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Properties of Minimal Charts and their Applications VI: The Graph Γ_{m+1} *in a Chart* Γ *of Type* $(m; 2, 3, 2)$

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Abstract. Let Γ be a chart, and we denote by Γ_m the union of all the edges of label m. A chart Γ is of type $(m; 2, 3, 2)$ if $w(\Gamma) = 7$, $w(\Gamma_m \cap \Gamma_{m+1}) = 2$, $w(\Gamma_{m+1} \cap \Gamma_{m+2}) = 3$, and $w(\Gamma_{m+2} \cap \Gamma_{m+3}) = 2$ where $w(G)$ is the number of white vertices in G. In this paper, we prove that if there is a minimal chart Γ of type $(m; 2, 3, 2)$, then each of Γ_{m+1} and Γ_{m+2} contains one of three kinds of graphs. In the next paper, we shall prove that there is no minimal chart of type $(m; 2, 3, 2)$.

1. Introduction

Charts are oriented labeled graphs in a disk (see [1],[5], and see Section 2 for the precise definition of charts). From a chart, we can construct an oriented closed surface embedded in 4-space $\mathbb{R}^4$ (see [5, Chapter 14, Chapter 18 and Chapter 23]). A C-move is a local modification between two charts in a disk (see Section 2 for C-moves). A C-move between two charts induces an ambient isotopy between oriented closed surfaces corresponding to the two charts.

We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat. In [16], we showed that there is no minimal chart with exactly five vertices (see Section 2 for the precise definition of minimal charts). Hasegawa proved that there exists a minimal chart with exactly six white vertices [2]. This chart represents a 2-twist spun trefoil. In [3] and [15], we investigated minimal charts with exactly four white vertices. In this paper, we investigate some properties of minimal charts which are to be used to prove our conjecture on nonexistence of minimal chart with exactly seven white vertices (see [6],[7], [8],[9], [10], [11], [12]).

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Let Γ be a chart. For each label m, we denote by Γ_m the union of all the edges of label m.

Now we define a type of a chart: Let Γ be a chart with at least one white vertex, and $n_1, n_2, \ldots, n_k$ integers. The chart Γ is of *type* $(n_1, n_2, \ldots, n_k)$ if there exists a label m of Γ satisfying the following three conditions:

(i) For each $i = 1, 2, \ldots, k$, the chart Γ contains exactly n_i white vertices in $\Gamma_{m+i-1} \cap \Gamma_{m+i}$.

(ii) If $i < 0$ or $i > k$, then Γ_{m+i} does not contain any white vertices.

(iii) Both of the two subgraphs Γ_m and Γ_{m+k} contain at least one white vertex.

If we want to emphasize the label m, then we say that Γ is of *type* $(m; n_1, n_2, \ldots, n_k)$. Note that $n_1 \geq 1$ and $n_k \geq 1$ by the condition (iii). We proved in [7, Theorem 1.1] that if there exists a minimal n-chart Γ with exactly seven white vertices, then Γ is a chart of type $(7), (5, 2), (4, 3), (3, 2, 2)$ or $(2, 3, 2)$ (if necessary we change the label i by $n-i$ for all label i). In [10], we showed that there is no minimal chart of type $(3, 2, 2)$. In this paper and [11], we shall show the following.

THEOREM 1.1 ([11, Theorem 1.1]). *There is no minimal chart of type* $(2, 3, 2)$.

In the future paper [12], we shall show there is no minimal chart of type $(7), (5, 2), (4, 3)$. Therefore we shall show that there is no minimal chart with exactly seven white vertices.

An edge in a chart is called a *terminal edge* if it has a white vertex and a black vertex.

In our argument we often construct a chart Γ. On the construction of a chart Γ, for a white vertex $w \in \Gamma_m$ for some label m, among the three edges of Γ_m containing w, if one of the three edges is a terminal edge (see Fig. 1(a) and (b)), then we remove the terminal edge and put a black dot at the center of the white vertex as shown in Fig. 1(c). Namely Fig. 1(c) means Fig. 1(a) or Fig. 1(b). We call the vertex in Fig. 1(c) a *BW-vertex* with respect to Γ_m.

For example, the graph as shown in Fig. 2(a) means one of the four graphs as shown in Fig. 2(b),(c),(d),(e).

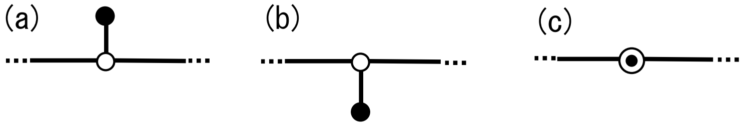

Fig. 1. (a),(b) White vertices in terminal edges. (c) A BW-vertex.

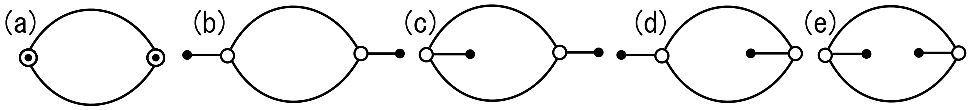

Fig. 2. Graphs with two white vertices.

The three graphs in Fig. 3 are examples of graphs in Γ_m for a chart Γ and a label m. We call a *θ-curve*, an *oval*, a *skew θ-curve* the three graphs as shown in Fig. 3(a),(b),(c) respectively.

Let X be a set in a chart Γ. Let

$$w(X) = \text{the number of white vertices in } X.$$

Let Γ be a chart of type $(m; 2, 3, 2)$. Then $w(\Gamma) = 7, w(\Gamma_m \cap \Gamma_{m+1}) = 2, w(\Gamma_{m+1} \cap \Gamma_{m+2}) = 3, w(\Gamma_{m+2} \cap \Gamma_{m+3}) = 2$. Thus $w(\Gamma_{m+1}) = 5$ and $w(\Gamma_{m+2}) = 5$. First we shall show the following lemma.

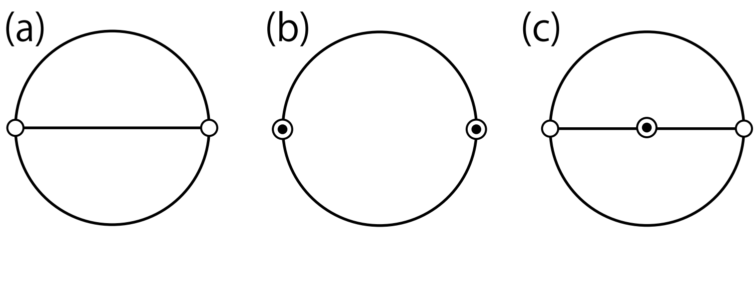

Fig. 3. (a) A θ-curve. (b) An oval. (c) A skew θ-curve.

Lemma 1.2. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then each of Γ_{m+1} and Γ_{m+2} contains one of nine graphs as shown in Fig. 4, or the union of a θ-curve and a skew θ-curve, or the union of an oval and a skew θ-curve.*

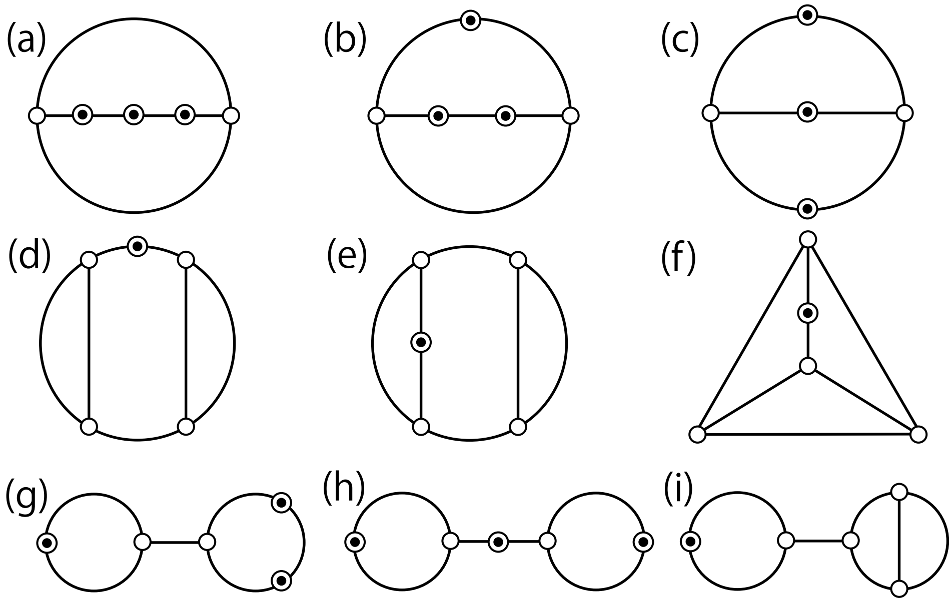

Fig. 4. (a),(b),(c) Graphs with three black vertices. (d),(e),(f) Graphs with one black vertex. (g),(h) Graphs with three black vertices. (i) A graph with one black vertex.

In this paper, the following is the main result.

Theorem 1.3. *If there exists a minimal chart Γ of type $(m; 2, 3, 2)$, then each of Γ_{m+1} and Γ_{m+2} contains either the union of an oval and a skew θ-curve, or one of two graphs as shown in Fig. 4(g),(h).*

The paper is organized as follows. In Section 2, we define charts and minimal charts. In Section 3, we investigate connected components of Γ_m with five white vertices for a minimal chart Γ. We shall show Lemma 1.2. In Section 4, we shall show that neither Γ_m nor Γ_{m+3} contains a θ-curve for any minimal chart Γ of type $(m; 2, 3, 2)$ (i.e. both of Γ_m and Γ_{m+3} contain ovals) (see Corollary 4.3). In Section 5, we investigate an oval of label m for a minimal chart Γ. In Section 6, we investigate white vertices in an oval of label m for a minimal chart Γ of type $(m; 2, 3, 2)$. In Section 7, we shall show that for any minimal chart Γ of type $(m; 2, 3, 2)$, the graph Γ_{m+1} contains none of the five graphs as shown in Fig. 4(a),(d),(e),(f),(i), and neither does Γ_{m+2}. Moreover we shall show that neither Γ_{m+1} nor Γ_{m+2} contains a θ-curve. In Section 8, we consider a minimal chart Γ of type $(m; 2, 3, 2)$ such that Γ_{m+1} contains either an oval, or one of the four graphs as shown in Fig. 4(b),(c),(g),(h). We investigate that the chart Γ contains what kind of pseudo charts. In Section 9, we shall show that neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(c) for any minimal chart Γ of type $(m; 2, 3, 2)$. In Section 10, we shall show that neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(b) for any minimal chart Γ of type $(m; 2, 3, 2)$. We obtain the main theorem (Theorem 1.3).

2. Preliminaries

In this section, we introduce the definition of charts and its related words.

Let n be a positive integer. An *n-chart* (a braid chart of degree n [1] or a surface braid chart of degree n [5]) is an oriented labeled graph in the interior of a disk, which may be empty or have closed edges without vertices satisfying the following four conditions (see Fig. 5):

(i) Every vertex has degree 1, 4, or 6.

(ii) The labels of edges are in $\{1, 2, \dots, n-1\}$.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

We call a vertex of degree 1 a *black vertex*, a vertex of degree 4 a *crossing*, and a vertex of degree 6 a *white vertex* respectively.

Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a *middle arc* at the white vertex (see Fig. 5(c)). For each white vertex v, there are two middle arcs at v in a small neighborhood of v. An edge is said to be *middle at* a white vertex v if it contains a middle arc at v.

Let e be an edge connecting v_1 and v_2. If e is oriented from v_1 to v_2, then we say that e is oriented *outward at* v_1 and *inward at* v_2

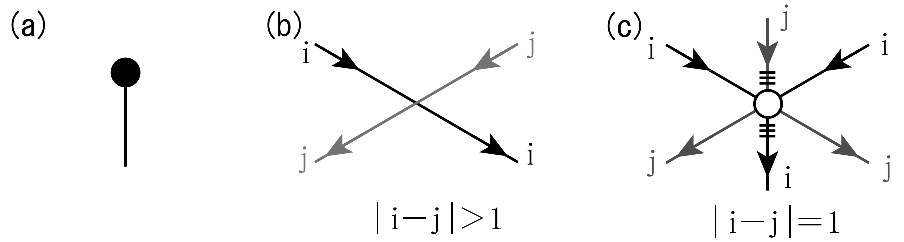

Fig. 5. (a) A black vertex. (b) A crossing. (c) A white vertex. Each arc with three transversal short arcs is a middle arc at the white vertex.

Now *C-moves* are local modifications of charts as shown in Fig. 6 (cf. [1], [5] and [17]). Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

An edge in a chart is called a *free edge* if it has two black vertices.

For each chart Γ, let $w(\Gamma)$ and $f(\Gamma)$ be the number of white vertices, and the number of free edges respectively. The pair $(w(\Gamma), -f(\Gamma))$ is called a *complexity* of the chart (see [4]). A chart Γ is called a *minimal chart* if its complexity is minimal among the charts C-move equivalent to the chart Γ with respect to the lexicographic order of pairs of integers.

We showed the difference of a chart in a disk and in a 2-sphere (see [6, Lemma 2.1]). This lemma follows from that there exists a natural one-to-one correspondence between $\{$charts in $S^2\}$/C-moves and $\{$charts in $D^2\}$/C-moves, conjugations ([5, Chapter 23 and Chapter 25]). To make the argument simple, we assume that the charts lie on the 2-sphere instead of the

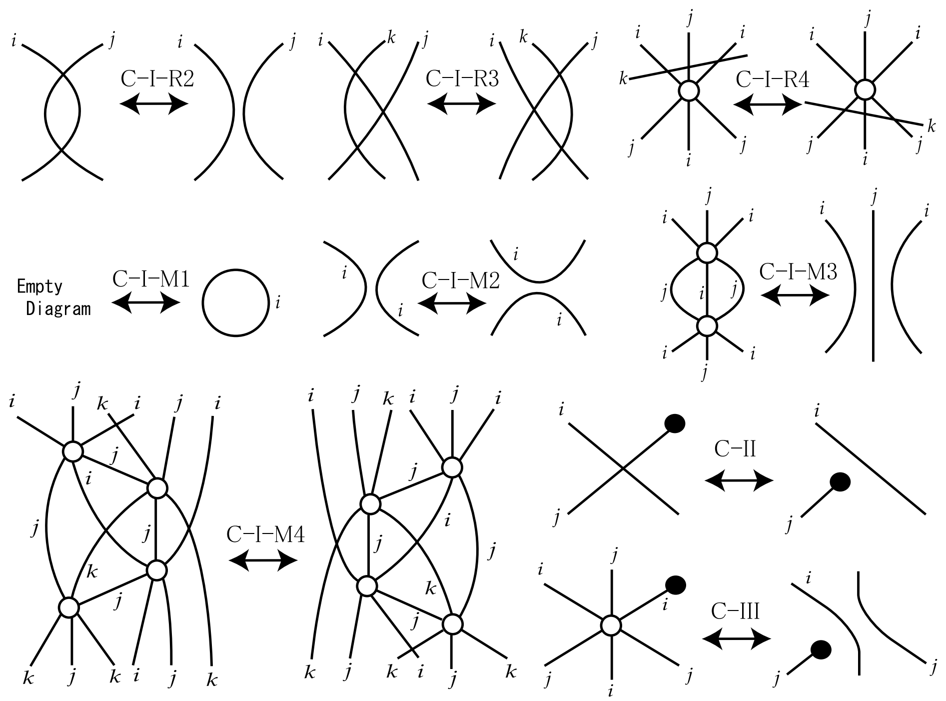

Fig. 6. For the C-III move, the edge with the black vertex is not middle at a white vertex in the left figure.

disk.

Assumption 1. *In this paper, all charts are contained in the* 2*-sphere* S^2.

We have the special point in the 2-sphere S^2, called the point at infinity, denoted by ∞. In this paper, all charts are contained in a disk such that the disk does not contain the point at infinity ∞.

Let Γ be a chart, and m a label of Γ. A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). A *ring* is a simple closed curve in Γ_m containing a crossing but not containing any white vertices. A hoop is said to be *simple* if one of the two complementary domains of the hoop does not contain any white vertices.

We can assume that all minimal charts Γ satisfy the following four conditions (see [6],[7],[8], [14]):

ASSUMPTION 2. *If an edge of* Γ *contains a black vertex, then the edge is a free edge or a terminal edge. Moreover any terminal edge contains a middle arc.*

ASSUMPTION 3. *All free edges and simple hoops in* Γ *are moved into a small neighborhood* U_∞ *of the point at infinity* ∞. *Hence we assume that* Γ *does not contain free edges nor simple hoops, otherwise mentioned.*

ASSUMPTION 4. *Each complementary domain of any ring and hoop must contain at least one white vertex.*

ASSUMPTION 5. *The point at infinity* ∞ *is moved in any complementary domain of* Γ.

In this paper for a set X in a space we denote the interior of X, the boundary of X and the closure of X by $\mathrm{Int}X$, ∂X and $Cl(X)$ respectively.

3. Connected Components of Γ_m

In this section, we investigate connected components of Γ_m with five white vertices for a minimal chart Γ. We shall show Lemma 1.2.

LEMMA 3.1 ([10, Lemma 3.1]). *In a minimal chart* Γ, *for each BW-vertex in* Γ_m, *the two edges of label* m *containing the BW-vertex are oriented inward or outward at the BW-vertex simultaneously if each of the two edges is not a terminal edge* (*see Fig. 7*).

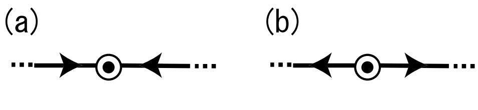

Fig. 7. BW-vertices.

Let Γ be a chart, and m a label of Γ. A *loop* is a simple closed curve in Γ_m with exactly one white vertex (possibly with crossings).

LEMMA 3.2 ([10, Lemma 3.2]). *Let Γ be a minimal chart, and m a label of Γ. Let G be a connected component of Γ_m. Then we have the following.*

(a) *If $1 \leq w(G)$, then $2 \leq w(G)$.*

(b) *If $1 \leq w(G) \leq 3$ and G does not contain any loop, then G is one of three graphs as shown in Fig. 3.*

The following lemma is easily shown. Thus we omit the proof.

LEMMA 3.3. *Let G be a 3-regular graph in S^2. Then we have the following.*

(a) *The graph G contains exactly an even number of vertices.*

(b) *If G has at most four vertices, then G is one of seven graphs as shown in Fig. 3(a) and Fig. 8.*

LEMMA 3.4. *Let Γ be a minimal chart, and m a label of Γ. Let G be a connected component of Γ_m. If $w(G) = 5$ and G has no loop, then G is one of nine graphs as shown in Fig. 4.*

PROOF. By Assumption 2, each terminal edge is middle at a white vertex. Thus each white vertex in Γ_m is contained in at most one terminal edge of label m. Hence

(1) the graph G is obtained from a simple closed curve or a 3-regular graph (possibly with loops) by adding BW-vertices.

Now we shall show that G contains at least one black vertex. If not, then the graph G is a 3-regular graph on S^2. By Lemma 3.3(a), the graph G contains exactly an even number of white vertices. This contradicts the fact $w(G) = 5$. Hence G contains at least one black vertex. Thus

(2) G contains at least one BW-vertex.

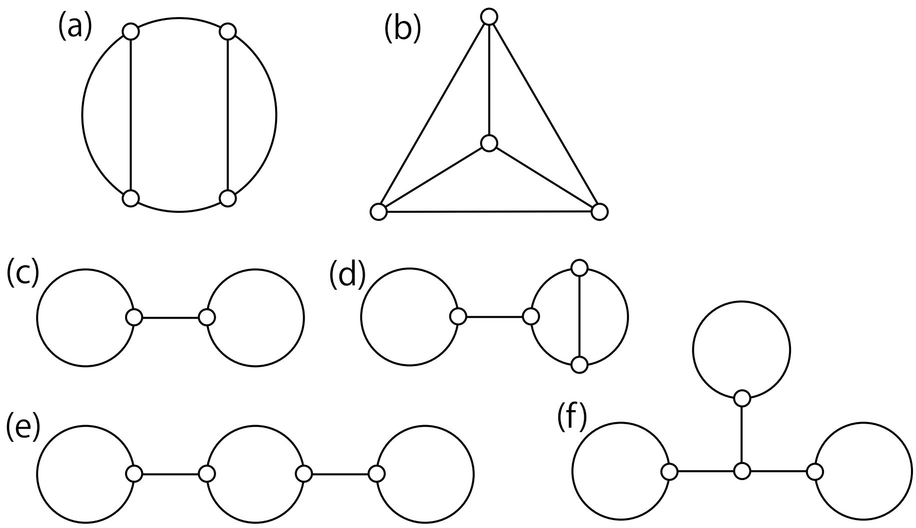

Fig. 8. (a),(b) Graphs without loops. (c),(d),(e),(f) Graphs with loops.

CLAIM. The graph G is obtained from a 3-regular graph by adding BW-vertices.

Suppose that all white vertices in G are BW-vertices. Then the graph G is obtained from a simple closed curve by adding BW-vertices. By Lemma 3.1, in a minimal chart, for each BW-vertex in Γ_m, the two edges of label m containing the BW-vertex are oriented inward or outward at the BW-vertex simultaneously if each of the two edges is not a terminal edge. Hence the orientation of edges must change at BW-vertices. Thus G contains exactly an even number of BW-vertices. This contradicts the fact $w(G) = 5$. Hence G contains a white vertex not a BW-vertex. Thus by (1), Claim holds.

By $w(G) = 5$, (2) and Claim, the graph G is obtained by adding BW-vertices from a 3-regular graph with at most four vertices. Hence by Lemma 3.3(b), the graph G is obtained by adding BW-vertices from one of seven graphs as shown in Fig. 3(a) and Fig. 8. Since G has no loop with $w(G) = 5$, the graph G is not obtained from the graphs as shown in Fig. 8(e),(f).

Now the graph G is on the 2-sphere S^2. Hence if G is obtained from the graph as shown in Fig. 3(a), then the graph G is one of three graphs as shown in Fig. 4(a),(b),(c). If G is obtained from the graph as shown in Fig. 8(a), then the graph G is one of two graphs as shown in Fig. 4(d),(e). If G is obtained from the graph as shown in Fig. 8(b), then the graph G is the graph as shown in Fig. 4(f). If G is obtained from the graph as shown in Fig. 8(c), then the graph G is one of two graphs as shown in Fig. 4(g),(h). If G is obtained from the graph as shown in Fig. 8(d), then the graph G is the graph as shown in Fig. 4(i). Therefore G is one of nine graphs as shown in Fig. 4. We complete the proof of Lemma 3.4. □

LEMMA 3.5. *Let* Γ *be a minimal chart, and* m *a label of* Γ. *If* $w(\Gamma_m) = 5$ *and* Γ_m *has no loop, then* Γ_m *contains one of the following graphs:*

(a) *one of nine graphs as shown in Fig. 4, or*

(b) *the union of a* θ*-curve and a skew* θ*-curve, or*

(c) *the union of an oval and a skew* θ*-curve.*

PROOF. First we shall show that there exist at most two connected components of Γ_m with white vertices. Suppose that there exist at least three connected components G_1, G_2, G_3 of Γ_m with $w(G_i) \geq 1$ for each $i = 1, 2, 3$. Then by Lemma 3.2(a), we have $w(G_i) \geq 2$ for each $i = 1, 2, 3$. Thus

$$5 = w(\Gamma_m) \geq w(G_1) + w(G_2) + w(G_3) \geq 2 + 2 + 2 = 6.$$

This is a contradiction. Hence there exist at most two connected components of Γ_m with white vertices.

Suppose that there exists a connected component G_1 of Γ_m with $w(G_1) = 5$. Since Γ_m has no loop, by Lemma 3.4 the graph G_1 is one of nine graphs as shown in Fig. 4.

Suppose that there exists two connected components G_1, G_2 of Γ_m with $w(G_1) \geq 1, w(G_2) \geq 1$ and $w(G_1) + w(G_2) = 5$. Then by Lemma 3.2(a), we have $w(G_1) \geq 2, w(G_2) \geq 2$.

Without loss of generality we can assume $2 \leq w(G_1) \leq w(G_2)$. Since $w(G_1) + w(G_2) = 5$, we have $w(G_1) = 2$ and $w(G_2) = 3$. Since Γ_m has no loop, by Lemma 3.2(b) the graph G_1 is a θ-curve or an oval, and the graph G_2 is a skew θ-curve. □

LEMMA 3.6 ([9, Theorem 1.1]). *There is no loop in any minimal chart with exactly seven white vertices.*

By Lemma 3.5 and Lemma 3.6, we have Lemma 1.2.

4. θ-Curves

In this section we shall show that neither Γ_m nor Γ_{m+3} contains a θ-curve for any minimal chart Γ of type $(m; 2, 3, 2)$ (i.e. both of Γ_m and Γ_{m+3} contain ovals) (see Corollary 4.3).

Let Γ be a chart, and m a label of Γ. Let L be the closure of a connected component of the set obtained by taking out all the white vertices from Γ_m. If L contains at least one white vertex but does not contain any black vertex, then L is called an *internal edge of label* m. Note that an internal edge may contain a crossing of Γ.

Let Γ be a chart. Let D be a disk such that

(1) the boundary ∂D consists of an internal edge e_1 of label m and an internal edge e_2 of label $m+1$, and

(2) any edge containing a white vertex in e_1 does not intersect the open disk $\mathrm{Int}D$.

Note that ∂D may contain crossings. Let w_1 and w_2 be the white vertices in e_1. If the disk D satisfies one of the following conditions, then D is called *a lens of type* $(m, m+1)$ (see Fig. 9):

(i) Neither e_1 nor e_2 contains a middle arc.

(ii) One of the two edges e_1 and e_2 contains middle arcs at both white vertices w_1 and w_2 simultaneously.

LEMMA 4.1 ([7, Corollary 1.3]). *There is no lens in any minimal chart with at most seven white vertices.*

LEMMA 4.2. *Let Γ be a minimal chart, and m a label of Γ. Suppose that Γ_m contains a θ-curve G. If Γ has no lens, and if the two white vertices in G are contained in $\Gamma_{m+\varepsilon}$ for some $\varepsilon \in \{+1, -1\}$, then $w(\Gamma_{m+\varepsilon}) \geq 6$.*

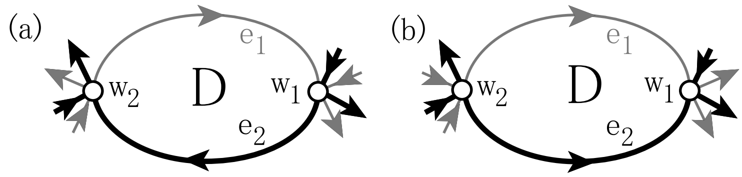

Fig. 9. Lenses.

Proof. Let w_1, w_2 be the white vertices in G, and e the internal edge of label m in G middle at w_1. Without loss of generality we can assume that

(1) the edge e is oriented from w_1 to w_2.

Then the other two internal edges in G are oriented from w_2 to w_1 (see Fig. 10(a)). Thus

(2) the edge e is middle at w_1, w_2.

The θ-curve G divides S^2 into three disks. Let D_1, D_2 be two of the three disks with $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = e$. Let e_1, e_2 be internal edges (possibly terminal edges) of label $m+\varepsilon$ at w_1 in D_1, D_2 respectively. Since e is middle at w_1 by (2), neither e_1 nor e_2 is middle at w_1. By Assumption 2, neither e_1 nor e_2 is a terminal edge. Hence e_1 and e_2 contain white vertices different from w_1, say w_3, w_4.

We shall show $w_3 \neq w_2$ and $w_4 \neq w_2$. If $w_3 = w_2$, then the edge e_1 separates the disk D_1 into two disks. One of the two disks contains the edge e. By (2), the disk is a lens. This contradicts the condition that Γ has no lens. Hence $w_3 \neq w_2$. Similarly we can show $w_4 \neq w_2$.

Let e_1', e_2' be internal edges (possibly terminal edges) of label $m+\varepsilon$ at w_2 in D_1, D_2 respectively. By using (2), we can show similarly that e_1', e_2' contain white vertices different from w_2, say w_3', w_4'.

We shall show that $w(\Gamma_{m+\varepsilon} \cap \mathrm{Int} D_1) \geq 2$. There are two cases: $w_3 \neq w_3'$ and $w_3 = w_3'$.

If $w_3 \neq w_3'$, then $w(\Gamma_{m+\varepsilon} \cap \mathrm{Int} D_1) \geq 2$.

Suppose $w_3 = w_3'$ (see Fig. 10(b)). Let e_1'' be an internal edge (possibly a terminal edge) of label $m+\varepsilon$ at w_3 different from e_1, e_1'. By (1) and (2), the edge e_1 is oriented from w_1 to w_3 and the edge e_1' is oriented from w_3 to w_2. Thus e_1'' is not middle at w_3. Hence by Assumption 2, the edge e_1'' is not a terminal edge. Thus the edge e_1'' contains a white vertex different from w_3. Thus $w(\Gamma_{m+\varepsilon} \cap \mathrm{Int}D_1) \geq 2$.

Similarly we can show $w(\Gamma_{m+\varepsilon} \cap \mathrm{Int}D_2) \geq 2$. Finally we have

$$\begin{aligned} w(\Gamma_{m+\varepsilon}) &\geq w(\Gamma_{m+\varepsilon} \cap G) + w(\Gamma_{m+\varepsilon} \cap \mathrm{Int}D_1) + w(\Gamma_{m+\varepsilon} \cap \mathrm{Int}D_2) \\ &\geq 2 + 2 + 2 = 6 \ \square \end{aligned}$$

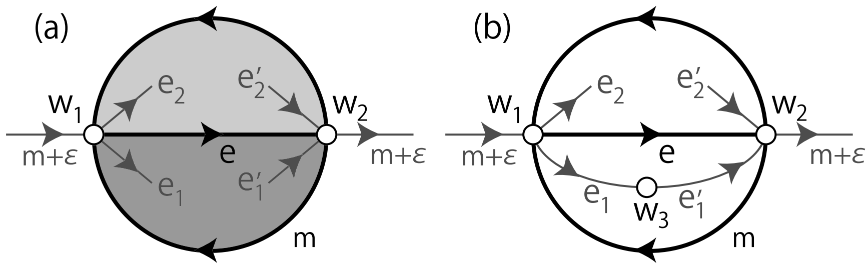

Fig. 10. (a) The dark gray region is the disk D_1, the light gray region is the disk D_2. (b) Both of e_1 and e_1' contain the white vertex w_3.

COROLLARY 4.3. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then both of Γ_m and Γ_{m+3} contain ovals.*

PROOF. Since Γ is of type $(m; 2, 3, 2)$, we have $w(\Gamma_m) = 2$. Since the graph Γ_m does not contain any loop by Lemma 3.6, Lemma 3.2(b) implies that the graph Γ_m contains one of the two graphs as shown in Fig. 3(a) and (b). Hence the graph Γ_m contains a θ-curve or an oval. If the graph Γ_m contains a θ-curve, then we have $w(\Gamma_{m+1}) \geq 6$ by Lemma 4.2.

On the other hand, since Γ is of type $(m; 2, 3, 2)$, we have $w(\Gamma_{m+1}) = 5$. This is a contradiction. Thus the graph Γ_m contains an oval.

Similarly we can show that the graph Γ_{m+3} contains an oval. $\square$

5. Ovals

In this section we investigate an oval of label m for a minimal chart Γ.

Let Γ be a chart, m a label of Γ, D a disk with $\partial D \subset \Gamma_m$, and k a positive integer. If ∂D contains exactly k white vertices, then D is called *a k-angled disk of* Γ_m. Note that the boundary ∂D may contain crossings.

LEMMA 5.1. *Let Γ be a minimal chart, and m a label of Γ. Let G be an oval of label m, and D a 2-angled disk of Γ_m with $\partial D \subset G$. Let E be a disk in D whose boundary consists of an internal edge in G and an internal edge of label $m + \varepsilon$ ($\varepsilon \in \{+1, -1\}$) connecting the two white vertices of G. If E does not contain the terminal edges in G, then E is a lens of Γ.*

PROOF. Let e be the internal edge of label $m + \varepsilon$ in ∂E. Let v_1, v_2 be the white vertices in G, and e_1, e_2 the terminal edges at v_1, v_2 in G respectively. By Assumption 2,

(1) both of the two edges e_1 and e_2 contain middle arcs.

There are three cases: (i) neither e_1 nor e_2 is contained in D (see Fig. 11(a)), (ii) only one of e_1 and e_2 is contained in D (see Fig. 11(b)), (iii) both of e_1 and e_2 are contained in D (see Fig. 11(c)).

Case (i). By (1), the edge e is middle at both white vertices v_1 and v_2 simultaneously. Thus the disk E is a lens.

Case (ii). Without loss of generality we can assume that

(2) the edge e_1 is oriented inward at v_1.

Then by (1), the other two internal edges in G are oriented from v_1 to v_2. Thus

(3) the edge e_2 is oriented outward at v_2.

If $e_1 \subset D$ (see Fig. 11(b)), then by (1) and (2), the edge e is oriented from v_2 to v_1. Hence e is oriented outward at v_2. On the other hand, by (1) and (3), the edge e is oriented inward at v_2. This is a contradiction.

Similarly if $e_2 \subset D$, then we have the same contradiction. Thus Case (ii) does not occur.

Case (iii). By (1), none of e and the two internal edges in G contain middle arcs. Thus the disk E is a lens. $\square$

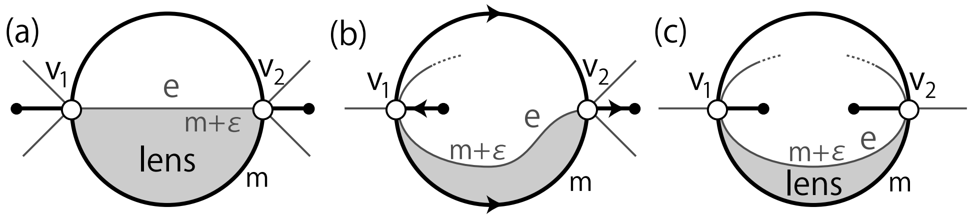

Fig. 11. The gray regions are disks E. (a) $e_1 \not\subset D, e_2 \not\subset D$. (b) $e_1 \subset D, e_2 \not\subset D$. (c) $e_1 \subset D, e_2 \subset D$.

Let Γ be a chart. Suppose that an object consists of some edges of Γ, arcs in edges of Γ and arcs around white vertices. Then the object is called a *pseudo chart.*

LEMMA 5.2. *Let Γ be a minimal chart, and m a label of Γ. Let G be an oval of label m. If for some $\varepsilon \in \{+1, -1\}$ there exists a 2-angled disk D of $\Gamma_{m+\varepsilon}$ with $G \cap \partial D$ two white vertices, then there exists a lens of Γ.*

PROOF. Let C be the simple closed curve in G. Let c_1, c_2, d_1, d_2 be internal edges with $c_1 \cup c_2 = C$ and $d_1 \cup d_2 = \partial D$. Let w be a white vertex in G. Then there are two cases: (i) The four edges c_1, c_2, d_1, d_2 (or c_1, c_2, d_2, d_1) lie around the white vertex w in this order (see Fig 12(a)). (ii) The four edges c_1, d_1, c_2, d_2 lie around the white vertex w in this order (see Fig 12(b)).

Case (i). Since $G \cap \partial D = C \cap \partial D$ consists of two white vertices, the union $C \cup \partial D$ separates S^2 into four disks. Let E, E' be two of the four disks such that each boundary consists of an internal edge of label m in G and an internal edge of label $m + \varepsilon$ in ∂D. By the condition of Case (i), neither E nor E' contains a terminal edge of label m in G (see Fig 12(a)). Thus by Lemma 5.1, both of E and E' are lenses.

Case (ii). Since $G \cap \partial D = C \cap \partial D$ consists of two white vertices, the union $C \cup \partial D$ separates S^2 into four disks. Let E, E' be two of the four disks with $E \cap E' = d_1$. Then $E \cup E'$ is a 2-angled disk of Γ_m whose boundary is contained in G.

If E or E' is a lens, then there exists a lens of Γ.

Suppose that neither E nor E' is a lens. Then by Lemma 5.1,

(1) each of E and E' contains a terminal edge in G.

Hence $E \cup E'$ contains one of the two pseudo charts as shown in Fig. 12(c),(d). Then $d_2 \not\subset E \cup E'$. Thus $d_2 \subset Cl(S^2 - (E \cup E'))$. Hence the edge d_2 separates the disk $Cl(S^2 - (E \cup E'))$ into two disks. Thus neither of the two disks contains a terminal edge in G, because $E \cup E'$ contains the two terminal edges of G by (1). Hence by Lemma 5.1, both of the two disks are lenses. □

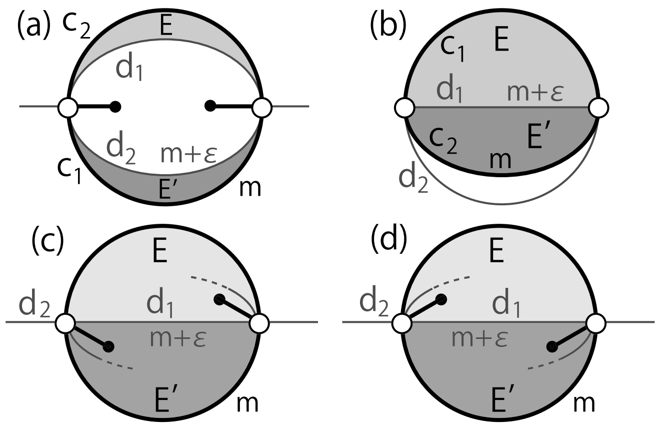

Fig. 12. The light gray region and the dark gray region are E and E'.

Lemma 5.3. *Let Γ be a chart, and m a label of Γ. Let G be an oval of label m, and v_1, v_2 the white vertices in G. Let D be a 2-angled disk of Γ_m with $\partial D \subset G$. If D satisfies one of the following two conditions, then Γ is not minimal.*

(a) *The disk D does not contain terminal edges of G, but contains two internal edges e_1, e_2 of label $m+\varepsilon$ at v_1, v_2 respectively* ($\varepsilon \in \{+1, -1\}$) *such that $e_1 \cap e_2$ is a BW-vertex with respect to $\Gamma_{m+\varepsilon}$ in* $\mathrm{Int}D$ (*see Fig. 13*(*a*)).

(b) *The disk D contains exactly one terminal edge of G, and contains three internal edges e_1, e_2, e_3 of label $m + \varepsilon$ at v_1, v_1, v_2 respectively*

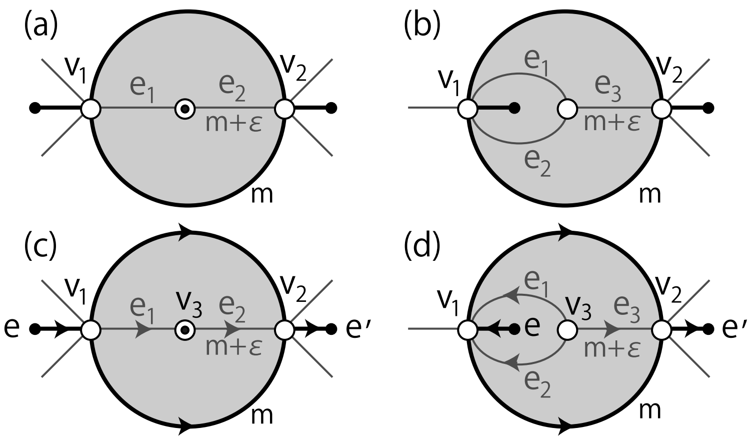

Fig. 13. (a),(c) The gray regions are 2-angled disks not containing terminal edges of G. (b),(d) The gray regions are 2-angled disks containing one terminal edge of G.

($\varepsilon \in \{+1, -1\}$) *such that* $e_1 \cap e_2 \cap e_3$ *is a white vertex in* $\mathrm{Int}D$ (*see Fig. 13*(*b*)).

PROOF. Suppose that Γ is minimal. Let e, e' be the terminal edges at v_1, v_2 in G respectively. By Assumption 2,

(1) each of the two edges e and e' is middle at a white vertex.

Without loss of generality we can assume that

(2) the edge e is oriented inward at v_1.

Then by (1), the other two internal edges in G are oriented from v_1 to v_2. Thus

(3) the edge e' is oriented outward at v_2.

If the disk D satisfies the condition (a), then $e_1 \cap e_2$ is a BW-vertex with respect to $\Gamma_{m+\varepsilon}$. Let $v_3 = e_1 \cap e_2$. By (2), the edge e_1 is oriented from v_1 to v_3. Thus the edge e_1 is oriented inward at the BW-vertex v_3.

On the other hand, by (3), the edge e_2 is oriented from v_3 to v_2. Thus the edge e_2 is oriented outward at the BW-vertex v_3. Hence for the BW-vertex v_3, the edge e_1 of label $m + \varepsilon$ is oriented inward at v_3, and the edge

e_2 of label $m+\varepsilon$ is oriented outward at v_3 (see Fig. 13(c)). This contradicts Lemma 3.1. Thus Γ is not minimal.

If the disk D satisfies the condition (b), then $e_1 \cap e_2 \cap e_3$ is a white vertex. Let $v_3 = e_1 \cap e_2 \cap e_3$. By (1) and (2), both of e_1 and e_2 are oriented from v_3 to v_1. Thus

(4) both of e_1 and e_2 are oriented outward at v_3.

On the other hand, by (3), the edge e_3 is oriented from v_3 to v_2. Thus the edge e_3 is oriented outward at v_3. Hence by (4), the three edges e_1, e_2, e_3 of label $m+\varepsilon$ are oriented outward at v_3 (see Fig. 13(d)). This contradicts the condition (iii) of the definition of charts. Thus Γ is not minimal. $\square$

6. White Vertices in the Graphs Γ_{m+1}

In this section, we investigate white vertices in an oval of label m for a minimal chart Γ of type $(m; 2, 3, 2)$.

Lemma 6.1. *Let Γ be a minimal chart, and m a label of Γ. Let w be a white vertex in a terminal edge of label m. Let e_1, e_2 be the two edges of label m at w different from the terminal edge. If w is contained in a terminal edge of label $m+\varepsilon$ for some $\varepsilon \in \{+1, -1\}$, then both edges e_1, e_2 are contained in the closure of the same connected component of $S^2 - \Gamma_{m+\varepsilon}$.*

Proof. Since w is contained in a terminal edge of label m and since w is contained in a terminal edge of label $m+\varepsilon$, by Assumption 2 we can show that in a neighborhood of the vertex w, the chart Γ contains the pseudo chart as shown in Fig. 14. Hence the edges e_1 and e_2 of label m are contained in the closure of the same connected component F of $S^2 - \Gamma_{m+\varepsilon}$. Thus we complete the proof of Lemma 6.1. $\square$

From the above lemma, we have the following lemma:

Lemma 6.2. *Let Γ be a minimal chart, and m a label of Γ. Let G be an oval of label m. If one of the two white vertices in G is a BW-vertex with respect to $\Gamma_{m+\varepsilon}$ for some $\varepsilon \in \{+1, -1\}$, then the two internal edges in G are contained in the closure of the same connected component of $S^2 - \Gamma_{m+\varepsilon}$.*

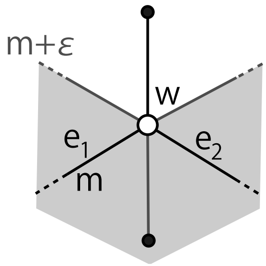

Fig. 14. A white vertex w is contained in two terminal edges. The gray region is F.

LEMMA 6.3. *Let* Γ *be a minimal chart of type* $(m; 2, 3, 2)$. *Suppose that* Γ_m *contains an oval* G. *Then we have the following.*

(a) *If* Γ_{m+1} *contains one of the three graphs as shown in Fig. 4(a),(b),(c), then either* G *contains two BW-vertices with respect to* Γ_{m+1}, *or* G *does not contain any BW-vertex with respect to* Γ_{m+1}.

(b) *If* Γ_{m+1} *contains one of the three graphs as shown in Fig. 4(d),(e),(f), then* G *does not contain any BW-vertex with respect to* Γ_{m+1}.

(c) *If* Γ_{m+1} *contains the union of a* θ*-curve and a skew* θ*-curve, then* G *does not contain any BW-vertex with respect to* Γ_{m+1}. *Moreover the two white vertices in* G *are contained in the* θ*-curve or the skew* θ*-curve simultaneously.*

(d) *If* Γ_{m+1} *contains the union of an oval and a skew* θ*-curve, then either* G *contains two BW-vertices with respect to* Γ_{m+1}, *or* G *does not contain any BW-vertex with respect to* Γ_{m+1}.

PROOF. Let e_1, e_2 be the internal edges of label m in G.

Statement (a). The graph Γ_{m+1} contains exactly three BW-vertices with respect to Γ_{m+1}, say w_1, w_2, w_3. Let w_4, w_5 be the other white vertices in Γ_{m+1}. It suffices to prove that if G contains one of BW-vertices w_1, w_2, w_3, then G contains two of w_1, w_2, w_3.

If G contains one of BW-vertices w_1, w_2, w_3, then by Lemma 6.2 there exists a connected component F of $S^2 - \Gamma_{m+1}$ with $e_1 \cup e_2 \subset Cl(F)$ (see Fig. 15(a)). Thus

(1) for each white vertex w in G, there exist *two* edges of label m at w contained in $Cl(F)$.

On the other hand, by the condition of Lemma 6.3(a), for each white vertex w_i $(i = 4, 5)$ there exists *at most one* edge label m at w_i in $Cl(F)$ (see Fig. 15(a)). Hence by (1), the oval G does not contain w_4 nor w_5. Thus G contains two of BW-vertices w_1, w_2, w_3. Hence Statement (a) holds.

Statement (b). The graph Γ_{m+1} contains exactly one BW-vertex with respect to Γ_{m+1}, say w_1. Let w_2, w_3, w_4, w_5 be the other white vertices in Γ_{m+1}.

Suppose that G contains the BW-vertex w_1. Then by Lemma 6.2 there exists a connected component F of $S^2 - \Gamma_{m+1}$ with $e_1 \cup e_2 \subset Cl(F)$ (see Fig. 15(b)). Thus for each white vertex in the oval G, there exist *two* edges of label m at the vertex in $Cl(F)$. However, by the condition of Lemma 6.3(b), for each white vertex w_i $(i = 2, 3, 4, 5)$ there exists *at most one* edge of label m at w_i in $Cl(F)$. This is a contradiction. Hence the oval G does not contain the vertex w_1. Hence Statement (b) holds.

Statement (c). Let w_1, w_2 be the white vertices of the θ-curve in Γ_{m+1}. Let w_3 be the BW-vertex of the skew θ-curve with respect to Γ_{m+1}, and w_4, w_5 the other white vertices of the skew θ-curve.

By a similar way of the proof of Statement (b), we can show that the oval G does not contain the BW-vertex w_3. Thus G contains two of w_1, w_2, w_4, w_5.

Suppose that the oval G contains one of the white vertices w_1, w_2, and one of the white vertices w_4, w_5. Without loss of generality we can assume $w_1, w_4 \in G$. Then

(2) the two edges e_1 and e_2 of label m connect the vertices w_1 and w_4.

Now, the θ-curve in Γ_{m+1} separates S^2 into three disks. One contains the skew θ-curve in Γ_{m+1}, say F. Hence the vertex w_4 in the skew θ-curve is contained in F. Thus by (2), we have $e_1 \cup e_2 \subset F$. Hence in F there exist *two* edges of label m at w_1. However, since w_1 is a white vertex of the θ-curve in Γ_{m+1}, there exists *at most one* edge of label m at w_1 in F (see Fig. 15(c)). This is a contradiction. Hence $w_1, w_2 \in G$ or $w_4, w_5 \in G$. Thus Statement (c) holds.

Similarly we can show Statement (d). $\square$

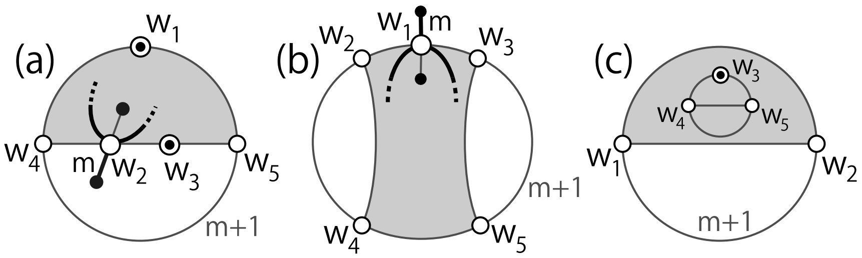

Fig. 15. The gray regions are F. (a) The graph as shown in Fig. 4(b) with $w_2 \in G$. (b) The graph as shown in Fig. 4(d) with $w_1 \in G$. (c) The skew θ-curve of label $m+1$ is contained in F.

7. The Graphs Γ_{m+1} and Γ_{m+2}

In this section, we shall show that for any minimal chart Γ of type $(m; 2, 3, 2)$, the graph Γ_{m+1} contains none of the five graphs as shown in Fig. 4(a),(d),(e),(f),(i), and neither does Γ_{m+2}. Moreover we shall show that neither Γ_{m+1} nor Γ_{m+2} contains a θ-curve.

LEMMA 7.1. *Let G be one of 12 graphs as shown in Fig. 3 and Fig. 4. If for any minimal chart Γ of type $(m; 2, 3, 2)$, the graph Γ_{m+1} does not contain the graph G, then the graph Γ_{m+2} does not contain the graph G.*

PROOF. Suppose that the graph Γ_{m+1} does not contain the graph G for any minimal chart Γ of type $(m; 2, 3, 2)$.

If there exists a minimal chart Γ' of type $(m; 2, 3, 2)$ with $\Gamma'_{m+2} \supset G$, then let Γ'' be the chart obtained from Γ' by changing labels $\cdots, m, m+1, m+2, m+3, \cdots$ into $\cdots, m+3, m+2, m+1, m, \cdots$, respectively. Then Γ'' is a chart of type $(m; 2, 3, 2)$ with $\Gamma''_{m+1} \supset G$. Hence Γ'' is not minimal. Thus Γ'' is C-move equivalent to a chart whose complexity is less than the complexity of Γ''. Hence by using the above C-moves, the chart Γ' is also C-move equivalent to a chart whose complexity is less than the complexity Γ'. Thus Γ' is not minimal. This is a contradiction. Hence if Γ' is a minimal chart of type $(m; 2, 3, 2)$, then $\Gamma'_{m+2} \not\supset G$. $\square$

LEMMA 7.2. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(a).*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(a). We use the notations as shown in Fig. 16(a) where w_3, w_4, w_5 are BW-vertices. By Corollary 4.3, the graph Γ_m contains an oval G. Thus by Lemma 6.3(a), there are three cases: (i) $w_1, w_2 \in G$, (ii) $w_3, w_4 \in G$ or $w_4, w_5 \in G$ (see Fig. 16(b)), (iii) $w_3, w_5 \in G$ (see Fig. 16(c)).

Case (i). Since there exist two internal edges of label $m+1$ connecting w_1 and w_2, there exists a 2-angled disk D of Γ_{m+1} with $w_1, w_2 \in \partial D$. Thus $w_1, w_2 \in G \cap \partial D$. Hence by Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). By Lemma 6.2, the two internal edges e_1, e_2 of label m in G are contained in the closure of the same connected component F of $S^2 - \Gamma_{m+1}$. Thus the curve $e_1 \cup e_2$ bounds a 2-angled disk of Γ_m in $Cl(F)$, say D. Hence $Cl(S^2 - D)$ is also a 2-angled disk of Γ_m, and by Lemma 5.1 the disk $Cl(S^2 - D)$ contains a lens (see Fig. 16(b)). This contradicts Lemma 4.1. Thus Case (ii) does not occur.

Case (iii). Let e_1, e_2 be the two internal edges of label m in G. Let e_3, e_5 be the internal edges of label $m+1$ at w_3, w_5 containing w_4 respectively (see Fig. 16(c)).

By Lemma 6.2, the two edges e_1, e_2 are contained in the closure of the same connected component F of $S^2 - \Gamma_{m+1}$ (see Fig. 16(c)). Without loss of generality we can assume that the terminal edge of label m at w_3 is oriented inward at w_3. By Assumption 2, the terminal edge is middle at w_3. Thus

(1) the edge e_3 is oriented inward at w_3,

and the two edges e_1, e_2 are oriented from w_3 to w_5. Hence the edge e_5 is oriented outward at w_5 (see Fig 16(d)). Thus e_5 is oriented inward at the BW-vertex w_4. However by (1) the edge e_3 is oriented outward at the BW-vertex w_4. This contradicts Lemma 3.1. Hence Case (iii) does not occur.

Therefore Γ_{m+1} does not contain the graph as shown in Fig. 4(a). By Lemma 7.1, we can show that Γ_{m+2} does not contain the graph as shown in Fig. 4(a). □

LEMMA 7.3. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(d).*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(d). We use the notations as shown in Fig. 17(a) where w_1 is a BW-vertex. By

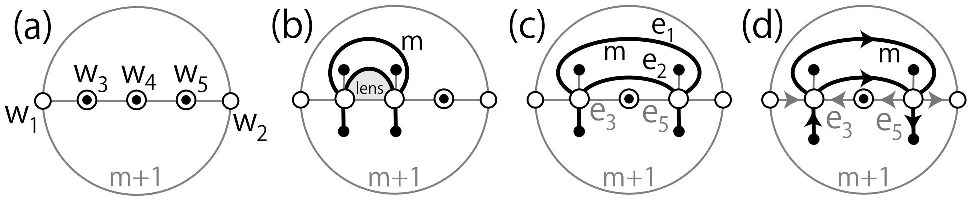

Fig. 16. (a) $w_1, w_2, \cdots, w_5$ are white vertices. (b) $w_3, w_4 \in G$. (c),(d) $w_3, w_5 \in G$.

Corollary 4.3, the graph Γ_m contains an oval G. Thus by Lemma 6.3(b), there are four cases: (i) $w_2, w_3 \in G$ (see Fig. 17(b)), (ii) $w_2, w_4 \in G$ or $w_3, w_5 \in G$, (iii) $w_2, w_5 \in G$ or $w_3, w_4 \in G$ (see Fig. 17(c)), (iv) $w_4, w_5 \in G$ (see Fig. 17(d)).

Case (i). By Lemma 5.3(a), the chart Γ is not minimal. This is a contradiction. Hence Case (i) does not occur.

Case (ii). By Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (ii) does not occur.

Case (iii). By Lemma 5.3(b), the chart Γ is not minimal. This is a contradiction. Hence Case (iii) does not occur.

Case (iv). By Lemma 5.1, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (iv) does not occur.

Therefore Γ_{m+1} does not contain the graph as shown in Fig. 4(d). By Lemma 7.1, we can show that Γ_{m+2} does not contain the graph as shown

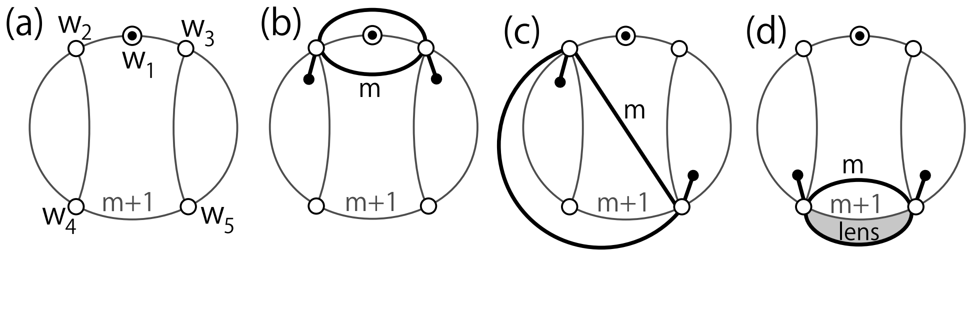

Fig. 17. (a) $w_1, w_2, \cdots, w_5$ are white vertices. (b) $w_2, w_3 \in G$. (c) $w_2, w_5 \in G$. (d) $w_4, w_5 \in G$.

in Fig. 4(d). □

LEMMA 7.4. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(e).*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(e). We use the notations as shown in Fig. 18(a) where w_1 is a BW-vertex. By Corollary 4.3, the graph Γ_m contains an oval G. Thus by Lemma 6.3(b), there are four cases: (i) $w_2, w_3 \in G$ (see Fig. 18(b),(c),(d)), (ii) $w_2, w_4 \in G$ or $w_3, w_5 \in G$ (see Fig. 18(e)), (iii) $w_2, w_5 \in G$ or $w_3, w_4 \in G$ (see Fig. 18(f)), (iv) $w_4, w_5 \in G$.

Case (i). Around the white vertex w_2, there are three internal edges of label $m+1$. Let e_1, e_2, e_3 be the three internal edges of label $m+1$ at w_2 containing w_1, w_3, w_4, respectively (see Fig. 18(a)). Let D be the 2-angled disk of Γ_m not containing the terminal edge of label m at w_2. Then there are three cases: $e_1 \subset D$ or $e_2 \subset D$ or $e_3 \subset D$.

If $e_1 \subset D$ (see Fig. 18(b)), then by Lemma 5.3(a) the chart Γ is not minimal. This is a contradiction. If $e_2 \subset D$ (see Fig. 18(c)), then by Lemma 5.1 there exists a lens of Γ. This contradicts Lemma 4.1. If $e_3 \subset D$ (see Fig. 18(d)), then by Lemma 5.1 there exists a lens in $Cl(S^2 - D)$. This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). By Lemma 5.1, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (ii) does not occur.

Case (iii). By Lemma 5.3(b), the chart Γ is not minimal. This is a contradiction. Hence Case (iii) does not occur.

Case (iv). By Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (iv) does not occur.

Therefore Γ_{m+1} does not contain the graph as shown in Fig. 4(e). By Lemma 7.1, we can show that Γ_{m+2} does not contain the graph as shown in Fig. 4(e). □

LEMMA 7.5. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(f).*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(f). We use the notations as shown in Fig. 19(a) where w_1 is a BW-vertex. By Corollary 4.3, the graph Γ_m contains an oval G. There are two cases: (i) $w_4 \in G$ or $w_5 \in G$, (ii) $w_4 \notin G$ and $w_5 \notin G$.

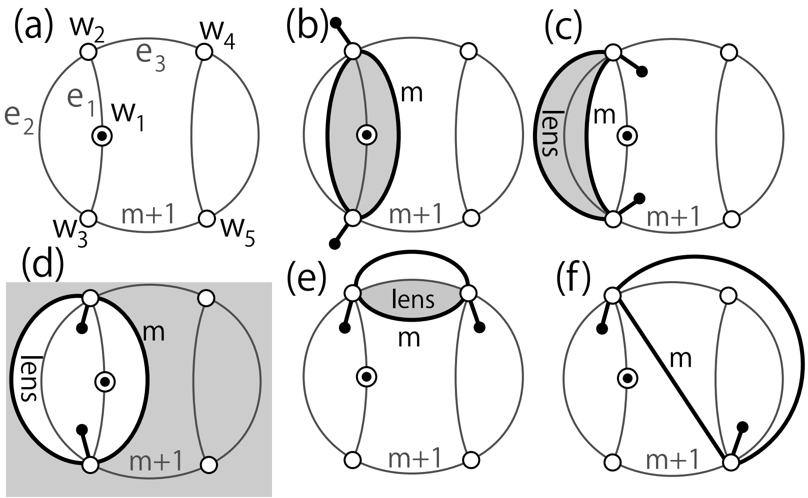

Fig. 18. (a) $w_1, w_2, \cdots, w_5$ are white vertices. (b), (c), (d) $w_2, w_3 \in G$, the gray regions are 2-angled disks of Γ_m not containing the terminal edges of G. (e) $w_2, w_4 \in G$. (f) $w_2, w_5 \in G$.

Case (i). If $w_4 \in G$, then by Lemma 6.3(b) the oval G contains one of w_2, w_3, w_5. Thus there exists an internal edge of label $m+1$ connecting the two white vertices of G (see Fig. 19(b)). Hence by Lemma 5.1, there exists a lens of Γ. This contradicts Lemma 4.1.

If $w_5 \in G$, then we have the same contradiction. Hence Case (i) does not occur.

Case (ii). By Lemma 6.3(b), we have $w_2, w_3 \in G$ (see Fig. 19(c)). By Lemma 5.3(a), the chart Γ is not minimal. This is a contradiction. Hence Case (ii) does not occur.

Therefore Γ_{m+1} does not contain the graph as shown in Fig. 4(f). By Lemma 7.1, we can show that Γ_{m+2} does not contain the graph as shown in Fig. 4(f). □

Lemma 7.6. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(i).*

Proof. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(i). We use the notations as shown in Fig. 20(a) where w_1 is a BW-vertex.

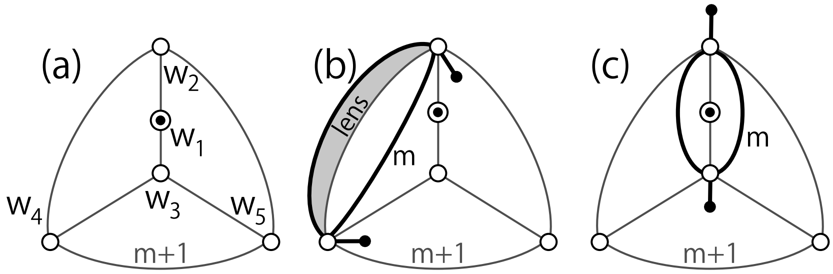

Fig. 19. (a) $w_1, w_2, \cdots, w_5$ are white vertices. (b) $w_2, w_4 \in G$. (c) $w_2, w_3 \in G$.

By Corollary 4.3, the graph Γ_m contains an oval G. There are four cases: (i) G contains the BW-vertex w_1 (see Fig. 20(b),(c)), (ii) $w_2, w_3 \in G$ (see Fig. 20(d)), (iii) $w_3, w_4 \in G$ or $w_3, w_5 \in G$ (see Fig. 20(e)), (iv) $w_4, w_5 \in G$.

Case (i). Let e_1, e_2 be the internal edges in G, and F_1, F_2 the connected components of $S^2 - \Gamma_{m+1}$ with $w_1 \in Cl(F_1) \cap Cl(F_2)$ and $w_3 \notin Cl(F_1)$ (see Fig. 20(a)).

We shall show that $e_1 \cup e_2 \subset Cl(F_2)$. By Lemma 6.2, we have $e_1 \cup e_2 \subset Cl(F_1)$ or $e_1 \cup e_2 \subset Cl(F_2)$. If $e_1 \cup e_2 \subset Cl(F_1)$, then in $Cl(F_1)$ there exist *two* edges of label m at w_2. However, since w_2 is a white vertex as shown in Fig. 20(a), there exists *at most one* edge of label m at w_2 in $Cl(F_1)$. This is a contradiction. Hence $e_1 \cup e_2 \subset Cl(F_2)$.

Thus there are two cases: $w_1, w_2 \in G$ or $w_1, w_3 \in G$. If $w_1, w_2 \in G$ (see Fig. 20(b)), then by Lemma 5.1 there exists a lens of Γ. This contradicts Lemma 4.1. If $w_1, w_3 \in G$ (see Fig. 20(c)), then by Lemma 5.3(b) the chart Γ is not minimal. This is a contradiction. Hence Case (i) does not occur.

Case (ii) and Case (iii). By Lemma 5.1, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (ii) and Case (iii) do not occur.

Case (iv). By Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (iv) does not occur.

Therefore Γ_{m+1} does not contain the graph as shown in Fig. 4(i). By Lemma 7.1, we can show that Γ_{m+2} does not contain the graph as shown in Fig. 4(i). □

LEMMA 7.7. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains a θ-curve.*

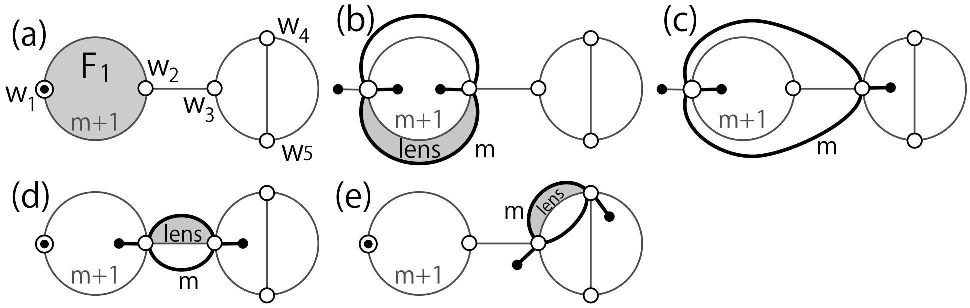

Fig. 20. (a) $w_1, w_2, \cdots, w_5$ are white vertices, the gray region is F_1. (b) $w_1, w_2 \in G$. (c) $w_1, w_3 \in G$. (d) $w_2, w_3 \in G$. (e) $w_3, w_4 \in G$.

PROOF. Suppose that Γ_{m+1} contains a θ-curve G_1. By Lemma 1.2, the graph Γ_{m+1} contains a skew θ-curve G_2. Let w_1, w_2 be the white vertices in G_1. Let w_3 be the BW-vertex in G_2, and w_4, w_5 the other white vertices in G_2.

By Corollary 4.3, the graph Γ_m contains an oval G. Thus by Lemma 6.3(c), there are two cases: $w_1, w_2 \in G$, or $w_4, w_5 \in G$. For both cases, there exist two internal edges of label $m+1$ connecting the two white vertices in G. Hence there exists a 2-angled disk D of Γ_{m+1} with $G \cap \partial D$ two white vertices. Thus by Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Therefore Γ_{m+1} does not contain a θ-curve.

By Lemma 7.1, we can show that Γ_{m+2} does not contain a θ-curve. $\square$

By using Lemma 1.2 and lemmata in this section, we obtain the following corollary:

COROLLARY 7.8. *If there exists a minimal chart* Γ *of type* $(m; 2, 3, 2)$, *then each of* Γ_{m+1} *and* Γ_{m+2} *contains either the union of an oval and a skew* θ*-curve, or one of four graphs as shown in Fig. 4(b),(c),(g),(h).*

8. RO-Families of Pseudo Charts

In this section, we consider a minimal chart Γ of type $(m; 2, 3, 2)$ such that Γ_{m+1} contains either an oval, or one of the four graphs as shown in

Fig. 4(b),(c),(g),(h). We investigate that the chart Γ contains what kind of pseudo charts.

Let Γ be a chart, D a disk, and G a pseudo chart with $G \subset D$. Let $r : D \to D$ be a reflection of D, and G^* the pseudo chart obtained from G by changing the orientations of all of the edges. Then the set $\{G, G^*, r(G), r(G^*)\}$ is called the *RO-family of the pseudo chart* G.

In our argument, we often need a name for an unnamed edge by using a given edge and a given white vertex. For the convenience, we use the following naming: Let e', e_i, e'' be three consecutive edges containing a white vertex w_j. Here, the two edges e' and e'' are unnamed edges. There are six arcs in a neighborhood U of the white vertex w_j. If the three arcs $e' \cap U$, $e_i \cap U$, $e'' \cap U$ lie anticlockwise around the white vertex w_j in this order, then e' and e'' are denoted by a_{ij} and b_{ij} respectively (see Fig. 21). There is a possibility $a_{ij} = b_{ij}$ if they are contained in a loop.

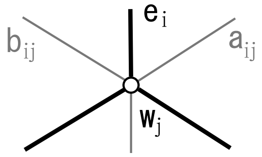

Fig. 21. The three edges a_{ij}, e_i, b_{ij} are consecutive edges around the white vertex w_j.

Lemma 8.1. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. If Γ_{m+1} contains the graph as shown in Fig. 4(c), then Γ contains one of the RO-family of the pseudo chart as shown in Fig. 22(a).*

Proof. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(c). We use the notations as shown in Fig. 22(b) where w_1, w_2, w_3 are BW-vertices. By Corollary 4.3, the graph Γ_m contains an oval G. Thus by Lemma 6.3(a), there are two cases: (i) $w_4, w_5 \in G$ (see Fig. 22(c)), (ii) the oval G contains two of BW-vertices w_1, w_2, w_3.

Case (i). By Lemma 5.3(a), the chart Γ is not minimal. This is a contradiction. Hence Case (i) does not occur.

Case (ii). Without loss of generality we can assume $w_1, w_2 \in G$. Let e_3 be the terminal edge of label $m+1$ at w_3. Let e_1', e_1'' be the internal edges

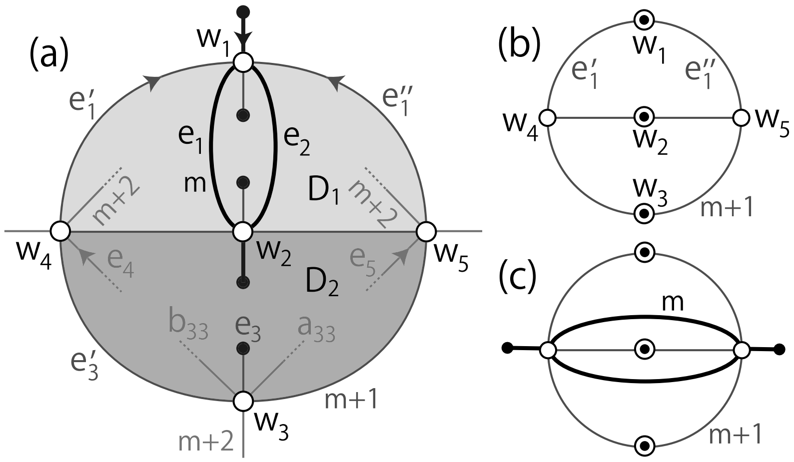

Fig. 22. (a) A pseudo chart containing the graph as shown in Fig. 4(c). The light gray region is D_1, and the dark gray region is D_2. (b) $w_1, w_2, \cdots, w_5$ are white vertices. (c) $w_4, w_5 \in G$.

of label $m+1$ at w_1 containing w_4, w_5 respectively (see Fig. 22(b)).

Now, the graph in Γ_{m+1} as shown in Fig. 4(c) separates S^2 into three disks. One of the three disks contains internal edges of label m in G, say D_1. One of the three disks contains the terminal edge e_3, say D_2. The last one is denoted by D_3. By Assumption 5, we can assume that the disk D_3 contains the point at infinity ∞.

If necessary we change the orientation of all the edges of Γ, we can assume that the terminal edge of label m at w_1 is oriented inward at w_1. Then by Assumption 2

(1) the two edges e_1', e_1'' are oriented inward at w_1.

Since $w_1, w_2 \in \Gamma_m \cap \Gamma_{m+1}$, $w_3, w_4, w_5 \in \Gamma_{m+1}$ and since Γ is of type $(m; 2, 3, 2)$, we have $w_3, w_4, w_5 \in \Gamma_{m+1} \cap \Gamma_{m+2}$. Let e_4, e_5 be the internal edges (possibly terminal edges) of label $m+2$ at w_4, w_5 in D_2 respectively. Then by (1), the two edges e_4, e_5 are oriented inward at w_4, w_5 respectively. Thus Γ contains the pseudo chart as shown in Fig. 22(a). $\square$

LEMMA 8.2. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. If Γ_{m+1}*

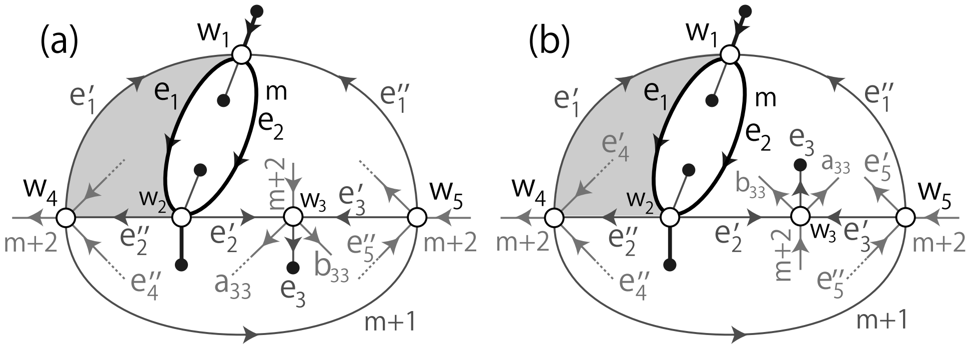

Fig. 23. Pseudo charts containing the graph as shown in Fig. 4(b). The gray regions are the disk D_1'.

contains the graph as shown in Fig. 4(b), then Γ *contains one of the RO-families of the two pseudo charts as shown in Fig. 23.*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(b). We use the notations as shown in Fig. 24(a) where w_1, w_2, w_3 are BW-vertices. By Corollary 4.3, the graph Γ_m contains an oval G. Thus by Lemma 6.3(a), there are three cases: (i) $w_4, w_5 \in G$ (see Fig. 24(b),(c),(d)), (ii) $w_1, w_2 \in G$ or $w_1, w_3 \in G$, (see Fig. 24(e)), (iii) $w_2, w_3 \in G$ (see Fig. 24(f)).

Case (i). Let e', e'', e''' be the internal edges of label $m+1$ at w_4 containing w_1, w_2, w_5, respectively (see Fig. 24(a)). Let D be the 2-angled disk of Γ_m not containing the terminal edge of label m at w_4. There are three cases: $e' \subset D$ or $e'' \subset D$ or $e''' \subset D$.

If $e' \subset D$ (see Fig. 24(b)), then by Lemma 5.3(a) the chart Γ is not minimal. This is a contradiction. If $e'' \subset D$ (see Fig. 24(c)), then by Lemma 5.1 there exists a lens in $Cl(S^2 - D)$. This contradicts Lemma 4.1. If $e''' \subset D$ (see Fig. 24(d)), then by Lemma 5.1 there exists a lens in D. This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). Without loss of generality we can assume that $w_1, w_2 \in G$. Let e_1, e_2 be the internal edges of label m in G.

Now, the graph in Γ_{m+1} as shown in Fig. 4(b) separates S^2 into three disks. One of the three disks contains both of e_1 and e_2.

Moreover, since $w_1, w_2 \in \Gamma_m \cap \Gamma_{m+1}$, $w_3, w_4, w_5 \in \Gamma_{m+1}$ and since Γ is of type $(m; 2, 3, 2)$, we have $w_3, w_4, w_5 \in \Gamma_{m+1} \cap \Gamma_{m+2}$. Hence the chart Γ contains the pseudo chart as shown in Fig. 24(e). We use the notations as shown in Fig. 24(e), where e'_1, e''_1 are internal edges of label $m+1$ at w_1, e'_2, e''_2 are internal edges of label $m+1$ at w_2 containing w_3, w_4 respectively, e'_3 is an internal edge of label $m+1$ connecting w_3, w_5.

Without loss of generality we can assume that the terminal edge of label m at w_1 is oriented inward at w_1. Thus by Assumption 2, the two edges e'_1, e''_1 are oriented inward at w_1, and the two edges e_1 and e_2 are oriented from w_1 to w_2. Hence the two edges e'_2, e''_2 are oriented outward at w_2. Thus the edge e'_2 is oriented from w_2 to the BW-vertex w_3. Hence by Lemma 3.1 the edge e'_3 is oriented from w_5 to w_3. Moreover, we have the orientation of other edges. Thus Γ contains one of the two pseudo charts as shown in Fig. 23.

Case (iii). By Lemma 6.2, the two internal edges e_1, e_2 of label m in G are contained in the closure of the same connected component F of $S^2 - \Gamma_{m+1}$ (see Fig. 24(f)). Thus the curve $e_1 \cup e_2$ bounds a 2-angled disk

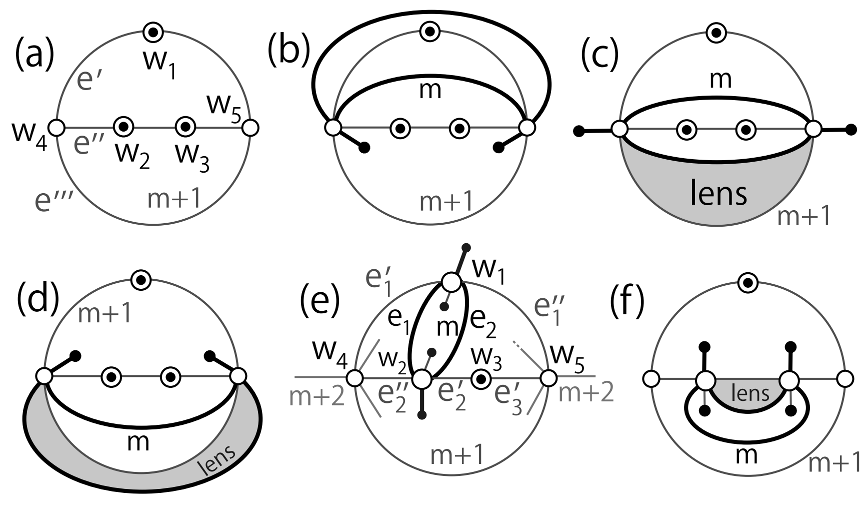

Fig. 24. (a) $w_1, w_2, \cdots, w_5$ are white vertices. (b),(c),(d) $w_4, w_5 \in G$. (e) $w_1, w_2 \in G$. (f) $w_2, w_3 \in G$.

of Γ_m in $Cl(F)$, say D. Hence $Cl(S^2 - D)$ is also a 2-angled disk of Γ_m, and by Lemma 5.1 the disk $Cl(S^2 - D)$ contains a lens. This contradicts Lemma 4.1. Hence Case (iii) does not occur.

Therefore Γ contains one of the RO-families of the two pseudo charts as shown in Fig. 23. □

The following lemma is not used in the this paper, but is used in the next paper [11].

LEMMA 8.3. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. If Γ_{m+1} contain an oval, then Γ contains one of the RO-families of the two pseudo charts as shown in Fig. 25(a) and (b).*

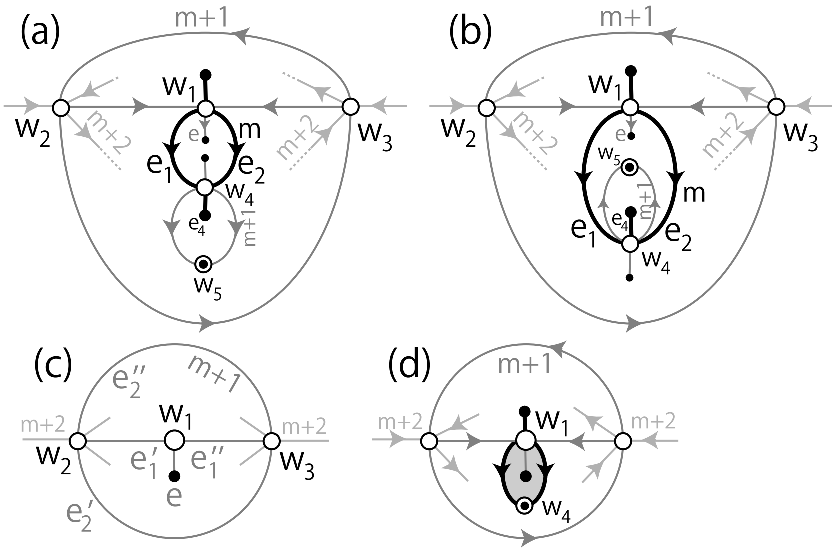

Fig. 25. (a),(b) Pseudo charts containing a skew θ-curve and an oval of label $m+1$. (c) The skew θ-curve of label $m+1$. (d) The gray region is the disk D.

PROOF. By Corollary 4.3, the graph Γ_m contains an oval G. Since Γ_{m+1} contain an oval, by Lemma 1.2 the graph Γ_{m+1} contains a skew θ-curve. Let w_1, w_2, w_3 be the white vertices in the skew θ-curve such that w_1 is a BW-vertex with respect to Γ_{m+1}. Let w_4, w_5 be the white vertices in the oval of label $m+1$. Then w_4, w_5 are BW-vertices with respect to Γ_{m+1}. Hence by Lemma 6.3(d), there are three cases: (i) $w_1, w_4 \in G$ or $w_1, w_5 \in G$, (ii) $w_2, w_3 \in G$, (iii) $w_4, w_5 \in G$.

Case (ii) and Case (iii). By Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (ii) and Case (iii) do not occur.

Case (i). Without loss of generality we can assume $w_1, w_4 \in G$. Since $w_1, w_4 \in \Gamma_m \cap \Gamma_{m+1}$, $w_2, w_3 \in \Gamma_{m+1}$ and since Γ is of type $(m; 2, 3, 2)$, we have $w_2, w_3 \in \Gamma_{m+1} \cap \Gamma_{m+2}$ (see Fig. 25(c)). We use the notations as shown in Fig. 25(c), where e is the terminal edge of label $m+1$ at w_1, e_1', e_1'' are internal edges of label $m+1$ at w_1, and e_2', e_2'' are internal edges of label $m+1$ connecting w_2 and w_3.

Without loss of generality, we can assume that

(1) the terminal edge e is oriented outward at w_1.

Since the terminal edge e is middle at w_1 by Assumption 2, the two edges e_1', e_1'' are oriented inward at w_1. If necessary we reflect the chart Γ, we can assume that the edge e_2' is oriented from w_2 to w_3. Looking at edges around w_2, the edge e_2'' is oriented from w_3 to w_2. Hence we have the orientation of the other edges of label $m+2$. Let e_1, e_2 be the internal edges of label m in G. Then by (1) the edges e_1, e_2 is oriented from w_1 to w_4. Hence Γ contains the pseudo chart as shown in Fig. 25(d).

Let D be the 2-angled disk of Γ_m with $\partial D \ni w_1, w_4$ and $D \not\ni w_2$ (see Fig. 25(d)). Let e_4 be the terminal edge of label m at w_4. There are two cases: $e_4 \not\subset D$ or $e_4 \subset D$. If $e_4 \not\subset D$, then the chart Γ contains the pseudo chart as shown in Fig. 25(a). If $e_4 \subset D$, then the chart Γ contains the pseudo chart as shown in Fig. 25(b). Therefore Γ contains one of the RO-families of the two pseudo charts as shown in Fig. 25(a),(b). $\square$

The following lemma is not used in the this paper, but is used in the next paper [11].

LEMMA 8.4. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. If Γ_{m+1} contains the graph as shown in Fig. 4(g), then Γ contains one of the RO-families of the two pseudo charts as shown in Fig. 26(a),(b).*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(g). We use the notations as shown in Fig. 26(c) where w_1, w_4, w_5 are BW-vertices. By Corollary 4.3, the graph Γ_m contains an oval G. There are seven

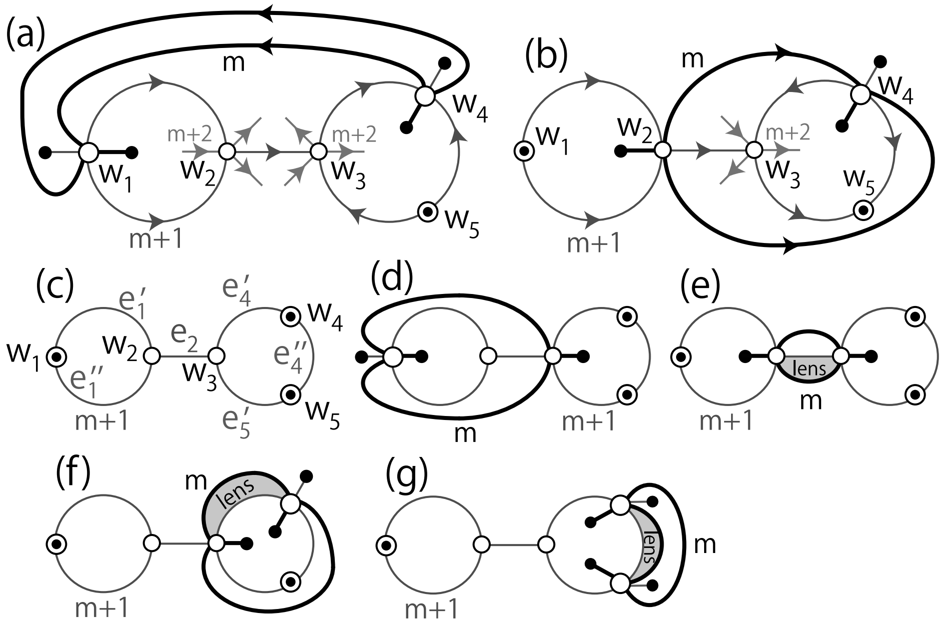

Fig. 26. (a),(b) Pseudo charts containing the graph as shown in Fig. 4(g). (c) $w_1, w_2, \cdots, w_5$ are white vertices. (d) $w_1, w_3 \in G$. (e) $w_2, w_3 \in G$. (f) $w_3, w_4 \in G$. (g) $w_4, w_5 \in G$.

cases: (i) $w_1, w_2 \in G$, (ii) $w_1, w_3 \in G$ (see Fig. 26(d)), (iii) $w_1, w_4 \in G$ or $w_1, w_5 \in G$ (see Fig. 26(a)), (iv) $w_2, w_3 \in G$ (see Fig. 26(e)), (v) $w_2, w_4 \in G$ or $w_2, w_5 \in G$ (see Fig. 26(b)), (vi) $w_3, w_4 \in G$ or $w_3, w_5 \in G$ (see Fig. 26(f)), (vii) $w_4, w_5 \in G$ (see Fig. 26(g)).

Case (i). By Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). By Lemma 5.3(b), the chart Γ is not minimal. This is a contradiction. Thus Case (ii) does not occur.

Case (iii). We use the notations as shown in Fig. 26(c) where e_1', e_1'' are two internal edges at w_1, e_2 is the internal edge connecting w_2 and w_3, e_4', e_4'' are two internal edges at w_4, and e_5' is the internal edge connecting w_3 and w_5.

If necessary we reflect the chart, we can assume that $w_1, w_4 \in G$. If necessary we change the orientation of all the edges, we can assume that the two edges e'_1, e''_1 are oriented from w_1 to w_2. Then the edge e_2 is oriented from w_2 to w_3, and the two internal edges of label m in G are oriented from w_4 to w_1. Thus the two edges e'_4, e''_4 are oriented inward at w_4. Hence the edge e''_4 is oriented from w_5 to w_4. Since w_5 is a BW-vertex with respect to Γ_{m+1}, by Lemma 3.1 the edge e'_5 is oriented from w_5 to w_3. Moreover we have the orientation of the other edges.

Since $w_1, w_4 \in \Gamma_m \cap \Gamma_{m+1}$, $w_2, w_3 \in \Gamma_{m+1}$ and since Γ is of type $(m; 2, 3, 2)$, we have $w_2, w_3 \in \Gamma_{m+1} \cap \Gamma_{m+2}$. Therefore Γ contains the pseudo chart as shown in Fig. 26(a).

Case (iv). By Lemma 5.1, there exists a lens. This contradicts Lemma 4.1. Thus Case (iv) does not occur.

Case (v). If necessary we reflect the chart, we can assume that $w_2, w_4 \in G$. If necessary we change the orientation of all the edges, we can assume that the two internal edges e'_1, e''_1 of label $m+1$ at w_1 are oriented from w_1 to w_2. By a similar way as Case (iii), we can show that Γ contains the pseudo chart as shown in Fig. 26(b).

Case (vi). By Lemma 6.2, the two internal edges e_1, e_2 of label m in G are contained in the closure of the same connected component F of $S^2 - \Gamma_{m+1}$. Thus the curve $e_1 \cup e_2$ bounds a 2-angled disk D of Γ_m with $w_5 \in D$ (see Fig. 26(f)), and the disk D contains a lens by Lemma 5.1. This contradicts Lemma 4.1. Thus Case (vi) does not occur.

Case (vii). By the similar way of Case (vi), there exists a lens. This contradicts Lemma 4.1. Thus Case (vii) does not occur.

Therefore Γ contains one of the RO-families of the two pseudo charts as shown in Fig. 26(a),(b). □

The following lemma is not used in the this paper, but is used in the next paper [11].

LEMMA 8.5. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. If Γ_{m+1} contains the graph as shown in Fig. 4(h), then Γ contains one of the RO-families of the two pseudo charts as shown in Fig. 27(a), (b).*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(h). We use the notations as shown in Fig. 27(c) where w_1, w_3, w_5 are BW-vertices, e'_1, e''_1 are two internal edges at w_1, e'_2 is the internal edge connecting

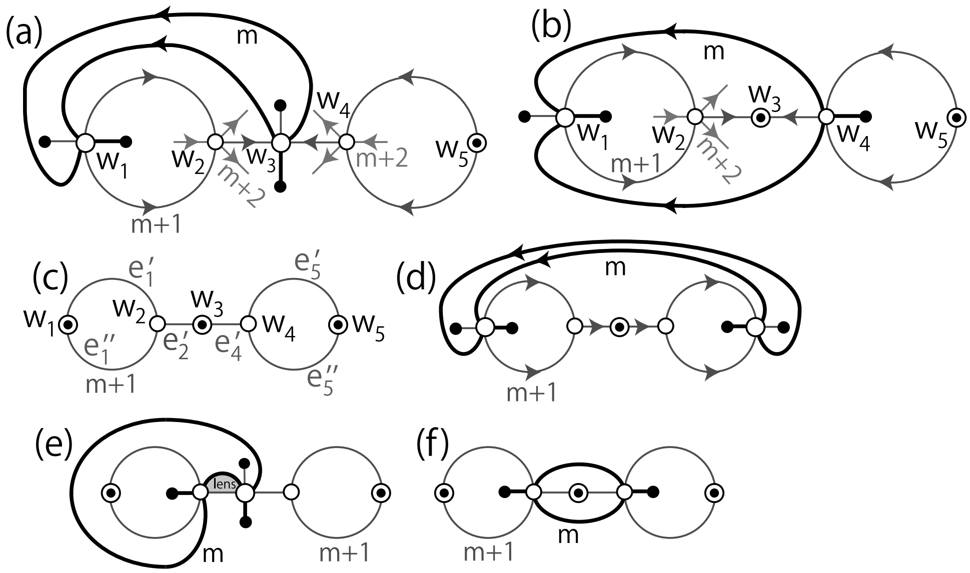

Fig. 27. (a),(b) Pseudo charts containing the graph as shown in Fig. 4(h). (c) $w_1, w_2, \cdots, w_5$ are white vertices. (d) $w_1, w_5 \in G$. (e) $w_2, w_3 \in G$. (f) $w_2, w_4 \in G$.

w_2 and w_3, e'_4 is the internal edge connecting w_3 and w_4, e'_5, e''_5 are two internal edges at w_5.

By Corollary 4.3, the graph Γ_m contains an oval G. There are six cases: (i) $w_1, w_2 \in G$ or $w_5, w_4 \in G$, (ii) $w_1, w_3 \in G$ or $w_5, w_3 \in G$ (see Fig. 27(a)), (iii) $w_1, w_4 \in G$ or $w_5, w_2 \in G$ (see Fig. 27(b)), (iv) $w_1, w_5 \in G$ (see Fig. 27(d)), (v) $w_2, w_3 \in G$ or $w_4, w_3 \in G$ (see Fig. 27(e)), (vi) $w_2, w_4 \in G$ (see Fig. 27(f)).

Case (i). By Lemma 5.2, there exists a lens of Γ. This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). If necessary we reflect the chart, we can assume that $w_1, w_3 \in G$. By Assumption 2, a neighborhood of w_3 contains the pseudo chart as shown in Fig. 14.

If necessary we change the orientation of all the edges, we can assume that the two edges e'_1, e''_1 are oriented from w_1 to w_2. Then the two internal edges of label m in G are oriented from w_3 to w_1, and the two edges e'_2, e'_4

are oriented inward at w_3. Thus the edge e'_4 is oriented outward at w_4. Hence by Lemma 3.1, the two edges e'_5, e''_5 are oriented from w_5 to w_4. Thus we have the orientation of the other edges.

Since $w_1, w_3 \in \Gamma_m \cap \Gamma_{m+1}$, $w_2, w_4 \in \Gamma_{m+1}$ and since Γ is of type $(m; 2, 3, 2)$, we have $w_2, w_4 \in \Gamma_{m+1} \cap \Gamma_{m+2}$. Therefore Γ contains the pseudo chart as shown in Fig. 27(a).

Case (iii). If necessary we reflect the chart, we can assume that $w_1, w_4 \in G$. If necessary we change the orientation of all the edges, we can assume that the two edges e'_1, e''_1 are oriented from w_1 to w_2. By a similar way as Case (ii), we can show that Γ contains the pseudo chart as shown in Fig. 27(b).

Case (iv). If necessary we change the orientation of all the edges, we can assume that the two edges e'_1, e''_1 are oriented from w_1 to w_2. Then

(1) the edge e'_2 is oriented from w_2 to w_3 (i.e. the edge e'_2 is oriented inward at w_3), and

the two internal edges of label m in G are oriented from w_5 to w_1. Thus the two edges e'_5, e''_5 are oriented from w_4 to w_5. Hence the edge e'_4 is oriented from w_3 to w_4. Thus the edge e'_4 is oriented outward at w_3 (see Fig. 27(d)). However by (1) and Lemma 3.1 we have a contradiction, because w_3 is a BW-vertex with respect to Γ_{m+1}. Thus Case (iv) does not occur.

Case (v). Without loss of generality we can assume $w_2, w_3 \in G$. If necessary we reflect the chart, by Assumption 2 (see Fig. 14) the chart Γ contains the pseudo chart as shown in Fig. 27(e). Hence by Lemma 5.1, there exists a lens. This contradicts Lemma 4.1. Thus Case (v) does not occur.

Case (vi). By Lemma 5.3(a), the chart Γ is not minimal. This is a contradiction. Thus Case (vi) does not occur.

Therefore Γ contains one of the RO-families of the two pseudo charts as shown in Fig. 27(a),(b). $\square$

9. IO-Calculation

In this section, we shall show that neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(c) for any minimal chart Γ of type $(m; 2, 3, 2)$.

Let Γ be a chart, and v a vertex. Let α be a short arc of Γ in a small neighborhood of v such that v is an endpoint of α. If the arc α is oriented to v, then α is called *an inward arc*, and otherwise α is called *an outward arc*.

Let Γ be an n-chart. Let F be a closed domain with $\partial F \subset \Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}$ for some label k of Γ, where $\Gamma_0 = \emptyset$ and $\Gamma_n = \emptyset$. By Condition (iii) for charts, in a small neighborhood of each white vertex, there are three inward arcs and three outward arcs. Also in a small neighborhood of each black vertex, there exists only one inward arc or one outward arc. We often use the following fact, when we fix (inward or outward) arcs near white vertices and black vertices:

($*$) *The number of inward arcs contained in $F \cap \Gamma_k$ is equal to the number of outward arcs in $F \cap \Gamma_k$.*

When we use this fact, we say that we use *IO-Calculation with respect to Γ_k in F*. For example, in a minimal chart Γ, consider the pseudo chart as shown in Fig. 28 where

(1) F is a 4-angled disk of Γ_{k-1},

(2) v_1, v_2, v_3, v_4 are white vertices in ∂F with $v_1 \in \Gamma_{k-2} \cap \Gamma_{k-1}$ and $v_2, v_3, v_4 \in \Gamma_{k-1} \cap \Gamma_k$,

(3) e_1 is a terminal edge of label $k-2$ at v_1,

(4) e_3 is a terminal edge of label $k-1$ oriented inward at v_3,

(5) for $i = 2, 4$, the edge e_i of label k is oriented inward at v_i.

Then we can show that $w(\Gamma \cap \text{Int}F) \geq 1$. Suppose $w(\Gamma \cap \text{Int}F) = 0$. By Assumption 2, the terminal edge e_3 contains a middle arc. Thus

(6) neither of edges a_{33}, b_{33} of label k is middle at v_3 (by Assumption 2, neither of them is a terminal edge).

Hence by (4),

(7) both of edges a_{33}, b_{33} of label k are oriented inward at v_3.

If both of e_2 and e_4 are terminal edges of label k, then by (5), (6), (7) the number of inward arcs in $F \cap \Gamma_k$ is four, but the number of outward arcs in $F \cap \Gamma_k$ is two. This contradicts the fact ($*$). Similarly if one of e_2 and e_4 is not a terminal edge of label k, then we have the same contradiction. Thus $w(\Gamma \cap \text{Int}F) \geq 1$. Instead of the above argument, we just say that

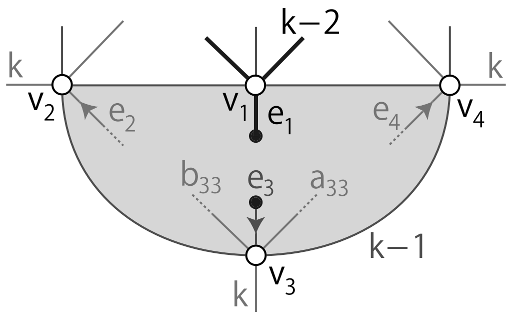

Fig. 28. The gray region is the disk F.

we have $w(\Gamma \cap \mathrm{Int}F) \geq 1$ by IO-Calculation with respect to Γ_k in F.

LEMMA 9.1 ([6, Lemma 5.4]). *If a minimal chart Γ contains the pseudo chart as shown in Fig. 29, then the interior of the disk D^* contains at least one white vertex, where D^* is the disk with the boundary $e_3^* \cup e_4^* \cup e^*$.*

LEMMA 9.2. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(c).*

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(c). By Lemma 8.1, the chart Γ contains one of the RO-family of the pseudo chart as shown in Fig. 22(a). We use the notations as shown in Fig. 22(a)

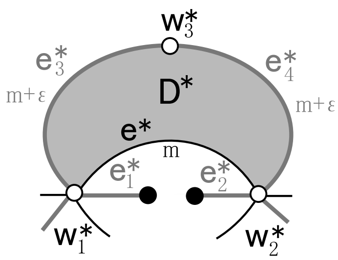

Fig. 29. The gray region is the disk D^*. The label of the edge e^* is m, and $\varepsilon \in \{+1, -1\}$.

where e_1, e_2 are the internal edges of label m with $e_1 \cap e_2 \ni w_1, w_2$, e_3 is the terminal edge of label $m + 1$ at w_3, and

(1) the two internal edges e_4, e_5 (possibly terminal edges) of label $m + 2$ are oriented inward at w_4, w_5, respectively.

Now, the graph in Γ_{m+1} as shown in Fig. 4(c) separates S^2 into three disks. One of them contains e_1, say D_1, and one of them contains the terminal edge e_3, say D_2.

Claim 1. $w(\Gamma \cap \text{Int}D_1) \geq 2$ and $w(\Gamma \cap \text{Int}D_2) = 0$.

Proof of Claim 1. The curve $e_1 \cup e_2$ separates the disk D_1 into three disks. One of them contains w_4, say D_1', and one of them contains w_5, say D_1''. Apply Lemma 9.1 considering as $D^* = D_1'$ and $w_3^* = w_4$, we have $w(\Gamma \cap \text{Int}D_1') \geq 1$. Similarly we can show that $w(\Gamma \cap \text{Int}D_1'') \geq 1$. Since $D_1 \supset D_1' \cup D_1''$ and $\text{Int}D_1' \cap \text{Int}D_1'' = \emptyset$, we have $w(\Gamma \cap \text{Int}D_1) \geq 2$.

Since Γ is of type $(m; 2, 3, 2)$, we have $w(\Gamma) = 7$ and $w(\Gamma_{m+1}) = 5$. Thus

$$7 = w(\Gamma) \geq w(\Gamma_{m+1}) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) \geq 5 + 2 + w(\Gamma \cap \text{Int}D_2).$$

Hence $w(\Gamma \cap \text{Int}D_2) = 0$. Thus Claim 1 holds. □

Claim 2. The terminal edge e_3 is oriented outward at w_3.

Proof of Claim 2. Suppose that e_3 is oriented inward at w_3. Considering as $F = D_2$ and $k = m + 2$ in the example of IO-Calculation in Section 9, the condition (1) implies that we have $w(\Gamma \cap \text{Int}D_2) \geq 1$. This contradicts the second equation of Claim 1. Hence the terminal edge e_3 is oriented outward at w_3. Thus Claim 2 holds. □

Let a_{33}, b_{33} be the internal edges of label $m + 2$ at w_3 in D_2 such that a_{33}, e_3, b_{33} lie anticlockwise around w_3 in this order (see Fig. 22(a)). By Assumption 2,

(2) the terminal edge e_3 is middle at w_3.

Claim 3. $a_{33} = e_5$ and $b_{33} = e_4$.

PROOF OF CLAIM 3. By (2) and Assumption 2, neither a_{33} nor b_{33} is a terminal edge. Moreover, by Claim 2, both of a_{33} and b_{33} are oriented outward at w_3. Thus by the second equation of Claim 1, we have $a_{33} = e_5$ and $b_{33} = e_4$. Hence Claim 3 holds. □

Finally we shall show that there exists a lens of Γ. Let e'_3 be the internal edge of label $m+1$ with $e'_3 \ni w_3, w_4$ (see Fig. 22(a)). By (2), neither e'_3 nor b_{33} is middle at w_3.

By (2) and Claim 2, the terminal edge e_3 is oriented outward at w_3 and middle at w_3. Hence by Claim 3, the edge $b_{33} = e_4$ is oriented from w_3 to w_4 and e'_3 is oriented from w_4 to w_3. Thus neither e'_3 nor b_{33} is middle at w_4. Hence the curve $e'_3 \cup b_{33}$ bounds a lens in D_2. This contradicts Lemma 4.1. Therefore Γ_{m+1} does not contain the graph as shown in Fig. 4(c).

By Lemma 7.1, we can show that Γ_{m+2} does not contain the graph as shown in Fig. 4(c). We complete the proof of Lemma 9.2. □

10. Shifting Lemma

In this section we shall show that neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(b) for any minimal chart Γ of type $(m; 2, 3, 2)$. Thus by Corollary 7.8 and Lemma 9.2, we obtain the main theorem.

LEMMA 10.1. *Let Γ be a chart of type $(m; 2, 3, 2)$. If Γ contains the pseudo chart as shown in Fig. 23(a), then Γ is not minimal.*

PROOF. Suppose that Γ is minimal. We use the notations as shown in Fig. 23(a). Here e_1, e_2 are internal edges of label m, and $e''_4, e''_5, a_{33}, b_{33}$ are internal edges (possibly terminal edges) of label $m+2$ such that

(1) e''_4, e''_5 are oriented inward at w_4, w_5, respectively,

(2) a_{33}, b_{33} are oriented outward at w_3.

Moreover none of $e''_4, e''_5, a_{33}, b_{33}$ are middle at w_3, w_4 or w_5. Thus by Assumption 2,

(3) none of $e''_4, e''_5, a_{33}, b_{33}$ are terminal edges.

Now, the graph Γ_{m+1} contains the graph as shown in Fig. 4(b). The graph in Γ_{m+1} separates S^2 into three disks. One of them contains the edges e_1 and e_2 of label m, say D_1, and one of them contains the edge e_4'', say D_2. Moreover, the curve $e_1 \cup e_2$ separates the disk D_1 into three disks. One of them contains w_4, say D_1'. Apply Lemma 9.1 considering as $D^* = D_1'$ and $w_3^* = w_4$, we have

(4) $w(\Gamma \cap \text{Int}D_1') \geq 1$.

There are three cases: (i) $w(\Gamma \cap \text{Int}D_2) = 0$, (ii) $w(\Gamma \cap \text{Int}D_2) = 1$, (iii) $w(\Gamma \cap \text{Int}D_2) \geq 2$.

Case (i). By using (1), (2) and (3), we have $a_{33} = e_4''$ and $b_{33} = e_5''$. Thus the curve $b_{33} \cup e_3'$ bounds a lens in D_2. This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). Let v be the white vertex in $\text{Int}D_2$. Since the five white vertices $w_1, w_2, \cdots, w_5$ are in Γ_{m+1} and Γ is of type $(m; 2, 3, 2)$, we have $v \in \Gamma_{m+2} \cap \Gamma_{m+3}$. Thus by using (1),(2) and (3), we have a contradiction by IO-Calculation with respect to Γ_{m+2} in D_2. Hence Case (ii) does not occur.

Case (iii). Since Γ is of type $(m; 2, 3, 2)$, we have $w(\Gamma) = 7$ and $w(\Gamma_{m+1}) = 5$. Thus by (4) and the condition $w(\Gamma \cap \text{Int}D_2) \geq 2$ of this case,

$$7 = w(\Gamma) \geq w(\Gamma_{m+1}) + w(\Gamma \cap \text{Int}D_1') + w(\Gamma \cap \text{Int}D_2) \geq 5 + 1 + 2 = 8.$$

This is a contradiction. Hence Case (iii) does not occur.

Therefore the three cases do not occur. Hence Γ is not minimal. $\square$

Let Γ and Γ' be C-move equivalent charts. Suppose that a pseudo chart X of Γ is also a pseudo chart of Γ'. Then we say that Γ is modified to Γ' by *C-moves keeping X fixed.* In Fig. 30, we give examples of C-moves keeping pseudo charts fixed.

Let Γ be a chart. Let α be an arc in an edge of Γ_m, and w a white vertex with $w \notin \alpha$. Suppose that there exists an arc β in Γ such that its end points are the white vertex w and an interior point p of the arc α. Then we say that *the white vertex w connects with the point p of α by the arc β.*

Let α be a simple arc, and p, q points in α. We denote by $\alpha[p, q]$ the subarc of α whose endpoints are p and q.

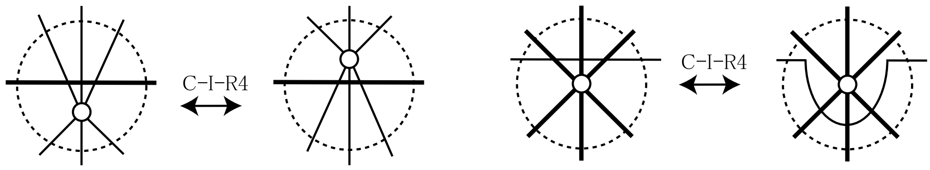

Fig. 30. C-moves keeping thicken figures fixed.

LEMMA 10.2 ([6, Lemma 4.2]). (*Shifting Lemma*) *Let Γ be a chart and α an arc in an edge of Γ_m. Let w be a white vertex of $\Gamma_k \cap \Gamma_h$ where $h = k + \varepsilon, \varepsilon \in \{+1, -1\}$. Suppose that the white vertex w connects with a point r of the arc α by an arc in an edge e of Γ_k. Suppose that one of the following two conditions is satisfied:*

(1) $h > k > m$.

(2) $h < k < m$.

Then for any neighborhood V of the arc $e[w, r]$ we can shift the white vertex w to $e - e[w, r]$ along the edge e by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves in V keeping $\bigcup_{i<0} \Gamma_{k+i\varepsilon}$ fixed (*see Fig. 31*).

PROPOSITION 10.3. *Let Γ be a minimal chart of type $(m; 2, 3, 2)$. Then neither Γ_{m+1} nor Γ_{m+2} contains the graph as shown in Fig. 4(b).*

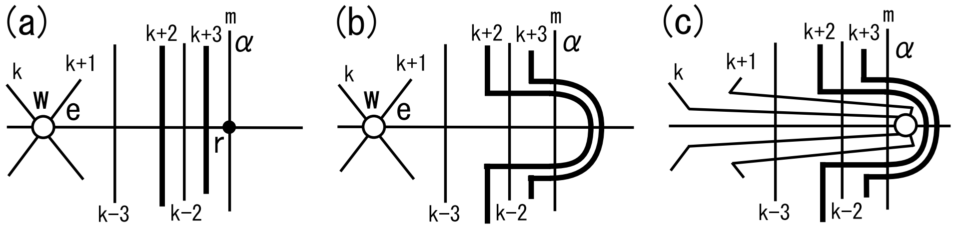

Fig. 31. $k > m$ and $\varepsilon = +1$.

PROOF. Suppose that Γ_{m+1} contains the graph as shown in Fig. 4(b). By Lemma 8.2 and Lemma 10.1, the chart Γ contains one of the RO-family of the pseudo chart as shown in Fig. 23(b). We use the notations as shown in Fig. 23(b). Here, e_1, e_2 are internal edges of label m, and $e_4'', e_5', e_5'', a_{33}, b_{33}$ are internal edges (possibly terminal edges) of label $m+2$ such that

(1) e_4'', e_5'' are oriented inward at w_4, w_5 respectively,

(2) e_5', a_{33}, b_{33} are oriented outward at w_5, w_3, w_3 respectively.

Moreover, none of $e_4'', e_5'', a_{33}, b_{33}$ are middle at w_3, w_4 or w_5. Thus by Assumption 2,

(3) none of $e_4'', e_5'', a_{33}, b_{33}$ are terminal edges.

Now, the graph Γ_{m+1} contains the graph as shown in Fig. 4(b). The graph in Γ_{m+1} separates S^2 into three disks. One of them contains the edges e_1 and e_2, say D_1, and one of them contains the edge e_4'', say D_2. Moreover, the curve $e_1 \cup e_2$ separates the disk D_1 into three disks. One of them contains w_4, say D_1'. Apply Lemma 9.1 considering as $D^* = D_1'$ and $w_3^* = w_4$, we have

(4) $w(\Gamma \cap \mathrm{Int}D_1') \geq 1$.

CLAIM 1. $w(\Gamma \cap \mathrm{Int}D_1) \geq 1$ and $w(\Gamma \cap \mathrm{Int}D_2) \geq 1$.

PROOF OF CLAIM 1. By (4) and $D_1' \subset D_1$, we have $w(\Gamma \cap \mathrm{Int}D_1) \geq 1$.

By (3), neither e_4'' nor e_5'' is a terminal edge. Since e_4'', e_5'' are oriented inward at w_4, w_5 respectively by (1), we have $w(\Gamma \cap \mathrm{Int}D_2) \geq 1$ by IO-Calculation with respect to Γ_{m+2} in D_2. $\square$

CLAIM 2. $w(\Gamma \cap \mathrm{Int}D_1) = 1$ and $w(\Gamma \cap \mathrm{Int}D_1') = 1$.

PROOF OF CLAIM 2. Suppose that $w(\Gamma \cap \mathrm{Int}D_1) \geq 2$. Since Γ is of type $(m; 2, 3, 2)$, we have $w(\Gamma) = 7$ and $w(\Gamma_{m+1}) = 5$. Thus by the second inequality of Claim 1

$$7 = w(\Gamma) \geq w(\Gamma_{m+1}) + w(\Gamma \cap \mathrm{Int}D_1) + w(\Gamma \cap \mathrm{Int}D_2) \geq 5 + 2 + 1 = 8.$$

This is a contradiction. Thus $w(\Gamma \cap \mathrm{Int}D_1) \leq 1$. Hence by the first inequality of Claim 1, we have $w(\Gamma \cap \mathrm{Int}D_1) = 1$.

Thus by (4) and $D'_1 \subset D_1$, we have $w(\Gamma \cap \mathrm{Int}D'_1) = 1$. Hence Claim 2 holds. □

Let w_6 be the white vertex in $\mathrm{Int}D'_1$. Since the five white vertices $w_1, w_2, \cdots, w_5$ are in Γ_{m+1} and Γ is of type $(m; 2, 3, 2)$, we have

(5) $w_6 \in \Gamma_{m+2} \cap \Gamma_{m+3}$.

CLAIM 3. $a_{33} \ni w_6$ or $b_{33} \ni w_6$.

PROOF OF CLAIM 3. First we shall show that $a_{33} = e'_4$ or $a_{33} \ni w_6$. By (3), the edge a_{33} is not a terminal edge. Moreover, by (2), we have $a_{33} \neq e'_5$ and $a_{33} \neq b_{33}$. Thus by Claim 2, we have $a_{33} = e'_4$ or $a_{33} \ni w_6$.

Similarly we can show that $b_{33} = e'_4$ or $b_{33} \ni w_6$.

If $a_{33} \not\ni w_6$ and $b_{33} \not\ni w_6$, then $a_{33} = e'_4$ and $b_{33} = e'_4$. This is a contradiction. Therefore $a_{33} \ni w_6$ or $b_{33} \ni w_6$. Thus Claim 3 holds. □

If $a_{33} \ni w_6$, let $e = a_{33}$, otherwise let $e = b_{33}$.

CLAIM 4. We can move the white vertex w_6 from the disk D'_1 to the outside of D'_1.

PROOF OF CLAIM 4. Since the edge e connects the vertex w_6 in $\mathrm{Int}D'_1$ and the vertex w_3 in the outside of D'_1, the edge e intersects the boundary $\partial D'_1$. Let x be the point in the edge e with $e[w_6, x] \cap \partial D'_1 = x$. Since the edge e is of label $m+2$ and since $\partial D'_1$ consists of the edge e_1 of label m and two internal edges of label $m+1$, we have

$$e[w_6, x] \cap e_1 = e[w_6, x] \cap \partial D'_1 = x.$$

Thus by (5), the white vertex $w_6 \in \Gamma_{m+2} \cap \Gamma_{m+3}$ connects with the point x in the edge e_1 of label m by the arc $e[w_6, x]$ of label $m+2$. Hence by Shifting Lemma (Lemma 10.2), we can shift the white vertex w_6 by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves in a neighborhood of the arc $e[w_6, x]$ keeping $\bigcup_{i<0} \Gamma_{m+2+i}$ fixed. Thus we can shift the white vertex w_6 to the outside of D'_1 by C-moves keeping $\partial D'_1$ fixed. Therefore Claim 4 holds. □

By Claim 2 and Claim 4, we have $w(\Gamma \cap \text{Int}D_1') = 0$. However we have a contradiction by Lemma 9.1 considering as $D^* = D_1'$ and $w_3^* = w_4$. Therefore Γ_{m+1} does not contain the graph as shown in Fig. 4(b).

By Lemma 7.1, we can show that Γ_{m+2} does not contain the graph as shown in Fig. 4(b). We complete the proof of Proposition 10.3. □

By Corollary 7.8, Lemma 9.2 and Proposition 10.3, we have the main theorem (Theorem 1.3).

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List of terminologies

List of notations