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On a Generalized Brauer Group in Mixed Characteristic Cases

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Abstract. We define a generalization of the Brauer group $\mathrm{H}^n_{\mathrm{B}}(X)$ for an equi-dimensional scheme X and $n > 0$. In the case where X is the spectrum of a local ring of a smooth algebra over a discrete valuation ring, $\mathrm{H}^n_{\mathrm{B}}(X)$ agrees with the étale motivic cohomology $\mathrm{H}^{n+1}_{\text{ét}}(X, \mathbb{Z}(n-1))$. We prove (a part of) the Gersten-type conjecture for the generalized Brauer group for a local ring of a smooth algebra over a mixed characteristic discrete valuation ring and an isomorphism $\mathrm{H}^n_{\mathrm{B}}(R) \simeq \mathrm{H}^n_{\mathrm{B}}(k)$ for a henselian local ring R of a smooth algebra over a mixed characteristic discrete valuation ring and the residue field k of R. As an application, we show local-global principles for Galois cohomology groups over function fields of smooth curves over a mixed characteristic excellent henselian discrete valuation ring.

1. Introdution

Let A be a Dedekind ring or field, X a smooth scheme over $\mathrm{Spec}(A)$, $\mathbb{Z}(n)_{\text{ét}}$ the Bloch's cycle complex for étale topology and $\mathbb{Z}/m(n)_{\text{ét}} = \mathbb{Z}(n)_{\text{ét}} \otimes \mathbb{Z}/m\mathbb{Z}$ for a positive integer m. Let $D^b(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ be the derived category of bounded complexes of étale $\mathbb{Z}/m\mathbb{Z}$-sheaves on X.

Then

(i) If $l \in \mathbb{N}$ is invertible in A, there is a quasi-isomorphism

$$\mathbb{Z}/l(n)_{\text{ét}} \xrightarrow{\sim} \mu_l^{\otimes n}[0]$$

in $D^b(X_{\text{ét}}, \mathbb{Z}/l\mathbb{Z})$ by ([2, p.774, Theorem 1.2.4], [22]). Here μ_l is the sheaf of l-th roots of unity.

(ii) If A is a field of characteristic $p > 0$, there is a quasi-isomorphism

$$\mathbb{Z}/p^r(n)_{\text{ét}} \xrightarrow{\sim} \mathrm{W}_r\,\Omega^n_{X,\log}[-n]$$

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in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ for any positive integer r by [2, p.787, §5, (12)]. Here $\mathrm{W}_r\,\Omega^n_{X,\log}$ is the logarithmic de Rham-Witt sheaf.

Assume that a positive integer m equals l in (i) or p^r in (ii). Let R be a local ring of a smooth algebra over A, $k(R)$ its fraction field and $\kappa(\mathfrak{p})$ the residue field of $\mathfrak{p} \in \operatorname{Spec} R$. Let n be a positive integer. Suppose that $n \geq N$ in the case where A is a field of characteristic $p > 0$ and $[k(R) : k(R)^p] = N$.

Then the sequence of étale hyper-cohomology groups

$$
\begin{aligned}
(1.1)\qquad 0 \to \mathrm{H}^{n+1}_{\text{ét}}(R, \mathbb{Z}/m(n)) &\to \mathrm{H}^{n+1}_{\text{ét}}(k(R), \mathbb{Z}/m(n)) \\
&\to \bigoplus_{\substack{\mathfrak{p}\in\operatorname{Spec} R\\ \operatorname{ht}(\mathfrak{p})=1}} \mathrm{H}^{n}_{\text{ét}}(\kappa(\mathfrak{p}), \mathbb{Z}/m(n-1))
\end{aligned}
$$

is exact by ([2, p.774, Theorem 1.2.(2, 4, 5)], [22]) and [21, p.608, Theorem 5.2].

Since $\mathrm{H}^{i}_{\text{Zar}}(R, \mathbb{Z}(n)) = 0$ for $i > n$ by [2, p.779, Theorem 3.2 b)] and [2, p.786, Corollary 4.4], we have

$$\mathrm{H}^{n+1}_{\text{ét}}(R, \mathbb{Z}(n)) = \mathrm{H}^{i}_{\text{ét}}(R, \mathbb{Q}(n)) = 0$$

for $i > n$ by [2, p.774, Theorem 1.2.2] and [2, p.781, Proposition 3.6]. Hence the étale motivic cohomology $\mathrm{H}^{n+2}_{\text{ét}}(R, \mathbb{Z}(n))$ is a torsion group and

$$\mathrm{H}^{n+1}_{\text{ét}}(R, \mathbb{Z}/m(n)) = \operatorname{Ker}\left(\mathrm{H}^{n+2}_{\text{ét}}(R, \mathbb{Z}(n)) \xrightarrow{\times m} \mathrm{H}^{n+2}_{\text{ét}}(R, \mathbb{Z}(n))\right)$$

for any positive integer m. Therefore we can regard the sequence (1.1) as (a part of) the Gersten type resolution for the étale motivic cohomology.

One of the objectives of this paper is to prove that an improved version of the sequence (1.1) is exact in the case where A is a discrete valuation ring of mixed-characteristic $(0, p)$ and $m = p^r$ as follows:

Theorem 1.1 (Proposition 3.4 and Theorem 4.6). *Let A be a discrete valuation ring of mixed-characteristic $(0, p)$, R a local ring of a smooth algebra over A, $k(R)$ the fraction field of R and $\kappa(\mathfrak{p})$ the residue field of $\mathfrak{p} \in \operatorname{Spec} R$.*

Then the sequence

$$
\begin{aligned}
(1.2)\qquad 0 \to \mathrm{H}^{n+1}_{\text{ét}}(R, \mathbb{Z}/p^r(n)) &\to \mathrm{H}^{n+1}_{\text{ét}}(k(R), \mathbb{Z}/p^r(n))' \\
&\to \bigoplus_{\substack{\mathfrak{p}\in\operatorname{Spec} R\\ \operatorname{ht}(\mathfrak{p})=1}} \mathrm{H}^{n}_{\text{ét}}(\kappa(\mathfrak{p}), \mathbb{Z}/p^r(n-1))
\end{aligned}
$$

is exact for integers $n \geq 0$ *and* $r > 0$ *where*

$$\begin{aligned}&\mathrm{H}^{n+1}_{\text{ét}}\left(k(R), \mathbb{Z}/p^r(n)\right)' \\ &= \operatorname{Ker}\left(\mathrm{H}^{n+1}_{\text{ét}}\left(k(R), \mathbb{Z}/p^r(n)\right) \to \prod_{\substack{\mathfrak{p}\in\operatorname{Spec} R \\ \operatorname{ht}(\mathfrak{p})=1}} \mathrm{H}^{n+1}_{\text{ét}}\left(k(R_{\bar{\mathfrak{p}}}), \mathbb{Z}/p^r(n)\right)\right)\end{aligned}$$

and $R_{\bar{\mathfrak{p}}}$ *is the strictly henselization of* $R_{\mathfrak{p}}$*.*

This paper is organized as follows. In §2 we prove Proposition 2.1 which is an improved version of the purity theorem of the Bloch's cycle complex for étale topology ([2, p.774, Theorem 1.2.1]) by improving the proof of [2, p.774, Theorem 1.2.1] and prove that the sequence (1.2) is exact in the case where R is a discrete valuation ring.

In §3 we define a generalization of the Brauer group. We show the reason why we can regard it as a generalization of the Brauer group (Proposition 3.3) and a relation between it and étale motivic cohomology (Proposition 3.4).

In §4 we prove (a part of) the Gersten-type conjecture for the generalized Brauer group for a local ring of a smooth algebra over a mixed characteristic discrete valuation ring (Theorem 4.6). In §5 we prove the rigidity theorem for étale motivic cohomology for a henselian local ring of a smooth algebra over a mixed-characteristic discrete valuation ring.

Namely we prove the following theorem:

THEOREM 1.2 (Theorem 5.6). *Let* R *be a henselian local ring of a smooth algebra over a mixed-characteristic discrete valuation ring* A *and* k *the residue field of* R*.*

Then we have an isomorphism

$$\mathrm{H}^{n+1}_{\text{ét}}\left(R, \mathbb{Z}/m(n)\right) \xrightarrow{\sim} \mathrm{H}^{n+1}_{\text{ét}}\left(k, \mathbb{Z}/m(n)\right)$$

for any positive integer m*.*

Theorem 1.2 is proved in the case where m is invertible in A by ([2, p.774, Theorem 1.2.3], [22]).

Finally we prove the following local-global principle in §6 by applying Theorem 1.1 and Theorem 1.2.

Theorem 1.3 (Theorem 6.3). *Let A be an excellent henselian discrete valuation ring of mixed characteristic $(0,p)$ and π a prime element of A. Let $\mathfrak{X}$ be a connected proper smooth curve over* $\operatorname{Spec} A$*, K the fraction field of $\mathfrak{X}$ and $K_{(\eta)}$ the fraction field of the henselization of $\mathcal{O}_{\mathfrak{X},\eta}$.*

Then the local-global map

$$\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mu_m^{\otimes n}\right) \to \prod_{\eta \in \mathfrak{X}^{(1)}} \mathrm{H}^{n+1}_{\text{ét}}\left(K_{(\eta)}, \mu_m^{\otimes n}\right) \tag{1.3}$$

is injective for integers $n \geq 0$ and $m = p^r$ where $\mathfrak{X}^{(1)}$ is the set of points of codimension 1*.*

Suppose that $\mathfrak{X}$ is a regular scheme which is flat of finite type over an excellent henselian discrete valuation ring A and whose fiber over the closed point of $\operatorname{Spec} A$ is reduced normal crossing divisors on $\mathfrak{X}$. Then the local-global map (1.3) is injective for $n \geq 0$ and m which is prime to $\operatorname{char}(A)$ (cf. [6, Theorem 3.3.6], [7, Theorem 1.2] and [18, Theorem 1.2]).

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Notations. Throughout this paper, p will be a fixed prime, unless otherwise stated. For a scheme X, $X_{\text{ét}}$, X_{Nis} and X_{Zar} denote the category of étale schemes over X equipped with the étale, Nisnevich and Zariski topology, respectively. For $t \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$, $\mathbb{S}_{X_t}$ denotes the category of sheaves on X_t. $X^{(i)}$ denotes the set of points of codimension i and $X_{(i)}$ denotes the set of points of dimension i. $k(X)$ denotes the ring of rational functions on X and $\kappa(x)$ denotes the residue field of $x \in X$. For scheme over $\mathbb{F}_p$, $\Omega^q_X = \Omega^q_{X/\mathbf{Z}}$ denotes the exterior algebra over $\mathcal{O}_X$ of the sheaf $\Omega^1_{X/\mathbf{Z}}$ of absolute differentials on X and $\Omega^q_{X,\log}$ the part of Ω^q_X generated étale locally by local sections of the forms

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}.$$

2. Étale Motivic Cohomology

Let $D_i = \mathbb{Z}[t_0, \cdots, t_i]/(\sum_{j=0}^{i} t_j - 1)$, and $\Delta^i = \operatorname{Spec} D_i$ be the algebraic i-simplex. For an equi-dimensional scheme X, let $z^n(X, i)$ be the free abelian group on closed integral subschemes of codimension n of $X \times \Delta^i$, which intersect all faces property. Intersecting with faces defines the structure of a simplicial abelian group, and hence gives a (homological) complex $z^n(X, *)$.

The complex of sheaves $\mathbb{Z}(n)_t$ on the site X_t, where $t \in \{\text{ét}, \text{Zar}\}$, is defined as the cohomological complex with $z^n(-, 2n-i)$ in degree i. For an abelian group A we define $A(n)$ to be $\mathbb{Z}(n) \otimes A$.

First we show an improved version of the purity theorem of the Bloch's cycle complex for étale topology ([2, p.774, Theorem 1.2.1]).

Proposition 2.1. *Let A be a regular local ring with $\dim(A) \leq 1$, $\mathfrak{X}$ a scheme which is essentially of finite type over $\operatorname{Spec} A$ and $i : Y \to \mathfrak{X}$ a closed subscheme of codimension c with open complement $j : X \to \mathfrak{X}$.*

Suppose that X is essentially smooth over a regular ring of dimension at most one. Then we have a quasi-isomorphism

$$\tau_{\leq n+2}\Big(\mathbb{Z}(n-c)_{\text{ét}}[-2c]\Big) \xrightarrow{\sim} \tau_{\leq n+2}\mathbf{R}i^!\mathbb{Z}(n)_{\text{ét}} \tag{2.1}$$

and a quasi-isomorphism

$$\tau_{\leq n+1}\Big(\mathbb{Z}/m(n-c)_{\text{ét}}[-2c]\Big) \xrightarrow{\sim} \tau_{\leq n+1}\mathbf{R}i^!\mathbb{Z}/m(n)_{\text{ét}} \tag{2.2}$$

for any positive integer m.

Proof (cf. The proof of [2, Theorem 1.2.1]). Let $X_{\text{ét}} \xrightarrow{\epsilon} X_{\text{Zar}}$ be the canonical map of sites. Then we have a quasi-isomorphism

$$\tau_{\leq n+1}\epsilon^*\mathbf{R}j_*\mathbb{Z}(n)_{\text{Zar}} \xrightarrow{\sim} \tau_{\leq n+1}\epsilon^*\mathbf{R}j_*\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}} \xrightarrow{\sim} \tau_{\leq n+1}\mathbf{R}j_*\mathbb{Z}(n)_{\text{ét}} \tag{2.3}$$

by ([2, p.774, Theorem 1.2.2], [22]) (cf. [2, p.787]). Since

$$\mathbf{R}^{n+2}j_*\left(\tau_{\leq n+1}\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}}\right) \to \mathbf{R}^{n+2}j_*\left(\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}}\right)$$

is injective by a distinguished triangle

$$\cdots \to \tau_{\leq n+1}\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}} \to \mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}} \to \tau_{\geq n+2}\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}} \to \cdots$$

and the equation

$$\mathbf{R}j^{n+1}(\tau_{\geq n+2}\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}}) = 0,$$

the composite map

$$\begin{aligned}&\epsilon^*\mathbf{R}^{n+2}j_*\mathbb{Z}(n)_{\text{Zar}} \xrightarrow{\sim} \epsilon^*\mathbf{R}^{n+2}j_*\left(\tau_{\leq n+1}\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}}\right)\\ \hookrightarrow &\epsilon^*\mathbf{R}^{n+2}j_*\mathbf{R}\epsilon_*\mathbb{Z}(n)_{\text{ét}} \xrightarrow{\sim} \mathbf{R}^{n+2}j_*\mathbb{Z}(n)_{\text{ét}}\end{aligned}$$

is injective by ([2, p.774, Theorem 1.2.2], [22]). Moreover we have the map of distinguished triangles

$$\begin{array}{ccccc}\epsilon^*\mathbb{Z}(n-c)_{\text{Zar}}[-2c] & \longrightarrow & \epsilon^*i^*\mathbb{Z}(n)_{\text{Zar}} & \longrightarrow & \epsilon^*i^*\mathbf{R}j_*\mathbb{Z}(n)_{\text{Zar}}\\ \downarrow & & \| & & \downarrow\\ \mathbf{R}i^!\mathbb{Z}(n)_{\text{ét}} & \longrightarrow & i^*\mathbb{Z}(n)_{\text{ét}} & \longrightarrow & i^*\mathbf{R}j_*\mathbb{Z}(n)_{\text{ét}}.\end{array}$$

Hence we have the quasi-isomorphism (2.1) by the five lemma. Moreover the quasi-isomorphism (2.2) follows from the quasi-isomorphism (2.1). Therefore the statement follows. □

REMARK 2.2. If A is a Dedekind ring and i is the inclusion of one of the closed fibers, we have the quasi-isomorphisms (2.1) and (2.2) as [2, p.774, Theorem 1.2.1].

LEMMA 2.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and let $\mathcal{A}$ be a Grothendieck category. Then*

$$\mathbf{R}^{n+1}F(\tau_{\leq n}A^\bullet) = \operatorname{Ker}\Big(\mathbf{R}^{n+1}F(A^\bullet) \to F(\mathcal{H}^{n+1}(A^\bullet))\Big)$$

for any complex $A^\bullet$.

PROOF. If $B^\bullet$ is bounded below, there is a convergent spectral sequence for the hyper-cohomology with

$$\mathbb{R}^pF(\mathcal{H}^q(B^\bullet)) \Rightarrow \mathbb{R}^{p+q}F(B^\bullet) \tag{2.4}$$

and

$$\mathbf{R}^nF(B^\bullet) = \mathbb{R}^nF(B^\bullet).$$

So

$$\mathbf{R}^n F(\tau_{\geq n+1} A^\bullet) = \mathbb{R}^n F(\tau_{\geq n+1} A^\bullet) = 0,$$
$$\mathbf{R}^{n+1} F(\tau_{\geq n+1} A^\bullet) = \mathbb{R}^{n+1} F(\tau_{\geq n+1} A^\bullet) \xrightarrow{\sim} F(\mathcal{H}^{n+1}(A^\bullet))$$

for any complex $A^\bullet$. By a distinguished triangle

$$\cdots \to \tau_{\leq n} A^\bullet \to A^\bullet \to \tau_{\geq n+1} A^\bullet \to \cdots ,$$

we have a distinguished triangle

$$\cdots \to \mathbf{R}F\left(\tau_{\leq n} A^\bullet\right) \to \mathbf{R}F\left(A^\bullet\right) \to \mathbf{R}F\left(\tau_{\geq n+1} A^\bullet\right) \to \cdots .$$

Therefore,

$$\mathbf{R}^{n+1} F(\tau_{\leq n} A^\bullet) = \operatorname{Ker}\Big(\mathbf{R}^{n+1} F(A^\bullet) \to F(\mathcal{H}^{n+1}(A^\bullet))\Big). \ \square$$

REMARK 2.4. Let $A^\bullet$ be a bounded below complex. Since the edge maps of spectral sequence are natural maps, the morphism

$$\mathbf{R}^{n+1} F(A^\bullet) \to \mathbf{R}^{n+1} F(\tau_{\geq n+1} A^\bullet)$$

corresponds to the edge map of $\mathbb{R}^p F(\mathcal{H}^q(A^\bullet)) \Rightarrow \mathbb{R}^{p+q} F(A^\bullet)$.

PROPOSITION 2.5. *Let A be a discrete valuation ring with the maximal ideal $\mathfrak{m}$ the fraction field K and the residue field k. Let $j : \operatorname{Spec} K \to \operatorname{Spec} A$ be the generic point and $K_{\bar{\mathfrak{m}}}$ the maximal unramified extension of K.*

Then

$$\mathrm{H}^{n+1}_{\text{ét}}\left(\operatorname{Spec} A, \tau_{\leq n} \mathbf{R} j_* \mathbb{Z}/m(n)\right) = \mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right)'$$

for integers $n \geq 0$ and $m > 0$ where

$$\begin{aligned} &\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right)' \\ &= \operatorname{Ker}\Big(\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(K_{\bar{\mathfrak{m}}}, \mathbb{Z}/m(n)\right)\Big). \end{aligned}$$

Moreover the sequence

$$\begin{aligned} (2.5) \qquad 0 &\to \mathrm{H}^{n+1}_{\text{ét}}\left(A, \mathbb{Z}/m(n)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right)' \\ &\to \mathrm{H}^{n}_{\text{ét}}\left(k, \mathbb{Z}/m(n-1)\right) \end{aligned}$$

is exact.

Proof. For each $x \in \operatorname{Spec} A$, we choose a geometric point $u_x : \bar{x} \to \operatorname{Spec} A$.

Then

$$F \to \prod_{x \in \operatorname{Spec} A} (u_x)_*(u_x)^* F$$

is injective for a sheaf F on $(\operatorname{Spec} A)_{\text{ét}}$ ([12, p.90, III, Remark 1.20 (c)]). Hence

$$\begin{aligned}
&\mathrm{H}^{n+1}_{\text{ét}}\left(\operatorname{Spec} A, \tau_{\leq n}\mathbf{R}j_*\mathbb{Z}/m(n)\right)\\
&= \operatorname{Ker}\Big(\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right) \to \Gamma\left(\operatorname{Spec} A, \mathbf{R}^{n+1}j_*\mathbb{Z}/m(n)\right)\Big)\\
&= \operatorname{Ker}\Big(\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right) \to \prod_{x \in \operatorname{Spec} A}\left(\mathbf{R}^{n+1}j_*\mathbb{Z}/m(n)\right)_{\bar{x}}\Big)\\
&= \operatorname{Ker}\Big(\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(K_{\bar{\mathfrak{m}}}, \mathbb{Z}/m(n)\right)\Big)
\end{aligned}$$

by Lemma 2.3.

Let $i : \operatorname{Spec} k \to \operatorname{Spec} A$ the closed immersion. Since we have a distinguished triangle

$$\cdots \to \mathbb{Z}/m(n)_{\text{ét}} \to \tau_{\leq n}\mathbf{R}j_*\mathbb{Z}/m(n)_{\text{ét}} \to i_*\mathbb{Z}/m(n-1)_{\text{ét}} \to \cdots$$

by Proposition 2.1 and [2, p.786, Corollary 4.4], the sequence

$$\begin{aligned}
\mathrm{H}^{n}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right) \to \mathrm{H}^{n-1}_{\text{ét}}\left(k, \mathbb{Z}/m(n-1)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(A, \mathbb{Z}/m(n)\right)\\
\to \mathrm{H}^{n+1}_{\text{ét}}\left(\operatorname{Spec} A, \tau_{\leq n}\mathbf{R}j_*\mathbb{Z}/m(n)\right) \to \mathrm{H}^{n}_{\text{ét}}\left(k, \mathbb{Z}/m(n-1)\right)
\end{aligned}$$

is exact.

By [2, p.774, Theorem 1.2.2] and [4, Lemma 3.2], the homomorphism

$$\mathrm{H}^{n}_{\text{ét}}\left(K, \mathbb{Z}/m(n)\right) \to \mathrm{H}^{n-1}_{\text{ét}}\left(k, \mathbb{Z}/m(n-1)\right) \tag{2.6}$$

agrees with the homomorphism

$$\mathrm{K}^M_n(K)/m \to \mathrm{K}^M_{n-1}(k)/m$$

which is induced by the symbol map of the Milnor K-group.

Hence the homomorphism (2.6) is surjective and the homomorphism

$$\mathrm{H}^{n+1}_{\text{ét}}\left(A, \mathbb{Z}/m(n)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(\operatorname{Spec} A, \tau_{\leq n}\mathbf{R}j_*\mathbb{Z}/m(n)\right)$$

is injective. Therefore the statement follows. □

PROPOSITION 2.6. *Let A be a henselian discrete valuation ring with the fraction field K and the residue field k. Let $j : \operatorname{Spec} K \to \operatorname{Spec} A$ be the generic point. Suppose that* $\operatorname{char}(k) = p > 0$. *Then*

$$\begin{aligned}&\mathrm{H}^{n+1}_{\text{ét}}\left(\operatorname{Spec} A, \tau_{\leq n}\left(\mathbf{R}j_*\mathbb{Z}/p^r(n)\right)\right)\\ =&\,\mathrm{H}^{1}\left(\mathrm{Gal}(k_s/k), \mathrm{K}^M_n(K_{\bar{\mathfrak{m}}})/p^r\right)\end{aligned}$$

for integers $n \geq 0$ and $r > 0$. Here k_s is the separable closure of k, $K_{\bar{\mathfrak{m}}}$ is the maximal unramified extension of K and $\mathrm{K}^M_n(K_{\bar{\mathfrak{m}}})$ is the n-th Milnor K-group of $K_{\bar{\mathfrak{m}}}$.

PROOF. Let $E_2^{l,m} \Rightarrow E^{l+m}$ be a spectral sequence. Suppose that $E_2^{l,m} = 0$ for $l < 0$ or $m < 0$. Then we have a filtration

$$0 \subset F^{l+m}_{l+m} \subset \cdots \subset F^{p+q}_1 \subset F^{p+q}_0 = E^{p+q}$$

such that

$$F^{l+m}_l / F^{l+m}_{l+1} \simeq E^{l,m}_\infty$$

and we can define the morphism

$$\text{(2.7)} \qquad \operatorname{Ker}\left(E^n \to E_2^{0,n}\right) \simeq F^n_1 \twoheadrightarrow F^n_1/F^n_2 \simeq E^{1,n-1}_\infty \hookrightarrow E_2^{1,n-1}.$$

Moreover, the morphism (2.7) is an isomorphism if $E_2^{l,m} = 0$ for $l \geq 2$.

Let $i : \operatorname{Spec} k \to \operatorname{Spec} A$ be the closed immersion. Since k has p-cohomological dimension at most 1,

$$\mathrm{H}^{l}_{\text{ét}}\left(\operatorname{Spec} A, \mathbf{R}^m j_*\mathbb{Z}/p^r(n)\right) = \mathrm{H}^{l}_{\text{ét}}\left(\operatorname{Spec} k, i^*\mathbf{R}^m j_*\mathbb{Z}/p^r(n)\right) = 0$$

for $l \geq 2$ by [2, p.777, The proof of Proposition 2.2 b)] and

$$\begin{aligned}&\operatorname{Ker}\Big(\mathrm{H}^{n+1}_{\text{ét}}(\operatorname{Spec} K, \mathbb{Z}/p^r(n)) \to \Gamma\left(\operatorname{Spec} A, \mathbf{R}^{n+1}j_*\mathbb{Z}/m(n)\right)\Big)\\ =&\,\mathrm{H}^{1}_{\text{ét}}\left(\operatorname{Spec} A, \mathbf{R}^n j_*\mathbb{Z}/p^r(n)\right)\\ =&\,\mathrm{H}^{1}_{\text{ét}}\left(\operatorname{Spec} k, i^*\mathbf{R}^n j_*\mathbb{Z}/p^r(n)\right)\end{aligned}$$

by the spectral sequence

$$\mathrm{H}^{l}_{\text{ét}}\left(\operatorname{Spec} A, \mathbf{R}^{m} j_{*}\mathbb{Z}/p^{r}(n)\right) \Rightarrow \mathrm{H}^{l+m}_{\text{ét}}\left(K, \mathbb{Z}/p^{r}(n)\right)$$

and [2, p.777, The proof of Proposition 2.2 b)]. Moreover

$$\left(i^{*}\mathbf{R}^{n} j_{*}\mathbb{Z}/p^{r}(n)\right)_{\bar{\mathfrak{m}}} = \mathrm{H}^{n}_{\text{ét}}\left(\operatorname{Spec} K_{\bar{\mathfrak{m}}}, \mathbb{Z}/p^{r}(n)\right) = \mathrm{K}^{M}_{n}(K_{\bar{\mathfrak{m}}})/p^{r}$$

by [12, p.88, III, Theorem 1.15] and [1, p.131, Theorem (5.12)]. Here $\bar{\mathfrak{m}}$ is a geometric point of $\operatorname{Spec} A$ such that $\kappa(\bar{\mathfrak{m}})$ is the separable closure of $k = \kappa(\mathfrak{m})$ and $\left(i^{*}\mathbf{R}^{n} j_{*}\mathbb{Z}/p^{r}(n)\right)_{\bar{\mathfrak{m}}}$ is the stalk of $i^{*}\mathbf{R}^{n} j_{*}\mathbb{Z}/p^{r}(n)$ at $\bar{\mathfrak{m}}$. Hence

$$\begin{aligned} &\operatorname{Ker}\Big(\mathrm{H}^{n+1}_{\text{ét}}(\operatorname{Spec} K, \mathbb{Z}/p^{r}(n)) \to \Gamma\left(\operatorname{Spec} A, \mathbf{R}^{n+1} j_{*}\mathbb{Z}/m(n)\right)\Big) \\ = &\,\mathrm{H}^{1}\left(\operatorname{Gal}(k_{s}/k), \mathrm{K}^{M}_{n}(K_{\bar{\mathfrak{m}}})/p^{r}\right). \end{aligned}$$

Therefore the statement follows from Lemma 2.3. □

Let A be a discrete valuation ring with the maximal ideal $\mathfrak{m}$ and the fraction field K. Suppose that $\operatorname{char}(k) = p > 0$.

Let $j : \operatorname{Spec} K \to \operatorname{Spec} A$ be the generic point, $i : \operatorname{Spec} k \to \operatorname{Spec} A$ the closed immersion, k_{s} the separable closure of k and $K_{\bar{\mathfrak{m}}}$ the maximal unramified extension of K.

Then

$$\begin{aligned} &\mathrm{H}^{n}_{\text{ét}}\left(\operatorname{Spec} A, i_{*}\mathbb{Z}/p^{r}(n-1)\right) \\ = &\,\mathrm{H}^{n}_{\text{ét}}\left(\operatorname{Spec} k, \mathbb{Z}/p^{r}(n-1)\right) \\ = &\,\mathrm{H}^{1}\left(\operatorname{Gal}(k_{s}/k), \mathrm{K}^{M}_{n-1}(k_{s})/p^{r}\right) \end{aligned}$$

by [4, Theorem 8.5] and [1, p.117, Corollary (2.8)].

PROPOSITION 2.7. *Let the notations be same as above. Suppose that A is a henselian discrete valuation ring. Then the homomorphism*

$$\mathrm{H}^{n+1}_{\text{ét}}\left(\operatorname{Spec} A, \tau_{\leq n}\left(\mathbf{R} j_{*}\mathbb{Z}/p^{r}(n)\right)\right) \to \mathrm{H}^{n}_{\text{ét}}\left(\operatorname{Spec} A, i_{*}\left(\mathbb{Z}/p^{r}(n-1)\right)\right)$$

which is induced by the map

$$\tau_{\leq n}\left(\mathbf{R} j_{*}\mathbb{Z}/p^{r}(n)_{\text{ét}}\right) \to i_{*}\left(\mathbb{Z}/p^{r}(n-1)_{\text{ét}}\right)[-1]$$

agrees with the homomorphism

$$\mathrm{H}^1\left(\mathrm{Gal}(k_s/k), \mathrm{K}_n^{\mathrm{M}}\left(K_{\bar{\mathfrak{m}}}\right)/p^r\right) \to \mathrm{H}^1\left(\mathrm{Gal}(k_s/k), \mathrm{K}_{n-1}^{\mathrm{M}}(k_s)/p^r\right) \tag{2.8}$$

which is induced by the symbol map of the Milnor K-group.

PROOF. We have the commutative diagram

$$\begin{array}{ccc}
 & & \mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/p^r(n)\right) \\
 & & \| \\
\mathrm{H}^1_{\text{ét}}\left(\mathrm{Spec}\, A, \mathbf{R}^n j_* \mathbb{Z}/p^r(n)\right) & \longrightarrow & \mathrm{H}^{n+1}_{\text{ét}}\left(\mathrm{Spec}\, A, \mathbf{R} j_* \mathbb{Z}/p^r(n)\right) \\
\downarrow & & \downarrow \\
\mathrm{H}^1_{\text{ét}}\left(\mathrm{Spec}\, A, i_* \mathbf{R}^{n+1} i^! \mathbb{Z}/p^r(n)\right) & \longrightarrow & \mathrm{H}^{n+1}_{\text{ét}}\left(\mathrm{Spec}\, A, i_* \mathbf{R} i^! \mathbb{Z}/p^r(n)[+1]\right) \\
 & & \| \\
 & & \mathrm{H}^{n+2}_{\mathfrak{m}}\left(\mathrm{Spec}\, A, \mathbb{Z}/p^r(n)\right)
\end{array} \tag{2.9}$$

where the horizontal maps are given by spectral sequences

$$\mathrm{H}^l_{\text{ét}}\left(\mathrm{Spec}\, A, \mathbf{R}^m j_* \mathbb{Z}/p^r(n)\right) \Rightarrow \mathrm{H}^{l+m}_{\text{ét}}\left(K, \mathbb{Z}/p^r(n)\right)$$

and

$$\mathrm{H}^l_{\text{ét}}\left(\mathrm{Spec}\, A, i_* \mathbf{R}^m i^! \mathbb{Z}/p^r(n-1)\right) \Rightarrow \mathrm{H}^{l+m}_{\mathfrak{m}}\left(\mathrm{Spec}\, A, \mathbb{Z}/p^r(n)\right).$$

Moreover, we have the commutative diagram

$$\begin{array}{ccc}
\epsilon^* i^* \mathbf{R} j_* \mathbb{Z}/p^r(n)_{\mathrm{Zar}} & \longrightarrow & \epsilon^* \mathbb{Z}/p^r(n-1)_{\mathrm{Zar}}[-1] \\
\downarrow & & \downarrow \\
i^* \tau_{\leq n}\left(\mathbf{R} j_* \mathbb{Z}/p^r(n)_{\text{ét}}\right) & \longrightarrow & \tau_{\leq n+1}\left(\mathbf{R} i^! \mathbb{Z}/p^r(n)_{\text{ét}}\right)[+1]
\end{array}$$

where the vertical maps are quasi-isomorphisms and the homomorphism

$$\left(\epsilon^* i^* \mathcal{H}^n\left(\mathbf{R} j_* \mathbb{Z}/p^r(n)_{\mathrm{Zar}}\right)\right)_{\bar{\mathfrak{m}}} \to \mathcal{H}^{n-1}\left(\epsilon^* \mathbb{Z}/p^r(n-1)_{\mathrm{Zar}}\right)_{\bar{\mathfrak{m}}}$$

agrees with the symbol map $\mathrm{K}_n^{\mathrm{M}}(K_{\bar{\mathfrak{m}}})/p^r \to \mathrm{K}_{n-1}^{\mathrm{M}}(k_s)/p^r$ by [4, Lemma 3.2]. Therefore the statement follows. □

REMARK 2.8. Let A be a discrete valuation ring with the fraction field K and the residue field k. Suppose that $\mathrm{char}(k) = p > 0$.

In the case where A is henselian and $[k : k^p] \leq n-1$, the homomorphism (2.8) is defined by K.Kato ([9, p.150, §1, (1.3) (ii)]).

In general, the homomorphism

$$\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/p^r(n)\right)' \to \mathrm{H}^{n}_{\text{ét}}\left(k, \mathbb{Z}/p^r(n-1)\right)$$

in the exact sequence (2.5) is the composition of homomorphisms

$$\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mathbb{Z}/p^r(n)\right)' \to \mathrm{H}^{n+1}_{\text{ét}}\left(\tilde{K}_{\mathfrak{m}}, \mathbb{Z}/p^r(n)\right)'$$

and (2.8) where $\tilde{K}_{\mathfrak{m}}$ is the henselization of K.

3. Definition of a Generalized Brauer Group

In this section, we define a generalization of the Brauer group.

Let $\epsilon : X_{\text{ét}} \to X_{\text{Zar}}$ be the canonical map of sites. Then we define a generalization of the Brauer group as follows.

DEFINITION 3.1. Let X be an equi-dimensional scheme. Then we define $\mathrm{H}^n_{\mathrm{B}}(X)$ as

$$\mathrm{H}^n_{\mathrm{B}}(X) = \Gamma\left(X, \mathbf{R}^{n+1}\epsilon_* \mathbb{Z}\left(n-1\right)_{\text{ét}}\right).$$

REMARK 3.2. Let X be an essentially smooth scheme over a Dedekind domain and m a positive integer. Since

$$\mathbf{R}^n\epsilon_*(n-1)_{\text{ét}} \simeq \mathcal{H}^n(\mathbb{Z}(n-1)_{\text{Zar}}) = 0$$

by [2, Theorem 1.2.2 and Corollary 4.4] and [22], we have

$$\begin{aligned}\mathrm{H}^n_{\mathrm{B}}(X)_m \overset{\text{def}}{=} & \mathrm{Ker}\left(\mathrm{H}^n_{\mathrm{B}}(X) \xrightarrow{\times m} \mathrm{H}^n_{\mathrm{B}}(X)\right) \\ = & \Gamma\left(X, \mathbf{R}^n\epsilon_*\mathbb{Z}/m(n-1)_{\text{ét}}\right)\end{aligned}$$

by a distinguished triangle

$$\cdots \to \mathbf{R}\epsilon_*\mathbb{Z}(n-1)_{\text{ét}} \xrightarrow{\times m} \mathbf{R}\epsilon_*\mathbb{Z}(n-1)_{\text{ét}} \to \mathbf{R}\epsilon_*\mathbb{Z}/m(n-1)_{\text{ét}} \to \cdots.$$

Therefore this cohomology group relates to Kato homology (cf. [10, p.160, (5.3)]).

For the following reason we can regard $\mathrm{H}^n_{\mathrm{B}}$ as a generalization of the Brauer group.

PROPOSITION 3.3. *Let X be an essentially smooth scheme over the spectrum of a Dedekind domain. Then*

$$\mathrm{H}^1_{\mathrm{B}}(X) = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}) \quad \text{and} \quad \mathrm{H}^2_{\mathrm{B}}(X) = \mathrm{Br}(X)$$

where $\mathrm{Br}(X)$ *is the cohomological Brauer group* $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$.

PROOF. We prove

$$\mathrm{H}^2_{\mathrm{B}}(X) = \mathrm{Br}(X). \tag{3.1}$$

Since X is a smooth scheme of finite type over the spectrum of a Dedekind domain, there is a quasi-isomorphism

$$\mathbb{Z}(1)_{\text{ét}} \simeq \mathbb{G}_m[-1]$$

by [3, pp.196–197] and we have the morphism

$$\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}(1)) \to \Gamma\left(X, \mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right) \tag{3.2}$$

which is induced by the morphism

$$\tau_{\leq 3}\left(\mathbf{R}\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right) \to \mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}.$$

Let $x \in X_{(0)}$ and $i_x : x \to X$ the closed immersion. Then the morphism

$$\Gamma\left(X, \mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right) \to \prod_{x \in X_{(0)}} \Gamma\left(x, (i_x)^*\mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right)$$

is injective and

$$\Gamma\left(x, (i_x)^*\mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right) = \mathrm{H}^3_{\text{ét}}(\operatorname{Spec}\mathcal{O}_{X,x}, \mathbb{Z}(1))$$

by [12, p.88, III, Proposition 1.13]. Hence

$$\Gamma\left(X, \mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right) \subset \bigcap_{x \in X_{(0)}} \mathrm{H}^3_{\text{ét}}(\operatorname{Spec}\mathcal{O}_{X,x}, \mathbb{Z}(1)).$$

On the other hand,

$$\mathrm{H}^3_{\text{ét}}\left(X, \mathbb{Z}(1)\right) = \bigcap_{x \in X_{(0)}} \mathrm{H}^3_{\text{ét}}\left(\operatorname{Spec} \mathcal{O}_{X,x}, \mathbb{Z}(1)\right)$$

by [16, Remark 7.18]. Therefore we have an injective homomorphism

$$\Gamma\left(X, \mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right) \to \mathrm{H}^3_{\text{ét}}\left(X, \mathbb{Z}(1)\right) \tag{3.3}$$

such that

$$\mathrm{H}^3_{\text{ét}}\left(X, \mathbb{Z}(1)\right) \xrightarrow{(3.2)} \Gamma\left(X, \mathbf{R}^3\epsilon_*\mathbb{Z}(1)_{\text{ét}}\right) \xrightarrow{(3.3)} \mathrm{H}^3_{\text{ét}}\left(X, \mathbb{Z}(1)\right) = \mathrm{id}$$

and the morphism (3.2) is an isomorphism. Hence the equation (3.1) follows.

Moreover we can show that

$$\mathrm{H}^1_{\mathrm{B}}\left(X\right) = \mathrm{H}^1_{\text{ét}}\left(X, \mathbb{Q}/\mathbb{Z}\right)$$

as above. Therefore the statement follows. □

In the following case, $\mathrm{H}^n_{\mathrm{B}}$ is expressed by étale motivic cohomology.

Proposition 3.4. *Let X be an essentially smooth scheme over the spectrum of a Dedekind domain and $\mathrm{H}^i_{\mathrm{Zar}}\left(X, \mathbb{Z}(n-1)\right) = 0$ for $i \geq n+1$.*

Then

$$\mathrm{H}^n_{\mathrm{B}}(X) = \mathrm{H}^{n+1}_{\text{ét}}\left(X, \mathbb{Z}(n-1)\right). \tag{3.4}$$

Especially, if A is a local ring of smooth algebra over a Dedekind domain and $X = \operatorname{Spec} A$, then the equation (3.4) holds and

$$\begin{aligned} \mathrm{H}^n_{\mathrm{B}}(X)_m &\stackrel{\text{def}}{=} \operatorname{Ker}\left(\mathrm{H}^n_{\mathrm{B}}(X) \xrightarrow{\times m} \mathrm{H}^n_{\mathrm{B}}(X)\right) \\ &= \mathrm{H}^n_{\text{ét}}\left(X, \mathbb{Z}/m(n-1)\right) \end{aligned} \tag{3.5}$$

for any positive integer m.

Proof. Since the canonical map induces a quasi-isomorphism

$$\mathbb{Z}(n-1)_{\mathrm{Zar}} \simeq \tau_{\leq n}\mathbf{R}\epsilon_*\mathbb{Z}(n-1)_{\text{ét}} \tag{3.6}$$

([2, p.774, Theorem 1.2.2], [22]), we have a distinguished triangle

$$\begin{aligned}(3.7)\qquad \cdots \to \mathbb{Z}(n-1)_{\mathrm{Zar}} &\to \tau_{\leq n+1}\mathbf{R}\epsilon_*\mathbb{Z}(n-1)_{\text{ét}} \\ &\to \mathbf{R}^{n+1}\epsilon_*\mathbb{Z}(n-1)_{\text{ét}}[-(n+1)] \to \cdots .\end{aligned}$$

Hence the sequence

$$\begin{aligned}0 &\to \mathrm{H}^{n+1}_{\mathrm{Zar}}(X, \mathbb{Z}(n-1)) \to \mathrm{H}^{n+1}_{\text{ét}}(X, \mathbb{Z}(n-1)) \\ &\to \Gamma\left(X, \mathbf{R}^{n+1}\epsilon_*\mathbb{Z}(n-1)_{\text{ét}}\right) \\ (3.8)\qquad &\to \mathrm{H}^{n+2}_{\mathrm{Zar}}(X, \mathbb{Z}(n-1)) \to \mathrm{H}^{n+2}_{\text{ét}}(X, \mathbb{Z}(n-1))\end{aligned}$$

is exact. Therefore the equation (3.4) holds.

Assume that A is a local ring of a smooth algebra over a Dedekind domain and $X = \operatorname{Spec} A$. Then

$$\mathrm{H}^{i}_{\mathrm{Zar}}(X, \mathbb{Z}(n-1)) = 0$$

for $i \geq n$ by [2, p.786, Corollary 4.4] and

$$\mathrm{H}^{n}_{\text{ét}}(X, \mathbb{Z}(n-1)) = \mathrm{H}^{n}_{\mathrm{Zar}}(X, \mathbb{Z}(n-1)) = 0$$

by the equation (3.6). Therefore the equations (3.4) and (3.5) hold. This completes the proof. □

Remark 3.5. In general,

$$\mathrm{H}^{n+1}_{\text{ét}}(X, \mathbb{Z}(n-1))_{\mathrm{tor}} \neq \Gamma\left(X, \mathbf{R}^{n+1}\epsilon_*\mathbb{Z}(n-1)_{\text{ét}}\right)_{\mathrm{tor}}.$$

Let K be a field and l a positive integer. Suppose that $\mu_l \subset K$. Then we have

$$\mathrm{H}^{5}_{\mathrm{Zar}}(\mathbf{P}^m_K, \mathbb{Z}(3)) = \mathrm{H}^{1}_{\mathrm{Zar}}(\operatorname{Spec} K, \mathbb{Z}(1)) = K^*$$

for an integer $m \geq 2$ by the relation

$$\mathrm{H}^{i}_{\mathrm{Zar}}(\mathbf{P}^m_K, \mathbb{Z}(n)) = \bigoplus_{j=0}^{m} \mathrm{H}^{i-2j}_{\mathrm{Zar}}(\operatorname{Spec} K, \mathbb{Z}(n-j)).$$

Hence

$$\mathrm{H}^{5}_{\mathrm{Zar}}(\mathbf{P}^m_K, \mathbb{Z}(3))_{\mathrm{tor}} \neq 0.$$

Since the sequence

$$\begin{aligned}0 \to &\mathrm{H}^5_{\mathrm{Zar}}\left(\mathbf{P}^m_K, \mathbb{Z}(3)\right)_{\mathrm{tor}} \to \mathrm{H}^5_{\text{ét}}\left(\mathbf{P}^m_K, \mathbb{Z}(3)\right)_{\mathrm{tor}} \\ \to &\Gamma\left(X, \mathbf{R}^5\epsilon_*\mathbb{Z}(3)_{\text{ét}}\right)_{\mathrm{tor}}\end{aligned}$$

is exact,

$$\mathrm{H}^5_{\text{ét}}\left(\mathbf{P}^m_K, \mathbb{Z}(3)\right)_{\mathrm{tor}} \neq \Gamma\left(X, \mathbf{R}^5\epsilon_*\mathbb{Z}(3)\right)_{\mathrm{tor}}.$$

PROPOSITION 3.6. *Let X be an essentially smooth scheme over the spectrum of a Dedekind domain. Let $\alpha : X_{\text{ét}} \to X_{\mathrm{Nis}}$ be the canonical map of sites. Then*

$$\mathrm{H}^{n+1}_{\mathrm{B}}(X) = \Gamma\left(X, \mathbf{R}^{n+2}\alpha_*\mathbb{Z}(n)_{\text{ét}}\right).$$

PROOF. Let $\beta : X_{\mathrm{Nis}} \to X_{\mathrm{Zar}}$ be the canonical map of sites. Since β^* is exact and

$$\beta^*\mathbb{Z}(n-1)_{\mathrm{Zar}} = \mathbb{Z}(n-1)_{\mathrm{Nis}},$$

we have a quasi-isomorphism

$$\mathbb{Z}(n-1)_{\mathrm{Nis}} \simeq \tau_{\leq n}\mathbf{R}\alpha_*\mathbb{Z}(n-1)_{\text{ét}}$$

by ([2, p.774, Theorem 1.2.2], [22]) and the sequence

$$\begin{aligned}0 &\to \mathrm{H}^{n+1}_{\mathrm{Nis}}\left(X, \mathbb{Z}(n-1)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(X, \mathbb{Z}(n-1)\right) \\ &\to \Gamma\left(X, \mathbf{R}^{n+1}\alpha_*\mathbb{Z}(n-1)_{\text{ét}}\right) \\ &\to \mathrm{H}^{n+2}_{\mathrm{Nis}}\left(X, \mathbb{Z}(n-1)\right) \to \mathrm{H}^{n+2}_{\text{ét}}\left(X, \mathbb{Z}(n-1)\right)\end{aligned}$$

is exact. Moreover the sequence (3.8) is exact and

$$\mathrm{H}^i_{\mathrm{Zar}}\left(X, \mathbb{Z}(n-1)\right) = \mathrm{H}^i_{\mathrm{Nis}}\left(X, \mathbb{Z}(n-1)\right)$$

for any i by [2, p.781, Proposition 3.6]. Therefore the statement follows from the five lemma. □

PROPOSITION 3.7. *Let X be an essentially smooth scheme over the spectrum of a Dedekind domain. Then*

$$\mathrm{H}_{\mathrm{B}}^{n+1}(X) = \Gamma\left(X, \mathbf{R}^{n+1}\epsilon_*\mathbb{Q}/\mathbb{Z}(n)_{\text{ét}}\right) \tag{3.9}$$

$$= \Gamma\left(X, \mathbf{R}^{n+1}\alpha_*\mathbb{Q}/\mathbb{Z}(n)_{\text{ét}}\right). \tag{3.10}$$

PROOF. We prove the equation (3.9). The sequence

$$\mathbf{R}^{n+1}\epsilon_*\mathbb{Q}(n)_{\text{ét}} \to \mathbf{R}^{n+1}\epsilon_*\mathbb{Q}/\mathbb{Z}(n)_{\text{ét}} \to \mathbf{R}^{n+2}\epsilon_*\mathbb{Z}(n)_{\text{ét}} \to \mathbf{R}^{n+2}\epsilon_*\mathbb{Q}(n)_{\text{ét}}$$

is exact. Thus, the canonical map

$$\mathbb{Q}(n)_{\mathrm{Zar}} \xrightarrow{\sim} \mathbf{R}\epsilon_*\mathbb{Q}(n)_{\text{ét}}$$

is a quasi-isomorphism by [2, p.781, Proposition 3.6], hence

$$\mathbf{R}^{n+1}\epsilon_*\mathbb{Q}(n)_{\text{ét}} = \mathbf{R}^{n+2}\epsilon_*\mathbb{Q}(n)_{\text{ét}} = 0$$

by [2, p.786, Corollary 4.4]. Therefore we have the equation (3.9). We can also prove the equation (3.10) as above. □

4. Purity

First we show the exactness of the following sequence in equi-characteristic cases.

PROPOSITION 4.1. *Let k be a field and $\mathfrak{X}$ an essentially smooth scheme over* $\operatorname{Spec} k$*. Suppose that $\mathfrak{X}$ is an integral quasi-compact scheme.*

Then the sequence

$$0 \to \mathrm{H}_{\mathrm{B}}^{n+1}(\mathfrak{X}) \to \operatorname{Ker}\left(\mathrm{H}_{\text{ét}}^{n+2}(k(\mathfrak{X}), \mathbb{Z}(n)) \to \prod_{x \in \mathfrak{X}^{(1)}} \mathrm{H}_{\text{ét}}^{n+2}(k(\mathcal{O}_{\mathfrak{X},\bar{x}}), \mathbb{Z}(n))\right)$$

$$\to \bigoplus_{x \in \mathfrak{X}^{(1)}} \mathrm{H}_{\text{ét}}^{n+1}(\kappa(x), \mathbb{Z}(n-1)) \tag{4.1}$$

is exact.

PROOF. Let $g : \operatorname{Spec} k(\mathfrak{X}) \to \mathfrak{X}$ be the generic point.

Since $\mathbf{R}^{n+1}\epsilon_*\mathbb{Z}/m(n)$ is the Zariski sheaf on $\mathfrak{X}$ associated to the presheaf

$$U \mapsto \mathrm{H}^{n+1}_{\text{ét}}(U, \mathbb{Z}/m(n))$$

for any positive integer m, we have homomorphisms

$$\mathbf{R}^{n+1}\epsilon_*\mathbb{Z}/m(n) \to g_*\left(\mathrm{H}^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n))\right)$$

and

$$g_*\left(\mathrm{H}^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n))\right)' \to \bigoplus_{x\in\mathfrak{X}^{(1)}} (i_x)_*\left(\mathrm{H}^{n+1}_{\text{ét}}(\kappa(x), \mathbb{Z}/m(n))\right).$$

Here

$$\begin{aligned} &g_*\left(\mathrm{H}^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n))\right)' \\ &= \mathrm{Ker}\left(g_*\left(\mathrm{H}^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n))\right) \to \prod_{x\in\mathfrak{X}^{(1)}} (i_x)_*\left(\mathrm{H}^{n+1}_{\text{ét}}(k(\mathcal{O}_{\mathfrak{X},x}), \mathbb{Z}/m(n))\right)\right). \end{aligned}$$

Then it suffices to show that the sequence

$$\begin{aligned} 0 \to \mathbf{R}^{n+1}\epsilon_*\mathbb{Z}/m(n) \to & g_*\left(\mathrm{H}^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n))\right)' \\ & \to \prod_{x\in\mathfrak{X}^{(1)}} (i_x)_*\left(\mathrm{H}^{n+1}_{\text{ét}}(k(\mathcal{O}_{\mathfrak{X},x}), \mathbb{Z}/m(n))\right) \end{aligned}$$

is exact. Hence it suffices to show that the sequence (4.1) is exact in the case where $\mathfrak{X}$ is the spectrum of a local ring of a smooth algebra over k.

If $m \in k^*$, then there is a quasi-isomorphism

$$\mathbb{Z}/m(n)_{\text{ét}} \xrightarrow{\sim} \mu_m^{\otimes n}[0]$$

in $D^b(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ by [5, Theorem 1.5]. Here μ_l is the sheaf of l-th roots of unity.

On the other hand, if k has characteristic p, then there is a quasi-isomorphism

$$\mathbb{Z}/p^r(n)_{\text{ét}} \xrightarrow{\sim} \mathrm{W}_r\,\Omega^n_{\mathfrak{X},\log}[-n]$$

in $D^b(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ for any positive integer r by [2, p.787, §5, (12)]. Here $\mathrm{W}_r\,\Omega^n_{\mathfrak{X},\log}$ is the logarithmic de Rham-Witt sheaf.

Therefore the statement follows from [13, 5.2. Theorem C] and (Proposition 2.5, [21, p.600, Theorem 4.1]). □

In the following, we consider $\mathrm{H}^n_{\mathrm{B}}(\mathfrak{X})$ in the case where $\mathfrak{X}$ has mixed-characteristic.

THEOREM 4.2. *Let A be a discrete valuation ring of mixed-characteristic, R a local ring of a smooth algebra over A and K the quotient field of R.*

Then the homomorphism

$$\mathrm{H}^{n+2}_{\text{ét}}\left(R, \mathbb{Z}(n)\right) \to \mathrm{H}^{n+2}_{\text{ét}}\left(K, \mathbb{Z}(n)\right)$$

is injective.

PROOF. Let $Y \to \operatorname{Spec} R$ be the inclusion of the closed fiber with open complement $j : U \to \operatorname{Spec} R$. Then Y is the spectrum of a local ring of smooth algebra over a field and

$$\mathrm{H}^{n}_{\text{ét}}\left(Y, \mathbb{Z}(n-1)\right) = \mathrm{H}^{n}_{\text{Zar}}\left(Y, \mathbb{Z}(n-1)\right)$$

by ([2, p.774, Theorem 1.2.2], [22]). Moreover

$$\mathrm{H}^{n}_{\text{Zar}}\left(Y, \mathbb{Z}(n-1)\right) = 0$$

by [2, p.779, Theorem 3.2] and [2, p.786, Corollary 4.4]. Hence

$$\mathrm{H}^{n+2}_{Y}\left(R_{\text{ét}}, \mathbb{Z}(n)\right) = \mathrm{H}^{n}_{\text{ét}}\left(Y, \mathbb{Z}(n-1)\right) = 0$$

by Proposition 2.1. Therefore the homomorphism

$$\mathrm{H}^{n+2}_{\text{ét}}\left(R, \mathbb{Z}(n)\right) \to \mathrm{H}^{n+2}_{\text{ét}}\left(U, \mathbb{Z}(n)\right)$$

is injective.

On the other hand,

$$\mathrm{H}^{i}_{\text{Zar}}\left(U, \mathbb{Z}(n)\right) = 0$$

for $i \geq n+2$ by [2, p.779, Theorem 3.2] and [2, p.786, Corollary 4.4].

Hence

$$\mathrm{H}^{n+2}_{\text{ét}}\left(U, \mathbb{Z}(n)\right) = \mathrm{H}^{n+1}_{\mathrm{B}}\left(U\right)$$

by Proposition 3.4 and the homomorphism

$$\mathrm{H}^{n+1}_{\mathrm{B}}\left(U\right) \to \mathrm{H}^{n+2}_{\text{ét}}\left(K, \mathbb{Z}(n)\right)$$

is injective by Proposition 4.1. Therefore the statement follows. □

Corollary 4.3. *Let A be a Dedekind domain of mixed-characteristic and $\mathfrak{X}$ an essentially smooth scheme over* $\operatorname{Spec} A$.

Then

$$\mathrm{H}^{n+1}_{\mathrm{B}}\left(\mathfrak{X}\right) = \bigcap_{x \in \mathfrak{X}_{(0)}} \mathrm{H}^{n+1}_{\mathrm{B}}\left(\operatorname{Spec} \mathcal{O}_{\mathfrak{X},x}\right).$$

Proof. Let $j : \mathfrak{X}_K \to \mathfrak{X}$ be the inclusion of the generic fiber and $i_a : \mathfrak{X}_a \to \mathfrak{X}$ the closed embedding of the fiber of $\mathfrak{X}$ over $a \in (\operatorname{Spec} A)^{(1)}$. Then we have a distinguished triangle

$$\cdots \to \bigoplus_{a \in (\operatorname{Spec} A)^{(1)}} i_{a*}\mathbf{R}i^{!}_{a}\left(\mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n)\right) \to \mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n)$$
$$\to \mathbf{R}j_{*}j^{*}\left(\mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n)\right) \to \cdots$$

by [2, p.778, Lemma 2.4]. On the other hand,

$$\left(j_{*}j^{*}\left(\mathbf{R}^{n+1}\mathbb{Z}/m(n)\right)\right)_{x} = \Gamma\left(\left(\operatorname{Spec} \mathcal{O}_{\mathfrak{X},x}\right)_{K}, \mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n)\right)$$

by [12, p.88, III, Proposition 1.13] and the homomorphism

$$\mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n) \to j_{*}j^{*}\left(\mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n)\right)$$

is injective by Theorem 4.2. Moreover, the homomorphism

$$\Gamma\left(\mathfrak{X}, i_{a*}\mathbf{R}^{1}i^{!}_{a}\left(\mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n)\right)\right) \to \prod_{x \in \mathfrak{X}_{(0)}} \mathrm{H}^{1}_{\left(\mathcal{O}_{\mathfrak{X},x}\right)_{a}}\left(\mathcal{O}_{\mathfrak{X},x}, \mathbf{R}^{n+1}\epsilon_{*}\mathbb{Z}/m(n)\right)$$

is injective where $(\mathcal{O}_{\mathfrak{X},x})_a = \mathcal{O}_{\mathfrak{X},x} \otimes_A A/a$. Hence the sequence

$$0 \to \mathrm{H}^{n+1}_{\mathrm{B}}(\mathfrak{X}) \to \mathrm{H}^{n+1}_{\mathrm{B}}(\mathfrak{X}_K) \to \prod_{\substack{x \in \mathfrak{X}_{(0)} \\ a \in (\operatorname{Spec} A)_{(0)}}} \mathrm{H}^1_{(\mathcal{O}_{\mathfrak{X},x})_a}\left(\mathcal{O}_{\mathfrak{X},x}, \mathbf{R}^{n+1}\epsilon_*\mathbb{Z}/m(n)\right)$$

is exact and

$$\mathrm{H}^{n+1}_{\mathrm{B}}(\mathfrak{X}_K) = \bigcap_{x \in (\mathfrak{X}_K)_{(0)}} \mathrm{H}^{n+1}_{\mathrm{B}}(\operatorname{Spec} \mathcal{O}_{\mathfrak{X}_K,x})$$

by Proposition 4.1. Therefore the statement follows. □

Lemma 4.4. *Let A be a discrete valuation ring of mixed-characteristic, R a local ring of a smooth algebra over A and $X = \operatorname{Spec} R$. Let $i : Z \to X$ be a regular closed subscheme of codimension 2 with open complement $j : U \to X$. Suppose that* $\operatorname{char}(Z) = p > 0$.

Then

$$\mathrm{H}^i_{\mathrm{Zar}}(U, \mathbb{Z}(n)) = 0$$

for $i \geq n+2$ and

$$\mathrm{H}^n_{\mathrm{B}}(U) = \mathrm{H}^{n+1}_{\text{ét}}(U, \mathbb{Z}(n-1)) \tag{4.2}$$

Proof. Let $Z = \operatorname{Spec} R'$. Then R' is a local ring of a regular ring of finite type over a field. By Quillen's method (cf.[15, §7, The proof of Theorem 5.11]),

$$R' = \varinjlim R'_i$$

where R'_i is a local ring of a smooth algebra over $\mathbb{F}_p$ and the maps $R'_i \to R'_j$ are flat. Hence

$$\mathrm{H}^i_{\mathrm{Zar}}(Z, \mathbb{Z}(n)) = 0 \tag{4.3}$$

for $i \geq n+1$ by [2, p.786, Corollary 4.4]. Therefore

$$\mathrm{H}^i_{\mathrm{Zar}}(U, \mathbb{Z}(n)) = 0$$

by [2, p.779, Theorem 3.2] and we have the equation (4.2) by Proposition 3.4. This completes the proof. □

PROPOSITION 4.5. *With the notations of Lemma 4.4, we have*

$$\mathrm{H}^n_{\mathrm{B}}(X) = \mathrm{H}^n_{\mathrm{B}}(U).$$

PROOF. We have

$$\mathrm{H}^n_{\mathrm{B}}(U) = \bigcap_{x \in U_{(0)}} \mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{U,x})$$

by Corollary 4.3.

Let l be a positive integer which is prime to $\mathrm{char}(Z) = p > 0$. Suppose that $\mu_l \subset A$. Then

$$\begin{aligned} &\mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{U,x})_l \left(\overset{\mathrm{def}}{=} \mathrm{Ker}\left(\mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{U,x}) \xrightarrow{\times l} \mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{U,x})\right)\right) \\ &= \bigcap_{y \in U^{(1)}} \mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{U,y})_l \left(= \bigcap_{y \in U^{(1)}} \mathrm{Ker}\left(\mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{U,y}) \xrightarrow{\times l} \mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{U,y})\right)\right) \end{aligned}$$

for $x \in U_{(0)}$ by [2, p.774, Theorem 1.2.(2, 4, 5)] and [22].

On the other hand, we have

$$\mathrm{H}^n_{\mathrm{B}}(X)_l = \bigcap_{x \in X^{(1)}} \mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{X,x})_l$$

by [2, p.774, Theorem 1.2.(2, 4, 5)] and [22].

Since $X^{(1)} = U^{(1)}$, we have

$$\mathrm{H}^n_{\mathrm{B}}(X)_l = \mathrm{H}^n_{\mathrm{B}}(U)_l. \tag{4.4}$$

Even if that is the case where $\mu_l \not\subset A$, we can show the equation (4.4) by a standard norm argument.

Therefore it is sufficient to prove that

$$\mathrm{H}^n_{\mathrm{B}}(X)_p = \mathrm{H}^n_{\mathrm{B}}(U)_p$$

in the case where A has mixed-characteristic $(0,p)$.

Assume that A is a discrete valuation of mixed-characteristic $(0,p)$. Then we have quasi-isomorphisms

$$\tau_{\leq n+1}\left(\mathbb{Z}(n-3)^{Z}_{\text{ét}}[-4]\right) \xrightarrow{\sim} \tau_{\leq n+1}\mathbf{R}i^{!}\mathbb{Z}(n-1)^{X}_{\text{ét}}$$

by Proposition 2.1 and

$$\mathbb{Z}/p(n-3)^{Z}_{\text{ét}}[-4] \xrightarrow{\sim} \tau_{\leq n+1}\mathbf{R}i^{!}\mathbb{Z}/p(n-1)^{X}_{\text{ét}}$$

by [19, p.540, Theorem 4.4.7] and [20, p.187, Remark 3.7].

Since the sequence

$$\mathrm{H}^{n-3}_{\text{ét}}\left(Z,\mathbb{Z}(n-3)\right) \xrightarrow{\times p} \mathrm{H}^{n-3}_{\text{ét}}\left(Z,\mathbb{Z}(n-3)\right) \to \mathrm{H}^{n-3}_{\text{ét}}\left(Z,\mathbb{Z}/p(n-3)\right) \to 0$$

is exact by ([2, p.774, Theorem 1.2.2], [22]) and the equation (4.3), the sequence

$$\mathrm{H}^{n+1}_{Z}\left(X_{\text{ét}},\mathbb{Z}(n-1)\right) \xrightarrow{\times p} \mathrm{H}^{n+1}_{Z}\left(X_{\text{ét}},\mathbb{Z}(n-1)\right) \to \mathrm{H}^{n+1}_{Z}\left(X_{\text{ét}},\mathbb{Z}/p(n-1)\right) \to 0$$

is exact and we have

$$\mathrm{H}^{n+2}_{Z}\left(X_{\text{ét}},\mathbb{Z}(n-1)\right)_{p} = 0.$$

Therefore we have

$$\mathrm{H}^{n+1}_{\text{ét}}\left(X,\mathbb{Z}(n-1)\right)_{p} = \mathrm{H}^{n+1}_{\text{ét}}\left(U,\mathbb{Z}(n-1)\right)_{p}$$

and the statement follows from Lemma 4.4 and Proposition 3.4. □

THEOREM 4.6. *Let A be a Dedekind domain of mixed-characteristic and $\mathfrak{X}$ an essentially smooth scheme over* $\operatorname{Spec} A$.

Then

$$\mathrm{H}^{n}_{\mathrm{B}}\left(\mathfrak{X}\right) = \bigcap_{x\in\mathfrak{X}^{(1)}} \mathrm{H}^{n}_{\mathrm{B}}\left(\mathcal{O}_{\mathfrak{X},x}\right) \tag{4.5}$$

and the sequence

$$0 \to \mathrm{H}^{n}_{\mathrm{B}}\left(\mathfrak{X}\right) \to \operatorname{Ker}\Big(\mathrm{H}^{n}_{\mathrm{B}}\left(k\left(\mathfrak{X}\right)\right) \to \prod_{x\in\mathfrak{X}^{(1)}} \mathrm{H}^{n}_{\mathrm{B}}\left(k\left(\mathcal{O}_{\mathfrak{X},\bar{x}}\right)\right)\Big) \to \bigoplus_{x\in\mathfrak{X}^{(1)}} \mathrm{H}^{n-1}_{\mathrm{B}}\left(\kappa(x)\right) \tag{4.6}$$

is exact.

PROOF. By Proposition 2.5, it suffices to prove the equation (4.5). We shall show the equation (4.5) by induction on $\dim(\mathfrak{X})$. In the case where $\dim(\mathfrak{X}) = 1$, we can prove the equations (4.5) and (4.6) by Proposition 2.5 and Corollary 4.3.

Assume that the equation (4.5) holds in the case where $\dim(\mathfrak{X}) \leq a-1$. Suppose that $\dim(\mathfrak{X}) = a$.

Let $x \in \mathfrak{X}_{(0)}$ and $i_x : Z_x \to \operatorname{Spec} \mathcal{O}_{\mathfrak{X},x}$ a regular closed subscheme of codimension 2 with open complement $j_x : U_x \to \operatorname{Spec} \mathcal{O}_{\mathfrak{X},x}$. Then we have

$$\mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{\mathfrak{X},x}) = \mathrm{H}^n_{\mathrm{B}}(U_x)$$

by Proposition 4.5. Since $(\operatorname{Spec} \mathcal{O}_{\mathfrak{X},x})^{(1)} = U_x^{(1)}$ and $\dim(U_x) = \dim(\mathfrak{X}) - 1$, we have

$$\mathrm{H}^n_{\mathrm{B}}(U_x) = \bigcap_{\substack{y \in \mathfrak{X}^{(1)} \\ x \in \overline{\{y\}}}} \mathrm{H}^n_{\mathrm{B}}(\mathcal{O}_{\mathfrak{X},y})$$

by induction hypothesis. Therefore the equation (4.5) follows from Corollary 4.3 and the sequence (4.6) is exact by Proposition 2.5. This completes the proof. □

COROLLARY 4.7. *Let R be a henselian local ring of a smooth algebra over a discrete valuation ring of mixed-characteristic $(0,p)$. Then the sequence*

$$\begin{aligned} 0 \to \mathrm{H}^{n+1}_{\text{ét}}(R, \mathbb{Z}/p^r(n)) &\to \mathrm{H}^{n+1}_{\text{ét}}(k(R), \mathbb{Z}/p^r(n))' \\ &\to \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{H}^n_{\text{ét}}(\kappa(\mathfrak{p}), \mathbb{Z}/p^r(n-1)) \end{aligned}$$

is exact for integers $n \geq 0$ and $r > 0$ where

$$\begin{aligned} &\mathrm{H}^{n+1}_{\text{ét}}(k(R), \mathbb{Z}/p^r(n))' \\ &= \operatorname{Ker}\left(\mathrm{H}^{n+1}_{\text{ét}}(k(R), \mathbb{Z}/p^r(n)) \to \prod_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{H}^{n+1}_{\text{ét}}(k(R_{\bar{\mathfrak{p}}}), \mathbb{Z}/p^r(n)) \right) \end{aligned}$$

and $R_{\bar{\mathfrak{p}}}$ is the strictly henselization of $R_{\mathfrak{p}}$.

PROOF. Let A be a regular local ring with $\dim(A) \leq 1$ and

$$R' = \varinjlim_{i \in I} A_i$$

where A_i are A-algebras essentially of finite type over A and the maps $A_i \to A_j$ are étale.

Then we have

$$\mathrm{H}^n_{\mathrm{B}}(R') = \varinjlim_{i \in I} \mathrm{H}^n_{\mathrm{B}}(A_i)$$

by [12, pp.88–89, III, Lemma 1.16]. Therefore the statement follows from Theorem 4.6 and Proposition 3.4. □

5. Étale Motivic Cohomology of Henselian Regular Local Rings

5.1. The equi-characteristic case

The objective of this subsection is to prove Theorem 1.2 in the case where R is a henselian local ring of a smooth algebra over a field of characteristic $p > 0$ (Theorem 5.3).

LEMMA 5.1. *Let A be a regular local ring over $\mathbb{F}_p$. Let t be a regular element of A, $\mathfrak{n} = (t)$ and $B = A/\mathfrak{n}$. Then we have an exact sequence*

$$0 \to (\mathfrak{n}\Omega^i_A + d\Omega^{i-1}_A)/d\Omega^{i-1}_A \to \Omega^i_A/d\Omega^{i-1}_A \xrightarrow{\bar{g}_i} \Omega^i_B/d\Omega^{i-1}_B \to 0$$

where the homomorphism $\bar{g}_i$ is induced by the natural homomorphism $g_i : \Omega^i_A \longrightarrow \Omega^i_B$.

PROOF. A can be written as a filtering inductive limit $\varinjlim_{\lambda} A_\lambda$ of finitely generated smooth algebras over $\mathbb{F}_p$ by Popescu's theorem ([14]). Let $\mathfrak{n}_\lambda$ be an ideal (t) of A_λ and $B_\lambda = A_\lambda/(t)$. Then we may assume that B_λ is 0-smooth over $\mathbb{F}_p$.

By [11, p.194, Theorem 25.2], we have a split exact sequence

$$0 \to \mathfrak{n}_\lambda/\mathfrak{n}^2_\lambda \xrightarrow{\delta_1} \Omega_{A_\lambda} \otimes_{A_\lambda} B_\lambda \xrightarrow{\alpha_1} \Omega_{B_\lambda} \to 0$$

where the maps are given by

$$\delta_1\left(\bar{a}\right) = da \otimes 1$$

and

$$\alpha_1(db \otimes \bar{c}) = \bar{c}d\bar{b}$$

for $a \in \mathfrak{n}_\lambda$ and $b, c \in A_\lambda$. Let γ_1 be a section of α_1. Then we have

$$\gamma_1(\bar{c}d\bar{b}) - db \otimes \bar{c} \in \operatorname{Im}(\delta_1)$$

for $b, c \in A_\lambda$. Therefore we have an exact sequence

$$0 \to \mathfrak{n}_\lambda/\mathfrak{n}_\lambda^2 \otimes_{B_\lambda} \Omega_{B_\lambda}^{i-1} \xrightarrow{\delta_i} \Omega_{A_\lambda}^i \otimes_{A_\lambda} B_\lambda \xrightarrow{\alpha_i} \Omega_{B_\lambda}^i \to 0$$

by [11, p.284, Theorem C.2]. Here

$$\delta_i\left(\bar{a} \otimes (d\bar{b}_1 \wedge \cdots \wedge d\bar{b}_{i-1})\right) = (da \wedge db_1 \wedge \cdots \wedge db_{i-1}) \otimes \bar{1}$$

for $a \in \mathfrak{n}_\lambda$, $b_1, \cdots, b_{i-1} \in A_\lambda$ and

$$\alpha_i\left((dc_1 \wedge \cdots \wedge dc_i) \otimes \bar{f}\right) = \bar{f}d\bar{c}_1 \wedge \cdots \wedge d\bar{c}_i$$

for $c_1, \cdots, c_i, f \in A_\lambda$. Hence the sequence

$$0 \to \mathfrak{n}/\mathfrak{n}^2 \otimes_B \Omega_B^{i-1} \xrightarrow{\delta_i} \Omega_A^i \otimes_A B \xrightarrow{\alpha_i} \Omega_B^i \to 0 \tag{5.1}$$

is exact.

On the other hand, we have

$$\operatorname{Ker}\left(\alpha_i\right) = \operatorname{Im}\left(\delta_i\right) \subset \left(d\Omega_A^{i-1} + \mathfrak{n}\Omega_A^i\right)/\mathfrak{n}\Omega_A^i$$

by the sequence (5.1). So we have

$$\operatorname{Ker}\left(g_i\right) \subset d\Omega_A^{i-1} + \mathfrak{n}\Omega_A^i.$$

Since

$$d\Omega_B^{i-1} = \operatorname{Im}\left(d\Omega_A^{i-1} \to \Omega_A^i \xrightarrow{g_i} \Omega_B^i\right),$$

we have

$$\mathrm{Ker}\,(\bar{g}_i) \subset \left(d\Omega_A^{i-1} + \mathfrak{n}\Omega_A^i\right)/d\Omega_A^{i-1}.$$

Therefore the statement follows. □

LEMMA 5.2. *Let A be a regular local ring and $X = \mathrm{Spec}\,A$. Suppose that* $\mathrm{char}(A) = p > 0$. *Then*

$$\mathrm{H}^j_{\text{ét}}\left(X, Z\Omega_X^i\right) = \mathrm{H}^j_{\text{ét}}\left(X, d\Omega_X^i\right) = 0$$

for all $j, i > 0$. Here $Z\Omega_X^i = \mathrm{Ker}\left(d : \Omega_X^i \to \Omega_X^{i+1}\right)$.

PROOF. $Z\Omega_X^i$ and $d\Omega_X^i$ are locally free $\mathcal{O}_X$-modules after twisting with Frobenius. Hence we have

$$\mathrm{H}^j_{\mathrm{Zar}}\left(X, Z\Omega_X^i\right) = \mathrm{H}^j_{\mathrm{Zar}}\left(X, d\Omega_X^i\right) = 0$$

for all $j > 0$ by [12, p.103, III, Lemma 2.15]. On the other hand,

$$\Gamma(U, Z\Omega_X^i \otimes_{\mathcal{O}_X} U) = Z\Omega_U^i \text{ and } \Gamma(U, d\Omega_X^i \otimes_{\mathcal{O}_X} U) = d\Omega_U^i$$

for U/X étale by [12, p.48, II, Proposition 1.3] and [21, p.574, Proposition 2.5]. Therefore we have

$$\mathrm{H}^j_{\text{ét}}\left(X, Z\Omega_X^i\right) = \mathrm{H}^j_{\text{ét}}\left(X, d\Omega_X^i\right) = 0$$

by [12, p.114, III, Remark 3.8]. This completes the proof. □

THEOREM 5.3. *Let A be a henselian regular local ring over $\mathbb{F}_p$ and k the residue field of A. Then the homomorphism*

$$\mathrm{H}^1_{\text{ét}}\left(\mathrm{Spec}\,A, \Omega^i_{A,\log}\right) \to \mathrm{H}^1_{\text{ét}}\left(\mathrm{Spec}\,k, \Omega^i_{k,\log}\right) \tag{5.2}$$

is an isomorphism.

Suppose that A is a henselian local of smooth algebra over a field of characteristic $p > 0$. Then we have an isomorphism

$$\mathrm{H}^{i+1}_{\text{ét}}\left(A, \mathbb{Z}/p(i)\right) \xrightarrow{\sim} \mathrm{H}^{i+1}_{\text{ét}}\left(k, \mathbb{Z}/p(i)\right)$$

by the isomorphism (5.2) and [2, p.787, §5, (12)].

Proof. Let t a regular element of A and $B = A/(t)$. Then it suffices to show that the homomorphism

$$\mathrm{H}^1_{\text{ét}}\left(\operatorname{Spec} A, \Omega^i_{A,\log}\right) \to \mathrm{H}^1_{\text{ét}}\left(\operatorname{Spec} B, \Omega^i_{B,\log}\right) \tag{5.3}$$

is an isomorphism.

We have the following commutative diagram.

$$\begin{array}{ccccccc}
\operatorname{Ker}(g_i) & \to & (\mathfrak{n}\Omega^i_A + d\Omega^{i-1}_A)/d\Omega^{i-1}_A & & & & \\
\downarrow & & \downarrow & & & & \\
\Omega^i_A & \xrightarrow{1-F} & \Omega^i_A/d\Omega^{i-1}_A & \longrightarrow & \mathrm{H}^1_{\text{ét}}\left(\operatorname{Spec} A, \Omega^i_{A,\log}\right) & \to & 0 \\
{\scriptstyle g_i}\downarrow & & \downarrow & & \downarrow & & \\
\Omega^i_B & \xrightarrow{1-F} & \Omega^i_B/d\Omega^{i-1}_B & \longrightarrow & \mathrm{H}^1_{\text{ét}}\left(\operatorname{Spec} B, \Omega^i_{B,\log}\right) & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array} \tag{5.4}$$

where F is the homomorphism which is induced by the Frobenius operator and $\mathfrak{n} = (t)$.

Then the horizontal arrows in (5.4) are exact by [21, p.576, Proposition 2.8] and Lemma 5.2. Moreover the vertical arrow in (5.4) is exact by Lemma 5.1.

If the upper homomorphism in (5.4) is surjective, we can show that the homomorphism (5.3) is injective by chasing diagram (5.4). We have surjective homomorphism

$$A \otimes (A^*)^{\otimes i} \to \Omega^i_A$$
$$a \otimes b_1 \otimes \cdots \otimes b_i \mapsto a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_i}{b_i}$$

by [1, p.122, Lemma (4.2)] and

$$F\left(a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_i}{b_i}\right) = a^p\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_i}{b_i}.$$

Since

$$\mathfrak{n}\Omega^i_A \subset \operatorname{Ker}(g_i),$$

it suffices to show that for any $a \in \mathfrak{n}$ there exists a $b \in \mathfrak{n}$ such that

$$b^p - b = a. \tag{5.5}$$

By the definition of Henselian, there exists a $b \in A \setminus A^*$ such that b is a solution of the equation (5.5) and $b+1, \cdots, b+p-1 \in A^*$ are also solutions of the equation (5.5). Hence $b \in \mathfrak{n}$ by the equation (5.5). Therefore the homomorphism (5.3) is injective. Moreover the homomorphism (5.3) is surjective by the diagram (5.4). This completes the proof. □

5.2. The mixed-characteristic case

In this subsection we show Theorem 1.2 in the case where R is mixed-characteristic (Theorem 5.6).

Let A be a mixed-characteristic henselian discrete valuation ring, K its fraction field and π a prime element of A.

We consider the following diagram of schemes.

$$\begin{array}{ccccc} \mathfrak{X} \otimes_A K & \xrightarrow{j} & \mathfrak{X} & \xleftarrow{i} & Y = \mathfrak{X} \otimes_A A/(\pi) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} A & \longleftarrow & \operatorname{Spec} A/(\pi) \end{array}$$

where the vertical arrows are smooth.

Suppose that $\operatorname{char}(A/(\pi)) = p > 0$. Then the filtration $\mathrm{U}^m \mathrm{M}_r^n$ of

$$\mathrm{M}_r^n = i^* \mathbf{R}^n j_* \mu_{p^r}^{\otimes n}$$

is defined and the structure of M_r^n is studied in [1].

In particular, the structure of M_1^n is as follows.

THEOREM 5.4 ([1, p.112, Corollary (1.4.1)]). *Let e be the absolute ramification index of K and $e' = \frac{ep}{p-1}$.*

Then the sheaf M_1^n has the following structure.

(i)

$$\mathrm{M}_1^n \,/\, \mathrm{U}^1 \mathrm{M}_1^n \simeq \Omega^n_{Y,\log} \oplus \Omega^{n-1}_{Y,\log}.$$

(ii) *If $1 \leq m < e'$ and m is prime to p,*

$$\mathrm{U}^m \mathrm{M}_1^n \,/\, \mathrm{U}^{m+1} \mathrm{M}_1^n \simeq \Omega_Y^{n-1}.$$

(iii) *If* $1 \leq m < e'$ *and* $p|m$,

$$\mathrm{U}^m \, \mathrm{M}_1^n \, / \, \mathrm{U}^{m+1} \, \mathrm{M}_1^n \simeq \mathrm{B}_1^{n-1} \oplus \mathrm{B}_1^{n-2}$$

where

$$\mathrm{B}_1^q = \mathrm{Image}\left(d : \Omega_Y^{q-1} \to \Omega_Y^q\right)$$

for an integer q.

(iv) *For* $m \geq e'$,

$$\mathrm{U}^m \, \mathrm{M}_1^n = 0.$$

[8, p.548, Corollary 1.7] and [20, pp.184–185, Theorem 3.3] are the improved versions of Theorem 5.4.

As an application, we have the following lemma.

Lemma 5.5. *Let* R *be a henselian local ring of a smooth scheme over a mixed-characteristic discrete valuation ring* A *and* π *a prime element of* A.

Then we have

$$\mathrm{H}^q_{\text{ét}}\left(R/(\pi), \mathrm{U}^1 \, \mathrm{M}_1^n\right) = 0$$

for $q \geq 1$.

Proof. We may assume that A is a henselian discrete valuation ring. Let $q \geq 1$. Since $R/(\pi)$ is a henselian local ring and $\mathrm{char}(R/(\pi)) > 0$, we have

$$\mathrm{H}^q_{\text{ét}}\left(R/(\pi), \mathrm{B}_1^{n-1}\right) = \mathrm{H}^q_{\text{ét}}\left(R/(\pi), \mathrm{B}_1^{n-2}\right) = 0$$

by Lemma 5.2. Moreover

$$\mathrm{H}^q_{\text{ét}}\left(R/(\pi), \Omega^{n-1}_{R/(\pi)}\right) = 0$$

by [12, p.103, III, Lemma 2.15] and [12, p.114, III, Remark 3.8]. Therefore the statement follows from Theorem 5.4. □

We prove Theorem 1.2 by computing the cohomology group

$$\mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), i_*\tau_{\leq n}\mathbf{R}j_*\mu_p^{\otimes n}\right)$$

as follows.

THEOREM 5.6. *Let R be a henselian local ring of a smooth scheme over a mixed-characteristic discrete valuation ring A and k the residue field of R.*

Then we have an isomorphism

$$\mathrm{H}^{n+1}_{\text{ét}}\left(R, \mathbb{Z}/m(n)\right) \xrightarrow{\sim} \mathrm{H}^{n+1}_{\text{ét}}\left(k, \mathbb{Z}/m(n)\right)$$

for any positive integer m.

PROOF. Let π be a prime element of A and $i : \operatorname{Spec} R/(\pi) \to \operatorname{Spec} R$ the closed subscheme with open complement $j : \operatorname{Spec} R[\pi^{-1}] \to \operatorname{Spec} R$.

Then the restriction map

$$\mathrm{H}^{n+1}_{\text{ét}}\left(R, \mathbb{Z}/m(n)\right) \xrightarrow{\sim} \mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), i^*\mathbb{Z}/m(n)\right)$$

is an isomorphism by [2, p.777, The proof of Proposition 2.2.b)]. Therefore it suffices to show that the homomorphism

$$\mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), i^*\mathbb{Z}/m(n)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), \mathbb{Z}/m(n)\right) \tag{5.6}$$

is an isomorphism in the case where $m = \operatorname{char}(k) = p > 0$ by Theorem 5.3.

We consider the cohomology group $\mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), i_*\tau_{\leq n}\mathbf{R}j_*\mu_p^{\otimes n}\right)$.

We have the spectral sequence

$$\mathrm{H}^{s}_{\text{ét}}\left(R/(\pi), \mathcal{H}^t\left(i^*\tau_{\leq n}\mathbf{R}j_*\mu_p^{\otimes n}\right)\right) \Rightarrow \mathbb{R}^{s+t}\Gamma_{\text{ét}}\left(i^*\tau_{\leq n}\mathbf{R}j_*\mu_p^{\otimes n}\right) \tag{5.7}$$

and

$$\mathbb{R}^{n+1}\Gamma_{\text{ét}}\left(i^*\tau_{\leq n}\mathbf{R}j_*\mu_p^{\otimes n}\right) = \mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), i^*\tau_{\leq n}\mathbf{R}j_*\mu_p^{\otimes n}\right)$$

where $\mathbb{R}^*\Gamma_{\text{ét}}$ is the right hyper-derived functor of the global sections functor $\Gamma_{\text{ét}}$ from $\mathbb{S}_{(\operatorname{Spec} R/(\pi))_{\text{ét}}}$.

Since p-cohomological dimension of k is at most 1,

$$\mathrm{H}^{s}_{\text{ét}}\left(R/(\pi), i^*\mathbf{R}^t j_*\mu_p^{\otimes n}\right) = 0 \tag{5.8}$$

for $s \geq 2$ by [2, p.777, The proof of Proposition 2.2.b)]. Hence we have

$$\mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), \tau_{\leq n}\left(i^{*}\mathbf{R}j_{*}\mu_{p}^{\otimes n}\right)\right) = \mathrm{H}^{1}_{\text{ét}}\left(R/(\pi), i^{*}\mathbf{R}^{n}j_{*}\mu_{p}^{\otimes n}\right)$$

by the spectral sequence (5.7) and

$$\mathrm{H}^{1}_{\text{ét}}\left(R/(\pi), i^{*}\mathbf{R}^{n}j_{*}\mu_{p}^{\otimes n}\right) = \mathrm{H}^{1}_{\text{ét}}\left(R/(\pi), \mathrm{M}^{n}_{1} / \mathrm{U}^{1}\,\mathrm{M}^{n}_{1}\right)$$

by Lemma 5.5. Therefore we have

$$\begin{aligned} &\mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), \tau_{\leq n}\left(i^{*}\mathbf{R}j_{*}\mu_{p}^{\otimes n}\right)\right) \\ &= \mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), \mathbb{Z}/p(n)\right) \oplus \mathrm{H}^{n}_{\text{ét}}\left(R/(\pi), \mathbb{Z}/p(n-1)\right) \end{aligned} \tag{5.9}$$

by Theorem 5.4 (i). On the other hand, the homomorphism

$$\mathrm{H}^{n}_{\text{ét}}\left(R/(\pi), i^{*}\tau_{\leq n}\mathbf{R}j_{*}\mu_{p}^{\otimes n}\right) \to \mathrm{H}^{0}_{\text{ét}}\left(R/(\pi), i^{*}\mathbf{R}^{n}j_{*}\mu_{p}^{\otimes n}\right)$$

is surjective by the spectral sequence (5.7) and the equation (5.8). Moreover the homomorphism

$$\mathrm{H}^{0}_{\text{ét}}\left(R/(\pi), i^{*}\mathbf{R}^{n}j_{*}\mu_{p}^{\otimes n}\right) \to \mathrm{H}^{0}_{\text{ét}}\left(R/(\pi), \mathrm{M}^{n}_{1} / \mathrm{U}^{1}\,\mathrm{M}^{n}_{1}\right)$$

is surjective by Lemma 5.5. Therefore the homomorphism

$$\mathrm{H}^{n}_{\text{ét}}\left(R/(\pi), i^{*}\tau_{\leq n}\mathbf{R}j_{*}\mu_{p}^{\otimes n}\right) \to \mathrm{H}^{n-1}_{\text{ét}}\left(R/(\pi), \mathbb{Z}/p(n-1)\right)$$

is surjective and the sequence

$$\begin{aligned} 0 \to \mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), i^{*}\mathbb{Z}/p(n)\right) &\to \mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), i^{*}\tau_{\leq n}\mathbf{R}j_{*}\mu_{p}^{\otimes n}\right) \\ &\to \mathrm{H}^{n}_{\text{ét}}\left(R/(\pi), \mathbb{Z}/p(n-1)\right) \to 0 \end{aligned} \tag{5.10}$$

is exact by the distinguished triangle

$$\cdots \to i^{*}\mathbb{Z}/p(n)_{\text{ét}} \to i^{*}\tau_{\leq n}\mathbf{R}j_{*}\mathbb{Z}/p(n)_{\text{ét}} \to \mathbb{Z}/p(n)_{\text{ét}} \to \cdots .$$

Therefore the homomorphism (5.6) is an isomorphism by (5.9) and (5.10). This completes the proof. □

6. Local-Global Principle

PROPOSITION 6.1. *Let A be a discrete valuation ring of mixed characteristic $(0,p)$ and π a prime element of A. Let R be a henselian local ring of a smooth algebra over A. Then the map*

$$(6.1)\qquad \begin{aligned}&\mathrm{H}^{n+1}_{\text{ét}}\left(k(R), \mu_{p^r}^{\otimes n}\right)\\ &\to \mathrm{H}^{n+1}_{\text{ét}}\left(k(\tilde{R_{(\pi)}}), \mu_{p^r}^{\otimes n}\right) \oplus \bigoplus_{\mathfrak{p}\in(\operatorname{Spec} R)^{(1)}\setminus(\pi)} \mathrm{H}^{n+2}_{\mathfrak{p}}\left((\tilde{R}_{\mathfrak{p}})_{\text{ét}}, \mu_{p^r}^{\otimes n}\right)\end{aligned}$$

is injective where $\tilde{R}_{\mathfrak{p}}$ is the henselization of R at $\mathfrak{p}$.

PROOF. By Theorem 4.6, it suffices to show that the homomorphism

$$(6.2)\qquad \mathrm{H}^{n+1}_{\text{ét}}\left(R, \mathbb{Z}/p^r(n)\right) \to \mathrm{H}^{n+1}_{\text{ét}}\left(\tilde{R_{(\pi)}}, \mathbb{Z}/p^r(n)\right)$$

is injective.

We consider the commutative diagram

$$(6.3)\qquad \begin{array}{ccc} \mathrm{H}^{n+1}_{\text{ét}}\left(R, \mathbb{Z}/p^r(n)\right) & \xrightarrow{(6.2)} & \mathrm{H}^{n+1}_{\text{ét}}\left(\tilde{R_{(\pi)}}, \mathbb{Z}/p^r(n)\right) \\ \downarrow & & \downarrow \\ \mathrm{H}^{n+1}_{\text{ét}}\left(R/(\pi), \mathbb{Z}/p^r(n)\right) & \longrightarrow & \mathrm{H}^{n+1}_{\text{ét}}\left(\tilde{R_{(\pi)}}/(\pi), \mathbb{Z}/p^r(n)\right). \end{array}$$

Then the left map in the diagram (6.3) is an isomorphism by Theorem 5.6. Moreover the lower map in the diagram (6.3) is injective by Theorem 4.2 and $R_{(\pi)}/(\pi) = \tilde{R_{(\pi)}}/(\pi)$. Therefore the homomorphism (6.2) is injective. This completes the proof. □

We review the K-theoretic fact before we prove the main result of this section (Theorem 6.3).

For a field F and an integer $n \geq 1$, $\mathrm{K}_n^M(F)$ denotes the n-th Milnor K-group of F. Then we have the following fact:

LEMMA 6.2 (cf.[7, Lemma 2.1]). *Let $v_1, \cdots, v_s$ be a finite collection of independent discrete valuations on a field F of characteristic* 0*. Denote by*

F_i the henselization of F at v_i for each i. Let $r \geq 1$ be an integer. Then for every $n \geq 1$, the natural map

$$\mathrm{K}_n^M(F)/r \to \bigoplus_i \mathrm{K}_n^M(F_i)/r$$

is surjective.

Proof. Since the sequence

$$\mathrm{K}_n^M(F)/r \xrightarrow{\times r'} \mathrm{K}_n^M(F)/rr' \to \mathrm{K}_n^M(F)/r' \to 0$$

is exact, it suffices to show the statement in the case where r is a prime number p.

Let

$$\mathrm{U}_{F_i}^m = \{x \in F_i, v_i(1-x) \geq m\}$$

for $m \geq 1$. If

$$\mathrm{U}_{F_i}^N \subset (F_i^*)^p$$

for any i and sufficient large N, we can show the statement by the weak approximation property.

Let $\kappa(v_i)$ be the residue field of v_i. In the case where $(\mathrm{char}(\kappa(v_i)), p) = 1$, we have

$$\mathrm{U}_{F_i}^1 \subset (F_i^*)^p.$$

In the case where $\mathrm{char}(\kappa(v_i)) = p$, we have

$$\mathrm{U}_{F_i}^m \subset (F_i^*)^p$$

for $m > \frac{v_i(p)\cdot p}{p-1}$ (cf. [1, p.124, Lemma (5.1)]). This completes the proof. $\square$

Theorem 6.3. *Let A be an excellent henselian discrete valuation ring of mixed characteristic $(0,p)$ and π a prime element of A. Let $\mathfrak{X}$ be a connected proper smooth curve over* $\mathrm{Spec}\, A$*, K the fraction field of $\mathfrak{X}$ and $K_{(\eta)}$ the fraction field of the henselization of $\mathcal{O}_{\mathfrak{X},\eta}$.*

Then

$$\begin{aligned} &\mathrm{H}^n_{\text{ét}}(K, \mu_{p^r}^{\otimes(n-1)}) \\ &\to \bigoplus_{\eta \in Y^{(0)}} \mathrm{H}^n_{\text{ét}}(K_{(\eta)}, \mu_{p^r}^{\otimes(n-1)}) \oplus \bigoplus_{x \in \mathfrak{X}^{(1)} \setminus Y^{(0)}} \mathrm{H}^{n-1}_{\text{ét}}(\kappa(x), \mu_{p^r}^{\otimes(n-2)}) \end{aligned}$$

is injective for $n \geq 2$.

Proof. The proof of the statement is same as [7, Theorem 2.5]. The statement follows from Proposition 6.1 and Lemma 6.2. □

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