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Superconducting Phase in the BCS Model with Imaginary Magnetic Field. II. Multi-Scale Infrared Analysis

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Abstract. We analyze the reduced BCS model with an imaginary magnetic field in a large domain of the temperature and the imaginary magnetic field. The magnitude of the attractive reduced BCS interaction is fixed to be small but independent of the temperature and the imaginary magnetic field unless the temperature is high. We impose a series of conditions on the free dispersion relation. These conditions are typically satisfied by free electron models with degenerate Fermi surface. For example, our theory applies to the model with nearest-neighbor hopping on 3 or 4-dimensional (hyper-)cubic lattice having degenerate free Fermi surface or the model with nearest-neighbor hopping on the honeycomb lattice with zero chemical potential. We prove that a spontaneous $U(1)$-symmetry breaking (SSB) and an off-diagonal long range order (ODLRO) occur in many areas of the parameter space. The SSB and the ODLRO are proved to occur in low temperatures arbitrarily close to zero in particular. However, it turns out that the SSB and the ODLRO are not present in the zero-temperature limit. The proof is based on Grassmann Gaussian integral formulations and a multi-scale infrared analysis of the formulations. We keep using notations and lemmas of our previous work [Kashima, Y., J. Math. Sci. Univ. Tokyo **28** (2021), 1–179] implementing the double-scale integration scheme. So the multi-scale analysis this paper presents is a continuation of the previous work.

Contents

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1. Introduction

1.1. Introduction

The Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity ([1]) has been a paradigm of modern physics. The BCS model Hamiltonian of interacting electrons lies at the core of the theory. A large amount of knowledge on how to analyze the BCS model have been accumulated. A history of mathematical development around the BCS model is summarized in e.g. [2]. However, it is still a fair remark that we have not yet achieved a consensus on the possibility of completely rigorous, explicit analysis of the full BCS model. Here we mean a Fermionic Hamiltonian consisting of a quadratic kinetic term and a quartic interacting term by the BCS model.

It is necessary to investigate in which parameter region the BCS model can be rigorously analyzed in order to clarify and increase our understanding of the model in its original definition as the Fermionic field operator.

To supplement overviews of the literature given in the introduction of our previous work [12], here let us comment on two of the most studied mathematical approaches to the theory of the BCS model. Analysis of the BCS functional has been vigorously developed by the authors of the review article [7] and their coauthors. The BCS functional is derived from the Gibbs variational principle as a functional of generalized one-body density matrices. Above all the derivation is based on an assumption that to characterize equilibrium states it suffices to minimize the pressure functional over a set of quasi-free states. To my knowledge, the equivalence between a quasi-free state minimizing the BCS functional and the Gibbs state of the BCS model has not been proved. This means that we cannot rigorously relate the superconducting order in terms of the minimizer of the BCS functional to that in the BCS model. At this point it is natural to consider that the recent papers summarized in [7] feature a well-recognized approach to the BCS theory, rather than analysis of the BCS model Hamiltonian itself. As for the BCS model Hamiltonian, it is known that its eigenstates can be constructed by using solutions to a system of nonlinear equations called Richardson's equations ([17], [18]). Nowadays Richardson's method is formulated within the framework of algebraic Bethe ansatz (see e.g. [19], [20]). Though there are many applications of this approach, Richardson's equations in principle need to be solved numerically. It seems that it has not been applied to rigorously prove existence of superconducting order in the form of finite-temperature correlation functions in the BCS model.

In our previous work [12] we studied the reduced BCS model, where the quartic interacting term is a product of the Cooper pair operators, at positive temperature by extending the external magnetic field to be purely imaginary. We reached the conclusion that under the imaginary magnetic field the BCS model is mathematically analyzable at positive temperatures and especially the superconducting phase characterized by spontaneous $U(1)$-symmetry breaking (SSB) and off-diagonal long range order (ODLRO) can be proven. Let us remark that the BCS model with the imaginary magnetic field is not Hermitian and thus it does not a priori define the Gibbs state. At present it seems that this model is not analyzable within the methods of

[7], [3] based on the Gibbs variational principle. One serious constraint in the previous work [12] is that the possible magnitude of the reduced BCS interaction heavily depends on the imaginary magnetic field and the temperature. In our previous construction, the closer the imaginary magnetic field is to the critical values or the lower the temperature is, the smaller the magnitude of the interaction must be. We have already mentioned in the introduction of [12] that the temperature-dependency of the allowed magnitude of the interaction should be improved by a multi-scale infrared integration. In line with this purpose, here we develop a theory where the magnitude of the interaction is allowed to be largely independent of the temperature and the imaginary magnetic field.

More precisely, in this paper we consider the reduced BCS model interacting with the imaginary magnetic field at positive temperature and prove the existence of SSB and ODLRO in the form of the infinite-volume limit of the thermal expectations over the full Fermionic Fock space under periodic boundary conditions. The magnitude of the attractive interaction must be small. However, the imaginary magnetic field and the temperature can take almost every value of a low temperature region of the parameter space without lowering the magnitude of the interaction. In order to substantially enlarge the possible parameter region, we need to impose restrictive assumptions on the free dispersion relation. Here, unlike in our previous paper, we construct the theory by assuming a series of conditions on the generalized free dispersion relation. These conditions are typically satisfied by a free dispersion relation with degenerate Fermi surface. Examples of the free Hamiltonian covered by our theory are the free electron model of nearest-neighbor hopping on 3 or 4-dimensional (hyper-)cubic lattice with a critical chemical potential or the free electron model of nearest-neighbor hopping on the honeycomb lattice with zero chemical potential. The free Hamiltonians with non-degenerate Fermi surface treated in [12] do not belong to the model class of this paper. See Remark 1.20 for a mathematical confirmation of this fact. As a new observation, we show that for a fixed small coupling constant and a non-zero imaginary magnetic field the SSB and the ODLRO occur in arbitrarily low temperatures. However, it turns out that the SSB and the ODLRO are not present in the zero-temperature, infinite-volume limit of the thermal expectations. Moreover, the zero-temperature limit of the free energy density is proved to be equal to that of the free electron model, which does not depend on either the coupling constant or the imaginary magnetic field. In terms of the superconducting order, the zero-temperature limits derived as a corollary of the main results at positive temperature seem plain and negative. However, if we think of the fact that the superconducting order exists in arbitrarily low temperatures, the whole scenario of the phase transitions in this system is unusual and counterintuitive. In Section 2 we study the nature of the phase transitions by focusing on the free energy density characterized in the main theorem and under a couple of reasonable additional assumptions on the free dispersion relation we prove that the phase transitions are of second order.

Though our free Hamiltonian is qualitatively different from that of the previous work, the basis of our approach is same. We formulate the grand canonical partition function into a time-continuum limit of finite-dimensional Grassmann Gaussian integration and perform mathematical analysis of the Grassmann integral formulation. Moreover we apply the key proposition [12, Proposition 4.16] concerning the uniform convergence of the Grassmann Gaussian integral having the modified interacting term in its action in order to deduce the convergence of the finite-volume thermal expectations to the infinite-volume limits in the final stage of the paper. While the previous analysis of the Grassmann Gaussian integral formulation was completed only by the double-scale integration, here we implement a multi-scale infrared integration with the aim of easing the temperature-dependency of the possible magnitude of the interaction. As in [12], we deal with the ultra-violet part with large Matsubara frequencies by simply applying Pedra-Salmhofer's determinant bound ([16]). Many general tools for the double-scale integration developed in the previous paper are applicable to our multi-scale integration. We need some more estimation tools to complete our scheme. We prepare them in accordance with the previous format of general lemmas. Therefore, from a technical view point of the constructive Fermionic field theory this work is seen as a continuation of the previous construction [12].

We should explain exceptional subsets of the parameter space of the temperature and the imaginary magnetic field where we are unable to construct our theory. If the imaginary magnetic field divided by 2 belongs to the set of Matsubara frequencies, the free covariance is not well-defined. This is because in this case the denominator of the free covariance in momentum space can be zero. As the free covariance is a central object in this approach, we have to exclude these points, which only amount to a 1-dimensional submanifold of the 2-dimensional parameter space. We claim the main results of this paper for the temperature and the imaginary magnetic field belonging to the complement of the union of these subsets. Also, we have to assume a nontrivial dependency of the possible magnitude of the coupling constant on the temperature and the imaginary magnetic field if the temperature is high. This constraint stems from a determinant bound of the full covariance and has no effect if the temperature is low. See Remark 1.7 for details of this constraint.

Taking the zero-temperature limit in interacting many-electron systems is still a challenging problem of mathematical physics. In the preceding examples of taking the zero-temperature limit in the systems with spatial dimension larger than 1 ([6], [5], [10], [11]) not only the degeneracy of the Fermi surface but also symmetries of the whole Hamiltonian are essential. In the infrared analysis of the Grassmann Gaussian integral of the correction term of the reduced BCS interaction, we have an advantage that quadratic Grassmann polynomials are always bounded by the inverse volume factor, which is incomparably smaller than any support size of infrared cut-off. We do not need to use symmetries to keep track of the zero set of the effective dispersion relation, the kernel of the quadratic Grassmann output, during the iterative infrared integration process. We only need a priori information of the infrared properties of the free dispersion relation in order to ensure that Grassmann polynomials of degree ≥ 4 remain bounded in the iterative scale-dependent norm estimations. For the above reason the free Hamiltonian can be chosen much more flexibly in this paper than in the preceding zero-temperature limit constructions based on multi-scale infrared integrations. The relative generality of the free Hamiltonian is one novelty of our low temperature analysis.

Here let us explain more about key ideas of our multi-scale analysis in order to help the readers proceed to the main technical sections and recognize technical novelties of this paper. Let us allow ourselves to use formulas informally and simplified notations in the following for illustrative purposes. As in [12] we begin with the Grassmann Gaussian integral formulation which has the correction term in its exponent.

$$\int e^{V^0(\psi)}d\mu_C(\psi), \tag{1.1}$$

where the Grassmann polynomial $V^0(\psi)$ denotes the correction term and C denotes the full covariance. The full Grassmann integral formulation is officially presented in Lemma 3.6. By using much simpler notations than those actually used in the main body of this paper we can write the correction term $V^0(\psi)$ as follows.

$$\begin{aligned}
&V^0(\psi) = V_s^0(\psi) + V_v^0(\psi),\\
&V_s^0(\psi) := \frac{\gamma}{L}\sum_{x=0}^{L-1}\sum_{t=0}^{n-1}\overline{\psi}_{xt}\psi_{xt},\\
&V_v^0(\psi) := \frac{\gamma}{L}\sum_{x,y=0}^{L-1}\sum_{t,u=0}^{n-1}\left(\delta_{t,u}-\frac{1}{n}\right)\overline{\psi}_{xt}\psi_{xt}\overline{\psi}_{yu}\psi_{yu}.
\end{aligned}$$

Here γ is a real number and L, n are positive integers. We should think of γ, $\{0,1,\cdots,L-1\}$, $\{0,1,\cdots,n-1\}$ as coupling constant, set of spatial lattice points, set of values of discretized imaginary time variable, respectively. In the following we sketch the analysis performed in Subsection 4.3, Subsection 4.4 and Subsection 4.6. A norm of $V_s^0(\psi)$ is bounded by the magnitude of the coupling constant $|\gamma|$ and the inverse volume factor L^{-1}.

$$\|V_s^0\| \le |\gamma|L^{-1}.$$

The norm $\|\cdot\|$ of a Grassmann polynomial is defined by summing its unique anti-symmetric kernel function over all but one variables. More explicitly, the above bound is derived as follows. Writing $\psi_{xt,1}$, $\psi_{xt,-1}$ in place of $\overline{\psi}_{xt}$, ψ_{xt}, respectively, the unique anti-symmetric kernel function of $V_s^0(\psi)$ is that

$$\begin{aligned}
&((x,t,\xi),(y,u,\zeta)) \mapsto \frac{\gamma}{2L}\delta_{x,y}\delta_{t,u}(\delta_{\xi,1}\delta_{\zeta,-1}-\delta_{\xi,-1}\delta_{\zeta,1})\\
&: (\{0,\cdots,L-1\}\times\{0,\cdots,n-1\}\times\{1,-1\})^2 \to \mathbb{C},
\end{aligned}$$

and thus

$$\|V_s^0\| = \sup_{(x,t,\xi)\in\{0,\cdots,L-1\}\times\{0,\cdots,n-1\}\times\{1,-1\}}\ \sum_{(y,u,\zeta)\in\{0,\cdots,L-1\}\times\{0,\cdots,n-1\}\times\{1,-1\}}$$

$$\cdot\left|\frac{\gamma}{2L}\delta_{x,y}\delta_{t,u}(\delta_{\xi,1}\delta_{\zeta,-1}-\delta_{\xi,-1}\delta_{\zeta,1})\right|$$
$$=\frac{1}{2}|\gamma|L^{-1}.$$

Though its norm cannot be bounded by L^{-1}, the Grassmann polynomial $V_v^0(\psi)$ has a particular vanishing property that

$$\int V_v^0(\psi)f(\psi)d\mu_{\hat{C}}(\psi)=0 \tag{1.2}$$

for any Grassmann polynomial $f(\psi)$ and covariance $\hat{C}:(\{0,\cdots,L-1\}\times\{0,\cdots,n-1\})^2\to\mathbb{C}$ satisfying that

(1.3)
$$\hat{C}(xt,yu)=\hat{C}(x0,y0),\quad (\forall x,y\in\{0,\cdots,L-1\},\ u,t\in\{0,\cdots,n-1\}).$$

In fact the equality (1.2) can be confirmed as follows.

$$\int V_v^0(\psi)f(\psi)d\mu_{\hat{C}}(\psi)$$
$$=\frac{\gamma}{L}\sum_{x,y=0}^{L-1}\sum_{t,u=0}^{n-1}\left(\delta_{t,u}-\frac{1}{n}\right)\int\overline{\psi}_{x0}\psi_{x0}\overline{\psi}_{y0}\psi_{y0}f(\psi)d\mu_{\hat{C}}(\psi)=0.$$

By inserting cut-off functions inside the integral over momentum we can write the full covariance as a sum of partial covariances. $C=\sum_{l=0}^{l_{end}}C_l$, where $l_{end}\in\mathbb{Z}_{\le 0}$ denotes the final scale of cut-off and C_l is the covariance containing the cut-off function of l-th scale. We remark that l_{end} is independent of L and proportional to $\log\beta^{-1}$ with the inverse temperature β if the temperature is low, i.e. $\beta\ge 1$. The multi-scale integration iterates as follows.

$$\int e^{V^0(\psi)}d\mu_C(\psi)=\int\int e^{V^0(\psi+\psi')}d\mu_{C_0}(\psi')d\mu_{\sum_{l=-1}^{l_{end}}C_l}(\psi)$$
$$=\int e^{V^{-1}(\psi)}d\mu_{\sum_{l=-1}^{l_{end}}C_l}(\psi)=\int e^{V^m(\psi)}d\mu_{\sum_{l=m}^{l_{end}}C_l}(\psi),$$

where

$$V^m(\psi)=\log\left(\int e^{V^{m+1}(\psi+\psi')}d\mu_{C_{m+1}}(\psi')\right),\ (m=-1,-2,\cdots,l_{end}).$$

At each step of the integration we can decompose the Grassmann polynomial $V^m(\psi)$ into 2 terms. $V^m(\psi) = V_s^m(\psi) + V_v^m(\psi)$, where the norm of $V_s^m(\psi)$ is bounded by L^{-1} and $V_v^m(\psi)$ satisfies the vanishing property (1.2). We can manipulate the support of the cut-off functions and perform a gauge transform so that the final covariance $C_{l_{end}}$, which has the most intense infrared singularity, satisfies (1.3). Thus, by the property (1.2) we reach that

$$\int e^{V^0(\psi)} d\mu_C(\psi) = \int e^{V_s^{l_{end}}(\psi)} d\mu_{C_{l_{end}}}(\psi).$$

The heavy contribution from $C_{l_{end}}$ can be effectively absorbed by the inverse volume factor L^{-1} which bounds the norm of $V_s^{l_{end}}(\psi)$. Also, the factor L^{-1} can be taken smaller than any power of the inverse temperature or l_{end} and thus any extra contribution from these parameters does not lower the possible magnitude of the coupling constant. This is where we take best advantage of the mean-field scaling property and the vanishing property (1.2) that the initial correction term $V^0(\psi)$ has. The integration with the covariances C_l $(l = 0, -1, \cdots, l_{end} + 1)$ is performed in Subsection 4.4 and the integration with the final covariance $C_{l_{end}}$ is specifically performed in Subsection 4.6. As the result the formulation (1.1) is proved to be uniformly bounded with respect to the coupling constant in a good neighborhood of the origin which is independent of the temperature and the imaginary magnetic field. In fact this mechanism was already implemented at the level of double-scale integration in [12], which did not require mathematical induction with the discrete energy scale. In this paper we implement this idea based on the classification of Grassmann polynomials inductively with respect to the scale index of infrared cut-off as described above. We also have to incorporate various scale-dependent bound properties into the classification of Grassmann polynomials. The mathematical justification of the whole inductive procedure is what this paper newly offers in terms of technical aspects.

Let us comment on key differences between this paper and [15] one by one, as both concern analysis of Grassmann integral formulations of BCS type-models. The paper [15] treats a quartic long range interaction which is derived from the reduced BCS interaction by inserting a Kac potential into the time integral. The essential goal of [15] is to ensure the solvability of the BCS gap equation in parameter regions where the correction part obtained after extracting the main reference model can be proved to vanish in the infinite-volume limit. The analysis of the correction part was done on the assumption that the parameter κ determining the range of the inserted Kac potential is bounded from above by some negative power of the coupling constant and the inverse temperature. This assumption does not affect the solvability of the BCS gap equation, since the gap equation is independent of the parameter κ, and thus the goal was achieved. We should add that the solvability of the gap equation is also due to that the free Fermi surface of the model in [15] is non-degenerate. The assumption on κ means that the modified BCS-type interaction depends on temperature and in particular it approaches to the doubly reduced BCS interaction which contains a double time integral, rather than to the original reduced BCS interaction in low temperatures. No multi-scale infrared integration was performed to improve the temperature-dependency of the interacting term. Conceptually this paper aims at completing the same story, though we have the reduced BCS interaction and the imaginary magnetic field from the beginning. We prove the solvability of a gap equation together with the fact that the correction part becomes negligible in the infinite-volume limit. However, we prove the irrelevance of the correction part without assuming that the interaction is temperature-dependent in low temperatures. In order to establish the temperature-independence of the interaction, we perform the multi-scale infrared integration which requires restrictive degeneracy of the free Fermi surface instead. Our gap equation explicitly depends on the imaginary magnetic field and thus admits a positive solution regardless of the degeneracy of the free Fermi surface. In summary, the properties of quartic interaction, the degeneracy of free Fermi surface and the presence of imaginary magnetic field are the key differences between [15] and the present paper. Among them, the temperature-dependency of interaction is considered as the main difference, since it largely affects the design of constructive theory of interacting Fermions.

If we face a question about whether SSB and ODLRO in the BCS model without imaginary magnetic field or in many-electron systems with realistic short range interaction can be proved by extending this paper's method, we realize that there are many essential problems to overcome. This paper's result implies that as long as the same free Hamiltonian is adopted, the BCS model without imaginary magnetic field can be analyzed down to zero temperature by keeping the magnitude of the coupling constant positive. However, we cannot prove that the allowed magnitude of the coupling constant is large enough to ensure the existence of a positive solution to the BCS gap equation and thus cannot prove SSB and ODLRO, either. See Remark 1.10 for a more detailed explanation of this issue. Because of the relatively simple form of the reduced BCS interaction, we can apply the Hubbard-Stratonovich transformation and reformulate the system into a hybrid of Grassmann Gaussian integral and Gaussian integral with a single classical field, where the quartic Grassmann field only appears as a controllable correction term. It is well known that one can also apply the Hubbard-Stratonovich transformation to derive a classical system with many degrees of freedom from the Grassmann integral formulation of a many-electron model with short range interaction. Since infinitely many classical fields come into play in the infinite-volume limit in the standard reformulation of a Hubbard-type short range interaction, it seems at present that its complete solution is beyond the reach of an immediate extension of this paper's methods. Let us remark that equivalence between the minimum configuration of an effective potential for many classical fields whose number can be proportional to the number of finite spatial lattice points and that of an approximate BCS-type potential, which is expressed as a truncated sum over the Matsubara frequencies, for a single classical field was proved in [13]. However, such a partial equivalence has not led to complete characterization of the thermodynamic limit of the original many-electron system with short range interaction, to the author's knowledge. For these reasons, possible new contributions of this paper may not be a construction of necessary steps toward complete solutions of the standard BCS model or realistic many-electron models with short range interaction, but should be a positive proposal for studying these models in a non-standard parameter region of complex plane by means of multi-scale analysis and a construction of its necessary tools. The proposal should make sense if a structurally rich phase transition can be proved as a result.

The outline of this paper is as follows. In the rest of this section we define the model Hamiltonian, state the main theorem concerning the superconducting phase at positive temperature and its corollary about the zero-temperature limit and present concrete examples of the model. In Section 2 we separately analyze the free energy density obtained in the main theorem, draw a schematic phase diagram on the plane of the inverse temperature and the imaginary magnetic field and prove that the phase transitions are of second order. In Section 3 we state the Grassmann Gaussian integral formulations of the grand canonical partition function. In Section 4 we perform the multi-scale infrared integration by assuming scale-dependent bound properties of generalized covariances. In Section 5 first we confirm that the actual covariance introduced as the free 2-point correlation function can be decomposed into a family of scale-dependent covariances satisfying the properties required in the general multi-scale analysis of Section 4. Then we prove the main theorem by applying the results of the general multi-scale analysis and its corollary. In Appendix A we summarize basic lemmas which are used to complete the proof of the main theorem in Section 5. In addition, we present a supplementary list of notations which are newly introduced in this paper or were introduced in the previous paper [12] with some different meaning. The list should be used together with that of [12], since many notations used in this paper are intentionally same or close to those in [12].

1.2. Models and the main results

Let us start by defining our model Hamiltonian. Throughout the paper the spatial dimension is represented by d. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_d$ be a basis of $\mathbb{R}^d$. Let $\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \cdots, \hat{\mathbf{v}}_d$ be vectors of $\mathbb{R}^d$ satisfying that $\langle \mathbf{v}_i, \hat{\mathbf{v}}_j \rangle = \delta_{i,j}$ ($i, j \in \{1, 2, \cdots, d\}$), where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product of $\mathbb{R}^d$. With $L \in \mathbb{N}(= \{1, 2, \cdots\})$ we define the spatial lattice Γ and the momentum lattice Γ^* as follows.

$$\Gamma := \left\{ \sum_{j=1}^{d} m_j \mathbf{v}_j \;\middle|\; m_j \in \{0, 1, \cdots, L-1\} \ (j = 1, 2, \cdots, d) \right\},$$

$$\Gamma^* := \left\{ \sum_{j=1}^{d} \hat{m}_j \hat{\mathbf{v}}_j \;\middle|\; \hat{m}_j \in \left\{0, \frac{2\pi}{L}, \frac{4\pi}{L}, \cdots, 2\pi - \frac{2\pi}{L}\right\} \ (j = 1, 2, \cdots, d) \right\}.$$

In the infinite-volume limit the finite sets Γ, Γ^* are replaced by the infinite sets Γ_∞, Γ_∞^* defined by

$$\Gamma_\infty := \left\{ \sum_{j=1}^{d} m_j \mathbf{v}_j \;\middle|\; m_j \in \mathbb{Z} \ (j = 1, 2, \cdots, d) \right\},$$

$$\Gamma_\infty^* := \left\{ \sum_{j=1}^d \hat{k}_j \hat{\mathbf{v}}_j \;\middle|\; \hat{k}_j \in [0, 2\pi]\ (j = 1, 2, \cdots, d) \right\}.$$

We plan to construct our theory by assuming a series of conditions on the free dispersion relation of the model Hamiltonian. We consider multi-band Hamiltonians since they can have a variety of free dispersion relations. The number of sites in the unit cell is denoted by $b(\in \mathbb{N})$. Set $\mathcal{B} := \{1, 2, \cdots, b\}$. A crystalline lattice having b sites per the unit cell is modeled by $\mathcal{B} \times \Gamma$. We define our b-band Hamiltonian on the Fermionic Fock space $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$. As in [12], we focus on the reduced BCS interaction defined by

$$\mathsf{V} := \frac{U}{L^d} \sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y}) \in \mathcal{B}\times\Gamma} \psi_{\rho\mathbf{x}\uparrow}^* \psi_{\rho\mathbf{x}\downarrow}^* \psi_{\eta\mathbf{y}\downarrow} \psi_{\eta\mathbf{y}\uparrow},$$

where $U(\in \mathbb{R}_{<0})$ is the negative coupling constant. Let us define the map $r_L : \Gamma_\infty \to \Gamma$ by

$$r_L\left(\sum_{j=1}^d m_j \mathbf{v}_j\right) := \sum_{j=1}^d m_j' \mathbf{v}_j,$$

where $m_j \in \mathbb{Z}$, $m_j' \in \{0, 1, \cdots, L-1\}$, $m_j = m_j' \pmod{L}$ $(\forall j \in \{1, 2, \cdots, d\})$. Throughout the paper we assume periodic boundary conditions so that for any $\mathbf{x} \in \Gamma_\infty$, $\psi_{\rho\mathbf{x}\sigma}^{(*)}$ is identified with $\psi_{\rho r_L(\mathbf{x})\sigma}^{(*)}$. We define the free Hamiltonian by giving a generalized hopping matrix. For $n \in \mathbb{N}$ let $\mathrm{Mat}(n, \mathbb{C})$ denote the set of all $n \times n$ complex matrices. For $A \in \mathrm{Mat}(n, \mathbb{C})$ let

$$\|A\|_{n\times n} := \sup_{\substack{\mathbf{v}\in\mathbb{C}^n \\ \text{with } \|\mathbf{v}\|_{\mathbb{C}^n}=1}} \|A\mathbf{v}\|_{\mathbb{C}^n},$$

where $\|\cdot\|_{\mathbb{C}^n}$ is the norm of $\mathbb{C}^n$ induced by the canonical Hermitian inner product. $\mathrm{Mat}(n, \mathbb{C})$ is a Banach space with the norm $\|\cdot\|_{n\times n}$. For any proposition P let $1_P := 1$ if P is true, 0 otherwise. We assume that the matrix-valued function $E : \mathbb{R}^d \to \mathrm{Mat}(b, \mathbb{C})$ satisfies the following conditions.

$$E \in C^\infty(\mathbb{R}^d, \mathrm{Mat}(b, \mathbb{C})),$$

$$E(\mathbf{k}) = E(\mathbf{k})^*, \ (\forall \mathbf{k} \in \mathbb{R}^d), \tag{1.4}$$

$$E(\mathbf{k}+2\pi\hat{\mathbf{v}}_j)=E(\mathbf{k}),\ (\forall\mathbf{k}\in\mathbb{R}^d,\ j\in\{1,2,\cdots,d\}),$$
$$E(\mathbf{k})=\overline{E(-\mathbf{k})},\ (\forall\mathbf{k}\in\mathbb{R}^d). \tag{1.5}$$

Moreover, there exist a function $e:\mathbb{R}^d\to\mathbb{R}_{\ge 0}$ and the constants $\mathsf{c}\in\mathbb{R}_{\ge 1}$, $\mathsf{n}_j\in\mathbb{N}$ $(j=1,2,\cdots,d)$, $\mathsf{a}\in\mathbb{R}_{>1}$ such that

- $$e(\mathbf{k})\le\inf_{\substack{\mathbf{v}\in\mathbb{C}^b\\ \text{with }\|\mathbf{v}\|_{\mathbb{C}^b}=1}}\|E(\mathbf{k})\mathbf{v}\|_{\mathbb{C}^b}\le\mathsf{c}e(\mathbf{k}),\ (\forall\mathbf{k}\in\mathbb{R}^d), \tag{1.6}$$
- $$\sup_{\mathbf{k}\in\mathbb{R}^d}e(\mathbf{k})\le\mathsf{c}, \tag{1.7}$$
- $$e\in C(\mathbb{R}^d,\mathbb{R}),\quad e^2\in C^\infty(\mathbb{R}^d,\mathbb{R}),$$
  $$e(\mathbf{k}+2\pi\hat{\mathbf{v}}_j)=e(\mathbf{k}),\ (\forall\mathbf{k}\in\mathbb{R}^d,\ j\in\{1,2,\cdots,d\}),$$
- $$\left|\left(\frac{\partial}{\partial\hat{k}_j}\right)^n e\left(\sum_{i=1}^d\hat{k}_i\hat{\mathbf{v}}_i\right)^2\right|\le\mathsf{c}\left(1_{n\le 2\mathsf{n}_j}e\left(\sum_{i=1}^d\hat{k}_i\hat{\mathbf{v}}_i\right)^{2-\frac{n}{\mathsf{n}_j}}+1_{2\mathsf{n}_j<n}\right), \tag{1.8}$$
  $$(\forall(\hat{k}_1,\hat{k}_2,\cdots,\hat{k}_d)\in\mathbb{R}^d,\ n\in\{0,1,\cdots,d+2\},\ j\in\{1,2,\cdots,d\}),$$
- (1.9)
  $$\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^n E\left(\sum_{i=1}^d\hat{k}_i\hat{\mathbf{v}}_i\right)\right\|_{b\times b}\le\mathsf{c}\left(1_{n\le\mathsf{n}_j}e\left(\sum_{i=1}^d\hat{k}_i\hat{\mathbf{v}}_i\right)^{1-\frac{n}{\mathsf{n}_j}}+1_{\mathsf{n}_j<n}\right),$$
  $$(\forall(\hat{k}_1,\hat{k}_2,\cdots,\hat{k}_d)\in\mathbb{R}^d,\ n\in\{1,2,\cdots,d+2\},\ j\in\{1,2,\cdots,d\}),$$
- $$\int_{\Gamma_\infty^*}d\mathbf{k}1_{e(\mathbf{k})\le R}\le\mathsf{c}\min\{R^{\mathsf{a}},1\},\ (\forall R\in\mathbb{R}_{>0}), \tag{1.10}$$

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$$\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1_{e(\mathbf{k})\le R}}{e(\mathbf{k})} \le \mathsf{c}\min\{R^{\mathsf{a}-1},1\},\ (\forall R\in\mathbb{R}_{>0}), \tag{1.11}$$

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$$\lim_{\varepsilon\searrow 0}\int_{\Gamma_\infty^*} d\mathbf{k}\frac{1}{e(\mathbf{k})^2+\varepsilon} = \infty. \tag{1.12}$$

Furthermore, we assume the following condition.

$$2\mathsf{a} - 1 - \sum_{j=1}^{d}\frac{1}{\mathsf{n}_j} > 0. \tag{1.13}$$

We define the free part of the Hamiltonian by

$$\mathsf{H}_0 := \frac{1}{L^d}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{\sigma\in\{\uparrow,\downarrow\}}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}E(\mathbf{k})(\rho,\eta)\psi_{\rho\mathbf{x}\sigma}^*\psi_{\eta\mathbf{y}\sigma}.$$

By the condition (1.4) H_0 is self-adjoint. The Hamiltonian H is defined by $\mathsf{H} := \mathsf{H}_0 + \mathsf{V}$, which is a self-adjoint operator on $F_f(L^2(\mathcal{B}\times\Gamma\times\{\uparrow,\downarrow\}))$. Because of the form of the interaction and the generality of the hopping matrix, we can consider that H represents a class of the reduced BCS model. As in [12], we analyze the system under the influence of imaginary magnetic field. Let S_z be the z-component of the spin operator, which is defined by

$$\mathsf{S}_z := \frac{1}{2}\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}(\psi_{\rho\mathbf{x}\uparrow}^*\psi_{\rho\mathbf{x}\uparrow} - \psi_{\rho\mathbf{x}\downarrow}^*\psi_{\rho\mathbf{x}\downarrow}).$$

With the parameter $\theta(\in\mathbb{R})$ we add the operator $i\theta\mathsf{S}_z$, which we formally consider as the interacting term with the imaginary magnetic field, to the Hamiltonian H and study the existence or non-existence of SSB and ODLRO in the infinite-volume limit of the thermal averages. To study SSB, we introduce the symmetry breaking external field F by

$$\mathsf{F} := \gamma\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}(\psi_{\rho\mathbf{x}\uparrow}^*\psi_{\rho\mathbf{x}\downarrow}^* + \psi_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow}),\ (\gamma\in\mathbb{R}).$$

Since the operator $\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}$ is not Hermitian, it is nontrivial that the partition function and the thermal expectations of our interest are real-valued. We should confirm these basic properties at this stage.

LEMMA 1.1. *For any* $\hat{\rho},\hat{\eta}\in\mathcal{B}$, $\hat{\mathbf{x}},\hat{\mathbf{y}}\in\Gamma_\infty$,

$$\begin{aligned}&\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})},\ \mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}),\\ &\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})\in\mathbb{R}\end{aligned}$$

and

$$\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})=\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow}).$$

PROOF. Observe that

$$\begin{aligned}&\mathrm{Tr}\, e^{-\beta(\mathsf{H}-i\theta\mathsf{S}_z+\mathsf{F})}=\overline{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}},\\ &\mathrm{Tr}(e^{-\beta(\mathsf{H}-i\theta\mathsf{S}_z+\mathsf{F})}\mathcal{O}^*)=\overline{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathcal{O})},\\ &\mathrm{Tr}(e^{-\beta(\mathsf{H}-i\theta\mathsf{S}_z+\mathsf{F})}\mathcal{O})=\overline{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathcal{O})},\\ &(\forall\mathcal{O}\in\{\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow},\ \psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow},\ \psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow}\}).\end{aligned}\tag{1.14}$$

To derive the third equality, one can use the property (1.5) and the periodicity of $E(\cdot)$. On the other hand, by using the transforms

$$\begin{aligned}&(\psi_{\rho\mathbf{x}\sigma},\psi^*_{\rho\mathbf{x}\sigma})\to(\psi_{\rho\mathbf{x}-\sigma},\psi^*_{\rho\mathbf{x}-\sigma}),\\ &(\psi_{\rho\mathbf{x}\sigma},\psi^*_{\rho\mathbf{x}\sigma})\to(-i\psi_{\rho\mathbf{x}\sigma},i\psi^*_{\rho\mathbf{x}\sigma}),\quad((\rho,\mathbf{x},\sigma)\in\mathcal{B}\times\Gamma\times\{\uparrow,\downarrow\})\end{aligned}$$

in this order we can show that

$$\begin{aligned}&\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}=\mathrm{Tr}\, e^{-\beta(\mathsf{H}-i\theta\mathsf{S}_z+\mathsf{F})},\\ &\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathcal{O})=\mathrm{Tr}(e^{-\beta(\mathsf{H}-i\theta\mathsf{S}_z+\mathsf{F})}\mathcal{O}),\\ &(\forall\mathcal{O}\in\{\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow},\ \psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow},\ \psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow}\}).\end{aligned}\tag{1.15}$$

The claims follow from (1.14) and (1.15). □

To state the main theorem, let us fix some notational conventions, which will be used throughout the paper. For $\mathbf{k}\in\mathbb{R}^d$ let $e_j(\mathbf{k})$ $(j=1,2,\cdots,b')$

be the eigenvalues of $E(\mathbf{k})$ satisfying $e_1(\mathbf{k}) > e_2(\mathbf{k}) > \cdots > e_{b'}(\mathbf{k})$. With the projection matrix $P_j(\mathbf{k})$ corresponding to the eigenvalue $e_j(\mathbf{k})$ ($j = 1, 2, \cdots, b'$) the spectral decomposition of $E(\mathbf{k})$ is that

$$E(\mathbf{k}) = \sum_{j=1}^{b'} e_j(\mathbf{k}) P_j(\mathbf{k}). \tag{1.16}$$

For any function $f : \mathbb{R} \to \mathbb{C}$ we define $f(E(\mathbf{k})) \in \mathrm{Mat}(b, \mathbb{C})$ by

$$f(E(\mathbf{k})) := \sum_{j=1}^{b'} f(e_j(\mathbf{k})) P_j(\mathbf{k}).$$

It is important in our applications that for $f \in C(\mathbb{R}, \mathbb{C})$ the function $\mathbf{k} \mapsto \mathrm{Tr}\, f(E(\mathbf{k})) : \mathbb{R}^d \to \mathbb{C}$ is continuous. This is essentially because the roots of the characteristic polynomial of $E(\mathbf{k})$ continuously depend on $\mathbf{k}$. Rouché's theorem ensures this fact.

The statements of our main theorem involve a solution to our gap equation. Let us confirm the unique solvability of our gap equation, which is written by the above convention. We admit that for any $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$, $x + \infty = \infty > x$, $y \times \infty = \infty$, $\infty^{-1} = 0$ and

$$\int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr} \left(\frac{\sinh(\beta |E(\mathbf{k})|)}{(-1 + \cosh(\beta E(\mathbf{k}))) |E(\mathbf{k})|} \right) = \infty,$$

which is consistent with the conditions (1.6), (1.12). In fact (1.6), (1.12) imply that

$$\lim_{\varepsilon \searrow 0} \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr} \left(\frac{\sinh(\beta |E(\mathbf{k})|)}{(\varepsilon - 1 + \cosh(\beta E(\mathbf{k}))) |E(\mathbf{k})|} \right) = \infty.$$

Set

$$D_d := |\det(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \cdots, \hat{\mathbf{v}}_d)|^{-1} (2\pi)^{-d}.$$

Lemma 1.2. *Let $U \in \mathbb{R}_{<0}$, $\beta \in \mathbb{R}_{>0}$, $\theta \in \mathbb{R}$. Then the following statements hold true. The equation*

(1.17)

$$-\frac{2}{|U|}$$

$$+ D_d \int_{\Gamma^*_\infty} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2})}{(\cos(\beta\theta/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2}))\sqrt{E(\mathbf{k})^2+\Delta^2}}\right)$$
$$= 0$$

has a solution Δ *in* $[0,\infty)$ *if and only if*

$$-\frac{2}{|U|} + D_d \int_{\Gamma^*_\infty} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta|E(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta E(\mathbf{k})))|E(\mathbf{k})|}\right) \ge 0.$$

Moreover, if a solution exists, it is unique.

Proof. Observe that the functions

$$x \mapsto \frac{\sinh x}{(\varepsilon+\cosh x)x} : [0,\infty) \to \mathbb{R},\ (\varepsilon \in (-1,1]),$$
$$x \mapsto \frac{\sinh x}{(-1+\cosh x)x} : (0,\infty) \to \mathbb{R}$$

are strictly monotone decreasing and converge to 0 as $x \to \infty$. See e.g. [12, Lemma 4.19] for hints of the proof. Thus the left-hand side of (1.17) is strictly monotone decreasing with Δ as the map from $\mathbb{R}_{\ge 0}$ to $\mathbb{R} \cup \{\infty\}$ and converges to $-2/|U|$ as $\Delta \to \infty$. Moreover, it is continuous with Δ as a real-valued function in $\mathbb{R}_{\ge 0}$ if $\cos(\beta\theta/2) \ne -1$, or in $\mathbb{R}_{>0}$ if $\cos(\beta\theta/2) = -1$. Furthermore, by (1.6) and (1.12)

$$\lim_{\Delta \searrow 0} \int_{\Gamma^*_\infty} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2})}{(-1+\cosh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2}))\sqrt{E(\mathbf{k})^2+\Delta^2}}\right) = \infty.$$

By using these facts we can deduce the claim. □

For a function $f : \Gamma_\infty \times \Gamma_\infty \to \mathbb{C}$ and $a \in \mathbb{C}$ we write $\lim_{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d}\to\infty} \cdot f(\mathbf{x},\mathbf{y}) = a$ if for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ such that $\|f(\mathbf{x},\mathbf{y})-a\|_{\mathbb{C}} < \varepsilon$ for any $\mathbf{x},\mathbf{y} \in \Gamma_\infty$ satisfying $\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d} > \delta$. Here $\|\cdot\|_{\mathbb{R}^d}$ denotes the Euclidean norm of $\mathbb{R}^d$.

For a sequence $(s_n)_{n=n_0}^{\infty}$ and an element s of a normed space with the norm $|||\cdot|||$ we write $\lim_{n\to\infty, n\in\mathbb{N}} s_n = s$ if for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $m \in \mathbb{N}$ such that $|||s_n - s||| < \varepsilon$ for any $n \in \mathbb{N}$ satisfying $n \ge m$. The point of this convention is that we write $\lim_{n\to\infty, n\in\mathbb{N}} s_n$ even if $s_1, s_2, \cdots, s_{n_0-1}$

are undefined. We use this convention especially when we consider the infinite-volume limit $L \to \infty$.

Our main result is stated as follows.

THEOREM 1.3. *We let $\Delta(\in \mathbb{R}_{\ge 0})$ be the solution to* (1.17) *if*

$$-\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\beta|E(\mathbf{k})|)}{(\cos(\beta\theta/2) + \cosh(\beta E(\mathbf{k})))|E(\mathbf{k})|}\right) \ge 0.$$

We let $\Delta := 0$ if

$$-\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\left(\frac{\sinh(\beta|E(\mathbf{k})|)}{(\cos(\beta\theta/2) + \cosh(\beta E(\mathbf{k})))|E(\mathbf{k})|}\right) < 0.$$

Then there exists a positive constant c_1 depending only on $d, b, (\hat{\mathbf{v}}_j)_{j=1}^d, \mathsf{a}$, $(\mathsf{n}_j)_{j=1}^d, \mathsf{c}$ such that the following statements hold for any $\beta \in \mathbb{R}_{>0}$, $\theta \in \mathbb{R}$ satisfying $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$ and

$$U \in \left(-c_1\left(1_{\beta \ge 1} + 1_{\beta<1} \max\left\{\beta^2, \min_{m\in\mathbb{Z}}\left|\frac{\beta\theta}{2} - \pi(2m+1)\right|^2\right\}\right), 0\right). \tag{1.18}$$

(i) There exists $L_0 \in \mathbb{N}$ such that

$$\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})} \in \mathbb{R}_{>0}, \quad (\forall L \in \mathbb{N} \text{ with } L \ge L_0,\ \gamma \in [0,1]).$$

(ii)

(1.19)

$$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})\right)$$
$$= \frac{\Delta^2}{|U|} - \frac{D_d}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\log\left(2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta E(\mathbf{k})} + e^{\beta(\sqrt{E(\mathbf{k})^2+\Delta^2}-E(\mathbf{k}))} + e^{-\beta(\sqrt{E(\mathbf{k})^2+\Delta^2}+E(\mathbf{k}))}\right).$$

(iii)

(1.20)

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}$$

$$=\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}$$

$$=-\frac{\Delta D_d}{2}\int_{\Gamma^*_\infty}d\mathbf{k}\frac{\sinh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2})}{(\cos(\beta\theta/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2}))\sqrt{E(\mathbf{k})^2+\Delta^2}}(\hat{\rho},\hat{\rho}),$$

$(\forall\hat{\rho}\in\mathcal{B},\ \hat{\mathbf{x}}\in\Gamma_\infty)$.

(iv) If

$$|U|\neq\left(\frac{D_d}{2}\int_{\Gamma^*_\infty}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta|E(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta E(\mathbf{k})))|E(\mathbf{k})|}\right)\right)^{-1},$$

(1.21)

$$\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}$$

$$=\Delta^2\prod_{\rho\in\{\hat{\rho},\hat{\eta}\}}$$

$$\cdot\left(\frac{D_d}{2}\int_{\Gamma^*_\infty}d\mathbf{k}\frac{\sinh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2})}{(\cos(\beta\theta/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+\Delta^2}))\sqrt{E(\mathbf{k})^2+\Delta^2}}(\rho,\rho)\right),$$

$(\forall\hat{\rho},\hat{\eta}\in\mathcal{B})$.

If

$$|U|=\left(\frac{D_d}{2}\int_{\Gamma^*_\infty}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta|E(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta E(\mathbf{k})))|E(\mathbf{k})|}\right)\right)^{-1},$$

$$\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty}\limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left|\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}\right|=0,\quad(\forall\hat{\rho},\hat{\eta}\in\mathcal{B}).$$

(v)

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{L^{2d}}\sum_{(\hat{\rho},\hat{\mathbf{x}}),(\hat{\eta},\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}=\frac{\Delta^2}{U^2}.$$

(vi) There exists $\delta\in\mathbb{R}_{>0}$ *such that if* $\min_{m\in\mathbb{Z}}|\beta\theta/2-\pi(2m+1)|<\delta$*, then*

$$-\frac{2}{|U|}+D_d\int_{\Gamma^*_\infty}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(\beta|E(\mathbf{k})|)}{(\cos(\beta\theta/2)+\cosh(\beta E(\mathbf{k})))|E(\mathbf{k})|}\right)>0$$

and $\Delta>0$*.*

In the rest of the paper except Section 2 we always assume that

$$\frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1).$$

This is because the free partition function can vanish if $\beta\theta/2\in\pi(2\mathbb{Z}+1)$ and thus we are unable to define the free covariance, which is indispensable for our construction. Only in Section 2 we lift this condition.

For $(x,y,z)\in\mathbb{R}_{>0}\times\mathbb{R}\times\mathbb{R}$ with $xy/2\notin\pi(2\mathbb{Z}+1)$ we define the matrix-valued function $G_{x,y,z}:\mathbb{R}^d\to\mathrm{Mat}(b,\mathbb{C})$ by

$$G_{x,y,z}(\mathbf{k}):=\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{(\cos(xy/2)+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))\sqrt{E(\mathbf{k})^2+z^2}}. \tag{1.22}$$

This notation helps to shorten formulas in subsequent arguments. We can prove the claim (vi) here. There uniquely exists $m_0\in\mathbb{Z}$ such that $\beta\theta/(2\pi)\in[2m_0,2m_0+2)$ and $\min_{m\in\mathbb{Z}}|\beta\theta/2-\pi(2m+1)|=|\beta\theta/2-\pi(2m_0+1)|$. By (1.6) and (1.7),

$$\begin{aligned}&\int_{\Gamma^*_\infty}d\mathbf{k}\,\mathrm{Tr}\,G_{\beta,\theta,0}(\mathbf{k})\\ &\geq\beta\int_{\Gamma^*_\infty}d\mathbf{k}\frac{1}{\cos(\beta\theta/2)+\cosh(\beta\mathsf{c}e(\mathbf{k}))}\\ &\geq\beta\int_{\Gamma^*_\infty}d\mathbf{k}\frac{1}{e^{\beta\mathsf{c}^2}\beta^2\mathsf{c}^2e(\mathbf{k})^2+1-\cos(|\beta\theta/2-\pi(2m_0+1)|)}\end{aligned}$$

$$\geq \frac{\beta}{\max\{1, e^{\beta \mathsf{c}^2}\beta^2\mathsf{c}^2\}} \int_{\Gamma_\infty^*} d\mathbf{k} \frac{1}{e(\mathbf{k})^2 + |\beta\theta/2 - \pi(2m_0+1)|^2}.$$

By (1.12) there exists $\delta \in \mathbb{R}_{>0}$ such that if $|\beta\theta/2 - \pi(2m_0+1)| < \delta$, the right-hand side of the above inequality is larger than $2D_d^{-1}|U|^{-1}$. Then Lemma 1.2 implies that $\Delta > 0$.

REMARK 1.4. The claim (i) ensures the well-definedness of the free energy density and the thermal expectations for $L \in \mathbb{N}$ with $L \geq L_0$. By following the above-mentioned convention we write $\lim_{L\to\infty, L\in\mathbb{N}}$ in Theorem 1.3, though these objects are not defined for $L \in \{1, 2, \cdots, L_0 - 1\}$.

REMARK 1.5. For any $\Delta \in \mathbb{R}_{\geq 0}$, $\hat{\rho} \in \mathcal{B}$,

$$\begin{aligned}
&\int_{\Gamma_\infty^*} d\mathbf{k} G_{\beta,\theta,\Delta}(\mathbf{k})(\hat{\rho}, \hat{\rho}) \\
&\geq D_d^{-1} \sinh\left(\beta\sqrt{\sup_{\mathbf{k}\in\mathbb{R}^d} \|E(\mathbf{k})\|_{b\times b}^2 + \Delta^2}\right) \\
&\quad \cdot \left(\cos(\beta\theta/2) + \cosh\left(\beta\sqrt{\sup_{\mathbf{k}\in\mathbb{R}^d} \|E(\mathbf{k})\|_{b\times b}^2 + \Delta^2}\right)\right)^{-1} \\
&\quad \cdot \left(\sqrt{\sup_{\mathbf{k}\in\mathbb{R}^d} \|E(\mathbf{k})\|_{b\times b}^2 + \Delta^2}\right)^{-1} \\
&> 0.
\end{aligned}$$

From this estimate we can see that the theorem implies the occurrence of SSB and ODLRO in the case

$$-\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k} \,\mathrm{Tr}\, G_{\beta,\theta,0}(\mathbf{k}) > 0.$$

REMARK 1.6. If $\theta \neq 0$, for any β_0, $\delta \in \mathbb{R}_{>0}$ there exists $\beta \in [\beta_0, \infty)$ such that $0 < \min_{m\in\mathbb{Z}} |\beta\theta/2 - \pi(2m+1)| < \delta$. Thus we can read from the claim (vi) that if $\theta \neq 0$, the SSB and the ODLRO occur in arbitrarily low temperatures.

REMARK 1.7. The β-dependency in the case $\beta < 1$ in (1.18) stems from a determinant bound on the full covariance, which is essentially governed by the integral

$$\frac{1}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\frac{1}{e(\mathbf{k})+\min_{m\in\mathbb{Z}}|\theta/2-\pi(2m+1)/\beta|} \tag{1.23}$$

if $\beta < 1$. See the proof of Lemma 5.7 (i). In fact a lower bound of the term

$$\left(\frac{1}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\frac{1}{e(\mathbf{k})+\min_{m\in\mathbb{Z}}|\theta/2-\pi(2m+1)/\beta|}\right)^{-2}$$

leads to the β-dependency in (1.18). If $\theta = 0$ (mod $4\pi/\beta$), the term (1.23) is bounded by a β-independent constant and the determinant bound on the full covariance becomes independent of β as usual. Thus we can explain that the nontrivial β-dependency in (1.18) is caused by the insertion of the imaginary magnetic field.

REMARK 1.8. For $\phi \in \mathbb{C}$ let us define the operator $\mathsf{F}(\phi)$ by

$$\mathsf{F}(\phi) := \sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}(\phi\psi^*_{\rho\mathbf{x}\uparrow}\psi^*_{\rho\mathbf{x}\downarrow}+\overline{\phi}\psi_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow}).$$

Take any $\hat{\mathbf{x}} \in \Gamma_\infty$, $\hat{\rho} \in \mathcal{B}$, $\xi \in \mathbb{R}$. It follows from the claim (iii) and the gauge transform $\psi_X^* \to e^{-i\frac{\xi}{2}}\psi_X^*$, $\psi_X \to e^{i\frac{\xi}{2}}\psi_X$ $(X \in \mathcal{B}\times\Gamma\times\{\uparrow,\downarrow\})$ that

$$\begin{aligned}
&\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\gamma e^{i\xi}))}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\gamma e^{i\xi}))}}\\
&=-\frac{e^{-i\xi}\Delta D_d}{2}\int_{\Gamma_\infty^*}d\mathbf{k}G_{\beta,\theta,\Delta}(\mathbf{k})(\hat{\rho},\hat{\rho}),\\
&\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\gamma e^{i\xi}))}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\gamma e^{i\xi}))}}\\
&=-\frac{e^{i\xi}\Delta D_d}{2}\int_{\Gamma_\infty^*}d\mathbf{k}G_{\beta,\theta,\Delta}(\mathbf{k})(\hat{\rho},\hat{\rho}).
\end{aligned}$$

These convergent properties imply that the limit

$$\lim_{\substack{\phi\to 0\\ \phi\in\mathbb{C}\backslash\{0\}}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}},$$

$$\lim_{\substack{\phi\to 0\\ \phi\in\mathbb{C}\backslash\{0\}}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}}$$

do not exist when $\Delta > 0$. However,

$$\begin{aligned}&\lim_{\substack{\phi\to 0\\ \phi\in\mathbb{C}\backslash\{0\}}}\left|\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}}\right|\\ &=\lim_{\substack{\phi\to 0\\ \phi\in\mathbb{C}\backslash\{0\}}}\left|\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F}(\phi))}}\right|\\ &=\frac{\Delta D_d}{2}\int_{\Gamma^*_\infty}d\mathbf{k}G_{\beta,\theta,\Delta}(\mathbf{k})(\hat{\rho},\hat{\rho}).\end{aligned}$$

REMARK 1.9. The claim (iv) does not imply the convergence of

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}$$

in the case

$$|U|=\left(\frac{D_d}{2}\int_{\Gamma^*_\infty}d\mathbf{k}\,\mathrm{Tr}\,G_{\beta,\theta,0}(\mathbf{k})\right)^{-1}. \tag{1.24}$$

In fact in this case we cannot prove the convergence of the finite-volume 4-point correlation function as $L\to\infty$. We can prove that the global maximum point of the function

$$\begin{aligned}x\mapsto -\frac{x^2}{|U|}&+\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh(\beta\sqrt{E(\mathbf{k})^2+x^2})\right)\\ &-\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta}{2}\right)+\cosh(\beta E(\mathbf{k}))\right):\mathbb{R}_{\geq 0}\to\mathbb{R}\end{aligned}$$

converges to 0 as $L \to \infty$. According to the proof of the theorem in Subsection 5.2 and Lemma A.2 in Appendix A, we must have more detailed information about how the maximum point and derivatives of the function at the maximum point converge as $L \to \infty$ to complete the proof. We are unable to extract the necessary information from our assumptions on $E(\cdot)$. On the contrary, the theorem guarantees the convergence of the thermal expectations

$$\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}},$$
$$\frac{1}{L^{2d}}\sum_{\substack{(\hat{\rho},\hat{\mathbf{x}}),(\hat{\eta},\hat{\mathbf{y}})\\ \in\mathcal{B}\times\Gamma}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}$$

and the free energy density

$$-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})$$

as $L \to \infty$ as long as $\beta(\in \mathbb{R}_{>0})$, $\theta(\in \mathbb{R})$ satisfies $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$ and U satisfies (1.18), whether (1.24) holds or not.

REMARK 1.10. As we can see from the claim (vi), the non-zero imaginary magnetic field is crucial to ensure the existence of a positive solution to the gap equation (1.17) for any small coupling constant and accordingly the existence of SSB and ODLRO in this regime. One natural question we face is whether we can prove SSB and ODLRO when the imaginary magnetic field is switched off. To find an answer to this question, let us examine the solvability of the gap equation when $\theta = 0$. By (1.6), (1.7), (1.11)

$$\int_{\Gamma^*_\infty} d\mathbf{k}\,\mathrm{Tr}\, G_{\beta,0,0}(\mathbf{k}) \le \int_{\Gamma^*_\infty} d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{|E(\mathbf{k})|}\right) \le b\int_{\Gamma^*_\infty} d\mathbf{k}\frac{1}{e(\mathbf{k})}$$
$$\le b\mathsf{c}\min\{\mathsf{c}^{\mathsf{a}-1}, 1\}.$$

Thus, a necessary condition for existence of a positive solution Δ to (1.17) with $\theta = 0$ is that

$$|U| > \frac{2}{D_d b\mathsf{c}\min\{\mathsf{c}^{\mathsf{a}-1}, 1\}}. \tag{1.25}$$

We can compute the thermal expectation values for some U independent of β and θ as described in (1.18), which is an advantageous result of the multi-scale integration. However, our multi-scale analysis has no advantage to make the allowed magnitude of U quantitatively explicit, as we need to go through a pile of calculations. Whether the necessary condition (1.25) holds in this regime is highly nontrivial and we cannot give an affirmative answer to the question at present.

Since the upper bound on $|U|$ does not depend on β if $\beta \geq 1$, we can consider the zero-temperature limit $\beta \to \infty$ of the free energy density and the thermal expectations. It turns out that in the weak coupling region where our construction is valid the zero-temperature limit does not exhibit the characteristics of superconductivity.

Corollary 1.11. *There exists a positive constant c_2 depending only on $d, b, (\hat{\mathbf{v}}_j)_{j=1}^d, \mathsf{a}, (\mathsf{n}_j)_{j=1}^d, \mathsf{c}$ such that the following statements hold for any $U \in (-c_2, 0)$, $\theta \in \mathbb{R}$.*

(i)

$$\Delta \leq \frac{1}{\beta}, \quad (\forall \beta \in \mathbb{R}_{\geq 1} \text{ with } \beta\theta/2 \notin \pi(2\mathbb{Z}+1)).$$

(ii)

$$\begin{aligned}
&\lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})\right)\\
&= D_d \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}(E(\mathbf{k}) - |E(\mathbf{k})|)\\
&= \lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{R}_{>0}}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \left(-\frac{1}{\beta L^d}\log(\mathrm{Tr}\, e^{-\beta\mathsf{H}_0})\right).
\end{aligned}$$

(iii)

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}} \lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}$$

$$= \lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}} \lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}$$

$$= \lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}} \lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}$$

$$= \lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}} \lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}} = 0,$$

$(\forall \hat{\rho}\in\mathcal{B},\ \hat{\mathbf{x}}\in\Gamma_\infty)$.

(iv)

$$\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty} \limsup_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}} \limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}} \left| \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}} \right| = 0,$$

$(\forall \hat{\rho},\hat{\eta}\in\mathcal{B})$.

(v)

$$\lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{1}{L^{2d}} \sum_{(\hat{\rho},\hat{\mathbf{x}}),(\hat{\eta},\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}$$
$$= 0.$$

Remark 1.12. Though it does not show the sign of superconductivity, it is interesting that the zero-temperature, infinite-volume limit of the free energy density claimed in (ii) is independent of both the coupling constant and the imaginary magnetic field.

Remark 1.13. Among the assumptions listed in the beginning, the smoothness of $E(\cdot)$, $e(\cdot)^2$ is assumed only for simplicity. In fact we only need to differentiate $E(\cdot)$, $e(\cdot)^2$ finite times depending only on the dimension d. Thus the smoothness condition can be relaxed to be continuous differentiability of certain degree.

Remark 1.14. See Remark 4.6 for the specific reason why we need to assume (1.13). Also, Remark 5.8 explains how we use the condition $\mathsf{a} > 1$.

1.3. Examples

In order to see the applicability of Theorem 1.3 and Corollary 1.11, we should examine which model satisfies the required conditions. We let 'c' denote a generic positive constant independent of any parameter not only in this section but in the rest of the paper. Also, I_n denotes the $n \times n$ unit matrix throughout the paper.

Example 1.15 (Nearest-neighbor hopping on the 3 or 4-dimensional (hyper-)cubic lattice with a critical chemical potential). Let $d = 3$ or 4 and $\{\mathbf{v}_j\}_{j=1}^d$, $\{\hat{\mathbf{v}}_j\}_{j=1}^d$ be the canonical basis of $\mathbb{R}^d$. In this case $\Gamma = \{0, 1, \cdots, L-1\}^d$, $\Gamma^* = \{0, \frac{2\pi}{L}, \cdots, 2\pi - \frac{2\pi}{L}\}^d$, $\Gamma_\infty^* = [0, 2\pi]^d$. Let $b = 1$ and set for $\mathbf{k} = (k_1, k_2, \cdots, k_d) \in \mathbb{R}^d$

$$E(\mathbf{k}) := (-1)^{hop} 2 \sum_{j=1}^{d} \cos k_j - 2d$$

with $hop \in \{0, 1\}$. In this case H_0 describes free electrons hopping to nearest-neighbor sites under the chemical potential $2d$. The role of the fixed parameter hop is to implement the negative and positive hopping at the same time. The applicability of the previous framework to this model was briefly studied in [12, Remark 1.9]. Define the function $e : \mathbb{R}^d \to \mathbb{R}$ by

$$e(\mathbf{k}) := 4 \sum_{j=1}^{d} \sin^2 \left(\frac{k_j}{2} + 1_{hop=1} \frac{\pi}{2} \right).$$

We can check that $e(\mathbf{k}) = |E(\mathbf{k})|$ for any $\mathbf{k} \in \mathbb{R}^d$. It is clear that (1.6), (1.7) hold with some $\mathsf{c}(\in \mathbb{R}_{\geq 1})$ and $e(\cdot)$ satisfies the required regularity and the periodicity. Moreover, the conditions (1.8), (1.9) hold with $\mathsf{n}_j = 2$ ($j = 1, 2, \cdots, d$) and the conditions (1.10), (1.11) hold with $\mathsf{a} = d/2$ and some $\mathsf{c}(\in \mathbb{R}_{\geq 1})$. By considering that $d = 3, 4$ we can confirm that the conditions (1.12), (1.13) hold as well.

Example 1.16 (Nearest-neighbor hopping on the honeycomb lattice). Many-Fermion systems on the honeycomb lattice with nearest neighbor hopping are well studied in a branch of mathematical physics based on Grassmann integral formulations. See e.g. [6]. Let us confirm that the free electron model on the honeycomb lattice with zero chemical potential

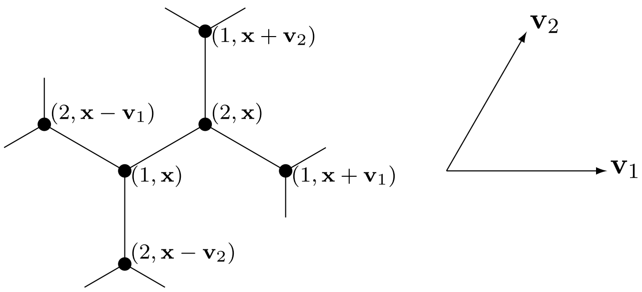

Fig. 1. A portion of the honeycomb lattice linked by the nearest-neighbor hopping.

can be dealt in this framework as the free Hamiltonian. Take the basis $\mathbf{v}_1 = (1,0)^T$, $\mathbf{v}_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})^T$ of $\mathbb{R}^2$. Then, $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$ are uniquely determined as follows. $\hat{\mathbf{v}}_1 = (1, -\frac{1}{\sqrt{3}})^T$, $\hat{\mathbf{v}}_2 = (0, \frac{2}{\sqrt{3}})^T$. The honeycomb lattice with a spatial cut-off is identified with the product set $\{1,2\} \times \Gamma$. The hopping matrix is given with momentum variables by

$$E(\mathbf{k}) := \begin{pmatrix} 0 & 1 + e^{-i\langle \mathbf{v}_1, \mathbf{k}\rangle} + e^{-i\langle \mathbf{v}_2, \mathbf{k}\rangle} \\ 1 + e^{i\langle \mathbf{v}_1, \mathbf{k}\rangle} + e^{i\langle \mathbf{v}_2, \mathbf{k}\rangle} & 0 \end{pmatrix}, \quad \mathbf{k} \in \mathbb{R}^2.$$

See Figure 1 for a portion of the honeycomb lattice linked by the nearest-neighbor hopping.

The eigenvalues of $E(\mathbf{k})$ are $(-1)^\delta |1 + e^{i\langle \mathbf{v}_1, \mathbf{k}\rangle} + e^{i\langle \mathbf{v}_2, \mathbf{k}\rangle}|$, $(\delta \in \{0,1\})$. Let us set $e(\mathbf{k}) := |1 + e^{i\langle \mathbf{v}_1, \mathbf{k}\rangle} + e^{i\langle \mathbf{v}_2, \mathbf{k}\rangle}|$. The validity of the inequalities (1.6), (1.7) and the regularity and the periodicity are clear. One can directly prove that (1.8), (1.9) hold with $\mathsf{n}_1 = \mathsf{n}_2 = 1$. Observe that $e(\mathbf{k}) = 0$ and $\mathbf{k} \in \Gamma_\infty^*$ if and only if $\mathbf{k} = \frac{2\pi}{3}\hat{\mathbf{v}}_1 + \frac{4\pi}{3}\hat{\mathbf{v}}_2$ or $\mathbf{k} = \frac{4\pi}{3}\hat{\mathbf{v}}_1 + \frac{2\pi}{3}\hat{\mathbf{v}}_2$. By making use of the expansions

$$\begin{aligned} &1 + \cos x + \cos y \\ &= -\frac{\sqrt{3}}{2}\left(x - \frac{2}{3}\pi\right) + \frac{\sqrt{3}}{2}\left(y - \frac{4}{3}\pi\right) \\ &\quad + \sum_{n=2}^{\infty} \frac{1}{n!}\left(\cos^{(n)}\left(\frac{2}{3}\pi\right)\left(x - \frac{2}{3}\pi\right)^n + \cos^{(n)}\left(\frac{4}{3}\pi\right)\left(y - \frac{4}{3}\pi\right)^n\right), \\ &\sin x + \sin y \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left(x - \frac{2}{3}\pi\right) - \frac{1}{2}\left(y - \frac{4}{3}\pi\right) \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{n!}\left(\sin^{(n)}\left(\frac{2}{3}\pi\right)\left(x - \frac{2}{3}\pi\right)^n + \sin^{(n)}\left(\frac{4}{3}\pi\right)\left(y - \frac{4}{3}\pi\right)^n\right), \\
&(x, y \in \mathbb{R}),
\end{aligned}
$$

we can prove that there exist constants δ_1, δ_2, $\delta_3 \in \mathbb{R}_{>0}$ such that for any $\hat{k}_1, \hat{k}_2 \in [0, 2\pi]$,

$$
\begin{aligned}
e(\hat{k}_1\hat{\mathbf{v}}_1 + \hat{k}_2\hat{\mathbf{v}}_2) \le \delta_1 \min\Bigg\{ &\left(\left(\hat{k}_1 - \frac{2}{3}\pi\right)^2 + \left(\hat{k}_2 - \frac{4}{3}\pi\right)^2\right)^{1/2}, \\
&\left(\left(\hat{k}_1 - \frac{4}{3}\pi\right)^2 + \left(\hat{k}_2 - \frac{2}{3}\pi\right)^2\right)^{1/2} \Bigg\}
\end{aligned} \tag{1.26}
$$

and if $e(\hat{k}_1\hat{\mathbf{v}}_1 + \hat{k}_2\hat{\mathbf{v}}_2) \le \delta_2$,

$$
\begin{aligned}
e(\hat{k}_1\hat{\mathbf{v}}_1 + \hat{k}_2\hat{\mathbf{v}}_2) \ge \delta_3 \min\Bigg\{ &\left(\left(\hat{k}_1 - \frac{2}{3}\pi\right)^2 + \left(\hat{k}_2 - \frac{4}{3}\pi\right)^2\right)^{1/2}, \\
&\left(\left(\hat{k}_1 - \frac{4}{3}\pi\right)^2 + \left(\hat{k}_2 - \frac{2}{3}\pi\right)^2\right)^{1/2} \Bigg\}.
\end{aligned} \tag{1.27}
$$

We can apply these properties to prove that the conditions (1.10), (1.11), (1.12), (1.13) hold with $\mathsf{a} = 2$, $d = 2$, $\mathsf{n}_1 = \mathsf{n}_2 = 1$.

Example 1.17 (Hopping on the square littice with additional sites). To demonstrate the applicability of the multi-band formulation, let us consider a model on the square lattice with additional lattice points. The basis $\mathbf{v}_1, \mathbf{v}_2$ are equal to the canonical basis $\mathbf{e}_1, \mathbf{e}_2$ of $\mathbb{R}^2$. The lattice of our interest is identified with $\{1, 2, 3, 4, 5, 6\} \times \Gamma$. So we are going to construct a 6-band model. We define the hopping matrix with momentum variables by

$$
E(\mathbf{k}) := \begin{pmatrix} 0 & E_0(\mathbf{k}) \\ E_0(\mathbf{k})^* & 0 \end{pmatrix}, \quad E_0(\mathbf{k}) := \begin{pmatrix} 2 & 1 + e^{ik_2} & 1 + e^{-ik_1} \\ 0 & 1 + e^{ik_1} & 1 + e^{-ik_2} \\ 1 & 0 & 1 + e^{-ik_1} \end{pmatrix},
$$

$\mathbf{k} \in \mathbb{R}^2$.

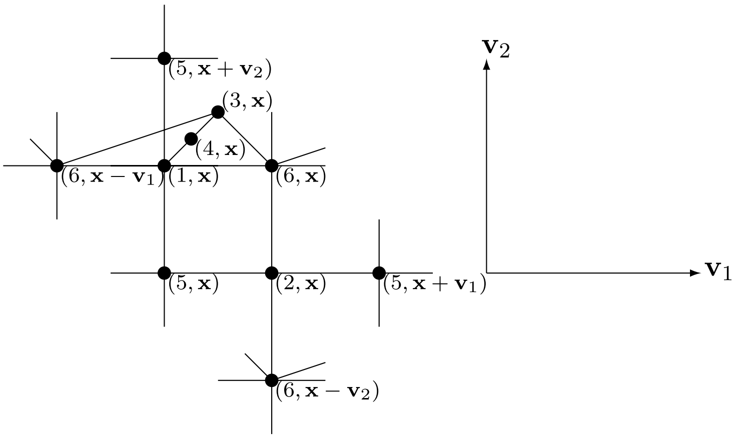

Fig. 2. A portion of the lattice $\{1, 2, 3, 4, 5, 6\} \times \Gamma$ linked by the hopping.

A portion of the lattice $\{1, 2, 3, 4, 5, 6\} \times \Gamma$ linked by the hopping is pictured in Figure 2.

To estimate the modulus of the eigenvalues of $E(\mathbf{k})$, it is efficient to estimate the eigenvalues of $E(\mathbf{k})^2$. Note that

$$E(\mathbf{k})^2 = \begin{pmatrix} E_0(\mathbf{k})E_0(\mathbf{k})^* & 0 \\ 0 & E_0(\mathbf{k})^*E_0(\mathbf{k}) \end{pmatrix},$$

$$\begin{aligned}
&\det(xI_3 - E_0(\mathbf{k})E_0(\mathbf{k})^*) = \det(xI_3 - E_0(\mathbf{k})^*E_0(\mathbf{k})) \\
&= x^3 - (5 + 3|1 + e^{ik_1}|^2 + 2|1 + e^{ik_2}|^2)x^2 \\
&\quad + \left(6\sum_{j=1}^{2}|1 + e^{ik_j}|^2 + 2|1 + e^{ik_1}|^4 + |1 + e^{ik_2}|^4 - |1 + e^{ik_1}|^2|1 + e^{ik_2}|^2\right)x \\
&\quad - \left(\sum_{j=1}^{2}|1 + e^{ik_j}|^2\right)^2.
\end{aligned}$$

Moreover, if $\sum_{j=1}^{2}|1 + e^{ik_j}|^2 \neq 0$,

$$\inf_{\substack{\mathbf{v}\in\mathbb{C}^6 \\ \text{with } \|\mathbf{v}\|_{\mathbb{C}^6}=1}} \|E(\mathbf{k})^2\mathbf{v}\|_{\mathbb{C}^6} \geq \frac{\det(E_0(\mathbf{k})E_0(\mathbf{k})^*)}{\sum_{l=1}^{3}\det(E_0(\mathbf{k})E_0(\mathbf{k})^*(i,j))_{\substack{1\leq i,j\leq 3 \\ i,j\neq l}}}$$

$$= \frac{\left(\sum_{j=1}^{2}|1+e^{ik_j}|^2\right)^2}{6\sum_{j=1}^{2}|1+e^{ik_j}|^2+2|1+e^{ik_1}|^4+|1+e^{ik_2}|^4-|1+e^{ik_1}|^2|1+e^{ik_2}|^2}$$

$$\ge \frac{\sum_{j=1}^{2}|1+e^{ik_j}|^2}{6+2\sum_{j=1}^{2}|1+e^{ik_j}|^2} \ge \frac{1}{22}\sum_{j=1}^{2}|1+e^{ik_j}|^2,$$

$$\inf_{\substack{\mathbf{v}\in\mathbb{C}^6\\ \text{with } \|\mathbf{v}\|_{\mathbb{C}^6}=1}} \|E(\mathbf{k})^2\mathbf{v}\|_{\mathbb{C}^6} \le \frac{\det(E_0(\mathbf{k})E_0(\mathbf{k})^*)}{\frac{1}{3}\sum_{l=1}^{3}\det(E_0(\mathbf{k})E_0(\mathbf{k})^*(i,j))_{\substack{1\le i,j\le 3\\ i,j\neq l}}}$$

$$\le \frac{1}{2}\sum_{j=1}^{2}|1+e^{ik_j}|^2.$$

Therefore, if we define the function $e:\mathbb{R}^2\to\mathbb{R}$ by

$$e(\mathbf{k}) := \left(\frac{1}{22}\sum_{j=1}^{2}|1+e^{ik_j}|^2\right)^{\frac{1}{2}},$$

the condition (1.6) holds with some positive constant c. It is apparent that $e(\cdot)$ satisfies (1.7) with some c and the required regularity and periodicity. There is no difficulty to confirm that $e(\cdot)^2$, $E(\cdot)$ satisfy (1.8), (1.9) with $\mathsf{n}_1=\mathsf{n}_2=1$. By using the inequalities

$$\frac{2}{\pi}\left(\sum_{j=1}^{2}|k_j-\pi|^2\right)^{\frac{1}{2}} \le \left(\sum_{j=1}^{2}|1+e^{ik_j}|^2\right)^{\frac{1}{2}} \le \left(\sum_{j=1}^{2}|k_j-\pi|^2\right)^{\frac{1}{2}}, \quad (\forall\mathbf{k}\in\Gamma_\infty^*), \tag{1.28}$$

we can check that (1.10), (1.11), (1.12), (1.13) hold with $\mathsf{a}=2$, $d=2$, $\mathsf{n}_1=\mathsf{n}_2=1$.

Example 1.18 (3-dimensional model with nonuniform exponents). Let us give a 3-dimensional model where the exponents $\mathsf{n}_1,\mathsf{n}_2,\mathsf{n}_3$ are not uniform. As in Example 1.15, we let $\{\mathbf{v}_j\}_{j=1}^3$, $\{\hat{\mathbf{v}}_j\}_{j=1}^3$ be the canonical basis of $\mathbb{R}^3$. Set

$$E(\mathbf{k}) = e(\mathbf{k}) := \sum_{j=1}^{2}\cos k_j + 2 + (\cos k_3+1)^2.$$

This is the dispersion relation of a one-band free electron model on the cubic lattice. The required regularity, periodicity, (1.6) and (1.7) are clearly satisfied by $E(\cdot)$, $e(\cdot)$. By making use of the form

$$e(\mathbf{k}) = 2\sum_{j=1}^{2}\sin^2\left(\frac{k_j-\pi}{2}\right) + 4\sin^4\left(\frac{k_3-\pi}{2}\right), \tag{1.29}$$

one can check that (1.8), (1.9) hold with $\mathsf{n}_1 = \mathsf{n}_2 = 2$, $\mathsf{n}_3 = 4$. Moreover, for $R \in \mathbb{R}_{>0}$,

$$\begin{aligned}\int_{\Gamma_\infty^*} d\mathbf{k} 1_{e(\mathbf{k})\le R} &\le c\int_{[0,1]^3} d\mathbf{k} 1_{\sum_{j=1}^{2} k_j^2 + k_3^4 \le R} \le c\int_0^{R^{1/2}} drr \int_0^1 dk_3 1_{r^2+k_3^4\le R} \\ &\le c\int_0^{R^{1/2}} drr(R-r^2)^{\frac{1}{4}} \le cR^{\frac{5}{4}}, \\ \int_{\Gamma_\infty^*} d\mathbf{k}\frac{1_{e(\mathbf{k})\le R}}{e(\mathbf{k})} &\le c\int_{[0,1]^3} d\mathbf{k}\frac{1_{\sum_{j=1}^{2} k_j^2 + k_3^4 \le R}}{\sum_{j=1}^{2} k_j^2 + k_3^4} \le c\int_0^{R^{1/4}} dk_3 \int_0^\infty dr \frac{r 1_{r^2+k_3^4\le R}}{r^2+k_3^4} \\ &\le c\int_0^{R^{1/4}} dk_3 \log\left(\frac{R}{k_3^4}\right) \le cR^{\frac{1}{4}}.\end{aligned}$$

These calculations lead to the conclusion that (1.10), (1.11) hold with $\mathsf{a} = \frac{5}{4}$. One can similarly confirm that (1.12) holds. Since

$$2\mathsf{a} - 1 - \sum_{j=1}^{3}\frac{1}{\mathsf{n}_j} = \frac{1}{4},$$

the condition (1.13) holds as well.

Example 1.19 (5-dimensional model whose Fermi surface does not degenerate into finite points). In the above examples the zero set of $e(\cdot)$ consists of finite points. Here let us give an example where the zero set of $e(\cdot)$ does not degenerate into finite points. Let $d = 5$ and let $\{\mathbf{v}_j\}_{j=1}^5$, $\{\hat{\mathbf{v}}_j\}_{j=1}^5$ be the canonical basis of $\mathbb{R}^5$. Define $E(\cdot) : \mathbb{R}^5 \to \mathbb{R}$ by

$$E(\mathbf{k}) := (\cos k_1 + \cos k_2)^2 + \sum_{j=3}^{5}\cos k_j + 3$$

and set $e(\mathbf{k}) := E(\mathbf{k})$. It is possible to make an interpretation of this model in terms of hopping and chemical potential. We can see from the equality

$$e(\mathbf{k}) = 4\cos^2\left(\frac{k_1+k_2}{2}\right)\cos^2\left(\frac{k_1-k_2}{2}\right) + 2\sum_{j=3}^{5}\sin^2\left(\frac{k_j-\pi}{2}\right) \tag{1.30}$$

that

$$\begin{aligned}
&\{\mathbf{k}\in\Gamma_\infty^* \mid e(\mathbf{k})=0\}\\
&= \left\{(k_1,k_2,\pi,\pi,\pi) \;\middle|\; \begin{array}{l} k_1,k_2\in[0,2\pi) \text{ satisfying} \\ k_1+k_2\in\{\pi,3\pi\} \text{ or } k_1-k_2\in\{-\pi,\pi\}\end{array}\right\}.
\end{aligned}$$

It is clear that $E(\cdot)$, $e(\cdot)$ satisfy (1.6), (1.7) and the required regularity and periodicity. By using the equality

$$e(\mathbf{k}) = (\cos k_1 + \cos k_2)^2 + 2\sum_{j=3}^{5}\sin^2\left(\frac{k_j-\pi}{2}\right)$$

we can check that (1.8), (1.9) hold with $\mathsf{n}_j = 2$ ($j = 1,2,3,4,5$). By using (1.30) and the inequality $\theta^2 \ge \sin^2\theta \ge \frac{2^2}{\pi^2}\theta^2$ ($\theta\in[0,\frac{\pi}{2}]$) and changing the variables we have that for $R\in(0,1]$,

$$\begin{aligned}
&\int_{\Gamma_\infty^*} d\mathbf{k}1_{e(\mathbf{k})\le R}\\
&\le c\int_{[0,\frac{\pi}{2}]^5} d\mathbf{k}1_{\frac{2^6}{\pi^4}k_1^2k_2^2+\frac{2^3}{\pi^2}\sum_{j=3}^5 k_j^2\le R} \le c\int_{[0,1]^5} d\mathbf{k}1_{k_1^2k_2^2+\sum_{j=3}^5 k_j^2\le R}\\
&\le cR^2\int_{[0,R^{-1/4}]^2} dk_1dk_2\int_{[0,R^{-1/2}]^3} dk_3dk_4dk_5 1_{k_1^2k_2^2+\sum_{j=3}^5 k_j^2\le 1}\\
&\le cR^2\int_0^1 drr^2\int_{[0,R^{-1/4}]^2} dk_1dk_2 1_{k_1k_2\le\sqrt{1-r^2}}\\
&\le cR^2\int_0^1 drr^2\left(\int_0^{R^{1/4}\sqrt{1-r^2}} dk_1R^{-\frac{1}{4}} + \int_{R^{1/4}\sqrt{1-r^2}}^{R^{-1/4}} dk_1\frac{\sqrt{1-r^2}}{k_1}\right)\\
&\le cR^2\log(R^{-1}+1),\\
&\int_{\Gamma_\infty^*} d\mathbf{k}\frac{1_{e(\mathbf{k})\le R}}{e(\mathbf{k})} \le c\int_{[0,1]^5} d\mathbf{k}\frac{1_{k_1^2k_2^2+\sum_{j=3}^5 k_j^2\le R}}{k_1^2k_2^2+\sum_{j=3}^5 k_j^2}
\end{aligned}$$

$$
\begin{aligned}
&\le cR\int_{[0,R^{-1/4}]^2}dk_1dk_2\int_{[0,R^{-1/2}]^3}dk_3dk_4dk_5\frac{1_{k_1^2k_2^2+\sum_{j=3}^5k_j^2\le 1}}{k_1^2k_2^2+\sum_{j=3}^5k_j^2}\\
&\le cR\int_0^1 dr\int_{[0,R^{-1/4}]^2}dk_1dk_2 1_{k_1k_2\le\sqrt{1-r^2}}\\
&\le cR\log(R^{-1}+1),\\
&\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{e(\mathbf{k})}\le c\int_{[0,1]^5}d\mathbf{k}\frac{1}{k_1^2k_2^2+\sum_{j=3}^5k_j^2}\le c.
\end{aligned}
$$

Since $\log(R^{-1}+1)\le cR^{-1/5}$ ($\forall R\in(0,1]$), the inequalities (1.10), (1.11) hold with $\mathsf{a}=\frac{9}{5}$. Let us check that the inequality (1.13) holds with $\mathsf{a}=\frac{9}{5}$, $d=5$, $\mathsf{n}_j=2$ ($j=1,2,3,4,5$). Moreover,

$$
\begin{aligned}
\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{e(\mathbf{k})^2+\varepsilon}&\ge c\int_{[0,1]^5}d\mathbf{k}\left(\left(k_1^2k_2^2+\sum_{j=3}^5k_j^2\right)^2+\varepsilon\right)^{-1}\\
&\ge c\int_0^1 dr\int_{[0,1]^2}dk_1dk_2\frac{r^2}{(k_1^2k_2^2+r^2)^2+\varepsilon}\\
&\ge c\int_0^1 dr\int_{[0,r^{-1/2}]^2}dk_1dk_2\frac{r^3}{(k_1^2k_2^2+1)^2r^4+\varepsilon}\to\infty,
\end{aligned}
$$

as $\varepsilon\searrow 0$. Thus the condition (1.12) holds as well.

In summary, Theorem 1.3 and Corollary 1.11 hold for the Hamiltonian H whose free part H_0 is defined with the hopping matrix $E(\cdot)$ given in Example 1.15 - Example 1.19.

Remark 1.20. Let us see that the free dispersion relation of nearest-neighbor hopping electrons on the (hyper-)cubic lattice with non-degenerate Fermi surface does not satisfy the condition (1.11), which will be essentially used to prove that $|U|$ can be taken independently of the temperature and the imaginary magnetic field in low temperature. Most of the necessary notations are defined in the same way as in Example 1.15, apart from that now $d\in\mathbb{N}$ and

$$E(\mathbf{k}):=(-1)^{hop}2\sum_{j=1}^d\cos k_j-\mu$$

with the chemical potential $\mu \in (-2d, 2d)$. This free model was treated in our previous work [12]. For any $R, \varepsilon \in \mathbb{R}_{>0}$ and a continuous function $e : \mathbb{R}^d \to \mathbb{R}_{\ge 0}$ satisfying (1.6) we can derive by the coarea formula that

$$\begin{aligned}\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1_{e(\mathbf{k}) \le R}}{e(\mathbf{k}) + \varepsilon} &\ge \int_{\Gamma_\infty^*} d\mathbf{k} \frac{1_{|E(\mathbf{k})| \le R}}{|E(\mathbf{k})| + \varepsilon} \ge \frac{1}{2\sqrt{d}} \int_{\Gamma_\infty^*} d\mathbf{k} \frac{1_{|E(\mathbf{k})| \le R} |\nabla E(\mathbf{k})|}{|E(\mathbf{k})| + \varepsilon} \\ &= \frac{1}{2\sqrt{d}} \int_{-R}^{R} d\eta \frac{\mathcal{H}^{d-1}(\{\mathbf{k} \in \Gamma_\infty^* \mid E(\mathbf{k}) = \eta\})}{|\eta| + \varepsilon},\end{aligned}$$

where $\mathcal{H}^{d-1}$ is the $d-1$ dimensional Hausdorff measure. Set $K := \frac{1}{2}(2d - |\mu|)$, $R' := \min\{K, R\}$. Then we can apply [12, Lemma 4.17] to derive that

$$\begin{aligned}&\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1_{e(\mathbf{k}) \le R}}{e(\mathbf{k}) + \varepsilon} \\ &\ge \frac{1}{2\sqrt{d}} \inf_{\eta \in [-K, K]} \mathcal{H}^{d-1}(\{\mathbf{k} \in \Gamma_\infty^* \mid E(\mathbf{k}) = \eta\}) \int_{-R'}^{R'} dx \frac{1}{|x| + \varepsilon} \\ &\ge \frac{1}{\sqrt{d}} \left(1_{d=1} + 1_{d \ge 2} \left(\frac{2d - |\mu|}{10(d-1)d} \right)^{d-1} \right) \log \left(\frac{R' + \varepsilon}{\varepsilon} \right).\end{aligned}$$

Since the right-hand side of the above inequality diverges to ∞ as $\varepsilon \searrow 0$, the condition (1.11) cannot be satisfied by this model.

2. Phase Transitions

In this section we analyze properties of the free energy density. We focus on the right-hand side of (1.19) as a function of $(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$ by fixing $U (\in \mathbb{R}_{<0})$ with small magnitude. Mathematical arguments in this section are essentially independent of the following sections, which aim at proving Theorem 1.3 and Corollary 1.11. Our aim here is to describe the nature of the phase transition happening in the system. The readers who want to prove Theorem 1.3 and Corollary 1.11 first can skip to Section 3 and come back to this section afterward. If we think of the right-hand side of (1.19) alone, we are free to substitute any large coupling constant and a hopping matrix with different properties. However, we restrict our attention not to deviate from the configuration where the derivation of (1.19) is justified. We simply assume that $E : \mathbb{R}^d \to \mathrm{Mat}(b, \mathbb{C})$, $e : \mathbb{R}^d \to \mathbb{R}$ satisfy the same conditions as listed in Subsection 1.2 and $|U|$ is small as described subsequently. We need to impose a couple more conditions on the function $e(\cdot)$. Assume that there exist $\mathsf{r}, \mathsf{s} \in \mathbb{R}_{>0}$, $\mathsf{c} \in \mathbb{R}_{\geq 1}$ such that $0 < \mathsf{r} \leq 1$, $1 + 2\mathsf{r} \leq \mathsf{s}$ and

$$\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1}{e(\mathbf{k})^2 + A^2} \leq \mathsf{c} A^{-\mathsf{r}}, \tag{2.1}$$

$$\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1_{e(\mathbf{k}) \leq B}}{(e(\mathbf{k})^2 + A^2)^2} \geq \mathsf{c}^{-1} A^{-\mathsf{s}}, \tag{2.2}$$

$$(\forall A, B \in (0, 1] \text{ with } 0 < A \leq B \leq 1).$$

The conditions (2.1), (2.2) are used only in this section and not required to prove Theorem 1.3 and Corollary 1.11. Moreover, we assume that $U (\in \mathbb{R}_{<0})$ satisfies

$$|U| < \frac{2}{bD_d \int_{\Gamma_\infty^*} d\mathbf{k} \frac{1}{e(\mathbf{k})}}. \tag{2.3}$$

We will replace the upper bound on $|U|$ by a smaller constant in the following.

Remark 2.1. In fact we do not use the conditions (1.5), (1.8), (1.9), (1.10), (1.13) in this section. Also, the regularity assumption of $E(\cdot)$, $e(\cdot)^2$ can be relaxed.

2.1. Study of the models

In order to see that the conditions (2.1), (2.2) are reasonable, let us check that the examples given in Subsection 1.3 satisfy these additional conditions.

For $x \in \mathbb{R}$ we let $\lfloor x \rfloor$ denote the largest integer which does not exceed x. This notation will be used in the rest of the paper.

In the model given in Example 1.15 with $d = 3$ the conditions (2.1), (2.2) hold with $\mathsf{r} = \frac{1}{2}$, $\mathsf{s} = \frac{5}{2}$ respectively. Since $1 + 2\mathsf{r} \leq \mathsf{s}$, the required conditions are fulfilled in this case. In the case $d = 4$ the condition (2.2) holds with $\mathsf{s} = 2$. Note that

$$\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1}{e(\mathbf{k})^2 + A^2} \leq c \log(A^{-1} + 1). \tag{2.4}$$

The condition (2.1) holds with e.g. $\mathsf{r} = \frac{1}{2}$ and thus $1 + 2\mathsf{r} \le \mathsf{s}$ in this case as well.

By using (1.26) we can check that $e(\cdot)$ introduced in Example 1.16 satisfies (2.2) with $\mathsf{s} = 2$. It follows from (1.27) that the function $e(\cdot)$ satisfies (2.4) and thus (2.1) with $\mathsf{r} = \frac{1}{2}$. Therefore, the additional conditions are met in this example.

By using (1.28) we can confirm without difficulty that $e(\cdot)$ introduced in Example 1.17 satisfies (2.1) with $\mathsf{r} = \frac{1}{2}$ and (2.2) with $\mathsf{s} = 2$ as well.

Let us study with the dispersion relation $e(\cdot)$ defined in Example 1.18. By using (1.29) and changing variables we have for $A, B \in (0, 1]$ with $0 < A \le B \le 1$ that

$$\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1}{e(\mathbf{k})^2 + A^2} \le c \int_{[0,1]^3} d\mathbf{k} \left(\left(\sum_{j=1}^{2} k_j^2 + k_3^4 \right)^2 + A^2 \right)^{-1}$$

$$\le cA^{-\frac{3}{4}} \int_{\mathbb{R}^3} d\mathbf{k} \left(\left(\sum_{j=1}^{2} k_j^2 + k_3^4 \right)^2 + 1 \right)^{-1} \le cA^{-\frac{3}{4}},$$

$$\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1_{e(\mathbf{k}) \le B}}{(e(\mathbf{k})^2 + A^2)^2}$$

$$\ge c \int_{[0,1]^3} d\mathbf{k} 1_{\sum_{j=1}^{2} k_j^2 + k_3^4 \le B} \left(\left(\sum_{j=1}^{2} k_j^2 + k_3^4 \right)^2 + A^2 \right)^{-2}$$

$$\ge cA^{-\frac{11}{4}} \int_{[0,1]^3} d\mathbf{k} 1_{\sum_{j=1}^{2} k_j^2 + k_3^4 \le 1} \left(\left(\sum_{j=1}^{2} k_j^2 + k_3^4 \right)^2 + 1 \right)^{-2} \ge cA^{-\frac{11}{4}}.$$

Thus the conditions (2.1), (2.2) hold with $\mathsf{r} = \frac{3}{4}$, $\mathsf{s} = \frac{11}{4}$ respectively. Check that the condition $1 + 2\mathsf{r} \le \mathsf{s}$ holds as well.

Finally let us consider the function $e(\cdot)$ introduced in Example 1.19. By (1.30) and change of variables,

$$\int_{\Gamma_\infty^*} d\mathbf{k} \frac{1}{e(\mathbf{k})^2 + A^2} \le c \int_{[0,1]^5} d\mathbf{k} \left(\left(k_1^2 k_2^2 + \sum_{j=3}^{5} k_j^2 \right)^2 + A^2 \right)^{-1}$$

$$\le c \int_{[0,A^{-1/4}]^2} dk_1 dk_2 \int_{[0,A^{-1/2}]^3} dk_3 dk_4 dk_5$$

$$
\begin{aligned}
&\cdot \left(1_{k_1k_2\le 1} + \sum_{l=0}^{\lfloor \log(A^{-1/2})\rfloor} 1_{e^l<k_1k_2\le e^{l+1}}\right)\left(\left(k_1^2k_2^2+\sum_{j=3}^5 k_j^2\right)^2+1\right)^{-1}\\
&\le c\int_{[0,A^{-1/4}]^2} dk_1dk_2 1_{k_1k_2\le 1}\int_{\mathbb{R}^3_{\ge 0}} dk_3dk_4dk_5\left(\left(\sum_{j=3}^5 k_j^2\right)^2+1\right)^{-1}\\
&\quad + c\sum_{l=0}^{\lfloor \log(A^{-1/2})\rfloor}\int_{[0,A^{-1/4}]^2} dk_1dk_2 1_{e^l<k_1k_2\le e^{l+1}}\\
&\qquad\qquad \cdot\int_{\mathbb{R}^3_{\ge 0}} dk_3dk_4dk_5\left(e^{2l}+\sum_{j=3}^5 k_j^2\right)^{-2}\\
&\le c\int_{[0,A^{-1/4}]^2} dk_1dk_2 1_{k_1k_2\le 1}\\
&\quad + c\sum_{l=0}^{\lfloor \log(A^{-1/2})\rfloor} e^{-l}\int_{[0,A^{-1/4}]^2} dk_1dk_2 1_{e^l<k_1k_2\le e^{l+1}}\\
&\le c\log(A^{-1}+1) + c\sum_{l=0}^{\lfloor \log(A^{-1/2})\rfloor} e^{-l}\int_{e^lA^{1/4}}^{A^{-1/4}} dk_1\int_{e^lk_1^{-1}}^{e^{l+1}k_1^{-1}} dk_2\\
&\le c(\log(A^{-1}+1))^2,\\
&\int_{\Gamma_\infty^*} d\mathbf{k}\frac{1_{e(\mathbf{k})\le B}}{(e(\mathbf{k})^2+A^2)^2}\\
&\ge c\int_{\Gamma_\infty^*} d\mathbf{k} 1_{\frac{1}{4}(k_1+k_2-\pi)^2(k_1-k_2-\pi)^2+\frac{1}{2}\sum_{j=3}^5(k_j-\pi)^2\le B}\\
&\quad\cdot\left(\left((k_1+k_2-\pi)^2(k_1-k_2-\pi)^2+\sum_{j=3}^5(k_j-\pi)^2\right)^2+A^2\right)^{-2}\\
&\ge c\int_{[0,\pi]^5} d\mathbf{k} 1_{(k_1+k_2)^2(k_1-k_2)^2+\sum_{j=3}^5 k_j^2\le B}\\
&\quad\cdot\left(\left((k_1+k_2)^2(k_1-k_2)^2+\sum_{j=3}^5 k_j^2\right)^2+A^2\right)^{-2}
\end{aligned}
$$

$$\begin{aligned}&\geq cA^{-2}\int_{[0,1]^5}d\mathbf{k}1_{(k_1+k_2)^2(k_1-k_2)^2+\sum_{j=3}^5 k_j^2\leq 1}\\&\quad\cdot\left(\left((k_1+k_2)^2(k_1-k_2)^2+\sum_{j=3}^5 k_j^2\right)^2+1\right)^{-2}\\&\geq cA^{-2}.\end{aligned}$$

Thus the inequalities (2.1), (2.2) hold with e.g. $\mathsf{r}=\frac{1}{2}$, $\mathsf{s}=2$.

We have seen that in each example of Subsection 1.3 the function $e(\cdot)$ satisfies the required conditions of this subsection.

2.2. Phase boundaries

Let us define the map $g:\mathbb{R}_{>0}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ by

$$g(x,y,z) := \begin{cases} -\frac{2}{|U|}+ \\ D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{(\cos(xy/2)+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))\sqrt{E(\mathbf{k})^2+z^2}}\right) \\ \quad \text{if } \frac{xy}{2}\notin\pi(2\mathbb{Z}+1) \text{ or } z\neq 0, \\ \infty \text{ if } \frac{xy}{2}\in\pi(2\mathbb{Z}+1) \text{ and } z=0. \end{cases}$$

We can check that the function $(x,y,z)\mapsto g(x,y,z)$ is C^∞-class in the open set

$$\left\{(x,y,z)\in\mathbb{R}_{>0}\times\mathbb{R}\times\mathbb{R}\ \Big|\ \frac{xy}{2}\notin\pi(2\mathbb{Z}+1)\text{ or }z\neq 0\right\}$$

of $\mathbb{R}^3$.

For $(\beta,\theta)\in\mathbb{R}_{>0}\times\mathbb{R}$ let $\Delta(\beta,\theta)$ be such that $\Delta(\beta,\theta)\geq 0$ and $g(\beta,\theta,\Delta(\beta,\theta))=0$ if $g(\beta,\theta,0)\geq 0$, $\Delta(\beta,\theta)=0$ if $g(\beta,\theta,0)<0$. This rule defines the function $\Delta:\mathbb{R}_{>0}\times\mathbb{R}\to\mathbb{R}_{\geq 0}$. The well-definedness of the function $\Delta(\cdot,\cdot)$ is guaranteed by Lemma 1.2.

The goal of this subsection is to characterize the set $\{(\beta,\theta)\in\mathbb{R}_{>0}\times\mathbb{R}\mid\Delta(\beta,\theta)>0\}$. We will see that this set consists of countable disjoint subsets. Let us call the boundaries of the disjoint subsets phase boundaries. Our goal here is equivalent to characterizing the phase boundaries.

Define the subsets O_+, O_- of $\mathbb{R}^2$ by

$$O_+ := \{(x,y) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g(x,y,0) > 0\},$$
$$O_- := \{(x,y) \in \mathbb{R}_{>0} \times \mathbb{R} \mid g(x,y,0) < 0\}.$$

We can see that O_+, O_- are open subsets of $\mathbb{R}^2$.

LEMMA 2.2.

$$\Delta \in C(\mathbb{R}_{>0} \times \mathbb{R}), \quad \Delta|_{O_+ \cup O_-} \in C^\infty(O_+ \cup O_-).$$

PROOF. It is trivial that $\Delta|_{O_-} \in C^\infty(O_-)$. We have observed that the functions

$$x \mapsto \frac{\sinh x}{(\varepsilon + \cosh x)x} : (0,\infty) \to \mathbb{R}, \quad (\varepsilon \in [-1,1])$$

are strictly monotone decreasing in the proof of Lemma 1.2. It follows that

$$\frac{\partial g}{\partial z}(x,y,z) < 0, \quad (\forall (x,y,z) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0}). \tag{2.5}$$

Thus

$$\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta)) < 0, \quad (\forall (\beta,\theta) \in O_+).$$

By the implicit function theorem we have that $\Delta|_{O_+} \in C^\infty(O_+)$, or $\Delta|_{O_+ \cup O_-} \in C^\infty(O_+ \cup O_-)$.

Let us prove that $\Delta \in C(\mathbb{R}_{>0} \times \mathbb{R})$. It is sufficient to prove the continuity at each point belonging to $\mathbb{R}_{>0} \times \mathbb{R} \backslash O_+ \cup O_-$. Let $(\beta_0,\theta_0) \in \mathbb{R}_{>0} \times \mathbb{R} \backslash O_+ \cup O_-$. By definition $g(\beta_0,\theta_0,0) = 0$, $\Delta(\beta_0,\theta_0) = 0$ and $\beta_0\theta_0/2 \notin \pi(2\mathbb{Z}+1)$. Suppose that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for any $\delta \in \mathbb{R}_{>0}$ there exists $(\beta_\delta,\theta_\delta) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $\|(\beta_0,\theta_0) - (\beta_\delta,\theta_\delta)\|_{\mathbb{R}^2} < \delta$ and $\Delta(\beta_\delta,\theta_\delta) \geq \varepsilon$. Then,

$$0 = g(\beta_\delta,\theta_\delta,\Delta(\beta_\delta,\theta_\delta)) \leq g(\beta_\delta,\theta_\delta,\varepsilon) \leq \sup_{\substack{(\beta,\theta) \in \mathbb{R}_{>0} \times \mathbb{R} \\ \text{with } \|(\beta_0,\theta_0) - (\beta,\theta)\|_{\mathbb{R}^2} < \delta}} g(\beta,\theta,\varepsilon).$$

By sending δ to 0 we have that $0 \le g(\beta_0, \theta_0, \varepsilon) < g(\beta_0, \theta_0, 0) = 0$, which is a contradiction. Thus $\lim_{(\beta,\theta)\to(\beta_0,\theta_0)} \Delta(\beta, \theta) = 0 = \Delta(\beta_0, \theta_0)$, which implies that $\Delta \in C(\mathbb{R}_{>0} \times \mathbb{R})$. $\square$

The next lemma states the existence of critical values of the imaginary magnetic field in $[0, 4\pi/\beta]$.

LEMMA 2.3. *For any $\beta \in \mathbb{R}_{>0}$ there uniquely exist $\theta_{c,1} \in (0, 2\pi/\beta)$, $\theta_{c,2} \in (2\pi/\beta, 4\pi/\beta)$ such that*

$$\begin{aligned} &g(\beta, \theta_{c,1}, 0) = g(\beta, \theta_{c,2}, 0) = 0, \\ &g(\beta, \theta, 0) > 0, \quad (\forall \theta \in (\theta_{c,1}, \theta_{c,2})), \\ &g(\beta, \theta, 0) < 0, \quad (\forall \theta \in [0, \theta_{c,1}) \cup (\theta_{c,2}, 4\pi/\beta]). \end{aligned}$$

PROOF. One can see from the definition that $\theta \mapsto g(\beta, \theta, 0)$ is strictly monotone increasing in $(0, 2\pi/\beta)$, strictly monotone decreasing in $(2\pi/\beta, 4\pi/\beta)$ and continuous in $(0, 2\pi/\beta) \cup (2\pi/\beta, 4\pi/\beta)$. By the assumption (2.3),

$$g(\beta, 0, 0) = g\left(\beta, \frac{4\pi}{\beta}, 0\right) \le -\frac{2}{|U|} + bD_d \int_{\Gamma_\infty^*} d\mathbf{k} \frac{1}{e(\mathbf{k})} < 0.$$

Also, by (1.12) $\lim_{\theta \to 2\pi/\beta} g(\beta, \theta, 0) = \infty$. We can deduce the claim from these properties. $\square$

By Lemma 2.3 we can define the functions $\theta_{c,1} : \mathbb{R}_{>0} \to (0, 2\pi/\beta)$, $\theta_{c,2} : \mathbb{R}_{>0} \to (2\pi/\beta, 4\pi/\beta)$.

For any parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$ we let $c(\alpha_1, \alpha_2, \cdots, \alpha_n)$ denote a positive constant depending only on $\alpha_1, \alpha_2, \cdots, \alpha_n$. This notational rule will be used not only in the proof of the next lemma but throughout the rest of the paper.

LEMMA 2.4. *There exist positive constants c_3, c_4 depending only on $b, D_d, \mathsf{c}, \mathsf{r}, \mathsf{s}$ such that the following statements hold for any $U \in (-c_3, 0)$.*

(i)

$$\left| \frac{\theta_{c,j}(\beta)}{2} - \frac{\pi}{\beta} \right| \le \frac{\pi}{2\beta}, \quad (\forall \beta \in \mathbb{R}_{>0},\ j \in \{1, 2\}).$$

(ii)

$$\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right|\le c_4\left(\frac{|U|}{\beta}\right)^{\frac{1}{\mathsf{r}}},\quad (\forall\beta\in\mathbb{R}_{>0},\ j\in\{1,2\}).$$

(iii)

$$\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right|\le c_4\left(\frac{|U|}{\beta}\right)^{\frac{1}{2}},\quad (\forall\beta\in(0,1],\ j\in\{1,2\}).$$

(iv) $\theta_{c,j}\in C^\infty(\mathbb{R}_{>0})$ *and*

$$\frac{d\theta_{c,j}}{d\beta}(\beta)<0,\quad (\forall\beta\in\mathbb{R}_{>0},\ j\in\{1,2\}).$$

Proof. (i): Take $\beta\in\mathbb{R}_{>0}$, $j\in\{1,2\}$. Assume that $|\theta_{c,j}(\beta)/2-\pi/\beta|>\pi/(2\beta)$. Then,

$$\begin{aligned}0=g(\beta,\theta_{c,j}(\beta),0)&\le-\frac{2}{|U|}+D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\tanh(\beta|E(\mathbf{k})|)}{|E(\mathbf{k})|}\right)\\&\le-\frac{2}{|U|}+bD_d\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{e(\mathbf{k})},\end{aligned}$$

which contradicts the condition (2.3). Thus the claim holds true.

(ii): By (1.11) and (2.1),

$$\begin{aligned}0&=g(\beta,\theta_{c,j}(\beta),0)\\&\le-\frac{2}{|U|}+bD_d\int_{\Gamma_\infty^*}d\mathbf{k}\frac{\sinh(\beta e(\mathbf{k}))}{(\cos(\beta\theta_{c,j}(\beta)/2)+\cosh(\beta e(\mathbf{k})))e(\mathbf{k})}\\&\le-\frac{2}{|U|}+c(b,D_d)\beta^{-1}\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1_{e(\mathbf{k})\le\beta^{-1}}}{e(\mathbf{k})^2+|\theta_{c,j}(\beta)/2-\pi/\beta|^2}\\&\quad+c(b,D_d)\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{e(\mathbf{k})}\\&\le-\frac{2}{|U|}+c(b,D_d,\mathsf{c})\\&\quad+c(b,D_d,\mathsf{c})\beta^{-1}\Bigg(1_{|\theta_{c,j}(\beta)/2-\pi/\beta|\le1}\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-\mathsf{r}}\end{aligned}$$

$$+ 1_{|\theta_{c,j}(\beta)/2-\pi/\beta|>1}\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-2}\Bigg)$$
$$\le -\frac{2}{|U|}+c(b,D_d,\mathsf{c})\left(\beta^{-1}\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-\mathsf{r}}+1\right).$$

To derive the last inequality, we also used that $0<\mathsf{r}\le 1$. If $|U|\le c(b,D_d,\mathsf{c})^{-1}$,

$$0\le -\frac{1}{|U|}+c(b,D_d,\mathsf{c})\beta^{-1}\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-\mathsf{r}}.$$

This leads to the result.

(iii): Since $\beta\le 1$,

$$\begin{aligned}0&=g(\beta,\theta_{c,j}(\beta),0)\\ &\le -\frac{2}{|U|}+c(b,D_d,\mathsf{c})\beta^{-1}\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{e(\mathbf{k})^2+|\theta_{c,j}(\beta)/2-\pi/\beta|^2}\\ &\le -\frac{2}{|U|}+c(b,D_d,\mathsf{c})\beta^{-1}\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-2},\end{aligned}$$

which implies the result.

(iv): For $x\in\mathbb{R}_{>0}$, $y\in(0,2\pi/x)\cup(2\pi/x,4\pi/x)$,

$$\begin{aligned}&\frac{\partial g}{\partial y}(x,y,0)\\ &=\frac{D_d}{2}x\sin\left(\frac{xy}{2}\right)\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(x|E(\mathbf{k})|)}{(\cos(xy/2)+\cosh(xE(\mathbf{k})))^2|E(\mathbf{k})|}\right).\end{aligned}\tag{2.6}$$

Thus

$$\begin{aligned}&\frac{\partial g}{\partial y}(x,y,0)>0,\quad (\forall y\in(0,2\pi/x)),\\ &\frac{\partial g}{\partial y}(x,y,0)<0,\quad (\forall y\in(2\pi/x,4\pi/x)).\end{aligned}\tag{2.7}$$

Therefore the implicit function theorem ensures that $\theta_{c,j}\in C^\infty(\mathbb{R}_{>0})$.

Let us determine the sign of $\frac{\partial g}{\partial x}(\beta,\theta_{c,1}(\beta),0)$, $\frac{\partial g}{\partial x}(\beta,\theta_{c,2}(\beta),0)$. For $x\in\mathbb{R}_{>0}$, $y\in\mathbb{R}$ with $xy/2\notin\pi(2\mathbb{Z}+1)$,

$$\frac{\partial g}{\partial x}(x,y,0)\tag{2.8}$$

$$
\begin{aligned}
&= \frac{D_d}{2}\sin\left(\frac{xy}{2}\right) y \int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(x|E(\mathbf{k})|)}{(\cos(xy/2)+\cosh(xE(\mathbf{k})))^2|E(\mathbf{k})|}\right)\\
&\quad + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\cosh(xE(\mathbf{k}))(1+\cos(xy/2))}{(\cos(xy/2)+\cosh(xE(\mathbf{k})))^2}\right)\\
&\quad - D_d \int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\cosh(xE(\mathbf{k}))-1}{(\cos(xy/2)+\cosh(xE(\mathbf{k})))^2}\right).
\end{aligned}
$$

By the result of (i),

$$
\theta_{c,1}(\beta)\sin\left(\frac{\beta\theta_{c,1}(\beta)}{2}\right) \ge c\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|, \tag{2.9}
$$

$$
\theta_{c,2}(\beta)\sin\left(\frac{\beta\theta_{c,2}(\beta)}{2}\right) \le -c\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|. \tag{2.10}
$$

Let us consider the case that $\beta \ge 1$. By the claim (ii) and $0 < \mathsf{r} \le 1$, if $|U| \le c_4^{-\mathsf{r}}$,

$$
\left|\frac{\theta_{c,j}(\beta)}{2}-\frac{\pi}{\beta}\right| \le \frac{1}{\beta} \le 1, \quad (\forall j \in \{1,2\}). \tag{2.11}
$$

Then by (2.1), (2.2), (2.8), (2.9), (2.11) and the claim (ii) again

$$
\begin{aligned}
&\frac{\partial g}{\partial x}(\beta,\theta_{c,1}(\beta),0)\\
&\ge cD_d\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{(\cos(\beta\theta_{c,1}(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2}\right)\\
&\quad - D_d\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{\cos(\beta\theta_{c,1}(\beta)/2)+\cosh(\beta E(\mathbf{k}))}\right)\\
&\ge c(D_d,\mathsf{c})\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta^{-3}\int_{\Gamma_\infty^*} d\mathbf{k}1_{e(\mathbf{k})\le\beta^{-1}}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-2}\\
&\quad - c(b,D_d)\beta^{-2}\int_{\Gamma_\infty^*} d\mathbf{k}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-1}\\
&\ge c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}} - c(b,D_d,\mathsf{c})\beta^{-2}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-\mathsf{r}}\\
&\ge c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-1+\mathsf{s}-\mathsf{r}}\right)
\end{aligned}
$$

$$\geq c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}\left(1-c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})\beta^{-\frac{\mathsf{s}-1-2\mathsf{r}}{\mathsf{r}}}|U|^{\frac{\mathsf{s}-1-\mathsf{r}}{\mathsf{r}}}\right)$$
$$\geq c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}\left(1-c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|^{\frac{\mathsf{s}-1-\mathsf{r}}{\mathsf{r}}}\right).$$

In the last inequality we used the conditions $\mathsf{s}-1-2\mathsf{r}\geq 0$, $\beta\geq 1$. Thus if $c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|^{(\mathsf{s}-1-\mathsf{r})/\mathsf{r}}<1$,

$$\frac{\partial g}{\partial x}(\beta,\theta_{c,1}(\beta),0)>0.$$

Similarly by using (2.1), (2.2), (2.10), (2.11), the claim (ii) and the conditions $\mathsf{s}-1-2\mathsf{r}\geq 0$, $\beta\geq 1$ we can derive from (2.8) that

$$\frac{\partial g}{\partial x}(\beta,\theta_{c,2}(\beta),0)$$
$$\leq -cD_d\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{(\cos(\beta\theta_{c,2}(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2}\right)$$
$$+D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\cosh(\beta E(\mathbf{k}))(1+\cos(\beta\theta_{c,2}(\beta)/2))}{(\cos(\beta\theta_{c,2}(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2}\right)$$
$$\leq -c(D_d,\mathsf{c})\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta^{-3}$$
$$\cdot\int_{\Gamma_\infty^*}d\mathbf{k}1_{e(\mathbf{k})\leq\beta^{-1}}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-2}$$
$$+D_d\int_{\Gamma_\infty^*}d\mathbf{k}1_{e(\mathbf{k})\leq\beta^{-1}}\,\mathrm{Tr}\left(\frac{\cosh(\beta E(\mathbf{k}))}{\cos(\beta\theta_{c,2}(\beta)/2)+\cosh(\beta E(\mathbf{k}))}\right)$$
$$+c(D_d)\int_{\Gamma_\infty^*}d\mathbf{k}1_{e(\mathbf{k})>\beta^{-1}}\,\mathrm{Tr}\left(\frac{1+\cos(\beta\theta_{c,2}(\beta)/2)}{\cos(\beta\theta_{c,2}(\beta)/2)+\cosh(\beta E(\mathbf{k}))}\right)$$
$$\leq -c(D_d,\mathsf{c})\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta^{-3}$$
$$\cdot\int_{\Gamma_\infty^*}d\mathbf{k}1_{e(\mathbf{k})\leq\beta^{-1}}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-2}$$
$$+c(b,D_d)\beta^{-2}\int_{\Gamma_\infty^*}d\mathbf{k}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-1}$$

$$\leq -c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}\left(1-c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|^{\frac{\mathsf{s}-1-\mathsf{r}}{\mathsf{r}}}\right).$$

To derive the third inequality, we also used that

$$x\mapsto\frac{\cosh x}{\cos(\beta\theta_{c,2}(\beta)/2)+\cosh x}:[0,\infty)\to\mathbb{R}$$

is non-increasing. Thus on the assumption $c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|^{(\mathsf{s}-1-\mathsf{r})/\mathsf{r}}<1$,

$$\frac{\partial g}{\partial x}(\beta,\theta_{c,2}(\beta),0)<0.$$

Next let us assume that $\beta<1$. By (2.1), (2.2), (2.8) and (2.9)

$$\begin{aligned}
&\frac{\partial g}{\partial x}(\beta,\theta_{c,1}(\beta),0)\\
&\geq cD_d\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{(\cos(\beta\theta_{c,1}(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2}\right)\\
&\quad-D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{\cos(\beta\theta_{c,1}(\beta)/2)+\cosh(\beta E(\mathbf{k}))}\right)\\
&\geq c(D_d,\mathsf{c})\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta^{-3}\int_{\Gamma_\infty^*}d\mathbf{k}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-2}\\
&\quad-c(b,D_d)\beta^{-2}\int_{\Gamma_\infty^*}d\mathbf{k}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-1}\\
&\geq 1_{|\theta_{c,1}(\beta)/2-\pi/\beta|\leq 1}\\
&\quad\cdot\left(c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}-c(b,D_d,\mathsf{c})\beta^{-2}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-\mathsf{r}}\right)\\
&\quad+1_{|\theta_{c,1}(\beta)/2-\pi/\beta|>1}\\
&\qquad\cdot\left(c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-3}-c(b,D_d)\beta^{-2}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-2}\right)\\
&\geq 1_{|\theta_{c,1}(\beta)/2-\pi/\beta|\leq 1}c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}\\
&\quad\cdot\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{\mathsf{s}-1-\mathsf{r}}\right)
\end{aligned}$$

$$+ 1_{|\theta_{c,1}(\beta)/2-\pi/\beta|>1} c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-3}$$
$$\cdot\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|\right).$$

By the claim (ii) and the condition $1+2\mathsf{r}\le\mathsf{s}$,

$$\begin{aligned}&1_{|\theta_{c,1}(\beta)/2-\pi/\beta|\le 1}\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{\mathsf{s}-1-\mathsf{r}}\right)\\&\ge 1_{|\theta_{c,1}(\beta)/2-\pi/\beta|\le 1}\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|^{\mathsf{r}}\right)\\&\ge 1_{|\theta_{c,1}(\beta)/2-\pi/\beta|\le 1}\left(1-c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|\right).\end{aligned}$$

Also, by the claim (iii) and the assumption $\beta<1$,

$$\begin{aligned}&1_{|\theta_{c,1}(\beta)/2-\pi/\beta|>1}\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,1}(\beta)}{2}-\frac{\pi}{\beta}\right|\right)\\&\ge 1_{|\theta_{c,1}(\beta)/2-\pi/\beta|>1}\left(1-c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|^{\frac{1}{2}}\right).\end{aligned}$$

Therefore if $|U|<c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})$,

$$\frac{\partial g}{\partial x}(\beta,\theta_{c,1}(\beta),0)>0.$$

Similarly we can derive from (2.1), (2.2), (2.8), (2.10), the claims (ii), (iii) and the conditions $1+2\mathsf{r}\le\mathsf{s}$, $\beta<1$ that

$$\begin{aligned}&\frac{\partial g}{\partial x}(\beta,\theta_{c,2}(\beta),0)\\&\le -cD_d\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{(\cos(\beta\theta_{c,2}(\beta)/2)+\cosh(\beta E(\mathbf{k})))^2}\right)\\&\quad+D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\cosh(\beta E(\mathbf{k}))}{\cos(\beta\theta_{c,2}(\beta)/2)+\cosh(\beta E(\mathbf{k}))}\right)\\&\le -c(D_d,\mathsf{c})\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|\beta^{-3}\int_{\Gamma_\infty^*}d\mathbf{k}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-2}\\&\quad+c(b,D_d,\mathsf{c})\beta^{-2}\int_{\Gamma_\infty^*}d\mathbf{k}\left(e(\mathbf{k})^2+\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^2\right)^{-1}\end{aligned}$$

$$\le -1_{|\theta_{c,2}(\beta)/2-\pi/\beta|\le 1} c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}$$
$$\cdot\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^{\mathsf{s}-1-\mathsf{r}}\right)$$
$$-1_{|\theta_{c,2}(\beta)/2-\pi/\beta|>1} c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-3}$$
$$\cdot\left(1-c(b,D_d,\mathsf{c})\beta\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|\right)$$
$$\le -1_{|\theta_{c,2}(\beta)/2-\pi/\beta|\le 1} c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^{1-\mathsf{s}}(1-c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|)$$
$$-1_{|\theta_{c,2}(\beta)/2-\pi/\beta|>1} c(D_d,\mathsf{c})\beta^{-3}\left|\frac{\theta_{c,2}(\beta)}{2}-\frac{\pi}{\beta}\right|^{-3}(1-c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})|U|^{\frac{1}{2}})$$
$$<0.$$

In the last inequality we assumed that $|U|<c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})$.

Thus we have proved that

$$\frac{\partial g}{\partial x}(\beta,\theta_{c,1}(\beta),0)>0,\quad \frac{\partial g}{\partial x}(\beta,\theta_{c,2}(\beta),0)<0,\quad (\forall\beta\in\mathbb{R}_{>0}). \tag{2.12}$$

Now by combining (2.7) with (2.12) we conclude that there exists a positive constant $c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})$ such that if $|U|<c(b,D_d,\mathsf{c},\mathsf{r},\mathsf{s})$,

$$\frac{d\theta_{c,j}}{d\beta}(\beta)=-\frac{\frac{\partial g}{\partial x}(\beta,\theta_{c,j}(\beta),0)}{\frac{\partial g}{\partial y}(\beta,\theta_{c,j}(\beta),0)}<0,\quad (\forall j\in\{1,2\},\ \beta\in\mathbb{R}_{>0}).\ \square$$

Let us assume that $U\in(-c_3,0)$ with the constant c_3 appearing in Lemma 2.4. For $m\in\mathbb{N}\cup\{0\}$, $j\in\{1,2\}$ we define the function $\theta_{c,j,m}:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ by

$$\theta_{c,j,m}(x):=\theta_{c,j}(x)+\frac{4\pi}{x}m.$$

By Lemma 2.4 (iv) the continuous function $\theta_{c,j,m}:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ is monotone decreasing and thus injective. By the fact that $0<\theta_{c,1}(\beta)<\frac{2\pi}{\beta}<\theta_{c,2}(\beta)<\frac{4\pi}{\beta}$ and Lemma 2.4 (i),

(2.13)

$$\frac{\pi}{\beta}\le\theta_{c,1,0}(\beta),$$

$$\frac{4\pi}{\beta}m < \theta_{c,1,m}(\beta) < \frac{2\pi}{\beta} + \frac{4\pi}{\beta}m < \theta_{c,2,m}(\beta) < \frac{4\pi}{\beta} + \frac{4\pi}{\beta}m, \quad (\forall \beta \in \mathbb{R}_{>0}).$$

This implies that the function $\theta_{c,j,m}$ is surjective and thus bijective. Let $\beta_{c,j,m}$ denote the inverse function of $\theta_{c,j,m}$.

The phase boundaries are characterized in the next proposition.

Proposition 2.5. *Let c_3 be the positive constant appearing in Lemma 2.4. Assume that $U \in (-c_3, 0)$. Then the following statements hold.*

(i) For any $m \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}_{>0}$,

$$\theta_{c,1,m}(x) \in \left(\frac{4\pi m}{x}, \frac{2\pi + 4\pi m}{x}\right), \quad \theta_{c,2,m}(x) \in \left(\frac{2\pi + 4\pi m}{x}, \frac{4\pi(m+1)}{x}\right).$$

(ii) For any $m \in \mathbb{N} \cup \{0\}$, $y \in \mathbb{R}_{>0}$,

$$\beta_{c,1,m}(y) \in \left(\frac{4\pi m}{y}, \frac{2\pi + 4\pi m}{y}\right), \quad \beta_{c,2,m}(y) \in \left(\frac{2\pi + 4\pi m}{y}, \frac{4\pi(m+1)}{y}\right).$$

(iii) Let $(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$. The following statements are equivalent to each other.

(a)

$$\Delta(\beta, \theta) > 0.$$

(b)

$$|\theta| \in \bigcup_{m \in \mathbb{N} \cup \{0\}} (\theta_{c,1,m}(\beta), \theta_{c,2,m}(\beta)).$$

(c) $\theta \neq 0$ and

$$\beta \in \bigcup_{m \in \mathbb{N} \cup \{0\}} (\beta_{c,1,m}(|\theta|), \beta_{c,2,m}(|\theta|)).$$

Proof. We have already seen the claim (i) in (2.13). The claim (ii) follows from the claim (i) and the definition of $\beta_{c,j,m}(\cdot)$ $(j = 1, 2)$. Take any $(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$. There uniquely exist $m' \in \mathbb{N} \cup \{0\}$, $\theta' \in [0, 4\pi/\beta)$ such that $|\theta| = \theta' + \frac{4\pi}{\beta}m'$. Let us confirm the claim (iii). The statement (a) is equivalent to $g(\beta, \theta, 0) > 0$, which is equivalent to $\theta' \in (\theta_{c,1}(\beta), \theta_{c,2}(\beta))$ since $g(\beta, \theta, 0) = g(\beta, \theta', 0)$. The inclusion $\theta' \in (\theta_{c,1}(\beta), \theta_{c,2}(\beta))$ is equivalent to the statement (b). Thus the equivalence between (a) and (b) is proved. The equivalence between (b) and (c) can be deduced from the definition of $\beta_{c,j,m}(\cdot)$ $(j = 1, 2)$. □

Based on Lemma 2.4 (iv) and Proposition 2.5, we can sketch the $\beta - |\theta|$ phase diagram as in Figure 3.

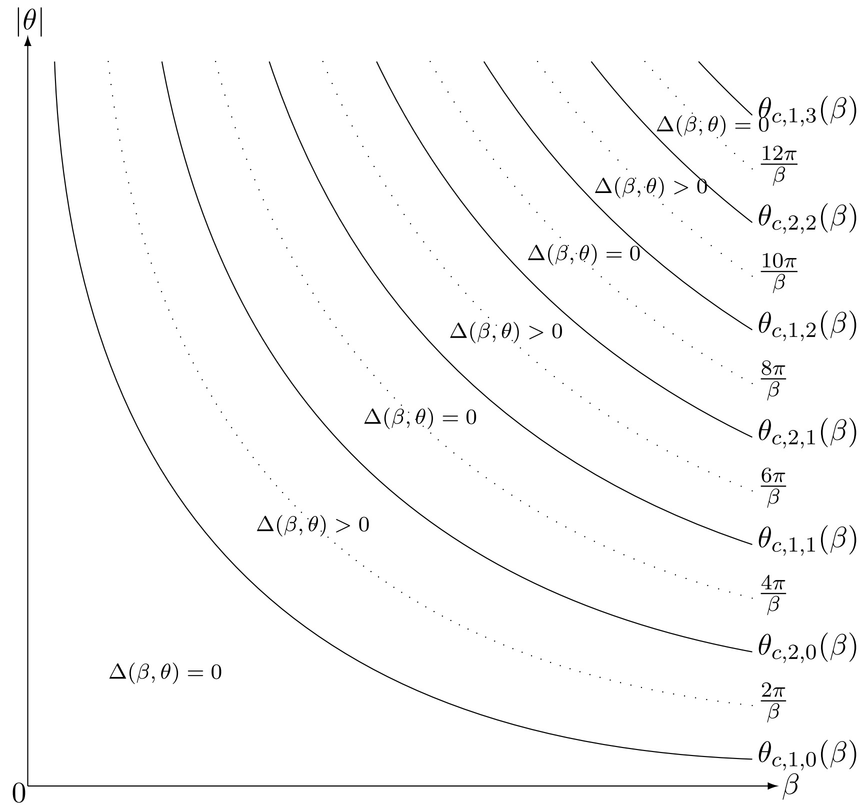

Fig. 3. The schematic $\beta - |\theta|$ phase diagram.

We can understand from Proposition 2.5 that for any fixed $\theta \in \mathbb{R}\backslash\{0\}$ the system repeatedly enters and exits a superconducting phase where $\Delta(\beta, \theta) > 0$ as β varies from 0 to ∞. It is notable that there are infinitely many critical temperatures.

2.3. The second order phase transitions

Using the function $\Delta : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}_{\ge 0}$, we define the function $F : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} F(x,y) :=& \frac{\Delta(x,y)^2}{|U|} \\ & - \frac{D_d}{x} \int_{\Gamma_\infty^*} d\mathbf{k} \,\mathrm{Tr}\log\Bigg(2\cos\left(\frac{xy}{2}\right)e^{-xE(\mathbf{k})} \\ & \qquad + e^{x(\sqrt{E(\mathbf{k})^2+\Delta(x,y)^2}-E(\mathbf{k}))} + e^{-x(\sqrt{E(\mathbf{k})^2+\Delta(x,y)^2}+E(\mathbf{k}))}\Bigg). \end{aligned}$$

Equally, we can write as follows.

$$\begin{aligned} F(x,y) =& \frac{\Delta(x,y)^2}{|U|} \\ & - \frac{D_d}{x} \int_{\Gamma_\infty^*} d\mathbf{k} \,\mathrm{Tr}\log\left(\cos\left(\frac{xy}{2}\right) + \cosh(x\sqrt{E(\mathbf{k})^2+\Delta(x,y)^2})\right) \\ & - \frac{b\log 2}{x} + D_d \int_{\Gamma_\infty^*} d\mathbf{k} \,\mathrm{Tr}\, E(\mathbf{k}). \end{aligned}$$

Since $\Delta(x,y) > 0$ if $xy/2 \in \pi(2\mathbb{Z}+1)$, F is well-defined. Recalling Theorem 1.3 (ii), we see that $F(\beta,\theta)$ is equal to the free energy density for $(\beta,\theta,U) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{<0}$ satisfying $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$ and (1.18).

We end this section by proving that the first order derivatives of F are globally continuous and the second order derivatives of F have jump discontinuities across the phase boundaries. Since these properties hold in the parameter region where F is proved to be equal to the free energy density by Theorem 1.3, we can consider that our many-electron system shows the second order phase transitions driven by the temperature and the imaginary magnetic field.

PROPOSITION 2.6. *Let c_3 be the positive constant appearing in Lemma 2.4 and $U \in (-c_3, 0)$. Then the following statements hold true.*

(i)

$\mathbb{R}_{>0} \times \mathbb{R} \backslash O_+ \cup O_-$

$$= \{(\beta, \delta\theta_{c,j,m}(\beta)) \mid \beta \in \mathbb{R}_{>0},\ j \in \{1,2\},\ m \in \mathbb{N} \cup \{0\},\ \delta \in \{1,-1\}\}$$
$$= \{(\beta_{c,j,m}(\theta), \delta\theta) \mid \theta \in \mathbb{R}_{>0},\ j \in \{1,2\},\ m \in \mathbb{N} \cup \{0\},\ \delta \in \{1,-1\}\}.$$

(ii)

$$F|_{O_+ \cup O_-} \in C^\infty(O_+ \cup O_-), \quad F \in C^1(\mathbb{R}_{>0} \times \mathbb{R}).$$

(iii) For any $\theta \in \mathbb{R}_{>0}$, $j \in \{1,2\}$, $m \in \mathbb{N} \cup \{0\}$, $\delta \in \{1,-1\}$, $\lim_{\beta \nearrow \beta_{c,j,m}(\theta)} \frac{\partial^2 F}{\partial \beta^2}(\beta, \delta\theta)$, $\lim_{\beta \searrow \beta_{c,j,m}(\theta)} \frac{\partial^2 F}{\partial \beta^2}(\beta, \delta\theta)$ *converge and*

$$\lim_{\beta \nearrow \beta_{c,1,m}(\theta)} \frac{\partial^2 F}{\partial \beta^2}(\beta, \delta\theta) > \lim_{\beta \searrow \beta_{c,1,m}(\theta)} \frac{\partial^2 F}{\partial \beta^2}(\beta, \delta\theta),$$
$$\lim_{\beta \nearrow \beta_{c,2,m}(\theta)} \frac{\partial^2 F}{\partial \beta^2}(\beta, \delta\theta) < \lim_{\beta \searrow \beta_{c,2,m}(\theta)} \frac{\partial^2 F}{\partial \beta^2}(\beta, \delta\theta).$$

(iv) For any $\beta \in \mathbb{R}_{>0}$, $j \in \{1,2\}$, $m \in \mathbb{N} \cup \{0\}$, $\delta \in \{1,-1\}$, $\lim_{\theta \nearrow \delta\theta_{c,j,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta)$, $\lim_{\theta \searrow \delta\theta_{c,j,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta)$ *converge and*

$$\lim_{\theta \nearrow \theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) > \lim_{\theta \searrow \theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta),$$
$$\lim_{\theta \nearrow \theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) < \lim_{\theta \searrow \theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta),$$
$$\lim_{\theta \nearrow -\theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) < \lim_{\theta \searrow -\theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta),$$
$$\lim_{\theta \nearrow -\theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) > \lim_{\theta \searrow -\theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta).$$

PROOF. (i): We can deduce the claim from Lemma 2.3, the definitions of $\theta_{c,j,m}(\cdot)$, $\beta_{c,j,m}(\cdot)$ and the fact that $g(\beta, \theta, 0) = g(\beta, |\theta| + \frac{4\pi}{\beta} n, 0)$ $(\forall n \in \mathbb{Z})$.

(ii): Set

$$D := \left\{(x, y, z) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R} \;\middle|\; \frac{xy}{2} \notin \pi(2\mathbb{Z} + 1) \text{ or } z > 0\right\},$$

which is an open set of $\mathbb{R}^3$. We define the function $\hat{F} : D \to \mathbb{R}$ by

$$\hat{F}(x, y, z) := \frac{z^2}{|U|} - \frac{D_d}{x} \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr} \log \left(\cos\left(\frac{xy}{2}\right) + \cosh(x\sqrt{E(\mathbf{k})^2 + z^2}) \right).$$

We can see that

$$\hat{F} \in C^{\infty}(D), \tag{2.14}$$

$$(\beta, \theta, \Delta(\beta, \theta)) \in D, \tag{2.15}$$

$$F(\beta, \theta) = \hat{F}(\beta, \theta, \Delta(\beta, \theta)) - \frac{b \log 2}{\beta} + D_d \int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr} E(\mathbf{k}), \tag{2.16}$$

$$(\forall (\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}).$$

Combined with Lemma 2.2, the functions $(\beta, \theta) \mapsto \hat{F}(\beta, \theta, \Delta(\beta, \theta))$, $(\beta, \theta) \mapsto F(\beta, \theta)$ are seen to be continuous in $\mathbb{R}_{>0} \times \mathbb{R}$ and C^{∞}-class in $O_+ \cup O_-$.

Let us prove that $F \in C^1(\mathbb{R}_{>0} \times \mathbb{R})$. For $(\beta, \theta) \in O_+ \cup O_-$

$$\frac{\partial \hat{F}}{\partial z}(\beta, \theta, \Delta(\beta, \theta)) = -g(\beta, \theta, \Delta(\beta, \theta))\Delta(\beta, \theta) = 0,$$

and thus

(2.17)
$$\begin{aligned} \frac{\partial}{\partial \beta} \hat{F}(\beta, \theta, \Delta(\beta, \theta)) &= \frac{\partial \hat{F}}{\partial x}(\beta, \theta, \Delta(\beta, \theta)) + \frac{\partial \hat{F}}{\partial z}(\beta, \theta, \Delta(\beta, \theta)) \frac{\partial \Delta}{\partial \beta}(\beta, \theta) \\ &= \frac{\partial \hat{F}}{\partial x}(\beta, \theta, \Delta(\beta, \theta)), \end{aligned}$$

(2.18)
$$\begin{aligned} \frac{\partial}{\partial \theta} \hat{F}(\beta, \theta, \Delta(\beta, \theta)) &= \frac{\partial \hat{F}}{\partial y}(\beta, \theta, \Delta(\beta, \theta)) + \frac{\partial \hat{F}}{\partial z}(\beta, \theta, \Delta(\beta, \theta)) \frac{\partial \Delta}{\partial \theta}(\beta, \theta) \\ &= \frac{\partial \hat{F}}{\partial y}(\beta, \theta, \Delta(\beta, \theta)). \end{aligned}$$

Note that

$$\Delta(\beta, \theta) = 0, \quad (\forall (\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R} \backslash O_+ \cup O_-). \tag{2.19}$$

It follows from the global continuity of $\Delta(\cdot, \cdot)$, (2.14), (2.15), (2.19) and the characterization of $\mathbb{R}_{>0} \times \mathbb{R} \backslash O_+ \cup O_-$ given in (i) that

$$\lim_{\substack{(\beta, \theta) \to (\beta', \theta') \\ (\beta, \theta) \in O_+ \cup O_-}} \frac{\partial^{m+n} \hat{F}}{\partial x^m \partial y^n}(\beta, \theta, \Delta(\beta, \theta)) = \frac{\partial^{m+n} \hat{F}}{\partial x^m \partial y^n}(\beta', \theta', 0), \tag{2.20}$$

$$(\forall(\beta',\theta')\in\mathbb{R}_{>0}\times\mathbb{R}\backslash O_+\cup O_-,\ m,n\in\mathbb{N}\cup\{0\}).$$

By (2.17), (2.18), (2.20) we can observe that for any $(\beta',\theta')\in\mathbb{R}_{>0}\times\mathbb{R}\backslash O_+\cup O_-$

$$\begin{aligned}\lim_{\substack{(\beta,\theta)\to(\beta',\theta')\\(\beta,\theta)\in O_+\cup O_-}}\frac{\partial}{\partial\beta}\hat{F}(\beta,\theta,\Delta(\beta,\theta))&=\lim_{\substack{(\beta,\theta)\to(\beta',\theta')\\(\beta,\theta)\in O_+\cup O_-}}\frac{\partial\hat{F}}{\partial x}(\beta,\theta,\Delta(\beta,\theta))\\&=\frac{\partial\hat{F}}{\partial x}(\beta',\theta',0),\\\lim_{\substack{(\beta,\theta)\to(\beta',\theta')\\(\beta,\theta)\in O_+\cup O_-}}\frac{\partial}{\partial\theta}\hat{F}(\beta,\theta,\Delta(\beta,\theta))&=\lim_{\substack{(\beta,\theta)\to(\beta',\theta')\\(\beta,\theta)\in O_+\cup O_-}}\frac{\partial\hat{F}}{\partial y}(\beta,\theta,\Delta(\beta,\theta))\\&=\frac{\partial\hat{F}}{\partial y}(\beta',\theta',0),\end{aligned}$$

which together with the characterization of $\mathbb{R}_{>0}\times\mathbb{R}\backslash O_+\cup O_-$ given in (i) implies that $(\beta,\theta)\mapsto\hat{F}(\beta,\theta,\Delta(\beta,\theta))$ is partially differentiable in $\mathbb{R}_{>0}\times\mathbb{R}$ and

$$\begin{aligned}&\frac{\partial}{\partial\beta}\hat{F}(\beta,\theta,\Delta(\beta,\theta))=\frac{\partial\hat{F}}{\partial x}(\beta,\theta,\Delta(\beta,\theta)),\\&\frac{\partial}{\partial\theta}\hat{F}(\beta,\theta,\Delta(\beta,\theta))=\frac{\partial\hat{F}}{\partial y}(\beta,\theta,\Delta(\beta,\theta)),\quad(\forall(\beta,\theta)\in\mathbb{R}_{>0}\times\mathbb{R}).\end{aligned}$$

Since $(\beta,\theta)\mapsto\frac{\partial\hat{F}}{\partial x}(\beta,\theta,\Delta(\beta,\theta))$, $(\beta,\theta)\mapsto\frac{\partial\hat{F}}{\partial y}(\beta,\theta,\Delta(\beta,\theta))$ are continuous in $\mathbb{R}_{>0}\times\mathbb{R}$, we can conclude that the function $(\beta,\theta)\mapsto\hat{F}(\beta,\theta,\Delta(\beta,\theta))$ belongs to $C^1(\mathbb{R}_{>0}\times\mathbb{R})$ and so does the function F.

(iii): By (2.17), for $(\beta,\theta)\in O_+\cup O_-$

$$\begin{aligned}&\frac{\partial^2}{\partial\beta^2}\hat{F}(\beta,\theta,\Delta(\beta,\theta))\\&=\frac{\partial}{\partial\beta}\frac{\partial\hat{F}}{\partial x}(\beta,\theta,\Delta(\beta,\theta))\\&=\frac{\partial^2\hat{F}}{\partial x^2}(\beta,\theta,\Delta(\beta,\theta))+\frac{\partial^2\hat{F}}{\partial z\partial x}(\beta,\theta,\Delta(\beta,\theta))\frac{\partial\Delta}{\partial\beta}(\beta,\theta).\end{aligned}\tag{2.21}$$

In particular for $(\beta,\theta)\in O_-$

$$\frac{\partial^2}{\partial\beta^2}\hat{F}(\beta,\theta,\Delta(\beta,\theta))=\frac{\partial^2\hat{F}}{\partial x^2}(\beta,\theta,\Delta(\beta,\theta)).\tag{2.22}$$

It follows from the claim (i) and (2.20) that for any $\theta \in \mathbb{R}_{>0}$, $j \in \{1,2\}$, $m \in \mathbb{N} \cup \{0\}$, $\delta \in \{1,-1\}$

$$\begin{aligned}\lim_{\beta \searrow \beta_{c,j,m}(\theta)} \frac{\partial^2 \hat{F}}{\partial x^2}(\beta, \delta\theta, \Delta(\beta,\delta\theta)) &= \lim_{\beta \nearrow \beta_{c,j,m}(\theta)} \frac{\partial^2 \hat{F}}{\partial x^2}(\beta, \delta\theta, \Delta(\beta,\delta\theta)) \\ &= \frac{\partial^2 \hat{F}}{\partial x^2}(\beta_{c,j,m}(\theta), \delta\theta, 0).\end{aligned} \tag{2.23}$$

For $(\beta,\theta) \in O_+$ one can derive that

$$\frac{\partial^2 \hat{F}}{\partial x \partial z}(\beta,\theta,\Delta(\beta,\theta)) = -\Delta(\beta,\theta)\frac{\partial g}{\partial x}(\beta,\theta,\Delta(\beta,\theta)). \tag{2.24}$$

Also by taking into account (2.5),

$$\frac{\partial \Delta}{\partial \beta}(\beta,\theta) = -\frac{\frac{\partial g}{\partial x}(\beta,\theta,\Delta(\beta,\theta))}{\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta))}. \tag{2.25}$$

By combining (2.24) with (2.25) we obtain that

$$\frac{\partial^2 \hat{F}}{\partial x \partial z}(\beta,\theta,\Delta(\beta,\theta))\frac{\partial \Delta}{\partial \beta}(\beta,\theta) = \Delta(\beta,\theta)\frac{(\frac{\partial g}{\partial x}(\beta,\theta,\Delta(\beta,\theta)))^2}{\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta))}. \tag{2.26}$$

Observe that for any $(x,y,z) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0}$

(2.27)

$$\begin{aligned}&z^{-1}\frac{\partial g}{\partial z}(x,y,z) \\ &= -D_d x \int_{\Gamma_\infty^*} d\mathbf{k} \\ &\quad \cdot \mathrm{Tr}\Bigg(\frac{1}{(\cos(xy/2) + \cosh(x\sqrt{E(\mathbf{k})^2+z^2}))^2(E(\mathbf{k})^2+z^2)} \\ &\quad\quad \cdot \Bigg(\sinh^2(x\sqrt{E(\mathbf{k})^2+z^2}) \\ &\quad\quad\quad - \left(\cosh(x\sqrt{E(\mathbf{k})^2+z^2}) - \frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{x\sqrt{E(\mathbf{k})^2+z^2}}\right)\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\cdot\left(\cos\left(\frac{xy}{2}\right)+\cosh(x\sqrt{E(\mathbf{k})^2+z^2})\right)\Bigg)\Bigg)\\
&\le -D_d x\int_{\Gamma_\infty^*}d\mathbf{k}\\
&\quad\cdot\mathrm{Tr}\Bigg(\frac{1}{(\cos(xy/2)+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))^2(E(\mathbf{k})^2+z^2)}\\
&\qquad\cdot(1+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))\left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{x\sqrt{E(\mathbf{k})^2+z^2}}-1\right)\Bigg)\\
&\le -\frac{D_d}{3!}x^3\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{\cos(xy/2)+\cosh(x\sqrt{E(\mathbf{k})^2+z^2})}\right)\\
&<0.
\end{aligned}
$$

By Proposition 2.5 (ii) for any $\theta\in\mathbb{R}_{>0}$, $m\in\mathbb{N}\cup\{0\}$, $\delta\in\{1,-1\}$, $j\in\{1,2\}$, $\delta\theta\beta_{c,j,m}(\theta)/2\notin\pi(2\mathbb{Z}+1)$. Thus we can see from Proposition 2.5 (iii), the global continuity of $\Delta(\cdot,\cdot)$ and (2.27) that

$$
\begin{aligned}
&\lim_{\beta\searrow\beta_{c,1,m}(\theta)}\Delta(\beta,\delta\theta)^{-1}\frac{\partial g}{\partial z}(\beta,\delta\theta,\Delta(\beta,\delta\theta)),\\
&\lim_{\beta\nearrow\beta_{c,2,m}(\theta)}\Delta(\beta,\delta\theta)^{-1}\frac{\partial g}{\partial z}(\beta,\delta\theta,\Delta(\beta,\delta\theta))
\end{aligned}
$$

converge to negative values. On the other hand, we can see from (2.8), (2.12) and Proposition 2.5 (i) that for any $\theta\in\mathbb{R}_{>0}$, $m\in\mathbb{N}\cup\{0\}$, $\delta\in\{1,-1\}$,

$$
\begin{aligned}
&\lim_{\beta\searrow\beta_{c,1,m}(\theta)}\frac{\partial g}{\partial x}(\beta,\delta\theta,\Delta(\beta,\delta\theta))=\frac{\partial g}{\partial x}(\beta_{c,1,m}(\theta),\theta,0)\\
&=\frac{\partial g}{\partial x}(\beta_{c,1,m}(\theta),\theta_{c,1,m}(\beta_{c,1,m}(\theta)),0)\ge\frac{\partial g}{\partial x}(\beta_{c,1,m}(\theta),\theta_{c,1}(\beta_{c,1,m}(\theta)),0)>0,\\
&\lim_{\beta\nearrow\beta_{c,2,m}(\theta)}\frac{\partial g}{\partial x}(\beta,\delta\theta,\Delta(\beta,\delta\theta))=\frac{\partial g}{\partial x}(\beta_{c,2,m}(\theta),\theta,0)\\
&=\frac{\partial g}{\partial x}(\beta_{c,2,m}(\theta),\theta_{c,2,m}(\beta_{c,2,m}(\theta)),0)\le\frac{\partial g}{\partial x}(\beta_{c,2,m}(\theta),\theta_{c,2}(\beta_{c,2,m}(\theta)),0)<0.
\end{aligned}
$$

It follows from Proposition 2.5 (iii), (2.21), (2.22), (2.23), (2.26) and the above convergence results that for any $\theta\in\mathbb{R}_{>0}$, $m\in\mathbb{N}\cup\{0\}$, $\delta\in\{1,-1\}$

$$
\lim_{\beta\searrow\beta_{c,1,m}(\theta)}\frac{\partial^2}{\partial\beta^2}\hat{F}(\beta,\delta\theta,\Delta(\beta,\delta\theta))
$$

$$
\begin{aligned}
&= \frac{\partial^2 \hat{F}}{\partial x^2}(\beta_{c,1,m}(\theta), \delta\theta, 0) + \frac{(\frac{\partial g}{\partial x}(\beta_{c,1,m}(\theta), \theta, 0))^2}{\lim_{\beta \searrow \beta_{c,1,m}(\theta)} \Delta(\beta, \delta\theta)^{-1} \frac{\partial g}{\partial z}(\beta, \delta\theta, \Delta(\beta, \delta\theta))} \\
&< \frac{\partial^2 \hat{F}}{\partial x^2}(\beta_{c,1,m}(\theta), \delta\theta, 0) \\
&= \lim_{\beta \nearrow \beta_{c,1,m}(\theta)} \frac{\partial^2}{\partial \beta^2} \hat{F}(\beta, \delta\theta, \Delta(\beta, \delta\theta)),
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\beta \nearrow \beta_{c,2,m}(\theta)} \frac{\partial^2}{\partial \beta^2} \hat{F}(\beta, \delta\theta, \Delta(\beta, \delta\theta)) \\
&= \frac{\partial^2 \hat{F}}{\partial x^2}(\beta_{c,2,m}(\theta), \delta\theta, 0) + \frac{(\frac{\partial g}{\partial x}(\beta_{c,2,m}(\theta), \theta, 0))^2}{\lim_{\beta \nearrow \beta_{c,2,m}(\theta)} \Delta(\beta, \delta\theta)^{-1} \frac{\partial g}{\partial z}(\beta, \delta\theta, \Delta(\beta, \delta\theta))} \\
&< \frac{\partial^2 \hat{F}}{\partial x^2}(\beta_{c,2,m}(\theta), \delta\theta, 0) \\
&= \lim_{\beta \searrow \beta_{c,2,m}(\theta)} \frac{\partial^2}{\partial \beta^2} \hat{F}(\beta, \delta\theta, \Delta(\beta, \delta\theta)),
\end{aligned}
$$

which together with the equality (2.16) implies the claim.

(iv): By (2.18), for $(\beta, \theta) \in O_+ \cup O_-$

$$
\begin{aligned}
\text{(2.28)} \qquad \frac{\partial^2}{\partial \theta^2} \hat{F}(\beta, \theta, \Delta(\beta, \theta)) &= \frac{\partial}{\partial \theta} \frac{\partial \hat{F}}{\partial y}(\beta, \theta, \Delta(\beta, \theta)) \\
&= \frac{\partial^2 \hat{F}}{\partial y^2}(\beta, \theta, \Delta(\beta, \theta)) + \frac{\partial^2 \hat{F}}{\partial z \partial y}(\beta, \theta, \Delta(\beta, \theta)) \frac{\partial \Delta}{\partial \theta}(\beta, \theta).
\end{aligned}
$$

For $(\beta, \theta) \in O_-$

$$
\text{(2.29)} \qquad \frac{\partial^2}{\partial \theta^2} \hat{F}(\beta, \theta, \Delta(\beta, \theta)) = \frac{\partial^2 \hat{F}}{\partial y^2}(\beta, \theta, \Delta(\beta, \theta)).
$$

By the claim (i) and (2.20), for any $\beta \in \mathbb{R}_{>0}$, $j \in \{1, 2\}$, $m \in \mathbb{N} \cup \{0\}$

$$
\begin{aligned}
\text{(2.30)} \qquad \lim_{\theta \searrow \theta_{c,j,m}(\beta)} \frac{\partial^2 \hat{F}}{\partial y^2}(\beta, \theta, \Delta(\beta, \theta)) &= \lim_{\theta \nearrow \theta_{c,j,m}(\beta)} \frac{\partial^2 \hat{F}}{\partial y^2}(\beta, \theta, \Delta(\beta, \theta)) \\
&= \frac{\partial^2 \hat{F}}{\partial y^2}(\beta, \theta_{c,j,m}(\beta), 0).
\end{aligned}
$$

For $(\beta, \theta) \in O_+$ we can derive in the same way as the derivation of (2.26) that

$$\frac{\partial^2 \hat{F}}{\partial z \partial y}(\beta,\theta,\Delta(\beta,\theta))\frac{\partial \Delta}{\partial \theta}(\beta,\theta) = \Delta(\beta,\theta)\frac{(\frac{\partial g}{\partial y}(\beta,\theta,\Delta(\beta,\theta)))^2}{\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta))}. \tag{2.31}$$

By Proposition 2.5 (i) for any $\beta \in \mathbb{R}_{>0}$, $m \in \mathbb{N} \cup \{0\}$, $j \in \{1,2\}$, $\beta\theta_{c,j,m}(\beta)/2 \notin \pi(2\mathbb{Z}+1)$. Thus we can deduce from Proposition 2.5 (iii), the global continuity of $\Delta(\cdot,\cdot)$ and (2.27) that

$$\lim_{\theta \searrow \theta_{c,1,m}(\beta)} \Delta(\beta,\theta)^{-1}\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta)),$$
$$\lim_{\theta \nearrow \theta_{c,2,m}(\theta)} \Delta(\beta,\theta)^{-1}\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta))$$

converge to negative values. On the other hand, it follows from (2.6), (2.7) that

$$\lim_{\theta \searrow \theta_{c,1,m}(\beta)} \frac{\partial g}{\partial y}(\beta,\theta,\Delta(\beta,\theta)) = \frac{\partial g}{\partial y}(\beta,\theta_{c,1,m}(\beta),0) = \frac{\partial g}{\partial y}(\beta,\theta_{c,1}(\beta),0) > 0,$$
$$\lim_{\theta \nearrow \theta_{c,2,m}(\beta)} \frac{\partial g}{\partial y}(\beta,\theta,\Delta(\beta,\theta)) = \frac{\partial g}{\partial y}(\beta,\theta_{c,2}(\beta),0) < 0.$$

By Proposition 2.5 (iii), (2.28), (2.29), (2.30), (2.31) and the above convergent properties, for any $\beta \in \mathbb{R}_{>0}$, $m \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}
&\lim_{\theta \searrow \theta_{c,1,m}(\beta)} \frac{\partial^2}{\partial \theta^2}\hat{F}(\beta,\theta,\Delta(\beta,\theta)) \\
&= \frac{\partial^2 \hat{F}}{\partial y^2}(\beta,\theta_{c,1,m}(\beta),0) + \frac{(\frac{\partial g}{\partial y}(\beta,\theta_{c,1}(\beta),0))^2}{\lim_{\theta \searrow \theta_{c,1,m}(\beta)} \Delta(\beta,\theta)^{-1}\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta))} \\
&< \frac{\partial^2 \hat{F}}{\partial y^2}(\beta,\theta_{c,1,m}(\beta),0) \\
&= \lim_{\theta \nearrow \theta_{c,1,m}(\beta)} \frac{\partial^2}{\partial \theta^2}\hat{F}(\beta,\theta,\Delta(\beta,\theta)), \\
&\lim_{\theta \nearrow \theta_{c,2,m}(\beta)} \frac{\partial^2}{\partial \theta^2}\hat{F}(\beta,\theta,\Delta(\beta,\theta)) \\
&= \frac{\partial^2 \hat{F}}{\partial y^2}(\beta,\theta_{c,2,m}(\beta),0) + \frac{(\frac{\partial g}{\partial y}(\beta,\theta_{c,2}(\beta),0))^2}{\lim_{\theta \nearrow \theta_{c,2,m}(\beta)} \Delta(\beta,\theta)^{-1}\frac{\partial g}{\partial z}(\beta,\theta,\Delta(\beta,\theta))}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\partial^2 \hat{F}}{\partial y^2}(\beta, \theta_{c,2,m}(\beta), 0) \\
&= \lim_{\theta \searrow \theta_{c,2,m}(\beta)} \frac{\partial^2}{\partial \theta^2} \hat{F}(\beta, \theta, \Delta(\beta, \theta)).
\end{aligned}
$$

Now recalling (2.16), we reach the conclusion that

$$
\begin{aligned}
&\lim_{\theta \searrow \theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) < \lim_{\theta \nearrow \theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta), \\
&\lim_{\theta \nearrow \theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) < \lim_{\theta \searrow \theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta), \quad (\forall \beta \in \mathbb{R}_{>0}).
\end{aligned}
$$

Note that for any $(\beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$, $\Delta(\beta, \theta) = \Delta(\beta, -\theta)$ and thus

$$
\begin{aligned}
&F(\beta, \theta) = F(\beta, -\theta), \\
&(\beta, \theta) \in O_+ \cup O_- \text{ if and only if } (\beta, -\theta) \in O_+ \cup O_-.
\end{aligned}
$$

Thus for $(\beta, \theta) \in O_+ \cup O_-$, $\frac{\partial^2 F}{\partial \theta^2}(\beta, -\theta) = \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta)$. Therefore

$$
\begin{aligned}
\lim_{\theta \nearrow -\theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) &= \lim_{\theta \searrow \theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) < \lim_{\theta \nearrow \theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) \\
&= \lim_{\theta \searrow -\theta_{c,1,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta), \\
\lim_{\theta \searrow -\theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) &= \lim_{\theta \nearrow \theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) < \lim_{\theta \searrow \theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta) \\
&= \lim_{\theta \nearrow -\theta_{c,2,m}(\beta)} \frac{\partial^2 F}{\partial \theta^2}(\beta, \theta).
\end{aligned}
$$

The claims have been proved. □

3. Formulation

In this section we derive Grassmann integral formulations of the grand canonical partition function of the model Hamiltonian. In essence the derivation can be completed by following [12, Section 2]. In order to support the readers, we state several lemmas leading to Lemma 3.6 step by step along the same lines as [12, Section 2]. One should be able to prove Lemma

3.6 by following the outline given in this section and the proofs presented in [12, Section 2]. We intend to adopt the notations used to formulate the 1-band problem in [12, Section 2] as much as possible so that the formulation procedure can be seen parallel.

Thanks to the next lemma, we can restrict the value of θ to prove the main results of this paper.

LEMMA 3.1. *Assume that $\theta' \in (-2\pi/\beta, 2\pi/\beta]$ and $\theta = \theta'$ (mod $4\pi/\beta$). Then*

$$\begin{aligned}
&\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})} = \mathrm{Tr}\, e^{-\beta(\mathsf{H}+i|\theta'|\mathsf{S}_z+\mathsf{F})},\\
&\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathcal{O}) = \mathrm{Tr}(e^{-\beta(\mathsf{H}+i|\theta'|\mathsf{S}_z+\mathsf{F})}\mathcal{O}),\\
&(\forall\mathcal{O} \in \{\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow},\ \psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow},\ \psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow}\}).
\end{aligned}$$

PROOF. This is essentially same as [12, Lemma 1.2]. By using the fact that S_z commutes with H, F, $\mathcal{O}$ and identifying the Fock space with the direct sum of the eigenspaces of S_z we can replace θ by θ' inside the trace operations. Then by (1.15) we can replace θ' by $|\theta'|$. □

In the rest of the paper for $\beta \in \mathbb{R}_{>0}$, $\theta \in \mathbb{R}$ we let $\theta(\beta)$ denote $|\theta'|$, where $\theta' \in (-2\pi/\beta, 2\pi/\beta]$ and $\theta = \theta'$ (mod $4\pi/\beta$). By the assumption $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$ we have that $\theta(\beta) \in [0, 2\pi/\beta)$.

We are going to formulate the normalized partition function

$$\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta(\beta)\mathsf{S}_z+\mathsf{F}+\mathsf{A})}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta(\beta)\mathsf{S}_z)}}$$

into a time-continuum limit of a finite-dimensional Grassmann Gaussian integral, where we set

$$\begin{aligned}
&\mathsf{A} := \lambda_1\mathsf{A}_1 + \lambda_2\mathsf{A}_2,\\
&\mathsf{A}_1 := \psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow},\\
&\mathsf{A}_2 := \psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow}
\end{aligned}$$

with the artificial parameters $\lambda_1, \lambda_2 \in \mathbb{C}$ and fixed sites $(\hat{\rho}, \hat{\mathbf{x}}), (\hat{\eta}, \hat{\mathbf{y}}) \in \mathcal{B} \times \Gamma_\infty$. The reason why we insert the operator A is that we can simply derive the thermal expectations of our interest by differentiating the partition function with the parameters λ_1, λ_2. We can compute the denominator and check that it is non-zero because of the property $\theta(\beta) \in [0, 2\pi/\beta)$.

Lemma 3.2.

$$\begin{aligned}
&\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0 + i\theta(\beta)\mathsf{S}_z)} = \prod_{\mathbf{k}\in\Gamma^*} \det\left(1 + 2\cos\left(\frac{\beta\theta(\beta)}{2}\right) e^{-\beta E(\mathbf{k})} + e^{-2\beta E(\mathbf{k})}\right)\\
&= e^{-\beta \sum_{\mathbf{k}\in\Gamma^*} \mathrm{Tr}\, E(\mathbf{k})} 2^{bL^d} \prod_{\mathbf{k}\in\Gamma^*} \det\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right) + \cosh(\beta E(\mathbf{k}))\right).
\end{aligned}$$

Proof. This is a b-band version of [12, Lemma 2.1]. One can diagonalize $\mathsf{H}_0 + i\theta(\beta)\mathsf{S}_z$ with respect to the band index and derive the result. □

To state the first Grassmann integral formulation, let us introduce the Grassmann algebra and the covariance. With the parameter $h \in \frac{2}{\beta}\mathbb{N}$, set $[0,\beta)_h := \{0, 1/h, 2/h, \cdots, \beta - 1/h\}$, which is a discretization of $[0,\beta)$. Define the index sets J_0, J by $J_0 := \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0,\beta)_h$, $J := J_0 \times \{1, -1\}$. Let $\mathcal{W}$ be the complex vector space spanned by the abstract basis $\{\psi_X\}_{X\in J}$. We let $\bigwedge \mathcal{W}$ denote the Grassmann algebra generated by $\{\psi_X\}_{X\in J}$. For $X \in J_0$ we also use the notation $\overline{\psi}_X$, ψ_X in place of $\psi_{(X,1)}$, $\psi_{(X,-1)}$ respectively. We do not restate the definitions and basic properties of finite-dimensional Grassmann algebra and Grassmann integrations in detail. The readers should refer to [12, Subsection 2.1] for the summary of them in line with our purposes or to [4] for more general statements. Let us introduce the Grassmann polynomials $\mathsf{V}(\psi)$, $\mathsf{F}(\psi)$, $\mathsf{A}^1(\psi)$, $\mathsf{A}^2(\psi)$, $\mathsf{A}(\psi)$ $(\in \bigwedge \mathcal{W})$ formulating the operators V, F, A^1, A^2, A respectively as follows.

$$\begin{aligned}
&\mathsf{V}(\psi) := \frac{U}{L^d h} \sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma} \sum_{s\in[0,\beta)_h} \overline{\psi}_{\rho\mathbf{x}\uparrow s} \overline{\psi}_{\rho\mathbf{x}\downarrow s} \psi_{\eta\mathbf{y}\downarrow s} \psi_{\eta\mathbf{y}\uparrow s},\\
&\mathsf{F}(\psi) := \frac{\gamma}{h} \sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma} \sum_{s\in[0,\beta)_h} (\overline{\psi}_{\rho\mathbf{x}\uparrow s} \overline{\psi}_{\rho\mathbf{x}\downarrow s} + \psi_{\rho\mathbf{x}\downarrow s} \psi_{\rho\mathbf{x}\uparrow s}),\\
&\mathsf{A}^1(\psi) := \frac{1}{h} \sum_{s\in[0,\beta)_h} \overline{\psi}_{\hat{\rho} r_L(\hat{\mathbf{x}})\uparrow s} \overline{\psi}_{\hat{\rho} r_L(\hat{\mathbf{x}})\downarrow s},\\
&\mathsf{A}^2(\psi) := \frac{1}{h} \sum_{s\in[0,\beta)_h} \overline{\psi}_{\hat{\rho} r_L(\hat{\mathbf{x}})\uparrow s} \overline{\psi}_{\hat{\rho} r_L(\hat{\mathbf{x}})\downarrow s} \psi_{\hat{\eta} r_L(\hat{\mathbf{y}})\downarrow s} \psi_{\hat{\eta} r_L(\hat{\mathbf{y}})\uparrow s},
\end{aligned}$$

$$\mathsf{A}(\psi) := \sum_{j=1}^{2} \lambda_j \mathsf{A}^j(\psi).$$

The covariance G for the Grassmann Gaussian integral is defined as the free 2-point correlation function. For $(\rho, \mathbf{x}, \sigma, s), (\eta, \mathbf{y}, \tau, t) \in \mathcal{B} \times \Gamma_\infty \times \{\uparrow, \downarrow\} \times [0, \beta)$

$$G(\rho\mathbf{x}\sigma s, \eta\mathbf{y}\tau t) := \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}_0 + i\theta(\beta)\mathsf{S}_z)}(1_{s \ge t}\psi^*_{\rho\mathbf{x}\sigma}(s)\psi_{\eta\mathbf{y}\tau}(t) - 1_{s<t}\psi_{\eta\mathbf{y}\tau}(t)\psi^*_{\rho\mathbf{x}\sigma}(s)))}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0 + i\theta(\beta)\mathsf{S}_z)}},$$

where $\psi^{(*)}_{\rho\mathbf{x}\sigma}(s) := e^{s(\mathsf{H}_0 + i\theta(\beta)\mathsf{S}_z)}\psi^{(*)}_{\rho\mathbf{x}\sigma}e^{-s(\mathsf{H}_0 + i\theta(\beta)\mathsf{S}_z)}$. According to the conventional definition, any covariance for Grassmann Gaussian integral on $\bigwedge \mathcal{W}$ is a map from J_0^2 to $\mathbb{C}$. If we follow the convention, we should introduce our covariance as the restriction $G|_{J_0^2}$. However, we call G covariance and omit the sign $|_{J_0^2}$ even when the argument is restricted to J_0^2 for simplicity.

For $r \in \mathbb{R}_{>0}$ let $D(r)$ denote the open disk $\{z \in \mathbb{C} \mid |z| < r\}$.

LEMMA 3.3. *For any* $r \in \mathbb{R}_{>0}$

$$\lim_{\substack{h \to \infty \\ h \in \frac{2}{\beta}\mathbb{N}}} \sup_{\boldsymbol{\lambda} \in \overline{D(r)}^2} \left| \int e^{-\mathsf{V}(\psi) - \mathsf{F}(\psi) - \mathsf{A}(\psi)} d\mu_G(\psi) - \frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H} + i\theta(\beta)\mathsf{S}_z + \mathsf{F} + \mathsf{A})}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0 + i\theta(\beta)\mathsf{S}_z)}} \right| = 0.$$

Here $\boldsymbol{\lambda}$ *denotes* (λ_1, λ_2).

PROOF. The proof is parallel to that of [12, Lemma 2.2]. □

The next step is to reformulate the Grassmann Gaussian integral given in Lemma 3.3 into a hybrid of a Gaussian integral with Grassmann variables and a Gaussian integral with real variables. Define $\mathsf{V}_+(\psi)$, $\mathsf{V}_-(\psi)$, $\mathsf{W}(\psi) \in \bigwedge \mathcal{W}$ by

$$\mathsf{V}_+(\psi) := \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}} h} \sum_{(\rho,\mathbf{x}) \in \mathcal{B} \times \Gamma} \sum_{s \in [0,\beta)_h} \overline{\psi}_{\rho\mathbf{x}\uparrow s} \overline{\psi}_{\rho\mathbf{x}\downarrow s},$$

$$\mathsf{V}_-(\psi) := \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}} h} \sum_{(\rho,\mathbf{x}) \in \mathcal{B} \times \Gamma} \sum_{s \in [0,\beta)_h} \psi_{\rho\mathbf{x}\downarrow s} \psi_{\rho\mathbf{x}\uparrow s},$$

$$\mathsf{W}(\psi) := \frac{U}{\beta L^d h^2} \sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma} \sum_{s,t\in[0,\beta)_h} \overline{\psi}_{\rho\mathbf{x}\uparrow s}\overline{\psi}_{\rho\mathbf{x}\downarrow s}\psi_{\eta\mathbf{y}\downarrow t}\psi_{\eta\mathbf{y}\uparrow t}.$$

Lemma 3.4.

$$\begin{aligned}&\int e^{-\mathsf{V}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)} d\mu_G(\psi)\\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-|\phi|^2} \int e^{-\mathsf{V}(\psi)+\mathsf{W}(\psi)-\mathsf{F}(\psi)-\mathsf{A}(\psi)+\phi\mathsf{V}_+(\psi)+\overline{\phi}\mathsf{V}_-(\psi)} d\mu_G(\psi),\end{aligned}$$

where $\phi = \phi_1 + i\phi_2$, $|\phi| = \|\phi\|_{\mathbb{C}}$.

Proof. The proof is same as that of [12, Lemma 2.3] based on the Hubbard-Stratonovich transformation. □

As the final step of the formulation, we introduce the index $\{1,2\}$ and derive the integral formulation on Grassmann algebra indexed by $\{1,2\}$ rather than by the spin $\{\uparrow,\downarrow\}$. The new index sets are defined by

$$I_0 := \{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)_h, \quad I := I_0\times\{1,-1\}.$$

Let $\mathcal{V}$ be the complex vector space spanned by the basis $\{\psi_X\}_{X\in I}$. We define the Grassmann polynomials $V(\psi)$, $W(\psi)$, $A^1(\psi)$, $A^2(\psi)$, $A(\psi)\in\bigwedge\mathcal{V}$ by

$$\begin{aligned}V(\psi) &:= \frac{U}{L^d h}\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{1\rho\mathbf{x}s}\\ &\quad + \frac{U}{L^d h}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{2\rho\mathbf{x}s}\overline{\psi}_{2\eta\mathbf{y}s}\psi_{1\eta\mathbf{y}s},\\ W(\psi) &:= \frac{U}{\beta L^d h^2}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{s,t\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{2\rho\mathbf{x}s}\overline{\psi}_{2\eta\mathbf{y}t}\psi_{1\eta\mathbf{y}t},\\ A^1(\psi) &:= \frac{1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\psi_{2\hat{\rho}r_L(\hat{\mathbf{x}})s},\\ A^2(\psi) &:= 1_{(\hat{\rho},r_L(\hat{\mathbf{x}}))=(\hat{\eta},r_L(\hat{\mathbf{y}}))}\frac{1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\psi_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\end{aligned}$$

$$+\frac{1}{h}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\hat{\rho}r_L(\hat{\mathbf{x}})s}\psi_{2\hat{\rho}r_L(\hat{\mathbf{x}})s}\overline{\psi}_{2\hat{\eta}r_L(\hat{\mathbf{y}})s}\psi_{1\hat{\eta}r_L(\hat{\mathbf{y}})s},$$

$$A(\psi):=\sum_{j=1}^{2}\lambda_j A^j(\psi). \tag{3.1}$$

Though the final formulation Lemma 3.6 does not explicitly involve any partition function of a Hamiltonian on the Fock space $F_f(L^2(\{1,2\}\times\mathcal{B}\times\Gamma))$, the final formulation can be systematically derived by relating such a partition function to the Grassmann Gaussian integral over $\bigwedge\mathcal{V}$. To this end, let us define a free Hamiltonian on $F_f(L^2(\{1,2\}\times\mathcal{B}\times\Gamma))$. For any $n\in\mathbb{N}$ let I_n denote the $n\times n$ unit matrix. For $\phi\in\mathbb{C}$, set

$$H_0(\phi):=\frac{1}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\cdot\left\langle\begin{pmatrix}\Psi^*_{1\mathbf{x}}\\ \Psi^*_{2\mathbf{x}}\end{pmatrix},\begin{pmatrix}i\frac{\theta(\beta)}{2}I_b+E(\mathbf{k}) & \phi I_b\\ \overline{\phi}I_b & i\frac{\theta(\beta)}{2}I_b-E(\mathbf{k})\end{pmatrix}\begin{pmatrix}\Psi_{1\mathbf{y}}\\ \Psi_{2\mathbf{y}}\end{pmatrix}\right\rangle,$$

where $\Psi^{(*)}_{\overline{\rho}\mathbf{x}}:=(\psi^{(*)}_{\overline{\rho}1\mathbf{x}},\psi^{(*)}_{\overline{\rho}2\mathbf{x}},\cdots,\psi^{(*)}_{\overline{\rho}b\mathbf{x}})^T$ and $\psi^*_{\overline{\rho}\rho\mathbf{x}}$ ($\psi_{\overline{\rho}\rho\mathbf{x}}$) is the creation (annihilation) operator on $F_f(L^2(\{1,2\}\times\mathcal{B}\times\Gamma))$ for $\overline{\rho}\in\{1,2\}$, $\rho\in\mathcal{B}$, $\mathbf{x}\in\Gamma$. The covariance in the final formulation is equal to the free 2-point correlation function $C(\phi):(\{1,2\}\times\mathcal{B}\times\Gamma_\infty\times[0,\beta))^2\to\mathbb{C}$ defined by

$$\begin{aligned}&C(\phi)(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t)\\ &:=\frac{\mathrm{Tr}(e^{-\beta H_0(\phi)}(1_{s\ge t}\psi^*_{\overline{\rho}\rho\mathbf{x}}(s)\psi_{\overline{\eta}\eta\mathbf{y}}(t)-1_{s<t}\psi_{\overline{\eta}\eta\mathbf{y}}(t)\psi^*_{\overline{\rho}\rho\mathbf{x}}(s)))}{\mathrm{Tr}\,e^{-\beta H_0(\phi)}},\end{aligned}$$

where $\psi^{(*)}_{\overline{\rho}\rho\mathbf{x}}(s):=e^{sH_0(\phi)}\psi^{(*)}_{\overline{\rho}\rho\mathbf{x}}e^{-sH_0(\phi)}$. For $\mathbf{x}\in\Gamma_\infty$ we identify $\psi^{(*)}_{\overline{\rho}\rho\mathbf{x}}$ with $\psi^{(*)}_{\overline{\rho}\rho r_L(\mathbf{x})}$. The next lemma ensures the well-definedness of $C(\phi)$ and gives its characterization and determinant bound. For any $\mathbf{k}\in\mathbb{R}^d$, $\phi\in\mathbb{C}$ we define $E(\phi)(\mathbf{k})\in\mathrm{Mat}(2b,\mathbb{C})$ by

$$E(\phi)(\mathbf{k}):=\begin{pmatrix}E(\mathbf{k}) & \overline{\phi}I_b\\ \phi I_b & -E(\mathbf{k})\end{pmatrix}.$$

Lemma 3.5.

(i)

$$\mathrm{Tr}\, e^{-\beta H_0(\phi)} = \prod_{\mathbf{k}\in\Gamma^*}\prod_{\delta\in\{1,-1\}} \det(1+e^{-\beta(i\frac{\theta(\beta)}{2}+\delta\sqrt{E(\mathbf{k})^2+|\phi|^2})})$$
$$= e^{-\frac{i}{2}\beta\theta(\beta)bL^d}2^{bL^d}\prod_{\mathbf{k}\in\Gamma^*}\det\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2})\right).$$

(ii) For any $(\overline{\rho},\rho,\mathbf{x},s),(\overline{\eta},\eta,\mathbf{y},t)\in\{1,2\}\times\mathcal{B}\times\Gamma_\infty\times[0,\beta)$*,*

(3.2)

$$\begin{aligned}&C(\phi)(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t)\\&=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}e^{(s-t)(i\frac{\theta(\beta)}{2}I_{2b}+E(\phi)(\mathbf{k}))}\\&\cdot\left(1_{s\ge t}(I_{2b}+e^{\beta(i\frac{\theta(\beta)}{2}I_{2b}+E(\phi)(\mathbf{k}))})^{-1}-1_{s<t}(I_{2b}+e^{-\beta(i\frac{\theta(\beta)}{2}I_{2b}+E(\phi)(\mathbf{k}))})^{-1}\right)\\&\cdot((\overline{\rho}-1)b+\rho,(\overline{\eta}-1)b+\eta).\end{aligned}$$

(iii)

$$\begin{aligned}&|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}C(\phi)(X_i,Y_j))_{1\le i,j\le n}|\\&\le\left(\frac{2^4b}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\left(1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}}\right.\right.\\&\qquad\left.\left.+e^{-2\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}\right)^n\\&\le\left(2^4b^2\left(1+\frac{\pi}{\beta}\left|\frac{\theta(\beta)}{2}-\frac{\pi}{\beta}\right|^{-1}\right)\right)^n,\\&(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m\ \textit{with}\ \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,\\&\quad X_i,Y_i\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)\ (i=1,2,\cdots,n),\ \phi\in\mathbb{C}).\end{aligned}$$

Here $\langle\cdot,\cdot\rangle_{\mathbb{C}^m}$ *denotes the canonical Hermitian inner product of* $\mathbb{C}^m$*.*

Proof. Let $e_\rho(\mathbf{k})$ ($\rho \in \mathcal{B}$) be the eigenvalues of $E(\mathbf{k})$. Then the eigenvalues of $E(\phi)(\mathbf{k})$ are $\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}$, $-\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}$ ($\rho \in \mathcal{B}$). There exists $U(\phi) \in \mathrm{Map}(\mathbb{R}^d, \mathrm{Mat}(2b,\mathbb{C}))$ such that $U(\phi)(\mathbf{k})$ is unitary and

$$
\begin{aligned}
(3.3)\qquad & U(\phi)(\mathbf{k})^* E(\phi)(\mathbf{k}) U(\phi)(\mathbf{k})((\overline{\rho}-1)b+\rho, (\overline{\eta}-1)b+\eta) \\
&= 1_{(\overline{\rho},\rho)=(\overline{\eta},\eta)}(-1)^{1_{\overline{\rho}=2}}\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}, \\
&(\forall (\overline{\rho},\rho),(\overline{\eta},\eta) \in \{1,2\}\times\mathcal{B},\ \mathbf{k}\in\mathbb{R}^d).
\end{aligned}
$$

Set $\alpha_{(\overline{\rho}-1)b+\rho}(\phi)(\mathbf{k}) := i\frac{\theta(\beta)}{2} + (-1)^{1_{\overline{\rho}=2}}\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}$. Remark that

$$
\begin{aligned}
& H_0(\phi) \\
&= \frac{1}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\left\langle \begin{pmatrix} \Psi^*_{1\mathbf{x}} \\ \Psi^*_{2\mathbf{x}} \end{pmatrix}, \left(i\frac{\theta(\beta)}{2}I_{2b} + E(\overline{\phi})(\mathbf{k})\right)\begin{pmatrix} \Psi_{1\mathbf{y}} \\ \Psi_{2\mathbf{y}} \end{pmatrix}\right\rangle.
\end{aligned}
$$

Then we can see that there is a unitary transform $\mathcal{U}(\phi)$ on $F_f(L^2(\{1,2\}\times \mathcal{B}\times\Gamma))$ such that

$$
\begin{aligned}
(3.4)\qquad & \mathcal{U}(\phi)\psi^*_{\overline{\rho}\rho\mathbf{x}}\mathcal{U}(\phi)^* \\
&= \frac{1}{L^d}\sum_{(\overline{\eta},\eta)\in\{1,2\}\times\mathcal{B}}\sum_{\mathbf{y}\in\Gamma}\sum_{\mathbf{k}\in\Gamma^*} e^{-i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle} \\
&\quad \cdot \overline{U(\overline{\phi})(\mathbf{k})((\overline{\rho}-1)b+\rho,(\overline{\eta}-1)b+\eta)}\psi^*_{\overline{\eta}\eta\mathbf{y}}, \\
(3.5)\qquad & \mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^* \\
&= \frac{1}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\sum_{(\overline{\rho},\rho)\in\{1,2\}\times\mathcal{B}}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle \mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\alpha_{(\overline{\rho}-1)b+\rho}(\phi)(\mathbf{k})\psi^*_{\overline{\rho}\rho\mathbf{x}}\psi_{\overline{\rho}\rho\mathbf{y}}.
\end{aligned}
$$

(i): Since $H_0(\phi)$ is diagonalized with respect to the band index in (3.5), a standard argument yields that

$$
\mathrm{Tr}\, e^{-\beta H_0(\phi)} = \prod_{\mathbf{k}\in\Gamma^*}\prod_{\overline{\rho}\in\{1,2\}}\prod_{\rho\in\mathcal{B}}(1+e^{-\beta\alpha_{(\overline{\rho}-1)b+\rho}(\phi)(\mathbf{k})}),
$$

which implies the claim.

(ii): Insertion of (3.4) gives that

(3.6)

$C(\phi)(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t)$

$$
\begin{aligned}
&= \frac{1}{L^{2d}} \sum_{\mathbf{x}',\mathbf{y}'\in\Gamma} \sum_{(\overline{\rho}',\rho'),(\overline{\eta}',\eta')\in\{1,2\}\times\mathcal{B}} \sum_{\mathbf{k},\mathbf{p}\in\Gamma^*} e^{-i\langle\mathbf{k},\mathbf{x}-\mathbf{x}'\rangle+i\langle\mathbf{p},\mathbf{y}-\mathbf{y}'\rangle} \\
&\quad \cdot \overline{U(\overline{\phi})(\mathbf{k})((\overline{\rho}-1)b+\rho,(\overline{\rho}'-1)b+\rho')} \\
&\quad \cdot U(\overline{\phi})(\mathbf{p})((\overline{\eta}-1)b+\eta,(\overline{\eta}'-1)b+\eta') \\
&\quad \cdot \frac{\mathrm{Tr}(e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}(1_{s\ge t}\tilde{\psi}^*_{\overline{\rho}'\rho'\mathbf{x}'}(s)\tilde{\psi}_{\overline{\eta}'\eta'\mathbf{y}'}(t)-1_{s<t}\tilde{\psi}_{\overline{\eta}'\eta'\mathbf{y}'}(t)\tilde{\psi}^*_{\overline{\rho}'\rho'\mathbf{x}'}(s)))}{\mathrm{Tr}\, e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}},
\end{aligned}
$$

where $\tilde{\psi}^*_{\overline{\rho}\rho\mathbf{x}}(s) := e^{s\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}\psi^{(*)}_{\overline{\rho}\rho\mathbf{x}}e^{-s\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}$. Since $\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*$ is diagonalized with the band index, an argument parallel to the proof of [8, Lemma B.10] yields that

$$
\begin{aligned}
&\frac{\mathrm{Tr}(e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}(1_{s\ge t}\tilde{\psi}^*_{\overline{\rho}'\rho'\mathbf{x}'}(s)\tilde{\psi}_{\overline{\eta}'\eta'\mathbf{y}'}(t)-1_{s<t}\tilde{\psi}_{\overline{\eta}'\eta'\mathbf{y}'}(t)\tilde{\psi}^*_{\overline{\rho}'\rho'\mathbf{x}'}(s)))}{\mathrm{Tr}\, e^{-\beta\mathcal{U}(\phi)H_0(\phi)\mathcal{U}(\phi)^*}} \\
&= \frac{1_{(\overline{\rho}',\rho')=(\overline{\eta}',\eta')}}{L^d}\sum_{\mathbf{q}\in\Gamma^*} e^{-i\langle\mathbf{q},\mathbf{x}'-\mathbf{y}'\rangle}e^{(s-t)\alpha_{(\overline{\rho}'-1)b+\rho'}(\phi)(\mathbf{q})} \\
&\quad \cdot\left(\frac{1_{s\ge t}}{1+e^{\beta\alpha_{(\overline{\rho}'-1)b+\rho'}(\phi)(\mathbf{q})}}-\frac{1_{s<t}}{1+e^{-\beta\alpha_{(\overline{\rho}'-1)b+\rho'}(\phi)(\mathbf{q})}}\right).
\end{aligned}
$$

We should remark that here we have the exponent $-i\langle\mathbf{q},\mathbf{x}'-\mathbf{y}'\rangle$ not $i\langle\mathbf{q},\mathbf{x}'-\mathbf{y}'\rangle$. By substituting this equality into (3.6) and using (3.3) we observe that

$$
\begin{aligned}
&C(\phi)(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t) \\
&= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*} e^{-i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}e^{(s-t)(i\frac{\theta(\beta)}{2}I_{2b}+E(\overline{\phi})(\mathbf{k}))} \\
&\quad \cdot\left(1_{s\ge t}(I_{2b}+e^{\beta(i\frac{\theta(\beta)}{2}I_{2b}+E(\overline{\phi})(\mathbf{k}))})^{-1}-1_{s<t}(I_{2b}+e^{-\beta(i\frac{\theta(\beta)}{2}I_{2b}+E(\overline{\phi})(\mathbf{k}))})^{-1}\right) \\
&\quad \cdot((\overline{\eta}-1)b+\eta,(\overline{\rho}-1)b+\rho).
\end{aligned}
$$

It follows from (1.4) and (1.5) that $E(\overline{\phi})(\mathbf{k})^T = E(\phi)(-\mathbf{k})$, $(\forall\mathbf{k}\in\mathbb{R}^d)$. By combining this equality with the above characterization of $C(\phi)$ and using periodicity we obtain (3.2).

(iii): In [12, Proposition 4.1] we stated a version of Pedra-Salmhofer's determinant bound [16, Theorem 1.3]. In [12, Appendix A] we gave a short proof of [12, Proposition 4.1]. By applying [12, Proposition 4.1] we derived the determinant bound [12, Proposition 4.2] which gives the claimed determinant bound in the case $b = 1$. Here let us use the proof of [12, Proposition 4.2] and [12, Lemma A.1], which is a simple application of the Cauchy-Binet formula, to derive the claimed determinant bound in the general case. It follows from (3.2) and (3.3) that for any $(\overline{\rho},\rho,\mathbf{x},s)$, $(\overline{\eta},\eta,\mathbf{y},t)\in\{1,2\}\times\mathcal{B}\times\Gamma_{\infty}\times[0,\beta)$

$$C(\phi)(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t)=\sum_{\rho'\in\mathcal{B}}C_{\rho'}(\phi)(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t), \tag{3.7}$$

where

$$\begin{aligned}&C_{\rho'}(\phi)(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t)\\&:=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\overline{\rho}'\in\{1,2\}}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\\&\quad\cdot U(\phi)(\mathbf{k})((\overline{\rho}-1)b+\rho,(\overline{\rho}'-1)b+\rho')\\&\quad\cdot U(\phi)(\mathbf{k})^*((\overline{\rho}'-1)b+\rho',(\overline{\eta}-1)b+\eta)\\&\quad\cdot e^{(s-t)\alpha_{(\overline{\rho}'-1)b+\rho'}(\phi)(\mathbf{k})}\left(\frac{1_{s\ge t}}{1+e^{\beta\alpha_{(\overline{\rho}'-1)b+\rho'}(\phi)(\mathbf{k})}}-\frac{1_{s<t}}{1+e^{-\beta\alpha_{(\overline{\rho}'-1)b+\rho'}(\phi)(\mathbf{k})}}\right).\end{aligned}$$

In the proof of [12, Proposition 4.2] we estimated the determinant bound of a covariance whose form is close to $C_{\rho'}(\phi)$. By following the proof of [12, Proposition 4.2] straightforwardly we can deduce that

$$\begin{aligned}&|\det(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}C_{\rho'}(\phi)(X_i,Y_j))_{1\le i,j\le n}|\\&\le\left(\frac{2^4}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\left(1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{e_{\rho'}(\mathbf{k})^2+|\phi|^2}}\right.\right.\\&\qquad\left.\left.+e^{-2\beta\sqrt{e_{\rho'}(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}\right)^n,\\&(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m\text{ with }\|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le1,\\&\quad X_i,Y_i\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)\ (i=1,2,\cdots,n),\ \phi\in\mathbb{C},\ \rho'\in\mathcal{B}).\end{aligned} \tag{3.8}$$

To support the readers, let us provide a guidance to derive (3.8). By using vectors of $\text{Map}(\{1,2\}\times\mathcal{B}\times\Gamma\times\mathbb{R},L^2(\Gamma^*\times\mathbb{R}))$ and the inner product of $L^2(\Gamma^*\times\mathbb{R})$ we can rewrite the regularized version of $C_{\rho'}(\phi)$ in a form close to [12, (4.4)]. The vectors satisfy a uniform bound similar to [12, (4.5)]. In this situation we can apply a close variant of [12, Proposition 4.1] to the regularized version of $C_{\rho'}(\phi)$. Then by sending the parameter used to regularize $C_{\rho'}(\phi)$ to zero we obtain (3.8). Since we have (3.7) and (3.8), we can repeatedly apply [12, Lemma A.1] to derive that

(3.9)

$$\begin{aligned}&|\det(\langle \mathbf{u}_i, \mathbf{v}_j\rangle_{\mathbb{C}^m} C(\phi)(X_i, Y_j))_{1\le i,j\le n}|\\ &\le \left(\sum_{\rho'\in\mathcal{B}}\left(\frac{2^4}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\right.\right.\\ &\qquad\left.\left.\cdot\left(1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{e_{\rho'}(\mathbf{k})^2+|\phi|^2}}+e^{-2\beta\sqrt{e_{\rho'}(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{2n},\\ &(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,\\ &\quad X_i, Y_i\in\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)\ (i=1,2,\cdots,n),\ \phi\in\mathbb{C}),\end{aligned}$$

which together with Schwarz' inequality yields the first inequality of the claim (iii). It is also possible to derive (3.9) by directly applying [16, Theorem 1.3]. In this case one should decompose $C(\phi)$ into a sum of $2b$ time-ordered covariances. Note that

$$\begin{aligned}&1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}}+e^{-2\beta\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}}\\ &\ge 1_{\theta(\beta)\in[0,\pi/\beta]}+1_{\theta(\beta)\in(\pi/\beta,2\pi/\beta)}\sin^2\left(\frac{\beta\theta(\beta)}{2}\right)\\ &\ge 1_{\theta(\beta)\in[0,\pi/\beta]}+1_{\theta(\beta)\in(\pi/\beta,2\pi/\beta)}\frac{\beta^2}{\pi^2}\left(\frac{\theta(\beta)}{2}-\frac{\pi}{\beta}\right)^2,\end{aligned}$$

which implies the second inequality. □

We finalize our formulation in the next lemma. We should remark that Lemma 3.3 and Lemma 3.4 will not see any application in the rest of the paper. We stated these lemmas in the hope that the readers can prove the next lemma by putting these lemmas together in the same manner as in the proof of [12, Lemma 2.5]. While it was quartic in [12], here the Grassmann polynomial $A^2(\psi)$ may contain quadratic terms. This is because we assumed $r_L(\hat{\mathbf{x}})\neq r_L(\hat{\mathbf{y}})$ in the previous construction and here we need to drop this assumption in order to study the Cooper pair density as claimed in Theorem 1.3 (v), Corollary 1.11 (v).

LEMMA 3.6. *The following statements hold true for any* $r \in \mathbb{R}_{>0}$.

(i)

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi)$$

converges in $C(\overline{D(r)}^3)$ *as a sequence of function with the variables* $(\boldsymbol{\lambda},\phi)\in\overline{D(r)}^3$.

(ii) The $C(\overline{D(r)}^2)$*-valued function*

$$(\phi_1,\phi_2)\mapsto e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2}\frac{\prod_{\mathbf{k}\in\Gamma^*}\det(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}\det(\cos(\beta\theta(\beta)/2)+\cosh(\beta E(\mathbf{k})))}$$
$$\cdot\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi)$$

belongs to $L^1(\mathbb{R}^2,C(\overline{D(r)}^2))$.

(iii)

$$\begin{aligned}&\frac{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta(\beta)\mathsf{S}_z+\mathsf{F}+\mathsf{A})}}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0+i\theta(\beta)\mathsf{S}_z)}}\\&=\frac{\beta L^d}{\pi|U|}\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2}\\&\quad\cdot\frac{\prod_{\mathbf{k}\in\Gamma^*}\det(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}\det(\cos(\beta\theta(\beta)/2)+\cosh(\beta E(\mathbf{k})))}\\&\quad\cdot\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi),\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta(\beta)\mathsf{S}_z+\mathsf{F})}\mathsf{A}_j)}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0+i\theta(\beta)\mathsf{S}_z)}}\\&=\frac{L^d}{\pi|U|}\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{-\frac{\beta L^d}{|U|}|\phi-\gamma|^2}\\&\quad\cdot\frac{\prod_{\mathbf{k}\in\Gamma^*}\det(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}\det(\cos(\beta\theta(\beta)/2)+\cosh(\beta E(\mathbf{k})))}\end{aligned}$$

$$\cdot \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi)+W(\psi)} A^j(\psi) d\mu_{C(\phi)}(\psi),$$

$$(j = 1, 2).$$

(iv) For any $\phi \in \mathbb{C}$,

$$\lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi) \in \mathbb{R}.$$

REMARK 3.7. Since we have obtained the ϕ-independent determinant bound in Lemma 3.5 (iii), we can readily prove that the integral with (ϕ_1, ϕ_2) and the limit operation $h \to \infty$ are interchangeable in the claim (iii). However, since we need to take large h depending on fixed (ϕ_1, ϕ_2) in the analysis of $C(\phi)$ in Subsection 5.1, taking the limit $h \to \infty$ after the integration with (ϕ_1, ϕ_2) has no application in this paper.

PROOF OF LEMMA 3.6. Based on Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5, the claims (i), (ii), (iii) can be proved in the same way as in the proof of [12, Lemma 2.5]. Note that the locally uniform convergence with $(\boldsymbol{\lambda}, \phi)$ is claimed in (i), while the convergence was claimed pointwise with ϕ in [12, Lemma 2.5 (i)]. The uniform convergence property with ϕ can be deduced by making use of the ϕ-independent determinant bound Lemma 3.5 (iii) in arguments parallel to the proof of [12, Lemma 2.5 (i)]. Concerning the form of the Grassmann polynomials $V(\psi)$, $A(\psi)$, which affects details of the forthcoming analysis, we should explain that they stem from the use of the unitary map $\mathcal{U} : F_f(L^2(\mathcal{B}\times\Gamma\times\{\uparrow,\downarrow\})) \to F_f(L^2(\{1,2\}\times \mathcal{B} \times \Gamma))$ satisfying that

$$\mathcal{U}\psi^*_{\rho\mathbf{x}\uparrow}\mathcal{U}^* = \psi^*_{1\rho\mathbf{x}}, \quad \mathcal{U}\psi^*_{\rho\mathbf{x}\downarrow}\mathcal{U}^* = \psi_{2\rho\mathbf{x}}, \quad (\forall(\rho,\mathbf{x}) \in \mathcal{B}\times\Gamma)$$

and thus

(3.10)

$$\mathcal{U}\mathsf{V}\mathcal{U}^* = \frac{U}{L^d}\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}\psi^*_{1\rho\mathbf{x}}\psi_{1\rho\mathbf{x}} - \frac{U}{L^d}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\psi^*_{1\rho\mathbf{x}}\psi^*_{2\eta\mathbf{y}}\psi_{2\rho\mathbf{x}}\psi_{1\eta\mathbf{y}},$$

(3.11)

$$\mathcal{U}\mathsf{A}\mathcal{U}^*$$

$$= \lambda_1 \psi^*_{1\hat{\rho} r_L(\hat{\mathbf{x}})} \psi_{2\hat{\rho} r_L(\hat{\mathbf{x}})} + \lambda_2 (1_{(\hat{\rho}, r_L(\hat{\mathbf{x}}))=(\hat{\eta}, r_L(\hat{\mathbf{y}}))} \psi^*_{1\hat{\rho} r_L(\hat{\mathbf{x}})} \psi_{1\hat{\rho} r_L(\hat{\mathbf{x}})} - \psi^*_{1\hat{\rho} r_L(\hat{\mathbf{x}})} \psi^*_{2\hat{\eta} r_L(\hat{\mathbf{y}})} \psi_{2\hat{\rho} r_L(\hat{\mathbf{x}})} \psi_{1\hat{\eta} r_L(\hat{\mathbf{y}})}).$$

The right-hand side of (3.10), (3.11) is formulated into $V(\psi)$, $A(\psi)$ respectively.

Let us prove the claim (iv). Define the Grassmann polynomials $W_+(\psi)$, $W_-(\psi) \in \bigwedge \mathcal{V}$ by

$$W_+(\psi) := \frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}} h} \sum_{(\rho,\mathbf{x},s)\in\mathcal{B}\times\Gamma\times[0,\beta)_h} \overline{\psi}_{1\rho\mathbf{x}s} \psi_{2\rho\mathbf{x}s},$$
$$W_-(\psi) := \frac{i|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}} h} \sum_{(\rho,\mathbf{x},s)\in\mathcal{B}\times\Gamma\times[0,\beta)_h} \overline{\psi}_{2\rho\mathbf{x}s} \psi_{1\rho\mathbf{x}s}.$$

Since $W(\psi) = W_+(\psi)W_-(\psi)$, the Hubbard-Stratonovich transformation yields that

$$\int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi) = \frac{1}{\pi} \int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \int e^{-V(\psi)+\xi W_+(\psi)+\overline{\xi} W_-(\psi)} d\mu_{C(\phi)}(\psi),$$

where $\xi = \xi_1 + i\xi_2$. See e.g. [12, Lemma 2.3] for the proof. By setting

$$D(b,\beta,\theta) := 2^4 b^2 \left(1 + \frac{\pi}{\beta}\left|\frac{\theta(\beta)}{2} - \frac{\pi}{\beta}\right|^{-1}\right)$$

and applying Lemma 3.5 (iii) we obtain that

$$\left|\int e^{-V(\psi)+\xi W_+(\psi)+\overline{\xi} W_-(\psi)} d\mu_{C(\phi)}(\psi)\right| \le e^{|U|b\beta D(b,\beta,\theta)+|U|b^2 L^d \beta D(b,\beta,\theta)^2 + 2|\xi||U|^{1/2} b L^{d/2} \beta^{1/2} D(b,\beta,\theta)}. \tag{3.12}$$

Let us define the operators V, W_+, W_- on $F_f(L^2(\{1,2\}\times\mathcal{B}\times\Gamma))$ by

$$V := \frac{U}{L^d} \sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma} \psi^*_{1\rho\mathbf{x}} \psi_{1\rho\mathbf{x}} - \frac{U}{L^d} \sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma} \psi^*_{1\rho\mathbf{x}} \psi^*_{2\eta\mathbf{y}} \psi_{2\rho\mathbf{x}} \psi_{1\eta\mathbf{y}},$$

$$W_+ := \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}}} \sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma} \psi^*_{1\rho\mathbf{x}}\psi_{2\rho\mathbf{x}}, \quad W_- := \frac{|U|^{\frac{1}{2}}}{\beta^{\frac{1}{2}} L^{\frac{d}{2}}} \sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma} \psi^*_{2\rho\mathbf{x}}\psi_{1\rho\mathbf{x}}.$$

In the same way as in the Grassmann integral formulation procedure [12, Lemma 2.2] or that of [8], [9], [10] we have that for any $r \in \mathbb{R}_{>0}$

$$\begin{aligned}(3.13)\qquad &\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \sup_{\xi\in\overline{D(r)}} \left| \int e^{-V(\psi)+\xi W_+(\psi)+\overline{\xi}W_-(\psi)} d\mu_{C(\phi)}(\psi) \right. \\ &\qquad\qquad \left. - \frac{\mathrm{Tr}\, e^{-\beta(H_0(\phi)+V-i\xi W_+-i\overline{\xi}W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(\phi)}} \right| \\ &= 0.\end{aligned}$$

By (3.12), (3.13) we can apply the dominated convergence theorem to conclude that

$$\begin{aligned}&\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi) \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \frac{\mathrm{Tr}\, e^{-\beta(H_0(\phi)+V-i\xi W_+-i\overline{\xi}W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(\phi)}}.\end{aligned}$$

To make clear the dependency on the parameter $\theta(\beta)$, let us write $H_0(\theta(\beta),\phi)$ in place of $H_0(\phi)$. Observe that

$$\begin{aligned}&\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} \overline{\int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi)} \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \frac{\mathrm{Tr}\, e^{-\beta(H_0(\theta(\beta),\phi)+V-i\xi W_+-i\overline{\xi}W_-)^*}}{\mathrm{Tr}\, e^{-\beta H_0(\theta(\beta),\phi)^*}} \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \frac{\mathrm{Tr}\, e^{-\beta(H_0(-\theta(\beta),\phi)+V+i\xi W_++i\overline{\xi}W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(-\theta(\beta),\phi)}} \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} d\xi_1 d\xi_2 e^{-|\xi|^2} \frac{\mathrm{Tr}\, e^{-\beta(H_0(-\theta(\beta),\phi)+V-i\xi W_+-i\overline{\xi}W_-)}}{\mathrm{Tr}\, e^{-\beta H_0(-\theta(\beta),\phi)}}.\end{aligned}$$

To derive the last equality, we performed the change of variables $\xi_j \to -\xi_j$ $(j=1,2)$.

There is a unitary transform $\mathcal{U}_0$ on $F_f(L^2(\{1,2\}\times\mathcal{B}\times\Gamma))$ satisfying that

$$\mathcal{U}_0\psi_{1\rho\mathbf{x}}\mathcal{U}_0^* = -\psi_{2\rho\mathbf{x}}^*, \quad \mathcal{U}_0\psi_{2\rho\mathbf{x}}\mathcal{U}_0^* = \psi_{1\rho\mathbf{x}}^*, \quad (\forall(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma).$$

We can check that

$$\begin{aligned}&\mathcal{U}_0H_0(-\theta(\beta),\phi)\mathcal{U}_0^* = -i\theta(\beta)L^db + H_0(\theta(\beta),\phi),\\ &\mathcal{U}_0V\mathcal{U}_0^* = V, \quad \mathcal{U}_0W_+\mathcal{U}_0^* = W_+, \quad \mathcal{U}_0W_-\mathcal{U}_0^* = W_-.\end{aligned}$$

In the derivation of the first equality we used (1.4) and (1.5). By using the unitary transform $\mathcal{U}_0$,

$$\begin{aligned}&\overline{\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi)}\\ &=\frac{1}{\pi}\int_{\mathbb{R}^2}d\xi_1d\xi_2e^{-|\xi|^2}\frac{\mathrm{Tr}\,e^{-\beta\mathcal{U}_0(H_0(-\theta(\beta),\phi)+V-i\xi W_+-i\overline{\xi}W_-)\mathcal{U}_0^*}}{\mathrm{Tr}\,e^{-\beta\mathcal{U}_0H_0(-\theta(\beta),\phi)\mathcal{U}_0^*}}\\ &=\frac{1}{\pi}\int_{\mathbb{R}^2}d\xi_1d\xi_2e^{-|\xi|^2}\frac{\mathrm{Tr}\,e^{-\beta(H_0(\theta(\beta),\phi)+V-i\xi W_+-i\overline{\xi}W_-)}}{\mathrm{Tr}\,e^{-\beta H_0(\theta(\beta),\phi)}}\\ &=\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(\phi)}(\psi),\end{aligned}$$

which implies the claim. □

4. Multi-Scale Integration

As one can expect from the formulation Lemma 3.6, the proof of the main theorem is based on analytical control of the Grassmann Gaussian integral

$$\int e^{-V(\psi)+W(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi).$$

We will achieve our purpose by means of multi-scale integration. In principle our analysis is an extension of the double-scale integration performed in the previous work [12]. We intend to keep using the previous framework as much as possible so that the readers can smoothly connect it to this extended version. As in the previous construction, after brief introductions or restatements of necessary notations concerning estimation of kernel functions we establish general bounds on Grassmann polynomials. Then by assuming scale-dependent bound properties of covariances we inductively construct a multi-scale integration process running from the largest scale to the smallest scale. In the next section we will confirm that our actual covariance satisfies the properties assumed in this section.

4.1. Necessary notions

Our multi-scale analysis needs a little more detailed notions of estimating kernel functions than the double-scale integration required in [12]. In order to avoid unnecessary repetitions, we use some terminology and notational convention without presenting the definitions in the following. The readers should refer to [12, Subsection 3.1] for their meaning. We will not use any terminology or notational rule which is not defined either in [12, Subsection 3.1] or in this section and the preceding sections of this paper. As in the previous paper, we define the norms $\|f\|_{1,\infty}$, $\|f\|_1$ of a function $f : I^n \to \mathbb{C}$ by

$$\|f\|_{1,\infty} := \sup_{j\in\{1,2,\cdots,n\}} \sup_{X_0\in I} \left(\frac{1}{h}\right)^{n-1} \sum_{\mathbf{X}\in I^{j-1}} \sum_{\mathbf{Y}\in I^{n-j}} |f(\mathbf{X}, X_0, \mathbf{Y})|,$$

$$\|f\|_1 := \left(\frac{1}{h}\right)^{n} \sum_{\mathbf{X}\in I^n} |f(\mathbf{X})|.$$

For $f_0 \in \mathbb{C}$ we let $\|f_0\|_{1,\infty} = \|f_0\|_1 := |f_0|$. This convention helps to organize formulas. We define the index set $I^0(\subset I)$ by

$$I^0 := \{1, 2\} \times \mathcal{B} \times \Gamma \times \{0\} \times \{1, -1\}.$$

Since we will frequently make use of bound properties of covariances, we need to introduce various norms on functions on I^2. For an anti-symmetric function $g : I^2 \to \mathbb{C}$ we define the norms $\|g\|'_{1,\infty}$, $\|g\|$ as follows.

$$\|g\|'_{1,\infty} := \sup_{\substack{X_0\in I\\ s\in[0,\beta)_h}} \sum_{X\in I^0} |g(X_0, X + s)|,$$

$$\|g\| := \|g\|'_{1,\infty} + (1 + \beta^{-1})\|g\|_{1,\infty}.$$

We should remark that the definition of the norm $\|\cdot\|$ is slightly different from that in [12]. We will also need to evaluate a function on $I^m \times I^n$

multiplied by another anti-symmetric function on I^2. More specifically, for a function $f_{m,n} : I^m \times I^n \to \mathbb{C}$ $(m, n \in \mathbb{N}_{\ge 2})$ and an anti-symmetric function $g : I^2 \to \mathbb{C}$ we set

$$\begin{aligned}
&[f_{m,n}, g]_{1,\infty} \\
&:= \max\Bigg\{ \sup_{\substack{X_0 \in I \\ j \in \{1,2,\cdots,m\}}} \left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{X} \in I^m} 1_{X_j = X_0} \\
&\qquad\qquad \cdot \left(\sup_{\substack{Y_0 \in I \\ k \in \{1,2,\cdots,n\}}} \left(\frac{1}{h}\right)^{n} \sum_{\mathbf{Y} \in I^n} |f_{m,n}(\mathbf{X},\mathbf{Y})||g(Y_0,Y_k)| \right), \\
&\qquad \sup_{\substack{Y_0 \in I \\ k \in \{1,2,\cdots,n\}}} \left(\frac{1}{h}\right)^{n-1} \sum_{\mathbf{Y} \in I^n} 1_{Y_k = Y_0} \\
&\qquad\qquad \cdot \left(\sup_{\substack{X_0 \in I \\ j \in \{1,2,\cdots,m\}}} \left(\frac{1}{h}\right)^{m} \sum_{\mathbf{X} \in I^m} |f_{m,n}(\mathbf{X},\mathbf{Y})||g(X_0,X_j)| \right) \Bigg\}, \\
&[f_{m,n}, g]_1 := \sup_{\substack{j \in \{1,2,\cdots,m\} \\ k \in \{1,2,\cdots,n\}}} \left(\frac{1}{h}\right)^{m+n} \sum_{\substack{\mathbf{X} \in I^m \\ \mathbf{Y} \in I^n}} |f_{m,n}(\mathbf{X},\mathbf{Y})||g(X_j,Y_k)|.
\end{aligned}$$

Since we do not assume that $f_{m,n}$ is bi-anti-symmetric, the forms of $[f_{m,n}, g]_{1,\infty}$, $[f_{m,n}, g]_1$ are more complex than those introduced in [12, Subsection 3.1]. If $f_{m,n}$ is bi-anti-symmetric, they become same as before.

$$\begin{aligned}
&[f_{m,n}, g]_{1,\infty} \\
&= \max\Bigg\{ \\
&\quad \sup_{X_0 \in I} \left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{X} \in I^{m-1}} \left(\sup_{Y_0 \in I} \left(\frac{1}{h}\right)^{n} \sum_{\mathbf{Y} \in I^n} |f_{m,n}((X_0,\mathbf{X}),\mathbf{Y})||g(Y_0,Y_1)| \right), \\
&\quad \sup_{Y_0 \in I} \left(\frac{1}{h}\right)^{n-1} \sum_{\mathbf{Y} \in I^{n-1}} \left(\sup_{X_0 \in I} \left(\frac{1}{h}\right)^{m} \sum_{\mathbf{X} \in I^m} |f_{m,n}(\mathbf{X},(Y_0,\mathbf{Y}))||g(X_0,X_1)| \right) \Bigg\}, \\
&[f_{m,n}, g]_1 = \left(\frac{1}{h}\right)^{m+n} \sum_{\substack{\mathbf{X} \in I^m \\ \mathbf{Y} \in I^n}} |f_{m,n}(\mathbf{X},\mathbf{Y})||g(X_1,Y_1)|.
\end{aligned}$$

Let $\bigwedge_{even}\mathcal{V}$ denote the subspace of $\bigwedge\mathcal{V}$ consisting of even polynomials. Each order term of the expansion of logarithm of a Grassmann Gaussian integral with respect to the effective interaction can be expressed as a finite sum over trees. Concerning the tree expansion, we can use the same notations as in [12, Subsection 3.1]. The only difference between the present setting and the previous setting is the definition of the index sets I_0, I. By keeping in mind that I_0, I count the band index $\mathcal{B}$ in this setting we can refer to [12, Subsection 3.1] for the meaning of the notations we use in the following. The tree formula is applied as follows. For any covariance $\mathcal{C}: I_0^2 \to \mathbb{C}$ and $f^j(\psi) \in \bigwedge_{even}\mathcal{V}$ $(j = 1, 2, \cdots, n)$,

$$\begin{aligned}(4.1)\qquad &\frac{1}{n!}\prod_{j=1}^{n}\left(\frac{\partial}{\partial z_j}\right)\log\left(\int e^{\sum_{j=1}^{n} z_j f^j(\psi+\psi^1)}d\mu_{\mathcal{C}}(\psi^1)\right)\Bigg|_{\substack{z_j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\ &= \frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C})\prod_{j=1}^{n} f^j(\psi+\psi^j)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.\end{aligned}$$

The major part of our analysis is devoted to estimating Grassmann polynomials produced by the operator $Tree(\{1, 2, \cdots, n\}, \mathcal{C})$.

4.2. General estimation

Here we summarize bound properties of Grassmann polynomials produced by the tree formula. Most of the necessary properties have essentially been prepared in [12, Subsection 3.2]. However, as we need to apply them repeatedly in the next subsection, let us present all the necessary inequalities so that the readers can follow the arguments without disruption.

Here we do not fix details of the covariance. We only assume that the covariance $\mathcal{C}: I_0^2 \to \mathbb{C}$ satisfies with a constant $D(\in \mathbb{R}_{>0})$ that

$$(4.2)\qquad \mathcal{C}(\mathcal{R}_\beta(\mathbf{X}+s)) = \mathcal{C}(\mathbf{X}), \quad \left(\forall \mathbf{X}\in I_0^2,\ s\in \frac{1}{h}\mathbb{Z}\right),$$

$$\begin{aligned}(4.3)\qquad &|\det(\langle \mathbf{u}_i, \mathbf{v}_j\rangle_{\mathbb{C}^m}\mathcal{C}(X_i, Y_j))_{1\le i,j\le n}| \le D^n,\\ &(\forall m, n\in\mathbb{N},\ \mathbf{u}_i, \mathbf{v}_i \in \mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{v}_i\|_{\mathbb{C}^m} \le 1,\\ &\quad X_i, Y_i \in I_0\ (i = 1, 2, \cdots, n)).\end{aligned}$$

Here the map $\mathcal{R}_\beta : (\{1, 2\} \times \mathcal{B} \times \Gamma \times \frac{1}{h}\mathbb{Z})^n \to I_0^n$ is defined by

$$\mathcal{R}_\beta((\overline{\rho}_1, \rho_1, \mathbf{x}_1, s_1), \cdots, (\overline{\rho}_n, \rho_n, \mathbf{x}_n, s_n))$$

$$:= ((\overline{\rho}_1, \rho_1, \mathbf{x}_1, r_\beta(s_1)), \cdots, (\overline{\rho}_n, \rho_n, \mathbf{x}_n, r_\beta(s_n))),$$

where for any $s \in \frac{1}{h}\mathbb{Z}$, $r_\beta(s) \in [0, \beta)_h$ and $r_\beta(s) = s$ in $\frac{1}{h}\mathbb{Z}/\beta\mathbb{Z}$. By abusing the notation we will sometimes consider $\mathcal{R}_\beta$ as a map from $(\{1,2\} \times \mathcal{B} \times \Gamma \times \frac{1}{h}\mathbb{Z} \times \{1,-1\})^n$ to I^n satisfying the same condition on the time variables. The precise meaning of the map $\mathcal{R}_\beta$ should be understood from the context.

As in (4.2) we will often impose the condition

$$F(\mathcal{R}_\beta(\mathbf{X} + s)) = F(\mathbf{X}), \quad \left(\forall \mathbf{X} \in I^m,\ s \in \frac{1}{h}\mathbb{Z}\right) \tag{4.4}$$

on a function $F : I^m \to \mathbb{C}$. For $j \in \mathbb{N}$ let $F^j(\psi) \in \bigwedge_{even} \mathcal{V}$ be such that its anti-symmetric kernels $F^j_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (4.4). The first lemma summarizes bound properties of $A^{(n)}(\psi)$ $(\in \bigwedge_{even} \mathcal{V})$ $(n \in \mathbb{N})$ defined by

$$A^{(n)}(\psi) := Tree(\{1, 2, \cdots, n\}, \mathcal{C}) \prod_{j=1}^{n} F^j(\psi^j + \psi)\Bigg|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}}.$$

Recall that the anti-symmetric extension $\tilde{\mathcal{C}} : I^2 \to \mathbb{C}$ of the covariance $\mathcal{C}$ is defined by

$$\begin{aligned} \tilde{\mathcal{C}}(X\xi, Y\zeta) &:= \frac{1}{2}(1_{(\xi,\zeta)=(1,-1)}\mathcal{C}(X,Y) - 1_{(\xi,\zeta)=(-1,1)}\mathcal{C}(Y,X)), \\ &(X, Y \in I_0,\ \xi, \zeta \in \{1,-1\}). \end{aligned} \tag{4.5}$$

Let N denote $4b\beta hL^d$, the cardinality of the index set I.

LEMMA 4.1. *For any $m \in \{2, 4, \cdots, N\}$, $n \in \mathbb{N}$ the anti-symmetric kernel $A^{(n)}_m(\cdot)$ satisfies* (4.4). *Moreover, the following inequalities hold for any $m \in \{0, 2, \cdots, N\}$, $n \in \mathbb{N}_{\ge 2}$.*

$$\|A^{(1)}_m\|_{1,\infty} \le \sum_{p=m}^{N} \left(\frac{N}{h}\right)^{1_{m=0 \wedge p \neq 0}} \binom{p}{m} D^{\frac{p-m}{2}} \|F^1_p\|_{1,\infty}. \tag{4.6}$$

$$\|A^{(1)}_m\|_1 \le \sum_{p=m}^{N} \binom{p}{m} D^{\frac{p-m}{2}} \|F^1_p\|_1. \tag{4.7}$$

$$\|A^{(n)}_m\|_{1,\infty} \le \left(\frac{N}{h}\right)^{1_{m=0}} (n-2)! D^{-n+1-\frac{m}{2}} 2^{-2m} \|\tilde{\mathcal{C}}\|^{n-1}_{1,\infty} \tag{4.8}$$

$$\cdot \prod_{j=1}^{n}\left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}} \|F^j_{p_j}\|_{1,\infty}\right) 1_{\sum_{j=1}^{n} p_j - 2(n-1)\ge m}.$$

$$\|A^{(n)}_m\|_1 \le (n-2)! D^{-n+1-\frac{m}{2}} 2^{-2m} \|\tilde{\mathcal{C}}\|^{n-1}_{1,\infty} \sum_{p_1=2}^{N} 2^{3p_1} D^{\frac{p_1}{2}} \|F^1_{p_1}\|_1 \cdot \prod_{j=2}^{n}\left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}} \|F^j_{p_j}\|_{1,\infty}\right) 1_{\sum_{j=1}^{n} p_j - 2(n-1)\ge m}. \tag{4.9}$$

PROOF. These are essentially same as [12, Lemma 3.1]. □

Next we deal with a Grassmann input with bi-anti-symmetric kernels. Let functions $F_{p,q} : I^p \times I^q \to \mathbb{C}$ $(p, q \in \{2, 4, \cdots, N\})$ be bi-anti-symmetric, satisfy (4.4) and the following property. For any functions $f : [0, \beta)^p_h \to \mathbb{C}$, $g : [0, \beta)^q_h \to \mathbb{C}$ satisfying

$$\begin{aligned}
&f(r_\beta(s_1 + s), r_\beta(s_2 + s), \cdots, r_\beta(s_p + s)) = f(s_1, s_2, \cdots, s_p)\\
&\left(\forall (s_1, s_2, \cdots, s_p) \in [0, \beta)^p_h,\ s \in \frac{1}{h}\mathbb{Z}\right),\\
&g(r_\beta(s_1 + s), r_\beta(s_2 + s), \cdots, r_\beta(s_q + s)) = g(s_1, s_2, \cdots, s_q)\\
&\left(\forall (s_1, s_2, \cdots, s_q) \in [0, \beta)^q_h,\ s \in \frac{1}{h}\mathbb{Z}\right),
\end{aligned}$$

(4.10)

$$\begin{aligned}
&\sum_{(s_1,\cdots,s_p)\in[0,\beta)^p_h} F_{p,q}((\overline{\rho}_1\rho_1\mathbf{x}_1 s_1\xi_1, \cdots, \overline{\rho}_p\rho_p\mathbf{x}_p s_p\xi_p), \mathbf{Y}) f(s_1, \cdots, s_p) = 0,\\
&(\forall \mathbf{Y} \in I^q,\ (\overline{\rho}_j, \rho_j, \mathbf{x}_j, \xi_j) \in \{1,2\} \times \mathcal{B} \times \Gamma \times \{1,-1\}\ (j = 1, 2, \cdots, p)),\\
&\sum_{(t_1,\cdots,t_q)\in[0,\beta)^q_h} F_{p,q}(\mathbf{X}, (\overline{\eta}_1\eta_1\mathbf{y}_1 t_1\zeta_1, \cdots, \overline{\eta}_q\eta_q\mathbf{y}_q t_q\zeta_q)) g(t_1, \cdots, t_q) = 0,\\
&(\forall \mathbf{X} \in I^p,\ (\overline{\eta}_j, \eta_j, \mathbf{y}_j, \zeta_j) \in \{1,2\} \times \mathcal{B} \times \Gamma \times \{1,-1\}\ (j = 1, 2, \cdots, q)).
\end{aligned}$$

Then let us define the Grassmann polynomials $B^{(n)}(\psi)$ $(\in \bigwedge_{even} \mathcal{V})$ $(n \in \mathbb{N})$

by

$$B^{(n)}(\psi) := \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} F_{p,q}(\mathbf{X},\mathbf{Y})Tree(\{1,2,\cdots,n+1\},\mathcal{C})$$
$$\cdot (\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\prod_{j=3}^{n+1}F^j(\psi^j+\psi)\Big|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n+1\})}}.$$

The kernels of the Grassmann polynomials $B^{(n)}(\psi)$ $(n\in\mathbb{N})$ are estimated as follows.

Lemma 4.2. *For any* $m\in\{2,4,\cdots,N\}$, $n\in\mathbb{N}$ *the anti-symmetric kernel* $B_m^{(n)}(\cdot)$ *satisfies* (4.4). *Moreover, for any* $m\in\{0,2,\cdots,N\}$, $n\in\mathbb{N}_{\ge 2}$,

(4.11)

$$\|B_m^{(1)}\|_{1,\infty} \le \left(\frac{N}{h}\right)^{1_{m=0}} D^{-1-\frac{m}{2}} \cdot \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{2p_1+2p_2} D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty} 1_{p_1+p_2-2\ge m}.$$

(4.12)

$$\|B_m^{(1)}\|_1 \le D^{-1-\frac{m}{2}} \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{2p_1+2p_2} D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_1 1_{p_1+p_2-2\ge m}.$$

(4.13)

$$\|B_m^{(n)}\|_{1,\infty} \le \left(\frac{N}{h}\right)^{1_{m=0}} (n-1)! D^{-n-\frac{m}{2}} 2^{-2m} \|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1} \cdot \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{3p_1+3p_2} D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty} \cdot \prod_{j=3}^{n+1}\left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}} \|F_{p_j}^j\|_{1,\infty}\right) 1_{\sum_{j=1}^{n+1} p_j - 2n\ge m}.$$

(4.14)

$$\begin{aligned}
&\|B_m^{(n)}\|_1\\
&\le (n-1)!D^{-n-\frac{m}{2}}2^{-2m}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1}\\
&\quad\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{3p_1+3p_2}D^{\frac{p_1+p_2}{2}}[F_{p_1,p_2},\tilde{\mathcal{C}}]_{1,\infty}\\
&\quad\cdot\prod_{j=3}^{n}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|F_{p_j}^{j}\|_{1,\infty}\right)\\
&\quad\cdot\sum_{p_{n+1}=2}^{N}2^{3p_{n+1}}D^{\frac{p_{n+1}}{2}}\|F_{p_{n+1}}^{n+1}\|_1 1_{\sum_{j=1}^{n+1}p_j-2n\ge m}.
\end{aligned}$$

PROOF. These are essentially proved in [12, Lemma 3.2]. □

REMARK 4.3. In fact the property (4.10) of $F_{p,q}(\cdot,\cdot)$ is not used to prove Lemma 4.2. It is only necessary to characterize kernels of the Grassmann polynomial denoted by $E^{(n)}(\psi)$ in Lemma 4.4. It is not directly used to derive estimates of the kernels in Lemma 4.4, either. However, we assume the property (4.10) of $F_{p,q}(\cdot,\cdot)$ throughout this subsection in order not to complicate the assumptions by unnecessary generalization of the lemmas. It is important to guarantee that some output polynomials inherit the property (4.10) from the input polynomial as claimed in Lemma 4.4, since it enables us to classify Grassmann polynomials with or without the property (4.10) during the multi-scale integration process. It turns out that those with the property (4.10) vanish at the final integration, which is one essential reason why we can construct this many-electron system by keeping the small coupling constant largely independent of the temperature and the imaginary magnetic field.

In the next lemma we claim several inequalities which were not proved in [12, Subsection 3.2]. Using the same input polynomials as in the above lemmas, we define $E^{(n)}(\psi)$ $(\in \bigwedge_{even}\mathcal{V})$ $(n\in\mathbb{N})$ as follows.

$$E^{(n)}(\psi):=\sum_{p,q=2}^{N}1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}F_{p,q}(\mathbf{X},\mathbf{Y})$$

$$\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C})(\psi^1+\psi)_{\mathbf{X}}\prod_{j=2}^{m+1}F^{s_j}(\psi^{s_j}+\psi)\Big|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}$$
$$\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C})(\psi^1+\psi)_{\mathbf{Y}}\prod_{k=2}^{n-m}F^{t_k}(\psi^{t_k}+\psi)\Big|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}},$$

where

$$\begin{aligned}&m\in\{0,1,\cdots,n-1\},\\&1=s_1<s_2<\cdots<s_{m+1}\le n,\quad 1=t_1<t_2<\cdots<t_{n-m}\le n,\\&\{s_j\}_{j=2}^{m+1}\cup\{t_k\}_{k=2}^{n-m}=\{2,3,\cdots,n\},\quad \{s_j\}_{j=2}^{m+1}\cap\{t_k\}_{k=2}^{n-m}=\emptyset.\end{aligned}$$

Here, unlike in [12, Lemma 3.3], we present the bi-anti-symmetric kernels of $E^{(n)}(\psi)$ beforehand. Let us define functions $E_{a,b}^{(n)}:I^a\times I^b\to\mathbb{C}$ ($n\in\mathbb{N}$, $a,b\in\{2,4,\cdots,N\}$) by

$$\begin{aligned}(4.15)\quad &E_{a,b}^{(n)}(\mathbf{X},\mathbf{Y})\\&:=\sum_{p_1,q_1=2}^{N}1_{p_1,q_1\in 2\mathbb{N}}\sum_{u_1=0}^{p_1}(1_{m=0}+1_{m\neq 0}1_{u_1\le p_1-1})\begin{pmatrix}p_1\\u_1\end{pmatrix}\\&\cdot\sum_{v_1=0}^{q_1}(1_{m=n-1}+1_{m\neq n-1}1_{v_1\le q_1-1})\begin{pmatrix}q_1\\v_1\end{pmatrix}\\&\cdot\prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N}\sum_{u_j=0}^{p_j-1}\begin{pmatrix}p_j\\u_j\end{pmatrix}\right)\prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N}\sum_{v_k=0}^{q_k-1}\begin{pmatrix}q_k\\v_k\end{pmatrix}\right)\\&\cdot f_m^n((p_j)_{1\le j\le m+1},(u_j)_{1\le j\le m+1},(q_j)_{1\le j\le n-m},(v_j)_{1\le j\le n-m})\\&\qquad((\mathbf{X}_1',\mathbf{X}_2',\cdots,\mathbf{X}_{m+1}'),(\mathbf{Y}_1',\mathbf{Y}_2',\cdots,\mathbf{Y}_{n-m}'))\\&\cdot 1_{\sum_{j=1}^{m+1}u_j=a}1_{\sum_{k=1}^{n-m}v_k=b}1_{\sum_{j=1}^{m+1}p_j-2m\ge a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b}\\&\cdot\frac{1}{a!b!}\sum_{\substack{\sigma\in\mathbb{S}_a\\\tau\in\mathbb{S}_b}}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)1_{(\mathbf{X}_1',\mathbf{X}_2',\cdots,\mathbf{X}_{m+1}')=\mathbf{X}_\sigma}1_{(\mathbf{Y}_1',\mathbf{Y}_2',\cdots,\mathbf{Y}_{n-m}')=\mathbf{Y}_\tau},\end{aligned}$$

where the function

$$f_m^n((p_j)_{1\le j\le m+1},(u_j)_{1\le j\le m+1},(q_j)_{1\le j\le n-m},(v_j)_{1\le j\le n-m})$$

$$: \prod_{j=1}^{m+1} I^{u_j} \times \prod_{k=1}^{n-m} I^{v_k} \to \mathbb{C}$$

is defined by

$$\begin{aligned}
& f^n_m((p_j)_{1\le j\le m+1}, (u_j)_{1\le j\le m+1}, (q_j)_{1\le j\le n-m}, (v_j)_{1\le j\le n-m}) \\
& \quad ((\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{m+1}), (\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_{n-m})) \\
& := \left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1} \sum_{\mathbf{W}_1\in I^{p_1-u_1}} \sum_{\mathbf{Z}_1\in I^{q_1-v_1}} F_{p_1,q_1}((\mathbf{W}_1,\mathbf{X}_1),(\mathbf{Z}_1,\mathbf{Y}_1)) \\
& \quad \cdot \prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j-u_j} \sum_{\mathbf{W}_j\in I^{p_j-u_j}} F^{s_j}_{p_j}(\mathbf{W}_j,\mathbf{X}_j)\right) \\
& \quad \cdot \prod_{k=2}^{n-m}\left(\left(\frac{1}{h}\right)^{q_k-v_k} \sum_{\mathbf{Z}_k\in I^{q_k-v_k}} F^{t_k}_{q_k}(\mathbf{Z}_k,\mathbf{Y}_k)\right) \\
& \quad \cdot Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{C}) \prod_{j=1}^{m+1} \psi^{s_j}_{\mathbf{W}_j}\Bigg|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}} \\
& \quad \cdot Tree(\{t_k\}_{k=1}^{n-m}, \mathcal{C}) \prod_{k=1}^{n-m} \psi^{t_k}_{\mathbf{Z}_k}\Bigg|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}} \\
& \quad \cdot (-1)^{\sum_{j=1}^{m} u_j \sum_{i=j+1}^{m+1}(p_i-u_i) + \sum_{k=1}^{n-m-1} v_k \sum_{i=k+1}^{n-m}(q_i-v_i)}.
\end{aligned}$$

LEMMA 4.4. *For any* $n \in \mathbb{N}$, $a, b \in \{2, 4, \cdots, N\}$, $E^{(n)}_{a,b}$ *is bi-anti-symmetric, satisfies* (4.4), (4.10) *and*

$$E^{(n)}(\psi) = \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{a+b} \sum_{\substack{\mathbf{X}\in I^a\\ \mathbf{Y}\in I^b}} E^{(n)}_{a,b}(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.$$

Moreover, the following inequalities hold for any $a, b \in \{2, 4, \cdots, N\}$, $n \in \mathbb{N}_{\ge 2}$ *and anti-symmetric function* $g : I^2 \to \mathbb{C}$.

(4.16)

$$\|E^{(1)}_{a,b}\|_{1,\infty} \le \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \begin{pmatrix} p \\ a \end{pmatrix} \begin{pmatrix} q \\ b \end{pmatrix} D^{\frac{1}{2}(p+q-a-b)} \|F_{p,q}\|_{1,\infty}.$$

(4.17)

$$\|E_{a,b}^{(1)}\|_1 \le \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \left(\begin{array}{c} p \\ a \end{array}\right)\left(\begin{array}{c} q \\ b \end{array}\right) D^{\frac{1}{2}(p+q-a-b)}\|F_{p,q}\|_1.$$

(4.18)

$$[E_{a,b}^{(1)}, g]_{1,\infty} \le \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \left(\begin{array}{c} p \\ a \end{array}\right)\left(\begin{array}{c} q \\ b \end{array}\right) D^{\frac{1}{2}(p+q-a-b)}[F_{p,q}, g]_{1,\infty}.$$

(4.19)

$$[E_{a,b}^{(1)}, g]_1 \le \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \left(\begin{array}{c} p \\ a \end{array}\right)\left(\begin{array}{c} q \\ b \end{array}\right) D^{\frac{1}{2}(p+q-a-b)}[F_{p,q}, g]_1.$$

(4.20)

$$\begin{aligned}
&\|E_{a,b}^{(n)}\|_{1,\infty}\\
&\le (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad \cdot 2^{-2a-2b} D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1} \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} 2^{3p_1+3q_1} D^{\frac{p_1+q_1}{2}}\|F_{p_1,q_1}\|_{1,\infty}\\
&\quad \cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}}\|F_{p_j}^{s_j}\|_{1,\infty}\right)\prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N} 2^{3q_k} D^{\frac{q_k}{2}}\|F_{q_k}^{t_k}\|_{1,\infty}\right)\\
&\quad \cdot 1_{\sum_{j=1}^{m+1} p_j - 2m \ge a} 1_{\sum_{k=1}^{n-m} q_k - 2(n-m-1)\ge b}.
\end{aligned}$$

(4.21)

$$\begin{aligned}
&\|E_{a,b}^{(n)}\|_1\\
&\le (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad \cdot 2^{-2a-2b} D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-1} \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} 2^{3p_1+3q_1} D^{\frac{p_1+q_1}{2}}\|F_{p_1,q_1}\|_{1,\infty}\\
&\quad \cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N} 2^{3p_j} D^{\frac{p_j}{2}}(1_{s_j\neq n}\|F_{p_j}^{s_j}\|_{1,\infty} + 1_{s_j=n}\|F_{p_j}^{s_j}\|_1)\right)\\
&\quad \cdot \prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N} 2^{3q_k} D^{\frac{q_k}{2}}(1_{t_k\neq n}\|F_{q_k}^{t_k}\|_{1,\infty} + 1_{t_k=n}\|F_{q_k}^{t_k}\|_1)\right)\\
&\quad \cdot 1_{\sum_{j=1}^{m+1} p_j - 2m \ge a} 1_{\sum_{k=1}^{n-m} q_k - 2(n-m-1)\ge b}.
\end{aligned}$$

(4.22)

$$
\begin{aligned}
&[E_{a,b}^{(n)}, g]_{1,\infty}\\
&\le (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\quad\cdot 2^{-2a-2b}D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-2}\\
&\quad\cdot \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}\\
&\quad\cdot D^{\frac{p_1+q_1}{2}}([F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty}+[F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty})\\
&\quad\cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}\|F_{p_j}^{s_j}\|_{1,\infty}\right)\prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N}2^{3q_k}D^{\frac{q_k}{2}}\|F_{q_k}^{t_k}\|_{1,\infty}\right)\\
&\quad\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\ge a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b}.
\end{aligned}
$$

(4.23)

$$
\begin{aligned}
&[E_{a,b}^{(n)}, g]_{1}\\
&\le (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\quad\cdot 2^{-2a-2b}D^{-n+1-\frac{1}{2}(a+b)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-2}\\
&\quad\cdot \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}\\
&\quad\cdot D^{\frac{p_1+q_1}{2}}([F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty}+[F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty})\\
&\quad\cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=2}^{N}2^{3p_j}D^{\frac{p_j}{2}}(1_{s_j\neq n}\|F_{p_j}^{s_j}\|_{1,\infty}+1_{s_j=n}\|F_{p_j}^{s_j}\|_{1})\right)\\
&\quad\cdot \prod_{k=2}^{n-m}\left(\sum_{q_k=2}^{N}2^{3q_k}D^{\frac{q_k}{2}}(1_{t_k\neq n}\|F_{q_k}^{t_k}\|_{1,\infty}+1_{t_k=n}\|F_{q_k}^{t_k}\|_{1})\right)\\
&\quad\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\ge a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b}.
\end{aligned}
$$

PROOF. The kernels $E_{a,b}^{(n)}$ $(a,b\in\{2,4,\cdots,N\})$ were essentially given in [12, (3.41),(3.39)]. The claimed properties of $E_{a,b}^{(n)}$ and the inequalities (4.16), (4.17), (4.20), (4.21) were essentially proved in [12, Lemma 3.3]. We need to show (4.18), (4.19), (4.22) and (4.23). In fact these inequalities can be proved in similar ways to the derivations of (4.16), (4.17), (4.20), (4.21). However, we provide the major part of the proof for completeness. Let $\mathbf{p} := (p_j)_{1\le j\le m+1}$, $\mathbf{u} := (u_j)_{1\le j\le m+1}$, $\mathbf{q} := (q_j)_{1\le j\le n-m}$, $\mathbf{v} := (v_j)_{1\le j\le n-m}$ for simplicity in the following. Since the norm bounds on $E_{a,b}^{(n)}$ follow from norm bounds on the function $f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})$, let us focus on estimating $f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})$.

First let us consider the case $n = 1$. By using the determinant bound (4.3) we have for any $\mathbf{X}_1 \in I^{u_1}$, $\mathbf{Y}_1 \in I^{v_1}$ that

$$\begin{aligned}&|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\mathbf{X}_1,\mathbf{Y}_1)|\\&\le \left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1}\sum_{\substack{\mathbf{W}_1\in I^{p_1-u_1}\\ \mathbf{Z}_1\in I^{q_1-v_1}}}|F_{p_1,q_1}((\mathbf{W}_1,\mathbf{X}_1),(\mathbf{Z}_1,\mathbf{Y}_1))|D^{\frac{1}{2}(p_1+q_1-u_1-v_1)},\end{aligned}$$

which implies that

$$[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_{index} \le D^{\frac{1}{2}(p_1+q_1-u_1-v_1)}[F_{p_1,q_1},g]_{index}, \tag{4.24}$$

for $index = `1,\infty'$ or $index = 1$ and any anti-symmetric function $g : I^2 \to \mathbb{C}$.

Let us consider the case $n \ge 2$. Let us take an anti-symmetric function $g : I^2 \to \mathbb{C}$ and estimate $[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_1$. If $n \in \{s_j\}_{j=1}^{m+1}$, we estimate the right-hand side of the following inequality.

(4.25)

$$\begin{aligned}&[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_1\\&\le \sup_{k_1\in\{1,2,\cdots,\sum_{k=1}^{n-m}v_k\}}\Bigg(\left(\frac{1}{h}\right)^{\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k}\sum_{\mathbf{X}\in\prod_{j=1}^{m+1}I^{u_j}}\\&\qquad\cdot\sup_{Y_0\in I}\sum_{\mathbf{Y}\in\prod_{k=1}^{n-m}I^{v_k}}|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\mathbf{X},\mathbf{Y})||g(Y_0,Y_{k_1})|\Bigg).\end{aligned}$$

For $k_1 \in \{1,2,\cdots,\sum_{k=1}^{n-m}v_k\}$ there uniquely exists $k_0 \in \{1,2,\cdots,n-m\}$ such that Y_{k_1} is a component of the variable of the function F_{p_1,q_1} if $k_0 = 1$ or $F_{q_{k_0}}^{t_{k_0}}$ if $k_0 \ne 1$. We consider the vertex t_{k_0} as the root of the tree $T \in \mathbb{T}(\{t_k\}_{k=1}^{n-m})$ and recursively estimate from the younger branches to the root

t_{k_0} along the lines of T. In this procedure we obtain especially

$$\sup_{Y_0\in I}\left(\frac{1}{h}\right)^{q_1}\sum_{\mathbf{Y}\in I^{q_1}}|F_{p_1,q_1}(\mathbf{X},\mathbf{Y})||g(Y_0,Y_1)|$$

if $k_0=1$,

$$\sup_{Y_0\in I}\left(\frac{1}{h}\right)^{q_1}\sum_{\mathbf{Y}\in I^{q_1}}|F_{p_1,q_1}(\mathbf{X},\mathbf{Y})||\tilde{\mathcal{C}}(Y_0,Y_1)|$$

if $k_0\neq 1$. Then we consider the vertex n as the root of $S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})$ and recursively estimate from the younger branches to the root n along the lines of S. In the end we obtain especially $[F_{p_1,q_1},g]_{1,\infty}$ if $k_0=1$, $[F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}$ if $k_0\neq 1$. In the case $n\in\{t_k\}_{k=1}^{n-m}$ we follow the other way round. We estimate the right-hand side of the following inequality.

(4.26)

$$\begin{aligned}&[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_1\\ &\le\sup_{j_1\in\{1,2,\cdots,\sum_{j=1}^{m+1}u_j\}}\Bigg(\left(\frac{1}{h}\right)^{\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k}\sum_{\mathbf{Y}\in\prod_{k=1}^{n-m}I^{v_k}}\\ &\qquad\cdot\sup_{X_0\in I}\sum_{\mathbf{X}\in\prod_{j=1}^{m+1}I^{u_j}}|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\mathbf{X},\mathbf{Y})||g(X_0,X_{j_1})|\Bigg).\end{aligned}$$

For $j_1\in\{1,2,\cdots,\sum_{j=1}^{m+1}u_j\}$ there uniquely exists $j_0\in\{1,2,\cdots,m+1\}$ such that X_{j_1} is a component of the variable of the function F_{p_1,q_1} if $j_0=1$ or $F_{p_{j_0}}^{s_{j_0}}$ if $j_0\neq 1$. We consider the vertex s_{j_0} as the root of the tree $S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})$ and recursively estimate along the lines of S. Then we consider the vertex n as the root of the tree $T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})$ and recursively estimate along the lines of T. By applying the determinant bound (4.3) we have that for any $\mathbf{X}_j\in I^{u_j}$ $(j=1,2,\cdots,m+1)$, $\mathbf{Y}_k\in I^{v_k}$ $(k=1,2,\cdots,n-m)$,

(4.27)

$$|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})((\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_{m+1}),(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_{n-m}))|$$

$$
\begin{aligned}
&\le 2^{n-1}\sum_{S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})}\sum_{T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})}\\
&\quad\cdot\left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1}\sum_{\substack{\mathbf{W}_1\in I^{p_1-u_1-d_1(S)}\\ \mathbf{W}_1'\in I^{d_1(S)}}}\sum_{\substack{\mathbf{Z}_1\in I^{q_1-v_1-d_1(T)}\\ \mathbf{Z}_1'\in I^{d_1(T)}}}\\
&\quad\cdot\begin{pmatrix} p_1-u_1\\ d_1(S)\end{pmatrix}\begin{pmatrix} q_1-v_1\\ d_1(T)\end{pmatrix}|F_{p_1,q_1}((\mathbf{W}_1,\mathbf{W}_1',\mathbf{X}_1),(\mathbf{Z}_1,\mathbf{Z}_1',\mathbf{Y}_1))|\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j-u_j}\sum_{\substack{\mathbf{W}_j\in I^{p_j-u_j-d_{s_j}(S)}\\ \mathbf{W}_j'\in I^{d_{s_j}(S)}}}\begin{pmatrix} p_j-u_j\\ d_{s_j}(S)\end{pmatrix}|F_{p_j}^{s_j}(\mathbf{W}_j,\mathbf{W}_j',\mathbf{X}_j)|\right)\\
&\quad\cdot\prod_{k=2}^{n-m}\left(\left(\frac{1}{h}\right)^{q_k-v_k}\sum_{\substack{\mathbf{Z}_k\in I^{q_k-v_k-d_{t_k}(T)}\\ \mathbf{Z}_k'\in I^{d_{t_k}(T)}}}\begin{pmatrix} q_k-v_k\\ d_{t_k}(T)\end{pmatrix}|F_{q_k}^{t_k}(\mathbf{Z}_k,\mathbf{Z}_k',\mathbf{Y}_k)|\right)\\
&\quad\cdot D^{\frac{1}{2}(\sum_{j=1}^{m+1}p_j-2m-\sum_{j=1}^{m+1}u_j)+\frac{1}{2}(\sum_{k=1}^{n-m}q_k-2(n-m-1)-\sum_{k=1}^{n-m}v_k)}\\
&\quad\cdot\left(1_{m=0}+1_{m\neq 0}\left|\prod_{\{p,q\}\in S}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{m+1}\psi_{\mathbf{W}_j'}^{s_j}\right|\right)\\
&\quad\cdot\left(1_{m=n-1}+1_{m\neq n-1}\left|\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\prod_{k=1}^{n-m}\psi_{\mathbf{Z}_k'}^{t_k}\right|\right)\\
&=2^{n-1}D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\sum_{S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})}\sum_{T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})}\\
&\quad\cdot\left(\frac{1}{h}\right)^{p_1+q_1-u_1-v_1}\sum_{\substack{\mathbf{W}_1\in I^{p_1-u_1-d_1(S)}\\ \mathbf{W}_1'\in I^{d_1(S)}}}\sum_{\substack{\mathbf{Z}_1\in I^{q_1-v_1-d_1(T)}\\ \mathbf{Z}_1'\in I^{d_1(T)}}}\\
&\quad\cdot\begin{pmatrix} p_1-u_1\\ d_1(S)\end{pmatrix}\begin{pmatrix} q_1-v_1\\ d_1(T)\end{pmatrix}D^{\frac{1}{2}(p_1+q_1)}|F_{p_1,q_1}((\mathbf{W}_1,\mathbf{W}_1',\mathbf{X}_1),(\mathbf{Z}_1,\mathbf{Z}_1',\mathbf{Y}_1))|\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j-u_j}\sum_{\substack{\mathbf{W}_j\in I^{p_j-u_j-d_{s_j}(S)}\\ \mathbf{W}_j'\in I^{d_{s_j}(S)}}}\begin{pmatrix} p_j-u_j\\ d_{s_j}(S)\end{pmatrix}D^{\frac{p_j}{2}}|F_{p_j}^{s_j}(\mathbf{W}_j,\mathbf{W}_j',\mathbf{X}_j)|\right)
\end{aligned}
$$

$$\cdot \prod_{k=2}^{n-m}\left(\left(\frac{1}{h}\right)^{q_k-v_k}\sum_{\substack{\mathbf{Z}_k\in I^{q_k-v_k-d_{t_k}(T)}\\ \mathbf{Z}'_k\in I^{d_{t_k}(T)}}}\begin{pmatrix} q_k-v_k\\ d_{t_k}(T)\end{pmatrix} D^{\frac{q_k}{2}}|F_{q_k}^{t_k}(\mathbf{Z}_k,\mathbf{Z}'_k,\mathbf{Y}_k)|\right)$$

$$\cdot\left(1_{m=0}+1_{m\neq 0}\left|\prod_{\{p,q\}\in S}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{m+1}\psi^{s_j}_{\mathbf{W}'_j}\right|\right)$$

$$\cdot\left(1_{m=n-1}+1_{m\neq n-1}\left|\prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\prod_{k=1}^{n-m}\psi^{t_k}_{\mathbf{Z}'_k}\right|\right).$$

Recall that for $S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})$, $d_{s_j}(S)$ denotes the degree of the vertex s_j in S. See [12, Subsection 3.1] for the definition of the operator $\Delta_{\{p,q\}}(\mathcal{C})$. By following the tactics of estimation explained in and after (4.25), (4.26) we can derive that

(4.28)

$$\begin{aligned}
&[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_1\\
&\le 2^{n-1}D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-2}\sum_{S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})}\sum_{T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})}\\
&\quad\cdot\begin{pmatrix} p_1-u_1\\ d_1(S)\end{pmatrix} d_1(S)!\begin{pmatrix} q_1-v_1\\ d_1(T)\end{pmatrix} d_1(T)!D^{\frac{p_1+q_1}{2}}\\
&\quad\cdot([F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty}+[F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty})\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\begin{pmatrix} p_j-u_j\\ d_{s_j}(S)\end{pmatrix} d_{s_j}(S)!D^{\frac{p_j}{2}}(1_{s_j=n}\|F_{p_j}^{s_j}\|_1+1_{s_j\neq n}\|F_{p_j}^{s_j}\|_{1,\infty})\right)\\
&\quad\cdot\prod_{k=2}^{n-m}\left(\begin{pmatrix} q_k-v_k\\ d_{t_k}(T)\end{pmatrix} d_{t_k}(T)!D^{\frac{q_k}{2}}(1_{t_k=n}\|F_{q_k}^{t_k}\|_1+1_{t_k\neq n}\|F_{q_k}^{t_k}\|_{1,\infty})\right).
\end{aligned}$$

In fact the inequality with the term

$$\sup\{[F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty},\ [F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty}\}$$

in place of

$$[F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty}+[F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty}$$

can hold. However, we choose to use the above inequality for simplicity. Also, we took into account the fact that

$$\prod_{\{p,q\}\in S}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{m+1}\psi_{\mathbf{W}_j'}^{s_j},\quad \prod_{\{p,q\}\in T}\Delta_{\{p,q\}}(\mathcal{C})\prod_{k=1}^{n-m}\psi_{\mathbf{Z}_k'}^{t_k}$$

create at most $\prod_{j=1}^{m+1}d_{s_j}(S)!$, $\prod_{k=1}^{n-m}d_{t_k}(T)!$ terms respectively.

In order to support the readers, let us present an intermediate step between (4.27) and (4.28). Assume that $n\in\{s_j\}_{j=1}^{m+1}$. Take any $k_1\in\{1,2,\cdots,\sum_{k=1}^{n-m}v_k\}$. In this case $m\neq 0$. By following the strategy explained after (4.25),

$$\begin{aligned}
&\left(\frac{1}{h}\right)^{\sum_{k=1}^{n-m}v_k}\sup_{Y_0\in I}\sum_{\mathbf{Y}\in\prod_{k=1}^{n-m}I^{v_k}}|f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})((\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_{m+1}),\mathbf{Y})|\\
&\qquad\qquad\cdot|g(Y_0,Y_{k_1})|\\
&\le 2^{n-1}D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\sum_{S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})}\sum_{T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})}\\
&\quad\cdot\left(\frac{1}{h}\right)^{p_1-u_1}\sum_{\substack{\mathbf{W}_1\in I^{p_1-u_1-d_1(S)}\\ \mathbf{W}_1'\in I^{d_1(S)}}}\begin{pmatrix}p_1-u_1\\ d_1(S)\end{pmatrix}\begin{pmatrix}q_1-v_1\\ d_1(T)\end{pmatrix}d_1(T)!D^{\frac{1}{2}(p_1+q_1)}\\
&\quad\cdot\Bigg(\left(\frac{1}{h}\right)^{q_1}\sup_{Y_0\in I}\sum_{\mathbf{Y}\in I^{q_1}}|F_{p_1,q_1}((\mathbf{W}_1,\mathbf{W}_1',\mathbf{X}_1),\mathbf{Y})||g(Y_0,Y_1)|\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-m-1}\\
&\qquad+1_{m\neq n-1}\left(\frac{1}{h}\right)^{q_1}\sup_{Y_0\in I}\sum_{\mathbf{Y}\in I^{q_1}}|F_{p_1,q_1}((\mathbf{W}_1,\mathbf{W}_1',\mathbf{X}_1),\mathbf{Y})||\tilde{\mathcal{C}}(Y_0,Y_1)|\\
&\qquad\cdot\|g\|_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-m-2}\Bigg)\\
&\quad\cdot\prod_{j=2}^{m+1}\left(\left(\frac{1}{h}\right)^{p_j-u_j}\sum_{\substack{\mathbf{W}_j\in I^{p_j-u_j-d_{s_j}(S)}\\ \mathbf{W}_j'\in I^{d_{s_j}(S)}}}\begin{pmatrix}p_j-u_j\\ d_{s_j}(S)\end{pmatrix}D^{\frac{p_j}{2}}|F_{p_j}^{s_j}(\mathbf{W}_j,\mathbf{W}_j',\mathbf{X}_j)|\right)\\
&\quad\cdot\prod_{k=2}^{n-m}\left(\begin{pmatrix}q_k-v_k\\ d_{t_k}(T)\end{pmatrix}d_{t_k}(T)!D^{\frac{q_k}{2}}\|F_{q_k}^{t_k}\|_{1,\infty}\right)\left|\prod_{\{p,q\}\in S}\Delta_{\{p,q\}}(\mathcal{C})\prod_{j=1}^{m+1}\psi_{\mathbf{W}_j'}^{s_j}\right|.
\end{aligned}$$

Then by integrating with the variable $(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{m+1})$ we obtain the right-hand side of (4.28). By following the strategy explained after (4.26) we can deal with the case $n \in \{t_k\}_{k=1}^{n-m}$ as well.

Now to restart with (4.28), let us recall the following estimate based on the well-known theorem on the number of trees with fixed degrees.

$$\begin{aligned}
&\sum_{S\in\mathbb{T}(\{s_j\}_{j=1}^{m+1})}\prod_{j=1}^{m+1}\left(\left(\begin{array}{c} p_j-u_j \\ d_{s_j}(S)\end{array}\right)d_{s_j}(S)!\right)\\
&\le (1_{m=0}+1_{m\neq 0}(m-1)!2^{-m-1})2^{2\sum_{j=1}^{m+1}(p_j-u_j)},\\
&\sum_{T\in\mathbb{T}(\{t_k\}_{k=1}^{n-m})}\prod_{k=1}^{n-m}\left(\left(\begin{array}{c} q_k-v_k \\ d_{t_k}(T)\end{array}\right)d_{t_k}(T)!\right)\\
&\le (1_{m=n-1}+1_{m\neq n-1}(n-m-2)!2^{-n+m})2^{2\sum_{k=1}^{n-m}(q_k-v_k)}.
\end{aligned}$$

See [12, (3.20),(3.21)]. By substituting these inequalities and using the inequality

$$\begin{aligned}
&2^{n-1}(1_{m=0}+1_{m\neq 0}(m-1)!2^{-m-1})(1_{m=n-1}+1_{m\neq n-1}(n-m-2)!2^{-n+m})\\
&\le (1_{m=0}+1_{m\neq 0}(m-1)!)(1_{m=n-1}+1_{m\neq n-1}(n-m-2)!)
\end{aligned}$$

we obtain that

$$\begin{aligned}
(4.29)\qquad &[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_1\\
&\le (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\quad\cdot 2^{-2\sum_{j=1}^{m+1}u_j-2\sum_{k=1}^{n-m}v_k}D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-2}\\
&\quad\cdot 2^{2p_1+2q_1}D^{\frac{p_1+q_1}{2}}([F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty}+[F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty})\\
&\quad\cdot\prod_{j=2}^{m+1}(2^{2p_j}D^{\frac{p_j}{2}}(1_{s_j=n}\|F_{p_j}^{s_j}\|_1+1_{s_j\neq n}\|F_{p_j}^{s_j}\|_{1,\infty}))\\
&\quad\cdot\prod_{k=2}^{n-m}(2^{2q_k}D^{\frac{q_k}{2}}(1_{t_k=n}\|F_{q_k}^{t_k}\|_1+1_{t_k\neq n}\|F_{q_k}^{t_k}\|_{1,\infty})).
\end{aligned}$$

Let us consider $[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_{1,\infty}$. To estimate

$$\sup_{\substack{X_0\in I\\ j_1\in\{1,2,\cdots,\sum_{j=1}^{m+1}u_j\}}}\left(\left(\frac{1}{h}\right)^{\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k-1}\sum_{\mathbf{X}\in\prod_{j=1}^{m+1}I^{u_j}}1_{X_{j_1}=X_0}\right.$$

$$\cdot \sup_{\substack{Y_0\in I\\ k_1\in\{1,2,\cdots,\sum_{k=1}^{n-m}v_k\}}} \sum_{\mathbf{Y}\in\prod_{k=1}^{n-m}I^{v_k}} |f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\mathbf{X},\mathbf{Y})||g(Y_0,Y_{k_1})|\Bigg),$$

we fix $j_1 \in \{1,2,\cdots,\sum_{j=1}^{m+1}u_j\}$, $k_1 \in \{1,2,\cdots,\sum_{k=1}^{n-m}v_k\}$. Then there uniquely exist $j_0 \in \{1,2,\cdots,m+1\}$, $k_0 \in \{1,2,\cdots,n-m\}$ such that X_{j_1} is a component of the variable of the function F_{p_1,q_1} if $j_0 = 1$ or $F_{p_{j_0}}^{s_{j_0}}$ if $j_0 \neq 1$ and Y_{k_1} is a component of the variable of the function F_{p_1,q_1} if $k_0 = 1$ or $F_{q_{k_0}}^{t_{k_0}}$ if $k_0 \neq 1$. We recursively estimate along the lines of $T \in \mathbb{T}(\{t_k\}_{k=1}^{n-m})$ by considering the vertex t_{k_0} as the root in the first place. Then we recursively estimate along the lines of $S \in \mathbb{T}(\{s_j\}_{j=1}^{m+1})$ by considering the vertex s_{j_0} as the root. On the other hand, we estimate

$$\sup_{\substack{Y_0\in I\\ k_1\in\{1,2,\cdots,\sum_{k=1}^{n-m}v_k\}}} \Bigg(\left(\frac{1}{h}\right)^{\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k-1} \sum_{\mathbf{Y}\in\prod_{k=1}^{n-m}I^{v_k}} 1_{Y_{k_1}=Y_0}$$
$$\cdot \sup_{\substack{X_0\in I\\ j_1\in\{1,2,\cdots,\sum_{j=1}^{m+1}u_j\}}} \sum_{\mathbf{X}\in\prod_{j=1}^{m+1}I^{u_j}} |f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v})(\mathbf{X},\mathbf{Y})||g(X_0,X_{j_1})|\Bigg)$$

by performing the recursive estimation along the lines of $S \in \mathbb{T}(\{s_j\}_{j=1}^{m+1})$ first and the recursive estimation along the lines of $T \in \mathbb{T}(\{t_k\}_{k=1}^{n-m})$ afterwards. Since the procedure is parallel to the estimation of $[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_1$, we only state the result.

$$\begin{aligned}(4.30)\quad &[f_m^n(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}),g]_{1,\infty}\\ &\le (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\ &\quad\cdot 2^{-2\sum_{j=1}^{m+1}u_j-2\sum_{k=1}^{n-m}v_k}D^{-n+1-\frac{1}{2}(\sum_{j=1}^{m+1}u_j+\sum_{k=1}^{n-m}v_k)}\|\tilde{\mathcal{C}}\|_{1,\infty}^{n-2}\\ &\quad\cdot 2^{2p_1+2q_1}D^{\frac{p_1+q_1}{2}}([F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty}+[F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty})\\ &\quad\cdot \prod_{j=2}^{m+1}(2^{2p_j}D^{\frac{p_j}{2}}\|F_{p_j}^{s_j}\|_{1,\infty})\prod_{k=2}^{n-m}(2^{2q_k}D^{\frac{q_k}{2}}\|F_{q_k}^{t_k}\|_{1,\infty}).\end{aligned}$$

Again we overestimated by replacing

$$\sup\{[F_{p_1,q_1},g]_{1,\infty}\|\tilde{\mathcal{C}}\|_{1,\infty},\ [F_{p_1,q_1},\tilde{\mathcal{C}}]_{1,\infty}\|g\|_{1,\infty}\}$$

by their sum for simplicity.

It follows from (4.15) that

$$
\begin{aligned}
(4.31)\quad & [E^{(n)}_{a,b}, g]_{index} \\
&\le \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} \sum_{u_1=0}^{p_1} (1_{m=0}+1_{m\neq 0}1_{u_1\le p_1-1}) \left(\begin{array}{c} p_1 \\ u_1 \end{array}\right) \\
&\quad \cdot \sum_{v_1=0}^{q_1} (1_{m=n-1}+1_{m\neq n-1}1_{v_1\le q_1-1}) \left(\begin{array}{c} q_1 \\ v_1 \end{array}\right) \\
&\quad \cdot \prod_{j=2}^{m+1} \left(\sum_{p_j=2}^{N} \sum_{u_j=0}^{p_j-1} \left(\begin{array}{c} p_j \\ u_j \end{array}\right) \right) \\
&\quad \cdot \prod_{k=2}^{n-m} \left(\sum_{q_k=2}^{N} \sum_{v_k=0}^{q_k-1} \left(\begin{array}{c} q_k \\ v_k \end{array}\right) \right) [f^n_m(\mathbf{p},\mathbf{u},\mathbf{q},\mathbf{v}), g]_{index} \\
&\quad \cdot 1_{\sum_{j=1}^{m+1} u_j = a} 1_{\sum_{k=1}^{n-m} v_k = b} 1_{\sum_{j=1}^{m+1} p_j - 2m \ge a} 1_{\sum_{k=1}^{n-m} q_k - 2(n-m-1) \ge b},
\end{aligned}
$$

where $index$ = '$1,\infty$' or $index = 1$. By substituting (4.24) for $index$ = '$1,\infty$', 1, (4.30), (4.29) into (4.31) we obtain (4.18), (4.19), (4.22), (4.23) respectively. □

4.3. Generalized covariances

We construct the general multi-scale integration process by assuming scale-dependent bound properties of covariances. Here let us list the properties of the generalized covariances. Assume that $N_\beta < \hat{N}_\beta$, $N_\beta, \hat{N}_\beta \in \mathbb{Z}$. These numbers represent the integration scales. We should think that at the scale $\hat{N}_\beta + 1$ we perform a single-scale ultra-violet (UV) integration and from $\hat{N}_\beta$ to N_β we perform a multi-scale infrared (IR) integration. Let c_0, $M \in \mathbb{R}_{\ge 1}$, $c_{end} \in \mathbb{R}_{>0}$. We assume that covariances $\mathcal{C}_l : I_0^2 \to \mathbb{C}$ $(l = N_\beta, N_\beta+1, \cdots, \hat{N}_\beta)$ satisfy the following properties.

- $\mathcal{C}_l$ $(l = N_\beta+1, N_\beta+2, \cdots, \hat{N}_\beta)$ satisfy (4.2).
-

$$
\begin{aligned}
(4.32)\qquad & \mathcal{C}_{N_\beta}(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) = \mathcal{C}_{N_\beta}(\overline{\rho}\rho\mathbf{x}0, \overline{\eta}\eta\mathbf{y}0), \\
& (\forall(\overline{\rho},\rho,\mathbf{x},s),(\overline{\eta},\eta,\mathbf{y},t) \in I_0).
\end{aligned}
$$

•

$$|\det(\langle \mathbf{u}_i, \mathbf{v}_j\rangle_{\mathbb{C}^m}\mathcal{C}_l(X_i,Y_j))_{1\le i,j\le n}| \le (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^n, \quad (4.33)$$
$$(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m}, \|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,$$
$$X_i, Y_i\in I_0\ (i=1,2,\cdots,n),\ l\in\{N_\beta, N_\beta+1,\cdots,\hat{N}_\beta\}).$$

•

$$\|\tilde{\mathcal{C}}_l\|_{1,\infty} \le c_0\left(1_{N_\beta+1\le l\le \hat{N}_\beta} M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l-\hat{N}_\beta)} + 1_{l=N_\beta} c_{end}\right), \quad (4.34)$$
$$(\forall l\in\{N_\beta, N_\beta+1,\cdots,\hat{N}_\beta\}).$$

•

$$\|\tilde{\mathcal{C}}_l\| \le c_0 M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l-\hat{N}_\beta)}, \quad (4.35)$$
$$(\forall l\in\{N_\beta+1, N_\beta+2,\cdots,\hat{N}_\beta\}).$$

Here $\tilde{\mathcal{C}}_l : I^2 \to \mathbb{C}$ is the anti-symmetric extension of $\mathcal{C}_l$ defined as in (4.5). In Subsection 5.1 we will explicitly define these scale-dependent covariances by decomposing the actual covariance appearing in the formulation Lemma 3.6.

4.4. Multi-scale integration without the artificial term

In the rest of this section we always extend the coupling constant to be a complex parameter. To distinguish, let u denote the extended coupling constant and set

$$\begin{aligned}
V(u)(\psi) &:= \frac{u}{L^d h}\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{1\rho\mathbf{x}s}\\
&\quad + \frac{u}{L^d h}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{s\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{2\rho\mathbf{x}s}\overline{\psi}_{2\eta\mathbf{y}s}\psi_{1\eta\mathbf{y}s},\\
W(u)(\psi) &:= \frac{u}{\beta L^d h^2}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{s,t\in[0,\beta)_h}\overline{\psi}_{1\rho\mathbf{x}s}\psi_{2\rho\mathbf{x}s}\overline{\psi}_{2\eta\mathbf{y}t}\psi_{1\eta\mathbf{y}t}.
\end{aligned}$$

In this subsection we construct a multi-scale integration for the Grassmann polynomial

$$\log\left(\int e^{-V(u)(\psi+\psi^1)+W(u)(\psi+\psi^1)} d\mu_{\sum_{l=N_\beta+1}^{\hat{N}_\beta}\mathcal{C}_l}(\psi^1)\right).$$

The well-definedness of this Grassmann polynomial is a priori guaranteed only for small u. We are going to construct an analytic continuation of this Grassmann polynomial. Uniform boundedness of the analytically continued polynomial is important in controlling the integrand of the Gaussian integrals in the final formulation Lemma 3.6 (iii). In the next subsection we will perform a multi-scale integration by adding the artificial term $A(\psi)$ to the input polynomial. We want to prove the analyticity of Grassmann polynomials with the variable u as a result of the multi-scale integration in this subsection. For this purpose it is natural to consider kernels of Grassmann polynomials as elements of the Banach space $C(\overline{D(r)}, \mathrm{Map}(I^m, \mathbb{C}))$ equipped with the norm $\|\cdot\|_{1,\infty,r}$ defined by

$$\|f\|_{1,\infty,r} := \sup_{u\in\overline{D(r)}} \|f(u)\|_{1,\infty}.$$

We also let $\|\cdot\|_{1,\infty,r}$ denote the uniform norm of $C(\overline{D(r)}, \mathbb{C})$ for notational consistency. Similarly for $f \in C(\overline{D(r)}, \mathrm{Map}(I^m, \mathbb{C}))$ and an anti-symmetric function $g : I^2 \to \mathbb{C}$ we set

$$[f, g]_{1,\infty,r} := \sup_{u\in\overline{D(r)}} [f(u), g]_{1,\infty}.$$

More generally, for any domain D of $\mathbb{C}^n$ and finite-dimensional complex Banach space B we let $C(\overline{D}, B)$, $C^\omega(D, B)$ denote the set of all continuous maps from $\overline{D}$ to B, the set of all analytic maps from D to B respectively. In practice we let B be $\bigwedge_{even} \mathcal{V}$ or $\mathrm{Map}(I^m, \mathbb{C})$, even though we do not always specify a norm on these complex vector spaces. The finite-dimensionality implies that every norm is equivalent to each other. Normally, we use $\|\cdot\|_{1,\infty}$ or $\|\cdot\|_1$ as a norm of $\mathrm{Map}(I^m, \mathbb{C})$ and induce a norm of $\bigwedge_{even} \mathcal{V}$ by measuring anti-symmetric kernels of a Grassmann polynomial by $\|\cdot\|_{1,\infty}$ or $\|\cdot\|_1$. The readers should understand which norm is being considered from the context. Observe that once a norm is defined in $\bigwedge_{even} \mathcal{V}$, $f \in C(\overline{D}, \bigwedge_{even} \mathcal{V})$ is equivalent to $f_0 \in C(\overline{D})$, $f(\cdot)_m \in C(\overline{D}, \mathrm{Map}(I^m, \mathbb{C}))$ $(m = 2, 4, \cdots, N)$, which is equivalent to $f_0 \in C(\overline{D})$, $f(\cdot)_m(\mathbf{X}) \in C(\overline{D})$ $(\mathbf{X} \in I^m,\ m = 2, 4, \cdots, N)$, where $f(u)_m$ $(m = 2, 4, \cdots, N)$ are anti-symmetric kernels of $f(u)(\psi)$ for $u \in \overline{D}$. The parallel statements can be made for $C^\omega(D, \bigwedge_{even} \mathcal{V})$. In order to systematically describe properties of Grassmann data in the multi-scale integration, we define several subsets of $C(\overline{D(r)}, \bigwedge_{even} \mathcal{V})$. In the following we let $l \in \{N_\beta + 1, N_\beta + 2, \cdots, \hat{N}_\beta\}$, $r \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{\geq 1}$.

We define the set $\mathcal{Q}(r,l)$ as follows. f belongs to $\mathcal{Q}(r,l)$ if and only if

- $$f \in C\left(\overline{D(r)}, \bigwedge_{even} \mathcal{V}\right) \cap C^{\omega}\left(D(r), \bigwedge_{even} \mathcal{V}\right).$$

- For any $u \in \overline{D(r)}$ the anti-symmetric kernels $f(u)_m : I^m \to \mathbb{C}$ ($m = 2, 4, \cdots, N$) satisfy (4.4) and

$$\frac{h}{N}\alpha^2 M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)}\|f_0\|_{1,\infty,r} \le L^{-d}, \tag{4.36}$$
$$\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m M^{\frac{m}{2}\mathsf{a}(l-\hat{N}_\beta)}\|f_m\|_{1,\infty,r} \le L^{-d}.$$

Simply speaking, we use the set $\mathcal{Q}(r,l)$ to collect Grassmann data bounded by the inverse volume factor.

We define the set $\mathcal{R}(r,l)$ as follows. f belongs to $\mathcal{R}(r,l)$ if and only if

- $$f \in C\left(\overline{D(r)}, \bigwedge_{even} \mathcal{V}\right) \cap C^{\omega}\left(D(r), \bigwedge_{even} \mathcal{V}\right).$$

- There exist $f_{p,q} \in C(\overline{D(r)}, \mathrm{Map}(I^p \times I^q, \mathbb{C}))$ ($p, q \in \{2, 4, \cdots, N\}$) such that for any $u \in \overline{D(r)}$, $p, q \in \{2, 4, \cdots, N\}$, $f_{p,q}(u) : I^p \times I^q \to \mathbb{C}$ is bi-anti-symmetric, satisfies (4.4), (4.10),

$$f(u)(\psi) = \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}} f_{p,q}(u)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}$$

  and

(4.37)
$$M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)}\sum_{p,q=2}^{N} c_0^{\frac{p+q}{2}}\alpha^{p+q}M^{\frac{p+q}{2}\mathsf{a}(l-\hat{N}_\beta)}\|f_{p,q}\|_{1,\infty,r} \le 1,$$

(4.38)
$$M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)}\sum_{p,q=2}^{N} c_0^{\frac{p+q}{2}}\alpha^{p+q}M^{\frac{p+q}{2}\mathsf{a}(l-\hat{N}_\beta)}[f_{p,q}, g]_{1,\infty,r} \le L^{-d}\|g\|,$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$.

In essence the set $\mathcal{R}(r,l)$ will be used to collect Grassmann data having the vanishing property (4.10).

Let us start explaining the inductive multi-scale integration process by explicitly defining the initial Grassmann data. Define $V_2^{0-1,\hat{N}_\beta} : \overline{D(r)} \to \mathrm{Map}(I^2,\mathbb{C})$ by

(4.39)
$$\begin{aligned}
&V_2^{0-1,\hat{N}_\beta}(u)(\overline{\rho}_1\rho_1\mathbf{x}_1 s_1\xi_1, \overline{\rho}_2\rho_2\mathbf{x}_2 s_2\xi_2)\\
&:= -\frac{1}{2}uL^{-d}h1_{(\rho_1,\mathbf{x}_1,s_1)=(\rho_2,\mathbf{x}_2,s_2)}1_{\overline{\rho}_1=\overline{\rho}_2=1}(1_{(\xi_1,\xi_2)=(1,-1)} - 1_{(\xi_1,\xi_2)=(-1,1)}).
\end{aligned}$$

Then we define $V^{0-1,\hat{N}_\beta} \in C(\overline{D(r)}, \bigwedge_{even}\mathcal{V})$ by

$$V^{0-1,\hat{N}_\beta}(u)(\psi) := \left(\frac{1}{h}\right)^2 \sum_{\mathbf{X}\in I^2} V_2^{0-1,\hat{N}_\beta}(u)(\mathbf{X})\psi_{\mathbf{X}}.$$

Let us define $V_{2,2}^{0-2,\hat{N}_\beta} : \overline{D(r)} \to \mathrm{Map}(I^2\times I^2,\mathbb{C})$ by

$$\begin{aligned}
(4.40)\quad &V_{2,2}^{0-2,\hat{N}_\beta}(u)(\overline{\rho}_1\rho_1\mathbf{x}_1 s_1\xi_1, \overline{\rho}_2\rho_2\mathbf{x}_2 s_2\xi_2, \overline{\eta}_1\eta_1\mathbf{y}_1 t_1\zeta_1, \overline{\eta}_2\eta_2\mathbf{y}_2 t_2\zeta_2)\\
&:= -\frac{1}{4}uL^{-d}h^2 1_{(\rho_1,\mathbf{x}_1,s_1,\eta_1,\mathbf{y}_1,t_1)=(\rho_2,\mathbf{x}_2,s_2,\eta_2,\mathbf{y}_2,t_2)}(h1_{s_1=t_1} - \beta^{-1})\\
&\quad\cdot \sum_{\sigma,\tau\in\mathbb{S}_2} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) 1_{(\overline{\rho}_{\sigma(1)},\overline{\rho}_{\sigma(2)},\overline{\eta}_{\tau(1)},\overline{\eta}_{\tau(2)})=(1,2,2,1)}\\
&\qquad\cdot 1_{(\xi_{\sigma(1)},\xi_{\sigma(2)},\zeta_{\tau(1)},\zeta_{\tau(2)})=(1,-1,1,-1)}.
\end{aligned}$$

Then we define $V^{0-2,\hat{N}_\beta} \in C(\overline{D(r)}, \bigwedge_{even}\mathcal{V})$ by

$$V^{0-2,\hat{N}_\beta}(u)(\psi) := \left(\frac{1}{h}\right)^4 \sum_{\mathbf{X},\mathbf{Y}\in I^2} V_{2,2}^{0-2,\hat{N}_\beta}(u)(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.$$

Observe that

$$V^{0-1,\hat{N}_\beta}(u)(\psi) + V^{0-2,\hat{N}_\beta}(u)(\psi) = -V(u)(\psi) + W(u)(\psi).$$

We give $V^{0-1,\hat{N}_\beta} + V^{0-2,\hat{N}_\beta}$ as the initial data to the multi-scale integration. Using the notations introduced above, let us inductively define $V^{0-1,l}$,

$V^{0-2,l} \in C(\overline{D(r)}, \bigwedge_{even} \mathcal{V})$ $(l = N_\beta + 1, N_\beta + 2, \cdots, \hat{N}_\beta)$ as follows. Assume that we have $V^{0-1,l+1} \in \mathcal{Q}(r, l+1)$, $V^{0-2,l+1} \in \mathcal{R}(r, l+1)$ and

$$V^{0-2,l+1}(u)(\psi) = \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p \\ \mathbf{Y}\in I^q}} V_{p,q}^{0-2,l+1}(u)(\mathbf{X}, \mathbf{Y}) \psi_{\mathbf{X}} \psi_{\mathbf{Y}},$$
$$(\forall u \in \overline{D(r)})$$

with $V_{p,q}^{0-2,l+1} \in C(\overline{D(r)}, \mathrm{Map}(I^p \times I^q, \mathbb{C}))$ $(p, q \in \{2, 4, \cdots, N\})$ satisfying the conditions required in $\mathcal{R}(r, l+1)$. Then, let us set for any $n \in \mathbb{N}_{\geq 1}$, $u \in \overline{D(r)}$,

$$\begin{aligned}
&V^{0-1-1,l,(n)}(u)(\psi) \\
&:= \prod_{j=1}^{n} \left(\sum_{a_j \in \{1,2\}} \right) 1_{\exists j(a_j=1)} \frac{1}{n!} Tree(\{1, 2, \cdots, n\}, \mathcal{C}_{l+1}) \\
&\quad \cdot \prod_{j=1}^{n} V^{0-a_j,l+1}(u)(\psi^j + \psi) \Big|_{\substack{\psi^j=0 \\ (\forall j\in\{1,2,\cdots,n\})}}, \\
&V^{0-1-2,l,(n)}(u)(\psi) \\
&:= \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p \\ \mathbf{Y}\in I^q}} V_{p,q}^{0-2,l+1}(u)(\mathbf{X}, \mathbf{Y}) \\
&\quad \cdot \frac{1}{n!} Tree(\{1, 2, \cdots, n+1\}, \mathcal{C}_{l+1}) \\
&\quad \cdot (\psi^1 + \psi)_{\mathbf{X}} (\psi^2 + \psi)_{\mathbf{Y}} \prod_{j=3}^{n+1} V^{0-2,l+1}(u)(\psi^j + \psi) \Big|_{\substack{\psi^j=0 \\ (\forall j\in\{1,2,\cdots,n+1\})}}, \\
&V^{0-2,l,(n)}(u)(\psi) \\
&:= \frac{1}{n!} \sum_{m=0}^{n-1} \sum_{(\{s_j\}_{j=1}^{m+1}, \{t_k\}_{k=1}^{n-m}) \in S(n,m)} \sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} \\
&\quad \cdot \left(\frac{1}{h}\right)^{p+q} \sum_{\substack{\mathbf{X}\in I^p \\ \mathbf{Y}\in I^q}} V_{p,q}^{0-2,l+1}(u)(\mathbf{X}, \mathbf{Y}) \\
&\quad \cdot Tree(\{s_j\}_{j=1}^{m+1}, \mathcal{C}_{l+1})
\end{aligned}$$

$$\cdot (\psi^{s_1}+\psi)_{\mathbf{X}} \prod_{j=2}^{m+1} V^{0-2,l+1}(u)(\psi^{s_j}+\psi)\Big|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}$$
$$\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C}_{l+1})$$
$$\cdot (\psi^{t_1}+\psi)_{\mathbf{Y}} \prod_{k=2}^{n-m} V^{0-2,l+1}(u)(\psi^{t_k}+\psi)\Big|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}},$$

where

$$S(n,m):=\left\{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\ \middle|\ \begin{array}{l} 1=s_1<s_2<\cdots<s_{m+1}\le n,\\ 1=t_1<t_2<\cdots<t_{n-m}\le n,\\ \{s_j\}_{j=2}^{m+1}\cup\{t_k\}_{k=2}^{n-m}=\{2,3,\cdots,n\},\\ \{s_j\}_{j=2}^{m+1}\cap\{t_k\}_{k=2}^{n-m}=\emptyset. \end{array}\right\}.$$

Set

$$V^{0-1-j,l}(u)(\psi):=\sum_{n=1}^{\infty}V^{0-1-j,l,(n)}(u)(\psi),\quad (j=1,2),$$
$$V^{0-1,l}:=\sum_{j=1}^{2}V^{0-1-j,l},$$
$$V^{0-2,l}(u)(\psi):=\sum_{n=1}^{\infty}V^{0-2,l,(n)}(u)(\psi)$$

on the assumption that these series converge in $\bigwedge_{even}\mathcal{V}$.

Let us see how these Grassmann data are derived during the process. We give $V^{0-1,l+1}+V^{0-2,l+1}$ to the single-scale integration with the covariance $\mathcal{C}_{l+1}$. By applying the formula (4.1) we can derive that

$$(4.41)\quad \frac{1}{n!}\left(\frac{d}{dz}\right)^n \log\left(\int e^{z\sum_{j=1}^{2}V^{0-j,l+1}(u)(\psi^1+\psi)}d\mu_{\mathcal{C}_{l+1}}(\psi^1)\right)\Big|_{z=0}$$
$$=V^{0-1-1,l,(n)}(u)(\psi)$$
$$+\sum_{p,q=2}^{N}1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}V_{p,q}^{0-2,l+1}(u)(\mathbf{X},\mathbf{Y})$$
$$\cdot\frac{1}{n!}\prod_{j=2}^{n}\left(\frac{\partial}{\partial z_j}\right)$$

$$\cdot \int (\psi^1+\psi)_{\mathbf{X}}(\psi^1+\psi)_{\mathbf{Y}} e^{\sum_{j=2}^n z_j V^{0-2,l+1}(u)(\psi^1+\psi)} d\mu_{\mathcal{C}_{l+1}}(\psi^1)$$
$$\cdot \left(\int e^{\sum_{j=2}^n z_j V^{0-2,l+1}(u)(\psi^1+\psi)} d\mu_{\mathcal{C}_{l+1}}(\psi^1)\right)^{-1}\Bigg|_{\substack{z_j=0\\(\forall j\in\{2,3,\cdots,n\})}}$$
$$= V^{0-1-1,l,(n)}(u)(\psi)+V^{0-1-2,l,(n)}(u)(\psi)+V^{0-2,l,(n)}(u)(\psi).$$

We should remark that the above transformation is essentially same as [12, (3.56)] and is based on the ideas of the earlier papers [15, (3.38)], [14, (IV.15)]. Also, we should remind us that the logarithm and the inverse of the even Grassmann polynomials are analytic with z, $(z_j)_{j=2}^n$ in a neighborhood of the origin and thus the above transformation is mathematically justified.

Let us explain the rule of the superscripts put on these Grassmann data. We use the label $0-1$, $0-2$ as the 1st superscript of Grassmann data independent of the artificial parameters λ_1, λ_2. In the next subsection we will use the label $1-j$, 2 as the 1st superscript of Grassmann data depending on λ_1, λ_2 linearly, at least quadratically respectively. The 2nd superscript stands for the scale of integration. The Grassmann data with the 2nd superscript l is to be integrated with the covariance $\mathcal{C}_l$. For example, $V^{0-1,l}$ is independent of λ_1, λ_2 and to be integrated with the covariance $\mathcal{C}_l$. $V^{1-1,l+1}$ is linearly dependent on λ_1, λ_2 and to be integrated with $\mathcal{C}_{l+1}$ and so on.

We can describe properties of these scale-dependent Grassmann data as follows.

LEMMA 4.5. *There exists a positive constant c_4 independent of any parameter such that if*

(4.42)
$$M^{\min\{1,2\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j}\}}\ge c_4,\quad \alpha\ge c_4 M^{\frac{\mathsf{a}}{2}},\quad L^d\ge M^{(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)},$$

then

$$V^{0-1,l}\in\mathcal{Q}(b^{-1}c_0^{-2}\alpha^{-4},l),$$
$$V^{0-2,l}\in\mathcal{R}(b^{-1}c_0^{-2}\alpha^{-4},l),\quad (\forall l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta\}).$$

Proof. During the proof we often replace a generic positive constant denoted by 'c' by a larger generic constant still denoted by the same symbol without commenting on the replacement. It should be understood in the end that these replacement do not violate the validity of the proof of the claims.

We can see from (4.39), (4.40) that $V_2^{0-1,\hat{N}_\beta}(u)(\cdot)$ is anti-symmetric, satisfies (4.4), $V_{2,2}^{0-2,\hat{N}_\beta}(u)(\cdot,\cdot)$ is bi-anti-symmetric, satisfies (4.4), (4.10) and

$$\|V_2^{0-1,\hat{N}_\beta}(u)\|_{1,\infty} \le |u|L^{-d}, \tag{4.43}$$

$$\|V_{2,2}^{0-2,\hat{N}_\beta}(u)\|_{1,\infty} \le b|u|, \tag{4.44}$$

$$[V_{2,2}^{0-2,\hat{N}_\beta}(u), g]_{1,\infty} \le |u|L^{-d}\|g\| \tag{4.45}$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$. Thus it follows that $V^{0-1,\hat{N}_\beta} \in \mathcal{Q}(b^{-1}c_0^{-2}\alpha^{-4}, \hat{N}_\beta)$, $V^{0-2,\hat{N}_\beta} \in \mathcal{R}(b^{-1}c_0^{-2}\alpha^{-4}, \hat{N}_\beta)$.

Set $r := b^{-1}c_0^{-2}\alpha^{-4}$. Assume that $l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta - 1\}$ and we have $V^{0-1,j} \in \mathcal{Q}(r,j)$, $V^{0-2,j} \in \mathcal{R}(r,j)$ ($j = l+1, l+2, \cdots, \hat{N}_\beta$). Let us show that $V^{0-1-1,l}$, $V^{0-1-2,l}$, $V^{0-2,l}$ are well-defined and $V^{0-1-1,l} + V^{0-1-2,l} \in \mathcal{Q}(r,l)$, $V^{0-2,l} \in \mathcal{R}(r,l)$. The following inequalities can be derived from the definition of $\mathcal{Q}(r,l+1)$, $\mathcal{R}(r,l+1)$ and the assumptions $\alpha \ge 2^3$, $M^{\mathsf{a}} \ge 2^4$.

(4.46)
$$\sum_{m=2}^{N} 2^{3m}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{0-1,l+1}\|_{1,\infty,r} \le c\alpha^{-2}L^{-d},$$

(4.47)
$$\sum_{m=2}^{N} 2^{m}\alpha^m(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{0-1,l+1}\|_{1,\infty,r} \le cM^{-\mathsf{a}}L^{-d},$$

(4.48)
$$\sum_{m=4}^{N} 2^{3m}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{0-2,l+1}\|_{1,\infty,r} \le c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)},$$

(4.49)
$$\sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} 2^{p+q}\alpha^{p+q}(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p+q}{2}} \|V_{p,q}^{0-2,l+1}\|_{1,\infty,r}$$

$$\le cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)},$$

(4.50)

$$\sum_{m=4}^{N}2^m\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{0-2,l+1}\|_{1,\infty,r}\le cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)},$$

(4.51)

$$\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}2^{3p+3q}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p+q}{2}}[V_{p,q}^{0-2,l+1},g]_{1,\infty,r}$$
$$\le c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}\|g\|,$$

(4.52)

$$\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}2^{2p+2q}\alpha^{p+q}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p+q}{2}}[V_{p,q}^{0-2,l+1},g]_{1,\infty,r}$$
$$\le cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}\|g\|,$$

for any anti-symmetric function $g:I^2\to\mathbb{C}$. In the derivation of (4.48), (4.50) we used the inequality

$$\|V_m^{0-2,l+1}\|_{1,\infty,r}\le\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}1_{p+q=m}\|V_{p,q}^{0-2,l+1}\|_{1,\infty,r},\tag{4.53}$$

which is based on the uniqueness of anti-symmetric kernel. By using these inequalities we will prove the claimed bound properties of the Grassmann data at l-th scale in the following.

First of all, let us consider $V^{0-1-1,l,(n)}$. By (4.6) and (4.33), for any $m\in\{0,2,\cdots,N\}$

$$\|V_m^{0-1-1,l,(1)}\|_{1,\infty,r}$$
$$\le\sum_{p=m}^{N}\left(\frac{N}{h}\right)^{1_{m=0\wedge p\neq0}}\left(\begin{array}{c}p\\m\end{array}\right)(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p-m}{2}}\|V_p^{0-1,l+1}\|_{1,\infty,r}.$$

Then by (4.46) and (4.36) for $l+1$

$$\|V_0^{0-1-1,l,(1)}\|_{1,\infty,r}\le\|V_0^{0-1,l+1}\|_{1,\infty,r}+\frac{N}{h}c\alpha^{-2}L^{-d}\tag{4.54}$$

$$\le c\frac{N}{h}\alpha^{-2}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}.$$

Also by (4.36) for $l+1$ and the assumptions that $\alpha \ge 2$, $M^{\mathsf{a}} \ge 2^4$,

$$\begin{aligned}(4.55)\quad &\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{0-1-1,l,(1)}\|_{1,\infty,r}\\ &\le\sum_{m=2}^{N}\sum_{p=m}^{N}\alpha^m 2^p M^{-\frac{\mathsf{a}m}{2}}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p}{2}}\|V_p^{0-1,l+1}\|_{1,\infty,r}\\ &\le\sum_{m=2}^{N}2^mM^{-\frac{\mathsf{a}m}{2}}\sum_{p=m}^{N}\alpha^p(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p}{2}}\|V_p^{0-1,l+1}\|_{1,\infty,r}\\ &\le cL^{-d}M^{-\mathsf{a}}.\end{aligned}$$

Assume that $n \in \mathbb{N}_{\ge 2}$. Observe that

$$\begin{aligned}&V^{0-1-1,l,(n)}(u)(\psi)\\ &=\sum_{q=1}^{n}\begin{pmatrix} n\\ q\end{pmatrix}\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})\\ &\quad\cdot\prod_{j=1}^{q}V^{0-1,l+1}(u)(\psi^j+\psi)\prod_{k=q+1}^{n}V^{0-2,l+1}(u)(\psi^k+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.\end{aligned}$$

It follows from (4.8), (4.33), (4.34) that for any $m \in \{0,2,\cdots,N\}$

$$\begin{aligned}(4.56)\quad &\|V_m^{0-1-1,l,(n)}\|_{1,\infty,r}\\ &\le\left(\frac{N}{h}\right)^{1_{m=0}}2^n\frac{(n-2)!}{n!}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n+1-\frac{m}{2}}2^{-2m}\\ &\quad\cdot(c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-1}\\ &\quad\cdot\sum_{p_1=2}^{N}2^{3p_1}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1}{2}}\|V_{p_1}^{0-1,l+1}\|_{1,\infty,r}\\ &\quad\cdot\prod_{j=2}^{n}\left(\sum_{p_j=2}^{N}2^{3p_j}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}\sum_{\delta\in\{1,2\}}\|V_{p_j}^{0-\delta,l+1}\|_{1,\infty,r}\right)\\ &\quad\cdot 1_{\sum_{j=1}^{n}p_j-2(n-1)\ge m}\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{N}{h}\right)^{1_{m=0}} 2^{-2m+n} c_0^{-\frac{m}{2}} M^{-\frac{m}{2}\mathsf{a}(l+1-\hat{N}_\beta)-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot \sum_{p_1=2}^{N} 2^{3p_1}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1}{2}} \|V_{p_1}^{0-1,l+1}\|_{1,\infty,r}\\
&\quad\cdot \prod_{j=2}^{n}\left(\sum_{p_j=2}^{N} 2^{3p_j}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}} \sum_{\delta\in\{1,2\}} \|V_{p_j}^{0-\delta,l+1}\|_{1,\infty,r}\right)\\
&\quad\cdot 1_{\sum_{j=1}^{n} p_j-2(n-1)\ge m}.
\end{aligned}
$$

By (4.46), (4.48) and the assumption $L^d \ge M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}$,

$$
\begin{aligned}
&\|V_0^{0-1-1,l,(n)}\|_{1,\infty,r}\\
&\le \frac{N}{h} M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)} c\alpha^{-2}L^{-d}\\
&\quad\cdot \left(c\alpha^{-2}L^{-d} + c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\right)^{n-1}\\
&\le \frac{N}{h}L^{-d}(c\alpha^{-2})^n,
\end{aligned}
$$

or by assuming that $\alpha \ge c$,

$$
\sum_{n=2}^{\infty} \|V_0^{0-1-1,l,(n)}\|_{1,\infty,r} \le c\frac{N}{h}\alpha^{-4}L^{-d}. \tag{4.57}
$$

Also, by substituting (4.47), (4.50) and the inequality $L^d \ge M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}$ and using the condition $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2^3$ we can derive from (4.56) that

$$
\begin{aligned}
&\sum_{m=2}^{N} \alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{0-1-1,l,(n)}\|_{1,\infty,r}\\
&\le c^n \alpha^{-2(n-1)} M^{\mathsf{a}(n-1)-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot \sum_{p_1=2}^{N} 2^{p_1}\alpha^{p_1}(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p_1}{2}} \|V_{p_1}^{0-1,l+1}\|_{1,\infty,r}
\end{aligned}
$$

$$\cdot\left(\sum_{p=2}^{N}2^p\alpha^p(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p}{2}}\sum_{\delta\in\{1,2\}}\|V_p^{0-\delta,l+1}\|_{1,\infty,r}\right)^{n-1}$$
$$\le c^n\alpha^{-2(n-1)}M^{\mathsf{a}(n-1)-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}M^{-\mathsf{a}}L^{-d}$$
$$\cdot\left(cM^{-\mathsf{a}}L^{-d}+cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\right)^{n-1}$$
$$\le M^{-\mathsf{a}}L^{-d}(c\alpha^{-2})^{n-1},$$

or by assuming that $\alpha\ge c$,

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\sum_{n=2}^{\infty}\|V_m^{0-1-1,l,(n)}\|_{1,\infty,r}\le c\alpha^{-2}M^{-\mathsf{a}}L^{-d}. \tag{4.58}$$

Next let us study $V^{0-1-2,l,(n)}$. By (4.11) and (4.33)

$$\|V_m^{0-1-2,l,(1)}\|_{1,\infty,r}$$
$$\le\left(\frac{N}{h}\right)^{1_{m=0}}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-1-\frac{m}{2}}$$
$$\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in2\mathbb{N}}2^{2p_1+2p_2}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+p_2}{2}}$$
$$\cdot[V_{p_1,p_2}^{0-2,l+1},\tilde{\mathcal{C}}_{l+1}]_{1,\infty,r}1_{p_1+p_2-2\ge m}.$$

Then by (4.35) and (4.51)

$$\begin{aligned}\|V_0^{0-1-2,l,(1)}\|_{1,\infty,r}&\le\frac{N}{h}c_0^{-1}M^{-\mathsf{a}(l+1-\hat{N}_\beta)}c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}\\&\quad\cdot c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}\\&\le c\frac{N}{h}\alpha^{-4}L^{-d}.\end{aligned}\tag{4.59}$$

Also by (4.35), (4.52) and the condition $\alpha M^{-\frac{\mathsf{a}}{2}}\ge 2$,

(4.60)

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{0-1-2,l,(1)}\|_{1,\infty,r}$$

$$
\begin{aligned}
&\le c\alpha^{-2}M^{\mathsf{a}}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-1}\\
&\quad\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{2p_1+2p_2}\alpha^{p_1+p_2}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p_1+p_2}{2}}[V^{0-2,l+1}_{p_1,p_2},\tilde{\mathcal{C}}_{l+1}]_{1,\infty,r}\\
&\le c\alpha^{-2}M^{\mathsf{a}}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-1}M^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}\\
&\quad\cdot c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}\\
&\le c\alpha^{-2}M^{-\mathsf{a}}L^{-d}.
\end{aligned}
$$

Let $n\in\mathbb{N}_{\ge 2}$. By substituting (4.33), (4.34) into (4.13) we have that

(4.61)

$$
\begin{aligned}
&\|V_m^{0-1-2,l,(n)}\|_{1,\infty,r}\\
&\le\left(\frac{N}{h}\right)^{1_{m=0}}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n-\frac{m}{2}}2^{-2m}(c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-1}\\
&\quad\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{3p_1+3p_2}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+p_2}{2}}[V^{0-2,l+1}_{p_1,p_2},\tilde{\mathcal{C}}_{l+1}]_{1,\infty,r}\\
&\quad\cdot\prod_{j=3}^{n+1}\left(\sum_{p_j=4}^{N}2^{3p_j}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}\|V^{0-2,l+1}_{p_j}\|_{1,\infty,r}\right)1_{\sum_{j=1}^{n+1}p_j-2n\ge m}.
\end{aligned}
$$

Then by (4.48), (4.51) and (4.35),

$$
\begin{aligned}
&\|V_0^{0-1-2,l,(n)}\|_{1,\infty,r}\\
&\le\frac{N}{h}c_0^{-1}M^{-\mathsf{a}(l+1-\hat{N}_\beta)n+(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}\\
&\quad\cdot c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}(c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)})^{n-1}\\
&\le\frac{N}{h}L^{-d}(c\alpha^{-4})^n,
\end{aligned}
$$

or by assuming that $\alpha\ge c$

$$
\sum_{n=2}^{\infty}\|V_0^{0-1-2,l,(n)}\|_{1,\infty,r}\le c\frac{N}{h}\alpha^{-8}L^{-d}. \tag{4.62}
$$

On the other hand, by using (4.50), (4.52), (4.35) and the condition $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2^3$ we can deduce from (4.61) that

$$\begin{aligned}
&\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{0-1-2,l,(n)}\|_{1,\infty,r}\\
&\le c^n\alpha^{-2n}c_0^{-1}M^{\mathsf{a}n-\mathsf{a}(l+1-\hat{N}_\beta)n+(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{p_1+p_2}\alpha^{p_1+p_2}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p_1+p_2}{2}}[V_{p_1,p_2}^{0-2,l+1},\tilde{\mathcal{C}}_{l+1}]_{1,\infty,r}\\
&\quad\cdot\left(\sum_{p=4}^{N}2^p\alpha^p(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p}{2}}\|V_p^{0-2,l+1}\|_{1,\infty,r}\right)^{n-1}\\
&\le c^n\alpha^{-2n}c_0^{-1}M^{\mathsf{a}n-\mathsf{a}(l+1-\hat{N}_\beta)n+(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot M^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\\
&\quad\cdot L^{-d}c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}(cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)})^{n-1}\\
&\le L^{-d}(c\alpha^{-2}M^{-\mathsf{a}})^n.
\end{aligned}$$

Thus on the assumption $\alpha \ge c$,

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\sum_{n=2}^{\infty}\|V_m^{0-1-2,l,(n)}\|_{1,\infty,r} \le c\alpha^{-4}M^{-2\mathsf{a}}L^{-d}. \tag{4.63}$$

By combining (4.54), (4.55), (4.57), (4.58), (4.59), (4.60), (4.62), (4.63) we have that

$$\begin{aligned}
&\frac{h}{N}\sum_{n=1}^{\infty}(\|V_0^{0-1-1,l,(n)}\|_{1,\infty,r}+\|V_0^{0-1-2,l,(n)}\|_{1,\infty,r})\\
&\le c(\alpha^{-2}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}+\alpha^{-4})L^{-d}\\
&\le c(M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)}+\alpha^{-2})\alpha^{-2}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)}L^{-d},\\
&\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\sum_{n=1}^{\infty}(\|V_m^{0-1-1,l,(n)}\|_{1,\infty,r}+\|V_m^{0-1-2,l,(n)}\|_{1,\infty,r})\\
&\le cM^{-\mathsf{a}}L^{-d}.
\end{aligned}$$

The above inequalities imply that if $M \ge c$ and $\alpha \ge c$,

$$V^{0-1-1,l} + V^{0-1-2,l} \in \mathcal{Q}(r,l).$$

In fact Lemma 4.1, Lemma 4.2 ensure that the anti-symmetric kernels of $V^{0-1-1,l,(n)} + V^{0-1-2,l,(n)}$ satisfy (4.4) and thus so do the anti-symmetric kernels of $V^{0-1-1,l} + V^{0-1-2,l}$. The above uniform convergent property implies the claimed regularity of $V^{0-1-1,l} + V^{0-1-2,l}$ with $u \in \overline{D(r)}$. Recall that we have also assumed $L^d \ge M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta - N_\beta)}$, $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2^3$ to reach this conclusion.

Next let us deal with $V^{0-2,l,(n)}$. The analysis is based on Lemma 4.4. The lemma ensures the existence of bi-anti-symmetric kernels satisfying (4.4), (4.10). We can see from (4.15) and the induction hypothesis that $V_{a,b}^{0-2,l,(n)} \in C(\overline{D(r)}, \text{Map}(I^a \times I^b, \mathbb{C})) \cap C^\omega(D(r), \text{Map}(I^a \times I^b, \mathbb{C}))$ ($n \in \mathbb{N}$, $a, b \in \{2, 4, \cdots, N\}$), which implies that $V^{0-2,l,(n)} \in C(\overline{D(r)}, \bigwedge_{even} \mathcal{V}) \cap C^\omega(D(r), \bigwedge_{even} \mathcal{V})$. We need to establish bound properties of $V^{0-2,l,(n)}$. It follows from (4.16) and (4.33) that for $a, b \in \{2, 4, \cdots, N\}$

$$\begin{aligned}
&\|V_{a,b}^{0-2,l,(1)}\|_{1,\infty,r} \\
&\le \sum_{p=a}^{N}\sum_{q=b}^{N} \begin{pmatrix} p \\ a \end{pmatrix} \begin{pmatrix} q \\ b \end{pmatrix} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{1}{2}(p+q-a-b)} \|V_{p,q}^{0-2,l+1}\|_{1,\infty,r}.
\end{aligned}$$

By (4.37) for $l+1$ and the assumption $\alpha \ge 2$, $M^{\mathsf{a}} \ge 2^6$,

$$\begin{aligned}
&\|V_{2,2}^{0-2,l,(1)}\|_{1,\infty,r} \\
&\le \|V_{2,2}^{0-2,l+1}\|_{1,\infty,r} \\
&\quad + \sum_{p=2}^{N}\sum_{q=2}^{N} 1_{p+q\ge 6} 2^{p+q} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{1}{2}(p+q-4)} \|V_{p,q}^{0-2,l+1}\|_{1,\infty,r} \\
&\le M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)} \alpha^{-4} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-2} \\
&\quad + c\alpha^{-6} M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-2} \\
&\le cM^{\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)} \alpha^{-4} (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{-2},
\end{aligned} \tag{4.64}$$

$$\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} 1_{a+b\ge 6} \alpha^{a+b} (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}} \|V_{a,b}^{0-2,l,(1)}\|_{1,\infty,r} \tag{4.65}$$

$$
\begin{aligned}
&\le \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} 1_{a+b\ge 6}\alpha^{a+b} M^{-\frac{\mathsf{a}}{2}(a+b)}\\
&\quad\cdot \sum_{p=a}^{N}\sum_{q=b}^{N} 2^{p+q}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p+q}{2}} \|V_{p,q}^{0-2,l+1}\|_{1,\infty,r}\\
&\le M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)} \sum_{a,b=2}^{N} 1_{a+b\ge 6} 2^{a+b} M^{-\frac{\mathsf{a}}{2}(a+b)}\\
&\le M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)} \sum_{m=6}^{\infty} 2^{2m} M^{-\frac{\mathsf{a}m}{2}}\\
&\le c M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)-3\mathsf{a}}.
\end{aligned}
$$

By applying (4.18), (4.38) in place of (4.16), (4.37) respectively and repeating a parallel argument to the above argument we can derive on the assumption $\alpha \ge 2$, $M^{\mathsf{a}} \ge 2^6$ that

$$
\begin{aligned}
(4.66)\quad & [V_{2,2}^{0-2,l,(1)}, g]_{1,\infty,r}\\
&\le c M^{\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)} \alpha^{-4}(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{-2} L^{-d}\|g\|,\\
(4.67)\quad & \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} 1_{a+b\ge 6}\alpha^{a+b}(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}} [V_{a,b}^{0-2,l,(1)}, g]_{1,\infty,r}\\
&\le c M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)-3\mathsf{a}} L^{-d}\|g\|,
\end{aligned}
$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$.

Let us take $n \in \mathbb{N}_{\ge 2}$. Observe that for any $m \in \{0, 1, \cdots, n-1\}$

$$
(4.68)\qquad \sharp S(n,m) = \begin{pmatrix} n-1 \\ m \end{pmatrix}.
$$

Combination of (4.20), (4.33), (4.34), (4.68) yields that for $a, b \in \{2, 4, \cdots, N\}$

$$
\begin{aligned}
&\|V_{a,b}^{0-2,l,(n)}\|_{1,\infty,r}\\
&\le \frac{1}{n!}\sum_{m=0}^{n-1} \begin{pmatrix} n-1 \\ m \end{pmatrix}
\end{aligned}
$$

$$\cdot (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})$$
$$\cdot 2^{-2a-2b}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n+1-\frac{1}{2}(a+b)}(c_0M^{(\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-1}$$
$$\cdot \sum_{p_1,q_1=2}^{N}1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+q_1}{2}}\|V_{p_1,q_1}^{0-2,l+1}\|_{1,\infty,r}$$
$$\cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=4}^{N}2^{3p_j}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}\|V_{p_j}^{0-2,l+1}\|_{1,\infty,r}\right)$$
$$\cdot \prod_{k=2}^{n-m}\left(\sum_{q_k=4}^{N}2^{3q_k}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{q_k}{2}}\|V_{q_k}^{0-2,l+1}\|_{1,\infty,r}\right)$$
$$\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\geq a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\geq b}.$$

Then by using (4.49), (4.50) and the assumption $\alpha M^{-\frac{\mathsf{a}}{2}}\geq 2^3$,

$$\begin{aligned}(4.69)\quad &\sum_{a,b=2}^{N}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\|V_{a,b}^{0-2,l,(n)}\|_{1,\infty,r}\\
&\leq c^n\alpha^{-2(n-1)}M^{\mathsf{a}(n-1)-\mathsf{a}(l+1-\hat{N}_\beta)(n-1)+(\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot\sum_{p_1,q_1=2}^{N}1_{p_1,q_1\in 2\mathbb{N}}2^{p_1+q_1}\alpha^{p_1+q_1}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p_1+q_1}{2}}\|V_{p_1,q_1}^{0-2,l+1}\|_{1,\infty,r}\\
&\quad\cdot\left(\sum_{p=4}^{N}2^p\alpha^p(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p}{2}}\|V_p^{0-2,l+1}\|_{1,\infty,r}\right)^{n-1}\\
&\leq c^n\alpha^{-2(n-1)}M^{-\mathsf{a}(l-\hat{N}_\beta)(n-1)+(\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot M^{-2\mathsf{a}n+(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)n}\\
&\leq M^{-2\mathsf{a}+(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}(c\alpha^{-2}M^{-\mathsf{a}})^{n-1}.\end{aligned}$$

Then on the assumption $\alpha\geq c$,

$$(4.70)\qquad \sum_{a,b=2}^{N}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\sum_{n=2}^{\infty}\|V_{a,b}^{0-2,l,(n)}\|_{1,\infty,r}$$

$$\leq c\alpha^{-2}M^{-3\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}.$$

Take any anti-symmetric function $g: I^2 \to \mathbb{C}$. We can derive from (4.22), (4.33), (4.34) that for $a, b \in \{2, 4, \cdots, N\}$

$$\begin{aligned}
&[V_{a,b}^{0-2,l,(n)}, g]_{1,\infty,r} \\
&\leq \frac{1}{n!}\sum_{m=0}^{n-1}\begin{pmatrix} n-1 \\ m \end{pmatrix} \\
&\quad \cdot (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1}) \\
&\quad \cdot 2^{-2a-2b}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n+1-\frac{1}{2}(a+b)}(c_0 M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-2} \\
&\quad \cdot \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} 2^{3p_1+3q_1}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+q_1}{2}} \\
&\quad \cdot \bigg([V_{p_1,q_1}^{0-2,l+1}, g]_{1,\infty,r} c_0 M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)} \\
&\qquad + [V_{p_1,q_1}^{0-2,l+1}, \tilde{\mathcal{C}}]_{1,\infty,r}\|g\|_{1,\infty}\bigg) \\
&\quad \cdot \prod_{j=2}^{m+1}\left(\sum_{p_j=4}^{N} 2^{3p_j}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}\|V_{p_j}^{0-2,l+1}\|_{1,\infty,r}\right) \\
&\quad \cdot \prod_{k=2}^{n-m}\left(\sum_{q_k=4}^{N} 2^{3q_k}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{q_k}{2}}\|V_{q_k}^{0-2,l+1}\|_{1,\infty,r}\right) \\
&\quad \cdot 1_{\sum_{j=1}^{m+1} p_j - 2m \geq a} 1_{\sum_{k=1}^{n-m} q_k - 2(n-m-1)\geq b}.
\end{aligned}$$

Then by using (4.35), (4.50), (4.52), (4.68), the inequality $\|g\|_{1,\infty} \leq \|g\|$ and the assumptions $\alpha M^{-\frac{\mathsf{a}}{2}} \geq 2^3$, $\alpha \geq c$ and repeating a parallel procedure to (4.69) we obtain that

$$\begin{aligned}
&\sum_{a,b=2}^{N}\alpha^{a+b}(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\sum_{n=2}^{\infty}[V_{a,b}^{0-2,l,(n)}, g]_{1,\infty,r} \\
&\leq c\alpha^{-2}M^{-3\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}\|g\|.
\end{aligned} \tag{4.71}$$

Here we can sum up (4.64), (4.65), (4.70) to deduce that

$$
\begin{aligned}
&M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)}\sum_{a,b=2}^{N}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\sum_{n=1}^{\infty}\|V_{a,b}^{0-2,l,(n)}\|_{1,\infty,r}\\
&\le c\left(M^{\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1-2\mathsf{a}}+\alpha^{-2}M^{\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1-3\mathsf{a}}\right).
\end{aligned}
$$

On the assumption $M^{2\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}}\ge c$ the right-hand side of the above inequality is less than 1. Because of the assumption (1.13), the condition $M^{2\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}}\ge c$ can be realized by taking M large. Similarly, it follows from (4.66), (4.67), (4.71) and the condition $M^{2\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}}\ge c$ that

$$
\begin{aligned}
&M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l-\hat{N}_\beta)}\sum_{a,b=2}^{N}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\sum_{n=1}^{\infty}[V_{a,b}^{0-2,l,(n)},g]_{1,\infty,r}\\
&\le c\left(M^{\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1-2\mathsf{a}}+\alpha^{-2}M^{\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1-3\mathsf{a}}\right)L^{-d}\|g\|\\
&\le L^{-d}\|g\|,
\end{aligned}
$$

for any anti-symmetric function $g: I^2\to\mathbb{C}$. Thus we conclude that on the assumption $M^{2\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}}\ge c$ that

$$
V^{0-2,l}\in\mathcal{R}(r,l).
$$

We needed to assume in total that

$$
M\ge c,\quad M^{2\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}}\ge c,\quad \alpha\ge cM^{\frac{\mathsf{a}}{2}},\quad L^d\ge M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}
$$

for a positive constant c independent of any parameter, in order to conclude the l-th step. The above assumptions can be summarized as in (4.42).

The induction with l proves that the claim holds true. □

REMARK 4.6. In the proof of the claim $V^{0-2,l}\in\mathcal{R}(r,l)$ we crucially used the condition (1.13).

4.5. Multi-scale integration with the artificial term

In this subsection we construct a multi-scale integration for

$$\log\left(\int e^{-V(u)(\psi+\psi^1)+W(u)(\psi+\psi^1)-A(\psi+\psi^1)}d\mu_{\sum_{l=N_\beta+1}^{\hat{N}_\beta}\mathcal{C}_l}(\psi^1)\right),$$

where $A(\psi)$ is the Grassmann polynomial defined in (3.1). Since the artificial term $A(\psi)$ is parameterized by $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)\in\mathbb{C}^2$, the Grassmann data in this process is parameterized by $(u,\boldsymbol{\lambda})$. We will classify them in terms of the degree with $\boldsymbol{\lambda}$. It is structurally natural to measure kernels of these Grassmann data by using a variant of the norm $\|\cdot\|_1$ defined as follows. For $f\in C(\overline{D(r)}\times\overline{D(r')}^2,\mathrm{Map}(I^m,\mathbb{C}))$ we set

$$\|f\|_{1,r,r'}:=\sup_{(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\overline{D(r')}^2}\|f(u,\boldsymbol{\lambda})\|_1.$$

Then $C(\overline{D(r)}\times\overline{D(r')}^2,\mathrm{Map}(I^m,\mathbb{C}))$ is a Banach space with the norm $\|\cdot\|_{1,r,r'}$. Also, to shorten subsequent formulas, we set

$$\|f_0\|_{1,r,r'}:=\sup_{(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\overline{D(r')}^2}|f_0(u,\boldsymbol{\lambda})|$$

for $f_0\in C(\overline{D(r)}\times\overline{D(r')}^2,\mathbb{C})$. Moreover, we introduce a variant of the measurement $[\cdot,\cdot]_1$ as follows. For $f\in C(\overline{D(r)}\times\overline{D(r')}^2,\mathrm{Map}(I^m\times I^n,\mathbb{C}))$ and an anti-symmetric function $g:I^2\to\mathbb{C}$,

$$[f,g]_{1,r,r'}:=\sup_{(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\overline{D(r')}^2}[f(u,\boldsymbol{\lambda}),g]_1.$$

To describe scale-dependent properties of Grassmann data during the multi-scale integration process, we introduce sets of $\bigwedge\mathcal{V}$-valued functions. Let $l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta\}$ and $r,r'\in\mathbb{R}_{>0}$. We define the subset $\mathcal{Q}'(r,r',l)$ of $\mathrm{Map}(\overline{D(r)}\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V})$ as follows. f belongs to $\mathcal{Q}'(r,r',l)$ if and only if

-
$$f\in C\left(\overline{D(r)}\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right)\cap C^\omega\left(D(r)\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right).$$

- For any $u\in\overline{D(r)}$, $\boldsymbol{\lambda}\mapsto f(u,\boldsymbol{\lambda})(\psi):\mathbb{C}^2\to\bigwedge_{even}\mathcal{V}$ is linear.
- For any $(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\mathbb{C}^2$ the anti-symmetric kernels $f(u,\boldsymbol{\lambda})_m: I^m\to\mathbb{C}$ $(m=2,4,\cdots,N)$ satisfy (4.4) and

$$\alpha^2\|f_0\|_{1,r,r'}\le L^{-d},\tag{4.72}$$
$$\sum_{m=2}^{N}c_0^{\frac{m}{2}}\alpha^m M^{\frac{m}{2}\mathsf{a}(l-\hat{N}_\beta)}\|f_m\|_{1,r,r'}\le L^{-d}.$$

We use the set $\mathcal{Q}'(r,r',l)$ to collect Grassmann data linearly dependent on $\boldsymbol{\lambda}$ and bounded by L^{-d}.

The set $\mathcal{R}'(r,r',l)$ is defined as follows. f belongs to $\mathcal{R}'(r,r',l)$ if and only if

- $$f\in C\left(\overline{D(r)}\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right)\cap C^{\omega}\left(D(r)\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right).$$
- For any $u\in\overline{D(r)}$, $\boldsymbol{\lambda}\mapsto f(u,\boldsymbol{\lambda})(\psi)$: $\mathbb{C}^2\to\bigwedge_{even}\mathcal{V}$ is linear.
- There exist $f_{p,q}\in C(\overline{D(r)}\times\mathbb{C}^2,\mathrm{Map}(I^p\times I^q,\mathbb{C}))$ $(p,q\in\{2,4,\cdots,N\})$ such that for any $(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\mathbb{C}^2$, $p,q\in\{2,4,\cdots,N\}$, $f_{p,q}(u,\boldsymbol{\lambda}): I^p\times I^q\to\mathbb{C}$ is bi-anti-symmetric, satisfies (4.4), (4.10),

$$f(u,\boldsymbol{\lambda})(\psi)=\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}f_{p,q}(u,\boldsymbol{\lambda})(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}$$

and

$$\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}c_0^{\frac{p+q}{2}}\alpha^{p+q}M^{\frac{p+q}{2}\mathsf{a}(l-\hat{N}_\beta)}\|f_{p,q}\|_{1,r,r'}\le 1,\tag{4.73}$$
$$\sum_{p,q=2}^{N}1_{p,q\in2\mathbb{N}}c_0^{\frac{p+q}{2}}\alpha^{p+q}M^{\frac{p+q}{2}\mathsf{a}(l-\hat{N}_\beta)}[f_{p,q},g]_{1,r,r'}\le L^{-d}\|g\|,\tag{4.74}$$

for any anti-symmetric function $g:I^2\to\mathbb{C}$.

The role of the set $\mathcal{R}'(r, r', l)$ is to collect Grassmann data linearly depending on $\boldsymbol{\lambda}$, having bi-anti-symmetric kernels satisfying the property (4.10).

It is also necessary to define a set which can contain descendants of the artificial term $-A(\psi)$. f belongs to $\mathcal{S}(r, r', l)$ if and only if

- $$f \in C\left(\overline{D(r)} \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V}\right) \cap C^\omega\left(D(r) \times \mathbb{C}^2, \bigwedge_{even} \mathcal{V}\right).$$

- For any $u \in \overline{D(r)}$, $\boldsymbol{\lambda} \mapsto f(u, \boldsymbol{\lambda})(\psi) : \mathbb{C}^2 \to \bigwedge_{even} \mathcal{V}$ is linear.

- For any $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \mathbb{C}^2$ the anti-symmetric kernels $f(u, \boldsymbol{\lambda})_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (4.4) and

$$\begin{aligned} &\alpha^2 \|f_0\|_{1,r,r'} \le 1, \\ &\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|f_m\|_{1,r,r'} \le 1. \end{aligned} \tag{4.75}$$

In fact the descendants of $-A(\psi)$ are independent of u. Thus the condition concerning the variable u assumed in $\mathcal{S}(r, r', l)$ is not necessary. However, by defining the set as above we can avoid introducing another norm.

Finally we define a set of Grassmann data depending on $\boldsymbol{\lambda}$ at least quadratically. f belongs to $\mathcal{W}(r, r', l)$ if and only if

- $$f \in C\left(\overline{D(r)} \times \overline{D(r')}^2, \bigwedge_{even} \mathcal{V}\right) \cap C^\omega\left(D(r) \times D(r')^2, \bigwedge_{even} \mathcal{V}\right).$$

- For any $u \in D(r)$, $j \in \{1, 2\}$,

$$f(u, \mathbf{0})(\psi) = \frac{\partial}{\partial \lambda_j} f(u, \mathbf{0})(\psi) = 0.$$

- For any $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r')}^2$ the anti-symmetric kernels $f(u, \boldsymbol{\lambda})_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, N)$ satisfy (4.4) and

$$\alpha^2\|f_0\|_{1,r,r'} \le 1,\qquad \sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m M^{\frac{m}{2}\mathsf{a}(l-\hat{N}_\beta)}\|f_m\|_{1,r,r'} \le 1. \tag{4.76}$$

In the following we inductively define a family of Grassmann polynomials, which are the scale-dependent input and output of the multi-scale integration process from $\hat{N}_\beta$ to $N_\beta+1$. We admit the results of Lemma 4.5 stating that $V^{0-1,l} \in \mathcal{Q}(b^{-1}c_0^{-2}\alpha^{-4}, l)$, $V^{0-2,l} \in \mathcal{R}(b^{-1}c_0^{-2}\alpha^{-4}, l)$ ($\forall l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta\}$) and define

$$V^{0,l} \in C\left(\overline{D(b^{-1}c_0^{-2}\alpha^{-4})}, \bigwedge_{even}\mathcal{V}\right) \quad (l = N_\beta, N_\beta+1, \cdots, \hat{N}_\beta)$$

by $V^{0,l} := V^{0-1,l} + V^{0-2,l}$. Recalling the definition (3.1), we define $V^{1-3,\hat{N}_\beta} \in C(\mathbb{C}^2, \bigwedge_{even}\mathcal{V})$ by

$$V^{1-3,\hat{N}_\beta}(\boldsymbol{\lambda})(\psi) := -A(\psi).$$

Moreover, set

$$V^{1-1,\hat{N}_\beta} = V^{1-2,\hat{N}_\beta} := 0, \quad V^{1,\hat{N}_\beta} := \sum_{j=1}^{3} V^{1-j,\hat{N}_\beta}, \quad V^{2,\hat{N}_\beta} := 0.$$

Let us assume that $l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta - 1\}$ and we have

$$\begin{aligned} &V^{1-1,l+1} \in \mathcal{Q}'(r, r', l+1), \quad V^{1-2,l+1} \in \mathcal{R}'(r, r', l+1),\\ &V^{1-3,l+1} \in \mathcal{S}(r, r', l+1), \quad V^{2,l+1} \in \mathcal{W}(r, r', l+1). \end{aligned}$$

Set $V^{1,l+1} := \sum_{j=1}^{3} V^{1-j,l+1}$. By recalling the formula (4.1) we can observe that

(4.77)

$$\frac{1}{n!}\left(\frac{d}{dz}\right)^n \log\left(\int e^{z\sum_{j=0}^{2} V^{j,l+1}(\psi^1+\psi)} d\mu_{\mathcal{C}_{l+1}}(\psi^1)\right)\Bigg|_{z=0}$$
$$= \frac{1}{n!}Tree(\{1,2,\cdots,n\}, \mathcal{C}_{l+1})\prod_{j=1}^{n} V^{0,l+1}(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

$$+ 1_{n=1} Tree(\{1\},\mathcal{C}_{l+1}) V^{1,l+1}(\psi^1+\psi)\Big|_{\psi^1=0}$$

$$+ 1_{n\ge 2}\frac{1}{(n-1)!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})$$

$$\cdot V^{1,l+1}(\psi^1+\psi)\prod_{j=2}^{n}\left(\sum_{a_j\in\{1,2\}}V^{0-a_j,l+1}(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

$$\cdot 1_{\exists j(a_j=1)}$$

$$+ 1_{n\ge 2}\frac{1}{(n-1)!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})$$

$$\cdot V^{1,l+1}(\psi^1+\psi)\prod_{j=2}^{n}V^{0-2,l+1}(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

$$+\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})\prod_{j=1}^{n}\left(\sum_{b_j=0}^{2}V^{b_j,l+1}(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

$$\cdot 1_{\sum_{j=1}^{n}b_j\ge 2}.$$

Let us decompose or rename each term of the right-hand side of (4.77) from top to bottom. In Subsection 4.4 we proved that if we set

$$V^{0,l,(n)}(\psi):=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})\prod_{j=1}^{n}V^{0,l+1}(\psi^j+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$

for $n\in\mathbb{N}_{\ge 1}$, then $V^{0,l}(\psi)=\sum_{n=1}^{\infty}V^{0,l,(n)}(\psi)$. Let us set

$$V^{1-1-1,l}(\psi):=Tree(\{1\},\mathcal{C}_{l+1})V^{1-1,l+1}(\psi^1+\psi)\Big|_{\psi^1=0},$$

$$V^{1-1-2,l}(\psi):=\sum_{p,q=2}^{N}1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}V_{p,q}^{1-2,l+1}(\mathbf{X},\mathbf{Y})$$

$$\cdot Tree(\{1,2\},\mathcal{C}_{l+1})(\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\Bigg|_{\psi^1=\psi^2=0},$$

$$V^{1-2-1,l}(\psi)$$

$$:=\sum_{p,q=2}^{N}1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}V_{p,q}^{1-2,l+1}(\mathbf{X},\mathbf{Y})$$

$$\cdot Tree(\{1\},\mathcal{C}_{l+1})(\psi^1+\psi)_{\mathbf{X}}\Big|_{\psi^1=0} Tree(\{1\},\mathcal{C}_{l+1})(\psi^1+\psi)_{\mathbf{Y}}\Big|_{\psi^1=0},$$

(4.78)

$$V^{1-3,l}(\psi) := Tree(\{1\},\mathcal{C}_{l+1})V^{1-3,l+1}(\psi^1+\psi)\Big|_{\psi^1=0}.$$

Then, for the same reason that the transformation (4.41) is valid, the following equality holds.

$$\frac{d}{dz}\log\left(\int e^{zV^{1-2,l+1}(\psi^1+\psi)}d\mu_{\mathcal{C}_{l+1}}(\psi^1)\right)\Bigg|_{z=0} = V^{1-1-2,l}(\psi)+V^{1-2-1,l}(\psi).$$

Thus it follows that

$$\begin{aligned}&Tree(\{1\},\mathcal{C}_{l+1})V^{1,l+1}(\psi^1+\psi)_{\mathbf{X}}\Big|_{\psi^1=0}\\&= V^{1-1-1,l}(\psi)+V^{1-1-2,l}(\psi)+V^{1-2-1,l}(\psi)+V^{1-3,l}(\psi).\end{aligned}$$

Moreover, set for $n\in\mathbb{N}_{\ge 2}$,

(4.79)

$$\begin{aligned}&V^{1-1-3,l,(n)}(\psi)\\&:= \frac{1}{(n-1)!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})\\&\quad\cdot V^{1,l+1}(\psi^1+\psi)\prod_{j=2}^{n}\left(\sum_{a_j\in\{1,2\}}V^{0-a_j,l+1}(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}} 1_{\exists j(a_j=1)},\end{aligned}$$

(4.80)

$$\begin{aligned}&V^{1-1-4,l,(n)}(\psi)\\&:= \sum_{p,q=2}^{N}1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}V^{0-2,l+1}_{p,q}(\mathbf{X},\mathbf{Y})\\&\quad\cdot\frac{1}{(n-1)!}Tree(\{1,2,\cdots,n+1\},\mathcal{C}_{l+1})(\psi^1+\psi)_{\mathbf{X}}(\psi^2+\psi)_{\mathbf{Y}}\\&\quad\cdot\prod_{j=3}^{n}V^{0-2,l+1}(\psi^j+\psi)\cdot V^{1,l+1}(\psi^{n+1}+\psi)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n+1\})}},\end{aligned}$$

(4.81)

$$
\begin{aligned}
&V^{1-2-2,l,(n)}(\psi)\\
&:=\frac{1}{(n-1)!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\\
&\quad\cdot\sum_{p,q=2}^{N}1_{p,q\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{p+q}\sum_{\substack{\mathbf{X}\in I^p\\ \mathbf{Y}\in I^q}}V_{p,q}^{0-2,l+1}(\mathbf{X},\mathbf{Y})\\
&\quad\cdot Tree(\{s_j\}_{j=1}^{m+1},\mathcal{C}_{l+1})(\psi^{s_1}+\psi)_{\mathbf{X}}\\
&\quad\cdot\prod_{j=2}^{m+1}(1_{s_j\neq n}V^{0-2,l+1}(\psi^{s_j}+\psi)+1_{s_j=n}V^{1,l+1}(\psi^{s_j}+\psi))\Big|_{\substack{\psi^{s_j}=0\\(\forall j\in\{1,2,\cdots,m+1\})}}\\
&\quad\cdot Tree(\{t_k\}_{k=1}^{n-m},\mathcal{C}_{l+1})(\psi^{t_1}+\psi)_{\mathbf{Y}}\\
&\quad\cdot\prod_{k=2}^{n-m}(1_{t_k\neq n}V^{0-2,l+1}(\psi^{t_k}+\psi)+1_{t_k=n}V^{1,l+1}(\psi^{t_k}+\psi))\Big|_{\substack{\psi^{t_k}=0\\(\forall k\in\{1,2,\cdots,n-m\})}}.
\end{aligned}
$$

It follows from the same argument as in (4.41) that

$$
\begin{aligned}
&\frac{1}{(n-1)!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})\\
&\cdot V^{1,l+1}(\psi^1+\psi)\prod_{j=2}^{n}V^{0-2,l+1}(\psi^j+\psi)\Big|_{\substack{\psi^{j}=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=\frac{1}{(n-1)!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})\\
&\quad\cdot\prod_{j=1}^{n-1}V^{0-2,l+1}(\psi^j+\psi)\cdot V^{1,l+1}(\psi^n+\psi)\Big|_{\substack{\psi^{j}=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=V^{1-1-4,l,(n)}(\psi)+V^{1-2-2,l,(n)}(\psi).
\end{aligned}
$$

Finally we set for $n\in\mathbb{N}_{\geq 1}$,

(4.82)

$$V^{2,l,(n)}(\psi)$$

$$:= \frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1})\prod_{j=1}^{n}\left(\sum_{b_j=0}^{2}V^{b_j,l+1}(\psi^j+\psi)\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}$$
$$\cdot 1_{\sum_{j=1}^{n}b_j\ge 2}.$$

By giving back these Grassmann polynomials to the expansion (4.77) we see that the following equality holds.

$$\frac{1}{n!}\left(\frac{d}{dz}\right)^n\log\left(\int e^{z\sum_{j=0}^{2}V^{j,l+1}(\psi^1+\psi)}d\mu_{\mathcal{C}_{l+1}}(\psi^1)\right)\Bigg|_{z=0}$$
$$= V^{0,l,(n)}(\psi)+1_{n=1}(V^{1-1-1,l}(\psi)+V^{1-1-2,l}(\psi)$$
$$+V^{1-2-1,l}(\psi)+V^{1-3,l}(\psi))$$
$$+1_{n\ge 2}(V^{1-1-3,l,(n)}(\psi)+V^{1-1-4,l,(n)}(\psi)+V^{1-2-2,l,(n)}(\psi))+V^{2,l,(n)}(\psi).$$

By assuming that these are convergent let us set

(4.83)

$$V^{1-1-3,l}(\psi):=\sum_{n=2}^{\infty}V^{1-1-3,l,(n)}(\psi),\quad V^{1-1-4,l}(\psi):=\sum_{n=2}^{\infty}V^{1-1-4,l,(n)}(\psi),$$
$$V^{1-2-2,l}(\psi):=\sum_{n=2}^{\infty}V^{1-2-2,l,(n)}(\psi),\quad V^{2,l}(\psi):=\sum_{n=1}^{\infty}V^{2,l,(n)}(\psi),$$
$$V^{1-1,l}(\psi):=\sum_{j=1}^{4}V^{1-1-j,l}(\psi),\quad V^{1-2,l}(\psi):=\sum_{j=1}^{2}V^{1-2-j,l}(\psi).$$

We are going to prove the convergence, regularity and bound properties of these Grassmann polynomials. Remind us that the data $V^{0,l}$ is independent of the artificial parameter $\boldsymbol{\lambda}$, the data $V^{1-j,l}$ $(j=1,2,3)$ are linear with $\boldsymbol{\lambda}$ and $V^{2,l}$ depends on $\boldsymbol{\lambda}$ at least quadratically. The 2nd superscript l indicates that these are to be integrated with the covariance $\mathcal{C}_l$.

Set

$$\varepsilon_\beta := M^{(\sum_{j=1}^{d}\frac{1}{\mathrm{n}_j}+1)(N_\beta-\hat{N}_\beta)}. \tag{4.84}$$

LEMMA 4.7. *Let c_4 be the constant appearing in Lemma 4.5. Then there exists a constant $c_5 \in [c_4, \infty)$ independent of any other parameters such that if*

(4.85)
$$M^{\min\{1,2\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}\}} \ge c_5, \quad \alpha \ge c_5 M^{\frac{\mathsf{a}}{2}}, \quad L^d \ge M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta - N_\beta)},$$
$$h \ge 2,$$

then

$$\begin{aligned}
&V^{1-1,l} \in \mathcal{Q}'(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta - l}\beta^{-1}c_0^{-2}\alpha^{-4}, l),\\
&V^{1-2,l} \in \mathcal{R}'(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta - l}\beta^{-1}c_0^{-2}\alpha^{-4}, l),\\
&V^{1-3,l} \in \mathcal{S}(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta - l}\beta^{-1}c_0^{-2}\alpha^{-4}, l),\\
&V^{2,l} \in \mathcal{W}(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}\beta^{-1}c_0^{-2}\alpha^{-4}, l),\\
&(\forall l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta\}).
\end{aligned}$$

REMARK 4.8. The radius $c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta - l}\beta^{-1}c_0^{-2}\alpha^{-4}$ of $\boldsymbol{\lambda}$ assumed on $V^{1-1,l}$, $V^{1-2,l}$, $V^{1-3,l}$ amounts to heavy β-dependent bounds on these Grassmann data. Also, the radius of analyticity of $V^{2,l}$ with $\boldsymbol{\lambda}$ depends not only on β but on h heavily. While we have to make best efforts to improve β-dependency of the possible magnitude of the variable u as the main focus of this paper, the β-dependency of the magnitude of $\boldsymbol{\lambda}$ does not affect our main results. Therefore we choose to simplify the following inductive estimation procedure at the expense of (β, h)-dependency of the magnitude of $\boldsymbol{\lambda}$ rather than to optimize it with some complications.

In the proof of Lemma 4.7 we will use the following lemma.

LEMMA 4.9. *Assume that $m,\ p,\ q \in \mathbb{N}_{\ge 2}$,*

$$\begin{aligned}
&f_0^1 \in C(\overline{D(r)} \times \mathbb{C}^2) \cap C^\omega(D(r) \times \mathbb{C}^2),\\
&f_m^1 \in C(\overline{D(r)} \times \mathbb{C}^2, \mathrm{Map}(I^m, \mathbb{C})) \cap C^\omega(D(r) \times \mathbb{C}^2, \mathrm{Map}(I^m, \mathbb{C})),\\
&f_{p,q}^1 \in C(\overline{D(r)} \times \mathbb{C}^2, \mathrm{Map}(I^p \times I^q, \mathbb{C})) \cap C^\omega(D(r) \times \mathbb{C}^2, \mathrm{Map}(I^p \times I^q, \mathbb{C})),
\end{aligned}$$

$$f_0^2 \in C(\overline{D(r)} \times \overline{D(r')}^2) \cap C^\omega(D(r) \times D(r')^2),$$
$$f_m^2 \in C(\overline{D(r)} \times \overline{D(r')}^2, \mathrm{Map}(I^m, \mathbb{C})) \cap C^\omega(D(r) \times D(r')^2, \mathrm{Map}(I^m, \mathbb{C})),$$
$$\boldsymbol{\lambda} \mapsto f_0^1(u, \boldsymbol{\lambda}) : \mathbb{C}^2 \to \mathbb{C}, \quad \boldsymbol{\lambda} \mapsto f_m^1(u, \boldsymbol{\lambda})(\mathbf{X}) : \mathbb{C}^2 \to \mathbb{C},$$
$$\boldsymbol{\lambda} \mapsto f_{p,q}^1(u, \boldsymbol{\lambda})(\mathbf{Y}, \mathbf{Z}) : \mathbb{C}^2 \to \mathbb{C} \text{ are linear and}$$
$$f_0^2(u, \mathbf{0}) = \frac{\partial}{\partial \lambda_j} f_0^2(u, \mathbf{0}) = 0, \quad f_m^2(u, \mathbf{0})(\mathbf{X}) = \frac{\partial}{\partial \lambda_j} f_m^2(u, \mathbf{0})(\mathbf{X}) = 0,$$
$$(\forall u \in D(r),\ \mathbf{X} \in I^m,\ (\mathbf{Y}, \mathbf{Z}) \in I^p \times I^q,\ j \in \{1, 2\}).$$

Then,

$$\begin{aligned} &\|f_n^j(u, \varepsilon\boldsymbol{\lambda})\|_1 \le \varepsilon \|f_n^j\|_{1,r,r'}, \quad \|f_n^j(u, \varepsilon\boldsymbol{\lambda})\|_{1,\infty} \le h\varepsilon \|f_n^j\|_{1,r,r'}, \\ &[f_{p,q}^1(u, \varepsilon\boldsymbol{\lambda}), g]_1 \le \varepsilon [f_{p,q}^1, g]_{1,r,r'}, \\ &(\forall u \in \overline{D(r)},\ \boldsymbol{\lambda} \in \overline{D(r')}^2,\ \varepsilon \in [0, 1/2],\ j \in \{1, 2\},\ n \in \{0, m\}). \end{aligned} \tag{4.86}$$

Here $g : I^2 \to \mathbb{C}$ is any anti-symmetric function.

Proof. These are essentially same as the inequalities [12, (3.91), (3.92)]. The inequalities (4.86) for $j = 1$ are trivial because of the linearity. To prove the inequalities for $j = 2$, one can use the following equality.

$$f_m^2(u, \varepsilon\boldsymbol{\lambda})(\mathbf{X}) = \frac{1}{2\pi i} \oint_{|z|=\delta} dz f_m^2(u, z\boldsymbol{\lambda})(\mathbf{X}) \frac{\varepsilon^2}{z^2(z - \varepsilon)},$$
$$(\forall u \in \overline{D(r)},\ \boldsymbol{\lambda} \in \overline{D(r')}^2,\ \varepsilon \in [0, 1/2],\ \delta \in (1/2, 1),\ \mathbf{X} \in I^m). \ \square$$

Proof of Lemma 4.7. During the proof we often omit the sign of dependency on the parameter $(u, \boldsymbol{\lambda})$ to shorten formulas. Since $V^{1-3,l}(\psi)$ $(l = N_\beta, N_\beta + 1, \cdots, \hat{N}_\beta)$ are defined independently of other polynomials, we can readily summarize their properties. Since $V_4^{1-3,l}(\psi) = -\lambda_2 A_4^2(\psi)$ for any $l \in \{N_\beta, N_\beta + 1, \cdots, \hat{N}_\beta\}$,

$$V_4^{1-3,l}(\overline{\rho}_1\rho_1\mathbf{x}_1 s_1\xi_1, \overline{\rho}_2\rho_2\mathbf{x}_2 s_2\xi_2, \overline{\rho}_3\rho_3\mathbf{x}_3 s_3\xi_3, \overline{\rho}_4\rho_4\mathbf{x}_4 s_4\xi_4)$$
$$= -\frac{\lambda_2 h^3}{4!} 1_{s_1=s_2=s_3=s_4} \sum_{\sigma \in \mathbb{S}_4} \mathrm{sgn}(\sigma)$$

$$\cdot 1_{\substack{((\overline{\rho}_{\sigma(1)},\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\overline{\rho}_{\sigma(2)},\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}),(\overline{\rho}_{\sigma(3)},\rho_{\sigma(3)},\mathbf{x}_{\sigma(3)},\xi_{\sigma(3)}),(\overline{\rho}_{\sigma(4)},\rho_{\sigma(4)},\mathbf{x}_{\sigma(4)},\xi_{\sigma(4)}))\\ =((1,\hat{\rho},r_L(\hat{\mathbf{x}}),1),(2,\hat{\rho},r_L(\hat{\mathbf{x}}),-1),(2,\hat{\eta},r_L(\hat{\mathbf{y}}),1),(1,\hat{\eta},r_L(\hat{\mathbf{y}}),-1))}},$$
$$(\forall(\overline{\rho}_j,\rho_j,\mathbf{x}_j,s_j,\xi_j)\in I\ (j=1,2,3,4)).$$

Thus

$$\|V_4^{1-3,l}\|_{1,r,r'}\le\beta r',\quad (\forall l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta\}).\tag{4.87}$$

Also,

$$\begin{aligned}&V_2^{1-3,\hat{N}_\beta}(\overline{\rho}_1\rho_1\mathbf{x}_1s_1\xi_1,\overline{\rho}_2\rho_2\mathbf{x}_2s_2\xi_2)\\ &=-\frac{h}{2}1_{s_1=s_2}\sum_{\sigma\in\mathbb{S}_2}\mathrm{sgn}(\sigma)\Big(1_{\substack{((\overline{\rho}_{\sigma(1)},\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\overline{\rho}_{\sigma(2)},\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}))\\=((1,\hat{\rho},r_L(\hat{\mathbf{x}}),1),(2,\hat{\rho},r_L(\hat{\mathbf{x}}),-1))}}\lambda_1\\ &\qquad+1_{\substack{((\overline{\rho}_{\sigma(1)},\rho_{\sigma(1)},\mathbf{x}_{\sigma(1)},\xi_{\sigma(1)}),(\overline{\rho}_{\sigma(2)},\rho_{\sigma(2)},\mathbf{x}_{\sigma(2)},\xi_{\sigma(2)}))\\=((1,\hat{\rho},r_L(\hat{\mathbf{x}}),1),(1,\hat{\rho},r_L(\hat{\mathbf{x}}),-1))}}\\ &\qquad\cdot 1_{(\hat{\rho},r_L(\hat{\mathbf{x}}))=(\hat{\eta},r_L(\hat{\mathbf{y}}))}\lambda_2\Big),\\ &\quad(\forall(\overline{\rho}_j,\rho_j,\mathbf{x}_j,s_j,\xi_j)\in I\ (j=1,2)).\end{aligned}$$

Thus

$$\|V_2^{1-3,\hat{N}_\beta}\|_{1,r,r'}\le 2\beta r'.\tag{4.88}$$

We can see from the definition that for $l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta-1\}$

(4.89)

$$\begin{aligned}&V_2^{1-3,l}(\psi)\\ &=V_2^{1-3,l+1}(\psi)\\ &\quad+\left(\frac{1}{h}\right)^2\sum_{\mathbf{X}\in I^2}\Bigg(\begin{pmatrix}4\\2\end{pmatrix}\\ &\qquad\cdot\left(\frac{1}{h}\right)^2\sum_{\mathbf{Y}\in I^2}V_4^{1-3,l+1}(\mathbf{X},\mathbf{Y})Tree(\{1\},\mathcal{C}_{l+1})\psi^1_{\mathbf{Y}}\Big|_{\psi^1=0}\Bigg)\psi_{\mathbf{X}}\\ &=V_2^{1-3,\hat{N}_\beta}(\psi)\\ &\quad+\left(\frac{1}{h}\right)^2\sum_{\mathbf{X}\in I^2}\Bigg(\begin{pmatrix}4\\2\end{pmatrix}\end{aligned}$$

$$\cdot\left(\frac{1}{h}\right)^2\sum_{\mathbf{Y}\in I^2}V_4^{1-3,\hat{N}_\beta}(\mathbf{X},\mathbf{Y})\sum_{j=l+1}^{\hat{N}_\beta}Tree(\{1\},\mathcal{C}_j)\psi_{\mathbf{Y}}^1\Big|_{\psi^1=0}\Bigg)\psi_{\mathbf{X}}.$$

By using (4.33), (4.87), (4.88) and the assumptions $c_0\ge 1$, $M\ge 2$, (4.90)

$$\|V_2^{1-3,l}\|_{1,r,r'}\le\|V_2^{1-3,\hat{N}_\beta}\|_{1,r,r'}+\left(\begin{array}{c}4\\2\end{array}\right)\|V_4^{1-3,\hat{N}_\beta}\|_{1,r,r'}\sum_{j=l+1}^{\hat{N}_\beta}c_0M^{\mathsf{a}(j-\hat{N}_\beta)}$$
$$\le c\beta c_0r'.$$

Moreover it follows from the definition and (4.89) that

$$\begin{aligned}
&V_0^{1-3,l}\\
&=V_0^{1-3,l+1}+Tree(\{1\},\mathcal{C}_{l+1})V_2^{1-3,l+1}(\psi^1)\Big|_{\psi^1=0}\\
&\quad+Tree(\{1\},\mathcal{C}_{l+1})V_4^{1-3,l+1}(\psi^1)\Big|_{\psi^1=0}\\
&=V_0^{1-3,l+1}+Tree(\{1\},\mathcal{C}_{l+1})V_2^{1-3,\hat{N}_\beta}(\psi^1)\Big|_{\psi^1=0}\\
&\quad+1_{l\le\hat{N}_\beta-2}\left(\frac{1}{h}\right)^2\sum_{\mathbf{X}\in I^2}\\
&\qquad\cdot\left(\left(\begin{array}{c}4\\2\end{array}\right)\left(\frac{1}{h}\right)^2\sum_{\mathbf{Y}\in I^2}V_4^{1-3,\hat{N}_\beta}(\mathbf{X},\mathbf{Y})\sum_{j=l+2}^{\hat{N}_\beta}Tree(\{1\},\mathcal{C}_j)\psi_{\mathbf{Y}}^1\Big|_{\psi^1=0}\right)\\
&\qquad\cdot Tree(\{1\},\mathcal{C}_{l+1})\psi_{\mathbf{X}}^1\Big|_{\psi^1=0}\\
&\quad+Tree(\{1\},\mathcal{C}_{l+1})V_4^{1-3,\hat{N}_\beta}(\psi^1)\Big|_{\psi^1=0}\\
&=\sum_{k=l+1}^{\hat{N}_\beta}Tree(\{1\},\mathcal{C}_k)V_2^{1-3,\hat{N}_\beta}(\psi^1)\Big|_{\psi^1=0}\\
&\quad+1_{l\le\hat{N}_\beta-2}\sum_{k=l+1}^{\hat{N}_\beta-1}\left(\frac{1}{h}\right)^2\sum_{\mathbf{X}\in I^2}\\
&\qquad\cdot\left(\left(\begin{array}{c}4\\2\end{array}\right)\left(\frac{1}{h}\right)^2\sum_{\mathbf{Y}\in I^2}V_4^{1-3,\hat{N}_\beta}(\mathbf{X},\mathbf{Y})\sum_{j=k+1}^{\hat{N}_\beta}Tree(\{1\},\mathcal{C}_j)\psi_{\mathbf{Y}}^1\Big|_{\psi^1=0}\right)
\end{aligned}$$

$$\cdot Tree(\{1\},\mathcal{C}_k)\psi^1_{\mathbf{X}}\Big|_{\psi^1=0}$$

$$+\sum_{k=l+1}^{\hat{N}_\beta} Tree(\{1\},\mathcal{C}_k)V_4^{1-3,\hat{N}_\beta}(\psi^1)\Big|_{\psi^1=0}.$$

Thus by (4.33), (4.87), (4.88) and $c_0\ge 1$, $M\ge 2$,

(4.91)

$$\begin{aligned}
&\|V_0^{1-3,l}\|_{1,r,r'}\\
&\le 2\beta r'\sum_{k=l+1}^{\hat{N}_\beta} c_0 M^{\mathsf{a}(k-\hat{N}_\beta)}+c\beta r' 1_{l\le \hat{N}_\beta-2}\sum_{k=l+1}^{\hat{N}_\beta-1} c_0 M^{\mathsf{a}(k-\hat{N}_\beta)}\sum_{j=k+1}^{\hat{N}_\beta} c_0 M^{\mathsf{a}(j-\hat{N}_\beta)}\\
&\quad+\beta r'\sum_{k=l+1}^{\hat{N}_\beta} c_0^2 M^{2\mathsf{a}(k-\hat{N}_\beta)}\\
&\le c\beta c_0^2 r'.
\end{aligned}$$

The inequalities (4.87), (4.90), (4.91) and $c_0\ge 1$, $\alpha\ge 1$ result in

$$\alpha^2\|V_0^{1-3,l}\|_{1,r,r'}\le c\beta c_0^2\alpha^2 r',\quad \sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m\|V_m^{1-3,l}\|_{1,r,r'}\le c\beta c_0^2\alpha^4 r'.$$

By definition $\boldsymbol{\lambda}\mapsto V^{1-3,l}(\boldsymbol{\lambda})(\psi)$ is linear for any $l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta\}$. The statement of Lemma 4.1 and the induction with l ensure that $V_m^{1-3,l}: I^m\to\mathbb{C}$ ($m=2,4,\ l=N_\beta,N_\beta+1,\cdots,\hat{N}_\beta$) satisfy (4.4). Combined with these basic properties, the above inequalities conclude that there exists a generic positive constant c' independent of any parameter such that

(4.92)

$$V^{1-3,l}\in\mathcal{S}(r,c'^{-1}\varepsilon_\beta^{\hat{N}_\beta-l}\beta^{-1}c_0^{-2}\alpha^{-4},l),\quad (\forall l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta\})$$

for any $\alpha\in\mathbb{R}_{\ge 1}$, $r\in\mathbb{R}_{>0}$. Here we also used that $\varepsilon_\beta\le 1$.

Let us set $r:=b^{-1}c_0^{-2}\alpha^{-4}$, $r':=c'^{-1}\beta^{-1}c_0^{-2}\alpha^{-4}$ with the constant c' appearing in (4.92). Let $l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta-1\}$ and assume that

$$V^{1-1,l+1}\in\mathcal{Q}'(r,\varepsilon_\beta^{\hat{N}_\beta-l-1}r',l+1),\quad V^{1-2,l+1}\in\mathcal{R}'(r,\varepsilon_\beta^{\hat{N}_\beta-l-1}r',l+1),$$

$$V^{2,l+1} \in \mathcal{W}(r, h^{l+1-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l-1}r', l+1)$$

as the induction hypothesis. Note that these inclusion trivially hold for $l = \hat{N}_\beta - 1$. Check that if $M \ge 2$, $\varepsilon_\beta \le M^{-1} \le \frac{1}{2}$. Thus we can apply the inequalities (4.86) with $\varepsilon = \varepsilon_\beta$. Let us list useful inequalities derived from the induction hypothesis, (4.86), (4.92) and the conditions $M^{\mathsf{a}} \ge 2^4$, $\alpha \ge 2^3$.

$$\sum_{m=2}^{N} 2^{3m}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta \alpha^{-2} L^{-d}, \tag{4.93}$$

$$\sum_{m=2}^{N} 2^{m}\alpha^m(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-\mathsf{a}} L^{-d}, \tag{4.94}$$

$$\sum_{m=4}^{N} 2^{3m}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-2,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta \alpha^{-4}, \tag{4.95}$$

$$\sum_{m=4}^{N} 2^{m}\alpha^m(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-2,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-2\mathsf{a}}, \tag{4.96}$$

$$\sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} 2^{3p+3q}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p+q}{2}} [V_{p,q}^{1-2,l+1}, g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta \alpha^{-4} L^{-d}\|g\|, \tag{4.97}$$

$$\sum_{p,q=2}^{N} 1_{p,q\in 2\mathbb{N}} 2^{2p+2q}\alpha^{p+q}(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p+q}{2}} [V_{p,q}^{1-2,l+1}, g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-2\mathsf{a}} L^{-d}\|g\|, \tag{4.98}$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$.

$$\sum_{m=2}^{N} 2^{3m}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-3,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta \alpha^{-2}, \tag{4.99}$$

$$\sum_{m=2}^{N} 2^{m}\alpha^m(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-3,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-\mathsf{a}}, \tag{4.100}$$

$$\sum_{m=2}^{N} 2^{3m}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{2,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta \alpha^{-2}, \tag{4.101}$$

$$\sum_{m=2}^{N} 2^m\alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{2,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-\mathsf{a}}. \tag{4.102}$$

To derive (4.95), (4.96), we used a variant of the inequality (4.53). We can derive from (4.93), (4.94), (4.95), (4.96), (4.99), (4.100) that

$$\sum_{m=2}^{N} 2^{3m} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta \alpha^{-2}, \tag{4.103}$$

$$\sum_{m=2}^{N} 2^m\alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-\mathsf{a}}. \tag{4.104}$$

Let us start the analysis of l-th step by studying $V^{1-1-1,l}$. By Lemma 4.1 its kernels satisfy (4.4). By (4.7), (4.33), (4.72) for $l+1$, (4.86) and the conditions $\alpha \ge 2$, $M^{\mathsf{a}} \ge 2^4$,

(4.105)

$$\begin{aligned}
&\|V_0^{1-1-1,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le \varepsilon_\beta \|V_0^{1-1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l-1}r'} + \varepsilon_\beta \sum_{p=2}^{N} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p}{2}} \|V_p^{1-1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l-1}r'}\\
&\le 2\varepsilon_\beta \alpha^{-2} L^{-d},
\end{aligned}$$

(4.106)

$$\begin{aligned}
&\sum_{m=2}^{N} \alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-1-1,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le \sum_{m=2}^{N} \alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \sum_{p=m}^{N} 2^p (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p-m}{2}} \|V_p^{1-1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le \varepsilon_\beta L^{-d} \sum_{m=2}^{N} 2^m M^{-\frac{\mathsf{a}}{2}m} \le c\varepsilon_\beta M^{-\mathsf{a}} L^{-d}.
\end{aligned}$$

Next let us consider $V^{1-1-2,l}$. By Lemma 4.2 the kernels of $V^{1-1-2,l}$ satisfy (4.4). By (4.12) and (4.33),

$$\|V_m^{1-1-2,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$

$$\le (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-1-\frac{m}{2}}$$
$$\cdot \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{2p_1+2p_2} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+p_2}{2}}$$
$$\cdot [V^{1-2,l+1}_{p_1,p_2}, \tilde{\mathcal{C}}_{l+1}]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} 1_{p_1+p_2-2\ge m}.$$

Then by using (4.35), (4.97), (4.98) and the condition $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2$ we observe that

$$\|V_0^{1-1-2,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \tag{4.107}$$
$$\le (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-1} c\varepsilon_\beta \alpha^{-4} c_0 M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)} L^{-d}$$
$$\le c\varepsilon_\beta \alpha^{-4} M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)} L^{-d},$$

$$\sum_{m=2}^{N} \alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{1-1-2,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \tag{4.108}$$
$$\le c(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-1} M^{\mathsf{a}} \alpha^{-2}$$
$$\cdot \sum_{p_1,p_2=2}^{N} 1_{p_1,p_2\in 2\mathbb{N}} 2^{2p_1+2p_2} \alpha^{p_1+p_2} (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p_1+p_2}{2}}$$
$$\cdot [V^{1-2,l+1}_{p_1,p_2}, \tilde{\mathcal{C}}_{l+1}]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$\le c\varepsilon_\beta M^{-\mathsf{a}-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)} \alpha^{-2} L^{-d}.$$

Next let us consider $V^{1-1-3,l,(n)}$ $(n \in \mathbb{N}_{\ge 2})$. By Lemma 4.1 the anti-symmetric kernels of $V^{1-1-3,l,(n)}$ satisfy (4.4). Thus if $\sum_{n=2}^{\infty} V^{1-1-3,l,(n)}$ converges, those of $V^{1-1-3,l}$ satisfy (4.4) too. Let us establish bound properties. Observe that

$$V^{1-1-3,l,(n)}(\psi)$$
$$= \sum_{q=1}^{n-1} \begin{pmatrix} n-1 \\ q \end{pmatrix} \frac{1}{(n-1)!} Tree(\{1,2,\cdots,n\},\mathcal{C}_{l+1}) V^{1,l+1}(\psi^1+\psi)$$
$$\cdot \prod_{j=2}^{q+1} V^{0-1,l+1}(\psi^j+\psi) \prod_{k=q+2}^{n} V^{0-2,l+1}(\psi^k+\psi)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.$$

By (4.9), (4.33) and (4.34),

(4.109)

$$
\begin{aligned}
&\|V_m^{1-1-3,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le (c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n+1-\frac{m}{2}}2^{-2m+n-1}(c_0M^{(\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-1}\\
&\quad\cdot\sum_{p_1=2}^N 2^{3p_1}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1}{2}}\|V_{p_1}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\quad\cdot\sum_{p_2=2}^N 2^{3p_2}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_2}{2}}\|V_{p_2}^{0-1,l+1}\|_{0,\infty,r}\\
&\quad\cdot\prod_{j=3}^n\left(\sum_{p_j=2}^N 2^{3p_j}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}(\|V_{p_j}^{0-1,l+1}\|_{0,\infty,r}+\|V_{p_j}^{0-2,l+1}\|_{0,\infty,r})\right)\\
&\quad\cdot 1_{\sum_{j=1}^n p_j-2(n-1)\ge m}.
\end{aligned}
$$

Substitution of (4.46), (4.48), (4.103) and the condition $L^d \ge M^{(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}$ yields that

$$
\begin{aligned}
&\|V_0^{1-1-3,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le M^{-(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}c\varepsilon_\beta\alpha^{-4}L^{-d}\\
&\quad\cdot\left(c\alpha^{-2}L^{-d}+c\alpha^{-4}M^{(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\right)^{n-2}\\
&\le \varepsilon_\beta M^{-(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}(c\alpha^{-2})^n.
\end{aligned}
$$

Thus on the assumption $\alpha\ge c$,

$$
\sum_{n=2}^\infty\|V_0^{1-1-3,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\le c\alpha^{-4}\varepsilon_\beta M^{-(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}. \tag{4.110}
$$

Also by using (4.47), (4.50), (4.104) and the assumptions $\alpha M^{-\frac{\mathsf{a}}{2}}\ge 2^3$, $L^d\ge M^{(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}$ we obtain from (4.109) that

$$
\sum_{m=2}^N\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{1-1-3,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}
$$

$$\leq c^n\alpha^{-2(n-1)}M^{\mathsf{a}(n-1)}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}\varepsilon_\beta M^{-2\mathsf{a}}L^{-d}$$
$$\cdot\left(cM^{-\mathsf{a}}L^{-d}+cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\right)^{n-2}$$
$$\leq M^{-\mathsf{a}-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\varepsilon_\beta L^{-d}(c\alpha^{-2})^{n-1}.$$

Therefore by assuming that $\alpha \geq c$,

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\sum_{n=2}^{\infty}\|V_m^{1-1-3,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$\leq c\alpha^{-2}M^{-\mathsf{a}-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\varepsilon_\beta L^{-d}. \tag{4.111}$$

Next let us deal with $V^{1-1-4,l,(n)}$ $(n \in \mathbb{N}_{\geq 2})$. By Lemma 4.2 its anti-symmetric kernels satisfy (4.4). Thus it suffices to prove the convergence of $\sum_{n=2}^{\infty}V^{1-1-4,l,(n)}$ in order to prove that the kernels of $V^{1-1-4,l}$ satisfy (4.4) as well. By (4.14), (4.33) and (4.34), for $m \in \{0, 2, \cdots, N\}$,

(4.112)

$$\|V_m^{1-1-4,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$\leq (c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n-\frac{m}{2}}2^{-2m}(c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-1}$$
$$\cdot\sum_{p_1,p_2=2}^{N}1_{p_1,p_2\in 2\mathbb{N}}2^{3p_1+3p_2}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+p_2}{2}}[V_{p_1,p_2}^{0-2,l+1},\tilde{\mathcal{C}}_{l+1}]_{1,\infty,r}$$
$$\cdot\prod_{j=3}^{n}\left(\sum_{p_j=4}^{N}2^{3p_j}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}\|V_{p_j}^{0-2,l+1}\|_{1,\infty,r}\right)$$
$$\cdot\sum_{p_{n+1}=2}^{N}2^{3p_{n+1}}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_{n+1}}{2}}\|V_{p_{n+1}}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}1_{\sum_{j=1}^{n+1}p_j-2n\geq m}.$$

Then by substituting (4.35), (4.48), (4.51), (4.103) we observe that

$$\|V_0^{1-1-4,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$\leq M^{-\mathsf{a}(l+1-\hat{N}_\beta)-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}$$

$$\cdot M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}(c\alpha^{-4}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)})^{n-2}\varepsilon_\beta\alpha^{-2}$$
$$\le M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\varepsilon_\beta L^{-d}\alpha^2(c\alpha^{-4})^n.$$

Thus on the assumption $\alpha \ge c$,

$$\sum_{n=2}^{\infty}\|V_0^{1-1-4,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\alpha^{-6}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\varepsilon_\beta L^{-d}. \tag{4.113}$$

Also by using (4.35), (4.50), (4.52), (4.104) and the assumption $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2^3$ we can derive from (4.112) that

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{1-1-4,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$\le c\alpha^{-2n}M^{\mathsf{a}n-\mathsf{a}(l+1-\hat{N}_\beta)n+(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)(n-1)}$$
$$\cdot M^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}L^{-d}$$
$$\cdot M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}(cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)})^{n-2}\varepsilon_\beta M^{-\mathsf{a}}$$
$$\le M^{\mathsf{a}-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\varepsilon_\beta L^{-d}(c\alpha^{-2}M^{-\mathsf{a}})^n,$$

or on the assumption $\alpha \ge c$,

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\sum_{n=2}^{\infty}\|V_m^{1-1-4,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\alpha^{-4}M^{-\mathsf{a}-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\varepsilon_\beta L^{-d}. \tag{4.114}$$

Let us sum up (4.105), (4.106), (4.107), (4.108), (4.110), (4.111), (4.113), (4.114).

(4.115)

$$\|V_0^{1-1-1,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} + \|V_0^{1-1-2,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} + \sum_{n=2}^{\infty}\|V_0^{1-1-3,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$+\sum_{n=2}^{\infty}\|V_0^{1-1-4,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$

$$\leq c\left(1+\alpha^{-2}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}\right)\varepsilon_\beta\alpha^{-2}L^{-d},$$

(4.116)

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\Bigg(\|V_m^{1-1-1,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}+\|V_m^{1-1-2,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$+\sum_{n=2}^{\infty}\|V_m^{1-1-3,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}+\sum_{n=2}^{\infty}\|V_m^{1-1-4,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\Bigg)$$
$$\leq c(M^{-\mathsf{a}}+\alpha^{-2}M^{-\mathsf{a}-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)})\varepsilon_\beta L^{-d}.$$

Recalling (4.84), one can see that under the assumptions $\alpha\geq c$, $M\geq c$ the right-hand side of (4.115), (4.116) is less than $\alpha^{-2}L^{-d}$, L^{-d} respectively. By setting $V^{1-1,l}:=\sum_{j=1}^{4}V^{1-1-j,l}$ we conclude that

$$V^{1-1,l}\in\mathcal{Q}'(r,\varepsilon_\beta^{\hat{N}_\beta-l}r',l).$$

Recall that we have also assumed that $\alpha M^{-\frac{\mathsf{a}}{2}}\geq 2^3$, $L^d\geq M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}$ to reach this conclusion.

Let us study $V^{1-2-1,l}$. By Lemma 4.4 there exist $V_{a,b}^{1-2-1,l}\in\mathrm{Map}(\overline{D(r)}\times\mathbb{C}^2,\mathrm{Map}(I^a\times I^b,\mathbb{C}))$ $(a,b=2,4,\cdots,N)$ such that for any $(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\mathbb{C}^2$, $V_{a,b}^{1-2-1,l}(u,\boldsymbol{\lambda}):I^a\times I^b\to\mathbb{C}$ is bi-anti-symmetric and satisfies (4.4), (4.10) and

$$V^{1-2-1,l}(u,\boldsymbol{\lambda})(\psi)=\sum_{a,b=2}^{N}1_{a,b\in 2\mathbb{N}}\left(\frac{1}{h}\right)^{a+b}\sum_{\substack{\mathbf{X}\in I^a\\ \mathbf{Y}\in I^b}}V_{a,b}^{1-2-1,l}(u,\boldsymbol{\lambda})(\mathbf{X},\mathbf{Y})\psi_{\mathbf{X}}\psi_{\mathbf{Y}}.$$

Moreover it follows from the definition and the induction hypothesis that

$$V^{1-2-1,l}\in C\left(\overline{D(r)}\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right)\cap C^\omega\left(D(r)\times\mathbb{C}^2,\bigwedge_{even}\mathcal{V}\right)$$

and $\boldsymbol{\lambda}\mapsto V^{1-2-1,l}(u,\boldsymbol{\lambda})(\psi):\mathbb{C}^2\mapsto\bigwedge_{even}\mathcal{V}$ is linear for any $u\in\overline{D(r)}$. Let us establish bound properties. By (4.17) and (4.33), for any $a,b\in\{2,4,\cdots,N\}$,

$$\|V_{a,b}^{1-2-1,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$

$$\le \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} \left(\begin{array}{c} p \\ a \end{array}\right)\left(\begin{array}{c} q \\ b \end{array}\right) (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{1}{2}(p+q-a-b)} \cdot \|V_{p,q}^{1-2,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}.$$

Then by using (4.73) for $l+1$, (4.86) and the conditions $\alpha \ge 2$, $M^{\mathsf{a}} \ge 2^4$ we have that

$$\begin{aligned}(4.117)\quad &\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \alpha^{a+b} (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}} \|V_{a,b}^{1-2-1,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \\ &\le \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} M^{-\frac{a+b}{2}\mathsf{a}} \alpha^{a+b}\varepsilon_\beta \\ &\quad \cdot \sum_{p=a}^{N}\sum_{q=b}^{N} 1_{p,q\in 2\mathbb{N}} 2^{p+q} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p+q}{2}} \|V_{p,q}^{1-2,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l-1}r'} \\ &\le \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} 2^{a+b} M^{-\frac{a+b}{2}\mathsf{a}}\varepsilon_\beta \le cM^{-2\mathsf{a}}\varepsilon_\beta.\end{aligned}$$

Based on (4.19), (4.33), (4.74) for $l+1$, (4.86) and the conditions $\alpha \ge 2$, $M^{\mathsf{a}} \ge 2^4$, an argument parallel to the above shows that

$$(4.118)\quad \sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}} \alpha^{a+b} (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}} [V_{a,b}^{1-2-1,l}, g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le cM^{-2\mathsf{a}}\varepsilon_\beta L^{-d}\|g\|,$$

for any anti-symmetric function $g : I^2 \to \mathbb{C}$.

Next let us consider $V^{1-2-2,l,(n)}$ ($n \in \mathbb{N}_{\ge 2}$). Lemma 4.4 ensures that $V^{1-2-2,l,(n)}$ can be written with bi-anti-symmetric kernels $V_{a,b}^{1-2-2,l,(n)} \in \mathrm{Map}(\overline{D(r)} \times \mathbb{C}^2, \mathrm{Map}(I^a \times I^b, \mathbb{C}))$ $(a, b = 2, 4, \cdots, N)$ and for any $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \mathbb{C}^2$, $a, b \in \{2, 4, \cdots, N\}$ the kernel $V_{a,b}^{1-2-2,l,(n)}(u, \boldsymbol{\lambda})$ satisfies (4.4) and (4.10). Moreover we can deduce from (4.15) and the induction hypothesis that

$$V_{a,b}^{1-2-2,l,(n)} \in C(\overline{D(r)} \times \mathbb{C}^2, \mathrm{Map}(I^a \times I^b, \mathbb{C})) \cap C^\omega(D(r) \times \mathbb{C}^2, \mathrm{Map}(I^a \times I^b, \mathbb{C}))$$

and $\boldsymbol{\lambda} \mapsto V^{1-2-2,l,(n)}(u,\boldsymbol{\lambda})(\psi) : \mathbb{C}^2 \to \bigwedge_{even}\mathcal{V}$ is linear for any $u \in \overline{D(r)}$. These properties must hold true for $V^{1-2-2,l}$, once $\sum_{n=2}^{\infty} V^{1-2-2,l,(n)}$ is proved to be uniformly convergent. By using (4.21), (4.33), (4.34) we obtain that

$$\begin{aligned}
&\|V_{a,b}^{1-2-2,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta - l} r'}\\
&\le \frac{1}{(n-1)!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\\
&\quad \cdot (1_{m\neq 0}(m-1)! + 1_{m=0})(1_{m\neq n-1}(n-m-2)! + 1_{m=n-1})\\
&\quad \cdot 2^{-2a-2b}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n+1-\frac{1}{2}(a+b)}(c_0 M^{(\mathsf{a}-1-\sum_{j=1}^d \frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-1}\\
&\quad \cdot \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} 2^{3p_1+3q_1}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+q_1}{2}} \|V_{p_1,q_1}^{0-2,l+1}\|_{1,\infty,r}\\
&\quad \cdot \prod_{j=2}^{m+1}\Bigg(\sum_{p_j=2}^{N} 2^{3p_j}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}\\
&\qquad\qquad \cdot \left(1_{s_j\neq n}\|V_{p_j}^{0-2,l+1}\|_{1,\infty,r} + 1_{s_j=n}\|V_{p_j}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\right)\Bigg)\\
&\quad \cdot \prod_{k=2}^{n-m}\Bigg(\sum_{q_k=2}^{N} 2^{3q_k}(c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{q_k}{2}}\\
&\qquad\qquad \cdot \left(1_{t_k\neq n}\|V_{q_k}^{0-2,l+1}\|_{1,\infty,r} + 1_{t_k=n}\|V_{q_k}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\right)\Bigg)\\
&\quad \cdot 1_{\sum_{j=1}^{m+1} p_j - 2m \ge a} 1_{\sum_{k=1}^{n-m} q_k - 2(n-m-1)\ge b}.
\end{aligned}$$

Then by (4.49), (4.50), (4.68), (4.104) and the assumption $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2^3$,

(4.119)

$$\begin{aligned}
&\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}\alpha^{a+b}(c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\|V_{a,b}^{1-2-2,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le c^n \alpha^{-2(n-1)} M^{\mathsf{a}(n-1)} M^{-\mathsf{a}(l+1-\hat{N}_\beta)(n-1)+(\mathsf{a}-1-\sum_{j=1}^d \frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)(n-1)}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \frac{1}{(n-1)!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m},\{t_k\}_{k=1}^{n-m})\in S(n,m)}\\
&\cdot (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\cdot \sum_{p_1,q_1=2}^{N} 1_{p_1,q_1\in 2\mathbb{N}} 2^{p_1+q_1}\alpha^{p_1+q_1}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p_1+q_1}{2}}\|V_{p_1,q_1}^{0-2,l+1}\|_{1,\infty,r}\\
&\cdot \prod_{j=2}^{m+1}\Bigg(\sum_{p_j=2}^{N} 2^{p_j}\alpha^{p_j}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{p_j}{2}}\\
&\qquad \cdot \left(1_{s_j\neq n}\|V_{p_j}^{0-2,l+1}\|_{1,\infty,r}+1_{s_j=n}\|V_{p_j}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\right)\Bigg)\\
&\cdot \prod_{k=2}^{n-m}\Bigg(\sum_{q_k=2}^{N} 2^{q_k}\alpha^{q_k}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{q_k}{2}}\\
&\qquad \cdot \left(1_{t_k\neq n}\|V_{q_k}^{0-2,l+1}\|_{1,\infty,r}+1_{t_k=n}\|V_{q_k}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\right)\Bigg)\\
&\le \alpha^{-2(n-1)}M^{\mathsf{a}(n-1)-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}\\
&\quad\cdot (cM^{-2\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)})^{n-1}\varepsilon_\beta M^{-\mathsf{a}}\\
&\le (c\alpha^{-2}M^{-\mathsf{a}})^{n-1}\varepsilon_\beta M^{-\mathsf{a}}.
\end{aligned}
$$

Thus on the assumption $\alpha\ge c$,

$$
\begin{aligned}
(4.120)\qquad &\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\sum_{n=2}^{\infty}\|V_{a,b}^{1-2-2,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le c\alpha^{-2}M^{-2\mathsf{a}}\varepsilon_\beta.
\end{aligned}
$$

On the other hand, one can apply (4.23), (4.33), (4.34) to derive that for any anti-symmetric function $g: I^2\to\mathbb{C}$,

$$
\begin{aligned}
&[V_{a,b}^{1-2-2,l,(n)},g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le \frac{1}{(n-1)!}\sum_{m=0}^{n-1}\sum_{(\{s_j\}_{j=1}^{m+1},\{t_k\}_{k=1}^{n-m})\in S(n,m)}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot (1_{m\neq 0}(m-1)!+1_{m=0})(1_{m\neq n-1}(n-m-2)!+1_{m=n-1})\\
&\cdot 2^{-2a-2b}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n+1-\frac{1}{2}(a+b)}\big(c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}\big)^{n-2}\\
&\cdot \sum_{p_1,q_1=2}^{N}1_{p_1,q_1\in 2\mathbb{N}}2^{3p_1+3q_1}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1+q_1}{2}}\\
&\cdot \Bigg([V_{p_1,q_1}^{0-2,l+1},g]_{1,\infty,r}c_0M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)}\\
&\quad +[V_{p_1,q_1}^{0-2,l+1},\tilde{\mathcal{C}}_{l+1}]_{1,\infty,r}\|g\|_{1,\infty}\Bigg)\\
&\cdot \prod_{j=2}^{m+1}\Bigg(\sum_{p_j=2}^{N}2^{3p_j}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}\\
&\qquad \cdot\Bigg(1_{s_j\neq n}\|V_{p_j}^{0-2,l+1}\|_{1,\infty,r}+1_{s_j=n}\|V_{p_j}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\Bigg)\Bigg)\\
&\cdot \prod_{k=2}^{n-m}\Bigg(\sum_{q_k=2}^{N}2^{3q_k}(c_0M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{q_k}{2}}\\
&\qquad \cdot\Bigg(1_{t_k\neq n}\|V_{q_k}^{0-2,l+1}\|_{1,\infty,r}+1_{t_k=n}\|V_{q_k}^{1,l+1}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\Bigg)\Bigg)\\
&\cdot 1_{\sum_{j=1}^{m+1}p_j-2m\ge a}1_{\sum_{k=1}^{n-m}q_k-2(n-m-1)\ge b}.
\end{aligned}
$$

Then by substituting (4.35), (4.50), (4.52), (4.68), (4.104), using the inequality $\|g\|_{1,\infty}\le\|g\|$ and the assumption $\alpha M^{-\frac{\mathsf{a}}{2}}\ge 2^3$ and computing in a parallel way to (4.119) one reaches that

$$
\begin{aligned}
&\sum_{a,b=2}^{N}1_{a,b\in 2\mathbb{N}}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}[V_{a,b}^{1-2-2,l,(n)},g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le (c\alpha^{-2}M^{-\mathsf{a}})^{n-1}\varepsilon_\beta M^{-\mathsf{a}}L^{-d}\|g\|,
\end{aligned}
$$

or on the assumption $\alpha\ge c$,

$$
\begin{aligned}
&\sum_{a,b=2}^{N}1_{a,b\in 2\mathbb{N}}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\sum_{n=2}^{\infty}[V_{a,b}^{1-2-2,l,(n)},g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\\
&\le c\alpha^{-2}M^{-2\mathsf{a}}\varepsilon_\beta L^{-d}\|g\|.
\end{aligned}
\tag{4.121}
$$

Here let us combine (4.117), (4.118) with (4.120), (4.121) respectively to derive that

$$\begin{aligned}(4.122)\quad &\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\\ &\cdot\left(\|V_{a,b}^{1-2-1,l}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}+\sum_{n=2}^{\infty}\|V_{a,b}^{1-2-2,l,(n)}\|_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\right)\\ &\le c(M^{-2\mathsf{a}}+\alpha^{-2}M^{-2\mathsf{a}})\varepsilon_\beta,\end{aligned}$$

$$\begin{aligned}(4.123)\quad &\sum_{a,b=2}^{N} 1_{a,b\in 2\mathbb{N}}\alpha^{a+b}(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{a+b}{2}}\\ &\cdot\left([V_{a,b}^{1-2-1,l},g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}+\sum_{n=2}^{\infty}[V_{a,b}^{1-2-2,l,(n)},g]_{1,r,\varepsilon_\beta^{\hat{N}_\beta-l}r'}\right)\\ &\le c(M^{-2\mathsf{a}}+\alpha^{-2}M^{-2\mathsf{a}})\varepsilon_\beta L^{-d}\|g\|.\end{aligned}$$

If we assume that $M \ge c$, the right-hand side of (4.122), (4.123) is less than 1, $L^{-d}\|g\|$ respectively. By setting $V^{1-2,l} := V^{1-2-1,l} + \sum_{n=2}^{\infty} V^{1-2-2,l,(n)}$ we conclude that

$$V^{1-2,l} \in \mathcal{R}'(r, \varepsilon_\beta^{\hat{N}_\beta-l}r', l).$$

Remind us that we have also assumed $\alpha \ge c$, $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2^3$ on the way to this result.

It remains to analyze $V^{2,l,(n)}$ ($n \in \mathbb{N}_{\ge 1}$). It can be seen from the definition, the induction hypothesis and the conditions $h \ge 1$, $\varepsilon_\beta \le 1$ that

(4.124)

$$\begin{aligned}&V^{2,l,(n)} \in C\left(\overline{D(r)}\times\overline{D(h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r')}^2, \bigwedge_{even}\mathcal{V}\right)\\ &\qquad\cap C^\omega\left(D(r)\times D(h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r')^2, \bigwedge_{even}\mathcal{V}\right),\\ &V^{2,l,(n)}(u,\mathbf{0})(\psi) = \frac{\partial}{\partial\lambda_j}V^{2,l,(n)}(u,\mathbf{0})(\psi) = 0, \quad (\forall j\in\{1,2\},\ u\in D(r)).\end{aligned}$$

For $(u,\boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r')}^2$ Lemma 4.1 implies that the anti-symmetric kernels of $V^{2,l,(n)}(u,\boldsymbol{\lambda})(\psi)$ satisfy (4.4). If $\sum_{n=1}^{\infty} V^{2,l,(n)}$ is uniformly convergent, then $V^{2,l}$ and its kernels automatically satisfy the properties (4.124), (4.4). By (4.7) and (4.33), for any $m \in \{0, 2, \cdots, N\}$,

$$\|V_m^{2,l,(1)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \le \sum_{p=m}^{N} \begin{pmatrix} p \\ m \end{pmatrix} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p-m}{2}} \|V_p^{2,l+1}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'}.$$

By (4.76) for $l+1$, (4.86) and the conditions $h \ge 1$, $M^{\mathsf{a}} \ge 2^4$, $\alpha \ge 2$,

$$\|V_0^{2,l,(1)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \le \sum_{p=0}^{N} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p}{2}} \|V_p^{2,l+1}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \le c\varepsilon_\beta \alpha^{-2}, \tag{4.125}$$

$$\begin{aligned} &\sum_{m=2}^{N} \alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{2,l,(1)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \\ &\le \sum_{m=2}^{N} \alpha^m M^{-\frac{\mathsf{a}}{2}m} \sum_{p=m}^{N} 2^p (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p}{2}} \|V_p^{2,l+1}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \\ &\le \varepsilon_\beta \sum_{m=2}^{N} 2^m M^{-\frac{\mathsf{a}}{2}m} \le c\varepsilon_\beta M^{-\mathsf{a}}. \end{aligned} \tag{4.126}$$

Let $n \in \mathbb{N}_{\ge 2}$. Take $b_j \in \{0, 1, 2\}$ ($j = 1, 2, \cdots, n$) satisfying $\sum_{j=1}^n b_j \ge 2$. Set

$$V^{2,l,(n),(b_j)_{j=1}^n}(\psi) := \frac{1}{n!} Tree(\{1, 2, \cdots, n\}, \mathcal{C}_{l+1}) \prod_{j=1}^{n} V^{b_j, l+1}(\psi^j + \psi) \Big|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}}.$$

There exists $\sigma \in \mathbb{S}_n$ such that $b_{\sigma(1)} \ne 0$. It follows from the definition of the tree expansion (4.1) that

$$V^{2,l,(n),(b_j)_{j=1}^n}(\psi) = \frac{1}{n!} Tree(\{1, 2, \cdots, n\}, \mathcal{C}_{l+1}) \prod_{j=1}^{n} V^{b_{\sigma(j)}, l+1}(\psi^j + \psi) \Big|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}}.$$

We can apply (4.9), (4.33), (4.34) and (4.86) to derive that for any $m \in \{0, 2, \cdots, N\}$

$$\|V_m^{2,l,(n),(b_j)_{j=1}^n}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \tag{4.127}$$
$$\le (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{-n+1-\frac{m}{2}} 2^{-2m} (c_0 M^{(\mathsf{a}-1-\sum_{j=1}^d \frac{1}{\mathsf{n}_j})(l+1-\hat{N}_\beta)})^{n-1}$$
$$\cdot \sum_{p_1=2}^N 2^{3p_1} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_1}{2}} \sum_{a_1=1}^2 \|V_{p_1}^{a_1,l+1}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'}$$
$$\cdot \prod_{j=2}^n \Bigg(\sum_{p_j=2}^N 2^{3p_j} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{p_j}{2}}$$
$$\cdot \Bigg(\|V_{p_j}^{0,l+1}\|_{1,\infty,r} + \sum_{a_j=1}^2 \|V_{p_j}^{a_j,l+1}\|_{1,r,h^{l+1-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \Bigg)\Bigg)$$
$$\cdot 1_{\sum_{j=1}^n p_j - 2(n-1) \ge m}.$$

Here we used the condition $h \ge 2$ to apply (4.86) with $\varepsilon = 1/h$. Since $V^{2,l,(n)}$ is the sum of $V^{2,l,(n),(b_j)_{j=1}^n}$ over possible $(b_j)_{j=1}^n$,

$$\|V_m^{2,l,(n)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'} \le 3^n \cdot (\text{R. H. S of (4.127)}). \tag{4.128}$$

We need to use the following inequalities which are derived from (4.46), (4.47), (4.48), (4.50) and the assumption $L^d \ge M^{(\sum_{j=1}^d \frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta - N_\beta)}$.

$$\sum_{m=2}^N 2^{3m} (c_0 M^{\mathsf{a}(l+1-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{0,l+1}\|_{1,\infty,r} \le c\alpha^{-2} M^{(\sum_{j=1}^d \frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}, \tag{4.129}$$

$$\sum_{m=2}^N 2^m \alpha^m (c_0 M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}} \|V_m^{0,l+1}\|_{1,\infty,r} \le c M^{-\mathsf{a}+(\sum_{j=1}^d \frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}. \tag{4.130}$$

By inserting (4.101), (4.103), (4.129) into (4.128) and recalling (4.84), $h \ge 1$ and $l + 1 - \hat{N}_\beta \le 0$ we have that

$$\|V_0^{2,l,(n)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - l}r'}$$

$$\le cM^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}\varepsilon_\beta\alpha^{-2}$$
$$\cdot\left(c\alpha^{-2}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}+c\varepsilon_\beta\alpha^{-2}\right)^{n-1}$$
$$\le \varepsilon_\beta(c\alpha^{-2})^n,$$

or by assuming that $\alpha \ge c$,

$$\sum_{n=2}^{\infty}\|V_0^{2,l,(n)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta\alpha^{-4}. \tag{4.131}$$

Also by combining (4.102), (4.104), (4.130) with (4.128), using the condition $\alpha M^{-\frac{\mathsf{a}}{2}} \ge 2^3$, $h \ge 1$, $l+1-\hat{N}_\beta \le 0$ and recalling (4.84) we see that

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{2,l,(n)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r'}$$
$$\le c\alpha^{-2(n-1)}M^{\mathsf{a}(n-1)-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)(n-1)}\varepsilon_\beta M^{-\mathsf{a}}$$
$$\cdot\left(cM^{-\mathsf{a}+(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(l+1-\hat{N}_\beta)}+c\varepsilon_\beta M^{-\mathsf{a}}\right)^{n-1}$$
$$\le \varepsilon_\beta M^{-\mathsf{a}}(c\alpha^{-2})^{n-1},$$

or by the condition $\alpha \ge c$,

$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\sum_{n=2}^{\infty}\|V_m^{2,l,(n)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-\mathsf{a}}\alpha^{-2}. \tag{4.132}$$

By summing up (4.125), (4.126), (4.131), (4.132) we obtain that

$$\sum_{n=1}^{\infty}\|V_0^{2,l,(n)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta\alpha^{-2}, \tag{4.133}$$
$$\sum_{m=2}^{N}\alpha^m(c_0M^{\mathsf{a}(l-\hat{N}_\beta)})^{\frac{m}{2}}\sum_{n=1}^{\infty}\|V_m^{2,l,(n)}\|_{1,r,h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r'} \le c\varepsilon_\beta M^{-\mathsf{a}}. \tag{4.134}$$

Under the assumption $M \ge c$, the right-hand side of (4.133), (4.134) becomes less than α^{-2}, 1 respectively. Thus by setting $V^{2,l} := \sum_{n=1}^{\infty}V^{2,l,(n)}$ we conclude that

$$V^{2,l} \in \mathcal{W}(r, h^{l-\hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta-l}r', l).$$

In the l-th step we needed the conditions

$$M \geq c, \quad \alpha \geq cM^{\frac{\mathsf{a}}{2}}, \quad L^d \geq M^{(\sum_{j=1}^{d} \frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta - N_\beta)}, \quad h \geq 2$$

for a positive constant c independent of any parameter. Since we have admitted the results of Lemma 4.5, we have to combine the above conditions with the conditions (4.42). All the conditions are summarized as in (4.85). The induction with l ensures that the claim holds true. □

4.6. The final integration

Here we study properties of an analytic continuation of the function

$$(u, \boldsymbol{\lambda}) \mapsto \log \left(\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)} d\mu_{\sum_{l=N_\beta}^{\hat{N}_\beta} \mathcal{C}_l}(\psi) \right).$$

Since we have constructed an analytic continuation of the Grassmann polynomial

$$\log \left(\int e^{-V(u)(\psi+\psi^1)+W(u)(\psi+\psi^1)-A(\psi+\psi^1)} d\mu_{\sum_{l=N_\beta+1}^{\hat{N}_\beta} \mathcal{C}_l}(\psi^1) \right),$$

we can use it as the input to the single-scale integration with the covariance $\mathcal{C}_{N_\beta}$. Since the constant c_5 is not less than c_4, we can deduce from Lemma 4.5, Lemma 4.7 that under the assumptions of Lemma 4.7,

$$\begin{aligned}
&V^{0-1,N_\beta} \in \mathcal{Q}(b^{-1}c_0^{-2}\alpha^{-4}, N_\beta),\\
&V^{0-2,N_\beta} \in \mathcal{R}(b^{-1}c_0^{-2}\alpha^{-4}, N_\beta),\\
&V^{1-1,N_\beta} \in \mathcal{Q}'(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta - N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4}, N_\beta),\\
&V^{1-2,N_\beta} \in \mathcal{R}'(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta - N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4}, N_\beta),\\
&V^{1-3,N_\beta} \in \mathcal{S}(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta - N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4}, N_\beta),\\
&V^{2,N_\beta} \in \mathcal{W}(b^{-1}c_0^{-2}\alpha^{-4}, c_5^{-1}h^{N_\beta - \hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4}, N_\beta).
\end{aligned} \tag{4.135}$$

Set

$$r := b^{-1}c_0^{-2}\alpha^{-4}, \quad \hat{r} := c_5^{-1}h^{N_\beta - \hat{N}_\beta}\varepsilon_\beta^{\hat{N}_\beta - N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4}.$$

Then let us define the functions $V^{end,(n)} : \overline{D(r)} \times \overline{D(\hat{r})}^2 \to \mathbb{C}$ ($n \in \mathbb{N}_{\geq 1}$), $V^{1-3,end} : \overline{D(\hat{r})}^2 \to \mathbb{C}$ by

$$\begin{aligned} &V^{end,(n)} \\ &:= \frac{1}{n!} Tree(\{1,2,\cdots,n\}, \mathcal{C}_{N_\beta}) \\ &\quad \cdot \prod_{j=1}^{n} \left(\sum_{p=1}^{2} V^{0-p,N_\beta}(\psi^j) + \sum_{q=1}^{3} V^{1-q,N_\beta}(\psi^j) + V^{2,N_\beta}(\psi^j) \right) \Bigg|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}}, \\ &V^{1-3,end} := Tree(\{1\}, \mathcal{C}_{N_\beta}) V^{1-3,N_\beta}(\psi^1) \Big|_{\psi^1 = 0}. \end{aligned}$$

We set $V^{end} := \sum_{n=1}^{\infty} V^{end,(n)}$ if it converges. We conclude this section by summarizing properties of V^{end} in a convenient way for applications in the next section.

LEMMA 4.10. *Let c_5 be the constant appearing in Lemma 4.7. Then there exists a constant $c_6 \in [c_5, \infty)$ independent of any other parameters such that if*

$$\begin{aligned} &M^{\min\{1, 2\mathsf{a} - 1 - \sum_{j=1}^{d} \frac{1}{\mathsf{n}_j}\}} \geq c_6, \quad \alpha \geq c_6 M^{\frac{\mathsf{a}}{2}}, \\ &L^d \geq (c_{end} + 1) M^{(\mathsf{a} + \sum_{j=1}^{d} \frac{1}{\mathsf{n}_j} + 1)(\hat{N}_\beta - N_\beta)}, \quad h \geq 2, \end{aligned} \tag{4.136}$$

the following statements hold true.

(i)

$$\begin{aligned} V^{end} \in &C \left(\overline{D(b^{-1} c_0^{-2} \alpha^{-4})} \times \overline{D(c_6^{-1} L^{-d} h^{N_\beta - \hat{N}_\beta - 1} \varepsilon_\beta^{\hat{N}_\beta - N_\beta} \beta^{-1} c_0^{-2} \alpha^{-4})}^2 \right) \\ &\cap C^\omega \left(D(b^{-1} c_0^{-2} \alpha^{-4}) \times D(c_6^{-1} L^{-d} h^{N_\beta - \hat{N}_\beta - 1} \varepsilon_\beta^{\hat{N}_\beta - N_\beta} \beta^{-1} c_0^{-2} \alpha^{-4})^2 \right). \end{aligned}$$

(ii)

$$\frac{h}{N} \sup_{u \in \overline{D(b^{-1} c_0^{-2} \alpha^{-4})}} |V^{end}(u, \mathbf{0})| \leq c_6 (1 + N^{-1}) \alpha^{-2} M^{(\sum_{j=1}^{d} \frac{1}{\mathsf{n}_j} + 1)(\hat{N}_\beta - N_\beta)} L^{-d}.$$

(iii) For $j \in \{1,2\}$

$$\sup_{u\in D(b^{-1}c_0^{-2}\alpha^{-4})}\left|\frac{\partial}{\partial\lambda_j}V^{end}(u,\mathbf{0})-\frac{\partial}{\partial\lambda_j}V^{1-3,end}(u,\mathbf{0})\right|$$
$$\le c_6\varepsilon_\beta^{N_\beta-\hat{N}_\beta}\beta c_0^2\left(\alpha^2+c_{end}M^{\mathsf{a}(\hat{N}_\beta-N_\beta)}\right)L^{-d}.$$

PROOF. The claims can be proved in the same way as the proof of [12, Lemma 3.8]. However, we provide a sketch of the proof for completeness of the paper. Observe that for any $c_6 \in [c_5,\infty)$ the conditions (4.136) imply (4.85). It follows from the property (4.32) and the property (4.10) of the kernels of V^{0-2,N_β}, V^{1-2,N_β} that for any $z \in \mathbb{C}$

$$\int e^{z\sum_{j=1}^2 V^{0-j,N_\beta}(\psi)+z\sum_{k=1}^3 V^{1-k,N_\beta}(\psi)+zV^{2,N_\beta}(\psi)}d\mu_{\mathcal{C}_{N_\beta}}(\psi)$$
$$=\int e^{zV^{0-1,N_\beta}(\psi)+zV^{1-1,N_\beta}(\psi)+zV^{1-3,N_\beta}(\psi)+zV^{2,N_\beta}(\psi)}d\mu_{\mathcal{C}_{N_\beta}}(\psi).$$

Thus by recalling the formula (4.1),

$$\begin{aligned}&V^{end,(n)}\\&=\frac{1}{n!}Tree(\{1,2,\cdots,n\},\mathcal{C}_{N_\beta})\\&\cdot\prod_{j=1}^n\big(V^{0-1,N_\beta}(\psi^j)+V^{1-1,N_\beta}(\psi^j)\\&\qquad+V^{1-3,N_\beta}(\psi^j)+V^{2,N_\beta}(\psi^j)\big)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.\end{aligned}\tag{4.137}$$

We can use (4.7), (4.9), (4.33), (4.34), (4.36), (4.72), (4.75), (4.76), (4.86) with $\varepsilon = h^{-1}L^{-d}(\le 1/2)$ and the condition $\alpha \ge 2^3$ to derive that

$$\|V^{end,(1)}\|_{1,r,h^{-1}L^{-d}\hat{r}}$$
$$\le \|V_0^{0-1,N_\beta}\|_{1,\infty,r}+h^{-1}L^{-d}\sum_{\delta\in\{1-1,1-3,2\}}\|V_0^{\delta,N_\beta}\|_{1,r,\hat{r}}$$
$$+\sum_{m=2}^N\left(c_0M^{\mathsf{a}(N_\beta-\hat{N}_\beta)}\right)^{\frac{m}{2}}$$

$$
\begin{aligned}
&\cdot \left(\frac{N}{h} \|V_m^{0-1,N_\beta}\|_{1,\infty,r} + h^{-1}L^{-d} \sum_{\delta\in\{1-1,1-3,2\}} \|V_m^{\delta,N_\beta}\|_{1,r,\hat{r}} \right)\\
&\le c\left(\frac{N}{h}\alpha^{-2}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(N_\beta-\hat{N}_\beta)}L^{-d} + h^{-1}L^{-d}\alpha^{-2}\right)\\
&\le ch^{-1}(N+1)\alpha^{-2}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(N_\beta-\hat{N}_\beta)}L^{-d},
\end{aligned}
$$

$$
\begin{aligned}
&\|V^{end,(n)}\|_{1,r,h^{-1}L^{-d}\hat{r}}\\
&\le (c_0M^{\mathsf{a}(N_\beta-\hat{N}_\beta)})^{-n+1}(c_0c_{end})^{n-1}\sum_{p_1=2}^{N}2^{3p_1}(c_0M^{\mathsf{a}(N_\beta-\hat{N}_\beta)})^{\frac{p_1}{2}}\\
&\qquad \cdot \left(\frac{N}{h}\|V_{p_1}^{0-1,N_\beta}\|_{1,\infty,r} + h^{-1}L^{-d}\sum_{\delta\in\{1-1,1-3,2\}}\|V_{p_1}^{\delta,N_\beta}\|_{1,r,\hat{r}}\right)\\
&\quad \cdot \Bigg(\sum_{p=2}^{N} 2^{3p}(c_0M^{\mathsf{a}(N_\beta-\hat{N}_\beta)})^{\frac{p}{2}}\\
&\qquad \cdot \left(\|V_p^{0-1,N_\beta}\|_{1,\infty,r} + L^{-d}\sum_{\delta\in\{1-1,1-3,2\}}\|V_p^{\delta,N_\beta}\|_{1,r,\hat{r}}\right)\Bigg)^{n-1}\\
&\le M^{-\mathsf{a}(N_\beta-\hat{N}_\beta)(n-1)}c_{end}^{n-1}(c\alpha^{-2})^n\left(\frac{N}{h}L^{-d}+h^{-1}L^{-d}\right)L^{-d(n-1)}\\
&\le ch^{-1}(N+1)\alpha^{-2}L^{-d}(cc_{end}\alpha^{-2}M^{-\mathsf{a}(N_\beta-\hat{N}_\beta)}L^{-d})^{n-1}
\end{aligned}
$$

for $n \in \mathbb{N}_{\ge 2}$, or under the assumptions of the lemma,

$$
\frac{h}{N}\sum_{n=1}^{\infty}\|V^{end,(n)}\|_{1,r,h^{-1}L^{-d}\hat{r}} \le c(1+N^{-1})\alpha^{-2}M^{-(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(N_\beta-\hat{N}_\beta)}L^{-d}.
$$

This implies the claims (i), (ii).

To prove the claim (iii), let us set

$$
\overline{r} := c_5^{-1}\varepsilon_\beta^{\hat{N}_\beta-N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4}.
$$

Recalling (4.135), we see that

$$
\frac{\partial}{\partial\lambda_j}V^{\delta,N_\beta}(u,\mathbf{0}) = \frac{1}{\overline{r}}V^{\delta,N_\beta}(u,\overline{r}\mathbf{e}_j), \tag{4.138}
$$

$$(\forall j \in \{1,2\},\ \delta \in \{1-1, 1-3\},\ u \in D(r)),$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are the canonical basis of $\mathbb{C}^2$. We can deduce from (4.7), (4.9), (4.33), (4.34), (4.36), (4.72), (4.75), (4.137), (4.138) and the assumptions of the lemma that for $j \in \{1,2\}$, $u \in D(r)$

$$\begin{aligned}
&\left|\frac{\partial}{\partial \lambda_j} V^{end}(u,\mathbf{0}) - \frac{\partial}{\partial \lambda_j} V^{1-3,end}(u,\mathbf{0})\right| \\
&\le \frac{1}{\overline{r}}\left| Tree(\{1\},\mathcal{C}_{N_\beta}) V^{1-1,N_\beta}(u,\overline{r}\mathbf{e}_j)(\psi^1)\Big|_{\psi^1=0}\right| \\
&\quad + \frac{1}{\overline{r}}\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\Bigg| Tree(\{1,2,\cdots,n\},\mathcal{C}_{N_\beta}) \\
&\qquad\qquad \cdot \sum_{\delta\in\{1-1,1-3\}} V^{\delta,N_\beta}(u,\overline{r}\mathbf{e}_j)(\psi^1) \\
&\qquad\qquad \cdot \prod_{k=2}^{n} V^{0-1,N_\beta}(u)(\psi^k)\Bigg|_{\substack{\psi^k=0\\(\forall k\in\{1,2,\cdots,n\})}}\Bigg| \\
&\le \frac{1}{\overline{r}}\left(\|V_0^{1-1,N_\beta}\|_{1,r,\overline{r}} + \sum_{m=2}^{N}(c_0 M^{\mathsf{a}(N_\beta-\hat{N}_\beta)})^{\frac{m}{2}}\|V_m^{1-1,N_\beta}\|_{1,r,\overline{r}}\right) \\
&\quad + \frac{1}{\overline{r}}\sum_{n=2}^{\infty} M^{-\mathsf{a}(N_\beta-\hat{N}_\beta)(n-1)} c_{end}^{n-1} \\
&\qquad \cdot \sum_{p_1=2}^{N} 2^{3p_1}(c_0 M^{\mathsf{a}(N_\beta-\hat{N}_\beta)})^{\frac{p_1}{2}}(\|V_{p_1}^{1-1,N_\beta}\|_{1,r,\overline{r}} + \|V_{p_1}^{1-3,N_\beta}\|_{1,r,\overline{r}}) \\
&\qquad \cdot \left(\sum_{p=2}^{N} 2^{3p}(c_0 M^{\mathsf{a}(N_\beta-\hat{N}_\beta)})^{\frac{p}{2}}\|V_p^{0-1,N_\beta}\|_{0,\infty,r}\right)^{n-1} \\
&\le \frac{c}{\overline{r}}\alpha^{-2}L^{-d} + \frac{c}{\overline{r}}\alpha^{-2}\sum_{n=2}^{\infty}(c c_{end}\alpha^{-2}M^{-\mathsf{a}(N_\beta-\hat{N}_\beta)}L^{-d})^{n-1} \\
&\le \frac{c}{\overline{r}}\left(\alpha^{-2} + \alpha^{-4}c_{end}M^{-\mathsf{a}(N_\beta-\hat{N}_\beta)}\right)L^{-d}.
\end{aligned}$$

Thus the claim holds true. □

5. Proof of the Theorem

In this section we complete the proof of Theorem 1.3 and Corollary 1.11. Since we have developed the multi-scale integration scheme in the previous section and we plan to apply the convergence result [12, Proposition 4.16], we have the main general tools at hand. We need to confirm that our actual covariance appearing in the formulation Lemma 3.6 can be decomposed into a family of covariances which fit in our framework. The way to complete the proof of Theorem 1.3 after the confirmation is essentially parallel to the proof of [12, Theorem 1.3]. Proving Corollary 1.11 requires some additional arguments which we provide in the end of this section.

From here we assume that

$$h \geq \max\{2, \mathsf{c}\}. \tag{5.1}$$

Since we send h to infinity first, we can assume that h is larger than any other parameter. As we proceed, we will replace (5.1) by stricter conditions.

5.1. Decomposition of the covariance

Let us decompose the covariance characterized in (3.2) into a sum of scale-dependent covariances. We begin with discretizing the time-variables. Let $\mathcal{M}_h$ denote the set of Matsubara frequency with cut-off

$$\left\{\omega \in \frac{\pi}{\beta}(2\mathbb{Z}+1) \;\middle|\; |\omega| < \pi h\right\}.$$

Lemma 5.1. *For any $(\overline{\rho}, \rho, \mathbf{x}, s)$, $(\overline{\eta}, \eta, \mathbf{y}, t) \in \{1,2\} \times \mathcal{B} \times \Gamma \times [0,\beta)_h$, $\phi \in \mathbb{C}$,*

$$\begin{aligned}
&C(\phi)(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t)\\
&= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\mathcal{M}_h} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y}\rangle + i\omega(s-t)}\\
&\qquad \cdot h^{-1}\left(I_{2b} - e^{-\frac{i}{h}(\omega - \frac{\theta(\beta)}{2})I_{2b} + \frac{1}{h}E(\phi)(\mathbf{k})}\right)^{-1}((\overline{\rho}-1)b+\rho, (\overline{\eta}-1)b+\eta).
\end{aligned}$$

Proof. According to [8, Lemma C.3],

$$e^{sA}\left(\frac{1_{s\ge 0}}{1+e^{\beta A}}-\frac{1_{s<0}}{1+e^{-\beta A}}\right)=\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h}\frac{e^{i\omega s}}{h(1-e^{-i\frac{\omega}{h}+\frac{A}{h}})},$$
$$(\forall s\in\{-\beta,-\beta+1/h,\cdots,\beta-1/h\},\ A\in\mathbb{C}\backslash i(\pi/\beta)(2\mathbb{Z}+1)).$$

By diagonalizing $E(\phi)(\mathbf{k})$ in (3.2) by a unitary matrix and substituting this formula we can derive the claimed equality. □

Next let us introduce a cut-off function. Let χ be a real-valued function on $\mathbb{R}$ satisfying the following properties.

$$\begin{aligned}
&\chi\in C^{\infty}(\mathbb{R}),\\
&\chi(x)=1,\quad (\forall x\in(-\infty,8/5]),\\
&\chi(x)\in(0,1),\quad (\forall x\in(8/5,2)),\\
&\chi(x)=0,\quad (\forall x\in[2,\infty)),\\
&\frac{d}{dx}\chi(x)\le 0,\quad (\forall x\in\mathbb{R}).
\end{aligned}$$

Using the function χ, we construct scale-dependent cut-off functions. We use the parameter $M\in\mathbb{R}_{\ge 2}$ to control the support size of the cut-off functions. Here we give explicit definitions of the numbers N_β, $\hat{N}_\beta$ introduced in Subsection 4.3. Let

$$N_\beta:=\left\lfloor\frac{\log(1/\beta)}{\log M}\right\rfloor,\quad \hat{N}_\beta:=1_{\beta\le 1}(N_\beta+1).$$

Moreover, set

$$N_h:=\left\lfloor\frac{\log(h)}{\log M}\right\rfloor+2.$$

Then by (5.1) and the condition $h\ge 1/\beta$ implied by $h\in\frac{2}{\beta}\mathbb{N}$,

$$\begin{aligned}
&N_\beta<\hat{N}_\beta<N_h,\\
&M^{-1}\beta^{-1}\le M^{N_\beta}\le\beta^{-1}.
\end{aligned}\tag{5.2}$$

Set $A(\beta, M) := \beta^{-1}M^{-N_\beta}$. It follows that $1 \le A(\beta, M) \le M$. With the function $e(\cdot) : \mathbb{R}^d \to \mathbb{R}$ we define the functions $\chi_l : \mathbb{R}^{d+1} \to \mathbb{R}$ ($l = N_\beta, N_\beta + 1, \cdots, N_h$) by

$$\begin{aligned}
&\chi_{N_\beta}(\omega, \mathbf{k}) := \chi\left(M^{-N_\beta}A(\beta, M)^{-1}\sqrt{h^2\sin^2\left(\frac{\omega - \pi/\beta}{2h}\right) + e(\mathbf{k})^2}\right),\\
&\chi_l(\omega, \mathbf{k}) := \chi\left(M^{-l}A(\beta, M)^{-1}\sqrt{h^2\sin^2\left(\frac{\omega - \pi/\beta}{2h}\right) + e(\mathbf{k})^2}\right)\\
&\qquad\qquad - \chi\left(M^{-(l-1)}A(\beta, M)^{-1}\sqrt{h^2\sin^2\left(\frac{\omega - \pi/\beta}{2h}\right) + e(\mathbf{k})^2}\right),\\
&((\omega, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d,\ l \in \{N_\beta + 1, N_\beta + 2, \cdots, N_h\}).
\end{aligned}$$

Keeping in mind that $2A(\beta, M)M^{l-1} < \frac{8}{5}A(\beta, M)M^l$, we observe that

$$\begin{aligned}
(5.3)\qquad &\chi_{N_\beta}(\omega, \mathbf{k})\\
&= \begin{cases}
1 & \text{if } \sqrt{h^2\sin^2\left(\frac{\omega-\pi/\beta}{2h}\right) + e(\mathbf{k})^2} \le \frac{8}{5}A(\beta, M)M^{N_\beta},\\
\in (0, 1) & \text{if } \frac{8}{5}A(\beta, M)M^{N_\beta} < \sqrt{h^2\sin^2\left(\frac{\omega-\pi/\beta}{2h}\right) + e(\mathbf{k})^2}\\
& \qquad < 2A(\beta, M)M^{N_\beta},\\
0 & \text{if } \sqrt{h^2\sin^2\left(\frac{\omega-\pi/\beta}{2h}\right) + e(\mathbf{k})^2} \ge 2A(\beta, M)M^{N_\beta},
\end{cases}\\
&\chi_l(\omega, \mathbf{k})\\
&= \begin{cases}
0 & \text{if } \sqrt{h^2\sin^2\left(\frac{\omega-\pi/\beta}{2h}\right) + e(\mathbf{k})^2} \le \frac{8}{5}A(\beta, M)M^{l-1},\\
\in (0, 1] & \text{if } \frac{8}{5}A(\beta, M)M^{l-1} < \sqrt{h^2\sin^2\left(\frac{\omega-\pi/\beta}{2h}\right) + e(\mathbf{k})^2}\\
& \qquad < 2A(\beta, M)M^{l},\\
0 & \text{if } \sqrt{h^2\sin^2\left(\frac{\omega-\pi/\beta}{2h}\right) + e(\mathbf{k})^2} \ge 2A(\beta, M)M^{l},
\end{cases}\\
&(\forall(\omega, \mathbf{k}) \in \mathbb{R}^{d+1},\quad l \in \{N_\beta + 1, N_\beta + 2, \cdots, N_h\}).
\end{aligned}$$

Basic properties of χ_l are summarized as follows.

Lemma 5.2. *Assume that*

$$L \ge \frac{\max_{j\in\{1,2,\cdots,d\}} M^{-N_\beta/\mathsf{n}_j}}{\min\{M^{\mathsf{a}N_\beta},1\}}. \tag{5.4}$$

Then there exists a positive constant $\hat{c}$ *depending only on* d, M, χ, c, a, $(\hat{\mathbf{v}}_j)_{j=1}^d$ *such that the following statements hold.*

(i)

$$\chi_l \in C^\infty(\mathbb{R}^{d+1}), \quad (\forall l \in \{N_\beta, N_\beta+1,\cdots,N_h\}).$$

(ii)

$$\sum_{l=N_\beta}^{N_h} \chi_l(\omega,\mathbf{k}) = 1, \quad (\forall(\omega,\mathbf{k})\in\mathbb{R}^{d+1}).$$

(iii)

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial\omega}\right)^n \chi_l\left(\omega,\sum_{j=1}^d \hat{k}_j\hat{\mathbf{v}}_j\right)\right| \le \hat{c}M^{-nl},\\
&\left|\left(\frac{\partial}{\partial\hat{k}_i}\right)^n \chi_l\left(\omega,\sum_{j=1}^d \hat{k}_j\hat{\mathbf{v}}_j\right)\right| \le \hat{c}(1_{l\ge 0}M^{-\frac{l}{\mathsf{n}_i}} + 1_{l<0}M^{-\frac{n}{\mathsf{n}_i}l}),\\
&(\forall n\in\{1,2,\cdots,d+2\},\ i\in\{1,2,\cdots,d\},\ (\omega,\hat{k}_1,\cdots,\hat{k}_d)\in\mathbb{R}^{d+1},\\
&\quad l\in\{N_\beta,N_\beta+1,\cdots,N_h\}).
\end{aligned}$$

(iv)

$$\begin{aligned}
&\frac{1}{\beta}\sup_{\mathbf{k}\in\mathbb{R}^d}\sum_{\omega\in\mathcal{M}_h} 1_{\sum_{j=N_\beta}^{l}\chi_j(\omega,\mathbf{k})\neq 0} \le \hat{c}M^l,\\
&\frac{1}{\beta L^d}\sup_{\substack{x\in\mathbb{R}\\ \mathbf{p}\in\mathbb{R}^d}}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h} 1_{\chi_l(\omega+x,\mathbf{k}+\mathbf{p})\neq 0} \le \hat{c}M^l\min\{M^{\mathsf{a}l},1\},\\
&(\forall l\in\{N_\beta,N_\beta+1,\cdots,N_h\}).
\end{aligned}$$

(v)

$$\chi_{N_\beta}(\omega, \mathbf{k}) = 0, \quad (\forall \omega \in \mathcal{M}_h \backslash \{\pi/\beta\},\ \mathbf{k} \in \mathbb{R}^d).$$

(vi) Let $l \in \{N_\beta + 1, N_\beta + 2, \cdots, N_h\}$, $(\omega, \mathbf{k}) \in \mathbb{R}^{d+1}$. *If*

$$\frac{8}{5} A(\beta, M) M^{l-1} \le \sqrt{h^2 \sin^2\left(\frac{\omega - \pi/\beta}{2h}\right) + e(\mathbf{k})^2},$$

then

$$\sqrt{\left(\omega - \frac{\theta(\beta)}{2}\right)^2 + e(\mathbf{k})^2} \ge \left(\frac{8}{5} - \frac{\pi}{2}\right) A(\beta, M) M^{l-1} > 0. \tag{5.5}$$

Especially if $\chi_l(\omega, \mathbf{k}) \neq 0$, (5.5) *holds.*

Let us prepare a useful inequality beforehand.

Lemma 5.3. *Let* $f \in C^1(\mathbb{R}^d, \mathbb{C})$ *satisfy*

$$\sup_{j \in \{1,2,\cdots,d\}} \sup_{\mathbf{k} \in \mathbb{R}^d} \left| \frac{\partial}{\partial k_j} f(\mathbf{k}) \right| < \infty.$$

Then the following inequality holds.

$$\begin{aligned}
&\left| D_d \int_{\Gamma_\infty^*} d\mathbf{p} f(\mathbf{p}) - \frac{1}{L^d} \sum_{\mathbf{k} \in \Gamma^*} f(\mathbf{k}) \right| \\
&\le L^{-1} 2\pi d^2 \sup_{i \in \{1,2,\cdots,d\}} \|\hat{\mathbf{v}}_i\|_{\mathbb{R}^d} \sup_{j \in \{1,2,\cdots,d\}} \sup_{\mathbf{k} \in \mathbb{R}^d} \left| \frac{\partial}{\partial k_j} f(\mathbf{k}) \right|.
\end{aligned}$$

Proof. Observe that

$$\begin{aligned}
&\Bigg| \int_0^{2\pi} dk_1 \int_{[0,2\pi]^{d-1}} dk_2 \cdots dk_d f\left(\sum_{j=1}^d k_j \hat{\mathbf{v}}_j\right) \\
&\quad - \frac{2\pi}{L} \sum_{m_1=0}^{L-1} \int_{[0,2\pi]^{d-1}} dk_2 \cdots dk_d f\left(\frac{2\pi}{L} m_1 \hat{\mathbf{v}}_1 + \sum_{j=2}^d k_j \hat{\mathbf{v}}_j\right) \Bigg|
\end{aligned}$$

$$\le \sum_{m_1=0}^{L-1} \int_{\frac{2\pi}{L}m_1}^{\frac{2\pi}{L}(m_1+1)} dk_1 \int_{[0,2\pi]^{d-1}} dk_2 \cdots dk_d \int_{\frac{2\pi}{L}m_1}^{k_1} dq_1$$

$$\cdot \left| \frac{\partial}{\partial q_1} f\left(q_1 \hat{\mathbf{v}}_1 + \sum_{j=2}^{d} k_j \hat{\mathbf{v}}_j \right) \right|$$

$$\le L^{-1}(2\pi)^{d+1} d \sup_{i\in\{1,2,\cdots,d\}} \|\hat{\mathbf{v}}_i\|_{\mathbb{R}^d} \sup_{j\in\{1,2,\cdots,d\}} \sup_{\mathbf{k}\in\mathbb{R}^d} \left| \frac{\partial}{\partial k_j} f(\mathbf{k}) \right|.$$

By repeating this type of estimate d times we have that

$$\left| \int_{[0,2\pi]^d} d\mathbf{k} f\left(\sum_{j=1}^{d} k_j \hat{\mathbf{v}}_j \right) - \left(\frac{2\pi}{L} \right)^d \sum_{\mathbf{k}\in\Gamma^*} f(\mathbf{k}) \right|$$

$$\le L^{-1}(2\pi)^{d+1} d^2 \sup_{i\in\{1,2,\cdots,d\}} \|\hat{\mathbf{v}}_i\|_{\mathbb{R}^d} \sup_{j\in\{1,2,\cdots,d\}} \sup_{\mathbf{k}\in\mathbb{R}^d} \left| \frac{\partial}{\partial k_j} f(\mathbf{k}) \right|.$$

By combining this inequality with the equality

$$\frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} d\mathbf{k} f\left(\sum_{j=1}^{d} k_j \hat{\mathbf{v}}_j \right) = D_d \int_{\Gamma_\infty^*} d\mathbf{p} f(\mathbf{p})$$

we obtain the result. □

PROOF OF LEMMA 5.2. (i): The claim follows from the assumptions $e^2 \in C^\infty(\mathbb{R}^d)$, $\chi \in C^\infty(\mathbb{R})$ and that $\chi(\cdot)$ is constant in a neighborhood of the origin.

(ii): By (1.7) and the assumption (5.1), for any $(\omega, \mathbf{k}) \in \mathbb{R}^{d+1}$

$$\sqrt{h^2 \sin^2\left(\frac{\omega - \pi/\beta}{2h} \right) + e(\mathbf{k})^2} \le \sqrt{2}h \le \frac{8}{5} A(\beta, M) M^{N_h}.$$

Thus

$$\sum_{l=N_\beta}^{N_h} \chi_l(\omega, \mathbf{k}) = \chi\left(M^{-N_h} A(\beta, M)^{-1} \sqrt{h^2 \sin^2\left(\frac{\omega - \pi/\beta}{2h} \right) + e(\mathbf{k})^2} \right) = 1.$$

(iii): We use the following formula. See e.g. [10, Lemma C.1] for the proof. Let Ω_1, Ω_2 be open sets of $\mathbb{R}$. Let $f_j \in C^\infty(\Omega_j, \mathbb{R})$ $(j = 1, 2)$ and $f_1(\Omega_1) \subset \Omega_2$. Then for $x_0 \in \Omega_1$, $n \in \mathbb{N}$,

$$\begin{aligned}(5.6)\qquad &\left(\frac{d}{dx}\right)^n f_2(f_1(x))\Big|_{x=x_0}\\ &= \sum_{m=1}^{n} \frac{n!}{m!} f_2^{(m)}(f_1(x_0)) \prod_{j=1}^{m} \left(\sum_{l_j=1}^{n} \frac{1}{l_j!} f_1^{(l_j)}(x_0) \right) 1_{\sum_{j=1}^{m} l_j = n}.\end{aligned}$$

Take $l \in \mathbb{Z}$ satisfying $l \le N_h$. Define the function $g : \mathbb{R}^{d+1} \to \mathbb{R}$ by

$$g(\omega, \mathbf{k}) := h^2 \sin^2\left(\frac{\omega - \pi/\beta}{2h}\right) + e(\mathbf{k})^2.$$

Since g is continuous, $g^{-1}(((\frac{8}{5}A(\beta, M)M^l)^2, (2A(\beta, M)M^l)^2))$ is an open set of $\mathbb{R}^{d+1}$. Assume that $(\omega, \mathbf{k}) \in g^{-1}(((\frac{8}{5}A(\beta, M)M^l)^2, (2A(\beta, M)M^l)^2))$. Take any $i \in \{1, 2, \cdots, d\}$, $n \in \{1, 2, \cdots, d + 2\}$. Let us estimate $|(\frac{\partial}{\partial \hat{k}_i})^n \sqrt{g(\omega, \mathbf{k})}|$, $|(\frac{\partial}{\partial \omega})^n \sqrt{g(\omega, \mathbf{k})}|$ where $\mathbf{k} = \sum_{j=1}^{d} \hat{k}_j \hat{\mathbf{v}}_j$. Since $h^{-1} \le M^{-N_h+2} \le M^{-l+2}$,

$$\left| \left(\frac{\partial}{\partial \omega}\right)^n g(\omega, \mathbf{k}) \right| \le c(d, M) M^{l(2-n)}.$$

Then by applying (5.6) we have that

$$\begin{aligned}(5.7)\qquad &\left| \left(\frac{\partial}{\partial \omega}\right)^n \sqrt{g(\omega, \mathbf{k})} \right| \le c(d, M) \sum_{m=1}^{n} M^{2l(\frac{1}{2}-m)} \prod_{j=1}^{m} \left(\sum_{l_j=1}^{n} M^{l(2-l_j)} \right) 1_{\sum_{j=1}^{m} l_j = n}\\ &\qquad \le c(d, M) M^{l(1-n)}.\end{aligned}$$

Note that by (1.8)

$$\left| \left(\frac{\partial}{\partial \hat{k}_i}\right)^n g(\omega, \mathbf{k}) \right| \le c(M, \mathsf{c}) \left(1_{n \le 2\mathsf{n}_i} M^{l(2-\frac{n}{\mathsf{n}_i})} + 1_{n > 2\mathsf{n}_i} \right).$$

Thus we can apply (5.6) to derive that

$$(5.8)\qquad \left| \left(\frac{\partial}{\partial \hat{k}_i}\right)^n \sqrt{g(\omega, \mathbf{k})} \right|$$

$$\le c(d,M,\mathsf{c})\sum_{m=1}^{n}M^{2l(\frac{1}{2}-m)}\prod_{j=1}^{m}$$
$$\cdot\left(\sum_{l_j=1}^{n}\left(1_{l_j\le 2\mathsf{n}_i}M^{l(2-\frac{l_j}{\mathsf{n}_i})}+1_{l_j>2\mathsf{n}_i}\right)\right)1_{\sum_{j=1}^{m}l_j=n}$$
$$\le c(d,M,\mathsf{c})\sum_{m=1}^{n}M^{2l(\frac{1}{2}-m)}\prod_{j=1}^{m}\left(\sum_{l_j=1}^{n}\right)1_{\sum_{j=1}^{m}l_j=n}$$
$$\cdot\left(1_{l\ge 0}M^{2ml-\frac{m}{\mathsf{n}_i}l}+1_{l<0}M^{l(2m-\frac{n}{\mathsf{n}_i})-\sum_{j=1}^{m}1_{l_j>2\mathsf{n}_i}l(2-\frac{l_j}{\mathsf{n}_i})}\right)$$
$$\le c(d,M,\mathsf{c})\sum_{m=1}^{n}M^{2l(\frac{1}{2}-m)}\left(1_{l\ge 0}M^{2ml-\frac{m}{\mathsf{n}_i}l}+1_{l<0}M^{l(2m-\frac{n}{\mathsf{n}_i})}\right)$$
$$\le c(d,M,\mathsf{c})\left(1_{l\ge 0}M^{(1-\frac{1}{\mathsf{n}_i})l}+1_{l<0}M^{(1-\frac{n}{\mathsf{n}_i})l}\right).$$

Moreover we can use (5.6), (5.7), (5.8) to deduce that

$$\left|\left(\frac{\partial}{\partial\omega}\right)^{n}\chi(M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega,\mathbf{k})})\right|$$
$$\le c(d,M,\chi)\sum_{m=1}^{n}M^{-ml}\prod_{j=1}^{m}\left(\sum_{l_j=1}^{n}M^{l(1-l_j)}\right)1_{\sum_{j=1}^{m}l_j=n}$$
$$\le c(d,M,\chi)M^{-nl},$$
$$\left|\left(\frac{\partial}{\partial\hat{k}_i}\right)^{n}\chi(M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega,\mathbf{k})})\right|$$
$$\le c(d,M,\chi,\mathsf{c})$$
$$\cdot\sum_{m=1}^{n}M^{-ml}\prod_{j=1}^{m}\left(\sum_{l_j=1}^{n}\left(1_{l\ge 0}M^{(1-\frac{1}{\mathsf{n}_i})l}+1_{l<0}M^{(1-\frac{l_j}{\mathsf{n}_i})l}\right)\right)1_{\sum_{j=1}^{m}l_j=n}$$
$$\le c(d,M,\chi,\mathsf{c})\left(1_{l\ge 0}M^{-\frac{l}{\mathsf{n}_i}}+1_{l<0}M^{-\frac{n}{\mathsf{n}_i}l}\right).$$

On the other hand,
if $(\omega,\mathbf{k})\in\mathbb{R}^{d+1}\backslash g^{-1}(((\frac{8}{5}A(\beta,M)M^{l})^2,(2A(\beta,M)M^{l})^2))$,

$$\left(\frac{\partial}{\partial\omega}\right)^{n}\chi(M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega,\mathbf{k})})$$

$$= \left(\frac{\partial}{\partial \hat{k}_i}\right)^n \chi(M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega,\mathbf{k})}) = 0.$$

Thus by summing up,

(5.9)

$$\left|\left(\frac{\partial}{\partial \omega}\right)^n \chi\left(M^{-l}A(\beta,M)^{-1}\sqrt{g\left(\omega,\sum_{j=1}^d \hat{k}_j\hat{\mathbf{v}}_j\right)}\right)\right| \le c(d,M,\chi)M^{-nl},$$

$$\left|\left(\frac{\partial}{\partial \hat{k}_i}\right)^n \chi\left(M^{-l}A(\beta,M)^{-1}\sqrt{g\left(\omega,\sum_{j=1}^d \hat{k}_j\hat{\mathbf{v}}_j\right)}\right)\right|$$

$$\le c(d,M,\chi,\mathsf{c})\left(1_{l\ge 0}M^{-\frac{l}{\mathsf{n}_i}} + 1_{l<0}M^{-\frac{n}{\mathsf{n}_i}l}\right),$$

$$(\forall(\omega,\hat{k}_1,\cdots,\hat{k}_d)\in\mathbb{R}^{d+1},\ i\in\{1,2,\cdots,d\},\ n\in\{1,2,\cdots,d+2\}).$$

This implies the claimed results.

(iv): By (5.2), the inequality $A(\beta,M)\le M$ and the support property of $\chi(\cdot)$,

$$\frac{1}{\beta}\sup_{\mathbf{k}\in\mathbb{R}^d}\sum_{\omega\in\mathcal{M}_h} 1_{\sum_{j=N_\beta}^{l}\chi_j(\omega,\mathbf{k})\neq 0} \le \frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h} 1_{|\omega-\pi/\beta|\le 2\pi M^{l+1}} \le c(M)M^l,$$

which is the first inequality. Note that by (5.3), the support property of $\chi(\cdot)$, $A(\beta,M)\le M$ and Lemma 5.3

$$\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h} 1_{\chi_l(\omega+x,\mathbf{k}+\mathbf{p})\neq 0}$$

$$\le \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h} \chi(2^{-1}M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega+x,\mathbf{k}+\mathbf{p})})$$

$$\le \frac{D_d}{\beta}\sum_{\omega\in\mathcal{M}_h}\int_{\Gamma_\infty^*} d\mathbf{k}\chi(2^{-1}M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega+x,\mathbf{k}+\mathbf{p})})$$

$$+\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h}\Bigg|D_d\int_{\Gamma_\infty^*} d\mathbf{k}\chi(2^{-1}M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega+x,\mathbf{k}+\mathbf{p})})$$

$$-\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\chi(2^{-1}M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega+x,\mathbf{k}+\mathbf{p})})\Bigg|$$

$$
\begin{aligned}
&\le c(d,D_d,(\hat{\mathbf{v}}_j)_{j=1}^d)\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h}1_{h|\sin(\frac{1}{2h}(\omega-\frac{\pi}{\beta}+x))|\le 2^2M^{l+1}}\\
&\quad\cdot\Bigg(\int_{\Gamma_\infty^*}d\mathbf{k}1_{e(\mathbf{k}+\mathbf{p})\le 2^2M^{l+1}}\\
&\qquad+L^{-1}\sup_{j\in\{1,2,\cdots,d\}}\sup_{(\omega,\mathbf{k})\in\mathbb{R}^{d+1}}\left|\frac{\partial}{\partial k_j}\chi(2^{-1}M^{-l}A(\beta,M)^{-1}\sqrt{g(\omega,\mathbf{k})})\right|\Bigg).
\end{aligned}
$$

Moreover by using (1.10), (5.2), (5.4) and a simple variant of (5.9) having $2^{-2}g(\cdot)$ in place of $g(\cdot)$ we have that

$$
\begin{aligned}
&\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h}1_{\chi_l(\omega+x,\mathbf{k}+\mathbf{p})\neq 0}\\
&\le c(d,M,\chi,\mathsf{c},D_d,(\hat{\mathbf{v}}_j)_{j=1}^d,\mathsf{a})M^l\left(\min\{M^{\mathsf{a}l},1\}+L^{-1}\max_{j\in\{1,2,\cdots,d\}}M^{-\frac{N_\beta}{\mathsf{n}_j}}\right)\\
&\le c(d,M,\chi,\mathsf{c},D_d,(\hat{\mathbf{v}}_j)_{j=1}^d,\mathsf{a})M^l\min\{M^{\mathsf{a}l},1\},
\end{aligned}
$$

which is the second inequality.

(v): Take any $\omega\in\mathcal{M}_h\backslash\{\pi/\beta\}$, $\mathbf{k}\in\mathbb{R}^d$. It follows from the inequality $|\sin x|\ge\frac{2}{\pi}|x|$, $(x\in[-\frac{\pi}{2},\frac{\pi}{2}])$ and the definition of $A(\beta,M)$ that

$$
M^{-N_\beta}A(\beta,M)^{-1}\sqrt{g(\omega,\mathbf{k})}\ge\beta h\left|\sin\left(\frac{\omega-\pi/\beta}{2h}\right)\right|\ge 2.
$$

Then we can deduce the claim from (5.3).

(vi): By the assumption, the definition of $A(\beta,M)$ and the triangle inequality of the norm $\|\cdot\|_{\mathbb{R}^2}$,

$$
\begin{aligned}
\frac{8}{5}A(\beta,M)M^{l-1}&\le\sqrt{g(\omega,\mathbf{k})}\le\left(\frac{1}{4}\left(\omega-\frac{\pi}{\beta}\right)^2+e(\mathbf{k})^2\right)^{\frac{1}{2}}\\
&\le\left(\frac{1}{4}\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2\right)^{\frac{1}{2}}+\frac{1}{2}\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|\\
&\le\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2\right)^{\frac{1}{2}}+\frac{\pi}{2\beta}
\end{aligned}
$$

$$\leq \left(\left(\omega - \frac{\theta(\beta)}{2}\right)^2 + e(\mathbf{k})^2\right)^{\frac{1}{2}} + \frac{\pi}{2}A(\beta, M)M^{l-1},$$

which implies the result. □

REMARK 5.4. In fact we set up the support of the function χ in order that we can explicitly prove Lemma 5.2 (v),(vi).

We fix $\phi \in \mathbb{C}$ in the following unless otherwise stated. Using the cut-off functions χ_l $(l = N_\beta, N_\beta + 1, \cdots, N_h)$, let us define the covariances $C_l : I_0^2 \to \mathbb{C}$ $(l = N_\beta, N_\beta + 1, \cdots, N_h)$ by

$$\begin{aligned}
&C_l(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) \\
&:= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\mathcal{M}_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle + i(\omega - \frac{\pi}{\beta})(s-t)} \chi_l(\omega, \mathbf{k}) \\
&\quad \cdot h^{-1}\left(I_{2b} - e^{-\frac{i}{h}(\omega - \frac{\theta(\beta)}{2})I_{2b} + \frac{1}{h}E(\phi)(\mathbf{k})}\right)^{-1}((\overline{\rho}-1)b + \rho, (\overline{\eta}-1)b + \eta).
\end{aligned}$$

By Lemma 5.1 and Lemma 5.2 (ii),

$$\sum_{l=N_\beta}^{N_h} C_l(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) = e^{-i\frac{\pi}{\beta}(s-t)} C(\phi)(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t), \tag{5.10}$$
$$(\forall (\overline{\rho}, \rho, \mathbf{x}, s), (\overline{\eta}, \eta, \mathbf{y}, t) \in I_0).$$

We collect basic properties of C_l in the next lemma. During the proof and in subsequent arguments we will need to consider a function on $(\{1,2\}\times\mathcal{B})^2$ as a $2b\times 2b$ matrix and measure the function by using the norm $\|\cdot\|_{2b\times 2b}$. Let us set the rule for this identification. For any $j \in \{1, 2, \cdots, 2b\}$ there uniquely exists $(\overline{\rho}, \rho) \in \{1,2\}\times\mathcal{B}$ such that $j = (\overline{\rho}-1)b+\rho$. This defines the bijection $\varphi : \{1, 2, \cdots, 2b\} \to \{1,2\}\times\mathcal{B}$. We identify a function $f : (\{1,2\}\times\mathcal{B})^2 \to \mathbb{C}$ with the $2b \times 2b$ matrix $(f(\varphi(i), \varphi(j)))_{1\leq i,j\leq 2b}$. We will apply this rule to $C_l(\cdot\mathbf{x}s, \cdot\mathbf{y}t) : (\{1,2\}\times\mathcal{B})^2 \to \mathbb{C}$ for fixed $(\mathbf{x}, s), (\mathbf{y}, t) \in \Gamma \times [0, \beta)_h$ in particular.

LEMMA 5.5. *Assume that*

$$h \geq \max\left\{2, \mathsf{c}, \sup_{\mathbf{k}\in\mathbb{R}^d} \|E(\phi)(\mathbf{k})\|_{2b\times 2b}\right\}, \tag{5.11}$$

$$L \ge \max\left\{\frac{\max_{j\in\{1,2,\cdots,d\}} M^{-N_\beta/\mathsf{n}_j}}{\min\{M^{\mathsf{a}N_\beta},1\}}, \frac{\max_{j\in\{1,2,\cdots,d\}} M^{-N_\beta/\mathsf{n}_j}|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}|^{-1}+|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}|^{-3}}{\min\{M^{(\mathsf{a}-1)N_\beta},1,|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}|^{-1}\}}\right\}. \tag{5.12}$$

Then there exists a positive constant $\hat{c}$ *depending only on* d, b, $(\hat{\mathbf{v}}_j)_{j=1}^d$, a, $(\mathsf{n}_j)_{j=1}^d$, c, M, χ *such that the following statements hold true.*

(i)

$$\begin{aligned} &|\det(\langle \mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m} C_l(X_i,Y_j))_{1\le i,j\le n}| \\ &\le \left(\hat{c}\left(1_{l\neq N_\beta}\min\{M^{\mathsf{a}l},1\} + 1_{l=N_\beta}\beta^{-1}\min\left\{M^{(\mathsf{a}-1)N_\beta},1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}\right)\right)^n, \end{aligned} \tag{5.13}$$

$$\begin{aligned} &\left|\det\left(\langle \mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}\sum_{p=N_\beta}^{\hat{N}_\beta-1} C_p(X_i,Y_j)\right)_{1\le i,j\le n}\right| \\ &\le \left(\hat{c}M^{\hat{N}_\beta}\min\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}\right)^n, \\ &(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m \text{ with } \|\mathbf{u}_i\|_{\mathbb{C}^m},\|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1, \\ &\quad X_i,Y_i\in I_0\ (i=1,2,\cdots,n),\ l\in\{N_\beta,N_\beta+1,\cdots,N_h\}). \end{aligned} \tag{5.14}$$

(ii)

$$\|\tilde{C}_l\|_{1,\infty} \le \hat{c}\left(1_{l\ge 0}M^{-l}+1_{l<0}M^{(\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})l}\right), \tag{5.15}$$

$$(\forall l\in\{N_\beta+1,N_\beta+2,\cdots,N_h\}),$$

$$\|\tilde{C}_{N_\beta}\|_{1,\infty} \le \hat{c}\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\prod_{j=1}^d\left(1+\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-\frac{1}{\mathsf{n}_j}}\right). \tag{5.16}$$

(iii)

$$\|\tilde{C}_l\| \le \hat{c}\left(1_{l\ge 0} + 1_{l<0}M^{(\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j})l}\right), \tag{5.17}$$
$$(\forall l \in \{N_\beta+1, N_\beta+2, \cdots, N_h\}).$$

$$\left\|\sum_{p=l'}^{N_h}\tilde{C}_p\right\| \le \hat{c}\left(\sup_{\substack{(\overline{\rho},\rho,s,\xi),(\overline{\eta},\eta,t,\zeta)\\ \in\{1,2\}\times\mathcal{B}\times[0,\beta)_h\times\{1,-1\}}}\left|\sum_{p=l'}^{N_h}\tilde{C}_p(\overline{\rho}\rho\mathbf{0}s\xi,\overline{\eta}\eta\mathbf{0}t\zeta)\right| + 1\right), \tag{5.18}$$
$$(\forall l' \in \{N_\beta+1, N_\beta+2, \cdots, N_h\}\cap\mathbb{Z}_{\ge 0}).$$

Remark 5.6. Here we need to assume that h is large depending on ϕ as stated in (5.11). This is a notable difference from [12, Lemma 4.10] where we had no condition depending on ϕ. We assume the ϕ-dependent condition (5.11) in order to simplify the proof of the lemma. We can see from Lemma 3.6 (iii) that this condition does not affect our goal since we take the limit $h \to \infty$ before the integration with ϕ in the final formulation.

Proof of Lemma 5.5. Let $x \in [-\pi h, \pi h]$, $\mathbf{k} \in \mathbb{R}^d$, $\delta \in \{1,-1\}$ and $e_\rho(\mathbf{k})$ be an eigenvalue of $E(\mathbf{k})$. By the condition (5.11), $h \ge \sqrt{e_\rho(\mathbf{k})^2 + |\phi|^2}$. Also, $|\frac{1}{2h}(x - \frac{\theta(\beta)}{2})| \le \frac{\pi}{2} + \frac{\pi}{2h\beta} \le \frac{3\pi}{4}$. By using these inequalities and (1.6),

(5.19)

$$\begin{aligned}
&\left|h\left(1 - e^{-\frac{i}{h}(x-\frac{\theta(\beta)}{2})+\frac{\delta}{h}\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}}\right)\right|^2\\
&\ge h^2\left(1 - e^{-\frac{1}{h}\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}}\right)^2 + 4h^2 e^{-\frac{1}{h}\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}}\sin^2\left(\frac{1}{2h}\left(x - \frac{\theta(\beta)}{2}\right)\right)\\
&\ge 1_{\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}>h}h^2(1-e^{-1})^2\\
&\quad + 1_{\sqrt{e_\rho(\mathbf{k})^2+|\phi|^2}\le h}\\
&\qquad\cdot\left(e^{-2}(e_\rho(\mathbf{k})^2+|\phi|^2) + 4h^2e^{-1}\sin^2\left(\frac{1}{2h}\left(x-\frac{\theta(\beta)}{2}\right)\right)\right)\\
&\ge c\left(e(\mathbf{k})^2 + |\phi|^2 + \left(x - \frac{\theta(\beta)}{2}\right)^2\right).
\end{aligned}$$

It follows from this inequality and (1.9) that

(5.20)

$$\left\| h^{-1}\left(I_{2b}-e^{-\frac{i}{h}(x-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}\right)^{-1}\right\|_{2b\times 2b}$$
$$\le c\left(\left(x-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}},$$
$$(\forall (x,\mathbf{k})\in[-\pi h,\pi h]\times\mathbb{R}^d \text{ satisfying } (x-\theta(\beta)/2)^2+e(\mathbf{k})^2\neq 0),$$

(5.21)

$$\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^n E(\phi)\left(\sum_{i=1}^d \hat{k}_i\hat{\mathbf{v}}_i\right)\right\|_{2b\times 2b}\le \mathsf{c}\left(1_{n\le \mathsf{n}_j}e\left(\sum_{i=1}^d\hat{k}_i\hat{\mathbf{v}}_i\right)^{1-\frac{n}{\mathsf{n}_j}}+1_{\mathsf{n}_j<n}\right),$$
$$(\forall(\hat{k}_1,\cdots,\hat{k}_d)\in\mathbb{R}^d,\ n\in\{1,2,\cdots,d+2\},\ j\in\{1,2,\cdots,d\}).$$

The following inequality will also be useful.

$$\left|\omega-\frac{\theta(\beta)}{2}\right|\ge h\left|\sin\left(\frac{\omega-\pi/\beta}{2h}\right)\right|,\quad (\forall\omega\in\mathcal{M}_h). \tag{5.22}$$

(i): Set the Hilbert space $\mathcal{H}$ by $\mathcal{H}:=L^2(\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathcal{M}_h)$. The inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ of $\mathcal{H}$ is defined by

$$\langle f,g\rangle_{\mathcal{H}}:=\frac{1}{\beta L^d}\sum_{K\in\{1,2\}\times\mathcal{B}\times\Gamma^*\times\mathcal{M}_h}\overline{f(K)}g(K),\quad (f,g\in\mathcal{H}).$$

We are going to apply Gram's inequality. Let us define f_X^l, g_X^l, $f_X^>$, $g_X^>\in\mathcal{H}$ $(X\in I_0,\ l\in\{N_\beta,N_\beta+1,\cdots,N_h\})$ by

$$f^\xi_{\overline{\rho}\rho\mathbf{x}s}(\overline{\eta},\eta,\mathbf{k},\omega)$$
$$:=e^{-i\langle\mathbf{k},\mathbf{x}\rangle-is(\omega-\frac{\pi}{\beta})}$$
$$\cdot\left(1_{\xi\in\{N_\beta,N_\beta+1,\cdots,N_h\}}\chi_\xi(\omega,\mathbf{k})^{\frac{1}{2}}+1_{\xi=>}\left(\sum_{j=N_\beta}^{\hat{N}_\beta-1}\chi_j(\omega,\mathbf{k})\right)^{\frac{1}{2}}\right)$$
$$\cdot 1_{(\overline{\rho},\rho)=(\overline{\eta},\eta)}\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{4}},$$

$$
\begin{aligned}
&g^{\xi}_{\overline{\rho}\rho\mathbf{x}s}(\overline{\eta},\eta,\mathbf{k},\omega)\\
&:= e^{-i\langle\mathbf{k},\mathbf{x}\rangle-is(\omega-\frac{\pi}{\beta})}\\
&\quad\cdot\left(1_{\xi\in\{N_\beta,N_\beta+1,\cdots,N_h\}}\chi_\xi(\omega,\mathbf{k})^{\frac{1}{2}}+1_{\xi=>}\left(\sum_{j=N_\beta}^{\hat{N}_\beta-1}\chi_j(\omega,\mathbf{k})\right)^{\frac{1}{2}}\right)\\
&\quad\cdot\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{\frac{1}{4}}\\
&\quad\cdot h^{-1}\left(I_{2b}-e^{-\frac{i}{h}(\omega-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}\right)^{-1}((\overline{\eta}-1)b+\eta,(\overline{\rho}-1)b+\rho),\\
&(\xi\in\{N_\beta,N_\beta+1,\cdots,N_h,>\}).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\langle f^l_X,g^l_Y\rangle_{\mathcal{H}}=C_l(X,Y),\quad \langle f^>_X,g^>_Y\rangle_{\mathcal{H}}=\sum_{p=N_\beta}^{\hat{N}_\beta-1}C_p(X,Y),\\
&(\forall X,Y\in I_0,\ l\in\{N_\beta,N_\beta+1,\cdots,N_h\}).
\end{aligned}
$$

It follows from Lemma 5.2 (iv),(v), (5.3), (5.20), (5.22) that for any $X\in I_0$, $l\in\{N_\beta+1,N_\beta+2,\cdots,N_h\}$,

(5.23)

$$
\begin{aligned}
&\|f^l_X\|^2_{\mathcal{H}},\ \|g^l_X\|^2_{\mathcal{H}}\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)\min\{M^{\mathsf{a}l},1\},\\
&\|f^{N_\beta}_X\|^2_{\mathcal{H}},\ \|g^{N_\beta}_X\|^2_{\mathcal{H}}\\
&\le\frac{c}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\chi(M^{-N_\beta}A(\beta,M)^{-1}e(\mathbf{k}))\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}},\\
&\|f^>_X\|^2_{\mathcal{H}},\ \|g^>_X\|^2_{\mathcal{H}}\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^{\hat{N}_\beta}\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}}.
\end{aligned}
$$

By (1.7), (1.8), (1.11), the support property of χ, $A(\beta,M)\le M$, (5.9), the assumption (5.12) and Lemma 5.3,

(5.24)

$$\begin{aligned}
&\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\chi(M^{-N_\beta}A(\beta,M)^{-1}e(\mathbf{k}))\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}}\\
&\le D_d\int_{\Gamma_\infty^*}d\mathbf{k}\chi(M^{-N_\beta}A(\beta,M)^{-1}e(\mathbf{k}))\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}}\\
&\quad+c(d,(\hat{\mathbf{v}}_j)_{j=1}^d)L^{-1}\sup_{j\in\{1,2,\cdots,d\}}\sup_{\mathbf{k}\in\mathbb{R}^d}\\
&\qquad\cdot\Bigg|\frac{\partial}{\partial k_j}\Bigg(\chi(M^{-N_\beta}A(\beta,M)^{-1}e(\mathbf{k}))\\
&\qquad\qquad\cdot\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}}\Bigg)\Bigg|\\
&\le c(D_d)\min\left\{\int_{\Gamma_\infty^*}d\mathbf{k}1_{e(\mathbf{k})\le 2M^{N_\beta+1}}e(\mathbf{k})^{-1},\quad\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}\\
&\quad+c(d,M,\chi,\mathsf{c},(\hat{\mathbf{v}}_j)_{j=1}^d)L^{-1}\\
&\qquad\cdot\left(\max_{j\in\{1,2,\cdots,d\}}M^{-\frac{N_\beta}{\mathsf{n}_j}}\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}+\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-3}\right)\\
&\le c(d,M,\chi,\mathsf{c},(\hat{\mathbf{v}}_j)_{j=1}^d,D_d,\mathsf{a})\min\left\{M^{(\mathsf{a}-1)N_\beta},1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\},\\
&\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}}\\
&\le D_d\int_{\Gamma_\infty^*}d\mathbf{k}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}}\\
&\quad+c(d,(\hat{\mathbf{v}}_j)_{j=1}^d)L^{-1}\\
&\qquad\cdot\sup_{j\in\{1,2,\cdots,d\}}\sup_{\mathbf{k}\in\mathbb{R}^d}\left|\frac{\partial}{\partial k_j}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}}\right|
\end{aligned}$$

$$\le c(d,D_d,\mathsf{c},(\hat{\mathbf{v}}_j)_{j=1}^d)\left(\min\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}+L^{-1}\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-3}\right)$$
$$\le c(d,D_d,\mathsf{c},(\hat{\mathbf{v}}_j)_{j=1}^d)\min\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}.$$

We can apply Gram's inequality in the Hilbert space $\mathbb{C}^m\otimes\mathcal{H}$ together with (5.23), (5.24) to derive the claimed bounds.

(ii): By Lemma 5.2 (i),(vi) and (5.19) the matrix-valued functions

$$(x,\mathbf{k})\mapsto\chi_l(x,\mathbf{k})\left(I_{2b}-e^{-\frac{i}{h}(x-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}\right)^{-1}:\mathbb{R}^{d+1}\to\mathrm{Mat}(2b,\mathbb{C}),$$
$$(l=N_\beta+1,N_\beta+2,\cdots,N_h)$$

are well-defined and C^∞-class. Indeed, the matrix-valued function with l is identically zero in the open set

(5.25)
$$\left\{(x,\mathbf{k})\in\mathbb{R}^{d+1}\ \Big|\ \sqrt{h^2\sin^2\left(\frac{x-\pi/\beta}{2h}\right)+e(\mathbf{k})^2}<\frac{8}{5}A(\beta,M)M^{l-1}\right\}.$$

By Lemma 5.2 (vi), (5.19) and the periodicity, for $(x,\mathbf{k})\in\mathbb{R}^{d+1}$ satisfying

$$\sqrt{h^2\sin^2\left(\frac{x-\pi/\beta}{2h}\right)+e(\mathbf{k})^2}\ge\frac{8}{5}A(\beta,M)M^{l-1},$$

$$\left|\det\left(I_{2b}-e^{-\frac{i}{h}(x-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}\right)\right|\ge\left(c\left(\frac{8}{5}-\frac{\pi}{2}\right)A(\beta,M)M^{l-1}\right)^{2b}>0.$$

This implies that at any point belonging to the complement of the set (5.25) the matrix $(I_{2b}-e^{-\frac{i}{h}(x-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})})^{-1}$ is well-defined and infinitely differentiable. Thus the claim follows. For any $\omega\in\mathcal{M}_h$, $\omega-\theta(\beta)/2\neq0$ (mod $2\pi h$) and thus the matrix-valued function

$$\mathbf{k}\mapsto\chi_{N_\beta}(\omega,\mathbf{k})\left(I_{2b}-e^{-\frac{i}{h}(\omega-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}\right)^{-1}:\mathbb{R}^d\to\mathrm{Mat}(2b,\mathbb{C})$$

is well-defined and C^{∞}-class. By keeping these basic facts in mind and using the periodicity we can derive that for $n \in \mathbb{N}$, $j \in \{1, 2, \cdots, d\}$,

(5.26)

$$\begin{aligned}&\left(\frac{\beta}{2\pi}(e^{-i\frac{2\pi}{\beta}(s-t)}-1)\right)^n C_l(\cdot\mathbf{x}s, \cdot\mathbf{y}t)\\&=\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle+i(\omega-\frac{\pi}{\beta})(s-t)}\prod_{m=1}^{n}\left(\frac{\beta}{2\pi}\int_0^{\frac{2\pi}{\beta}}dr_m\right)\\&\quad\cdot\left(\frac{\partial}{\partial r}\right)^n \chi_l(r,\mathbf{k})h^{-1}\left(I_{2b}-e^{-\frac{i}{h}(r-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}\right)^{-1}\Bigg|_{r=\omega+\sum_{m=1}^{n}r_m},\\&\quad(l\in\{N_\beta+1, N_\beta+2,\cdots,N_h\}),\end{aligned}$$

(5.27)

$$\begin{aligned}&\left(\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\right)^n C_{l'}(\cdot\mathbf{x}s, \cdot\mathbf{y}t)\\&=\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle+i(\omega-\frac{\pi}{\beta})(s-t)}\prod_{m=1}^{n}\left(\frac{L}{2\pi}\int_0^{\frac{2\pi}{L}}dp_m\right)\\&\quad\cdot\left(\frac{\partial}{\partial \hat{k}_j}\right)^n \chi_{l'}(\omega,\mathbf{k}+\hat{k}_j\hat{\mathbf{v}}_j)\\&\qquad\qquad\cdot h^{-1}\left(I_{2b}-e^{-\frac{i}{h}(\omega-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k}+\hat{k}_j\hat{\mathbf{v}}_j)}\right)^{-1}\Bigg|_{\hat{k}_j=\sum_{m=1}^{n}p_m},\\&\quad(l'\in\{N_\beta, N_\beta+1,\cdots,N_h\}).\end{aligned}$$

For $(\omega, \mathbf{k}) \in \mathbb{R}^{d+1}$, set

$$B(\omega,\mathbf{k}) := h\left(I_{2b}-e^{-\frac{i}{h}(\omega-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\phi)(\mathbf{k})}\right).$$

Observe that

(5.28)

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^n B(\omega,\mathbf{k})^{-1}\right\|_{2b\times 2b}\\&\le c(d)\sum_{m=1}^{n}\prod_{u=1}^{m}\left(\sum_{l_u=1}^{n}\right)1_{\sum_{u=1}^{m}l_u=n}\end{aligned}$$

$$\cdot\prod_{p=1}^{m}\left\|B(\omega,\mathbf{k})^{-1}\left(\frac{\partial}{\partial\hat{k}_j}\right)^{l_p}B(\omega,\mathbf{k})\right\|_{2b\times 2b}\|B(\omega,\mathbf{k})^{-1}\|_{2b\times 2b},$$
$$(\forall n\in\{1,2,\cdots,d+2\},\ \omega\in[-\pi h,\pi h],\ (\hat{k}_1,\cdots,\hat{k}_d)\in\mathbb{R}^d \text{ satisfying } (\omega-\theta(\beta)/2)^2+e(\mathbf{k})^2\neq 0,\ j\in\{0,1,\cdots,d\}),$$

where we set $\mathbf{k}:=\sum_{j=1}^{d}\hat{k}_j\hat{\mathbf{v}}_j$, $\partial/\partial\hat{k}_0:=\partial/\partial\omega$. This inequality follows from e.g. the formula [10, (C.1)]. By using the assumption $h\geq\sup_{\mathbf{k}\in\mathbb{R}^d}\|E(\phi)(\mathbf{k})\|_{2b\times 2b}$ and (5.20) we can derive from (5.28) that

$$(5.29)\qquad \left\|\left(\frac{\partial}{\partial\omega}\right)^{n}B(\omega,\mathbf{k})^{-1}\right\|_{2b\times 2b}\leq c(d)\sum_{m=1}^{n}h^{m-n}\|B(\omega,\mathbf{k})^{-1}\|_{2b\times 2b}^{m+1}$$
$$\leq c(d)\sum_{m=1}^{n}h^{m-n}\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{m+1}{2}},$$
$$(\forall n\in\{1,2,\cdots,d+2\},\ \omega\in[-\pi h,\pi h],\ \mathbf{k}\in\mathbb{R}^d \text{ satisfying } (\omega-\theta(\beta)/2)^2+e(\mathbf{k})^2\neq 0).$$

On the other hand, it follows from the inequality

$$\left(\frac{e(\mathbf{k})}{\mathsf{c}}\right)^{s}\leq\left(\frac{e(\mathbf{k})}{\mathsf{c}}\right)^{t},\quad (\forall\mathbf{k}\in\mathbb{R}^d,\ s,t\in\mathbb{R}_{\geq 0}\text{ satisfying } t\leq s)$$

and (5.21) that

$$(5.30)\qquad \left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^{m}E(\phi)\left(\sum_{i=1}^{d}\hat{k}_i\hat{\mathbf{v}}_i\right)\right\|_{2b\times 2b}$$
$$\leq c(\mathsf{c})\left(1_{n\leq\mathsf{n}_j}e\left(\sum_{i=1}^{d}\hat{k}_i\hat{\mathbf{v}}_i\right)^{1-\frac{n}{\mathsf{n}_j}}+1_{\mathsf{n}_j<n}\right),$$
$$(\forall(\hat{k}_1,\cdots,\hat{k}_d)\in\mathbb{R}^d,\ m,n\in\{1,2,\cdots,d+2\}\text{ satisfying } m\leq n,\ j\in\{1,2,\cdots,d\}).$$

Let us admit that $\mathbf{k}=\sum_{j=1}^{d}\hat{k}_j\hat{\mathbf{v}}_j$, $(\hat{k}_1,\cdots,\hat{k}_d)\in\mathbb{R}^d$ in the following arguments. By using (5.30), the assumption (5.11) and the formula

$$\frac{\partial}{\partial\hat{k}_j}e^{\frac{1}{h}E(\phi)(\mathbf{k})}=\frac{1}{h}\int_0^1 ds e^{\frac{s}{h}E(\phi)(\mathbf{k})}\frac{\partial}{\partial\hat{k}_j}E(\phi)(\mathbf{k})e^{\frac{1-s}{h}E(\phi)(\mathbf{k})}$$

repeatedly we obtain that

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial \hat{k}_j}\right)^n B(\omega,\mathbf{k})\right\|_{2b\times 2b}\\&\le c(d,\mathsf{c})\sum_{m=1}^{n}\left\|\left(\frac{\partial}{\partial \hat{k}_j}\right)^m E(\phi)(\mathbf{k})\right\|_{2b\times 2b}\le c(d,\mathsf{c})\left(1_{n\le \mathsf{n}_j}e(\mathbf{k})^{1-\frac{n}{\mathsf{n}_j}}+1_{\mathsf{n}_j<n}\right)\\&\le c(d,\mathsf{c})\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{\frac{1}{2}(1-\frac{n}{\mathsf{n}_j})-1_{\mathsf{n}_j<n}\frac{1}{2}(1-\frac{n}{\mathsf{n}_j})},\\&(\forall j\in\{1,2,\cdots,d\},\ n\in\{1,2,\cdots,d+2\}).\end{aligned}$$

By substituting this inequality and (5.20) into (5.28) we have that

(5.31)

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial \hat{k}_j}\right)^n B(\omega,\mathbf{k})^{-1}\right\|_{2b\times 2b}\\&\le c(d,\mathsf{c})\sum_{m=1}^{n}\prod_{u=1}^{m}\left(\sum_{l_u=1}^{n}\right)1_{\sum_{u=1}^m l_u=n}\\&\quad\cdot\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}(m+1)+\sum_{p=1}^m(\frac{1}{2}(1-\frac{l_p}{\mathsf{n}_j})-1_{\mathsf{n}_j<l_p}\frac{1}{2}(1-\frac{l_p}{\mathsf{n}_j}))}\\&=c(d,\mathsf{c})\sum_{m=1}^{n}\prod_{u=1}^{m}\left(\sum_{l_u=1}^{n}\right)1_{\sum_{u=1}^m l_u=n}\\&\quad\cdot\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}-\frac{n}{2\mathsf{n}_j}+\sum_{p=1}^m 1_{\mathsf{n}_j<l_p}\frac{1}{2}(\frac{l_p}{\mathsf{n}_j}-1)}\\&=c(d,\mathsf{c})\sum_{m=1}^{n}\prod_{u=1}^{m}\left(\sum_{l_u=1}^{n}\right)1_{\sum_{u=1}^m l_u=n}\\&\quad\cdot\left(\left(\omega-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2+|\phi|^2\right)^{-\frac{1}{2}-\sum_{p=1}^m(1_{l_p\le\mathsf{n}_j}\frac{l_p}{2\mathsf{n}_j}+1_{\mathsf{n}_j<l_p}\frac{1}{2})},\end{aligned}$$

$(\forall n\in\{1,2,\cdots,d+2\},\ \omega\in[-\pi h,\pi h],\ \mathbf{k}\in\mathbb{R}^d$ satisfying

$(\omega-\theta(\beta)/2)^2+e(\mathbf{k})^2\neq 0,\ j\in\{1,2,\cdots,d\})$.

By using Lemma 5.2 (iii),(iv),(vi), (5.20), (5.29) we can derive from (5.26) that for $l\in\{N_\beta+1,N_\beta+2,\cdots,N_h\}$

(5.32)

$$
\begin{aligned}
&\left\|\left(\frac{\beta}{2\pi}(e^{-i\frac{2\pi}{\beta}(s-t)}-1)\right)^{d+2}C_l(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\right\|_{2b\times 2b}\\
&\le\frac{1}{\beta L^d}\sup_{x\in\mathbb{R}}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h}1_{\chi_l(\omega+x,\mathbf{k})\neq 0}\\
&\quad\cdot\sup_{\substack{r\in[-\pi h,\pi h]\\ \mathbf{p}\in\mathbb{R}^d}}\left\|\left(\frac{\partial}{\partial r}\right)^{d+2}\chi_l(r,\mathbf{p})B(r,\mathbf{p})^{-1}\right\|_{2b\times 2b}\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^l\min\{M^{\mathsf{a}l},1\}\\
&\quad\cdot\sum_{m=0}^{d+2}\sup_{\substack{r\in[-\pi h,\pi h]\\ \mathbf{p}\in\mathbb{R}^d}}\left|\left(\frac{\partial}{\partial r}\right)^{m}\chi_l(r,\mathbf{p})\right|\left\|\left(\frac{\partial}{\partial r}\right)^{d+2-m}B(r,\mathbf{p})^{-1}\right\|_{2b\times 2b}\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^l\min\{M^{\mathsf{a}l},1\}\\
&\quad\cdot\left(\sum_{m=0}^{d+1}\sup_{\substack{r\in[-\pi h,\pi h]\\ \mathbf{p}\in\mathbb{R}^d}}\left(\left|\left(\frac{\partial}{\partial r}\right)^{m}\chi_l(r,\mathbf{p})\right|\right.\right.\\
&\qquad\qquad\left.\cdot\sum_{u=1}^{d+2-m}h^{u-(d+2-m)}\left(\left(r-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{p})^2+|\phi|^2\right)^{-\frac{u+1}{2}}\right)\\
&\qquad\left.+\sup_{\substack{r\in[-\pi h,\pi h]\\ \mathbf{p}\in\mathbb{R}^d}}\left|\left(\frac{\partial}{\partial r}\right)^{d+2}\chi_l(r,\mathbf{p})\right|\left(\left(r-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{p})^2+|\phi|^2\right)^{-\frac{1}{2}}\right)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^l\min\{M^{\mathsf{a}l},1\}\\
&\quad\cdot\left(\sum_{m=0}^{d+1}M^{-ml}\sum_{u=1}^{d+2-m}h^{u-(d+2-m)}M^{-(u+1)l}+M^{-(d+2)l-l}\right)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^l\min\{M^{\mathsf{a}l},1\}
\end{aligned}
$$

$$
\cdot\left(\sum_{m=0}^{d+1}M^{-ml}M^{-(d+3-m)l}+M^{-(d+3)l}\right)
\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^{-(d+2)l}\min\{M^{\mathsf{a}l},1\},
$$

where we also used that $h\ge M^{N_h-2}$. By combining Lemma 5.2 (iii),(iv),(vi), (5.20), (5.31) with (5.27) we deduce that for $l\in\{N_\beta+1,N_\beta+2,\cdots,N_h\}$, $n\in\{1,2,\cdots,d+2\}$,

(5.33)

$$
\begin{aligned}
&\left\|\left(\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\right)^nC_l(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\right\|_{2b\times 2b}\\
&\le\frac{1}{\beta L^d}\sup_{\mathbf{p}\in\mathbb{R}^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{\omega\in\mathcal{M}_h}1_{\chi_l(\omega,\mathbf{k}+\mathbf{p})\neq 0}\\
&\quad\cdot\sup_{\substack{\omega\in\mathcal{M}_h\\ \mathbf{k}\in\mathbb{R}^d}}\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^n\chi_l(\omega,\mathbf{k})B(\omega,\mathbf{k})^{-1}\right\|_{2b\times 2b}\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^l\min\{M^{\mathsf{a}l},l\}\\
&\quad\cdot\left(\sup_{\substack{\omega\in\mathcal{M}_h\\ \mathbf{k}\in\mathbb{R}^d}}\left|\left(\frac{\partial}{\partial\hat{k}_j}\right)^n\chi_l(w,\mathbf{k})\right|\|B(\omega,\mathbf{k})^{-1}\|_{2b\times 2b}\right.\\
&\qquad\left.+\sum_{m=0}^{n-1}\sup_{\substack{\omega\in\mathcal{M}_h\\ \mathbf{k}\in\mathbb{R}^d}}\left|\left(\frac{\partial}{\partial\hat{k}_j}\right)^m\chi_l(w,\mathbf{k})\right|\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^{n-m}B(\omega,\mathbf{k})^{-1}\right\|_{2b\times 2b}\right)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^l\min\{M^{\mathsf{a}l},1\}\\
&\quad\cdot\left(1_{l\ge 0}\left(M^{-(\frac{1}{\mathsf{n}_j}+1)l}+\sum_{m=0}^{n-1}\left(1_{m=0}+1_{m\ge 1}M^{-\frac{l}{\mathsf{n}_j}}\right)\sum_{k=1}^{n-m}\prod_{u=1}^{k}\left(\sum_{l_u=1}^{n-m}\right)\right.\right.\\
&\qquad\qquad\left.\cdot 1_{\sum_{u=1}^k l_u=n-m}M^{2l(-\frac{1}{2}-\sum_{p=1}^k(1_{l_p\le\mathsf{n}_j}\frac{l_p}{2\mathsf{n}_j}+1_{\mathsf{n}_j<l_p}\frac{1}{2}))}\right)\\
&\qquad+1_{l<0}\left(M^{-(\frac{n}{\mathsf{n}_j}+1)l}+\sum_{m=0}^{n-1}M^{-\frac{m}{\mathsf{n}_j}l}\sum_{k=1}^{n-m}\prod_{u=1}^{k}\left(\sum_{l_u=1}^{n-m}\right)1_{\sum_{u=1}^k l_u=n-m}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \cdot M^{2l(-\frac{1}{2}-\frac{n-m}{2\mathsf{n}_j}+\sum_{p=1}^{k}1_{\mathsf{n}_j<l_p}\frac{1}{2}(\frac{l_p}{\mathsf{n}_j}-1))}\Bigg)\Bigg)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)M^l\min\{M^{\mathsf{a}l},1\}\\
&\quad\cdot\left(1_{l\ge0}\left(M^{-(\frac{1}{\mathsf{n}_j}+1)l}+\sum_{m=0}^{n-1}\left(1_{m=0}+1_{m\ge1}M^{-\frac{l}{\mathsf{n}_j}}\right)M^{2l(-\frac{1}{2}-\frac{1}{2\mathsf{n}_j})}\right)\right.\\
&\qquad\left.+1_{l<0}\left(M^{-(\frac{n}{\mathsf{n}_j}+1)l}+\sum_{m=0}^{n-1}M^{-\frac{m}{\mathsf{n}_j}l}M^{2l(-\frac{1}{2}-\frac{n-m}{2\mathsf{n}_j})}\right)\right)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)\min\{M^{\mathsf{a}l},1\}\left(1_{l\ge0}M^{-\frac{l}{\mathsf{n}_j}}+1_{l<0}M^{-\frac{n}{\mathsf{n}_j}l}\right).
\end{aligned}
$$

Here we estimated for all $n\in\{1,2,\cdots,d+2\}$ not only for $n=d+2$ so that we can use the result to prove Lemma 5.7 (ii) later. Moreover by combining Lemma 5.2 (iii),(v), (5.20), (5.31) with (5.27) and using that $M^{-N_\beta}\le M\beta\le\pi M|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}|^{-1}$ we have that

(5.34)

$$
\begin{aligned}
&\left\|\left(\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\right)^{d+1}C_{N_\beta}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\right\|_{2b\times2b}\\
&\le\frac{1}{\beta}\sup_{\mathbf{k}\in\mathbb{R}^d}\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^{d+1}\chi_{N_\beta}\left(\frac{\pi}{\beta},\mathbf{k}\right)B\left(\frac{\pi}{\beta},\mathbf{k}\right)^{-1}\right\|_{2b\times2b}\\
&\le\frac{c(d)}{\beta}\left(\sup_{\mathbf{k}\in\mathbb{R}^d}\left|\left(\frac{\partial}{\partial\hat{k}_j}\right)^{d+1}\chi_{N_\beta}\left(\frac{\pi}{\beta},\mathbf{k}\right)\right|\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right.\\
&\quad\left.+\sum_{m=0}^{d}\sup_{\mathbf{k}\in\mathbb{R}^d}\left|\left(\frac{\partial}{\partial\hat{k}_j}\right)^{m}\chi_{N_\beta}\left(\frac{\pi}{\beta},\mathbf{k}\right)\right|\left\|\left(\frac{\partial}{\partial\hat{k}_j}\right)^{d+1-m}B\left(\frac{\pi}{\beta},\mathbf{k}\right)^{-1}\right\|_{2b\times2b}\right)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)\beta^{-1}\\
&\quad\cdot\left(\left(1_{N_\beta\ge0}M^{-\frac{N_\beta}{\mathsf{n}_j}}+1_{N_\beta<0}M^{-\frac{d+1}{\mathsf{n}_j}N_\beta}\right)\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right.\\
&\qquad+\sum_{m=0}^{d}\left(1_{m=0}+1_{m\ge1}\left(1_{N_\beta\ge0}M^{-\frac{N_\beta}{\mathsf{n}_j}}+1_{N_\beta<0}M^{-\frac{m}{\mathsf{n}_j}N_\beta}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \sum_{k=1}^{d+1-m} \prod_{u=1}^{k} \left(\sum_{l_u=1}^{d+1-m} \right) 1_{\sum_{u=1}^{k} l_u = d+1-m} \\
&\cdot \Bigg(1_{|\pi/\beta - \theta(\beta)/2| > 1} \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1-\sum_{p=1}^{k}(1_{l_p \le \mathsf{n}_j} \frac{l_p}{\mathsf{n}_j} + 1_{\mathsf{n}_j < l_p})} \\
&\quad + 1_{|\pi/\beta - \theta(\beta)/2| \le 1} \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1-\frac{d+1-m}{\mathsf{n}_j} + \sum_{p=1}^{k} 1_{\mathsf{n}_j < l_p}(\frac{l_p}{\mathsf{n}_j} - 1)} \Bigg) \Bigg) \\
&\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^{d}) \beta^{-1} \\
&\cdot \Bigg(\left(1 + \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-\frac{d+1}{\mathsf{n}_j}} \right) \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1} \\
&+ \sum_{m=0}^{d} \left(1_{m=0} + 1_{m \ge 1} \left(1 + \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-\frac{m}{\mathsf{n}_j}} \right) \right) \\
&\cdot \Bigg(1_{|\pi/\beta - \theta(\beta)/2| > 1} \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1} \\
&\quad + 1_{|\pi/\beta - \theta(\beta)/2| \le 1} \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1-\frac{d+1-m}{\mathsf{n}_j}} \Bigg) \Bigg) \\
&\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^{d}) \beta^{-1} \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1} \left(1 + \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-\frac{1}{\mathsf{n}_j}} \right)^{d+1}.
\end{aligned}
$$

By summing up (5.13) for $n = 1$, (5.32), (5.33), (5.34) we reach that for $l \in \{N_\beta + 1, N_\beta + 2, \cdots, N_h\}$, $\mathbf{x}, \mathbf{y} \in \Gamma$, $s, t \in [0, \beta)_h$,

(5.35)

$$
\begin{aligned}
&\|C_l(\cdot \mathbf{x} s, \cdot \mathbf{y} t)\|_{2b \times 2b} \\
&\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^{d}) (1_{l \ge 0} + 1_{l < 0} M^{\mathsf{a} l}) \\
&\cdot \Bigg(1 + M^{(d+2)l} \left| \frac{\beta}{2\pi} \left(e^{i \frac{2\pi}{\beta}(s-t)} - 1 \right) \right|^{d+2} \\
&\quad + \sum_{j=1}^{d} \left(1_{l \ge 0} M^{\frac{l}{\mathsf{n}_j}} + 1_{l < 0} M^{\frac{d+2}{\mathsf{n}_j} l} \right) \left| \frac{L}{2\pi} (e^{i \frac{2\pi}{L} \langle \mathbf{x} - \mathbf{y}, \hat{\mathbf{v}}_j \rangle} - 1) \right|^{d+2} \Bigg)^{-1},
\end{aligned}
$$

$$\begin{aligned}
&\|C_{N_\beta}(\cdot\mathbf{x}s,\cdot\mathbf{y}t)\|_{2b\times 2b}\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)\beta^{-1}\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\\
&\quad\cdot\left(1+\sum_{j=1}^d\left(1+\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-\frac{1}{\mathsf{n}_j}}\right)^{-d-1}\left|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\right|^{d+1}\right)^{-1}.
\end{aligned}$$

These inequalities together with the fact $h \ge M^{N_h-2}$ imply the claimed bounds.

(iii): By (5.2) and (5.35), for $l \in \{N_\beta+1, N_\beta+2,\cdots,N_h\}$,

$$\begin{aligned}
\|\tilde{C}_l\| &\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d,b)\\
&\quad\cdot\Bigg(1_{l\ge 0}+1_{l<0}M^{(\mathsf{a}-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})l}\\
&\qquad+(1+M^{N_\beta})\left(1_{l\ge 0}M^{-l}+1_{l<0}M^{(\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})l}\right)\Bigg)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d,b)\left(1_{l\ge 0}+1_{l<0}M^{(\mathsf{a}-1-\sum_{j=1}^d\frac{1}{\mathsf{n}_j})l}\right),
\end{aligned}$$

which is (5.17). Note that by (5.2) and (5.35), for any $l' \in \{N_\beta+1, N_\beta+2,\cdots,N_h\}\cap\mathbb{Z}_{\ge 0}$,

$$\begin{aligned}
\left\|\sum_{p=l'}^{N_h}\tilde{C}_p\right\| &\le 4b\sup_{\substack{(\overline{\rho},\rho,s,\xi),(\overline{\eta},\eta,t,\zeta)\\ \in\{1,2\}\times\mathcal{B}\times[0,\beta)_h\times\{1,-1\}}}\left|\sum_{p=l'}^{N_h}\tilde{C}_p(\overline{\rho}\rho\mathbf{0}s\xi,\overline{\eta}\eta\mathbf{0}t\zeta)\right|\\
&\quad+\sum_{p=l'}^{N_h}\sum_{\substack{\mathbf{x}\in\Gamma\\ \mathbf{x}\neq\mathbf{0}}}\frac{c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d,b)}{\sum_{j=1}^d M^{\frac{p}{\mathsf{n}_j}}|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle\mathbf{x},\hat{\mathbf{v}}_j\rangle}-1)|^{d+2}}\\
&\quad+c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d,b)(1+M^{N_\beta})\sum_{p=l'}^{N_h}M^{-p}\\
&\le 4b\sup_{\substack{(\overline{\rho},\rho,s,\xi),(\overline{\eta},\eta,t,\zeta)\\ \in\{1,2\}\times\mathcal{B}\times[0,\beta)_h\times\{1,-1\}}}\left|\sum_{p=l'}^{N_h}\tilde{C}_p(\overline{\rho}\rho\mathbf{0}s\xi,\overline{\eta}\eta\mathbf{0}t\zeta)\right|\\
&\quad+c(d,M,\chi,\mathsf{c},(\mathsf{n}_j)_{j=1}^d,\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d,b)(1+(1+M^{N_\beta})M^{-l'}),
\end{aligned}$$

which gives (5.18). □

Using the covariances C_l $(l = N_\beta, N_\beta+1, \cdots, N_h)$, we define the covariances $\mathcal{C}_l$ $(l = N_\beta, N_\beta+1, \cdots, \hat{N}_\beta)$ as follows.

$$\mathcal{C}_{\hat{N}_\beta} := \sum_{l=\hat{N}_\beta}^{N_h} C_l,$$
$$\mathcal{C}_l := C_l, \quad (\forall l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta - 1\}).$$

The claim (i) of the next lemma states that the covariances $\mathcal{C}_l$ $(l = N_\beta, N_\beta+1, \cdots, \hat{N}_\beta)$ satisfy the conditions assumed in Subsection 4.3. The claim (ii) of the lemma will be used to prove Corollary 1.11 (iv) in the next subsection.

LEMMA 5.7. *Assume that* (5.11), (5.12) *hold. Then there exists a constant* $\hat{c}(\in \mathbb{R}_{\geq 1})$ *depending only on* d, b, $(\hat{\mathbf{v}}_j)_{j=1}^d$, a, $(\mathsf{n}_j)_{j=1}^d$, c, M, χ *such that the following statements hold true.*

(i) $\mathcal{C}_l$ $(l = N_\beta+1, N_\beta+2, \cdots, \hat{N}_\beta)$ *satisfy* (4.2) *and* $\mathcal{C}_{N_\beta}$ *satisfies* (4.32). *Moreover* $\mathcal{C}_l$ $(l = N_\beta, N_\beta+1, \cdots, \hat{N}_\beta)$ *satisfy* (4.33), (4.34), (4.35) *with*

$$c_0 = \hat{c}\left(\beta^{-1}\min\left\{1, \left|\frac{\pi}{\beta} - \frac{\theta(\beta)}{2}\right|^{-1}\right\} + 1\right),$$
$$c_{end} = \left|\frac{\pi}{\beta} - \frac{\theta(\beta)}{2}\right|^{-1} \prod_{j=1}^{d}\left(1 + \left|\frac{\pi}{\beta} - \frac{\theta(\beta)}{2}\right|^{-\frac{1}{\mathsf{n}_j}}\right).$$

(ii)

$$\sum_{j=1}^{d} \frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \hat{\mathbf{v}}_j\rangle} - 1| \left\| \sum_{l=N_\beta+1}^{\hat{N}_\beta} \mathcal{C}_l(\cdot\mathbf{x}s, \cdot\mathbf{y}t) \right\|_{2b\times 2b} \leq \hat{c},$$
$$(\forall \mathbf{x}, \mathbf{y} \in \Gamma,\ s, t \in [0, \beta)_h).$$

PROOF. (i): The claim concerning (4.2) is clear. Lemma 5.2 (v) ensures that $\mathcal{C}_{N_\beta}$ satisfies (4.32). Let us derive the determinant bound on $\mathcal{C}_{\hat{N}_\beta}$. Let us improve the second inequality in Lemma 3.5 (iii). For any $x \in \mathbb{R}_{\ge 0}$,

$$\begin{aligned}
&\left(1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta x}+e^{-2\beta x}\right)^{-\frac{1}{2}} \\
&\le c\left(1_{\beta x>1}+1_{\beta x\le 1}\beta^{-1}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+x^2\right)^{-\frac{1}{2}}\right) \\
&\le c\left(1+\beta^{-1}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+x^2\right)^{-\frac{1}{2}}\right).
\end{aligned} \tag{5.36}$$

Thus by (1.6), (1.11), (5.12) and Lemma 5.3,

$$\begin{aligned}
&\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\left(1+2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}}+e^{-2\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}}\right)^{-\frac{1}{2}} \\
&\le \frac{c(b)\beta^{-1}}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2\right)^{-\frac{1}{2}}+c(b) \\
&\le c(D_d,b)\beta^{-1}\int_{\Gamma_\infty^*}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k})^2\right)^{-\frac{1}{2}} \\
&\quad + c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,\mathsf{c},b)\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-3}L^{-1}\beta^{-1}+c(b) \\
&\le c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,\mathsf{c},b)\left(\beta^{-1}\min\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}+1\right).
\end{aligned}$$

Then by this inequality, Lemma 3.5 (iii) and (5.10),

$$\begin{aligned}
&\left|\det\left(\langle\mathbf{u}_i,\mathbf{v}_j\rangle_{\mathbb{C}^m}\sum_{l=N_\beta}^{N_h}C_l(X_i,Y_j)\right)_{1\le i,j\le n}\right| \\
&\le\left(c(d,(\hat{\mathbf{v}}_j)_{j=1}^d,\mathsf{c},b)\left(\beta^{-1}\min\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}+1\right)\right)^n, \\
&(\forall m,n\in\mathbb{N},\ \mathbf{u}_i,\mathbf{v}_i\in\mathbb{C}^m\ (i=1,2,\cdots,n)\text{ with }\|\mathbf{u}_i\|_{\mathbb{C}^m},\ \|\mathbf{v}_i\|_{\mathbb{C}^m}\le 1,
\end{aligned} \tag{5.37}$$

$X_i, Y_i \in I_0$ $(i = 1, 2, \cdots, n))$.

We can apply [12, Lemma A.1], which is based on the Cauchy-Binet formula, together with (5.14), (5.37), the equality $\mathcal{C}_{\hat{N}_\beta} = \sum_{l=N_\beta}^{N_h} C_l - \sum_{l=N_\beta}^{\hat{N}_\beta - 1} C_l$ and the inequality $M^{\hat{N}_\beta} \le M\beta^{-1} + 1$ to derive the claim determinant bound on $\mathcal{C}_{\hat{N}_\beta}$. If $N_\beta + 1 \le \hat{N}_\beta - 1$, then $\hat{N}_\beta = 0$. Thus the claimed determinant bound on $\mathcal{C}_l$ $(l = N_\beta + 1, N_\beta + 2, \cdots, \hat{N}_\beta - 1)$ follows from (5.13). If $\beta > 1$, $N_\beta < 0 = \hat{N}_\beta$. Thus by (5.2)

$$\beta^{-1} \min\left\{ M^{(\mathsf{a}-1)N_\beta}, 1, \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1} \right\} \le \beta^{-1} M^{(\mathsf{a}-1)N_\beta} \le M^{1+\mathsf{a}(N_\beta - \hat{N}_\beta)}.$$

If $\beta \le 1$,

$$\begin{aligned} &\beta^{-1} \min\left\{ M^{(\mathsf{a}-1)N_\beta}, 1, \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1} \right\} \\ &\le \beta^{-1} \min\left\{ 1, \left| \frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right|^{-1} \right\} M^{\mathsf{a}+\mathsf{a}(N_\beta - \hat{N}_\beta)}. \end{aligned}$$

Thus the claimed determinant bound on $\mathcal{C}_{N_\beta}$ follows from (5.13).

Note that $\hat{N}_\beta \ge 0$ $(\forall \beta > 0)$. Thus by (5.15),

$$\|\tilde{\mathcal{C}}_{\hat{N}_\beta}\|_{1,\infty} \le \sum_{l=\hat{N}_\beta}^{N_h} \|\tilde{C}_l\|_{1,\infty} \le \hat{c} \sum_{l=0}^{\infty} M^{-l} \le 2\hat{c}.$$

The inclusion $l \in \{N_\beta + 1, N_\beta + 2, \cdots, \hat{N}_\beta - 1\}$ implies that $\hat{N}_\beta = 0$ and $l \le -1$. Thus by (5.15) the claimed bound on $\|\tilde{\mathcal{C}}_l\|_{1,\infty}$ holds true for $l \in \{N_\beta + 1, N_\beta + 2, \cdots, \hat{N}_\beta\}$. The claimed bound on $\|\tilde{\mathcal{C}}_{N_\beta}\|_{1,\infty}$ follows from (5.16). For the same reason as above (5.17) gives the claimed bound on $\|\tilde{\mathcal{C}}_l\|$ for $l \in \{N_\beta + 1, N_\beta + 2, \cdots, \hat{N}_\beta - 1\}$. Since $\hat{N}_\beta \in \{N_\beta + 1, \cdots, N_h\} \cap \mathbb{Z}_{\ge 0}$, we can derive the claimed bound on $\|\tilde{\mathcal{C}}_{\hat{N}_\beta}\|$ by combining (5.18) with the determinant bound on $\mathcal{C}_{\hat{N}_\beta}$ for $n = 1$.

(ii): By (5.33) for $n = 1$ and the inequality $\mathsf{a} > 1/\mathsf{n}_j$ $(j = 1, 2, \cdots, d)$ implied by the original assumptions $\mathsf{a} > 1$, $\mathsf{n}_j \ge 1$ $(j = 1, 2, \cdots, d)$,

$$\sum_{j=1}^{d} \frac{L}{2\pi} |e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \hat{\mathbf{v}}_j \rangle} - 1| \left\| \sum_{l=N_\beta+1}^{\hat{N}_\beta} \mathcal{C}_l(\cdot \mathbf{x} s, \cdot \mathbf{y} t) \right\|_{2b \times 2b}$$

$$\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^d) \sum_{j=1}^{d} \left(\sum_{l=0}^{\infty} M^{-\frac{l}{\mathsf{n}_j}} + \sum_{l=-1}^{-\infty} M^{(\mathsf{a}-\frac{1}{\mathsf{n}_j})l} \right)$$
$$\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^d, (\mathsf{n}_j)_{j=1}^d),$$

which is the claimed bound. □

REMARK 5.8. We use the condition $\mathsf{a} > 1$ to prove Lemma 5.7 (ii), which will be used only to prove Corollary 1.11 (iv).

Since we have confirmed that the real covariance derived from the free Hamiltonian can be decomposed into a family of covariances satisfying the desired properties, we can apply the general result Lemma 4.10 to analyze the Grassmann Gaussian integral appearing in the formulation Lemma 3.6.

PROPOSITION 5.9. *Let c_6 be the positive constant appearing in Lemma 4.10. Let c_0, c_{end} be those set in Lemma 5.7* (i). *Fix M, $\alpha \in \mathbb{R}_{\ge 1}$ satisfying*

$$M^{\min\{1, 2\mathsf{a}-1-\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}\}} \ge c_6, \quad \alpha \ge c_6 M^{\frac{\mathsf{a}}{2}}.$$

Then the following statements hold for any $h \in \frac{2}{\beta}\mathbb{N}$, $L \in \mathbb{N}$ satisfying (5.11) *and*

(5.38)
$$L \ge \max\Bigg\{ (c_{end}+1)^{\frac{1}{d}} M^{\frac{1}{d}(\mathsf{a}+\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta - N_\beta)}, \frac{\max_{j\in\{1,2,\cdots,d\}} M^{-N_\beta/\mathsf{n}_j}}{\min\{M^{\mathsf{a}N_\beta}, 1\}},$$
$$\frac{\max_{j\in\{1,2,\cdots,d\}} M^{-N_\beta/\mathsf{n}_j} |\frac{\pi}{\beta} - \frac{\theta(\beta)}{2}|^{-1} + |\frac{\pi}{\beta} - \frac{\theta(\beta)}{2}|^{-3}}{\min\{M^{(\mathsf{a}-1)N_\beta}, 1, |\frac{\pi}{\beta} - \frac{\theta(\beta)}{2}|^{-1}\}} \Bigg\}.$$

(i)

$$e^{-8c_6 b\beta\alpha^{-2} M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta - N_\beta)}} \le \left| \int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{C(\phi)}(\psi) \right|$$
$$\le e^{8c_6 b\beta\alpha^{-2} M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta - N_\beta)}},$$

$(\forall u \in \overline{D(b^{-1}c_0^{-2}\alpha^{-4})})$.

(ii)

$$\sup_{u\in D(b^{-1}c_0^{-2}\alpha^{-4})}\left|\frac{\int e^{-V(u)(\psi)+W(u)(\psi)}A^j(\psi)d\mu_{C(\phi)}(\psi)}{\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)}-\int A^j(\psi)d\mu_{C(\phi)}(\psi)\right|$$
$$\le c_6\varepsilon_\beta^{N_\beta-\hat{N}_\beta}\beta c_0^2\left(\alpha^2+c_{end}M^{\mathsf{a}(\hat{N}_\beta-N_\beta)}\right)L^{-d},\quad (\forall j\in\{1,2\}).$$

PROOF. Observe that under the condition of this proposition the claims of Lemma 4.10 and Lemma 5.7 hold true. By the definition and (5.10),

$$\sum_{l=N_\beta}^{\hat{N}_\beta}\mathcal{C}_l(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t)=e^{-i\frac{\pi}{\beta}(s-t)}C(\phi)(\overline{\rho}\rho\mathbf{x}s,\overline{\eta}\eta\mathbf{y}t),\tag{5.39}$$
$$(\forall(\overline{\rho},\rho,\mathbf{x},s),(\overline{\eta},\eta,\mathbf{y},t)\in I_0).$$

Thus by the gauge transform $\psi_{\overline{\rho}\rho\mathbf{x}s\xi}\to e^{-\xi i\frac{\pi}{\beta}s}\psi_{\overline{\rho}\rho\mathbf{x}s\xi}$,

$$\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi)$$
$$=\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{\sum_{l=N_\beta}^{\hat{N}_\beta}\mathcal{C}_l}(\psi).$$

Thus the function V^{end} studied in Lemma 4.10 coincides with the function

$$(u,\boldsymbol{\lambda})\mapsto\log\left(\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi)\right)$$

if $|u|$, $\|\boldsymbol{\lambda}\|_{\mathbb{C}^2}$ are sufficiently small. This claim can be confirmed by an elementary argument close to a part of the proof of [12, Lemma 4.13] or the proof of [10, Proposition 6.4 (3)]. However, we provide a sketch for the readers' convenience. With the constant c_6 appearing in Lemma 4.10, set $r:=b^{-1}c_0^{-2}\alpha^{-4}$, $r'':=c_6^{-1}L^{-d}h^{N_\beta-\hat{N}_\beta-1}\varepsilon_\beta^{\hat{N}_\beta-N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4}$. For $l\in\{N_\beta,N_\beta+1,\cdots,\hat{N}_\beta\}$ we define $V^l\in C(\overline{D(r)}\times\overline{D(r'')}^2,\bigwedge_{even}\mathcal{V})\cap C^\omega(D(r)\times D(r'')^2,\bigwedge_{even}\mathcal{V})$ by $V^l:=\sum_{j=1}^2V^{0-j,l}+\sum_{j=1}^3V^{1-j,l}+V^{2,l}$. By Lemma 4.5 and Lemma 4.7,

$$\sum_{m=0}^N\|V_m^l\|_{1,r,r''}\le c(\beta,L^d,h,(\mathsf{n}_j)_{j=1}^d,b,M,\mathsf{a})\alpha^{-2},$$

$$(\forall l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta\}).$$

This together with (4.33) implies that there exists a positive constant $c'(\beta, L^d, h, (\mathsf{n}_j)_{j=1}^d, b, M, \mathsf{a}, c_0)$ such that if $\alpha \ge c'(\beta, L^d, h, (\mathsf{n}_j)_{j=1}^d, b, M, \mathsf{a}, c_0)$,

$$\text{(5.40)} \quad \mathrm{Re} \int e^{zV^l(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_l}(\psi) > 0,$$
$$(\forall z \in \overline{D(2)},\ (u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r'')}^2,\ l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta\}).$$

Let us fix such a large α. Then for any $(u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r'')}^2$, $l \in \{N_\beta, N_\beta + 1, \cdots, \hat{N}_\beta\}$

$$z \mapsto \log\left(\int e^{zV^l(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_l}(\psi^1)\right),$$
$$z \mapsto \log\left(\int e^{zV^{N_\beta}(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_{N_\beta}}(\psi)\right)$$

are analytic in $D(2)$. Then by definition,

(5.41)
$$V^l(u, \boldsymbol{\lambda})(\psi) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d}{dz}\right)^n \Big|_{z=0} \log\left(\int e^{zV^{l+1}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_{l+1}}(\psi^1)\right)$$
$$= \log\left(\int e^{V^{l+1}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_{l+1}}(\psi^1)\right),$$

(5.42)
$$V^{end}(u, \boldsymbol{\lambda}) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d}{dz}\right)^n \Big|_{z=0} \log\left(\int e^{zV^{N_\beta}(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_{N_\beta}}(\psi)\right)$$
$$= \log\left(\int e^{V^{N_\beta}(u,\boldsymbol{\lambda})(\psi)} d\mu_{\mathcal{C}_{N_\beta}}(\psi)\right),$$
$$(\forall (u, \boldsymbol{\lambda}) \in \overline{D(r)} \times \overline{D(r'')}^2,\ l \in \{N_\beta, N_\beta+1, \cdots, \hat{N}_\beta - 1\}).$$

By (5.40) for $l = \hat{N}_\beta$,

$$\int e^{V^{\hat{N}_\beta - 1}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_{\hat{N}_\beta - 1}}(\psi^1)$$

$$
\begin{aligned}
&= \int\int e^{V^{\hat{N}_\beta}(u,\boldsymbol{\lambda})(\psi+\psi^1+\psi^2)} d\mu_{\mathcal{C}_{\hat{N}_\beta}}(\psi^2) d\mu_{\mathcal{C}_{\hat{N}_\beta-1}}(\psi^1)\\
&= \int e^{V^{\hat{N}_\beta}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_{\hat{N}_\beta-1}+\mathcal{C}_{\hat{N}_\beta}}(\psi^1),\\
&(\forall (u,\boldsymbol{\lambda})\in \overline{D(r)}\times \overline{D(r'')}^2).
\end{aligned}
$$

Here we used the fact that $e^{\log f(\psi)} = f(\psi)$ ($\forall f \in \bigwedge \mathcal{V}$ with $\mathrm{Re}\, f_0 > 0$) (see e.g. [9, Lemma C.2]) and the division formula of Grassmann Gaussian integral (see e.g. [4, Proposition I.21]). Assume that $l \in \{N_\beta+2, N_\beta+3, \cdots, \hat{N}_\beta\}$ and

(5.43)
$$
\begin{aligned}
&\int e^{V^{l-1}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_{l-1}}(\psi^1) = \int e^{V^{\hat{N}_\beta}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\sum_{j=l-1}^{\hat{N}_\beta}\mathcal{C}_j}(\psi^1),\\
&(\forall (u,\boldsymbol{\lambda})\in \overline{D(r)}\times \overline{D(r'')}^2).
\end{aligned}
$$

Then by using (5.41), (5.40), (5.43) in this order,

$$
\begin{aligned}
&\int e^{V^{l-2}(u,\boldsymbol{\lambda})(\psi+\psi^2)} d\mu_{\mathcal{C}_{l-2}}(\psi^2)\\
&= \int\int e^{V^{l-1}(u,\boldsymbol{\lambda})(\psi+\psi^1+\psi^2)} d\mu_{\mathcal{C}_{l-1}}(\psi^1) d\mu_{\mathcal{C}_{l-2}}(\psi^2)\\
&= \int e^{V^{\hat{N}_\beta}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\sum_{j=l-2}^{\hat{N}_\beta}\mathcal{C}_j}(\psi^1),\\
&(\forall (u,\boldsymbol{\lambda})\in \overline{D(r)}\times \overline{D(r'')}^2).
\end{aligned}
$$

Thus by induction with l we have that

$$
\begin{aligned}
&\int e^{V^{N_\beta}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\mathcal{C}_{N_\beta}}(\psi^1) = \int e^{V^{\hat{N}_\beta}(u,\boldsymbol{\lambda})(\psi+\psi^1)} d\mu_{\sum_{j=N_\beta}^{\hat{N}_\beta}\mathcal{C}_j}(\psi^1),\\
&(\forall (u,\boldsymbol{\lambda})\in \overline{D(r)}\times \overline{D(r'')}^2).
\end{aligned}
$$

By combining this equality with (5.42) we obtain that

$$
\begin{aligned}
&V^{end}(u,\boldsymbol{\lambda}) = \log\left(\int e^{V^{\hat{N}_\beta}(u,\boldsymbol{\lambda})(\psi^1)} d\mu_{\sum_{j=N_\beta}^{\hat{N}_\beta}\mathcal{C}_j}(\psi^1)\right),\\
&(\forall (u,\boldsymbol{\lambda})\in \overline{D(r)}\times \overline{D(r'')}^2),
\end{aligned}
$$

which implies the claim.

By the identity theorem and continuity,

(5.44)

$$\int e^{-V(u)(\psi)+W(u)(\psi)-A(\psi)}d\mu_{C(\phi)}(\psi) = e^{V^{end}(u,\boldsymbol{\lambda})},$$
$$\left(\forall(u,\boldsymbol{\lambda}) \in \overline{D(b^{-1}c_0^{-2}\alpha^{-4})} \times \overline{D(c_6^{-1}L^{-d}h^{N_\beta-\hat{N}_\beta-1}\varepsilon_\beta^{\hat{N}_\beta-N_\beta}\beta^{-1}c_0^{-2}\alpha^{-4})}^2\right).$$

By Lemma 4.10 (ii),

$$\sup_{u\in\overline{D(b^{-1}c_0^{-2}\alpha^{-4})}}|V^{end}(u,\mathbf{0})| \le 8c_6 b\beta\alpha^{-2}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}. \tag{5.45}$$

Then the claim (i) follows from (5.44) and (5.45). By the same gauge transform as above and the division formula of Grassmann Gaussian integral,

$$-\int A(\psi)d\mu_{C(\phi)}(\psi) = -\int A(\psi)d\mu_{\sum_{l=N_\beta}^{\hat{N}_\beta}C_l}(\psi) = V^{1-3,end}.$$

Then Lemma 4.10 (iii) and (5.44) ensure the claim (ii). □

5.2. Proof of the main results

Let us move toward the proofs of Theorem 1.3 and Corollary 1.11. In order to prove the existence of the limit $L \to \infty$ claimed in Theorem 1.3, we use the following proposition.

PROPOSITION 5.10. *Assume that M, α satisfy the same conditions as in Proposition 5.9 and L ($\in \mathbb{N}$) satisfies (5.38). Then for any non-empty compact set Q of $\mathbb{C}$*

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi),$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)$$

converge in $C(Q\times\overline{D(2^{-1}b^{-1}c_0^{-2}\alpha^{-4})})$ as sequences of function of the variable (ϕ, u).

PROOF. In essence the claim can be proved in the same way as the proof of [12, Proposition 4.16]. Our model depends on the band index $\rho \in \mathcal{B}$, while the model in [12] does not. However, the band index makes no essential difference to prove the claim. We have to comment on one notable difference from the situation of [12, Proposition 4.16]. Here we do not know whether the function

$$u \mapsto \log\left(\int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{C(\phi)}(\psi)\right)$$

is analytic in $D(b^{-1}c_0^{-2}\alpha^{-4})$, while we knew the analyticity of the function in the domain of interest in [12, Proposition 4.16]. In the following we outline the proof of the proposition without using the above-mentioned analyticity.

For $\phi \in \mathbb{C}$ we define the function $\mathcal{G}(\phi) : (\{1,2\} \times \mathcal{B} \times \Gamma_\infty \times [0,\beta))^2 \to \mathbb{C}$ by

$$\mathcal{G}(\phi)(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t) := e^{-i\frac{\pi}{\beta}(s-t)} C(\phi)(\overline{\rho}\rho\mathbf{x}s, \overline{\eta}\eta\mathbf{y}t).$$

It follows from the gauge transform that

$$\begin{aligned}
&\int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{\mathcal{G}(\phi)}(\psi) \\
&= \int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{C(\phi)}(\psi), \quad (\forall u, \phi \in \mathbb{C}).
\end{aligned}$$

For $n \in \mathbb{N}$, $\phi \in Q$, set

$$\alpha_{n,L,h}(\phi) := \frac{1}{n!}\left(\frac{d}{du}\right)^n \log\left(\int e^{-V(u)(\psi)+W(u)(\psi)} d\mu_{\mathcal{G}(\phi)}(\psi)\right)\Bigg|_{u=0}.$$

We consider $\alpha_{n,L,h}$ as a function of ϕ on Q. The major part of the proof of [12, Proposition 4.16] was devoted to proving the uniform convergence of the function denoted by the same symbol '$\alpha_{n,L,h}$'. Despite the presence of the band index, the essentially same argument as the corresponding part of the proof of [12, Proposition 4.16] shows that for any $n \in \mathbb{N}$

$$\lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \alpha_{n,L,h}, \quad \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \alpha_{n,L,h}$$

converge in $C(Q)$. In place of [12, (4.50)], here we need to use a spatial decay property of $\mathcal{G}(\phi)$ such as

$$\|\mathcal{G}(\phi)(\cdot \mathbf{x}s, \cdot \mathbf{y}t)\|_{2b\times 2b} \le \frac{c(d,\beta,\theta,b,E,Q)}{1+\sum_{j=1}^{d}|\frac{L}{2\pi}(e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)|^{d+1}}, \tag{5.46}$$
$$(\forall \mathbf{x},\mathbf{y}\in\Gamma_\infty,\ s,t\in[0,\beta),\ \phi\in Q)$$

with a positive constant $c(d,\beta,\theta,b,E,Q)$ depending only on d, β, θ, b, E, Q. The above inequality can be directly derived from (3.2). Let $V^{end}(\phi)$ be the function studied in Lemma 4.10. Here we indicate that it depends on the variable ϕ. Set $r := b^{-1}c_0^{-2}\alpha^{-4}$. Take any $h \in \frac{2}{\beta}\mathbb{N}$ satisfying

$$h \ge \max\left\{2, \mathsf{c}, \sup_{\phi\in Q}\sup_{\mathbf{k}\in\mathbb{R}^d}\|E(\phi)(\mathbf{k})\|_{2b\times 2b}\right\}. \tag{5.47}$$

Since

$$V^{end}(\phi)(u,\mathbf{0}) = \log\left(\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{\mathcal{G}(\phi)}(\psi)\right)$$

for small u (see the proof of Proposition 5.9 for this claim) and $u \mapsto V^{end}(\phi)(u,\mathbf{0})$ is analytic in $D(r)$,

$$V^{end}(\phi)(u,\mathbf{0}) = \sum_{n=1}^{\infty}\alpha_{n,L,h}(\phi)u^n, \quad (\forall u\in\overline{D(r/2)}), \tag{5.48}$$
$$\alpha_{n,L,h}(\phi) = \frac{1}{2\pi i}\oint_{|z|=(1+\varepsilon)2^{-1}r}dz\frac{V^{end}(\phi)(z,\mathbf{0})}{z^{n+1}}, \quad (\forall\varepsilon\in(0,1)).$$

Here we should remark that the condition (5.11) was replaced by (5.47). By (5.45),

$$\sup_{\phi\in Q}|\alpha_{n,L,h}(\phi)| \le \frac{8c_6 b\beta\alpha^{-2}M^{(\sum_{j=1}^{d}\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}}{(1+\varepsilon)^n 2^{-n}r^n} \tag{5.49}$$

for $h\in\frac{2}{\beta}\mathbb{N}$ satisfying (5.47) and $L\in\mathbb{N}$ satisfying (5.38). By the convergent properties of $\alpha_{n,L,h}$, (5.48), (5.49) and the dominated convergence theorem in $l^1(\mathbb{N}, C(Q\times\overline{D(r/2)}))$ we conclude that

$$\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} V^{end}(\cdot)(\cdot,\mathbf{0}), \quad \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}} V^{end}(\cdot)(\cdot,\mathbf{0})$$

converge in $C(Q\times\overline{D(\frac{r}{2})})$. Then the equality (5.44) implies the claim. □

Here let us characterize the covariance with zero time-variable so that it can be used to compute the thermal expectations of our interest.

LEMMA 5.11. *For any* $\mathbf{x},\mathbf{y}\in\Gamma$, $\rho\in\mathcal{B}$, $\overline{\rho},\overline{\eta}\in\{1,2\}$ *with* $\overline{\rho}\neq\overline{\eta}$,

(5.50)

$$\begin{aligned}&C(\phi)(\overline{\rho}\cdot\mathbf{x}0,\overline{\rho}\cdot\mathbf{y}0)\\&=\frac{1}{2L^d}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\\&\quad\cdot\left(\frac{e^{-i\frac{\beta\theta(\beta)}{2}}+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2})}{\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2})}\right.\\&\qquad\left.+\frac{(-1)^{\overline{\rho}}\sinh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2})E(\mathbf{k})}{(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}))\sqrt{E(\mathbf{k})^2+|\phi|^2}}\right),\\&C(\phi)(\overline{\rho}\cdot\mathbf{x}0,\overline{\eta}\cdot\mathbf{y}0)\\&=\frac{-1}{2L^d}(1_{(\overline{\rho},\overline{\eta})=(1,2)}\overline{\phi}+1_{(\overline{\rho},\overline{\eta})=(2,1)}\phi)\\&\quad\cdot\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\frac{\sinh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2})}{(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{E(\mathbf{k})^2+|\phi|^2}))\sqrt{E(\mathbf{k})^2+|\phi|^2}},\end{aligned}$$

(5.51)

$$\begin{aligned}&|C(\phi)(\overline{\rho}\rho\mathbf{0}0,\overline{\eta}\rho\mathbf{0}0)|\\&\geq|\phi|\sinh\left(\beta\sqrt{\sup_{\mathbf{k}\in\mathbb{R}^d}\|E(\mathbf{k})\|_{b\times b}^2+|\phi|^2}\right)\\&\quad\cdot\left(2\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\cosh\left(\beta\sqrt{\sup_{\mathbf{k}\in\mathbb{R}^d}\|E(\mathbf{k})\|_{b\times b}^2+|\phi|^2}\right)\right)\right)^{-1}\\&\quad\cdot\left(\sqrt{\sup_{\mathbf{k}\in\mathbb{R}^d}\|E(\mathbf{k})\|_{b\times b}^2+|\phi|^2}\right)^{-1}.\end{aligned}$$

PROOF. For any $A\in\mathrm{Mat}(2,\mathbb{C})$, $B\in\mathrm{Mat}(b,\mathbb{C})$ let us define $A\otimes B\in$

$\mathrm{Mat}(2b,\mathbb{C})$ by

$$A\otimes B := \begin{pmatrix} A(1,1)B & A(1,2)B \\ A(2,1)B & A(2,2)B \end{pmatrix}.$$

Fix $\mathbf{k}\in\mathbb{R}^d$, $\phi\in\mathbb{C}$. Using the notations used in the spectral decomposition (1.16), let us set

$$E_j(\phi)(\mathbf{k}) := \begin{pmatrix} e_j(\mathbf{k}) & \overline{\phi} \\ \phi & -e_j(\mathbf{k}) \end{pmatrix}, \quad (j=1,2,\cdots,b')$$

and observe that

$$E(\phi)(\mathbf{k}) = \sum_{j=1}^{b'} E_j(\phi)(\mathbf{k})\otimes P_j(\mathbf{k}).$$

Then it follows from (3.2) that

(5.52)

$$\begin{aligned}&C(\phi)(\overline{\rho}\rho\mathbf{x}0,\overline{\eta}\eta\mathbf{y}0)\\ &= \frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle}\sum_{j=1}^{b'}\\ &\quad\cdot\left(I_2+e^{\beta(i\frac{\theta(\beta)}{2}I_2+E_j(\phi)(\mathbf{k}))}\right)^{-1}\otimes P_j(\mathbf{k})((\overline{\rho}-1)b+\rho,(\overline{\eta}-1)b+\eta).\end{aligned}$$

The following equalities are essentially same as what we computed in [12, Lemma 4.20]. For $\overline{\rho},\overline{\eta}\in\{1,2\}$ with $\overline{\rho}\neq\overline{\eta}$,

$$\begin{aligned}(5.53)\quad &\left(I_2+e^{\beta(i\frac{\theta(\beta)}{2}I_2+E_j(\phi)(\mathbf{k}))}\right)^{-1}(\overline{\rho},\overline{\rho})\\ &= \frac{e^{-i\frac{\beta\theta(\beta)}{2}}+\cosh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2})}{2(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2}))}\\ &\quad+\frac{(-1)^{\overline{\rho}}\sinh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2})e_j(\mathbf{k})}{2(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2}))\sqrt{e_j(\mathbf{k})^2+|\phi|^2}},\\ &\left(I_2+e^{\beta(i\frac{\theta(\beta)}{2}I_2+E_j(\phi)(\mathbf{k}))}\right)^{-1}(\overline{\rho},\overline{\eta})\end{aligned}$$

$$= \frac{-(1_{(\overline{\rho},\overline{\eta})=(1,2)}\overline{\phi} + 1_{(\overline{\rho},\overline{\eta})=(2,1)}\phi)\sinh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2})}{2(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2}))\sqrt{e_j(\mathbf{k})^2+|\phi|^2}}.$$

By combining (5.52) with (5.53) we can derive (5.50).

Note that

$$\begin{aligned}&|C(\phi)(\overline{\rho}\rho\mathbf{0}0,\overline{\eta}\rho\mathbf{0}0)|\\&=\frac{|\phi|}{2L^d}\sum_{\mathbf{k}\in\Gamma^*}\sum_{j=1}^{b'}\frac{\sinh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2})}{(\cos(\beta\theta(\beta)/2)+\cosh(\beta\sqrt{e_j(\mathbf{k})^2+|\phi|^2}))\sqrt{e_j(\mathbf{k})^2+|\phi|^2}}\\&\qquad\cdot P_j(\mathbf{k})(\rho,\rho).\end{aligned}$$

Then the inequality (5.51) follows from that $P_j(\mathbf{k})(\rho,\rho) \ge 0$, $\sum_{j=1}^{b'} P_j(\mathbf{k})(\rho,\rho) = 1$ and the fact that

$$x \mapsto \frac{\sinh x}{(\cos(\beta\theta(\beta)/2)+\cosh x)x} : [0,\infty) \to \mathbb{R}$$

is strictly monotone decreasing. See e.g. [12, Lemma 4.19] for the proof of this fact. □

By admitting general lemmas proved in Appendix A we can prove Theorem 1.3 here.

Proof of Theorem 1.3. On the whole, the structure of the proof is parallel to the proof of [12, Theorem 1.3]. However, we should keep in mind that in the present case the parameter h must be taken large depending on ϕ and for this reason we cannot change the order of the limit operation $h \to \infty$ and the integral with the variable ϕ, while they were interchangeable in the proof of [12, Theorem 1.3].

Note that

$$\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right| = \min_{m\in\mathbb{Z}}\left|\frac{\theta}{2}-\frac{\pi(2m+1)}{\beta}\right|.$$

If $\beta < 1$,

$$\beta^{-1}\min\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\} \ge \pi^{-1}.$$

By using these properties,

$$\begin{aligned}
&\left(\beta^{-1}\min\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^{-1}\right\}+1\right)^{-2}\\
&\geq 1_{\beta\geq 1}2^{-2}+1_{\beta<1}(1+\pi)^{-2}\beta^2\max\left\{1,\left|\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right|^2\right\}\\
&=1_{\beta\geq 1}2^{-2}+1_{\beta<1}(1+\pi)^{-2}\max\left\{\beta^2,\min_{m\in\mathbb{Z}}\left|\frac{\beta\theta}{2}-\pi(2m+1)\right|^2\right\}.
\end{aligned}$$

Thus by recalling the definition of c_0 stated in Lemma 5.7 we see that there exists $c'\in(0,1]$ depending only on d, b, $(\hat{\mathbf{v}}_j)_{j=1}^d$, a, $(\mathsf{n}_j)_{j=1}^d$, c, M, χ, α such that

$$c'\left(1_{\beta\geq 1}+1_{\beta<1}\max\left\{\beta^2,\min_{m\in\mathbb{Z}}\left|\frac{\beta\theta}{2}-\pi(2m+1)\right|^2\right\}\right)\leq 2^{-1}b^{-1}c_0^{-2}\alpha^{-4}.$$

In the following we always assume that $U\in\mathbb{R}_{<0}$ and

$$|U|<c'\left(1_{\beta\geq 1}+1_{\beta<1}\max\left\{\beta^2,\min_{m\in\mathbb{Z}}\left|\frac{\beta\theta}{2}-\pi(2m+1)\right|^2\right\}\right)$$

so that the claims of Proposition 5.9, Proposition 5.10 hold with this U and $h\in\frac{2}{\beta}\mathbb{N}$, $L\in\mathbb{N}$ satisfying (5.11), (5.38). By Lemma 3.6 (i) and Proposition 5.9 (i), for any $\phi\in\mathbb{C}$, $L\in\mathbb{N}$ satisfying (5.38) and $u\in[-2^{-1}b^{-1}c_0^{-2}\alpha^{-4},2^{-1}b^{-1}c_0^{-2}\alpha^{-4}]$

$$\left|\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\right|\geq e^{-8c_6b\beta\alpha^{-2}M^{(\sum_{j=1}^d\frac{1}{\mathsf{n}_j}+1)(\hat{N}_\beta-N_\beta)}}.$$

By Lemma 3.6 (iv) and Proposition 5.10, the real-valued function

$$\begin{aligned}
&u\mapsto\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(u)(\psi)+W(u)(\psi)}d\mu_{C(\phi)}(\psi)\\
&:[-2^{-1}b^{-1}c_0^{-2}\alpha^{-4},2^{-1}b^{-1}c_0^{-2}\alpha^{-4}]\to\mathbb{R}
\end{aligned}$$

is continuous. Since this function takes 1 at $u = 0$, we conclude that

$$\lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \int e^{-V(\psi)+W(\psi)} d\mu_{C(\phi)}(\psi) \ge e^{-8c_6 b\beta\alpha^{-2} M^{(\sum_{j=1}^{d}\frac{1}{\mathrm{n}_j}+1)(\hat{N}_\beta - N_\beta)}} \tag{5.54}$$

for any $\phi \in \mathbb{C}$ and $L \in \mathbb{N}$ satisfying (5.38). Therefore we see from Lemma 3.1, Lemma 3.2 and Lemma 3.6 (iii) that the claim (i) holds.

Let us prove the claim (iii). Assume that $\gamma \in (0, 1]$. Let us define the functions $F : \mathbb{R}^2 \to \mathbb{R}$, $F_L : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned}
&F(\mathbf{x}) \\
&:= -\frac{1}{|U|}((x_1-\gamma)^2 + x_2^2) \\
&\quad + \frac{D_d}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right) + \cosh\left(\beta\sqrt{E(\mathbf{k})^2 + \|\mathbf{x}\|_{\mathbb{R}^2}^2}\right)\right) \\
&\quad - \frac{D_d}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right) + \cosh(\beta E(\mathbf{k}))\right), \\
&F_L(\mathbf{x}) \\
&:= -\frac{1}{|U|}((x_1-\gamma)^2 + x_2^2) \\
&\quad + \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right) + \cosh\left(\beta\sqrt{E(\mathbf{k})^2 + \|\mathbf{x}\|_{\mathbb{R}^2}^2}\right)\right) \\
&\quad - \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right) + \cosh(\beta E(\mathbf{k}))\right).
\end{aligned}$$

Let us recall the definition (1.22) of the matrix-valued function $G_{x,y,z}(\cdot)$. By making use of the monotone decreasing property of the function

$$x \mapsto \frac{\sinh x}{(\cos(\beta\theta(\beta)/2) + \cosh x)x} : [0, \infty) \to \mathbb{R}$$

we can prove that there uniquely exist $a(\gamma) \in (\Delta, \infty)$, $a_L(\gamma) \in (0, \infty)$ such that

$$a(\gamma)\left(-\frac{2}{|U|} + D_d\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\,G_{\beta,\theta(\beta),a(\gamma)}(\mathbf{k})\right) = -\frac{2\gamma}{|U|}, \tag{5.55}$$

$$a_L(\gamma)\left(-\frac{2}{|U|}+\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\,G_{\beta,\theta(\beta),a_L(\gamma)}(\mathbf{k})\right)=-\frac{2\gamma}{|U|},$$

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}} a(\gamma)=\Delta. \tag{5.56}$$

Set $\mathbf{a}_L := (a_L(\gamma), 0)$, $\mathbf{a} := (a(\gamma), 0)$. By computing Hessians one can check that $\mathbf{a}_L$, $\mathbf{a}$ are the unique global maximum point of F_L, F respectively.

Let us define the functions g_L, $u_{1,L} : \mathbb{R}^2 \to \mathbb{C}$ by

$$g_L(\mathbf{x}) := \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(x_1+ix_2)}(\psi),$$

$$u_{1,L}(\mathbf{x}) := \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}A^1(\psi)d\mu_{C(x_1+ix_2)}(\psi).$$

It follows from Lemma 3.6 (i) that g_L, $u_{1,L} \in C(\mathbb{R}^2)$. Moreover by Proposition 5.9 (i),(ii) and the determinant bound Lemma 3.5 (iii),

$$\sup_{\substack{L\in\mathbb{N}\\ \text{satisfying (5.38)}}}\sup_{\mathbf{x}\in\mathbb{R}^2}|g_L(\mathbf{x})|<\infty,\qquad \sup_{\substack{L\in\mathbb{N}\\ \text{satisfying (5.38)}}}\sup_{\mathbf{x}\in\mathbb{R}^2}|u_{1,L}(\mathbf{x})|<\infty.$$

Furthermore by Proposition 5.10 and Proposition 5.9 (ii) there exists $g \in C(\mathbb{R}^2)$ such that g_L converges to g locally uniformly as $L \to \infty$ ($L \in \mathbb{N}$) and if we set

$$u_1(x_1,x_2) := \beta g(x_1,x_2)\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(x_1+ix_2)(1\hat{\rho}\mathbf{0}0,2\hat{\rho}\mathbf{0}0),\quad (x_1,x_2\in\mathbb{R}),$$

$u_{1,L}$ converges to u_1 locally uniformly as $L \to \infty$ ($L \in \mathbb{N}$). Also, let us remark that by Proposition 5.9 (i), $g(\mathbf{a}) \neq 0$. We can check that the assumptions of Lemma A.1 are satisfied. We can apply the lemma to ensure that

$$\begin{aligned}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\int_{\mathbb{R}^2} d\mathbf{x}e^{\beta L^d F_L(\mathbf{x})}u_{1,L}(\mathbf{x})}{\int_{\mathbb{R}^2} d\mathbf{x}e^{\beta L^d F_L(\mathbf{x})}g_L(\mathbf{x})} &= \frac{\beta g(\mathbf{a})\lim_{L\to\infty, L\in\mathbb{N}} C(a(\gamma))(1\hat{\rho}\mathbf{0}0,2\hat{\rho}\mathbf{0}0)}{g(\mathbf{a})}\\ &= \beta\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(a(\gamma))(1\hat{\rho}\mathbf{0}0,2\hat{\rho}\mathbf{0}0).\end{aligned} \tag{5.57}$$

By Lemma 5.11,

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(a(\gamma))(1\hat{\rho}\mathbf{0}0,2\hat{\rho}\mathbf{0}0)=-\frac{a(\gamma)D_d}{2}\int_{\Gamma_\infty^*}d\mathbf{k}G_{\beta,\theta(\beta),a(\gamma)}(\mathbf{k})(\hat{\rho},\hat{\rho}). \tag{5.58}$$

By combining (5.56), (5.57), (5.58), Lemma 3.1 with Lemma 3.6 (iii) we have that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1)}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}=-\frac{a(\gamma)D_d}{2}\int_{\Gamma_\infty^*}d\mathbf{k}G_{\beta,\theta(\beta),a(\gamma)}(\mathbf{k})(\hat{\rho},\hat{\rho}), \tag{5.59}$$
$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1)}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}=-\frac{\Delta D_d}{2}\int_{\Gamma_\infty^*}d\mathbf{k}G_{\beta,\theta,\Delta}(\mathbf{k})(\hat{\rho},\hat{\rho}).$$

This concludes the proof of the claim (iii).

Let us show the claim (iv). Define the functions $f:\mathbb{R}\to\mathbb{R}$, $f_L:\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}
&f(x)\\
&:=-\frac{x^2}{|U|}+\frac{D_d}{\beta}\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\cosh\left(\beta\sqrt{E(\mathbf{k})^2+x^2}\right)\right)\\
&\quad-\frac{D_d}{\beta}\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\cosh\left(\beta E(\mathbf{k})\right)\right),\\
&f_L(x)\\
&:=-\frac{x^2}{|U|}+\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\cosh\left(\beta\sqrt{E(\mathbf{k})^2+x^2}\right)\right)\\
&\quad-\frac{1}{\beta L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\cosh\left(\beta E(\mathbf{k})\right)\right).
\end{aligned}$$

We let Δ_L ($\in[0,\infty)$) be the solution to

$$-\frac{2}{|U|}+\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\,G_{\beta,\theta(\beta),\Delta_L}(\mathbf{k})=0, \tag{5.60}$$

if

$$-\frac{2}{|U|}+\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\,G_{\beta,\theta(\beta),0}(\mathbf{k})\ge 0.$$

We let $\Delta_L := 0$ if

$$-\frac{2}{|U|}+\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\,G_{\beta,\theta(\beta),0}(\mathbf{k})<0.$$

The well-definedness of Δ_L is guaranteed by the parallel consideration to Lemma 1.2. Note that Δ, Δ_L are the unique maximum point of $f|_{\mathbb{R}_{\ge 0}}$, $f_L|_{\mathbb{R}_{\ge 0}}$ respectively.

First let us consider the case that

$$|U|\neq\left(\frac{D_d}{2}\int_{\Gamma^*_\infty}d\mathbf{k}\,\mathrm{Tr}\,G_{\beta,\theta(\beta),0}(\mathbf{k})\right)^{-1}. \tag{5.61}$$

It follows that

$$-\frac{2}{|U|}+\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{Tr}\,G_{\beta,\theta(\beta),0}(\mathbf{k})\neq 0$$

for sufficiently large $L\in\mathbb{N}$. Moreover, $\frac{d^2f}{dx^2}(\Delta)<0$. If $\Delta=0$, $\Delta_L=0$ for sufficiently large $L\in\mathbb{N}$. Let us define the functions $v_L:\mathbb{R}\to\mathbb{C}$, $u_{2,L}:\mathbb{R}\to\mathbb{C}$ by

$$\begin{aligned}
v_L(x)&:=\int_0^{2\pi}d\xi\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}d\mu_{C(xe^{i\xi})}(\psi)\\
&=\int_0^{2\pi}d\xi g_L(x\cos\xi,x\sin\xi),\\
u_{2,L}(x)&:=\int_0^{2\pi}d\xi\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C(xe^{i\xi})}(\psi).
\end{aligned}$$

By Lemma 3.6 (i), v_L, $u_{2,L}\in C(\mathbb{R})$. By Lemma 3.5 (iii), Proposition 5.9 (i),(ii) and Proposition 5.10, for any $r\in\mathbb{R}_{>0}$

$$\sup_{\substack{L\in\mathbb{N}\\ \text{satisfying (5.38)}}}\sup_{x\in\mathbb{R}}|v_L(x)|<\infty,\qquad \sup_{\substack{L\in\mathbb{N}\\ \text{satisfying (5.38)}}}\sup_{x\in\mathbb{R}}|u_{2,L}(x)|<\infty,$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[-r,r]}\left|v_L(x)-\int_0^{2\pi}d\xi g(x\cos\xi,x\sin\xi)\right|=0,$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[-r,r]}\Bigg|u_{2,L}(x)$$
$$-\beta\int_0^{2\pi}d\xi g(x\cos\xi,x\sin\xi)\Bigg(1_{(\hat{\rho},\hat{\mathbf{x}})=(\hat{\eta},\hat{\mathbf{y}})}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}C(xe^{i\xi})(1\hat{\rho}\mathbf{0}0,1\hat{\rho}\mathbf{0}0)$$
$$-\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\det\begin{pmatrix}C(xe^{i\xi})(1\hat{\rho}\hat{\mathbf{x}}0,1\hat{\eta}\hat{\mathbf{y}}0) & C(xe^{i\xi})(1\hat{\rho}\mathbf{0}0,2\hat{\rho}\mathbf{0}0)\\ C(xe^{i\xi})(2\hat{\eta}\mathbf{0}0,1\hat{\eta}\mathbf{0}0) & C(xe^{i\xi})(2\hat{\eta}\hat{\mathbf{y}}0,2\hat{\rho}\hat{\mathbf{x}}0)\end{pmatrix}\Bigg)\Bigg|=0.$$

Also, note that by (5.54)

$$\int_0^{2\pi}d\xi g(\Delta\cos\xi,\Delta\sin\xi)>0.$$

Moreover by changing variables we can see that

$$\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{\beta L^df_L(|\phi|)}g_L(\phi_1,\phi_2)=\int_0^\infty dxxe^{\beta L^df_L(x)}v_L(x),$$
$$\int_{\mathbb{R}^2}d\phi_1d\phi_2e^{\beta L^df_L(|\phi|)}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C(\phi)}(\psi)$$
$$=\int_0^\infty dxxe^{\beta L^df_L(x)}u_{2,L}(x).$$

In this situation we can apply Lemma A.2. As the result,

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\int_0^\infty dxxe^{\beta L^df_L(x)}u_{2,L}(x)}{\int_0^\infty dxxe^{\beta L^df_L(x)}v_L(x)}=\frac{\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}u_{2,L}(\Delta)}{\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}v_L(\Delta)}$$
$$=\beta\int_0^{2\pi}d\xi g(\Delta\cos\xi,\Delta\sin\xi)\Bigg(1_{(\hat{\rho},\hat{\mathbf{x}})=(\hat{\eta},\hat{\mathbf{y}})}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}C(\Delta e^{i\xi})(1\hat{\rho}\mathbf{0}0,1\hat{\rho}\mathbf{0}0)$$
$$-\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\det\begin{pmatrix}C(\Delta e^{i\xi})(1\hat{\rho}\hat{\mathbf{x}}0,1\hat{\eta}\hat{\mathbf{y}}0) & C(\Delta e^{i\xi})(1\hat{\rho}\mathbf{0}0,2\hat{\rho}\mathbf{0}0)\\ C(\Delta e^{i\xi})(2\hat{\eta}\mathbf{0}0,1\hat{\eta}\mathbf{0}0) & C(\Delta e^{i\xi})(2\hat{\eta}\hat{\mathbf{y}}0,2\hat{\rho}\hat{\mathbf{x}}0)\end{pmatrix}\Bigg)$$
$$\cdot\left(\int_0^{2\pi}d\xi g(\Delta\cos\xi,\Delta\sin\xi)\right)^{-1}$$
$$=\beta\Bigg(1_{(\hat{\rho},\hat{\mathbf{x}})=(\hat{\eta},\hat{\mathbf{y}})}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}C(\Delta)(1\hat{\rho}\mathbf{0}0,1\hat{\rho}\mathbf{0}0)$$

$$- \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \det \begin{pmatrix} C(\Delta)(1\hat{\rho}\hat{\mathbf{x}}0, 1\hat{\eta}\hat{\mathbf{y}}0) & C(\Delta)(1\hat{\rho}\mathbf{0}0, 2\hat{\rho}\mathbf{0}0) \\ C(\Delta)(2\hat{\eta}\mathbf{0}0, 1\hat{\eta}\mathbf{0}0) & C(\Delta)(2\hat{\eta}\hat{\mathbf{y}}0, 2\hat{\rho}\hat{\mathbf{x}}0) \end{pmatrix} \Bigg).$$

We derived the last equality by recalling Lemma 5.11. Therefore by Lemma 3.1 and Lemma 3.6 (iii),

$$\begin{aligned} &\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\mathsf{A}_2)}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}} \\ &= 1_{(\hat{\rho},\hat{\mathbf{x}})=(\hat{\eta},\hat{\mathbf{y}})} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(\Delta)(1\hat{\rho}\mathbf{0}0, 1\hat{\rho}\mathbf{0}0) \\ &\quad - \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \det \begin{pmatrix} C(\Delta)(1\hat{\rho}\hat{\mathbf{x}}0, 1\hat{\eta}\hat{\mathbf{y}}0) & C(\Delta)(1\hat{\rho}\mathbf{0}0, 2\hat{\rho}\mathbf{0}0) \\ C(\Delta)(2\hat{\eta}\mathbf{0}0, 1\hat{\eta}\mathbf{0}0) & C(\Delta)(2\hat{\eta}\hat{\mathbf{y}}0, 2\hat{\rho}\hat{\mathbf{x}}0) \end{pmatrix}. \end{aligned} \tag{5.62}$$

By using the fact that for any compact set K of $\mathbb{C}$ and $(\overline{\rho}, \rho)$, $(\overline{\eta}, \eta) \in \{1,2\} \times \mathcal{B}$

$$\lim_{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d}\to\infty} \sup_{\phi\in K} \left| \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(\phi)(\overline{\rho}\rho\mathbf{x}0, \overline{\eta}\eta\mathbf{y}0) \right| = 0 \tag{5.63}$$

and recalling Lemma 5.11 we observe that

$$\begin{aligned} &\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\mathsf{A}_2)}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}} \\ &= \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(\Delta)(1\hat{\rho}\mathbf{0}0, 2\hat{\rho}\mathbf{0}0) C(\Delta)(2\hat{\eta}\mathbf{0}0, 1\hat{\eta}\mathbf{0}0) = (\text{R. H. S of (1.21)}). \end{aligned}$$

We can show the property (5.63) by establishing a decay bound such as (5.46).

Let us assume that

$$|U| = \left(\frac{D_d}{2} \int_{\Gamma_\infty^*} d\mathbf{k}\, \mathrm{Tr}\, G_{\beta,\theta(\beta),0}(\mathbf{k}) \right)^{-1}. \tag{5.64}$$

In this case we apply Lemma A.3 to prove the claim. By (5.54) $v_L \in C(\mathbb{R}, \mathbb{R})$ and

$$\inf_{\substack{L\in\mathbb{N}\\ \text{satisfying (5.38)}}} \inf_{x\in\mathbb{R}} v_L(x) > 0. \tag{5.65}$$

Let us define the function $u_{3,L}:\mathbb{R}\to\mathbb{C}$ by

$$\begin{aligned}u_{3,L}(x):=&\beta 1_{(\hat{\rho},r_L(\hat{\mathbf{x}}))=(\hat{\eta},r_L(\hat{\mathbf{y}}))}C(x)(1\hat{\rho}\mathbf{0}0,1\hat{\rho}\mathbf{0}0)\\&-\beta\det\begin{pmatrix}C(x)(1\hat{\rho}\hat{\mathbf{x}}0,1\hat{\eta}\hat{\mathbf{y}}0) & C(x)(1\hat{\rho}\mathbf{0}0,2\hat{\rho}\mathbf{0}0)\\ C(x)(2\hat{\eta}\mathbf{0}0,1\hat{\eta}\mathbf{0}0) & C(x)(2\hat{\eta}\hat{\mathbf{y}}0,2\hat{\rho}\hat{\mathbf{x}}0)\end{pmatrix}.\end{aligned}$$

We can see from (5.50) that

$$\int A^2(\psi)d\mu_{C(xe^{i\xi})}(\psi)=u_{3,L}(x),\quad (\forall x,\xi\in\mathbb{R}).$$

We have to prove that there exists $n_0\in 2\mathbb{N}$ such that

$$\frac{d^nf}{dx^n}(0)=0,\quad (\forall n\in\{1,2,\cdots,n_0-1\}),\quad \frac{d^{n_0}f}{dx^{n_0}}(0)<0. \tag{5.66}$$

We define the function q in a neighborhood of the origin by

$$\begin{aligned}&q(z)\\&:=-\frac{z^2}{|U|}+\frac{D_d}{\beta}\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\sum_{n=0}^{\infty}\frac{\beta^{2n}}{(2n)!}(E(\mathbf{k})^2+z^2)^n\right)\\&\quad-\frac{D_d}{\beta}\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\log\left(\cos\left(\frac{\beta\theta(\beta)}{2}\right)+\cosh(\beta E(\mathbf{k}))\right).\end{aligned}$$

Since $\cos(\beta\theta(\beta)/2)>-1$, q is analytic in a neighborhood of the origin. Moreover $q(x)=f(x)$ if $x\in\mathbb{R}$. Since $0=f(0)>f(x)$ $(\forall x\in\mathbb{R}\backslash\{0\})$, q is not identically 0. Thus there exists $n_0\in\mathbb{N}$ such that $q^{(n_0)}(0)\neq 0$ and $q(z)=\sum_{n=n_0}^{\infty}\frac{1}{n!}q^{(n)}(0)z^n$ in a neighborhood of the origin. Thus $f^{(n_0)}(0)\neq 0$ and $f(x)=\sum_{n=n_0}^{\infty}\frac{1}{n!}f^{(n)}(0)x^n$ for any $x\in\mathbb{R}$ close to 0. Since f takes the maximum value 0 at $x=0$, n_0 must be even and $f^{(n_0)}(0)<0$. Therefore the claim (5.66) holds true. We can check that all the other conditions required in Lemma A.3 are satisfied by the functions f_L, f, v_L, $u_{3,L}$. Thus the lemma ensures that

$$\limsup_{\substack{L\to\infty\\L\in\mathbb{N}}}\left|\frac{\int_0^\infty dxxe^{\beta L^df_L(x)}v_L(x)u_{3,L}(x)}{\int_0^\infty dxxe^{\beta L^df_L(x)}v_L(x)}\right|\le\left|\lim_{\substack{L\to\infty\\L\in\mathbb{N}}}u_{3,L}(0)\right|.$$

Moreover by Lemma 3.1, Lemma 3.6 (iii), Proposition 5.9 (i),(ii) and (5.65),

$$\limsup_{\substack{L\to\infty\\L\in\mathbb{N}}}\left|\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\mathsf{A}_2)}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}\right| \tag{5.67}$$

$$
\begin{aligned}
&\le \frac{1}{\beta}\limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left|\frac{\int_0^\infty dx x e^{\beta L^d f_L(x)} u_{2,L}(x)}{\int_0^\infty dx x e^{\beta L^d f_L(x)} v_L(x)}\right|\\
&\le \frac{1}{\beta}\limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left|\frac{\int_0^\infty dx x e^{\beta L^d f_L(x)} v_L(x) u_{3,L}(x)}{\int_0^\infty dx x e^{\beta L^d f_L(x)} v_L(x)}\right|\\
&\le \Bigg|1_{(\hat\rho,\hat{\mathbf{x}})=(\hat\eta,\hat{\mathbf{y}})}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(0)(1\hat\rho\mathbf{0}0,1\hat\rho\mathbf{0}0)\\
&\quad -\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} C(0)(1\hat\rho\hat{\mathbf{x}}0,1\hat\eta\hat{\mathbf{y}}0)C(0)(2\hat\eta\hat{\mathbf{y}}0,2\hat\rho\hat{\mathbf{x}}0)\Bigg|.
\end{aligned}
$$

Then we can apply (5.63) to conclude the claimed convergent property.

Let us prove the claim (v). First let us consider the case that (5.61) holds. Define the function $u_{4,L}:\mathbb{R}\to\mathbb{C}$ by

$$
\begin{aligned}
&u_{4,L}(x)\\
&:= \frac{1}{L^{2d}}\sum_{(\hat\rho,\hat{\mathbf{x}}),(\hat\eta,\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma}\int_0^{2\pi} d\xi \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int e^{-V(\psi)+W(\psi)}A^2(\psi)d\mu_{C(xe^{i\xi})}(\psi).
\end{aligned}
$$

By Lemma 3.6 (i), $u_{4,L}\in C(\mathbb{R})$. Moreover by Lemma 3.5 (iii), Proposition 5.9 (i),(ii), Proposition 5.10 and Lemma 5.11, for any $r\in\mathbb{R}_{>0}$

$$
\begin{aligned}
&\sup_{\substack{L\in\mathbb{N}\\ \text{satisfying } (5.38)}}\sup_{x\in\mathbb{R}}|u_{4,L}(x)|<\infty,\\
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[-r,r]}\Bigg|u_{4,L}(x)\\
&\quad -\frac{1}{L^{2d}}\sum_{(\hat\rho,\hat{\mathbf{x}}),(\hat\eta,\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma}\int_0^{2\pi} d\xi \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int A^2(\psi)d\mu_{C(xe^{i\xi})}(\psi)g_L(x\cos\xi,x\sin\xi)\Bigg|\\
&=0,\\
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[-r,r]}\\
&\quad\cdot\Bigg|\frac{1}{L^{2d}}\sum_{(\hat\rho,\hat{\mathbf{x}}),(\hat\eta,\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma}\int_0^{2\pi} d\xi \lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\int A^2(\psi)d\mu_{C(xe^{i\xi})}(\psi)g_L(x\cos\xi,x\sin\xi)
\end{aligned}
$$

$$- \beta x^2 \left(\frac{D_d}{2} \int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr} G_{\beta,\theta(\beta),x}(\mathbf{k}) \right)^2 \int_0^{2\pi} d\xi g(x\cos\xi, x\sin\xi) \Bigg| = 0.$$

Here we can apply Lemma A.2 to derive that

$$\begin{aligned}
\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \frac{\int_0^\infty dx x e^{\beta L^d f_L(x)} u_{4,L}(x)}{\int_0^\infty dx x e^{\beta L^d f_L(x)} v_L(x)} &= \frac{\lim_{L\to\infty, L\in\mathbb{N}} u_{4,L}(\Delta)}{\lim_{L\to\infty, L\in\mathbb{N}} v_L(\Delta)} \\
&= \beta\Delta^2 \left(\frac{D_d}{2} \int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr} G_{\beta,\theta(\beta),\Delta}(\mathbf{k}) \right)^2 = \frac{\beta\Delta^2}{U^2},
\end{aligned}$$

which combined with Lemma 3.1, Lemma 3.6 (iii) ensures the claimed result in this case.

Next let us assume that (5.64) holds. Define the function $u_{5,L} : \mathbb{R} \to \mathbb{C}$ by

$$u_{5,L}(x) := \frac{1}{L^{2d}} \sum_{(\hat{\rho},\hat{\mathbf{x}}),(\hat{\eta},\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma} \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \int A^2(\psi) d\mu_{C(x)}(\psi).$$

Then by Lemma 3.5 (iii) and Lemma 5.11, for any $r \in \mathbb{R}_{>0}$

$$\begin{aligned}
&\sup_{L\in\mathbb{N}} \sup_{x\in\mathbb{R}} |u_{5,L}(x)| < \infty, \\
&\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \sup_{x\in[-r,r]} \left| u_{5,L}(x) - \beta x^2 \left(\frac{D_d}{2} \int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr} G_{\beta,\theta(\beta),x}(\mathbf{k}) \right)^2 \right| = 0.
\end{aligned}$$

Thus by Lemma A.3

$$\limsup_{\substack{L\to\infty \\ L\in\mathbb{N}}} \left| \frac{\int_0^\infty dx x e^{\beta L^d f_L(x)} v_L(x) u_{5,L}(x)}{\int_0^\infty dx x e^{\beta L^d f_L(x)} v_L(x)} \right| \le \left| \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} u_{5,L}(0) \right| = 0,$$

which together with Lemma 3.1, Lemma 3.6 (iii), Proposition 5.9 (i),(ii) and (5.65) gives that

$$\limsup_{\substack{L\to\infty \\ L\in\mathbb{N}}} \left| \frac{1}{L^{2d}} \sum_{(\hat{\rho},\hat{\mathbf{x}}),(\hat{\eta},\hat{\mathbf{y}})\in\mathcal{B}\times\Gamma} \frac{\operatorname{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\mathsf{A}_2)}{\operatorname{Tr} e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}} \right| = 0.$$

This implies the claim in this case as well.

Finally let us prove the claim (ii). Whether (5.61) holds or not, we can readily apply Lemma A.4 to derive that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{L^d}\log\left(\int_0^\infty dx x e^{\beta L^d f_L(x)}v_L(x)\right)=\beta f(\Delta). \tag{5.68}$$

On the other hand, by Lemma 3.1, Lemma 3.6 (iii) and Lemma 3.2

$$\begin{aligned}&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{L^d}\log\left(\int_0^\infty dx x e^{\beta L^d f_L(x)}v_L(x)\right)\\&=\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{L^d}\log\left(\frac{\pi|U|}{\beta L^d}\frac{\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}{\mathrm{Tr}\, e^{-\beta(\mathsf{H}_0+i\theta\mathsf{S}_z)}}\right)\\&=\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{L^d}\log(\mathrm{Tr}\, e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)})\\&\quad-D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\log\left(1+2\cos\left(\frac{\beta\theta}{2}\right)e^{-\beta E(\mathbf{k})}+e^{-2\beta E(\mathbf{k})}\right).\end{aligned} \tag{5.69}$$

By coupling (5.68) with (5.69) we obtain the claimed equality.

Since the claim (vi) has been proved right after the statement of Theorem 1.3, the proof of the theorem is complete. □

In the rest of this section we prove Corollary 1.11.

PROOF OF COROLLARY 1.11. Let c_1 be the constant introduced in Theorem 1.3. Let us assume that

$$U\in\left(-\min\left\{c_1,\frac{2(\cosh(1)-1)}{\cosh(1)D_d b\mathsf{c}}\right\},0\right)$$

in the following.

(i): Assume that there exists $\beta\in\mathbb{R}_{\ge 1}$ with $\beta\theta/2\notin\pi(2\mathbb{Z}+1)$ such that $\Delta>1/\beta$. Then by (1.11),

$$\begin{aligned}D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\, G_{\beta,\theta,\Delta}(\mathbf{k})&\le\left(1-\frac{1}{\cosh(1)}\right)^{-1}D_d b\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{e(\mathbf{k})}\\&\le\left(1-\frac{1}{\cosh(1)}\right)^{-1}D_d b\mathsf{c}.\end{aligned}$$

Thus

$$|U| \geq \frac{2(\cosh(1)-1)}{\cosh(1)D_d b\mathsf{c}},$$

which contradicts the assumption. Thus the claim (i) holds with

$$c_2 = \min\left\{c_1, \frac{2(\cosh(1)-1)}{\cosh(1)D_d b\mathsf{c}}\right\}.$$

(v): The claim follows from Theorem 1.3 (v) and the claim (i) of this corollary.

(ii): Let $\beta \geq 1$. Observe that

$$\begin{aligned}
&(\text{R. H. S of (1.19)})\\
&= \frac{\Delta^2}{|U|} - D_d \int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}(\sqrt{E(\mathbf{k})^2+\Delta^2} - E(\mathbf{k}))\\
&\quad - \frac{D_d}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\\
&\qquad\qquad \cdot \mathrm{Tr}\log\left(1 + 2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{E(\mathbf{k})^2+\Delta^2}} + e^{-2\beta\sqrt{E(\mathbf{k})^2+\Delta^2}}\right).
\end{aligned}$$

By a calculation similar to (5.36)

$$\begin{aligned}
b\log 4 &\geq \mathrm{Tr}\log\left(1 + 2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{E(\mathbf{k})^2+\Delta^2}} + e^{-2\beta\sqrt{E(\mathbf{k})^2+\Delta^2}}\right)\\
&\geq b\log\left(c\min\left\{1, \beta^2\left(e(\mathbf{k})^2 + \left(\frac{\theta(\beta)}{2} - \frac{\pi}{\beta}\right)^2\right)\right\}\right), \quad (\forall \mathbf{k} \in \mathbb{R}^d).
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|\frac{D_d}{\beta}\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\log\left(1 + 2\cos\left(\frac{\beta\theta(\beta)}{2}\right)e^{-\beta\sqrt{E(\mathbf{k})^2+\Delta^2}} + e^{-2\beta\sqrt{E(\mathbf{k})^2+\Delta^2}}\right)\right|\\
&\leq \frac{c(D_d,b)}{\beta}\left(1 + \int_{\Gamma_\infty^*} d\mathbf{k}\left|\log\left(\beta^2\left(e(\mathbf{k})^2 + \left(\frac{\theta(\beta)}{2} - \frac{\pi}{\beta}\right)^2\right)\right)\right|\right)\\
&\leq \frac{c(D_d,b)}{\beta}\left(1 + \log\beta + \int_{\Gamma_\infty^*} d\mathbf{k}\left(e(\mathbf{k})^2 + \left(\frac{\theta(\beta)}{2} - \frac{\pi}{\beta}\right)^2\right)^{\frac{1}{2}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Gamma_\infty^*} d\mathbf{k}\left(e(\mathbf{k})^2+\left(\frac{\theta(\beta)}{2}-\frac{\pi}{\beta}\right)^2\right)^{-\frac{1}{2}}\Bigg)\\
&\le \frac{c(D_d,b,\mathsf{c})}{\beta}(1+\log\beta).
\end{aligned}
$$

In the last inequality we used (1.11). Then by using the claim (i) of this corollary we can deduce the first convergent property. The second convergent property can be derived from Lemma 3.2 and the same calculation as above.

(iii): Observe that the modulus of the right-hand side of (1.20) is less than or equal to $\Delta/|U|$. Thus it is clear from the claim (i) of this corollary that the expectation value converges to zero if we take the limit $\beta\to\infty$ after sending γ to 0. Let us prove the claims concerning the limit $\gamma\searrow 0$ after sending $\beta\to\infty$. Recall the equality (5.55). To make clear the dependency on β, let us write $a(\beta,\gamma)$ instead of $a(\gamma)$. Let us define the function $f:\mathbb{R}_{>0}\times[-1,1]\times\mathbb{R}_{>0}\to\mathbb{R}$ by

$$
\begin{aligned}
&f(x,y,z)\\
&:= z\Bigg(-\frac{2}{|U|}\\
&\qquad + D_d\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{(y+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))\sqrt{E(\mathbf{k})^2+z^2}}\right)\Bigg)\\
&\quad +\frac{2\gamma}{|U|}.
\end{aligned}
$$

For any $(x,y)\in\mathbb{R}_{>0}\times[-1,1]$ there uniquely exists $z(x,y)\in\mathbb{R}_{>0}$ such that $f(x,y,z(x,y))=0$. Set

$$
\begin{aligned}
S:=\Bigg\{&(x,y,z)\in\mathbb{R}_{>0}\times(-1,1)\times\mathbb{R}_{>0}\ \Bigg|\ -\frac{2}{|U|}\\
&+D_d\int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{(y+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))\sqrt{E(\mathbf{k})^2+z^2}}\right)<0\Bigg\}.
\end{aligned}
$$

The set S is an open set of $\mathbb{R}^3$ and $f\in C^\infty(S)$. If $(x,y)\in\mathbb{R}_{>0}\times(-1,1)$ and $f(x,y,z(x,y))=0$, then $(x,y,z(x,y))\in S$. Observe that for any

$(x,y,z)\in S$ $\frac{\partial f}{\partial y}(x,y,z)<0$, $\frac{\partial f}{\partial z}(x,y,z)<0$. Thus by the implicit function theorem, $z(\cdot,\cdot)\in C^\infty(\mathbb{R}_{>0}\times(-1,1))$ and

$$\frac{\partial z}{\partial y}(x,y)=-\frac{\frac{\partial f}{\partial y}(x,y,z(x,y))}{\frac{\partial f}{\partial z}(x,y,z(x,y))}<0,\quad (\forall (x,y)\in\mathbb{R}_{>0}\times(-1,1)).$$

Fix $x\in\mathbb{R}_{>0}$. Since $y\mapsto z(x,y):(-1,1)\to\mathbb{R}_{>0}$ is monotone decreasing and bounded from below, $\lim_{y\nearrow 1}z(x,y)$ exists in $\mathbb{R}_{\ge 0}$. We can take the limit $y\nearrow 1$ in the equality $f(x,y,z(x,y))=0$. Then by the uniqueness of the solution to the equation $f(x,1,z)=0$, $\lim_{y\nearrow 1}z(x,y)=z(x,1)$. Since $\lim_{z\to\infty}\sup_{y\in[-1,1]}f(x,y,z)=-\infty$, $y\mapsto z(x,y):(-1,1)\to\mathbb{R}_{>0}$ is bounded from above. Thus $\lim_{y\searrow -1}z(x,y)$ exists in $\mathbb{R}_{\ge 0}$. Since $\lim_{y\searrow -1}z(x,y)\ge z(x,1)>0$, we can take the limit $y\searrow -1$ in the equality $f(x,y,z(x,y))=0$ and by the uniqueness of the solution we conclude that $\lim_{y\searrow -1}z(x,y)=z(x,-1)$. Thus we have proved that

$$0<z(x,1)\le z(x,y)\le z(x,-1),\quad (\forall (x,y)\in\mathbb{R}_{>0}\times[-1,1]). \tag{5.70}$$

For $\delta\in\{1,-1\}$, set

$$\begin{aligned}S_\delta:=\Bigg\{&(x,z)\in\mathbb{R}_{>0}\times\mathbb{R}_{>0}\ \Bigg|\ -\frac{2}{|U|}\\ &+D_d\int_{\Gamma_\infty^*}d\mathbf{k}\,\mathrm{Tr}\left(\frac{\sinh(x\sqrt{E(\mathbf{k})^2+z^2})}{(\delta+\cosh(x\sqrt{E(\mathbf{k})^2+z^2}))\sqrt{E(\mathbf{k})^2+z^2}}\right)<0\Bigg\}.\end{aligned}$$

The set S_δ is an open set of $\mathbb{R}^2$, $f(\cdot,\delta,\cdot)\in C^\infty(S_\delta)$ and $(x,z(x,\delta))\in S_\delta$ for any $x\in\mathbb{R}_{>0}$, $\delta\in\{1,-1\}$. Bearing in mind the fact that the functions

$$x\mapsto\frac{\sinh x}{-1+\cosh x}:\mathbb{R}_{>0}\to\mathbb{R},\quad x\mapsto\frac{\sinh x}{1+\cosh x}:\mathbb{R}_{>0}\to\mathbb{R}$$

are strictly monotone decreasing, strictly monotone increasing respectively, we see that $\frac{\partial f}{\partial x}(x,-1,z(x,-1))<0$, $\frac{\partial f}{\partial x}(x,1,z(x,1))>0$, $(\forall x\in\mathbb{R}_{>0})$. As we considered in the proof of Lemma 1.2, the functions

$$x\mapsto\frac{\sinh x}{(\delta+\cosh x)x}:\mathbb{R}_{>0}\to\mathbb{R},\quad (\delta\in\{1,-1\})$$

are strictly monotone decreasing. Based on this fact, we can also verify that $\frac{\partial f}{\partial z}(x,\delta,z(x,\delta)) < 0$, $(\forall x \in \mathbb{R}_{>0},\ \delta \in \{1,-1\})$. Therefore by the implicit function theorem, $z(\cdot,\delta) \in C^\infty(\mathbb{R}_{>0})$ $(\delta = 1,-1)$ and

$$\begin{aligned}&\frac{\partial z}{\partial x}(x,1) = -\frac{\frac{\partial f}{\partial x}(x,1,z(x,1))}{\frac{\partial f}{\partial z}(x,1,z(x,1))} > 0,\\ &\frac{\partial z}{\partial x}(x,-1) = -\frac{\frac{\partial f}{\partial x}(x,-1,z(x,-1))}{\frac{\partial f}{\partial z}(x,-1,z(x,-1))} < 0, \quad (\forall x \in \mathbb{R}_{>0}),\end{aligned}$$

which implies that the functions $x \mapsto z(x,1) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $x \mapsto z(x,-1) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ are strictly monotone increasing, strictly monotone decreasing respectively. Then by the boundedness (5.70) we see that $\lim_{x\to\infty} z(x,1)$, $\lim_{x\to\infty} z(x,-1)$ converge in $\mathbb{R}_{>0}$. Set $z_\infty(\delta) := \lim_{x\to\infty} z(x,\delta)$ for $\delta \in \{1,-1\}$. We can take the limit $x \to \infty$ in the equality $f(x,\delta,z(x,\delta)) = 0$ to derive that

$$(5.71)\qquad z_\infty(\delta)\left(-\frac{2}{|U|} + D_d \int_{\Gamma_\infty^*} d\mathbf{k}\,\mathrm{Tr}\left(\frac{1}{\sqrt{E(\mathbf{k})^2 + z_\infty(\delta)^2}}\right)\right) = -\frac{2\gamma}{|U|}$$

for $\delta \in \{1,-1\}$. Since the solution to this equation is unique in $\mathbb{R}_{>0}$, we have that $z_\infty(1) = z_\infty(-1)$. We can read from (5.70) that

$$\begin{aligned}&z(\beta,1) \le z\left(\beta, \cos\left(\frac{\beta\theta(\beta)}{2}\right)\right) = a(\beta,\gamma) \le z(\beta,-1),\\ &(\forall \beta \in \mathbb{R}_{>0} \text{ with } \beta\theta/2 \notin \pi(2\mathbb{Z}+1)).\end{aligned}$$

Thus it follows that $a(\beta,\gamma)$ converges to the unique positive solution of the equation (5.71) as $\beta \to \infty$ with $\beta \in \mathbb{R}_{>0}$ satisfying $\beta\theta/2 \notin \pi(2\mathbb{Z}+1)$. Set $a_\infty(\gamma) := \lim_{\beta\to\infty, \beta\in\mathbb{R}_{>0} \text{ with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)} a(\beta,\gamma)$. We can derive from (5.71), (1.6) and (1.11) that

$$-\frac{2\gamma}{|U|} \le a_\infty(\gamma)\left(-\frac{2}{|U|} + D_d b\mathsf{c}\right),$$

which combined with the inequality $|U| < 2/(D_d b\mathsf{c})$ implies that

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in(0,1]}} a_\infty(\gamma) = 0.$$

It follows from (5.59) that

$$\begin{aligned}&\lim_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}\mathsf{A}_1)}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z+\mathsf{F})}}\\ &= -a_\infty(\gamma)\frac{D_d}{2}\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{\sqrt{E(\mathbf{k})^2+a_\infty(\gamma)^2}}(\hat{\rho},\hat{\rho}).\end{aligned}$$

Then by recalling (1.11) and sending γ to 0 we reach the claimed equality.

(iv): By Lemma 5.11, (5.62), (5.67) and (1.17)

$$\begin{aligned}(5.72)\qquad &\limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left|\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\uparrow}\psi^*_{\hat{\rho}\hat{\mathbf{x}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}\right|\\ &\le 1_{(\hat{\rho},\hat{\mathbf{x}})=(\hat{\eta},\hat{\mathbf{y}})}\left|\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}C(\Delta)(1\hat{\rho}\mathbf{0}0,1\hat{\rho}\mathbf{0}0)\right|+\frac{\Delta^2}{U^2}\\ &\quad+\left|\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}C(\Delta)(1\hat{\rho}\hat{\mathbf{x}}0,1\hat{\eta}\hat{\mathbf{y}}0)C(\Delta)(2\hat{\eta}\hat{\mathbf{y}}0,2\hat{\rho}\hat{\mathbf{x}}0)\right|.\end{aligned}$$

Let us prove that

$$\limsup_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}}\left\|\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}C(\Delta)(\cdot\hat{\mathbf{x}}0,\cdot\hat{\mathbf{y}}0)\right\|_{2b\times 2b}$$

decays as $\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty$. The β-dependent bound of the form (5.46) has no use here. Remind us the relation (5.39). We have seen a β-independent decay property of $\sum_{l=N_\beta+1}^{\hat{N}_\beta}\mathcal{C}_l$ in Lemma 5.7 (ii). Let us establish a spatial decay property of

$$\limsup_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with } \frac{\beta\theta}{2}\in\pi(2\mathbb{Z}+1)}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\|\mathcal{C}_{N_\beta}(\Delta)(\cdot\mathbf{x}0,\cdot\mathbf{y}0)\|_{2b\times 2b}.$$

Take any $j\in\{1,2,\cdots,d\}$, $\mathbf{x},\mathbf{y}\in\Gamma$. Assume that (5.11) with $\phi=\Delta$ and (5.12) hold so that we can use Lemma 5.2 and inequalities established in the proof of Lemma 5.5. By (5.27) for $l' = N_\beta$, (5.20), (5.31) and Lemma 5.2 (iii),(v)

$$\begin{aligned}
&\left\|\frac{L}{2\pi}(e^{-\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle}-1)\mathcal{C}_{N_\beta}(\Delta)(\cdot\mathbf{x}0,\cdot\mathbf{y}0)\right\|_{2b\times 2b}\\
&\le \frac{c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)}{\beta L^d}\sup_{\mathbf{p}\in\mathbb{R}^d}\sum_{\mathbf{k}\in\Gamma^*}1_{\chi_{N_\beta}(\frac{\pi}{\beta},\mathbf{k}+\mathbf{p})\neq 0}\\
&\quad\cdot\Bigg(M^{-\frac{N_\beta}{\mathsf{n}_j}}\left\|h^{-1}(I_{2b}-e^{-\frac{i}{h}(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\Delta)(\mathbf{k}+\mathbf{p})})^{-1}\right\|_{2b\times 2b}\\
&\qquad+\left\|\frac{\partial}{\partial\hat{k}_j}h^{-1}(I_{2b}-e^{-\frac{i}{h}(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2})I_{2b}+\frac{1}{h}E(\Delta)(\mathbf{k}+\mathbf{p})})^{-1}\right\|_{2b\times 2b}\Bigg)\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)\\
&\quad\cdot\Bigg(\frac{1}{L^d}\sup_{\mathbf{p}\in\mathbb{R}^d}\sum_{\mathbf{k}\in\Gamma^*}\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k}+\mathbf{p})^2+\Delta^2\right)^{-\frac{1}{2}}\\
&\qquad+\frac{1}{\beta L^d}\sup_{\mathbf{p}\in\mathbb{R}^d}\sum_{\mathbf{k}\in\Gamma^*}1_{\chi_{N_\beta}(\frac{\pi}{\beta},\mathbf{k}+\mathbf{p})\neq 0}\\
&\qquad\quad\cdot\left(\left(\frac{\pi}{\beta}-\frac{\theta(\beta)}{2}\right)^2+e(\mathbf{k}+\mathbf{p})^2+\Delta^2\right)^{-\frac{1}{2}-\frac{1}{2\mathsf{n}_j}}\Bigg).
\end{aligned}$$

In the last inequality we also used (5.2) and the assumption $\beta \ge 1$. By (5.3) and the support property of $\chi(\cdot)$

$$1_{\chi_{N_\beta}(\frac{\pi}{\beta},\mathbf{k}+\mathbf{p})\neq 0}\le \chi(2^{-1}M^{-N_\beta}A(\beta,M)^{-1}e(\mathbf{k}+\mathbf{p})).$$

By substituting this inequality and using periodicity and (1.11), for any $\mathbf{x},\mathbf{y}\in\Gamma_\infty$

$$\begin{aligned}
&|\langle \mathbf{x}-\mathbf{y},\hat{\mathbf{v}}_j\rangle|\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in\frac{2}{\beta}\mathbb{N}}}\|\mathcal{C}_{N_\beta}(\Delta)(\cdot\mathbf{x}0,\cdot\mathbf{y}0)\|_{2b\times 2b}\\
&\le c(d,M,\chi,\mathsf{c},\mathsf{a},(\hat{\mathbf{v}}_j)_{j=1}^d)\Bigg(\int_{\Gamma_\infty^*}d\mathbf{k}\frac{1}{e(\mathbf{k})}\\
&\quad+\frac{1}{\beta}\int_{\Gamma_\infty^*}d\mathbf{k}\chi(2^{-1}M^{-N_\beta}A(\beta,M)^{-1}e(\mathbf{k}))
\end{aligned}$$

$$
\cdot \left(\left(\left(\frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right)^2 + e(\mathbf{k})^2 + \Delta^2 \right)^{-\frac{1}{2}-\frac{1}{2\mathsf{n}_j}} \right)
$$

$$
\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^d)
$$

$$
\cdot \left(1 + \frac{1}{\beta} \int_{\Gamma_\infty^*} d\mathbf{k} 1_{e(\mathbf{k}) \le 4\beta^{-1}} \left(\left(\left(\frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right)^2 + e(\mathbf{k})^2 + \Delta^2 \right)^{-\frac{1}{2}-\frac{1}{2\mathsf{n}_j}} \right) \right).
$$

In the second inequality we also used the support property of $\chi(\cdot)$ and recalled the definition of $A(\beta, M)$. By the claim (i) of this corollary, if $e(\mathbf{k}) \le 4\beta^{-1}$ and β is large, $(\pi/\beta - \theta(\beta)/2)^2 + e(\mathbf{k})^2 + \Delta^2 \le (\pi^2+17)\beta^{-2} < 1$. Thus by the condition $\mathsf{n}_j \ge 1$ and Lemma 1.2,

$$
|\langle \mathbf{x}-\mathbf{y}, \hat{\mathbf{v}}_j \rangle| \limsup_{\substack{\beta\to\infty, \beta\in\mathbb{R}_{>0} \\ \text{with } \frac{\beta\theta}{2}\in\pi(2\mathbb{Z}+1)}} \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty \\ h\in\frac{2}{\beta}\mathbb{N}}} \|\mathcal{C}_{N_\beta}(\Delta)(\cdot\mathbf{x}0, \cdot\mathbf{y}0)\|_{2b\times 2b}
$$

$$
\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^d) \limsup_{\substack{\beta\to\infty, \beta\in\mathbb{R}_{>0} \\ \text{with } \frac{\beta\theta}{2}\in\pi(2\mathbb{Z}+1)}}
$$

$$
\cdot \left(1 + \frac{1}{\beta} \int_{\Gamma_\infty^*} d\mathbf{k} 1_{e(\mathbf{k}) \le 4\beta^{-1}} \left(\left(\left(\frac{\pi}{\beta} - \frac{\theta(\beta)}{2} \right)^2 + e(\mathbf{k})^2 + \Delta^2 \right)^{-1} \right) \right)
$$

$$
\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^d) \left(1 + \limsup_{\substack{\beta\to\infty, \beta\in\mathbb{R}_{>0} \\ \text{with } \frac{\beta\theta}{2}\in\pi(2\mathbb{Z}+1)}} \int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr} G_{\beta,\theta,\Delta}(\mathbf{k}) \right)
$$

$$
\le c(d, M, \chi, \mathsf{c}, \mathsf{a}, (\hat{\mathbf{v}}_j)_{j=1}^d) \left(1 + \frac{1}{|U|} \right).
$$

To make clear, let us remark that the inequality

$$
\int_{\Gamma_\infty^*} d\mathbf{k} \operatorname{Tr} G_{\beta,\theta,\Delta}(\mathbf{k}) \le \frac{2}{D_d |U|}
$$

ensured by Lemma 1.2 and the definition of Δ was used. By combining the above inequality with Lemma 5.7 (ii) and recalling (5.39) we have that for any $\mathbf{x}, \mathbf{y} \in \Gamma_\infty$

$$
\sum_{j=1}^d |\langle \mathbf{x}-\mathbf{y}, \hat{\mathbf{v}}_j \rangle| \limsup_{\substack{\beta\to\infty, \beta\in\mathbb{R}_{>0} \\ \text{with } \frac{\beta\theta}{2}\in\pi(2\mathbb{Z}+1)}} \left\| \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} C(\Delta)(\cdot\mathbf{x}0, \cdot\mathbf{y}0) \right\|_{2b\times 2b}
$$

$$\leq c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,\mathsf{a},(\mathsf{n}_j)_{j=1}^d,\mathsf{c},M,\chi)\left(1+\frac{1}{|U|}\right).$$

Now coming back to (5.72) and using the claim (i) of this corollary again, we conclude that

$$\begin{aligned}&\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty}\ \limsup_{\substack{\beta\to\infty,\beta\in\mathbb{R}_{>0}\\ \text{with }\frac{\beta\theta}{2}\notin\pi(2\mathbb{Z}+1)}}\ \limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left|\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}\psi_{\hat{\rho}\hat{\mathbf{x}}\uparrow}^*\psi_{\hat{\rho}\hat{\mathbf{x}}\downarrow}^*\psi_{\hat{\eta}\hat{\mathbf{y}}\downarrow}\psi_{\hat{\eta}\hat{\mathbf{y}}\uparrow})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}+i\theta\mathsf{S}_z)}}\right|\\
&\leq c(d,b,(\hat{\mathbf{v}}_j)_{j=1}^d,\mathsf{a},(\mathsf{n}_j)_{j=1}^d,\mathsf{c},M,\chi)\left(1+\frac{1}{|U|}\right)^2\\
&\quad\cdot\lim_{\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^d}\to\infty}\left(\sum_{j=1}^d|\langle\hat{\mathbf{x}}-\hat{\mathbf{y}},\hat{\mathbf{v}}_j\rangle|\right)^{-2}\\
&=0.\ \square\end{aligned}$$

Appendix A. General Lemmas for the Infinite-Volume Limit

Here we state general lemmas which we use to take the infinite-volume limit of the thermal expectations and the free energy density of our many-electron systems. We use these lemmas in the proof of Theorem 1.3 in Subsection 5.2. The first lemma enables us to take the infinite-volume limit of the thermal expectation of the Cooper pair operator.

LEMMA A.1. *Let f_L, $f\in C^2(\mathbb{R}^2,\mathbb{R})$, g_L, g, u_L, $u\in C(\mathbb{R}^2,\mathbb{C})$ $(L\in\mathbb{N})$. Assume that the following conditions hold.*

(i) For any non-empty compact set Q of $\mathbb{R}^2$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{\mathbf{x}\in Q}\left|\frac{\partial^{i+j}}{\partial x_1^i\partial x_2^j}f_L(\mathbf{x})-\frac{\partial^{i+j}}{\partial x_1^i\partial x_2^j}f(\mathbf{x})\right|=0,\tag{A.1}$$

$$(\forall i,j\in\mathbb{N}\cup\{0\}\text{ satisfying }i+j\leq 2),$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{\mathbf{x}\in Q}|g_L(\mathbf{x})-g(\mathbf{x})|=0,\quad \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{\mathbf{x}\in Q}|u_L(\mathbf{x})-u(\mathbf{x})|=0.$$

(ii)

$$\sup_{L\in\mathbb{N}}\sup_{\mathbf{x}\in\mathbb{R}^2}|u_L(\mathbf{x})|<\infty,\quad \sup_{L\in\mathbb{N}}\sup_{\mathbf{x}\in\mathbb{R}^2}|g_L(\mathbf{x})|<\infty.$$

(iii) There exist $R,\ c\in\mathbb{R}_{>0}$ such that

$$f_L(\mathbf{x})\le -c\|\mathbf{x}\|_{\mathbb{R}^2},\quad (\forall\mathbf{x}\in\mathbb{R}^2 \text{ with } \|\mathbf{x}\|_{\mathbb{R}^2}\ge R,\ L\in\mathbb{N}). \tag{A.2}$$

(iv) There exist $\mathbf{a}_L,\mathbf{a}\in\mathbb{R}^2$ $(L\in\mathbb{N})$ such that

$$f_L(\mathbf{a}_L)>f_L(\mathbf{x}),\quad (\forall\mathbf{x}\in\mathbb{R}^2\backslash\{\mathbf{a}_L\},\ L\in\mathbb{N}), \tag{A.3}$$

$$f(\mathbf{a})>f(\mathbf{x}),\quad (\forall\mathbf{x}\in\mathbb{R}^2\backslash\{\mathbf{a}\}), \tag{A.4}$$

$$H(f)(\mathbf{a})<0, \tag{A.5}$$

$$g(\mathbf{a})\neq 0.$$

Here $H(f)(\mathbf{x})$ denotes the Hessian of f.

Then

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\int_{\mathbb{R}^2}d\mathbf{x}e^{L^d f_L(\mathbf{x})}u_L(\mathbf{x})}{\int_{\mathbb{R}^2}d\mathbf{x}e^{L^d f_L(\mathbf{x})}g_L(\mathbf{x})}=\frac{u(\mathbf{a})}{g(\mathbf{a})}.$$

PROOF. The proof below is essentially a digest of the part concerning SSB of the proof of [12, Theorem 1.3]. By basic arguments based on the assumptions one can prove the following properties.

•

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\mathbf{a}_L=\mathbf{a}. \tag{A.6}$$

• There exist $\delta\in\mathbb{R}_{>0}$, $L_0\in\mathbb{N}$ such that for any $L\in\mathbb{N}$ with $L\ge L_0$,

(A.7)

$$f_L(\mathbf{x})=f_L(\mathbf{a}_L)+\int_0^1 dt(1-t)\langle\mathbf{x}-\mathbf{a}_L,H(f_L)(t(\mathbf{x}-\mathbf{a}_L)+\mathbf{a}_L)(\mathbf{x}-\mathbf{a}_L)\rangle,$$
$$(\forall\mathbf{x}\in\overline{B_\delta(\mathbf{a}_L)}),$$

(A.8)

$$H(f_L)(t(\mathbf{x}-\mathbf{a}_L)+\mathbf{a}_L) \le \frac{1}{2}H(f)(\mathbf{a}), \quad (\forall \mathbf{x} \in \overline{B_\delta(\mathbf{a}_L)},\ t \in [0,1]),$$

(A.9)

$$f_L(\mathbf{x}) - f_L(\mathbf{a}_L) \le \frac{1}{2} \sup_{\mathbf{y}\in\mathbb{R}^2\backslash\overline{B_{\delta/2}(\mathbf{a})}} (f(\mathbf{y}) - f(\mathbf{a})) < 0, \quad (\forall \mathbf{x} \in \mathbb{R}^2\backslash\overline{B_\delta(\mathbf{a}_L)}).$$

Here $B_r(\mathbf{b})$ denotes $\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}-\mathbf{b}\|_{\mathbb{R}^2} < r\}$ for $\mathbf{b} \in \mathbb{R}^2$, $r \in \mathbb{R}_{>0}$. In fact by using (A.1) with $i = j = 0$, (A.2), (A.3), (A.4) we can prove (A.6). Taylor's theorem gives (A.7) for any $\delta \in \mathbb{R}_{>0}$ and $L \in \mathbb{N}$. Then by (A.1) with (i,j) satisfying $i+j = 2$, (A.5), (A.6) and the continuity of the 2nd order derivatives of f we can prove (A.8) with some δ and any $L \in \mathbb{N}$ satisfying $L \ge L_0$ for some L_0. For the fixed δ the property (A.6) ensures that $\|\mathbf{a}-\mathbf{a}_L\|_{\mathbb{R}^2} \le \delta/2$ for any $L \in \mathbb{N}$ satisfying $L \ge L_0$, if we take L_0 larger if necessary. This implies that $\mathbb{R}^2\backslash\overline{B_\delta(\mathbf{a}_L)} \subset \mathbb{R}^2\backslash\overline{B_{\delta/2}(\mathbf{a})}$. Then for the fixed δ, by taking L_0 larger if necessary we can apply (A.1) with $i+j = 0$, (A.2), (A.4), (A.6) and the continuity of f to prove (A.9).

For any $L \in \mathbb{N}$ with $L \ge L_0$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} d\mathbf{x} e^{L^d f_L(\mathbf{x})} g_L(\mathbf{x}) \\
&= e^{L^d f_L(\mathbf{a}_L)} L^{-d} \Bigg(\int_{\overline{B_{L^{\frac{d}{2}}\delta}(\mathbf{0})}} d\mathbf{x} e^{\int_0^1 dt(1-t)\langle \mathbf{x}, H(f_L)(tL^{-\frac{d}{2}}\mathbf{x}+\mathbf{a}_L)\mathbf{x}\rangle} g_L(L^{-\frac{d}{2}}\mathbf{x}+\mathbf{a}_L) \\
&\qquad + L^d \int_{\mathbb{R}^2\backslash\overline{B_\delta(\mathbf{a}_L)}} d\mathbf{x} e^{L^d(f_L(\mathbf{x})-f_L(\mathbf{a}_L))} g_L(\mathbf{x}) \Bigg).
\end{aligned}$$

It follows from the assumptions, the properties listed above and the dominated convergence theorem that

$$\begin{aligned}
&\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \Bigg(\int_{\overline{B_{L^{\frac{d}{2}}\delta}(\mathbf{0})}} d\mathbf{x} e^{\int_0^1 dt(1-t)\langle \mathbf{x}, H(f_L)(tL^{-\frac{d}{2}}\mathbf{x}+\mathbf{a}_L)\mathbf{x}\rangle} g_L(L^{-\frac{d}{2}}\mathbf{x}+\mathbf{a}_L) \\
&\qquad + L^d \int_{\mathbb{R}^2\backslash\overline{B_\delta(\mathbf{a}_L)}} d\mathbf{x} e^{L^d(f_L(\mathbf{x})-f_L(\mathbf{a}_L))} g_L(\mathbf{x}) \Bigg) \\
&= g(\mathbf{a}) \int_{\mathbb{R}^2} d\mathbf{x} e^{\frac{1}{2}\langle \mathbf{x}, H(f)(\mathbf{a})\mathbf{x}\rangle} \neq 0.
\end{aligned}$$

The numerator can be dealt in the same way. As the result,

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\int_{\mathbb{R}^2}d\mathbf{x}e^{L^2f_L(\mathbf{x})}u_L(\mathbf{x})}{\int_{\mathbb{R}^2}d\mathbf{x}e^{L^2f_L(\mathbf{x})}g_L(\mathbf{x})}=\frac{u(\mathbf{a})\int_{\mathbb{R}^2}d\mathbf{x}e^{\frac{1}{2}\langle\mathbf{x},H(f)(\mathbf{a})\mathbf{x}\rangle}}{g(\mathbf{a})\int_{\mathbb{R}^2}d\mathbf{x}e^{\frac{1}{2}\langle\mathbf{x},H(f)(\mathbf{a})\mathbf{x}\rangle}}=\frac{u(\mathbf{a})}{g(\mathbf{a})}. \square$$

Next let us prove a lemma which is used to prove the existence of the infinite-volume limit of the correlation function in the case that the physical parameters are not on the phase boundary.

Lemma A.2. *Let f_L, $f\in C^2(\mathbb{R},\mathbb{R})$, g_L, g, u_L, $u\in C(\mathbb{R},\mathbb{C})$ $(L\in\mathbb{N})$. Assume that the following conditions hold.*

(i) For any $r\in\mathbb{R}_{>0}$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[-r,r]}\left|\frac{d^i}{dx^i}f_L(x)-\frac{d^i}{dx^i}f(x)\right|=0,\quad (\forall i\in\{0,1,2\}),$$
$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[-r,r]}|u_L(x)-u(x)|=0,\quad \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[-r,r]}|g_L(x)-g(x)|=0.$$

(ii)

$$\sup_{L\in\mathbb{N}}\sup_{x\in\mathbb{R}}|u_L(x)|<\infty,\quad \sup_{L\in\mathbb{N}}\sup_{x\in\mathbb{R}}|g_L(x)|<\infty.$$

(iii) There exist R, $c\in\mathbb{R}_{>0}$ such that

$$f_L(x)\le -c|x|,\quad (\forall x\in\mathbb{R}\text{ with }|x|\ge R,\ L\in\mathbb{N}).$$

(iv) There exist $a_L,a\in\mathbb{R}_{\ge 0}$ $(L\in\mathbb{N})$ such that

$$f_L(a_L)>f_L(x),\quad (\forall x\in\mathbb{R}_{\ge 0}\backslash\{a_L\},\ L\in\mathbb{N}),$$
$$f(a)>f(x),\quad (\forall x\in\mathbb{R}_{\ge 0}\backslash\{a\}),$$
$$\frac{d}{dx}f_L(a_L)=0,\quad (\forall L\in\mathbb{N}),$$
$$\frac{d^2}{dx^2}f(a)<0,\quad g(a)\neq 0.$$

Moreover if $a=0$, there exists $L_0\in\mathbb{N}$ such that $a_L=0$ $(\forall L\in\mathbb{N}$ with $L\ge L_0)$.

Then

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\int_0^{\infty}dxxe^{L^df_L(x)}u_L(x)}{\int_0^{\infty}dxxe^{L^df_L(x)}g_L(x)}=\frac{u(a)}{g(a)}. \tag{A.10}$$

Proof. The following argument is a generalization of the part concerning ODLRO of the proof of [12, Theorem 1.3]. The assumptions imply the following statements.

- $$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}a_L=a.$$
- There exist $\delta\in\mathbb{R}_{>0}$, $L_1\in\mathbb{N}$ such that for any $L\in\mathbb{N}$ with $L\ge L_1$,

$$\begin{aligned}
&f_L(x)=f_L(a_L)+\int_0^1dt(1-t)\frac{d^2f_L}{dx^2}(t(x-a_L)+a_L)(x-a_L)^2,\\
&\quad(\forall x\in[a_L-\delta,a_L+\delta]),\\
&\frac{d^2f_L}{dx^2}(t(x-a_L)+a_L)\le\frac{1}{2}\frac{d^2f}{dx^2}(a)<0,\quad(\forall x\in[a_L-\delta,a_L+\delta],\ t\in[0,1]),\\
&f_L(x)-f_L(a_L)\le\frac{1}{2}\sup_{y\in\mathbb{R}_{\ge0}\backslash[a-\frac{\delta}{2},a+\frac{\delta}{2}]}(f(y)-f(a))<0,\\
&\quad(\forall x\in\mathbb{R}_{\ge0}\backslash[a_L-\delta,a_L+\delta]).
\end{aligned}$$

Then let us observe that for $L\in\mathbb{N}$ satisfying $L\ge 1_{a>0}L_1+1_{a=0}\max\{L_0,L_1\}$,

$$\begin{aligned}
&\int_0^{\infty}dxxe^{L^df_L(x)}g_L(x)\\
&=e^{L^df_L(a_L)}\\
&\quad\cdot\Bigg(L^{-\frac{d}{2}}\int_{-L^{\frac{d}{2}}\min\{a_L,\delta\}}^{L^{\frac{d}{2}}\delta}dx(L^{-\frac{d}{2}}x+a_L)e^{\int_0^1dt(1-t)f_L^{(2)}(tL^{-\frac{d}{2}}x+a_L)x^2}\\
&\qquad\qquad\cdot g_L(L^{-\frac{d}{2}}x+a_L)\\
&\qquad+\int_{\mathbb{R}_{\ge0}\backslash[a_L-\delta,a_L+\delta]}dxxe^{L^d(f_L(x)-f_L(a_L))}g_L(x)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= e^{L^d f_L(a_L)}(1_{a>0}L^{-\frac{d}{2}} + 1_{a=0}L^{-d}) \\
&\quad \cdot \Bigg(\int_{-L^{\frac{d}{2}}\min\{a_L,\delta\}}^{L^{\frac{d}{2}}\delta} dx(1_{a>0}(L^{-\frac{d}{2}}x + a_L) + 1_{a=0}x) \\
&\qquad\qquad \cdot e^{\int_0^1 dt(1-t)f_L^{(2)}(tL^{-\frac{d}{2}}x+a_L)x^2} g_L(L^{-\frac{d}{2}}x + a_L) \\
&\qquad + (1_{a>0}L^{\frac{d}{2}} + 1_{a=0}L^d)\int_{\mathbb{R}_{\ge 0}\setminus[a_L-\delta,a_L+\delta]} dxxe^{L^d(f_L(x)-f_L(a_L))}g_L(x)\Bigg).
\end{aligned}
$$

By using the above properties we can take the limit $L \to \infty$ as follows.

$$
\begin{aligned}
&\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \Bigg(\int_{-L^{\frac{d}{2}}\min\{a_L,\delta\}}^{L^{\frac{d}{2}}\delta} dx(1_{a>0}(L^{-\frac{d}{2}}x + a_L) + 1_{a=0}x) \\
&\qquad\qquad \cdot e^{\int_0^1 dt(1-t)f_L^{(2)}(tL^{-\frac{d}{2}}x+a_L)x^2} g_L(L^{-\frac{d}{2}}x + a_L) \\
&\qquad + (1_{a>0}L^{\frac{d}{2}} + 1_{a=0}L^d)\int_{\mathbb{R}_{\ge 0}\setminus[a_L-\delta,a_L+\delta]} dxxe^{L^d(f_L(x)-f_L(a_L))}g_L(x)\Bigg) \\
&= 1_{a>0}ag(a)\int_{-\infty}^{\infty} dxe^{\frac{1}{2}f^{(2)}(a)x^2} + 1_{a=0}g(a)\int_0^{\infty} dxe^{\frac{1}{2}f^{(2)}(a)x^2} \neq 0.
\end{aligned}
$$

The numerator can be decomposed and sent to the limit in the same way. Consequently,

$$
\begin{aligned}
&\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \frac{\int_0^\infty dxxe^{L^d f_L(x)}u_L(x)}{\int_0^\infty dxxe^{L^d f_L(x)}g_L(x)} \\
&= \left(1_{a>0}au(a)\int_{-\infty}^{\infty} dxe^{\frac{1}{2}f^{(2)}(a)x^2} + 1_{a=0}u(a)\int_0^{\infty} dxxe^{\frac{1}{2}f^{(2)}(a)x^2}\right) \\
&\quad \cdot \left(1_{a>0}ag(a)\int_{-\infty}^{\infty} dxe^{\frac{1}{2}f^{(2)}(a)x^2} + 1_{a=0}g(a)\int_0^{\infty} dxxe^{\frac{1}{2}f^{(2)}(a)x^2}\right)^{-1} \\
&= \frac{u(a)}{g(a)}. \ \square
\end{aligned}
$$

We need to estimate the correlation function in the case that the physical parameters are on the phase boundary. We need the next lemma for the purpose.

Lemma A.3. *Let* $n_0 \in 2\mathbb{N}$, f_L, $f \in C^{n_0}(\mathbb{R},\mathbb{R})$, $g_L \in C(\mathbb{R},\mathbb{R})$, u_L, $u \in C(\mathbb{R},\mathbb{C})$ $(L \in \mathbb{N})$. *Assume that the following conditions hold.*

(i) For any $r \in \mathbb{R}_{>0}$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \sup_{x\in[-r,r]} \left| \frac{d^n}{dx^n} f_L(x) - \frac{d^n}{dx^n} f(x) \right| = 0, \quad (\forall n \in \{0,1,\cdots,n_0\}),$$
$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \sup_{x\in[-r,r]} |u_L(x) - u(x)| = 0.$$

(ii)

$$\sup_{L\in\mathbb{N}} \sup_{x\in\mathbb{R}} |u_L(x)| < \infty, \quad \sup_{L\in\mathbb{N}} \sup_{x\in\mathbb{R}} |g_L(x)| < \infty.$$

(iii) There exist R, $c \in \mathbb{R}_{>0}$ *such that*

$$f_L(x) \le -c|x|, \quad (\forall x \in \mathbb{R} \text{ with } |x| \ge R,\ L \in \mathbb{N}).$$

(iv) There exist $a_L, a \in \mathbb{R}_{\ge 0}$ $(L \in \mathbb{N})$ *such that*

$$f_L(a_L) > f_L(x), \quad (\forall x \in \mathbb{R}_{\ge 0}\backslash\{a_L\},\ L \in \mathbb{N}),$$
$$f(a) > f(x), \quad (\forall x \in \mathbb{R}_{\ge 0}\backslash\{a\}),$$
$$\frac{d^n}{dx^n} f(a) = 0, \quad (\forall n \in \{1,2,\cdots,n_0-1\}),$$
$$\frac{d^{n_0}}{dx^{n_0}} f(a) < 0.$$

(v)

$$\inf_{L\in\mathbb{N}} \inf_{x\in\mathbb{R}} g_L(x) > 0.$$

Then

$$\limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}} \left| \frac{\int_0^\infty dx x e^{L^d f_L(x)} g_L(x) u_L(x)}{\int_0^\infty dx x e^{L^d f_L(x)} g_L(x)} \right| \le \sup_{x\in[0,a]} |u(x)|.$$

Proof. Since $f^{(n_0)}(a) < 0$ and $f^{(n_0)}(\cdot)$ is continuous, there exists $\delta_0 \in \mathbb{R}_{>0}$ such that

$$f^{(n_0)}(x) \le \frac{2}{3} f^{(n_0)}(a), \quad (\forall x \in [a-\delta_0, a+\delta_0]). \tag{A.11}$$

Take any $\delta \in (0, \delta_0)$. By using the assumptions and (A.11) we can prove the following statements.

- $$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} a_L = a.$$

- There exists $L_0 \in \mathbb{N}$ such that for any $L \in \mathbb{N}$ with $L \ge L_0$,

$$\begin{aligned}
f_L(x) = &\sum_{n=0}^{n_0-1} \frac{1}{n!} f_L^{(n)}(a_L)(x-a_L)^n \\
&+ \int_0^1 dt \frac{(1-t)^{n_0-1}}{(n_0-1)!} f_L^{(n_0)}(t(x-a_L)+a_L)(x-a_L)^{n_0}, \\
&(\forall x \in [a_L-\delta, a_L+\delta]), \\
f_L^{(n_0)}(t(x-a_L)+a_L) &\le \frac{1}{2} f^{(n_0)}(a) < 0, \quad (\forall x \in [a_L-\delta, a_L+\delta],\ t \in [0,1]), \\
f_L(x) - f_L(a_L) &\le \frac{1}{2} \sup_{y\in\mathbb{R}_{\ge 0}\backslash[a-\frac{\delta}{2}, a+\frac{\delta}{2}]} (f(y)-f(a)) < 0, \\
&(\forall x \in \mathbb{R}_{\ge 0}\backslash[a_L-\delta, a_L+\delta]).
\end{aligned}$$

- $$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} f_L^{(n)}(a_L) = f^{(n)}(a) = 0, \quad (\forall n \in \{1, 2, \cdots, n_0-1\}).$$

Let us observe that

(A.12)

$$\int_0^\infty dx x e^{L^d f_L(x)} u_L(x) g_L(x)$$
$$= e^{L^d f_L(a_L)} L^{-\frac{d}{n_0}}$$

$$\cdot\Bigg(\int_{-L^{\frac{d}{n_0}}\min\{a_L,\delta\}}^{L^{\frac{d}{n_0}}\delta}dx(L^{-\frac{d}{n_0}}x+a_L)$$
$$\cdot e^{\sum_{n=1}^{n_0-1}\frac{1}{n!}L^{(1-\frac{n}{n_0})d}f_L^{(n)}(a_L)x^n+\int_0^1dt\frac{1}{(n_0-1)!}(1-t)^{n_0-1}f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x+a_L)x^{n_0}}$$
$$\cdot u_L(L^{-\frac{d}{n_0}}x+a_L)g_L(L^{-\frac{d}{n_0}}x+a_L)$$
$$+L^{\frac{d}{n_0}}\int_{\mathbb{R}_{\ge0}\backslash[a_L-\delta,a_L+\delta]}dxxe^{L^d(f_L(x)-f_L(a_L))}u_L(x)g_L(x)\Bigg).$$

By decomposing the denominator in the same way as above we can derive that

$$\left|\frac{\int_0^\infty dxxe^{L^df_L(x)}g_L(x)u_L(x)}{\int_0^\infty dxxe^{L^df_L(x)}g_L(x)}\right|$$
$$\le\Bigg(\int_{-L^{\frac{d}{n_0}}\min\{a_L,\delta\}}^{L^{\frac{d}{n_0}}\delta}dx(L^{-\frac{d}{n_0}}x+a_L)$$
$$\cdot e^{\sum_{n=1}^{n_0-1}\frac{1}{n!}L^{(1-\frac{n}{n_0})d}f_L^{(n)}(a_L)x^n+\int_0^1dt\frac{1}{(n_0-1)!}(1-t)^{n_0-1}f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x+a_L)x^{n_0}}$$
$$\cdot|u_L(L^{-\frac{d}{n_0}}x+a_L)|g_L(L^{-\frac{d}{n_0}}x+a_L)$$
$$+L^{\frac{d}{n_0}}\int_{\mathbb{R}_{\ge0}\backslash[a_L-\delta,a_L+\delta]}dxxe^{L^d(f_L(x)-f_L(a_L))}|u_L(x)|g_L(x)\Bigg)$$
$$\cdot\Bigg(\int_{-L^{\frac{d}{n_0}}\min\{a_L,\delta\}}^{L^{\frac{d}{n_0}}\delta}dx(L^{-\frac{d}{n_0}}x+a_L)$$
$$\cdot e^{\sum_{n=1}^{n_0-1}\frac{1}{n!}L^{(1-\frac{n}{n_0})d}f_L^{(n)}(a_L)x^n+\int_0^1dt\frac{1}{(n_0-1)!}(1-t)^{n_0-1}f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x+a_L)x^{n_0}}$$
$$\cdot g_L(L^{-\frac{d}{n_0}}x+a_L)\Bigg)^{-1}$$
$$\le\sup_{x\in[0,\delta+a_L]}|u_L(x)|$$
$$+\Bigg(L^{2\frac{d}{n_0}}e^{\sum_{n=1}^{n_0-1}\frac{L^d}{n!}|f_L^{(n)}(a_L)|\delta^n}$$

$$
\begin{aligned}
&\cdot \int_{\mathbb{R}_{\geq 0}\backslash[a_L-\delta,a_L+\delta]} dx x e^{L^d(f_L(x)-f_L(a_L))}|u_L(x)|g_L(x)\Bigg)\\
&\cdot\Bigg(\int_0^{L^{\frac{d}{n_0}}\delta} dx x e^{\int_0^1 dt \frac{1}{(n_0-1)!}(1-t)^{n_0-1}f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x+a_L)x^{n_0}}\\
&\qquad\cdot g_L(L^{-\frac{d}{n_0}}x+a_L)\Bigg)^{-1}\\
\leq & \sup_{x\in[0,\delta+a_L]}|u_L(x)|\\
&+\sup_{L\in\mathbb{N}}\sup_{x\in\mathbb{R}}|u_L(x)g_L(x)|\left(\inf_{L\in\mathbb{N}}\inf_{x\in\mathbb{R}} g_L(x)\right)^{-1}\\
&\cdot\Bigg(L^{2\frac{d}{n_0}}e^{\sum_{n=1}^{n_0-1}\frac{L^d}{n!}|f_L^{(n)}(a_L)|\delta^n}\int_{\mathbb{R}_{\geq 0}\backslash[a_L-\delta,a_L+\delta]} dx x e^{L^d(f_L(x)-f_L(a_L))}\Bigg)\\
&\cdot\Bigg(\int_0^{L^{\frac{d}{n_0}}\delta} dx x e^{\int_0^1 dt \frac{1}{(n_0-1)!}(1-t)^{n_0-1}f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x+a_L)x^{n_0}}\Bigg)^{-1}.
\end{aligned}
$$

Then by using the properties listed in the beginning of the proof we can deduce that

$$
\begin{aligned}
\limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}}\left|\frac{\int_0^\infty dx x e^{L^d f_L(x)}g_L(x)u_L(x)}{\int_0^\infty dx x e^{L^d f_L(x)}g_L(x)}\right| &\leq \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\sup_{x\in[0,2\delta+a]}|u_L(x)|\\
&=\sup_{x\in[0,2\delta+a]}|u(x)|.
\end{aligned}
$$

The arbitrariness of δ implies the result. □

Finally let us show a lemma which ensures the convergence of the free energy density in the infinite-volume limit when it is applied in practice.

LEMMA A.4. *Let $n_0\in 2\mathbb{N}$, f_L, $f\in C^{n_0}(\mathbb{R},\mathbb{R})$, $g_L\in C(\mathbb{R},\mathbb{R})$, $(L\in\mathbb{N})$. Assume that these functions satisfy the same conditions as in Lemma A.3. Then*

$$
\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{L^d}\log\left(\int_0^\infty dx x e^{L^d f_L(x)}g_L(x)\right)=f(a).
$$

PROOF. Since the assumptions are same, we can transform the integral inside the logarithm in the same way as in the proof of Lemma A.3. We use the following equality close to (A.12). For $\varepsilon \in \{1,-1\}$

$$\begin{aligned}
&\int_0^\infty dx x e^{L^d f_L(x)} g_L(x)\\
&= L^{-\frac{d}{n_0}} e^{L^d f_L(a_L)+\varepsilon\sum_{n=1}^{n_0-1}\frac{L^d}{n!}|f_L^{(n)}(a_L)|\delta^n}\\
&\quad\cdot\Bigg(\int_{-L^{\frac{d}{n_0}}\min\{a_L,\delta\}}^{L^{\frac{d}{n_0}}\delta} dx(L^{-\frac{d}{n_0}}x+a_L)\\
&\qquad\cdot e^{\sum_{n=1}^{n_0-1}\frac{L^d}{n!}(L^{-\frac{n}{n_0}d}f_L^{(n)}(a_L)x^n-\varepsilon|f_L^{(n)}(a_L)|\delta^n)}\\
&\qquad\cdot e^{\int_0^1 dt\frac{1}{(n_0-1)!}(1-t)^{n_0-1}f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x+a_L)x^{n_0}}\\
&\qquad\cdot g_L(L^{-\frac{d}{n_0}}x+a_L)\\
&\quad+L^{\frac{d}{n_0}}\int_{\mathbb{R}_{\ge 0}\backslash[a_L-\delta,a_L+\delta]} dx x\\
&\qquad\cdot e^{-\varepsilon\sum_{n=1}^{n_0-1}\frac{L^d}{n!}|f_L^{(n)}(a_L)|\delta^n+L^d(f_L(x)-f_L(a_L))}g_L(x)\Bigg).
\end{aligned}$$

By taking $\varepsilon = 1$ this implies that

(A.13)

$$\begin{aligned}
&\frac{1}{L^d}\log\left(\int_0^\infty dx x e^{L^d f_L(x)} g_L(x)\right)\\
&\le f_L(a_L)+\sum_{n=1}^{n_0-1}\frac{1}{n!}|f_L^{(n)}(a_L)|\delta^n\\
&\quad+\frac{1}{L^d}\log\Bigg(\int_{-L^{\frac{d}{n_0}}\min\{a_L,\delta\}}^{L^{\frac{d}{n_0}}\delta} dx(L^{-\frac{d}{n_0}}x+a_L)\\
&\qquad\cdot e^{\int_0^1 dt\frac{1}{(n_0-1)!}(1-t)^{n_0-1}f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x+a_L)x^{n_0}} g_L(L^{-\frac{d}{n_0}}x+a_L)\\
&\qquad+\int_{\mathbb{R}_{\ge 0}\backslash[a_L-\delta,a_L+\delta]} dx x e^{L^d(f_L(x)-f_L(a_L))}g_L(x)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\le f_L(a_L) + \sum_{n=1}^{n_0-1} \frac{1}{n!} |f_L^{(n)}(a_L)| \delta^n \\
&\quad + \frac{1}{L^d} \log \Bigg((\delta + a_L) \sup_{L\in\mathbb{N}} \sup_{y\in\mathbb{R}} g_L(y) \\
&\qquad\qquad \cdot \int_{-L^{\frac{d}{n_0}}\delta}^{L^{\frac{d}{n_0}}\delta} dx e^{\int_0^1 dt \frac{1}{(n_0-1)!}(1-t)^{n_0-1} f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x + a_L) x^{n_0}} \\
&\qquad\qquad + \sup_{L\in\mathbb{N}} \sup_{y\in\mathbb{R}} g_L(y) \int_{\mathbb{R}_{\ge 0}\backslash[a_L-\delta, a_L+\delta]} dx e^{L^d(f_L(x)-f_L(a_L))} \Bigg).
\end{aligned}
$$

On the other hand, by taking $\varepsilon = -1$

(A.14)

$$
\begin{aligned}
&\frac{1}{L^d} \log \left(\int_0^\infty dx e^{L^d f_L(x)} g_L(x) \right) \\
&\ge f_L(a_L) - \sum_{n=1}^{n_0-1} \frac{1}{n!} |f_L^{(n)}(a_L)| \delta^n + \frac{1}{L^d} \log(L^{-\frac{2d}{n_0}}) \\
&\quad + \frac{1}{L^d} \log \Bigg(\inf_{L\in\mathbb{N}} \inf_{y\in\mathbb{R}} g_L(y) \\
&\qquad\qquad \cdot \int_0^{L^{\frac{d}{n_0}}\delta} dx e^{\int_0^1 dt \frac{1}{(n_0-1)!}(1-t)^{n_0-1} f_L^{(n_0)}(tL^{-\frac{d}{n_0}}x + a_L) x^{n_0}} \Bigg).
\end{aligned}
$$

By using the properties listed in the beginning of the proof of Lemma A.3 we can show that both the right-hand side of (A.13) and that of (A.14) converge to $f(a)$ as $L \to \infty$. □

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Supplementary List of Notations

Parameters and constants

Notation	Description	Reference
b	number of sites in unit cell	Subsection 1.2
c	positive constant (≥ 1) appearing in bounds on $E(\cdot)$ and $e(\cdot)$	Subsection 1.2
n_j $(j = 1, \cdots, d)$	positive numbers ($\in \mathbb{N}$) appearing in bounds on derivatives of $e(\cdot)^2$ and $E(\cdot)$	(1.8), (1.9)
a	positive constant (> 1) appearing in bounds on integrals of $e(\cdot)$	(1.10), (1.11)
D_d	$\lvert \det(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \cdots, \hat{\mathbf{v}}_d)\rvert^{-1}(2\pi)^{-d}$	Subsection 1.2
$\theta(\beta)$	projection of θ to $[0, \frac{2\pi}{\beta})$	beginning of Section 3
N	$4b\beta hL^d$, cardinality of I	beginning of Subsection 4.2
$\hat{N}_\beta$	largest scale in IR integration	Subsection 4.3 and beginning of Subsection 5.1
N_β	smallest scale in IR integration	Subsection 4.3 and beginning of Subsection 5.1
M	parameter to control support size of cut-off	Subsection 4.3
c_0	positive constant (≥ 1) appearing in bounds on scale-dependent covariances	Subsection 4.3
c_{end}	positive constant appearing in $\lVert\cdot\rVert_{1,\infty}$-norm bound of covariance of scale N_β	(4.34)

Sets and spaces

Notation	Description	Reference
Γ	$\{\sum_{j=1}^d m_j\mathbf{v}_j \mid m_j \in \{0, 1, \cdots, L-1\}, (j = 1, 2, \cdots, d)\}$	Subsection 1.2
Γ^*	$\{\sum_{j=1}^d \hat{m}_j\hat{\mathbf{v}}_j \mid \hat{m}_j \in \{0, \frac{2\pi}{L}, \cdots, 2\pi - \frac{2\pi}{L}\}, (j = 1, 2, \cdots, d)\}$	Subsection 1.2
Γ_∞	$\{\sum_{j=1}^d m_j\mathbf{v}_j \mid m_j \in \mathbb{Z}\ (j = 1, 2, \cdots, d)\}$	Subsection 1.2
Γ_∞^*	$\{\sum_{j=1}^d \hat{k}_j\hat{\mathbf{v}}_j \mid \hat{k}_j \in [0, 2\pi]\ (j = 1, 2, \cdots, d)\}$	Subsection 1.2
$\mathcal{B}$	$\{1, 2, \cdots, b\}$	Subsection 1.2
$\mathrm{Mat}(n, \mathbb{C})$	set of $n \times n$ complex matrices	Subsection 1.2

I_0	$\{1,2\}\times\mathcal{B}\times\Gamma\times[0,\beta)_h$	Section 3
I	$I_0\times\{1,-1\}$	Section 3
$\mathcal{V}$	complex vector space spanned by $\{\psi_X\}_{X\in I}$	Section 3
$\bigwedge\mathcal{V}$	Grassmann algebra generated by $\{\psi_X\}_{X\in I}$	Section 3
I^0	$\{1,2\}\times\mathcal{B}\times\Gamma\times\{0\}\times\{1,-1\}$	Subsection 4.1
$\bigwedge_{even}\mathcal{V}$	subspace of $\bigwedge\mathcal{V}$ consisting of even polynomials	Subsection 4.1
$C(\overline{D},\bigwedge_{even}\mathcal{V})$	set of continuous maps from $\overline{D}$ to $\bigwedge_{even}\mathcal{V}$	Subsection 4.4
$C^{\omega}(D,\bigwedge_{even}\mathcal{V})$	set of analytic maps from D to $\bigwedge_{even}\mathcal{V}$	Subsection 4.4

Functions and maps

Notation	Description	Reference
V	$\frac{U}{L^d}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\psi^*_{\rho\mathbf{x}\uparrow}\psi^*_{\rho\mathbf{x}\downarrow}\psi_{\eta\mathbf{y}\downarrow}\psi_{\eta\mathbf{y}\uparrow}$	Subsection 1.2
r_L	map from Γ_∞ to Γ	Subsection 1.2
$E(\cdot)$	map from $\mathbb{R}^d$ to $\mathrm{Mat}(b,\mathbb{C})$, hopping matrix in momentum space	Subsection 1.2
$e(\cdot)$	non-negative function on $\mathbb{R}^d$	Subsection 1.2
H_0	$\frac{1}{L^d}\sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma}\sum_{\sigma\in\{\uparrow,\downarrow\}}\sum_{\mathbf{k}\in\Gamma^*}$ $\cdot e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}E(\mathbf{k})(\rho,\eta)\psi^*_{\rho\mathbf{x}\sigma}\psi_{\eta\mathbf{y}\sigma}$	Subsection 1.2
S_z	$\frac{1}{2}\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}(\psi^*_{\rho\mathbf{x}\uparrow}\psi_{\rho\mathbf{x}\uparrow}-\psi^*_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\downarrow})$	Subsection 1.2
F	$\gamma\sum_{(\rho,\mathbf{x})\in\mathcal{B}\times\Gamma}(\psi^*_{\rho\mathbf{x}\uparrow}\psi^*_{\rho\mathbf{x}\downarrow}+\psi_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow})$	Subsection 1.2
$G_{x,y,z}(\cdot)$	map from $\mathbb{R}^d$ to $\mathrm{Mat}(b,\mathbb{C})$ parameterized by x,y,z	(1.22)
$C(\phi)(\cdot)$	function on $(\{1,2\}\times\mathcal{B}\times\Gamma_\infty\times[0,\beta))^2$ parameterized by ϕ, full covariance	Section 3
$E(\phi)(\cdot)$	map from $\mathbb{R}^d$ to $\mathrm{Mat}(2b,\mathbb{C})$ parameterized by ϕ	Section 3
$\mathcal{R}_\beta$	map from $(\{1,2\}\times\mathcal{B}\times\Gamma\times\frac{1}{h}\mathbb{Z})^n$ to I_0^n or from $(\{1,2\}\times\mathcal{B}\times\Gamma\times\frac{1}{h}\mathbb{Z}\times\{1,-1\})^n$ to I^n	beginning of Subsection 4.2

Other notations

Notation	Description	Reference
$\mathbf{v}_j$ $(j=1,\cdots,d)$	basis of $\mathbb{R}^d$	Subsection 1.2
$\hat{\mathbf{v}}_j$	dual basis of $\{\mathbf{v}_j\}_{j=1}^d$	Subsection 1.2

$(j = 1, \cdots, d)$		
I_n	$n \times n$ unit matrix	beginning of Subsection 1.3
$V(\psi)$	polynomial of $\bigwedge \mathcal{V}$ consisting of quadratic part and quartic part	Section 3
$W(\psi)$	quartic polynomial of $\bigwedge \mathcal{V}$	Section 3
$A^1(\psi)$	quadratic polynomial of $\bigwedge \mathcal{V}$	Section 3
$A^2(\psi)$	polynomial of $\bigwedge \mathcal{V}$ consisting of quadratic part and quartic part	Section 3
$A(\psi)$	$\lambda_1 A^1(\psi) + \lambda_2 A^2(\psi)$	(3.1)
$V(u)(\psi)$	same as $V(\psi)$, apart from having $u(\in \mathbb{C})$ in place of U	beginning of Subsection 4.4
$W(u)(\psi)$	same as $W(\psi)$, apart from having $u(\in \mathbb{C})$ in place of U	beginning of Subsection 4.4
$[\cdot,\cdot]_{1,\infty,r}$	$\sup_{u\in\overline{D(r)}}[f(u), g]_{1,\infty}$	beginning of Subsection 4.4
$[\cdot,\cdot]_{1,r,r'}$	$\sup_{(u,\boldsymbol{\lambda})\in\overline{D(r)}\times\overline{D(r')}^2}[f(u,\boldsymbol{\lambda}), g]_1$	beginning of Subsection 4.5

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