the discussion of our base change trick which ensures the dlt model existence satisfying the condition (ii).

*Step 5.* As a next step, we wish to improve further the obtained model  $(\mathcal{X}^{(N)}, \mathcal{D}^{(N)})$  to make it satisfy (ii) i.e., the weak pointed open K-polystability as desired. Here, we use the same trick as previous Theorem 4.5 of “gluing in” test configuration; we take an arbitrary ample extension  $\mathcal{L}^{(N)}$  which extends that of (base change of)  $\mathcal{L}^*$ . If  $((\mathcal{X}_0^{(N)}, \mathcal{D}_0^{(N)}), \mathcal{L}^{(N)}|_{\mathcal{X}_0^{(N)}})$  is not weakly pointed open K-polystable with respect to the limit  $p := \lim_{t \rightarrow 0} s(t)$  of meromorphic section  $s$  in  $\mathcal{X}^{(N)}$ , then there is a log test configuration of  $((\mathcal{X}_0^{(N)}, \mathcal{D}_0^{(N)}), \mathcal{L}^{(N)}|_{\mathcal{X}_0^{(N)}}) \ni p$  of plt-type  $((\mathcal{X}', \mathcal{D}'), \mathcal{L}')$  such that the limit of  $p$  is inside klt locus of  $(\mathcal{X}_0^{(N)}, \mathcal{D}_0^{(N)})$ , the log Donaldson-Futaki invariant vanishes  $\text{DF}(\mathcal{X}_0^{(N)}, \mathcal{D}_0^{(N)}) = 0$ , and not of product type. More precisely, completely similarly to the Steps 2,3,4 of the proof of Theorem 4.5, we take a sufficiently high degree ramifying base change of  $(\mathcal{X}^{(N)}, \mathcal{D}^{(N)})$  and glue them along the fiber over  $0 \in \Delta$  with the above log plt-type test configuration  $((\mathcal{X}', \mathcal{D}'), \mathcal{L}')$ . Therefore we obtain  $((\mathcal{X}, \mathcal{D}), \mathcal{L})$  which satisfies (ii).

*Step 6.* Finally we want to ensure the dlt property, and we slightly modify as follows: we first take (semi-)normalization of the total space  $\mathcal{X}$  which must be bijective to  $\mathcal{X}$ , then the inversion of adjunction tells us that  $(\mathcal{X}, \mathcal{D} + \mathcal{X}_0)$  is log canonical. Hence we can take log crepant dlt blow up as in e.g., [FG14, OX12, Kol13], which is desired model. The condition (ii) is automatically satisfied because of the glueing construction which preserves the same property for  $((\mathcal{X}', \mathcal{D}'), \mathcal{L}')$ .  $\square$

We explore the above Step 2 more systematically and thoroughly in another paper in preparation. If we replace the use of weak *pointed* open K-polystability in above 4.8, 4.9, by the weakly open K-polystability in the sense of Definition 2.19 (i) (unpointed version), we also obtain the following. Since the proof goes same way as that of Theorem 4.5 just by replacing the definition of stability, hence easier than 4.8, we omit the proof.

**THEOREM 4.10** (Weak (proper unpointed) stable reduction). *Take an arbitrary flat  $\mathbb{Q}$ -Gorenstein family  $((\mathcal{X}^*, \mathcal{D}^*), \mathcal{L}^*) \rightarrow \Delta^*$  of polarized projective klt log Calabi-Yau varieties over a punctured germ of smooth curve  $0 \in \Delta$ , with  $K_{\mathcal{X}^*/\Delta^*} + \mathcal{D}^* \equiv 0$ <sup>14</sup>, possibly after a finite base change of  $\Delta \ni 0$ ,*

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<sup>14</sup>again, equivalent to  $K_{\mathcal{X}^*/\Delta^*} + \mathcal{D}^* \sim_{\mathbb{Q}} 0$  by [Fjn00, Gon11a]



there is a dlt minimal model  $(\mathcal{X}, \mathcal{D} + \mathcal{X}_0) \rightarrow \Delta$  such that the following additional conditions hold:

- (i)  $(\mathcal{X}_0, \mathcal{D}|_{\mathcal{X}_0})$  is semi-dlt and  $K_{\mathcal{X}_0} + \mathcal{D}|_{\mathcal{X}_0} \sim_{\mathbb{Q}} 0$ ,
- (ii)  $((\mathcal{X}_0, \mathcal{D}|_{\mathcal{X}_0}), \mathcal{L}|_{\mathcal{X}_0})$  is weakly open K-polystable (in the sense of Definition 2.19 (i)).

REMARK 4.11. After above Theorem 3.1, Corollary 3.7 etc, we expect that the observation by Gross-Hacking-Keel [GHK] that any Type III one parameter algebraic degeneration can be vertically birationally transformed into a toric degeneration, can be seen as a special case of strongly open K-polystable reduction.

In another paper [Od20c], we discuss a continuation of the above discussion in the proof.

REMARK 4.12. Consider a log Calabi-Yau lc pair  $(\mathbb{P}^2, \frac{3}{4}C)$  where  $C$  is a “cat-eye” i.e., a union of two smooth conics  $C_1$  and  $C_2$  intersecting at two tacnodes, i.e., 1-dimensional  $A_3$ -singularities (cf., [HL10, OSS16, ADL19]). Take the blow up of both tacnodes  $C_1 \cap C_2$  and further blow up the two singular points of the strict transform of  $C$ . Then we obtain a birational ruled surface  $X \rightarrow \mathbb{P}^1$  with two sections  $D_1$  and  $D_2$ . We can make this into a log crepant resolution  $(X, D_1 + D_2 + \Delta) \rightarrow \mathbb{P}^2$ . This log Calabi-Yau  $(X, D_1 + D_2 + \Delta)$  is dlt and we can easily see its weakly open K-polystability by its simple structure of log canonical centers by Theorem 2.7. However, in the wall crossing of [ADL19], they replace this by  $(\mathbb{P}(1, 1, 4), \frac{3}{4}[x^4y^4 = z^2])$  (cf., [ADL19], also [OSS16]). This gives an example of further reduction process. See [Od20c] for further discussion.

### 4.3. Strong open stable reduction

This subsection shows another stable reduction type theorem in the case approximated by log Fano pairs with conical singular weak Kähler-Einstein metrics. We start with a subtle remark.

REMARK 4.13. Consider a punctured family  $((\mathcal{X}^*, \mathcal{D}^*), \mathcal{L}^*) \rightarrow \Delta^*$  of strongly open K-polystable log Calabi-Yau pairs, whose fibers  $\mathcal{X}_t (t \neq 0)$

are  $\mathbb{Q}$ -Fano varieties. Even if they are smooth Fano varieties with Kähler-Einstein metrics, whose existence ensures the strongly open K-polystabilities of  $((\mathcal{X}_t, \mathcal{D}_t), \mathcal{L}_t)$  for any  $t \neq 0$  (cf., [LS14], [OS15]), in general, we can-*not* get strongly open K-polystable central fiber after possible finite base change, simply by applying the K-stable reduction to  $\mathcal{X}_t$ s by [DonSun14].

For instance, take a family of Tian-Yau metric on the complements of quartic K3 surfaces which degenerates to the doubled quadric surface.

Nevertheless, by making the conical angles rather small (acute), we prove the following.

**THEOREM 4.14** (Strongly open K-polystable reduction). *Consider a punctured polarized proper family  $((\mathcal{X}^*, \mathcal{D}^*), \mathcal{L}^*) \rightarrow \Delta^* = \Delta \setminus \{0\}$  of strongly open K-polystable log Calabi-Yau pairs. We suppose the fibres are asymptotic log Fano pairs in the sense of [CR15].*

*Then, possibly after finite base change, there is its model  $(\mathcal{X}, \mathcal{D}) \rightarrow \Delta \ni 0$  such that the central fiber  $((\mathcal{X}_0, \mathcal{D}_0), \mathcal{L}_0)$  is a strongly open K-polystable asymptotic log Fano pair. Moreover, such filling is unique.*

**PROOF.** By the Zariski openness of ampleness of line bundles for flat variation, there is a uniform constant  $0 < c_0 \ll 1$  such that  $(\mathcal{X}_t, c\mathcal{D}_t)$  are (klt) log Fano pairs for any  $(0 <)c < c_0$  and any  $t \in \Delta^*$ .

By [OS15] and the usual Skoda type estimate (cf., e.g., [Od12]), there is a uniform  $\beta_0 > 0$  such that  $(\mathcal{X}_t, (1 - \beta)\mathcal{D}_t)$  is log K-polystable for any  $(0 <)\beta < \beta_0$  and any  $t$ .

For any such fixed  $\beta$ , from [BHLLX21, LXZ21] (see also [CDS15],[Tia15] for earlier partial results via Cheeger-Colding theory), there is a model  $(\mathcal{X}, \mathcal{D})$  such that  $(\mathcal{X}_t, (1 - \beta)\mathcal{D}_t)$  is a log K-polystable  $\mathbb{Q}$ -Fano pair.

What remains is to show its independence of small  $\beta$ . Note that the log canonical threshold of above obtained  $\mathcal{X}_0$  with respect to  $\mathcal{D}_t$  is at least  $1 - \beta$ , as it follows from the klt property of the obtained pair  $(\mathcal{X}_0, (1 - \beta)\mathcal{D}_0)$ . Therefore, from [HMX14], for small enough  $\beta \ll 1$ , the concerned log canonical threshold should be at least 1. In other words,  $(\mathcal{X}_0, \mathcal{D}_0)$  is log canonical so that it is log K-semistable with respect to any polarization, by [OS15] (=Theorem 1.3). Therefore, from the affine-linearity of the log Donaldson-Futaki invariant with respect to the coefficient of the boundary, it follows that  $((\mathcal{X}_0, \mathcal{D}_0), \mathcal{L}_0)$  is strongly open K-polystable. The uniqueness assertion follows from [BX19].  $\square$

REMARK 4.15. For general punctured family of polarized log Calabi-Yau pairs  $((\mathcal{X}^*, \mathcal{D}^*), \mathcal{L}^*) \rightarrow \Delta^* = \Delta \setminus \{0\}$ , the uniqueness of strongly open K-polystable reduction as in the above Theorem 4.14 case does not necessarily hold.

We discuss related developments in [Od20c].

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