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# *On Regularizable Birational Maps*

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**Abstract.** Bedford asked if there exists a birational self map $f$ of the complex projective plane such that for any automorphism $A$ of the complex projective plane $A \circ f$ is not conjugate to an automorphism. In this article we give such an $f$ of degree 5.

## 1. Introduction

Let us consider $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ the group of all birational self maps of $\mathbb{P}^k_{\mathbb{C}}$, also called the *k-dimensional Cremona group*. A birational map $f \in \mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ is *regularizable* if there there exist a smooth projective variety $V$ and a birational map $g\colon V \dashrightarrow \mathbb{P}^k_{\mathbb{C}}$ such that $g^{-1} \circ f \circ g$ is an automorphism of $V$. For instance any $f \in \mathrm{Aut}(\mathbb{P}^k_{\mathbb{C}}) = \mathrm{PGL}(k+1, \mathbb{C})$ is regularizable but any $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ such that $(\deg f^n)_n$ grows linearly is not regularizable ([6]). To any element $f$ of $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ we associate the set $\mathrm{Reg}(f)$ defined by

$$\mathrm{Reg}(f) := \big\{A \in \mathrm{Aut}(\mathbb{P}^k_{\mathbb{C}}) \mid A \circ f \ \text{ is regularizable}\big\}.$$

On the one hand Dolgachev asked[1] whether there exists a birational self map of $\mathbb{P}^k_{\mathbb{C}}$ of degree $> 1$ such that $\mathrm{Reg}(f) = \mathrm{Aut}(\mathbb{P}^k_{\mathbb{C}})$. In [4] we give a negative answer to this question; more precisely we prove:

THEOREM 1.1 ([4]). *Let $\mathbb{k}$ be an uncountable, algebraically closed field. Let $f$ be a birational self map of $\mathbb{P}^m_{\mathbb{k}}$ of degree $d \geq 2$. The set of automorphisms $A$ of $\mathbb{P}^m_{\mathbb{k}}$ such that $\deg\big((A \circ f)^n\big) \neq \big(\deg(A \circ f)\big)^n$ for some $n > 0$ is a countable union of proper Zariski closed subsets of $\mathrm{PGL}(m+1, \mathbb{k})$.*

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[1]Summer school "Complex geometry and beyond", june 2016, Toulouse, France.

*Let $\mathbb{k}$ be a field of characteristic zero. Let $f$ be a birational transformation of $\mathbb{P}^m$ which is defined over the field $\mathbb{k}$, i.e. the formulas defining $f$ have coefficients in $\mathbb{k}$. Then, there exists an element $A$ of* $\mathrm{PGL}(m+1,\mathbb{k})$ *such that* $\deg((A\circ f)^n)=\deg(f)^n$ *for all* $n\geq 1$.

On the other hand Bedford asked[2]: does there exist a birational map $f$ of $\mathbb{P}^k_{\mathbb{C}}$ such that $\mathrm{Reg}(f)=\emptyset$ ? We will focus on the case $k=2$. According to [1, 5] if $f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ and $\deg f=2$, then $\mathrm{Reg}(f)\neq\emptyset$. Furthermore Blanc proves that the set

$$\big\{f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\mid \deg f=3,\ \mathrm{Reg}(f)\neq\emptyset,\ \lim_{n\to+\infty}(\deg(f^n))^{1/n}>1\big\}$$

is dense in $\big\{f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\mid\deg f=3\big\}$ and that its complement has codimension 1 (*see* [2]). Blanc also gives a positive answer to Bedford question in dimension 2: if $\chi\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is given by

$$\chi\colon(x:y:z)\dashrightarrow\big(xz^5+(yz^2+x^3)^2:yz^5+x^3z^3:z^6\big)$$

then $\mathrm{Reg}(\chi)=\emptyset$. Note that $\chi=(x+y^2,y)\circ(x,y+x^3)$ in the affine chart $z=1$. Indeed Blanc example can be generalized as follows: the birational map of degree $np$ given in the affine chart $z=1$ by

$$\chi_{n,p}=(x+y^n,y)\circ(x,y+x^p)=\big(x+(y+x^p)^n,y+x^p\big)$$

satisfies $\mathrm{Reg}(\chi_{n,p})=\emptyset$. Finally Blanc asked ([2, Question 1.6]): does there exists $f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ such that $\deg f<6$ and $\mathrm{Reg}(f)=\emptyset$ ? In this article we give a positive answer:

THEOREM A. *If $\psi$ is the birational self map of $\mathbb{P}^2_{\mathbb{C}}$ given by*

$$\psi\colon(x:y:z)\dashrightarrow\big(x^2yz^2-z^5+x^5:x^2(x^2y-z^3):xz(x^2y-z^3)\big),$$

*then* $\mathrm{Reg}(\psi)=\emptyset$.

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[2]Private communication, 2010.

## 2. Proof of Theorem A

Let $S$ be a smooth projective surface. Let $\phi\colon S \dashrightarrow S$ be a birational map. This map admits a resolution

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \arrow[rr, dashed, "\phi"'] & & S, \end{tikzcd}$$

where $\pi_1\colon Z \to S$ and $\pi_2\colon Z \to S$ are finite sequences of blow-ups. The resolution is *minimal* if and only if no $(-1)$-curve of $Z$ is contracted by both $\pi_1$ and $\pi_2$. Assume $\phi$ is minimal; the *base-points* $\mathrm{Base}(\phi)$ of $\phi$ are the points blown-up by $\pi_1$, which can be points of $S$ or infinitely near points. Finally we denote by $\mathrm{Exc}(\phi)$ the set of curves contracted by $\phi$.

Denote by $\mathfrak{b}(\phi)$ the number of base-points of $\phi$; note that $\mathfrak{b}(\phi)$ is equal to the difference of the ranks of $\mathrm{Pic}(Z)$ and $\mathrm{Pic}(S)$ and thus equal to $\mathfrak{b}(\phi^{-1})$. Let us introduce the *dynamical number of the base-points of* $\phi$

$$\mu(\phi) = \lim_{k\to+\infty} \frac{\mathfrak{b}(\phi^k)}{k}.$$

Since $\mathfrak{b}(\phi\circ\varphi) \leq \mathfrak{b}(\phi) + \mathfrak{b}(\varphi)$ for any birational self map $\varphi$ of $S$, $\mu(\phi)$ is a non-negative real number. As $\mathfrak{b}(\phi) = \mathfrak{b}(\phi^{-1})$ one gets $\mu(\phi^k) = |k\,\mu(\phi)|$ for any $k \in \mathbb{Z}$. Furthermore if $Z$ is a smooth projective surface and $\varphi\colon S \dashrightarrow Z$ a birational map, then for all $n \in \mathbb{Z}$

$$-2\mathfrak{b}(\varphi) + \mathfrak{b}(\phi^n) \leq \mathfrak{b}(\varphi\circ\phi^n\circ\varphi^{-1}) \leq 2\mathfrak{b}(\varphi) + \mathfrak{b}(\phi^n);$$

hence $\mu(\phi) = \mu(\varphi\circ\phi\circ\varphi^{-1})$. One can thus state the following result:

Lemma 2.1 ([3]). *The dynamical number of base-points is an invariant of conjugation. In particular if $\phi$ is a regularizable birational self map of a smooth projective surface, then $\mu(\phi) = 0$.*

A base-point $p$ of $\phi$ is a *persistent base-point* if there exists an integer $N$ such that for any $k \geq N$

- $\diamond$ $p \in \mathrm{Base}(\phi^k)$

$\diamond$ and $p \notin \mathrm{Base}(\phi^{-k})$.

Let $p$ be a point of $S$ or a point infinitely near $S$ such that $p \notin \mathrm{Base}(\phi)$. Consider a minimal resolution of $\phi$

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \arrow[rr, dashed, "\phi"'] & & S. \end{tikzcd}$$

Because $p$ is not a base-point of $\phi$ it corresponds via $\pi_1$ to a point of $Z$ or infinitely near; using $\pi_2$ we view this point on $S$ again maybe infinitely near and denote it $\phi^\bullet(p)$. For instance if $S = \mathbb{P}^2_{\mathbb{C}}$, $p = (1:0:0)$ and $f$ is the birational self map of $\mathbb{P}^2_{\mathbb{C}}$ given by

$$(z_0 : z_1 : z_2) \dashrightarrow (z_1z_2 + z_0^2 : z_0z_2 : z_2^2)$$

the point $f^\bullet(p)$ is not equal to $p = f(p)$ but is infinitely near to it. Note that if $\varphi$ is a birational self map of $S$ and $p$ is a point of $S$ such that $p \notin \mathrm{Base}(\phi)$, $\phi(p) \notin \mathrm{Base}(\varphi)$, then $(\varphi\circ\phi)^\bullet(p) = \varphi^\bullet(\phi^\bullet(p))$. One can put an equivalence relation on the set of points of $S$ or infinitely near $S$: the point $p$ is *equivalent* to the point $q$ if there exists an integer $k$ such that $(\phi^k)^\bullet(p) = q$; in particular $p \notin \mathrm{Base}(\phi^k)$ and $q \notin \mathrm{Base}(\phi^{-k})$. Remark that the equivalence class is the generalization of set of orbits for birational maps.

Let us give the relationship between the dynamical number of base-points and the equivalence classes of persistent base-points:

Proposition 2.2 ([3]). *Let $S$ be a smooth projective surface. Let $\phi$ be a birational self map of $S$.*

*Then $\mu(\phi)$ coincides with the number of equivalence classes of persistent base-points of $\phi$. In particular $\mu(\phi)$ is an integer.*

This interpretation of the dynamical number of base-points allows to prove the following result that gives a characterization of regularizable birational maps:

Theorem 2.3 ([3]). *Let $\phi$ be a birational self map of a smooth projective surface. Then $\phi$ is regularizable if and only if $\mu(\phi) = 0$.*

## 2.1. Base-points of $\psi$

The birational map

$$\psi\colon (x:y:z) \dashrightarrow \big(x^2yz^2 - z^5 + x^5 : x^2(x^2y - z^3) : xz(x^2y - z^3)\big)$$

has only one base-point in $\mathbb{P}^2_{\mathbb{C}}$, namely $p_1 = (0:1:0)$, and all its base-points are in tower that is: the nine base-points of $\psi$ that we denote $p_1$, $p_2$, $\ldots$, $p_9$ are such that $p_i$ is infinitely near to $p_{i-1}$ for $2 \leq i \leq 9$. We denote by $\pi\colon S \to \mathbb{P}^2_{\mathbb{C}}$ the blow up of the 9 base-points, and still write $L_x$ (resp. $\mathcal{C}$) the strict transform of the line $L_x \subset \mathbb{P}^2_{\mathbb{C}}$ of equation $x = 0$ (resp. the curve of equation $x^2y - z^3 = 0$) which is contracted by $\psi$. We denote by $E_i \subset S$ the strict transform of the curve obtained by blowing up $p_i$. The configuration of the curves $E_1$, $E_2$, $\ldots$, $E_9$, $L_x$ and $\mathcal{C}$ is

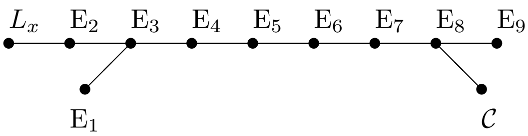


Fig. 1.

Two curves are connected by an edge if their intersection is positive. Let us write $\psi_A = A \circ \psi$ where $A$ is an automorphism of $\mathbb{P}^2_{\mathbb{C}}$. Because $\pi$ is the blow-up of the base-points of $\psi$, which are also the base-points of $\psi_A$, the map $\eta = \psi_A \circ \pi$ is a birational morphism $S \to \mathbb{P}^2_{\mathbb{C}}$ which is the blow-up of the base-points of $\psi_A^{-1}$. In fact this diagram

$$\begin{tikzcd} & S \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2_{\mathbb{C}} \arrow[rr, dashed, "\psi_A"'] & & \mathbb{P}^2_{\mathbb{C}} \end{tikzcd}$$

is the minimal resolution of $\psi_A$.

The morphism $\eta$ contracts $L_x$ and $\mathcal{C}$ as well as the union of seven other irreducible curves which are among the curves $E_1$, $E_2$, $\ldots$, $E_9$. The configuration of Figure 1 shows that $\eta$ contracts the curves $L_x$, $E_2$, $E_3$, $E_4$, $E_5$, $E_6$, $E_7$, $E_8$, $\mathcal{C}$ following this order.

We can see $\eta\colon S \to \mathbb{P}^2_{\mathbb{C}}$ as a sequence of nine blow-ups in the same way as we did for $\pi$. We denote by $q_1$, $q_2$, $\ldots$, $q_9$ the base-points of $\psi_A^{-1}$ (or equivalently the points blown up by $\eta$) so that $q_1 \in \mathbb{P}^2_{\mathbb{C}}$ and $q_i$ is infinitely near to $q_{i-1}$ for $2 \leq i \leq 9$. We denote by $D \subset \mathbb{P}^2_{\mathbb{C}}$ (resp. $\mathcal{C}' \subset \mathbb{P}^2_{\mathbb{C}}$) the line contracted by $\psi_A^{-1}$ which is the image by $A$ of the line $y = 0$ (resp. of the conic $z^2 - xy = 0$). We denote by $F_i \subset S$ the strict transform of the curve obtained by blowing up $q_i$. Because of the order of the curves contracted by $\eta$ we get equalities between $L_x$, $\mathcal{C}$, $E_1$, $E_2$, $\ldots$, $E_9$ and $D$, $\mathcal{C}'$, $F_1$, $F_2$, $\ldots$, $F_9$ as follows

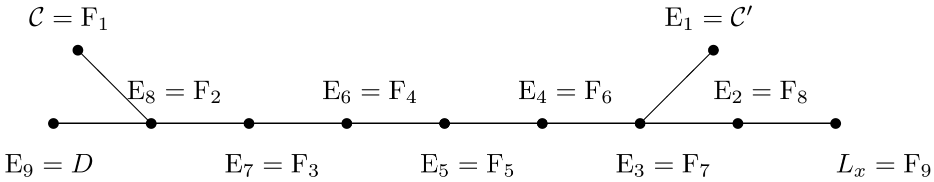


Fig. 2.

In particular we see that the configuration of the points $q_1$, $q_2$, $\ldots$, $q_9$ is not the same as that of the points $p_1$, $p_2$, $\ldots$, $p_9$. Saying that a point $m$ is proximate to a point $m'$ if $m$ is infinitely near to $m'$ and that it belongs to the strict transform of the curve obtained by blowing up $m'$ the configurations of the points $p_i$ and $q_i$ are

$$\begin{tikzcd}
p_1 & p_2 \arrow[l] & p_3 \arrow[l] \arrow[ll, bend left] & p_4 \arrow[l] & p_5 \arrow[l] & p_6 \arrow[l] & p_7 \arrow[l] & p_8 \arrow[l] & p_9 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
q_1 & q_2 \arrow[l] & q_3 \arrow[l] & q_4 \arrow[l] & q_5 \arrow[l] & q_6 \arrow[l] & q_7 \arrow[l] & q_8 \arrow[l] & q_9 \arrow[l]
\end{tikzcd}$$

Fig. 3. An arrow corresponds to the relation "is proximate to".

We will prove that for any integer $i > 0$ the point $p_3$ belongs to $\mathrm{Base}(\psi_A^i)$ and does not belong to $\mathrm{Base}(\psi_A^{-i})$. It implies that $\mu(\psi_A) > 0$ and that $\psi_A$

is not regularizable.

Denote by $k$ the lowest positive integer such that $p_1$ belongs to $\mathrm{Base}(\psi_A^{-k})$. If no such integer exists we write $k=\infty$. For any $1\leq i<k$ the point $p_1$ does not belong to $\mathrm{Base}(\psi_A^{-i})$ so $\psi_A$ and $\psi_A^{-1}$ have no common base-point. As a consequence the set of base-points of the map $\psi_A^{i+1}=\psi_A\circ\psi_A^i$ is the union of the base-points of $\psi_A^i$ and of the points $(\psi_A^{-i})^\bullet(p_j)$ for $1\leq j\leq 9$. Since the map $\psi_A^{-i}$ is defined at $p_1$ the point $(\psi_A^{-i})^\bullet(p_j)$ is proximate to the point $(\psi_A^{-i})^\bullet(p_k)$ if and only if $p_j$ is proximate to $p_k$. Proceeding by induction on $i$ we get the following assertions:

- ◇ $\mathrm{Base}(\psi_A^i)=\{(\psi_A^{-m})^\bullet(p_j)\,|\,1\leq j\leq 9,\ 0\leq m\leq i-1\}$ for any $1\leq i\leq k$;
- ◇ for any $0\leq -\ell\leq k$ the configuration of the points $\{(\psi_A^\ell)^\bullet(p_j)\,|\,1\leq j\leq 9\}$ is given by

$$\begin{tikzcd}[column sep=small]
(\psi_A^\ell)^\bullet(p_1) & (\psi_A^\ell)^\bullet(p_2) \arrow[l] & (\psi_A^\ell)^\bullet(p_3) \arrow[l] \arrow[ll, bend left] & (\psi_A^\ell)^\bullet(p_4) \arrow[l] & (\psi_A^\ell)^\bullet(p_5) \arrow[l] & (\psi_A^\ell)^\bullet(p_6) \arrow[l] & (\psi_A^\ell)^\bullet(p_7) \arrow[l] & (\psi_A^\ell)^\bullet(p_8) \arrow[l] & (\psi_A^\ell)^\bullet(p_9) \arrow[l]
\end{tikzcd}$$

Hence the point $p_3$ belongs to $\mathrm{Base}(\psi_A^i)$ for any $1\leq i\leq k$.

If $k=\infty$, then $p_3$ belongs to $\mathrm{Base}(\psi_A^i)$ for any $i>0$ and by definition of $k$ the point $p_1$ does not belong to $\mathrm{Base}(\psi_A^{-i})$ for any $i>0$, and so neither $p_3$. We can thus assume that $k$ is a positive integer.

Assume that $q_1$ belongs to $\mathrm{Base}(\psi_A^i)$ for some $1\leq i\leq k-1$. Then $q_1$ is equal to $(\psi_A^{-m})^\bullet(p_j)$ for some $0\leq m\leq k-2$ and $1\leq j\leq 9$. This implies that $p_j$ belongs to $\mathrm{Base}(\psi_A^{m+1})$ which is impossible because $m+1\leq k-1$. Hence $q_1$ does not belong to $\mathrm{Base}(\psi_A^i)$ for any $1\leq i\leq k-1$.

We thus see that $\psi_A^{-1}$ has no common base-point with $\psi_A^i$ for $1\leq i\leq k-1$. In particular if $B$ denotes $\mathrm{Base}(\psi_A^{-1})\cap\mathrm{Base}(\psi_A^k)$, then

$$B=\{(\psi_A^{-(k-1)})^\bullet(p_j)\,|\,1\leq j\leq 9\}\cap\{q_j\,|\,1\leq j\leq 9\}.$$

Let us remark that $p_1$ belongs to $\mathrm{Base}(\psi_A^{-k})$ and $p_1$ does not belong to $\mathrm{Base}(\psi_A^{-(k-1)})$; as a result $(\psi_A^{-(k-1)})^\bullet(p_1)$, which is a base-point of $\psi_A^k$, is also a base-point of $\psi_A^{-1}$. The set $B$ is thus not empty.

The configurations of the two sets of points $\{(\psi_A^{-(k-1)})^\bullet(p_j)\,|\,1\leq j\leq 9\}$ and $\{q_j\,|\,1\leq j\leq 9\}$ imply that $q_1=(\psi_A^{-(k-1)})^\bullet(p_1)$.

Moreover either $B=\{q_1\}$, or $B=\{q_1,\,q_2\}$. Indeed $(\psi_A^{-(k-1)})^\bullet(p_3)$ is proximate to $(\psi_A^{-(k-1)})^\bullet(p_2)$ and $(\psi_A^{-(k-1)})^\bullet(p_1)$ whereas $q_3$ is proximate to $q_2$ but not to $q_1$.

The point $(\psi_A^{-(k-1)})^\bullet(p_3)$ is thus a point infinitely near to $q_1$ in the second neighborhood which is maybe infinitely near to $q_2$ but not equal to $q_3$. Recalling that $\eta$ is the blow up of $q_1,\,q_2,\,\ldots,\,q_9$ the point $(\eta^{-1}\circ\psi_A^{-(k-1)})^\bullet(p_3)$ corresponds to a point that belongs, as a proper or infinitely near point, to one of the curves $F_1,\,F_2\subset S$. So $(\pi\circ\eta^{-1}\circ\psi_A^{-(k-1)})^\bullet(p_3)$ is a point infinitely near to $p_3$. For any $1\leq i\leq k$ the point $p_3$ does not belong to $\mathrm{Base}(\psi_A^{-i})$; therefore there is no base-point of $\psi_A^{-i}$ which is infinitely near to $p_3$. As a result $(\psi_A^{-k})^\bullet(p_3)$ does not belong to $\mathrm{Base}(\psi_A^{-i})$ and $p_3$ does not belong to $\mathrm{Base}(\psi_1^{-(k+i)})$. Moreover $(\psi_A^{-(k+i)})^\bullet(p_3)$ is infinitely near to $(\psi_A^{-i})^\bullet(p_3)$. Choosing $i=k$ we see that $(\psi_A^{-2k})^\bullet(p_3)$ is infinitely near to $(\psi_A^{-k})^\bullet(p_3)$ which is infinitely near to $p_3$. Continuing like this we get

$$\forall\, i\geq 1 \qquad p_3\notin \mathrm{Base}(\psi_A^{-i}).$$

To get the result it remains to show that $p_3$ belongs to $\mathrm{Base}(\psi_A^{i})$ for any $i\geq 1$. Reversing the order of $\psi_A$ and $\psi_A^{-1}$ we prove as previously that

$$\forall\, i\geq 1 \qquad q_3\notin \mathrm{Base}(\psi_A^{i}).$$

Let us now see that

$$\big(\forall\, i\geq 1 \quad q_3\notin\mathrm{Base}(\psi_A^i)\big)\Rightarrow\big(\forall\, i\geq 1\quad p_3\in\mathrm{Base}(\psi_A^i)\big).$$

For $i=1$ it is obvious. Assume $i>1$; let us decompose

- ◇ $\psi_A^i$ into $\psi_A^{i-1}\circ\psi_A$,
- ◇ $\pi\colon S\to\mathbb{P}^2_{\mathbb{C}}$ into $\pi_{12}\circ\pi_{39}$ where $\pi_{12}\colon Y\to\mathbb{P}^2_{\mathbb{C}}$ is the blow up of $p_1$, $p_2$ and $\pi_{39}\colon S\to\ Y$ is the blow up of $p_3,\,p_4,\,\ldots,\,p_9$,
- ◇ $\eta\colon S\to\mathbb{P}^2_{\mathbb{C}}$ into $\eta_{12}\circ\eta_{39}$ where $\eta_{12}\colon Z\to\mathbb{P}^2_{\mathbb{C}}$ is the blow up of $q_1$, $q_2$ and $\eta_{39}\colon S\to\ Z$ is the blow up of $q_3,\,q_4,\,\ldots,\,q_9$.

Note that $\eta_{39}$ contracts $F_9,\,F_8,\,\ldots,\,F_3$ onto the point $Z\ni q_3\notin\mathrm{Base}(\psi_A^{i-1}\circ\eta_{12})$. Consider the system of conics of $\mathbb{P}^2_{\mathbb{C}}$ passing through $p_1$, $p_2$ and $p_3$. Denote by $\Lambda$ its lift on $Y$; it is a system of smooth curves passing through

$q_3$ with movable tangents and $\dim\Lambda = 2$. The strict transform of $\Lambda$ on $S$ is a system of curves intersecting $E_3$ at a general movable point. The map $\eta_{39}$ contracts the curves $L_x$, $E_2$, $E_3$, $E_4$, $E_5$, $E_6$, $E_7$. As the curve $E_3$ is contracted and is not the last one, the image of the system by $\eta_{39}$ passes through $q_3$ with a fixed tangent corresponding to the point $q_4$. Since $q_3 \notin \mathrm{Base}(\psi_A^{i-1} \circ \eta_{12})$ the image of $\Lambda \subset Y$ by $\psi_A^{i-1} \circ \eta \circ (\pi_{39})^{-1}$ has a fixed tangent at the point $(\psi_A^{i-1} \circ \eta_{12})(q_3)$. As a consequence $p_3$ belongs to $\mathrm{Base}(\psi_A^{i-1} \circ \eta \circ (\pi_{39})^{-1})$ and thus to $\mathrm{Base}(\underbrace{\psi_A^{i-1} \circ \eta \circ (\pi_{39})^{-1} \circ (\pi_{12})^{-1}}_{\psi_A^i})$.

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