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On a Chern Number Inequality in Dimension 3

By Jheng-Jie Chen

Abstract. We prove $c_1(X) \cdot c_2(X) < c_1(X^+) \cdot c_2(X^+)$ if $X \dashrightarrow X^+$ is a 3-fold terminal flip (resp. $c_1(X) \cdot c_2(X) \leq c_1(Y) \cdot c_2(Y)$ if $X \to Y$ is a 3-fold elementary contraction contracting a divisor to a curve), where $c_1(X)$ and $c_2(X)$ denote the Chern classes. These provide affirmative answers to two questions by Xie in [Xie].

1. Introduction

In this article, we work over complex number field $\mathbb{C}$.

Based on the works of Mori, Kollár, Reid, Kawamata, Shokurov, and many others, minimal model conjecture in dimension three was proved (see [KMM1, M82, M88, KM98]). Further detailed study of three dimensional elementary birational maps in the minimal model program (MMP) is expected to be generally useful in three dimensional geometry.

One prominent concern is developing an understanding of Chern numbers or Chern classes in the MMP. Recall that $c_1(X) \cdot c_2(X)$ equals to $24\chi(\mathcal{O}_X)$ for a smooth 3-fold X. More generally, they are related in [Kaw86] and [YPG] as follows.

Theorem 1 (Kawamata, Reid). *Let X be a normal compact complex analytic (resp. normal projective) 3-fold with at worst canonical singularities. Then*

$$\chi(\mathcal{O}_X) = \frac{1}{24} c_1(X) \cdot c_2(X) + \frac{1}{24} \sum_i (r_i - 1/r_i),$$

where r_i is the index for the virtue singularity $\frac{1}{r_i}(1, -1, b_i)$.

The invariant *difficulty* is known to decrease through a 3-fold terminal flip and an elementary contraction contracting a divisor to a curve.

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Moreover, Chen provided analogous inequalities for the invariant *depth* in [CJK15, Proposition 2.1]. Define $F(X) := \sum_i (r_i - 1/r_i)$. (Similarly, define the integer $\Xi(X) := \sum_i r_i$). In this study, we aim to establish the following inequalities for F.

THEOREM 2. *Let $f\colon X \to Y$ be an elementary contraction contracting a divisor to a curve. Then $F(X) \geq F(Y)$. Moreover, if the exceptional divisor of f contains some non-Gorenstein points of X, then $F(X) > F(Y)$.*

THEOREM 3. *If $X \dashrightarrow X^+$ is a 3-fold terminal flip, then $F(X) > F(X^+)$.*

Note that $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Z)$ where X is a $\mathbb{Q}$-factorial terminal normal projective 3-fold and Z is any birational model in the MMP of X. Thus, the inequality $F(X) \geq F(Y)$ is equivalent to the inequality $c_1(X) \cdot c_2(X) \leq c_1(Y) \cdot c_2(Y)$ of Chern numbers. This gives positive answers to Problem 3.12 and Problem 3.13 in [Xie].

Elementary contractions contracting a divisor to a non-Gorenstein terminal singularity were completely classified by Kawamata, Hayakawa and Kawakita in [Kaw96, Haya99, Haya00, Kwk05, Kwk12]. It is not hard to observe the following weaker version according their explicit classifications.

THEOREM 4 (Kawamata, Hayakawa, Kawakita). *Let $f\colon X \to Y$ be a 3-fold elementary contraction contracting a divisor to a point. Then $\Xi(X) - \Xi(Y) \geq -2$ and $F(X) - F(Y) \geq -3/2$.*

Combining Theorem 4 with Theorem 2, Theorem 3 and Xie's result in [Xie, Theorem 1.4], we obtain that the second Chern class $c_2(X)$ is pseudo-effective when the Picard number is relatively small (See Corollary 5). Notice that the pseudo-effectivity of $c_2(X)$ implies the effective non-vanishing Conjecture for terminal projective threefolds (cf. [Xie, Proposition 4.3]).

COROLLARY 5. *Let X be a $\mathbb{Q}$-factorial terminal projective 3-fold whose anti-canonical divisor $-K_X$ is nef. If the Picard number $\rho(X) \leq 2 + (2c_1(X) \cdot c_2(X))/3$, then the second Chern class $c_2(X)$ is pseudo-effective. In particular, if X is a $\mathbb{Q}$-factorial Gorenstein terminal projective 3-fold with $-K_X$ nef and $\rho(X) \leq 18$, then $c_2(X)$ is pseudo-effective.*

Remark 6. During the preparation of this study, Chen Jiang pointed out that the recent research by Ou [Ou, Corollary 0.5] implies the pseudo-effectivity of c_2. Regardless, we think the comparison of Chern number $c_1 \cdot c_2$ in the MMP is a very interesting question for its own sake. Note that Cascini and Tasin investigated the difference of c_1^3 via elementary divisorial contractions in [CT, Theorem 1.3]. Then, I think that it is interesting to consider how $3c_1 \cdot c_2 - c_1^3$ varies via elementary contractions in dimension 3.

The article is organized as follows. In section 2, we review some basic results, the classification of non-Gorenstein terminal singularities (Table 1), and the classification of *extremal neighborhoods* of Kollár and Mori (Table 2). In section 3, we establish the inequality $F(X) \geq F(Y)$ where $f\colon X \to Y$ is an analytic elementary contraction contracting a divisor to a curve. In fact, we use Tables 1 and 2 to determine the possible non-Gorenstein singularities on Y. For the majority of the cases, we directly derive $F(X) \geq F(Y)$ according to the classification. One case requires us to use Mori's result on semistable extremal neighborhood in [M02]. In section 4, we prove $F(X) > F(X^+)$ for any analytic 3-fold terminal flip $X \dashrightarrow X^+$. In some cases, we are required to apply Mori's research on semistable flips in [M02] and Chen-Hacon's factorization in [CH11, Theorem 3.3]. We prove Theorem 2, Theorem 3 and Corollary 5 at the end of this article.

2. Preliminaries and Notations

In this section, we recall various notions derived from three dimensional terminal singularities and some basic properties.

We fix X to be an (algebraic or analytic) normal 3-fold with at worst terminal singularities. Suppose $X \dashrightarrow Z$ is a birational map where Z is a normal variety. Let D be a prime divisor on X. We denote D_Z the strict transform of D on Z. X is called $\mathbb{Q}$-factorial if every Weil divisor D is $\mathbb{Q}$-Cartier.

Every terminal 3-fold singular point $P \in X$ is known to be a quotient of isolated compound Du Val singularity by Reid in [Reid83]. The index of $P \in X$ is the smallest positive integer r such that rK_X is Cartier at P. All singular points of index $r \geq 2$ are called non-Gorenstein points and are classified explicitly by Mori [M85]. Let $P \in X$ be given by the equation $\phi(x_1, x_2, x_3, x_4) = 0$ in $\mathbb{C}^4$ with action $\frac{1}{r}(a_1, a_2, a_3, a_4)$. If P is not $cAx/4$, up to a permutation of x_1, x_2, x_3, x_4, there exists exactly one invariant, say x_4, satisfying $wt(x_4) \equiv wt(\phi) \equiv 0 \pmod r$ where $wt(x_1, x_2, x_3, x_4) := (a_1, a_2, a_3, a_4)$ and

$$wt(\phi) := \min\{ \sum_{i=1}^{4} a_i l_i \mid \text{the monomial } x_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} \text{ appears in } \phi \}.$$

If P is of type $cAx/4$, the action can be assumed to be $\frac{1}{4}(1, 3, 1, 2)$. The axial multiplicity $am(P \in X)$ is defined (in [M88, 1a.5]) by

$$am(P \in X) := \max\{ j \in \mathbb{N} \mid x_4^j \text{ divides the polynomial } \phi(0, 0, 0, x_4)\}.$$

We shall use the axial weight $aw(P \in X)$ instead which is defined (see [Haya99]) by

$$aw(P \in X) := \begin{cases} am(P \in X) & \text{if } p \in X \text{ is not of type } cAx/4 \\ (am(P \in X) + 1)/2 & \text{if } p \in X \text{ is of type } cAx/4 \end{cases}$$

Suppose $aw(P \in X) = k > 0$. If P is not $cAx/4$, then $P \in X$ can be locally deformed into k cyclic quotient points $\frac{1}{r}(a, -a, 1)$ (See [YPG, Section 6]). If P is $cAx/4$, it can be deformed into one cyclic quotient point $\frac{1}{4}(1, -1, 1)$ and $k - 1$ cyclic quotient points $\frac{1}{2}(1, 1, 1)$. This collection of cyclic quotient terminal singularities is called the basket of $P \in X$. Define $F(P \in X)$ (resp. $\Xi(P \in X)$) to be the rational number $k(r - \frac{1}{r})$ (resp. kr). Note that for each non-Gorenstein point $P \in X$, the dual graph $\Delta(E)$ of a general member $E \in |-K_X|$ in a neighborhood of P is determined by Reid in [YPG, Section 6]. We list in Table 1 for the classification and numerical invariants of various types. Note that if $P \in X$ is of type $cAx/2$, then $\Delta(E) = D_{m+2}$ where $m \geq 2$ is an integer which may be different from the axial weight 2.

For a terminal 3-fold X, define

$$\Xi(X) := \sum_{P \in \mathrm{Sing}(X)} \Xi(P \in X), \quad F(X) := \sum_{P \in \mathrm{Sing}(X)} F(P \in X).$$

A proper birational morphism $f\colon X \to Y$ is called an elementary contraction contracting a divisor to a point Q (resp. a curve Γ) if $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, the exceptional set $\mathrm{Exc}(f) = F$ is an irreducible divisor on X, relative Picard number $\rho(X/Y) = 1$, and $-K_X$ is f-ample such that $f(F)$ is a point Q (resp. a curve Γ).

Similarly, a proper birational morphism $f\colon X \to Y$ is called a flipping contraction (resp. flopping contraction) if $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, $\mathrm{Exc}(f)$ is a curve, $\rho(X/Y) = 1$ and $-K_X$ is f-ample (resp. f-trivial). In this case, the flip (resp. a flop) of f is a birational morphism $f^+\colon X^+ \to Y$ where X^+ is a terminal 3-fold such that ${f^+}_*(\mathcal{O}_{X^+}) = \mathcal{O}_Y$, $\mathrm{Exc}(f^+)$ is a curve, $\rho(X^+/Y) = 1$ and K_{X^+} is f-ample (resp. f^+-trivial). f^+ is called the flipped contraction (resp. a flopped contraction). A curve C in the exceptional set $\mathrm{Exc}(f)$ is called a flipping (resp. flopping) curve. A curve C^+ in the exceptional set $\mathrm{Exc}(f^+)$ is called a flipped (resp. flopped) curve. Note that C (resp. C^+) might be reducible.

We recall some definitions in [KM92, CH11].

Definition 7. An extremal neighborhood is a proper bimeromorphic morphism $f\colon X \supset C \to Y \ni Q$ satisfying the following properties.

1. X is an analytic 3-fold with at worst terminal singularities.
2. Y is normal and Q is the distinguished point.
3. $f^{-1}(Q) = C$ is isomorphic to $\mathbb{P}^1$.
4. $K_X \cdot C < 0$.

Denote by $E_X \in |-K_X|$ a general element in the extremal neighborhood $X \supset C$ and $E_Y := f(E_X)$. Then E_X and E_Y are normal Du Val surfaces and the restriction $f|_{E_X}\colon E_X \to E_Y$ is a partial resolution by Kollár and Mori [KM92, Theorem 2.2]. Moreover, they gave the classification of the possible extremal neighborhoods which we summarize in Table 2 where $\Delta(E_X)$ and $\Delta(E_Y)$ are the corresponding dual graphs, $\mu_{C\subset X} := \max\{\text{index } r(P) \mid P \in C\}$, and IA, IIA, IC, IIB, III (resp. $IA^\vee, II^\vee$) denote the local structures of extremal neighborhoods of primitive points (resp. imprimitive points) (cf. [M88, Appendix A]).

The extremal neighborhood $X \supset C$ is called semistable if $\Delta(E_Y)$ is A-type. Otherwise, it is called non-semistable. From Table 2, only cases 2.2.1.1 and 2.2.4 are semistable. The extremal neighborhood $X \supset C$ is called isolated if $f|_{X-C}\colon X - C \to Y - \{Q\}$ is an isomorphism. Otherwise, it is called divisorial. If $f\colon X \to Y$ is an isolated extremal neighborhood (i.e., a flipping contraction) and $f^+\colon X^+ \to Y$ is the flipped contraction, we define $\mu_{C^+\subset X^+} := \max\{\text{index } r(P^+) \mid P^+ \in C^+\}$.

Table 1. 3-dimensional Non-Gorenstein Terminal Singularities.

type	type of action	aw	$\Delta(E)$	basket	$\Xi(P \in X)$	$F(P \in X)$
cA/r	$\frac{1}{r}(a,-a,1,0)$	k	A_{rk-1}	$k \times (b,r)$	rk	$rk - \frac{k}{r}$
$cAx/2$	$\frac{1}{2}(0,1,1,1)$	2	$D_{m+2}(m \geq 2)$	$2 \times (1,2)$	4	3
$cAx/4$	$\frac{1}{4}(1,1,3,2)$	k	D_{2k+1}	$\{(1,4),(k-1) \times (1,2)\}$	$2k+2$	$\frac{6k+9}{4}$
$cD/2$	$\frac{1}{2}(1,0,1,1)$	k	D_{2k}	$k \times (1,2)$	$2k$	$\frac{3k}{2}$
$cD/3$	$\frac{1}{3}(0,2,1,1)$	2	E_6	$2 \times (1,3)$	6	$\frac{16}{3}$
$cE/2$	$\frac{1}{2}(0,1,1,1)$	3	E_7	$3 \times (1,2)$	6	$\frac{9}{2}$

Table 2. Extremal Neighborhoods.

ref in [KM92]	type	$\mu_{C \subset X}$	$\Delta(E_X)$	$\Delta(E_Y)$	remark
2.2.1.1	$cA/r(+III)$	r	A_{rk-1}	A_{rk-1}	
2.2.1.2	$cD/3(+III)$	3	E_6	E_6	
2.2.1.3	$IIA(cAx/4)(+III)$	4	D_{2k+1}	D_{2k+1}	
2.2.1′.1	$cAx/2$	2	D_4	D_4	
2.2.1′.2	$cD/2$	2	D_{2k}	D_{2k}	
2.2.1′.3	$cE/2$	2	E_7	E_7	
2.2.1′.4	$II^{\vee}(cAx/4)$	4	D_{2k+1}	D_{2k+1}	
2.2.2	$IC(cyclic\ quotient)$	r	A_{r-1}	D_r	r is odd
2.2.2′	IIB	4	D_5	E_6	
2.2.3	$IA + IA$	r	$A_{r-1} + D_{2k}$	D_{r+2k}	r is odd
2.2.3′	$IA + IA + III$	r	$A_{r-1} + A_1$	D_{r+2}	r is odd
2.2.4	$ssIA + IA$	$\max\{r_1, r_2\}$	$A_{r_1k_1-1} + A_{r_2k_2-1}$	$A_{r_1k_1+r_2k_2-1}$	
2.2.5	III	1	smooth	smooth	

DEFINITION 8. Suppose $P \in X$ is a terminal 3-fold singular point with index $r > 1$. We say that $g\colon W \to X$ is a w-morphism if it is an elementary contraction contracting a divisor to the point P with minimal discrepancy $1/r$.

2.1. Cartier index

In this subsection, we collect some known results.

LEMMA 9. *Let $f\colon X \to Y$ be an elementary contraction contracting a divisor to a curve Γ. If $Q \in \Gamma$ is a non-Gorenstein point of Y with index r, then X has a singular point of index $r_i \geq 2r$ such that r_i is divisible by r.*

PROOF. Denote by F the exceptional divisor of f. Let $g\colon W \to X$ be a resolution of X obtained by successive weighted blowups over singular points on $f^{-1}(\Gamma)$. Then we may write

$$K_W = g^* K_X + \sum_{i=1}^{s} \frac{a_i}{r_i} F_i \quad \text{and} \quad g^* F = F_W + \sum_{i=1}^{s} \frac{\alpha_i}{r_i} F_i,$$

where all a_i, α_i and r_i are positive integers. Therefore,

$$K_W = g^* f^* K_Y + F_W + \sum_{i=1}^{s} \frac{a_i + \alpha_i}{r_i} F_i.$$

Now $f \circ g : W \to Y$ is a resolution of Y. There exists an exceptional divisor over Y with discrepancy $\frac{1}{r}$ by [Haya99, Haya00]. Hence for some i, we have

$$\frac{1}{r} = \frac{a_i + \alpha_i}{r_i} \geq \frac{2}{r_i}. \ \square$$

LEMMA 10. *If $X \dashrightarrow X^+$ is a 3-fold terminal flip, then $\mu_{C \subset X} > \mu_{C^+ \subset X^+}$.*

PROOF. Let $f\colon X \to Y$ be a flipping contraction and $f^+\colon X^+ \to Y$ be the flipped contraction. Let W be a common resolution of X and X^+ and

$g\colon W \to X$ and $g^+\colon W \to X^+$ be the corresponding morphisms. Then, we write

$$K_W = g^* K_X + \sum_{i=1}^{s} \frac{a_i}{r_i} F_i = {g^+}^* K_{X^+} + \sum_{i=1}^{s} b_i F_i$$

where all a_i, and r_i (resp. b_i) are positive integers (resp. rational numbers). By the negativity lemma, each $\frac{a_i}{r_i} < b_i$ (cf. [Kol92, Lemma 2.19]).

There exists an exceptional divisor over X^+ with discrepancy $\frac{1}{\mu_{C^+ \subset X^+}}$. Hence $\frac{1}{\mu_{C^+ \subset X^+}} = b_i$ for some i and it follows that

$$\frac{1}{\mu_{C^+ \subset X^+}} = b_i > \frac{a_i}{r_i} \geq \frac{1}{r_i} \geq \frac{1}{\mu_{C \subset X}}. \ \square$$

From Lemma 9 and Lemma 10, we easily derive the following assertion.

Corollary 11. *Let $f\colon X \to Y$ be an elementary contraction contracting a divisor E to a curve Γ. If Γ contains a non-Gorenstein point of Y, then E contains at least one non-Gorenstein point (of X) of index greater than* 3. *Similarly, if $X \dashrightarrow X^+$ is a 3-fold terminal flip with $\mu_{C \subset X} = 2$, then X^+ has only Gorenstein points on the flipped curves.*

The following easy result will be used frequently in our computations.

Lemma 12. *Let $P \in X$ be a terminal singular point and let $D \in |-K_X|$ be an element. Suppose that D is of type E_n then the general member is of type $E_m (m \leq n)$, $D_m (m < n)$, or $A_m (m < n)$. Similarly, if D is of type D_n, then the general member is of type $D_m (m \leq n)$ or $A_m (m < n)$. Also, if D is of type A_n, then the general member is of type $A_m (m \leq n)$.*

Proof. This is the case since corank and Milnor number are semicontinuous. See [GLS, Corollary 2.49, 2.52, 2.54] for details. $\square$

3. Inequalities for Analytic Elementary Contractions Contracting a Divisor to a Curve

In this section, we verify the desired inequalities $\Xi(X) \geq \Xi(Y)$ and $F(X) \geq F(Y)$ for every divisorial extremal neighborhood $f\colon X \supset C \to Y \ni Q$. The computations base on the classification in Table 2.

For our purpose, we may assume that Q is a non-Gorenstein point of index $r' > 1$ and axial weight k'. Furthermore, we have the following observation.

Proposition 13. *Let $f\colon X \supset C \to Y \ni Q$ be an extremal neighborhood that contracts a divisor to a curve. Then the point Q cannot be of type $cE/2$, $cD/3$ nor $cAx/4$.*

Proof. Denote by $E \in |-K_Y|$ a general member near $Q \in Y$.

Suppose that $Q \in Y$ is of type $cE/2$. Since the dual graph $\Delta(E)$ is of type E_7 by Table 1, it follows from Lemma 12 that the extremal neighborhood must be of type 2.2.1′.3. By Lemma 9, $\mu_{C\subset X} \geq 4$, which is impossible. Similar argument shows that Q cannot be of type $cD/3$.

Suppose $Q \in Y$ is of type $cAx/4$. Then $\Delta(E)$ is D-type. By Lemma 12 and Table 2, the extremal neighborhood is non-semistable and each non-Gorenstein point on X has index 2, 4 or an odd integer $r \geq 3$. By Lemma 9, the fiber $f^{-1}(Q) = C$ contains a non-Gorenstein point whose index is greater than 7 and is divisible by 4, which is impossible. □

Proposition 14. *Let $f\colon X \supset C \to Y \ni Q$ be an extremal neighborhood that contracts a divisor to a curve. Then $\Xi(X) \geq \Xi(Y)$ and $F(X) \geq F(Y)$. Moreover, if C contains a non-Gorenstein point of X, then $F(X) > F(Y)$.*

Proof. Since f is an isomorphism outside the exceptional divisor, by abusing of notations, one may assume

$$\Xi(X) = \sum_{P\in C\cap \mathrm{Sing}(X)} \Xi(P\in X) \quad (\text{resp. } F(X) = \sum_{P\in C\cap \mathrm{Sing}(X)} F(P\in X)\ ),$$

and $\Xi(Y) = \Xi(Q \in Y)$ (resp. $F(Y) = F(Q \in Y)$). Furthermore, we may assume that $Q \in Y$ is a non-Gorenstein point. From Proposition 13 and Table 1, we divide the proof into three parts according to the types of $Q \in Y$.

Denote r' (resp. k') to be the index (resp. axial weight) of $Q \in Y$.

Case $Q \in Y$ is of type $cAx/2$: By Lemma 9 and Table 1, we have $\mu_{X\supset C} \geq 4$. In particular, $\Xi(X) \geq 4 = \Xi(Y)$ and $F(X) > 3 = F(Y)$.

Case $Q \in Y$ is of type $cD/2$: Since the dual graph of a general member in $|-K_Y|$ near $Q \in Y$ is of type $D_{2k'}$, the extremal neighborhood is non-semistable. Since $\mu_{X\supset C} \geq 4$, it remains to consider the extremal neighborhood in the cases 2.2.1.3, 2.2.1′.4, 2.2.2, 2.2.2′, 2.2.3 and 2.2.3′. In the cases 2.2.1.3 and 2.2.1′.4, we have $E_X \simeq E_Y \in |-K_Y|$ near Q. Since $\Delta(E_X) = \Delta(E_Y) = D_{2k+1}$, by Lemma 12, this gives $2k+1 \geq 2k'$. Thus, $\Xi(X) = 2k+2 > 2k' = \Xi(Y)$ and $F(X) = (6k+9)/4 > 3k'/2 = F(Y)$. In the case 2.2.3, we have $E_X \not\simeq E_Y \in |-K_Y|$ and $m \geq 3$ is odd. By Lemma 12, one sees $\Xi(X) = r + 2k \geq 2k' = \Xi(Y)$ and so

$$F(X) = r - \frac{1}{r} + \frac{3k}{2} > \frac{3r}{4} + \frac{3k}{2} \geq \frac{3k'}{2} = F(Y).$$

Computations in the cases 2.2.2, 2.2.2′ and 2.2.3′ are similar to the above case 2.2.3 and we omit it.

Case $Q \in Y$ is of type cA/r': If the extremal neighborhood $X \supset C$ contains exactly one non-Gorenstein point, by Lemma 9, the computations are similar to previous cases and we leave it to the reader. From now on, we may assume $X \supset C$ contains at least two non-Gorenstein points.

Suppose $X \supset C$ is the semistable case 2.2.4. The dual graphs are $\Delta(E_X) = A_{r_1k_1-1} + A_{r_2k_2-1}, \Delta(E_Y) = A_{r_1k_1+r_2k_2-1}$. By Lemma 12, it follows that $\Xi(Y) = r'k' \leq r_1k_1 + r_2k_2 = \Xi(X)$. From [M02, Theorem 4.5] and Lemma 9, one has $r' = \gcd(r_1, r_2) \leq \min\{r_1, r_2\}$ and $r_1 \neq r_2$. If $k_1 + k_2 > k'$, then

$$\begin{aligned} F(Y) - F(X) &< (k_1+k_2)\left(r' - \frac{1}{r'}\right) - k_1\left(r_1 - \frac{1}{r_1}\right) - k_2\left(r_2 - \frac{1}{r_2}\right) \\ &= k_1\left(r' - \frac{1}{r'} - r_1 + \frac{1}{r_1}\right) + k_2\left(r' - \frac{1}{r'} - r_2 + \frac{1}{r_2}\right) \leq 0. \end{aligned}$$

We may assume that $k_1 + k_2 \leq k'$. Then $\frac{k_1}{r_1} + \frac{k_2}{r_2} < \frac{k_1}{r'} + \frac{k_2}{r'} \leq \frac{k'}{r'}$ and

$$F(Y) - F(X) = \Xi(Y) - \Xi(X) + \left(\frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{k'}{r'}\right) < 0.$$

Finally, we claim that $X \supset C$ is neither in the case 2.2.3 nor in the case 2.2.3′. Suppose not. C contains two non-Gorenstein points of indices r and 2 (and probably a Gorenstein point) where r is odd. Note that $-K_X \cdot C = -1/2r$ by [KM92, (2.12)] and [M07]. Thus, by [M88, Corollary 1.10], the extremal neighborhood $X \supset C$ is locally primitive. By [M88, Corollary 1.12], $1 < r' = gcd(r, 2) = 1$ gives a contradiction. This completes the proof of Proposition 14. □

4. Inequalities for Analytic 3-Fold Terminal Flips

In this section, we prove the inequalities $\Xi(X) \geq \Xi(X^+)$ and $F(X) > F(X^+)$ for any analytic 3-fold terminal flip $X \dashrightarrow X^+$.

We start with the following useful result which can be viewed as an application of [KM92, Theorem 2.2] (cf. [MP2, Proposition 2.3]).

THEOREM 15. *Suppose $X \supset C$ is an isolated extremal neighborhood and $X \dashrightarrow X^+$ is the flip. Let $E_X \in |-K_X|$ be a general element and $E_{X^+} \in |-K_{X^+}|$ be its strict transform. Then E_{X^+} is normal near the flipped curve and has at worst Du Val singularities. Moreover, if S is the minimal resolution of E_X, then the induced rational map $S \dashrightarrow E_{X^+}$ is a morphism.*

PROOF. Let $f \colon X \to Y$ be a flipping contraction and $f^+ \colon X^+ \to Y$ be the flipped contraction of f. By [KM92, Theorem 2.2], the surfaces E_X and E_Y are normal and have at worst Du Val singularities and the restriction morphism $f|_{E_X} : E_X \to E_Y$ is crepant. By inversion of adjunction, the pair (X, E_X) is canonical. Since the birational map $X \dashrightarrow X^+$ is $(K_X + E_X)$-flop, the pair (X^+, E_{X^+}) is canonical. By adjunction, the surface E_{X^+} is normal and has at worst Du Val singularities.

Because $K_{E_{X^+}} = \mathcal{O}_{E_{X^+}}$ is $f^+|_{E_{X^+}}$-trivial, the restriction morphisms $E_X \to E_Y$ and $f^+|_{E_{X^+}} : E_{X^+} \to E_Y$ are both crepant. Hence S is also the minimal resolution of E_Y. □

The isolated extremal neighborhoods are classified by Kollár and Mori in Table 3. They are named $k1A, cD/3, IIA, IC, kAD, k2A$ according to the general element $E \in |-K_X|$ of $X \supset C$. This classification enables us to study the non-Gorenstein points on the flipped curve C^+ when $X \dashrightarrow X^+$ is an analytic 3-fold terminal flip.

To simplify the notions, we denote P the non-Gorenstein point (resp. P_1, P_2 the non-Gorenstein points) on the flipping curve C with index r and axial weight k (resp. indices r_1, r_2 and axial weights k_1, k_2). We denote P^+

Table 3. Isolated Extremal Neighborhoods.

ref in [KM92]	type	$\mu_{C\subset X}$	$\Delta(E_X)$	$\Delta(E_Y)$	remark
$k1A$ (= 2.2.1.1)	$cA/r(+III)$	r	A_{rk-1}	A_{rk-1}	
$cD/3$ (= 2.2.1.2)	$cD/3(+III)$	3	E_6	E_6	
IIA (= 2.2.1.3)	$IIA(cAx/4)(+III)$	4	D_{2k+1}	D_{2k+1}	
IC (= 2.2.2)	$IC(cyclic\ quotient)$	r	A_{r-1}	D_r	r is odd
kAD (= 2.2.3)	$IA+IA+III$	r	$A_{r-1}+D_{2k}$	D_{r+2k}	r is odd
$k2A$ (= 2.2.4)	$ssIA+IA$	$\max\{r_1,r_2\}$	$A_{r_1k_1-1}+A_{r_2k_2-1}$	$A_{r_1k_1+r_2k_2-1}$	

the non-Gorenstein point (resp. $P_1^+, P_2^+, ..., P_n^+$ the non-Gorenstein points) on the flipped curve C^+ with index r^+ and axial weight k^+ (resp. indices $r_1^+, r_2^+, \cdots, r_n^+$ and axial weights $k_1^+, k_2^+, ..., k_n^+$).

PROPOSITION 16. *Suppose $X \supset C$ is an isolated extremal neighborhood and C^+ is the flipped curve of the flip $X \dashrightarrow X^+$. Then any point on the flipped curve $C^+ \subset X^+$ cannot be of type $cE/2$, $cD/3$ nor $cAx/4$.*

PROOF. If $P^+ \in C^+ \subset X^+$ is of type $cE/2$ (resp. $cD/3$), then the dual graph of a general member of $P^+ \in X^+$ is of type E_7 (resp. E_6) by Table 1. The dual graph $\Delta(E_Y)$ is at worst E_6 by Table 3. As C^+ corresponds to one vertex of $\Delta(E_Y)$, $\Delta(E_{X^+})$ is better than $\Delta(E_Y)$. This contradicts to Lemma 12.

In the semistable cases $k1A, k2A$, every non-Gorenstein point $P^+ \in C^+ \subset X^+$ is of type cA/r^+ by Lemma 12. In the non-semistable cases $cD/3, IIA, IC, kAD$, every non-Gorenstein point on the flipped curve C^+ has index 2 or 3 by [KM92, Theorem 13.17, Theorem 13.18]. Thus C^+ cannot contain singular points of type $cAx/4$. □

PROPOSITION 17. *If the extremal neighborhood $X \supset C$ is isolated and $X \dashrightarrow X^+$ is the flip, then $\Xi(X) \geq \Xi(X^+)$ and $F(X) > F(X^+)$.*

PROOF. Note that $X \dashrightarrow X^+$ is an isomorphism outside C. For our purpose, one may assume that X has only Gorenstein singularities outside C. We divide it into two cases.

We first deal with the non-semistable extremal neighborhoods.

Case $cD/3$**:** The dual graphs are $\Delta(E_X) = \Delta(E_Y) = E_6$. There are at most one singular point P^+ of index 2 on C^+ by [KM92, Theorem 13.17, Appendix]. Since E_{X^+} is a partial resolution of E_Y and $\Delta(E_Y) = E_6$, by Lemma 12, the dual graph of a general member in $|-K_{X^+}|$ near $P^+ \in X^+$ is $A_n (n < 6)$ or $D_n (n < 6)$. By Proposition 16, $P^+ \in X^+$ is of type $cA/2$, $cAx/2$ or $cD/2$ of axial weights $\leq 3, = 2, < 3$ respectively. It is then easy to verify the inequalities for Ξ and F.

The computations in the cases IC and kAD are similar.

Case IIA**:** The dual graphs are $\Delta(E_X) = \Delta(E_Y) = D_{2k+1}$. If C^+ contains only one singular point of higher index, we derive the inequalities as in the previous cases. We may assume that C^+ contains precisely two singular points P_1^+, P_2^+ with $r_1^+ = 2$ and $r_2^+ = 3$ by [KM92, Theorem 13.17, Appendix]. By Proposition 16, $P_1^+ \in X^+$ is one of type $cA/2$ or $cAx/2$ or $cD/2$ and $P_2^+ \in X^+$ is of type $cA/3$. By Theorem 15, the minimal resolution of E_Y dominants E_{X^+} and so C^+ corresponds to one vertex of $\Delta(E_Y)$. By Lemma 12, $\Xi(X) = 2k + 2 > 2k_1^+ + 3k_2^+ = \Xi(X^+)$. Together with the following claim, we derive $F(X) = \frac{6k+9}{4} > \frac{3k_1^+}{2} + \frac{8k_2^+}{3} = F(X^+)$.

Claim 18. $k_2^+ = 1$.

Proof. Since the flipped curve C^+ contains two non-Gorenstein points, by classification in [KM92, Theorem 7.2, Theorem 13.17, and Appendix A.2], the extremal neighborhood $X \supset C \ni P$ is in [KM92, Appendix (A.2.2.1)]. That is,

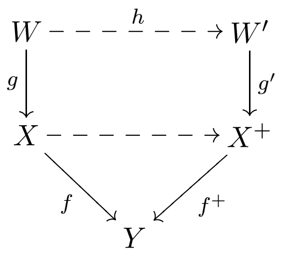
$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_4(1, 1, 3, 2; 2) \supset C = y_1\text{-axis}/\mathbb{Z}_4,$$

$$\text{and } \alpha = 0 \cdot y_4 + y_3^2 + g(y_1, y_2)y_2 + \cdots \in (y_2,\ y_3,\ y_4)$$

where $g(y_1, y_2)$ is a nonzero linear form in y_1, y_2 with the condition

$$\alpha \equiv y_1 y_2 \mod (y_2,\ y_3,\ y_4)^2.$$

By a coordinate change, we may assume that $\alpha = y_1^2 + y_3^2 + f(y_2, y_4)$ where y_2^2 appears in $f(y_2, y_4)$. If we put $\tau\text{-}wt(y_2) = 1/4$, and $\tau\text{-}wt(y_4) = 2/4$, then $\tau\text{-}wt(f(y_2, y_4)) = \tau\text{-}wt(y_2^2) = 1/2$. From [CH11, Theorem 3.3], the flip $X \dashrightarrow X^+$ can be factored into the diagram



where g is a w-morphism, g' is an elementary divisorial contraction, and h is a composition of flips and probably a flop. Denote by G the exceptional divisor of g. By [Haya99, Theorem 7.4, Theorem 7.9], the w-morphism $g\colon W \to X$ with center P is actually the weighted blowup with weight

$$wt(y_1, y_2, y_3, y_4) = (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{2}{4}) \text{ or } (\frac{5}{4}, \frac{1}{4}, \frac{3}{4}, \frac{2}{4}),$$

and the non-Gorenstein points (of W) on G consist of a cyclic quotient point of index ≤ 5 and at worst a point $cD/2$. Let k'' be axial weight of the singular point $cD/2$. We have $\Xi(X) = \Xi(W) + 1$. Note that singularities are unchanged under a flop [Kol89, Theorem 2.4]. By Lemma 10, the strict transform $G_{W'}$ contains no non-Gorenstein point (of W') of index greater than 5. Since P_2^+ is a point of index 3 in the flipped curve C^+, by Lemma 9, the center $g'(G_{W'})$ is a point.

Case $k'' < 2$**:** We have $6+2k'' \geq \Xi(W)+1 = \Xi(X) > \Xi(X^+) = 2k_1^+ + 3k_2^+$. This gives $k_2^+ = 1$.

Case $k'' \geq 2$**:** If $W_j \dashrightarrow W_{j+1}$ is a flip which factors through $h\colon W \dashrightarrow W'$, then the flipping curve $C_j \subset W_j$ contains no point of types $cAx/4$, $cD/3$ and $cE/2$ (resp. $cD/2$) of W_j by Proposition 16 (resp. by Table 3 and [M07, Remark 1]). In particular, W is isomorphic to W' in an open neighborhood of the singular point $cD/2$. If the center $g'(G_{W'})$ of g' has index 3, it must be P_2^+. From Kawakita's classification in [Kwk05, Theorem 1.2], g' is a weighted blow up and every non-Gorenstein point of W' on the exceptional divisor $G_{W'}$ is either a cyclic quotient or $cA/3$. This is impossible. Thus, we may assume that the center of g' has index < 3. Similar to the computations in the above case $cD/3$, the inequality for Ξ holds every isolated extremal neighborhood. In particular, each $\Xi(W_j) \geq \Xi(W_{j+1})$. This gives

$$5 \geq \Xi(W) - 2k'' \geq \Xi(W') - 2k'' \geq \Xi_{>2}(W') \geq \Xi_{>2}(X^+) = 3k_2^+,$$

where $\Xi_{>2}(X)$ is temporarily defined by

$$\Xi_{>2}(X) := \sum_{\substack{P \in \mathrm{Sing}(X) \\ \text{index } r(P)>2}} \Xi(P \in X).$$

So $k_2^+ = 1$ and the proof of claim is completed. $\square$

Next, we deal with semistable cases.

Case $k1A$**:** The dual graphs are $\Delta(E_X) = \Delta(E_Y) = A_{rk-1}$. Let $P_1^+, P_2^+, ..., P_n^+$ be the non-Gorenstein singularities on C^+. By Lemma 12, they are of types $cA/r_1^+, cA/r_2^+, ..., cA/r_n^+$ respectively. By Lemma 12 again, $\Xi(X) = rk \geq \sum_{i=1}^n r_i^+ k_i^+ = \Xi(X^+)$. From Lemma 10, all $r_i^+ < r$.

Suppose that $k > \sum_{i=1}^{n} k_i^+$. Then

$$F(X) - F(X^+) > \left(\sum_{i=1}^{n} k_i^+\right)\left(r - \frac{1}{r}\right) - \sum_{i=1}^{n} k_i^+ \left(r_i^+ - \frac{1}{r_i^+}\right)$$
$$= \sum_{i=1}^{n} k_i^+ \left(r - \frac{1}{r} - r_i^+ + \frac{1}{r_i^+}\right) > 0.$$

Suppose that $k \leq \sum_{i=1}^{n} k_i^+$. Since each $r_i^+ < r$, we see that

$$r_1^+ r_2^+ \cdots r_n^+ k \leq r_1^+ r_2^+ \cdots r_n^+ \sum_{i=1}^{n} k_i^+ < r r_2^+ r_3^+ \cdots r_n^+ k_1^+ + \cdots + r_1^+ \cdots r_{n-1}^+ r k_n^+.$$

We obtain

$$F(X) - F(X^+) = \Xi(X) - \Xi(X^+) - \left(\frac{k}{r} - \sum_{i=1}^{n} \frac{k_i^+}{r_i^+}\right) > 0.$$

Case $k2A$: By Mori's classification in [M02, Theorem 4.7], the singularities on C^+ consist of two points P_1^+ and P_2^+ of types cA/r_1^+ and cA/r_2^+. We have $\Delta(E_X) = A_{r_1k_1-1} + A_{r_2k_2-1}$, $\Delta(E_Y) = A_{r_1k_1+r_2k_2-1}$ and $\Xi(X) = r_1k_1 + r_2k_2 \geq r_1^+k_1^+ + r_2^+k_2^+ = \Xi(X^+)$. Then, it follows from Lemma 19 that $r_1 \geq r_1^+, r_2 \geq r_2^+, k_1 \leq k_1^+$, and $k_2 \leq k_2^+$. Furthermore, one sees either $r_1 > r_1^+$ or $r_2 > r_2^+$ by Lemma 10. So $r_1^+k_1 \leq r_1k_1^+$ and $r_2^+k_2 \leq r_2k_2^+$ and

$$F(X) - F(X^+) = \Xi(X) - \Xi(X^+) - \left(\frac{k_1}{r_1} - \frac{k_1^+}{r_1^+} + \frac{k_2}{r_2} - \frac{k_2^+}{r_2^+}\right) > 0.$$

This completes the proof of Theorem 3. □

In order to give the inequality $F(X) > F(X^+)$ in the case $k2A$, we need the following key relations of indices and axial weights by using Mori's study in [M02]. See also [MP2, (2.3.5)].

LEMMA 19. *Suppose $X \supset C$ is in the case $k2A$. Let the singular points on the flipping curve C (resp. flipped curve C^+) be of types cA/r_1 and cA/r_2 with the corresponding axial weights k_1 and k_2 (resp. cA/r_1^+ and cA/r_2^+ with the corresponding axial weights k_1^+ and k_2^+). Then, by rearranging the subindices 1 and 2, we have $r_1 \geq r_1^+, r_2 \geq r_2^+, k_1 \leq k_1^+$, and $k_2 \leq k_2^+$.*

PROOF. By classification of Mori in [M02, Theorem 4.7], there are precisely two singularities P_1^+, P_2^+ of types $cA/r_1^+, cA/r_2^+$ on C^+. We adopt the notations in [M02] to prove the inequalities. Put $d(i) = m_i = r_i$ and $\alpha_i = k_i$ for $i = 1, 2$. From [M02, Definition 3.2], Mori defined the sequences $d(n), e(n) \in \mathbb{Z}$ by

$$\begin{aligned} &d(n+1) + d(n-1) = \delta\rho_n d(n) \text{ and} \\ &e(n+1) + e(n-1) = \delta\rho_n e(n) + \delta\alpha_{n-2} - \alpha_{n-1,2}. \end{aligned}$$

From [M02, Definition 3.2, Corollary 4.1, Definition 4.2, Theorem 4.7], there exists a smallest positive integer $k \geq 3$ satisfying the indices $m_1^+ = d(k-1) > 0$ and $m_2^+ = -d(k) > 0$ and the corresponding axial weights $\alpha_{k-1} + \rho_{k-1}e(k+1)$ and $\alpha_{k-2} + \rho_{k-2}e(k)$, respectively. Here $\alpha_3 = \alpha_1(\rho_1 - 1)$, $\alpha_4 = \alpha_2(\rho_2 - 1)$ and each $\rho_i = \rho_{i+2j}$, $\alpha_i = \alpha_{i+4j}$ for all integers i, j. Note that $e(k), e(k+1) > 0$ if $k \geq 4$ by [M02, Corollary 3.8]. From [M02, Lemma 3.3.1, Corollary 3.4], it follows that

$$\begin{aligned} m_1^+ &= d(k-1) < d(k-3) < \cdots < d(1) \text{ (or } d(2) \text{) and} \\ m_2^+ &= -d(k) = d(k-2) - \delta\rho_{k-1}d(k-1) \\ &< d(k-2) < d(k-4) < \cdots < d(2) \text{ (or } d(1) \text{).} \end{aligned}$$

Case $k \geq 7$: By [M02, Lemma 3.5, Corollary 3.7], we see $e(n) \geq \alpha_1 + \alpha_2$ for all $n \geq 7$. In particular, we have the indices $m_1^+ = d(k-1)$, $m_2^+ = -d(k)$ with the corresponding axial weights $\alpha_{k-1} + \rho_{k-1}e(k+1) \geq \alpha_1 + \alpha_2$, and $\alpha_{k-2} + \rho_{k-2}e(k) \geq \alpha_1 + \alpha_2$, respectively.
Case $k = 6$: By the equality $\alpha_4 = \alpha_2(\rho_2 - 1)$, we may assume that $\rho_2 = 1$. By [M02, Remark 3.6.1, Corollary 3.8], we have

$$(\delta^2\rho_2 + \rho_1 - 3)\delta\rho_1\alpha_1 + (\delta^2\rho_1 - 1)\alpha_2 = e(6) > 0.$$

So either $\delta > 1$ or $\rho_1 > 1$. This gives the inequality $e(6) \geq \alpha_2$. We have the indices $m_1^+ = d(5) < d(1)$, $m_2^+ = -d(6) < d(2)$ with the corresponding axial weights $\alpha_5 + \rho_5 e(7) \geq 2\alpha_1 + \alpha_2$, and $\alpha_4 + \rho_4 e(6) \geq \alpha_2$, respectively.
Case $k = 5$: We have the indices $m_1^+ = d(4) < d(2)$, $m_2^+ = -d(5) < d(1)$ with the corresponding axial weights $\alpha_4 + \rho_4 e(6)$, and $\alpha_3 + \rho_3 e(5)$, respectively. We see $e(6) \geq \alpha_2$ as in the case $k = 6$. So $\alpha_4 + \rho_4 e(6) \geq \alpha_2$. From the equality $\alpha_3 = \alpha_1(\rho_1 - 1)$, we may assume that $\rho_1 = 1$. By [M02,

Corollary 3.8], $e(6) > 0$. So either $\delta > 1$ or $\rho_2 > 1$. From [M02, Remark 3.6.1], we have

$$\alpha_3 + \rho_3 e(5) = e(5) = (\delta^2\rho_2 - 1)\alpha_1 + \delta\alpha_2 \geq \alpha_1 + \delta\alpha_2 > \alpha_1.$$

Case $k = 4$**:** Suppose $\rho_1 > 1$. We have the indices $m_1^+ = d(3) < d(1)$, $m_2^+ = -d(4) < d(2)$ with the corresponding axial weights $\alpha_3 + \rho_3 e(5) > \alpha_1 + \alpha_2$, and $\alpha_2 + \rho_2 e(4) \geq \alpha_1 + \alpha_2$, respectively.

Suppose that $\rho_1 = 1$ and $e(5) < \alpha_1$. In this case, we see $\delta = 1$ and $\rho_2 = 1$. We have the indices $m_1^+ = d(3) = d(2) - d(1) < d(2)$, $m_2^+ = -d(4) = d(1)$ with the corresponding axial weights $\alpha_3 + \rho_3 e(5) = e(5) = \alpha_2$, and $\alpha_2 + \rho_2 e(4) = \alpha_1 + \alpha_2 > \alpha_1$, respectively.
Case $k = 3$**:** We have the indices $m_1^+ = d(2)$, $m_2^+ = -d(3) < d(1)$ and the axial weights $\alpha_2 + \rho_2 e(4) > \alpha_2$, $\alpha_1 + \rho_1 e(3) = \alpha_1$, respectively. □

5. Proofs of Theorem 2, 3 and Corollary 5

Proof of Theorem 2. Suppose $f\colon X \to Y$ is an elementary contraction contracting a divisor a curve Γ (possibly with reducible fibers). Let Q be a point in Γ and $X^q \to X$ be an analytic $\mathbb{Q}$-factorialization of the pair $(X, f^{-1}(Q))$ as in [Kaw88, Corollary 4.5']. Run the MMP from X^q over the germ (Y, Q) in the analytic category: $X^q \dashrightarrow X'$ where X' is a minimal model over (Y, Q). Since the composition $X^q \to X \to Y$ has one dimension fibers, the birational map $X^q \dashrightarrow X'$ is factored into a sequence of flips, extremal divisorial contractions that contract a divisor to a curve (cf. [MP2, 3.1.2]). The map $X' \to Y$ is either the identity or a small crepant contraction. The inequality $F(X) \geq F(Y)$ follows from Proposition 14, Proposition 17 and Remark 20. □

Remark 20. If $\nu\colon X_1 \to X_2$ is a small proper bimeromorphic (resp. small projective birational) morphism between normal compact complex analytic (resp. normal projective) 3-folds with only canonical singularites, then $\nu_* c_2(X_1) = c_2(X_2)$, $c_1(X_1) \cdot c_2(X_1) = c_1(X_2) \cdot c_2(X_2)$ and $F(X_1) = F(X_2)$ by Theorem 1.

Proof of Theorem 3. The argument is similar to that of Theorem 2. Suppose $f\colon X \to Y$ is a flipping contraction (possibly with reducible flipping curves) and $X \dashrightarrow X^+$ is the 3-fold terminal flip in the algebraic category. Let $X^q \to X$ be an analytic $\mathbb{Q}$-factorization (cf. [Kaw88, Corollary 4.5']) and run the MMP from X^q over Y in the analytic category. By a finite sequence of analytic flips, we obtain a minimal model X' and a small crepant morphism $X' \to X^+$ (cf. [Kaw88, argument of Proposition 8.4] or [MP2, (3.1.2)]). The inequality $F(X) > F(X^+)$ follows from Proposition 17 and Remark 20. □

Proof of Corollary 5. Researchs by Miyaoka, Kollár, Mori, Takagi, Keel, Matsuki, McKernan and Xie [Miy, KMMT, KMM04, Xie] show that $c_2(X)$ is pseudo-effective unless the numerical dimension $\nu(-K_X)$ is 2 and the irregularity $q(X)$ is 0. We may assume that X is a $\mathbb{Q}$-factorial terminal projective 3-fold with $-K_X$ nef, $\nu(-K_X) = 2$, $q(X) = 0$ and $\rho(X) \leq 2 + (2c_1(X) \cdot c_2(X))/3$.

Recall that the conditions $-K_X$ nef and $\nu(-K_X) = 2$ imply $c_1(X) \cdot c_2(X) \geq 0$ by [KMM04, Corollary 6.2]. Run the MMP from X and let $X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_t$ be a sequence of elementary divisorial contractions and flips such that $X_t \to S$ is a Mori fiber space. Suppose j is any nonnegative integer less than $t+1$. If $\rho(X_j) \geq 2$, we see that $c_1(X_j) \cdot c_2(X_j) \geq 0$ by Theorems 2, 3 and 4. If $\rho(X_j) = 1$, so is $\rho(X_t)$. In particular, X_t is a Fano 3-fold. We have $j = t$ and $c_1(X_j).c_2(X_j) = c_1(X_t).c_2(X_t) \geq 0$ by [KMMT]. Applying to [Xie, Theorem 1.2], we obtain the pseudo-effectivity of $c_2(X)$. Furthermore, if X is $\mathbb{Q}$-factorial and Gorenstein, then $c_1(X).c_2(X) = 24\chi(\mathcal{O}_X) = 24$ and thus $2 + (2c_1(X) \cdot c_2(X))/3 = 18$. This completes the proof of Corollary 5. □

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Department of Mathematics
National Central University
No. 300, Zhongda Rd.
Zhongli District, Taoyuan City 32001
Taiwan
E-mail: jhengjie@math.ncu.edu.tw