J. Math. Sci. Univ. Tokyo
26 (2019), 361–389.

Linearization of Quotient Families

By Shigeru TAKAMURA

Abstract. Degeneration of Riemann surfaces is a subject studied by many researchers from various viewpoints — our viewpoint herein is the mixture of algebro-geometric one and topological one. In the present paper, motivated by degenerations of Riemann surfaces, we take the next step towards working in a wider context: after introducing the notion of linear quotient family, we show a linear approximation theorem. Consider a proper submersion between manifolds on which a Lie group (or a discrete group, a finite group) acts equivariantly and properly such that every stabilizer is finite. We show that the quotient of this submersion under the group action is locally orbi-diffeomorphic to a linear quotient family (*Linearization Theorem*). This has an application to universal families over various moduli spaces (e.g. of Riemann surfaces), enabling us to determine the configuration of singular fibers in universal families and describe how they crash, simply by means of linear algebra and group action.

1. Introduction

The aim of this paper is two-fold: first to introduce linear quotient families and second to show linearization theorem in terms of them. The notion of *linear quotient family* is, besides complex geometry, defined in algebraic geometry, differential topology, and even in topology. It is related to group actions, representations of groups, subspace arrangements (e.g. hyperplane arrangements), combinatorics, quotient singularities, vector bundles, etc. Many important families (e.g. the universal families over moduli spaces of Riemann surfaces, and the universal families of Abelian varieties over Shimura varieties) are locally approximated by linear quotient families (*Linearization Theorem*), where by universal families, we mean coarse ones (which are different from fine universal families in the sense of

2010 *Mathematics Subject Classification.* Primary 14D05, 14H37, 30F30; Secondary 14H15.
Key words: Riemann surface, complex variety, degeneration, monodromy, group action, representation, moduli space, universal family.

Grothendieck). The description of complicated families thus reduces to linear ones, which are very tractable and computable in terms of linear algebra and group action. This result enables to determine the topological types of various families, and moreover, brings their "visual description" (as in Figure 2). Our strategy to describe various families in complex geometry is divided into two steps: 1. Locally linearly approximate them by linear quotient families. 2. Consider the moduli problem of the complex structures on linear quotient families (note: a linear quotient family is *a priori* holomorphic such that the complex structures on its generic fibers are all equal, i.e. "constant moduli"). This paper provides the fundamental notions and tools for 1. (2. requires the theory of higher-order quotient families and will be discussed elsewhere).

We begin with motivation. The prototype of a linear quotient family is a degeneration of Riemann surfaces, which has been intensively studied by many researchers (see surveys [AsEn], [AsKo]): a *degeneration of Riemann surfaces of genus* g is a proper surjective holomorphic map $\eta : M \to \Delta$ from a complex surface M to a disk $\Delta := \{s \in \mathbb{C} : |s| < r\}$ (possibly $r = \infty$) such that $\eta^{-1}(s)$ for $s \neq 0$ is a Riemann surface of genus g while $\eta^{-1}(0)$ is singular. Here M may be assumed to be smooth by resolving its singularities.

Example 1.1. Let Y be a Riemann surface and $f : Y \to Y$ be a periodic automorphism of order l, that is, $f^l = 1$. Then $F : Y \times \mathbb{C} \to Y \times \mathbb{C}$ defined by $(y, t) \mapsto (f(y), e^{-2\pi \mathrm{i}/l} t)$ is also a periodic automorphism of order l, where the negative sign of the exponent in $e^{-2\pi \mathrm{i}/l}$ is chosen to ensure that the monodromy of the subsequently-constructed degeneration is f. The cyclic group of order l generated by F is denoted by $\mathbb{Z}_l$. The holomorphic map $\Phi : Y \times \mathbb{C} \to \mathbb{C}$ given by $\Phi(y, t) = t^l$ is $\mathbb{Z}_l$-invariant, so a holomorphic map $\eta := \overline{\Phi} : (Y \times \mathbb{C})/\mathbb{Z}_l \to \mathbb{C}$ is defined. Here $\eta^{-1}(s)$ $(s \neq 0)$ is Y (smooth), while $\eta^{-1}(0) = Y/\mathbb{Z}_l$ is singular — non-reduced with multiplicity l.

The *monodromy* of η measures how the total space $(Y \times \mathbb{C})/\mathbb{Z}_l$ is twisted around the singular fiber of η. In the above example, it is f and of finite order. There is a degeneration with monodromy of infinite order:

Example 1.2. The singular fiber of a Lefschetz fibration over a disk Δ is a Riemann surface with one *node* — which results from pinching a simple closed curve on a smooth fiber. The monodromy of the Lefschetz fibration is the right Dehn twist along this curve, and its order is infinite.

Roughly speaking: *The degenerations of Riemann surfaces are families corresponding to cyclic groups.* What kind of families correspond to more general finite groups? This is the starting point of our work. The first step is the following observation: In Example 1.1, $t \mapsto e^{-2\pi \mathrm{i}/l}t$ may be regarded as a 1-dimensional representation of the cyclic group $\mathbb{Z}_l$. The degeneration of Riemann surfaces thus results from a cyclic group action on a Riemann surface together with a 1-dimensional representation of the cyclic group. We ask:

> *What kind of families correspond to actions and representations of more general finite groups?*

Such families are linear quotient families:

Definition 1.3. Suppose that a finite group G acts on a complex analytic variety Y (possibly with singularities) holomorphically. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation, via which G acts on $\mathbb{C}^n$ as $t \mapsto \rho(g)t$, $g \in G$, accordingly G acts on $Y \times \mathbb{C}^n$ as $(y,t) \mapsto (gy, \rho(g)t)$. The projection $\mathrm{pr} : Y \times \mathbb{C}^n \to \mathbb{C}^n$, $(y,t) \mapsto t$ is then compatible with the G-actions on $Y \times \mathbb{C}^n$ and $\mathbb{C}^n$, that is, $\mathrm{pr}\big(g(y,t)\big) = g\,\mathrm{pr}(y,t)$. Thus pr descends to a holomorphic map $\eta := \overline{\mathrm{pr}} : (Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$, which is called a *linear quotient family* of Y associated with ρ.

Remark 1.4. Linear quotient families are similarly defined in other categories — algebro-geometric, smooth, topological: for which Y is an algebraic variety, a smooth manifold, or a topological space and the G-action is algebraic, smooth, or continuous. We also point out that the idea of linearization of degenerations already appeared in the theory of splitting deformations developed in [Ta1].

We give several methods to construct linear quotient families:

(Co)homological construction Suppose that a finite group G acts on a complex analytic variety Y. Then G naturally acts on the homology group $H_i(Y,\mathbb{C})$ and the cohomology group $H^i(Y,\mathbb{C})$. We thus obtain representations $G \to GL(H_i(Y,\mathbb{C}))$ and $G \to GL(H^i(Y,\mathbb{C}))$, from which we obtain linear quotient families:

$$\begin{cases} \textit{homological quotient family} \quad (Y \times H_i(Y,\mathbb{C}))/G \to H_i(Y,\mathbb{C})/G, \\ \textit{cohomological quotient family} \quad (Y \times H^i(Y,\mathbb{C}))/G \to H^i(Y,\mathbb{C})/G. \end{cases}$$

Canonical families Suppose that a finite group G acts on a *nonsingular* algebraic variety Y. Let $K := \det(\Omega^1)$ be the canonical bundle on Y. Then G naturally acts on $H^0(Y, K^{\otimes i})$ $(i = 1, 2, \dots)$, so a representation $G \to GL(H^0(Y, K^{\otimes i}))$ is obtained, accordingly a linear quotient family $\eta_i : (Y \times H^0(Y, K^{\otimes i}))/G \to H^0(Y, K^{\otimes i})/G$ is obtained. If the automorphism group $\mathrm{Aut}(Y)$ of Y is finite, then taking it as G, this is called the *ith canonical linear quotient family* (or simply *canonical family*) associated with the *ith canonical representation* of $\mathrm{Aut}(Y)$.

Cabling construction Let P be a regular polyhedron. Thickening its edges yields a real surface Y (the *cable surface* of P), on which the polyhedral group $G := \mathrm{Aut}(P)$ acts. See Figure 1. By Kerckhoff's theorem, there exists a complex structure on Y for which the G-action is holomorphic. Now Y is a Riemann surface on which G acts. To each representation $G \to GL_n(\mathbb{C})$, a linear quotient family $(Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$ is thus associated. For examples, see [HiTa1], [HiTa2].

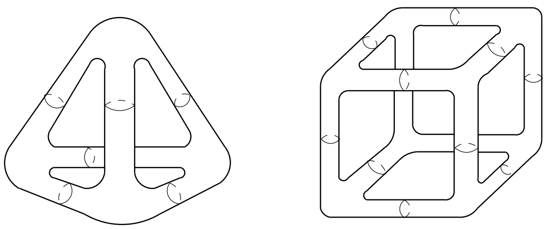

Fig. 1. The cable surfaces of the tetrahedron and the hexahedron.

Description of fibers

Each fiber $\eta^{-1}(s)$ of a linear quotient family $\eta : (Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$ (where Y may be singular) is described by the quotient fiber theorem (Theorem 2.1): *Take a lift $\widetilde{s} \in \mathbb{C}^n$ of $s \in \mathbb{C}^n/G$ and let $H_{\widetilde{s}}$ be its stabilizer for the G-action on $\mathbb{C}^n$, then $\eta^{-1}(s) = Y/H_{\widetilde{s}}$.* In particular if $s = \overline{0}$ (i.e. $\widetilde{s} = 0$) then $\eta^{-1}(\overline{0}) = Y/G$, which is called the *crystal fiber* of η.

Definition 1.5. For a G-stable (i.e. G-invariant) open neighborhood B of 0 in $\mathbb{C}^n$, replacing $\mathbb{C}^n$ with B in the construction of linear quotient family yields a family $(Y \times B)/G \to B/G$, which is called a *crystal fiber neighborhood* of $(Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$. (Sometimes it is also called a linear quotient family.)

Remark 1.6. "Fiber neighborhoods" are more generally defined. Let $f : V \to W$ be a family, that is, a proper flat surjective holomorphic map between complex analytic varieties V and W. For a point $w \in W$, take an open neighborhood U and set $U' := f^{-1}(U)$. Then the restriction $f|_{U'} : U' \to U$ is a *fiber neighborhood* of $f : V \to W$ around the fiber $X := f^{-1}(w)$. Note: The *fiber germ* of X in $f : V \to W$ is the equivalence class of fiber neighborhoods around X.

Let $\eta : (Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$ be a linear quotient family, say, associated with a representation $\rho : G \to GL_n(\mathbb{C})$. *To simplify discussion, in what follows the G-action on Y is assumed to be effective* (while that on $\mathbb{C}^n$ via ρ may not be). If $H_{\tilde{s}} = \{1\}$, the fiber $\eta^{-1}(s) = Y$ is called a *pure fiber*. If $H_{\tilde{s}} \neq \{1\}$, the fiber $\eta^{-1}(s) = Y/H_{\tilde{s}}\,(\neq Y)$ is called a *kaleido fiber* (named after kaleidoscope: in one direction (a slice of the total space $(Y \times \mathbb{C}^n)/G$), this fiber may be a smooth fiber — the slice is a family of smooth fibers —, but in another direction, it may be a degenerate fiber). We hesitate to use the term "singular fiber", because if Y itself is singular then in general all fibers of η are singular. We also avoid the term "special fiber" instead of kaleido fiber, because possibly $H_{\tilde{s}} \neq \{1\}$ for *every* $s \in \mathbb{C}^n/G$, in which case all fibers are kaleido fibers (not special at all).

The *covering multiplicity* of a fiber $\eta^{-1}(s) = Y/H_{\tilde{s}}$ is defined as the covering degree of the quotient map $Y \to Y/H_{\tilde{s}}$, which is equal to the order $|H_{\tilde{s}}|$. In drawing the figure of a *kaleido* fiber, we attach this number on it (for a pure fiber, this number is always 1). Covering multiplicity is viewed as a generalization of "algebro-geometric" multiplicity from the viewpoint of a covering map; note that for a degenerating family of algebraic varieties, the multiplicity of an irreducible component of a singular fiber is algebraically defined, while the covering multiplicity is more geometrically defined as "covering degree". Note: If $n \neq 1$, the fiber $\eta^{-1}(s)$ is not of codimension 1 in $(Y \times \mathbb{C}^n)/G$, thus *not* a divisor, and its algebro-geometric multiplicity is not defined.

The 'discriminant' locus $\mathrm{KL} := \{s \in \mathbb{C}^n/G : H_{\tilde{s}} \neq \{1\}\}$ is called the *kaleido locus* of η (over which the kaleido fibers lie). Its complementary domain $\mathrm{PD} := \{s \in \mathbb{C}^n/G : H_{\tilde{s}} = \{1\}\}$ is called the *pure domain* (over which the pure fibers lie); possibly $\mathrm{PD} = \emptyset$ and $\mathrm{KL} = \mathbb{C}^n/G$. Note that KL is obtained by dividing the *prekaleido locus* $\widetilde{\mathrm{KL}} := \{t \in \mathbb{C}^n : H_t \neq \{1\}\}$ by the G-action on $\mathbb{C}^n$: $\mathrm{KL} = \widetilde{\mathrm{KL}}/G$. By definition, $\widetilde{\mathrm{KL}} = \bigcup_{g \in G \setminus \{1\}} \mathrm{Fix}(g)$, where the fixed point set $\mathrm{Fix}(g)$ of the linear transformation g of $\mathbb{C}^n$ is a vector subspace of $\mathbb{C}^n$, so $\widetilde{\mathrm{KL}}$ *is a subspace arrangement and describable by linear algebra, afterward its quotient* KL *is describable by the G-action on* $\widetilde{\mathrm{KL}}$.

Singular loci of linear quotient families

For a linear quotient family $\eta : (Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$, the singular loci of its base space $\mathbb{C}^n/G$ and total space $(Y \times \mathbb{C}^n)/G$ are described in terms of stabilizers: First let $\mathcal{P} = \{H_t\}_{t \in \mathbb{C}^n}$ be the set of all *nontrivial* stabilizers for the G-action on $\mathbb{C}^n$. For each $H \in \mathcal{P}$, the locus of $\mathbb{C}^n$ consisting of $t \in \mathbb{C}^n$ such that $H = H_t$ (up to conjugation) is called the *H-prevein*. Its image under the quotient map $\mathbb{C}^n \to \mathbb{C}^n/G$ is called the *H-vein*, over which the same kaleido fibers Y/H lie. By definition,

$$\mathrm{KL} = \bigcup_{H \in \mathcal{P}} H\text{-vein}. \tag{1.1}$$

Now the singular locus of $\mathbb{C}^n/G$ consists of quotient singularities, so each stratum of its singular locus corresponds to some nontrivial stabilizer: If a nontrivial stabilizer H is *not* a pseudo-reflection group, then the H-vein is a singularity of $\mathbb{C}^n/G$, in fact it is isomorphic to the singularity of $\mathbb{C}^n/H$. If the H-action on $\mathbb{C}^n$ is a pseudo-reflection group, then the H-vein is not a singularity of $\mathbb{C}^n/G$, as $\mathbb{C}^n/H \cong \mathbb{C}^n$ by Chevalley–Shephard–Todd theorem [Che], [Hil] p.77.

Consider next the total space $(Y \times \mathbb{C}^n)/G$. For the G-action on $Y \times \mathbb{C}^n$, let $\mathcal{Q} = \{K_{(y,t)}\}_{(y,t) \in Y \times \mathbb{C}^n}$ be the set of all *nontrivial* stabilizers. For each $K \in \mathcal{Q}$, the locus of $Y \times \mathbb{C}^n$ consisting of points (y, t) such that $K = K_{(y,t)}$ (up to conjugation) is called the *K-preridge*. Its image under the quotient map $Y \times \mathbb{C}^n \to (Y \times \mathbb{C}^n)/G$ is called the *K-ridge*. Suppose now that Y is a complex *manifold*; then $Y \times \mathbb{C}^n$ is also a complex manifold. Recall that any quotient singularity (a germ of the quotient of a complex manifold under a finite group action) is isomorphic to the germ of a quotient $\mathbb{C}^m/F$ for some finite subgroup F of $GL_m(\mathbb{C})$ ([BeRi] p.5 Proposition 1.3). Thus: *As long as we are concerned with quotient singularities, a finite group action on a manifold may be locally regarded as a linear action.*

We return to our situation. If the action of K on $Y \times \mathbb{C}^n$ is, around the K-preridge, *not* a pseudo-reflection group, then the K-ridge is a singularity of $(Y \times \mathbb{C}^n)/G$, which is isomorphic to the singularity of $(Y \times \mathbb{C}^n)/K$. Otherwise the K-ridge is not a singularity of $(Y \times \mathbb{C}^n)/G$.

Example 1.7. The plane algebraic curve $Y : xy^3 + yz^3 + zx^3 = 0$ in $\mathbb{CP}^2$ is a Riemann surface of genus 3, called the *Klein curve*. Its automorphism group is $PSL_2(\mathbb{F}_7)$ and has an irreducible 3-dimensional representation $PSL_2(\mathbb{F}_7) \to GL_3(\mathbb{C})$ to which the associated linear quotient family $\eta : (Y \times \mathbb{C}^3)/PSL_2(\mathbb{F}_7) \to \mathbb{C}^3/PSL_2(\mathbb{F}_7)$ is as illustrated in Figure 2 (this family will be described in detail in a joint work with K. Sasaki). Its kaleido locus in $\mathbb{C}^3/PSL_2(\mathbb{F}_7)$ consists of two 'nontrivial' veins ($\mathbb{Z}_3$- and $\mathbb{Z}_4$-veins) and one 'trivial' vein consisting only of $\overline{0}$ (the $PSL_2(\mathbb{F}_7)$-vein). The singular

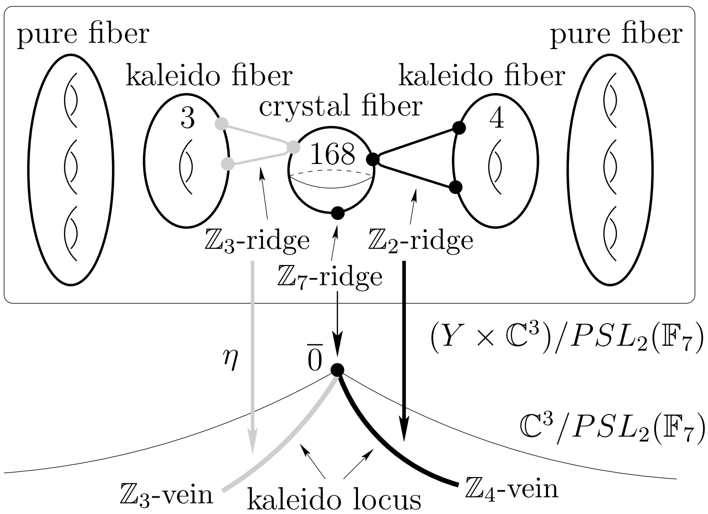

Fig. 2. The linear quotient family constructed from the Klein curve: The kaleido fiber over $\overline{0}$ is called the *crystal fiber*. The positive integer on a kaleido/crystal fiber indicates its covering multiplicity (168 is the order of $PSL_2(\mathbb{F}_7)$). *Note*: The $\mathbb{Z}_2$-ridge lies over the $\mathbb{Z}_4$-vein.

locus of $(Y \times \mathbb{C}^3)/PSL_2(\mathbb{F}_7)$ consists of three ridges: $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}_7$-ridges, where each of the first two is isomorphic to a smooth complex line, while the last one is a single point.

(Non-linear) quotient families

A linear quotient family is actually a special case of a more general quotient family, which is still a special case of a more general concept "quota" — defined in many categories (complex analytic, algebraic geometric, topological, etc.). We here introduce its "simplified" version. Let Γ be a complex Lie group, a discrete group, or a finite group. Let $\varphi : S \to T$ be a Γ-*equivariant* holomorphic map between complex analytic varieties, i.e. $\varphi(\gamma s) = \gamma\varphi(s)$ for any $\gamma \in \Gamma$, $s \in S$, such that the Γ-actions on S and T are holomorphic and proper. Here recall that a Γ-action on a space X is *proper* if the map $\Gamma \times X \to X \times X$, $(\gamma, x) \mapsto (x, \gamma x)$ is proper (note: for discrete group action, properness is equivalent to proper discontinuity, while *any* finite group action is proper). Then:

(i) The properness of the Γ-actions on S and T ensures that S/Γ and T/Γ are complex analytic varieties (Holmann's theorem [Hol]). In the special case that Γ is a discrete group, this is due to H. Cartan.

(ii) The Γ-equivariance of φ implies that φ descends to a holomorphic map $\overline{\varphi} : S/\Gamma \to T/\Gamma$ between complex analytic varieties.

We say that $\overline{\varphi} : S/\Gamma \to T/\Gamma$ is a *quota* (precisely speaking, a quota is not merely a map but equipped with additional structures (vein, ridge, shelf, etc.), however we do not go into details; see [Ta3].

DEFINITION 1.8. Let Γ be a complex Lie group, a discrete group, or a finite group acting on complex analytic varieties S and T holomorphically and properly. Let $\varphi : S \to T$ be a Γ-equivariant holomorphic map.

(1) If $\varphi : S \to T$ is a family (i.e. proper, flat and surjective), then the quota $\overline{\varphi} : S/\Gamma \to T/\Gamma$ is called a *quotient family.* In the special case that S and T are complex manifolds and φ is a proper submersion, we say that $\overline{\varphi}$ is a *submersive quotient family.*

(2) For a submersive quotient family $\overline{\varphi}$, if every stabilizer of the Γ-actions on S and T is finite, then S/Γ and T/Γ are orbifolds, and $\overline{\varphi}$ is called an *orbi-quotient family.*

Example 1.9. Let T_g be the Teichmüller space of marked Riemann surfaces of genus $g \geq 2$. Let $\varphi : S_g \to T_g$ be the universal (i.e. tautological) family: for each point $[(X, \mu)] \in T_g$, its fiber is a marked Riemann surface (X, μ) (see [Nag] p.322 §5.2.2), where μ is a marking of X, that is, an orientation-preserving homeomorphism from X to a fixed closed oriented surface Σ_g of genus g. The mapping class group Γ_g acts on both S_g and T_g holomorphically and properly discontinuously (so, every stabilizer is finite), and $\varphi : S_g \to T_g$ is a Γ_g-equivariant proper submersion. The orbi-quotient family $\overline{\varphi} : S_g/\Gamma_g \to T_g/\Gamma_g$ is nothing but the (coarse) universal family $\eta : U_g \to M_g$ over the moduli space $M_g := T_g/\Gamma_g$ (e.g., see [ACG]). Here note that the coarse universal family is different from the fine universal family in the sense of Grothendieck.

Main results

Many submersive quotient families may be locally approximated by linear ones. To show this, the following plays a key role:

Linearization Lemma (Lemma 3.5). *Let $\overline{\varphi} : S/\Gamma \to T/\Gamma$ be a submersive quotient family. For a point $o \in T$, let $G := \mathrm{Stab}_o$ be its stabilizer for the Γ-action on T and set $X := \varphi^{-1}(o)$. If G is compact* (e.g. *finite*), *then there exist* (i) *a G-stable open neighborhood U of o in T and an open ball B centered at 0 in $\mathbb{C}^n$ $(n := \dim T)$ on which G acts linearly and* (ii) *G-equivariant diffeomorphisms $\varphi^{-1}(U) \xrightarrow{\cong} X \times B$ and $U \xrightarrow{\cong} B$ that make the following diagram commute:*

$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow[G\text{-equiv}]{\cong} & X \times B \\ {\scriptstyle \varphi}\big\downarrow {\scriptstyle G\text{-equiv}} & & {\scriptstyle G\text{-equiv}}\big\downarrow {\scriptstyle \mathrm{pr}} \\ U & \xrightarrow[G\text{-equiv}]{\cong} & B. \end{array} \tag{1.2}$$

Remark 1.10.

(i) The compactness of G is used in constructing G-invariant Riemannian metrics on S and T. From these metrics, G-equivariant exponential maps are defined and used in our construction.

(ii) By Ehresmann's fibration theorem, any proper submersion is locally trivial. The above lemma is considered as a G-equivariant version of this theorem.

In Linearization Lemma, if G is finite, then from the G-equivariant projection $X \times B \to B$ a linear quotient family $\overline{\mathrm{pr}} : (X \times B)/G \to B/G$ is defined, and the diagram (1.2) descends to the following commutative diagram:

$$\begin{array}{ccc} \varphi^{-1}(U)/G & \xrightarrow{\cong} & (X \times B)/G \\ \overline{\varphi}\downarrow & & \downarrow\overline{\mathrm{pr}} \\ U/G & \xrightarrow{\cong} & B/G. \end{array} \tag{1.3}$$

DEFINITION 1.11. The quotient family $\overline{\varphi} : \varphi^{-1}(U)/G \to U/G$ is said to be *orbi-diffeomorphic* to the linear quotient family $\overline{\mathrm{pr}} : (X \times B)/G \to B/G$. In this case, $\varphi^{-1}(U)/G \cong (X \times B)/G$ and $U/G \cong B/G$ are *orbi-diffeomorphisms.*

In Linearization Lemma, if moreover every stabilizer of the Γ-action on T is finite, the compactness condition is satisfied at any point of T and the following holds:

LINEARIZATION THEOREM (THEOREM 4.3). *Any orbi-quotient family* $\overline{\varphi} : S/\Gamma \to T/\Gamma$ *is locally orbi-diffeomorphic to a linear quotient family: the local quotient family* $\overline{\varphi} : \varphi^{-1}(U)/G \to U/G$ *is orbi-diffeomorphic to the linear quotient family* $\overline{\mathrm{pr}} : (X \times B)/G \to B/G$ *such that the orbi-diffeomorphism between* U/G *and* B/G *is biholomorphic.*

Comparison. Our linearization is completely different from that in Kodaira–Spencer theory of deformations of the complex structure of a complex manifold X, for which linearization is given by $H^1(X, TX)$ (see [Kod]), while in our case it is not a vector space but a linear quotient family.

Application to Riemann surfaces

The following result enables us to describe the universal family over the moduli space in terms of linear quotient families:

LINEARIZATION THEOREM OF UNIVERSAL FAMILIES (COROLLARY 4.4). *Let M_g be the moduli space of Riemann surfaces of genus $g \geq 2$. Then the universal family $U_g \to M_g$ is locally orbi-diffeomorphic to a canonical family: Around $[X] \in M_g$, it is orbi-diffeomorphic to a crystal fiber neighborhood of the 2-canonical family $\big(X \times H^0(X, K^{\otimes 2})\big)/\mathrm{Aut}(X) \to H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$.*

	Local linear approximation
T_g	$H^0(X, K^{\otimes 2})$ $(\cong H^1(X, TX))$
$S_g \to T_g$	projection $X \times H^0(X, K^{\otimes 2}) \to H^0(X, K^{\otimes 2})$

	Local linear approximation
M_g	$H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$
$U_g \to M_g$	$\big(X \times H^0(X, K^{\otimes 2})\big)/\mathrm{Aut}(X) \to H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$

REMARK 1.12. In the above linearization theorem of universal families, $H^0(X, K^{\otimes 2})$ may be replaced with $H^1(X, TX)$ via the Serre duality. Note that in terms of the Kodaira–Spencer theory, the latter is regarded as the tangent space of T_g at the point corresponding to X (with a marking). However in the practical computation to describe the 2-canonical family, $H^0(X, K^{\otimes 2})$ is advantageous. See Remark 4.5.

Thanks to the quotient fiber theorem, one may describe, by means of linear algebra and group action, the singular fibers (the kaleido fibers) of a linear quotient family and the discriminant locus (the kaleido locus) in the base space. *A powerful tool to describe the universal family is thus obtained.* Our subsequent work will apply this to give the explicit description of the universal family.

Observation. For a linear quotient family of a complex *manifold*, the complex structures of pure fibers (in this case, smooth fibers) are identical, so the image of them under the moduli map is a single point. For the 2-canonical family of a Riemann surface X, it is $[X] \in M_g$. Nevertheless the 2-canonical family contains geometric information *around* $[X]$ in M_g: without investigating the deformation of the complex structure on X, the geometric information around $[X]$ is obtained. In some sense, a 2-canonical family is a *stacky object*: although its image under the moduli map is a single point, this point 'contains' information on group action (the $\mathrm{Aut}(X)$-action on X) and representation (the 2-canonical representation of $\mathrm{Aut}(X)$) — these two depend only on the point $[X]$, *but* carry information around $[X]$ in M_g.

Acknowledgments. We would like to thank Tadashi Ashikaga for valuable discussions and comments. We also would like to thank Ryota Hirakawa, Takayuki Okuda, and Kenjiro Sasaki for fruitful discussions.

2. Quotient Fiber Theorem

Suppose that a finite group G acts on a complex analytic variety Y (holomorphically). Let $\eta : (Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$ be a linear quotient family associated with a representation. $\rho : G \to GL_n(\mathbb{C})$. Each fiber $\eta^{-1}(s)$ $(s \in \mathbb{C}^n/G)$ — a set-theoretic fiber (not a scheme-theoretic one) — is determined by the following:

THEOREM 2.1 (Quotient fiber theorem). *Let $q : \mathbb{C}^n \to \mathbb{C}^n/G$ be the quotient map and take a lift $\widetilde{s} \in q^{-1}(s)$ of s. Let $H_{\widetilde{s}}$ be the stabilizer of $\widetilde{s}$ for the G-action on $\mathbb{C}^n$ (note $H_{\widetilde{s}}\,(\subset G)$ also acts on Y). Then $\eta^{-1}(s) = Y/H_{\widetilde{s}}$.*

REMARK 2.2. The isomorphism class of $Y/H_{\widetilde{s}}$ does not depend on the choice of a lift $\widetilde{s}$ of s. Indeed if $\widetilde{s}'$ is another lift, then $H_{\widetilde{s}}$ and $H_{\widetilde{s}'}$ are conjugate in G, accordingly $Y/H_{\widetilde{s}}$ and $Y/H_{\widetilde{s}'}$ are biholomorphic.

PROOF. Consider the following commutative diagram (q' and q are quotient maps):

$$\begin{array}{ccc} Y \times \mathbb{C}^n & \xrightarrow{\quad q' \quad} & (Y \times \mathbb{C}^n)/G \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathbb{C}^n & \xrightarrow{\quad q \quad} & \mathbb{C}^n/G. \end{array} \tag{2.1}$$

The commutativity of this diagram implies $\eta^{-1}(s) = q'\mathrm{pr}^{-1}q^{-1}(s)$. Say that $q^{-1}(s) = \{\widetilde{s}_1, \widetilde{s}_2, \dots, \widetilde{s}_l\}$, then $\mathrm{pr}^{-1}q^{-1}(s) = \mathrm{pr}^{-1}(\widetilde{s}_1) \amalg \mathrm{pr}^{-1}(\widetilde{s}_2) \amalg \cdots \amalg \mathrm{pr}^{-1}(\widetilde{s}_l)$ (disjoint union), so

$$\eta^{-1}(s) = q'\Big(\mathrm{pr}^{-1}(\widetilde{s}_1) \amalg \mathrm{pr}^{-1}(\widetilde{s}_2) \amalg \cdots \amalg \mathrm{pr}^{-1}(\widetilde{s}_l)\Big).$$

Here $\mathrm{pr}^{-1}(\widetilde{s}_1), \mathrm{pr}^{-1}(\widetilde{s}_2), \dots, \mathrm{pr}^{-1}(\widetilde{s}_l)$ are mapped to each other by the G-action, hence $\eta^{-1}(s) = q'\big(\mathrm{pr}^{-1}(\widetilde{s}_i)\big)$, where i may be any of $1, 2, \dots, l$. Since $\mathrm{pr}^{-1}(\widetilde{s}_i) = Y$ we have $\eta^{-1}(s) = q'(Y) = Y/H_{\widetilde{s}_i}$, confirming the assertion. □

Recall that if $H_{\widetilde{s}} \neq \{1\}$, then $\eta^{-1}(s) = Y/H_{\widetilde{s}}$ is called a *kaleido fiber* of η.

Example 2.3. Where $\overline{0} \in \mathbb{C}^n/G$ denotes the image of $0 \in \mathbb{C}^n$, $\eta^{-1}(\overline{0}) = Y/H_0$ by the quotient fiber theorem. Here $H_0 = G$, so $\eta^{-1}(\overline{0}) = Y/G$ is a kaleido fiber. This is the most 'folded' among the kaleido fibers of η and is called the *crystal fiber* of η.

The locus of $\mathbb{C}^n/G$ over which kaleido fibers lie is the *kaleido locus* of η, denoted by KL. It is the quotient of the *prekaleido locus* $\widetilde{\mathrm{KL}} := \{t \in \mathbb{C}^n \,:\, H_t \neq \{1\}\}$ by the G-action on $\mathbb{C}^n$: $\mathrm{KL} = \widetilde{\mathrm{KL}}/G$. By definition, $\widetilde{\mathrm{KL}} = \bigcup_{g \in G \setminus \{1\}} \mathrm{Fix}(g)$. Here the determination of $\mathrm{Fix}(g)$ is simply a matter of linear algebra. Note that G permutes $\{\mathrm{Fix}(g) \,:\, g \in G \setminus \{1\}\}$: indeed $h \in G$ maps $\mathrm{Fix}(g)$ to $\mathrm{Fix}(hgh^{-1})$. Let $\mathrm{Fix}(g_1), \mathrm{Fix}(g_2), \dots, \mathrm{Fix}(g_l)$ be representatives of the orbits of this action. For each g_i, let L_{g_i} denote the subgroup of G consisting of h that maps $\mathrm{Fix}(g_i)$ to itself (note L_{g_i} contains the centralizer $C_{g_i} = \{h \in G \,:\, hg_ih^{-1} = g_i\}$). The locus KL is then given by

$$\mathrm{KL} = \bigcup_{i=1}^{l} \mathrm{Fix}(g_i)/L_{g_i}. \tag{2.2}$$

The determination of KL *is thus done simply by linear algebra and group action.*

3. Diagonalization and Linearization of Actions

In this section, unless otherwise mentioned, manifolds and maps are smooth (i.e. C^∞), *and group actions are also smooth*; complex manifolds are regarded as underlying smooth manifolds (but the results obtained here are later applied to complex analytic category). Γ *denotes a Lie group, a discrete group, or a finite group* (note: a discrete/finite group may be regarded as a 0-dimensional Lie group).

Definition 3.1. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A Γ-action on a product space $X \times V$ is *semi-diagonal* if it is of the form: for $g \in \Gamma$, $(x, v) \in X \times V \mapsto (gx, \rho_x(g)v) \in X \times V$, where $\rho_x : \Gamma \to GL(V)$ is a family of representations depending on x *smoothly* (i.e. the map $\Gamma \times X \to GL(V)$, $(g, x) \mapsto \rho_x(g)$ is smooth). If ρ_x is constant, i.e. independent of $x \in X$, then the Γ-action on $X \times V$ is called *diagonal.* (More generally for an arbitrary manifold V, these terms are used for a family of homomorphisms $\rho_x : \Gamma \to \mathrm{Aut}(V)$ to the automorphism group of V.)

Note. (i) A semi-diagonal action of Γ amounts to an action of Γ as a bundle automorphism group on the trivial bundle $X \times V \to X$ (a *bundle action* of Γ). (ii) If the Γ-action on $X \times V$ is diagonal, then the projection $\mathrm{pr} : X \times V \to V$ is Γ-equivariant, thus, if moreover Γ is a finite group, then a linear quotient family $\overline{\mathrm{pr}} : (X \times V)/\Gamma \to V/\Gamma$ is defined.

Let $\varphi : M \to N$ be a Γ-equivariant proper surjective submersion between (real or complex) manifolds M and N. The subsequent discussion works for both real and complex manifolds, and we only consider the case that M and N are complex manifolds (in the case of real manifolds, a tangent space is a real vector space, and for instance, $\mathbb{C}^n$ in Lemma 3.3 should be replaced with $\mathbb{R}^n$).

Let $K \subset \Gamma$ be the stabilizer of a point $o \in N$, that is, $K = \{g \in \Gamma : go = o\}$. Then K maps $\varphi^{-1}(o)$ to itself (indeed from the Γ-equivariance of φ, for any $x \in \varphi^{-1}(o)$ and $k \in K$ we have $\varphi(kx) = k\varphi(x) = ko = o$, thus $kx \in \varphi^{-1}(o)$). Now since Γ acts on M, Γ also acts on the tangent bundle TM ($g \in \Gamma$ acts as its differential g_*). Since K preserves $X := \varphi^{-1}(o)$, K preserves the restriction $TM|_X$.

In the sequel, suppose that K is compact. There then exists a K-invariant Riemannian metric on N (and also on M): First take an arbitrary Riemannian metric $\langle\ ,\ \rangle$ on N, then integrating it over K (with respect to the Haar measure μ of K) gives a K-invariant metric $\langle\ ,\ \rangle_N$ on N, i.e. $\langle v, w\rangle_N := \int_K \langle k_* v, k_* w\rangle d\mu(k)$ for $v, w \in T_pN$. Similarly there exists a K-invariant Riemannian metric $\langle\ ,\ \rangle_M$ on M. Note that any ε-neighborhood

U of o in N (with respect to $\langle\ ,\ \rangle_N$) is K-stable, i.e. K maps U to itself. Take sufficiently small ε so that U is a small n-dimensional open ball, where $n := \dim N$.

LEMMA 3.2. *$\varphi^{-1}(U)$ is K-stable.* (In the extreme case $\varepsilon = 0$, we have $U = o$ and $\varphi^{-1}(U) = X$.)

PROOF. Let $z \in \varphi^{-1}(U)$. From the K-equivariance, $\varphi(kz) = k\varphi(z)$ for any $k \in K$. Here $\varphi(z) \in U$ and $kU = U$, thus $\varphi(kz) \in U$, so $kz \in \varphi^{-1}(U)$. $\square$

Now we summarize our cast:

- U is a K-stable open ball centered at a fixed point o of the K-action.
- K preserves both U and $\varphi^{-1}(U)$.
- K preserves $X := \varphi^{-1}(o)$.

With respect to the K-invariant metric $\langle\ ,\ \rangle_M$, let $TM|_X \cong TX \oplus NX$ be the orthogonal decomposition of $TM|_X$ into the tangent bundle TX of X and the normal bundle NX of X in M (the orthogonal complements $(T_xX)^\perp$, $x \in X$ together form NX). Since the K-action preserves X, it preserves $TM|_X$ and also TX, accordingly it preserves the orthogonal complement NX of TX in $TM|_X$ (because the metric $\langle\ ,\ \rangle_M$ is K-invariant). The K-action thus preserves the decomposition $TM|_X \cong TX \oplus NX$.

Semi-diagonalization. *The K-action on NX is considered as a linearization of the K-action on $\varphi^{-1}(U)$*: the exponential map $\exp : NX \to M$ defined from the K-invariant metric $\langle\ ,\ \rangle_M$ is K-equivariant ([BCO] p.36) (moreover diffeomorphic around the zero section ([Kos] p.44 Theorem 2.2)). So the K-action on $\varphi^{-1}(U)$ is transformed to a bundle action on NX. Here NX is trivial: $NX = X \times \mathbb{C}^n$, because $\varphi^{-1}(U) \cong X \times U$ by Ehresmann's fibration theorem (where U is sufficiently small). The K-action on it, being a bundle action, is necessarily semi-diagonal, that is, of the form: for $k \in K$, $(x, v) \mapsto (kx, \rho_x(k)v)$, where ρ_x is a representation of K depending on $x \in X$ smoothly.

We summarize the above as follows:

Lemma 3.3. *The following hold:*

(1) *The normal bundle NX of X in M is trivial: $NX = X \times \mathbb{C}^n$, where $n = \dim N$.*

(2) *The K-action on $NX = X \times \mathbb{C}^n$ is semi-diagonal.*

(3) *Via* $\exp : NX \to M$, *a small tubular neighborhood of the zero section in NX is K-equivariantly diffeomorphic to a tubular neighborhood of X in $\varphi^{-1}(U)$.*

Now by Ehresmann's fibration theorem, the following diagram commutes (U is sufficiently small):

$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow{\;\cong\;} & X \times U \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & U. & \end{array} \tag{3.1}$$

(Note: Ehresmann's fibration theorem is irrelevant to group action, and this diagram is generally not K-equivariant.)

In what follows, we identify $\varphi^{-1}(U)$ with $X \times U$ and $\varphi : \varphi^{-1}(U) \to U$ with the projection $\mathrm{pr} : X \times U \to U$.

Diagonalization. We now have a semi-diagonal K-action on $NX = X \times \mathbb{C}^n$. Since $\varphi : M \to N$ is proper, if necessary shrinking U, we assume that a tubular neighborhood NX' of the zero section in NX is diffeomorphic to $\varphi^{-1}(U)$ under exp (see Lemma 3.3 (3)):

$$\exp : \ NX' \xrightarrow{\cong} \varphi^{-1}(U) = X \times U.$$

Note that the (semi-diagonal) K-action on NX preserves NX', because K preserves $\varphi^{-1}(U)$ and exp is K-equivariant. We will make the semi-diagonal K-action on NX' diagonal under some coordinate change.

We begin with preparation. Since $\varphi : \varphi^{-1}(U) = X \times U \to U$ is a projection, for $u \in U$ we have $\varphi^{-1}(u) = X \times \{u\}$ ('vertical' in $X \times U$; see Figure 3). On the other hand, the composition $\varphi' := \varphi \circ \exp : NX' \to U$ is generally *not* a projection: for $u \in U$, the preimage $(\varphi')^{-1}(u) = \exp^{-1}(X \times \{u\})$ is not necessarily vertical in $NX' \subset X \times \mathbb{C}^n$ — not of the form $X \times \{v\}$.

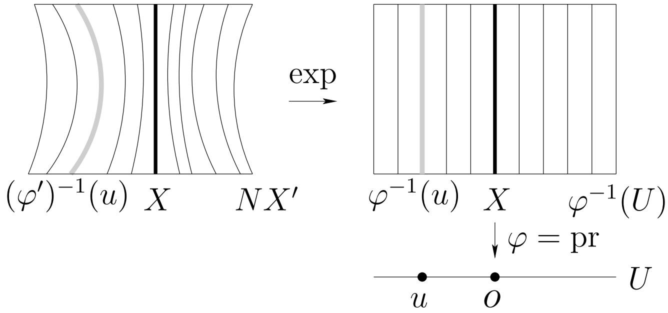

Fig. 3. $(\varphi')^{-1}(u)$ is not vertical.

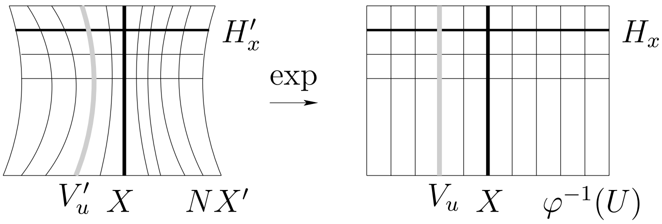

Fig. 4. Leaves.

However composing another diffeomorphism with φ', we will make it into a projection and moreover make the K-action on NX' diagonal.

We introduce notation as in Figure 4:

- On $\varphi^{-1}(U) = X \times U$: For each $x \in X$, set $H_x := \{x\} \times U$ (a horizontal leaf) and for each $u \in U$, set $V_u := X \times \{u\}$ (a vertical leaf).
- On NX' ($\subset X \times \mathbb{C}^n$): For each $x \in X$, set $H'_x := (\{x\} \times \mathbb{C}^n) \cap NX'$ (a horizontal leaf) and for each $u \in U$, set $V'_u := \exp^{-1}(V_u)$ (not necessarily a vertical leaf).

Besides, for each $x \in X$ consider a leaf $L_x := \exp(H'_x)$ in $X \times U$. Here note that X is a compact manifold (because $\varphi : M \to N$ is a proper submersion). We claim that we may take a sufficiently small tubular neighborhood of X in $\varphi^{-1}(U) = X \times U$ such that L_x intersects transversally each vertical leaf V_u at one point (where u is close to o, i.e. V_u is close to $X = X \times \{o\}$).

To see this, note first that since $\exp : TN|_X \to N$ is the identity map on X, around each $p \in X$ it 'preserves' the decomposition of T_pN into the horizontal and vertical subspaces. Consequently the leaf L_p is transversal to V_o. Since "transversality" is an open condition, L_p is transversal to any V_u (at one point) for u close to o, say, in an open ball O_p centered at p in $\varphi^{-1}(U)$. Consider an open cover $X \subset \bigcup_{p\in X} O_p$. As X is compact, this admits a finite subcover, say $X \subset \bigcup_{i=1}^{l} O_{p_i}$. Set $r := \mathrm{dist}\big(X, \partial(\bigcup_{i=1}^{l} O_{p_i})\big)$, the distance between X and the boundary $\partial(\bigcup_{i=1}^{l} O_{p_i})$ of $\bigcup_{i=1}^{l} O_{p_i}$, that is, the infimum of the distances between the points of X and $\partial(\bigcup_{i=1}^{l} O_{p_i})$ (note that $r > 0$ and that since both X and $\partial(\bigcup_{i=1}^{l} O_{p_i})$ are compact, "infimum" is actually "minimum"). For a positive number ε such that $\varepsilon < r$, the ε-tubular neighborhood of X in $\varphi^{-1}(U)$ is contained in this finite open cover and satisfies the desired property (note: if X is non-compact, this is generally not the case, as possibly $\mathrm{dist}\big(X, \partial(\bigcup_{p\in X} O_p)\big) = 0$ — the radius of O_p may approach 0 as $p \in X$ goes to an 'infinity' of X). We denote the ε-tubular neighborhood by P. See Figure 5.

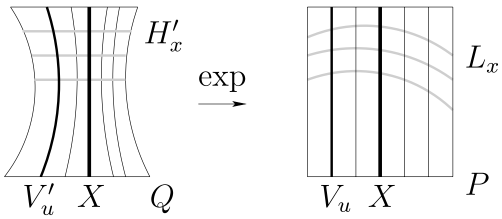

Fig. 5.

In the tubular neighborhood $Q := \exp^{-1}(P)$ of X in NX',

$$(*) \qquad V'_u \text{ intersects each } H'_x \text{ transversally at one point.}$$

For simplicity, rewrite Q as NX'. We construct a diffeomorphism defined on NX' that 'straightens' the leaves V'_u ($u \in U$) in NX'. First fix $x_0 \in X$. From $(*)$, we may define a projection $\Pi : NX' \to H'_{x_0}$ by transforming points of NX' to points of H'_{x_0} along the leaves V'_u as illustrated in Figure 6. Write $\Pi : p = (x, y) \mapsto p' = (x_0, \pi(p))$. Set $NX'' := X \times H'_{x_0}$ and define a diffeomorphism $\psi : NX' \to NX''$ by $\psi :\ p = (x, y) \mapsto q = (x, \pi(p))$, which straightens the leaves V'_u as illustrated in Figure 7.

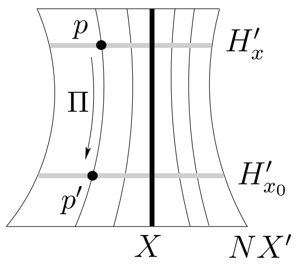

Fig. 6. The projection of p along H'_{x_0}.

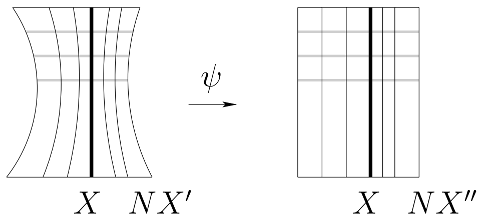

Fig. 7. Straightening.

The composition of the diffeomorphism $\psi^{-1} : NX'' \to NX'$ with $\varphi' := \varphi \circ \exp : NX' \to U$ yields a projection $\Phi := \varphi' \circ \psi^{-1} : NX'' = X \times H_{x_0} \to U$ (note that each fiber is straight — vertical). Here Φ may not be surjective, in which case shrinking U we assume $U = H_{x_0}$, so $\Phi : NX'' = X \times U \to U$ (surjective).

Consider now the K-action on NX'' induced via ψ from the K-action on NX'. Since $\exp : NX' \to U$ is K-equivariant, the map $\Phi : NX'' \to U$ is K-equivariant. Note also that the K-action on NX'' is semi-diagonal (because ψ maps each horizontal leaf to another and the K-action on NX' ($\subset X \times \mathbb{C}^n$) is semi-diagonal). Write this action as

$$(x, u) \longmapsto (kx, \rho_x(k)u), \quad k \in K, \ (x, u) \in X \times U,$$

where ρ_x is a representation of K depending on $x \in X$. Actually the K-equivariance of $\Phi : NX'' \to U$ implies that ρ_x does *not* depend on x. In fact, from $\Phi(k(x, u)) = k\Phi(x, u)$ we have $\Phi\big(kx, \rho_x(k)u\big) = k\Phi(x, u)$, that is,

$\rho_x(k)u = ku$ (recall Φ is a projection). Thus ρ_x is independent of x, and this K-action is diagonal. We summarize these results as follows:

LEMMA 3.4 (Diagonalization lemma). *Let Γ be a Lie group and $\varphi : M \to N$ be a Γ-equivariant proper surjective submersion. Suppose that the stabilizer K of a point $o \in N$ is compact. Then there exist* (i) *a K-stable open neighborhood U of o in N and* (ii) *a K-equivariant diffeomorphism $\varphi^{-1}(U) \xrightarrow{\cong} X \times U$ (where $X := \varphi^{-1}(o)$ and the K-action on $X \times U$ is diagonal) such that the following diagram commutes:*

(3.2)
$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow{\cong} & X \times U \\ & \searrow^{\varphi} \quad \swarrow_{\mathrm{pr}} & \\ & U. & \end{array}$$

(Note: K may be replaced with any compact subgroup of G fixing o.)

Linearization. The map $\exp : T_oU \to U$ is a local diffeomorphism around o, and under this correspondence, the K-action on U becomes a linear K-action on the vector space T_oU (the *isotropy representation* of K). For a sufficiently small open neighborhood B of the origin in T_oU, if necessary shrinking U we may assume that $\exp : B \xrightarrow{\cong} U$ (diffeomorphic). Lemma 3.4 is then refined as follows:

LEMMA 3.5 (Linearization lemma). *Let Γ be a Lie group and $\varphi : M \to N$ be a Γ-equivariant proper surjective submersion. Suppose that the stabilizer K of a point $o \in N$ is compact. Then there exist* (i) *a K-stable open neighborhood U of o in N and an open ball B centered at the origin in $\mathbb{C}^n$ $(n := \dim N)$ on which K acts linearly and* (ii) *K-equivariant diffeomorphisms $U \xrightarrow{\cong} B$ and $\varphi^{-1}(U) \xrightarrow{\cong} X \times B$ (where $X := \varphi^{-1}(o)$ and the K-action on $X \times B$ is diagonal) such that the following diagram commutes:*

(3.3)
$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow{\cong} & X \times B \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}} \\ U & \xrightarrow{\cong} & B. \end{array}$$

(Note: K may be replaced with any compact subgroup of G fixing o.)

Holomorphic linearization

We next consider the complex analytic case: M and N are *complex* manifolds, Γ is a *complex* Lie group acting on them *holomorphically*, and φ is *holomorphic*. Note that if a stabilizer K is compact, K is rarely a complex Lie group: Any compact complex Lie group of dim ≥ 1 is necessarily a complex torus (Picard's theorem). Except for this special case, K is a real Lie group (its action on N cannot be holomorphic but merely smooth) or a finite group (its action on N is holomorphic). In the sequel, we consider the finite case. K is then denoted by G.

We shall show that after a *holomorphic* coordinate change, the G-action on U becomes linear. Note first that the exponential map cannot be used: recall that in the smooth case (M, N are smooth manifolds), we used a G-equivariant $\exp : T_oU \to U$ made from a G-invariant metric on U, however this is generally *not* holomorphic (even if the G-invariant metric is Kähler). The construction without using a G-invariant metric and its associated exponential map proceeds as follows: Take a G-stable open neighborhood U of o in N (note: since G is finite, we may take such a neighborhood without using a G-invariant metric, indeed for an arbitrary open neighborhood W of o in N, set $U := \bigcap_{g\in G} gW$). We identify U with a coordinate chart and regard $U \subset \mathbb{C}^n$, where o corresponds to the origin. We may then identify the tangent space T_oU at o with $\mathbb{C}^n$, so that $U \subset T_oU = \mathbb{C}^n$. Now for $h \in G$, let $h_* : T_oU \to T_oU$ denote the induced linear map. Consider the composition $h_*^{-1} \circ h : U \xrightarrow{h} U \xrightarrow{h_*^{-1}} \mathbb{C}^n$ and then their *average*, i.e. the holomorphic map $F : U \to \mathbb{C}^n$ defined by $F(x) := \frac{1}{|G|}\sum_{h\in G} h_*^{-1} \circ h(x)$, where the division by $|G|$ ensures that the Jacobian matrix J_o of F at o is the identity matrix (indeed for any $v \in T_oU$, we have $dF_o(v) = \frac{1}{|G|}\sum_{h\in G} v = \frac{1}{|G|}|G|v = v$). Since J_o is invertible, F is biholomorphic around $o \in U$. If necessary shrinking U we assume that $F : U \to B := F(U)$ is biholomorphic.

Lemma 3.6. *The biholomorphic map $F : U \to B$ is G-equivariant: for any $g \in G$, the following diagram commutes.*

$$\begin{array}{ccc} U & \xrightarrow{\quad g \quad} & U \\ {\scriptstyle F}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle F} \\ B & \xrightarrow{\quad g_* \quad} & B. \end{array}$$

Proof. It suffices to show $Fg = g_*F$, that is, $g_*^{-1}Fg = F$. This is confirmed as follows:

$$g_*^{-1}Fg(x) = \frac{1}{|G|}g_*^{-1}\sum_{h\in G} h_*^{-1}\circ hg(x) = \frac{1}{|G|}\sum_{h\in G} g_*^{-1}h_*^{-1}\circ hg(x)$$
$$= \frac{1}{|G|}\sum_{h\in G}(h_*g_*)^{-1}\circ (hg)(x) = F(x). \square$$

Lemma 3.6 means that in the new coordinate chart B of U, the G-action becomes linear (g acts as a linear map g_*).

Definition 3.7. By abuse of terminology, the G-action on B is called the *isotropy representation* of G.

Now — in the complex analytic case — the assumption of Lemma 3.5 is modified as follows: Γ is a *complex* Lie group, $\varphi : M \to N$ is a Γ-equivariant proper *holomorphic* surjective submersion between *complex* manifolds such that the Γ-actions on M and N are *holomorphic*. Then by Lemma 3.6 the following holds:

Lemma 3.8 (Holomorphic linearization lemma). *In the complex analytic case, if the stabilizer of o is finite, the diffeomorphism $U \to B$ in Lemma 3.5 may be taken to be a biholomorphic map.*

Alternative approach for diagonalizing semi-diagonal actions

The set $\mathrm{Hom}(G, GL_n(\mathbb{C}))$ of homomorphisms from a finite group G to $GL_n(\mathbb{C})$ forms an algebraic variety: Say G is generated by $g_1, g_2, \ldots, g_l$ with relations $r_i(g_1, g_2, \ldots, g_l) = 1$, $i = 1, 2, \ldots, k$. Then $\mathrm{Hom}(G, GL_n(\mathbb{C}))$ is identified with an algebraic variety V in $GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C})$ (l times) defined by $r_i(X_1, \ldots, X_l) = I$, $i = 1, 2, \ldots, k$, under the correspondence $\rho \in \mathrm{Hom}(G, GL_n(\mathbb{C})) \mapsto (\rho(g_1), \ldots, \rho(g_l)) \in V$. The representation variety $R(G, GL_n(\mathbb{C}))$ is the quotient of $\mathrm{Hom}(G, GL_n(\mathbb{C}))$ by the conjugation action of $GL_n(\mathbb{C})$.

Recall that the number of irreducible representations of a finite group is finite. Let $\mu_1, \mu_2, \ldots, \mu_l$ be the irreducible representations of G, then any representaton μ of G is expressed as $\mu = \mu_1^{m_1} \oplus \mu_2^{m_2} \oplus \cdots \oplus \mu_l^{m_l}$ (up to conjugation), where m_i are nonnegative integers. Consequently the number of representations of G to $GL_n(\mathbb{C})$ (for fixed n) is finite up to conjugation, that is, $R(G, GL_n(\mathbb{C}))$ is a finite set, or the number of $GL_n(\mathbb{C})$-orbits in $\mathrm{Hom}(G, GL_n(\mathbb{C}))$ is finite.

We turn to a semi-diagonal G-action on $X \times \mathbb{C}^n$ given by $(x, v) \mapsto (gx, \rho_x(g)v)$, where ρ_x depends on $x \in X$ smoothly (or holomorphically in the complex analytic case — "smooth" below is also replaced with "holomorphic"). Consider a smooth map

$$f : X \to \mathrm{Hom}(G, GL_n(\mathbb{C})), \quad x \mapsto \rho_x, \tag{3.4}$$

and the composite map $\overline{f} := q \circ f : X \to R(G, GL_n(\mathbb{C}))$, where $q : \mathrm{Hom}(G, GL_n(\mathbb{C})) \to R(G, GL_n(\mathbb{C}))$ is the quotient map. Suppose that X is connected, then the image $f(X)$, and also $\overline{f}(X)$, is connected. Here $R(G, GL_n(\mathbb{C}))$ consists of a finite number of points, so $\overline{f}(X)$ is necessarily one of these points. Namely $f(X)$ is contained in a single $GL_n(\mathbb{C})$-orbit. Thus fixing a base point $x_0 \in X$, for each ρ_x there exists $A_x \in GL_n(\mathbb{C})$ such that $\rho_x = A_x \rho_{x_0} A_x^{-1}$. Here the choice of A_x for each x is *not* unique (for any $B \in GL_n(\mathbb{C})$ such that $\rho_{x_0} = B\rho_{x_0}B^{-1}$, we have $\rho_x = (A_xB)\rho_{x_0}(A_xB)^{-1}$). As explained in Remark 3.9 below, we may choose A_x for each $x \in X$ such that the map $\alpha : X \to GL_n(\mathbb{C})$, $x \mapsto A_x$ is smooth and locally single-valued — but may not be globally single-valued on X. *If* it is globally single-valued on X, then a coordinate change $(x, v) \mapsto (x, A_xv)$ of $X \times \mathbb{C}^n$ is defined, and under which the semi-diagonal G-action $(x, v) \mapsto (gx, \rho_x(g)v)$ becomes a diagonal action $(x, v) \mapsto (gx, \rho_{x_0}(g)v)$. Since this is not always the case, we adopted another method for diagonalization, which also works for compact groups.

Remark 3.9. Since $f(X)$ is contained in a single $GL_n(\mathbb{C})$-orbit, the $GL_n(\mathbb{C})$-action on $f(X)$ is transitive. Let S be the subgroup of $GL_n(\mathbb{C})$ consisting of $A \in GL_n(\mathbb{C})$ such that $A\rho_{x_0}A^{-1} \in f(X)$ and let H be the subgroup of S consisting of $A \in GL_n(\mathbb{C})$ such that $A\rho_{x_0}A^{-1} = \rho_{x_0}$. Then $f(X) = S/H$, and a principal H-bundle $H \to S \to S/H = f(X)$ is obtained. Consider its pullback $H \to f^*S \to X$ via f. Since $f^*S \to X$ is a fiber bundle, around each point of X it admits a *local* section, accordingly we may take the map $\alpha : X \to GL_n(\mathbb{C})$, $x \mapsto A_x$ as locally a single-valued smooth map (globally α may not be single-valued). If α is globally single-valued on X, then it is regarded as a section of $f^*S \to X$. However this bundle may not have a section, in which case such a single-valued map does *not* exist.

4. Application

Let $\varphi : M \to N$ be a Γ-equivariant proper surjective submersion between manifolds such that the Γ-actions on M and N are proper; so $\overline{\varphi} : M/\Gamma \to N/\Gamma$ is a submersive quotient family. Suppose that the stabilizer G of a point $o \in N$ is finite. Taking a sufficiently small G-stable neighborhood U of o in N, the inclusions $\varphi^{-1}(U) \hookrightarrow M$ and $U \hookrightarrow N$ induce embeddings $\varphi^{-1}(U)/G \hookrightarrow M/\Gamma$ and $U/G \hookrightarrow N/\Gamma$, and the following diagram commutes:

$$\begin{array}{ccc} \varphi^{-1}(U)/G & \hookrightarrow & M/\Gamma \\ \overline{\varphi}\downarrow & & \downarrow\overline{\varphi} \\ U/G & \hookrightarrow & N/\Gamma. \end{array} \tag{4.1}$$

The quotient family $\overline{\varphi} : \varphi^{-1}(U)/G \to U/G$ is thus a *subfamily of* $\overline{\varphi} : M/\Gamma \to N/\Gamma$ *around* $\overline{o} \in N/\Gamma$.

Now applying Lemma 3.5 to the case that K is a finite group, we obtain the following (below, K is denoted by G):

Proposition 4.1 (Linearization). *Let $\varphi : M \to N$ be a Γ-equivariant proper surjective submersion between manifolds such that the Γ-actions on M and N are proper. If the stabilizer G of a point $o \in N$ is finite, there exist a G-stable neighborhood U of o in N and an n-dimensional open ball B centered at 0 in $\mathbb{C}^n$ $(n := \dim N)$ on which G acts linearly such that the local quotient family $\overline{\varphi} : \varphi^{-1}(U)/G \to U/G$ is orbi-diffeomorphic to the linear quotient family $\overline{\mathrm{pr}} : (X \times B)/G \to B/G$, where $X := \varphi^{-1}(o)$.* (Note: For "orbi-diffeomorphic", see Definition 1.11.)

In the complex analytic case, the assumption of Proposition 4.1 is modified as follows: Γ is a *complex* Lie group, $\varphi : M \to N$ is a Γ-equivariant proper *holomorphic* surjective submersion between *complex* manifolds such that the *holomorphic* Γ-actions on M and N are proper. Then both $\overline{\varphi} : \varphi^{-1}(U)/G \to U/G$ and $\overline{\mathrm{pr}} : (X \times B)/G \to B/G$ are holomorphic maps between complex analytic varieties. Moreover by Lemma 3.8 the following holds:

Proposition 4.2 (Holomorphic linearization). *In the complex analytic case, the orbi-diffeomorphism between U/G and B/G in Proposition 4.1 may be taken to be a biholomorphic map.*

Quotient as orbifold. Let Γ be a Lie group acting on a smooth manifold M properly such that the stabilizer of every point of M is finite (in the extreme case, Γ itself is a finite group or a discrete group acting properly discontinuously). Then M/Γ is an orbifold, which is shown by using slice theorem.

Descent to orbi-map. Let Γ be a Lie group acting on smooth manifolds M and N properly such that the stabilizer of every point of M and N is finite. Let $\varphi : M \to N$ be a Γ-equivariant map. Then it descends to an orbi-map $\overline{\varphi} : M/\Gamma \to N/\Gamma$ between orbifolds M/Γ and N/Γ. If moreover φ is a proper surjective submersion, $\overline{\varphi} : M/\Gamma \to N/\Gamma$ is called an *orbi-quotient family.*

Theorem 4.3 (Linearization theorem). *Any orbi-quotient family $\overline{\varphi} : M/\Gamma \to N/\Gamma$ is locally orbi-diffeomorphic to a linear quotient family as in Proposition 4.1 (and in the complex analytic case, the orbi-diffeomorphism between the base spaces may be taken to be a biholomorphic map by Proposition 4.2).*

Theorem 4.3 enables one to describe, in terms of linear quotient families, many important families, e.g. the (coarse) universal families of Abelian varieties over Shimura varieties and the (coarse) universal families over moduli spaces of Riemann surfaces. We here apply our result to the latter families. Let T_g be the Teichmüller space of marked Riemann surfaces of genus $g \geq 2$ and let $\varphi : S_g \to T_g$ be the universal (i.e. tautological) family: for each point $[(X, \mu)] \in T_g$, its fiber is a marked Riemann surface (X, μ), where μ is a marking of X. The mapping class group Γ_g acts on both S_g and T_g properly discontinuously, and $\varphi : S_g \to T_g$ is Γ_g-equivariant. The quotient $\overline{\varphi} : S_g/\Gamma_g \to T_g/\Gamma_g$ is the universal family $\eta : U_g \to M_g$ over the moduli space $M_g := T_g/\Gamma_g$. Now the stabilizer of every point of T_g for the Γ_g-action is finite (as this action is properly discontinuous): for a point $[(X, \mu)] \in T_g$, its stabilizer is given by $\mathrm{Aut}(X)$, which linearly acts on the tangent space of

T_g at $[(X,\mu)]$ (the *isotropy representation*). Here T_g around $[(X,\mu)]$ is identified with the Kuranishi space of X, and by the Kodaira–Spencer theory, its tangent space at $[(X,\mu)]$ is given by $H^1(X,TX)$.

In what follows, we identify $H^1(X,TX)$ *with the vector space* $H^0(X,K^{\otimes 2})$ *of holomorphic quadratic differentials via the Serre duality*, where K is the canonical bundle on X; alternatively in terms of the Bers embedding of T_g into $H^0(X,K^{\otimes 2})$ as a ball around the origin (e.g., see [Nag]), the tangent space of T_g at $[(X,\mu)]$ is naturally identified with $H^0(X,K^{\otimes 2})$. (For the reason why we use $H^0(X,K^{\otimes 2})$ rather than $H^1(X,TX)$, see Remark 4.5 below.) We now apply the linearization theorem to $\overline{\varphi}: S_g/\Gamma_g \to T_g/\Gamma_g$, obtaining the following result:

Corollary 4.4 (Linearization theorem of universal families). *Let X be a Riemann surface of genus $g \geq 2$. Then the* (*coarse*) *universal family $\overline{\varphi}: U_g \to M_g$ around $[X]$ is orbi-diffeomorphic to a crystal fiber neighborhood of the 2-canonical family*

$$\big(X \times H^0(X,K^{\otimes 2})\big)/\mathrm{Aut}(X) \to H^0(X,K^{\otimes 2})/\mathrm{Aut}(X)$$

such that the orbi-diffeomorphism between the base spaces is biholomorphic.

Remark 4.5. The advantages to using $H^0(X,K^{\otimes 2})$ instead of $H^1(X,TX)$ lie in the facts that sections are easier to treat than 1-cocyles, and moreover, in the practical computation to describe the $\mathrm{Aut}(X)$-action with respect to some basis (this description is needed for describing the 2-canonical family), it is easier to find a basis of $H^0(X,K^{\otimes 2})$ than that of $H^1(X,TX)$, as we may use Max Noether's theorem: for a nonhyperelliptic curve X, the natural map $\mathrm{Sym}^2 H^0(X,K) \to H^0(X,K^{\otimes 2})$ is surjective (see [ACGH] p.117); thus for a basis $\alpha_1,\alpha_2,\dots,\alpha_g$ of $H^0(X,K)$, a basis of $H^0(X,K^{\otimes 2})$ may be taken from $\{\alpha_i\alpha_j\}_{1\leq i\leq j\leq g}$.

The Kuranishi space of X is realized as an open neighborhood of the origin in $H^0(X,K^{\otimes 2})$ such that the origin corresponds to the complex structure of X. The automorphism group $\mathrm{Aut}(X)$ acts on the Kuranishi space (which is the restriction of the linear action of $\mathrm{Aut}(X)$ on $H^0(X,K^{\otimes 2})$ induced from the action of $\mathrm{Aut}(X)$ on X). The quotient of the Kuranshi space under the $\mathrm{Aut}(X)$-action is an open neighborhood of the image of the origin in $H^0(X,K^{\otimes 2})/\mathrm{Aut}(X)$; this quotient is regarded as an open neighborhood of $[X]$ in the moduli space M_g. In fact, M_g is obtained by patching such quotients (see [ACG]). This construction fits the following consequence of Corollary 4.4:

Corollary 4.6. *The (coarse) universal family $U_g \to M_g$ over the moduli space M_g of Riemann surfaces of genus $g \geq 2$ is topologically obtained by patching crystal fiber neighborhoods of the 2-canonical families $\big(X \times H^0(X, K^{\otimes 2})\big)/\mathrm{Aut}(X) \to H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$. Here in the resulting family obtained by patching, while the total space is merely orbi-diffeomorphic to U_g, the base space is biholomorphic to M_g.*

Remark 4.7. For the automorphism groups of Riemann surfaces, see the survey [BCG], and for the characters of their representations, see [Br].

We point out that while for the case $g \geq 3$ the kaleido fibers of the 2-canonical family $\eta : \big(X \times H^0(X, K^{\otimes 2})\big)/\mathrm{Aut}(X) \to H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$ coincide with its special (or singular) fibers, for the case $g = 2$ the situation is slightly different. In the former case, the action of $\mathrm{Aut}(X)$ on $H^0(X, K^{\otimes 2})$ is effective, and the genus of a fiber $\eta^{-1}(s) = X/H_{\widetilde{s}}$ (where $H_{\widetilde{s}}$ is the stabilizer of $\widetilde{s} \in H^0(X, K^{\otimes 2})$ for the $\mathrm{Aut}(X)$-action; see Theorem 2.1) is strictly less than g precisely when s lies in the kaleido locus of η. In the latter case, the action of $\mathrm{Aut}(X)$ on $H^0(X, K^{\otimes 2})$ is *not* effective: any Riemann surface of genus 2 is hyperelliptic, having a hyperelliptic involution ι, which acts on $H^0(X, K^{\otimes 2})$ trivially (in fact as an element of the mapping class group, ι fixes all points of the Teichmüller space T_2). The non-effectivity of the action of $\mathrm{Aut}(X)$ on $H^0(X, K^{\otimes 2})$ implies that all fibers of η are kaleido, that is, the kaleido locus is the whole of $H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$, and a generic fiber is not X but a projective line $X/\langle\iota\rangle$ of covering multiplicity 2.

It is however possible to turn a generic fiber into a pure fiber by "reduction" replacing the $\mathrm{Aut}(X)$*-action on* $H^0(X, K^{\otimes 2})$ *with the* $\mathrm{Aut}(X)/\langle\iota\rangle$*-action on* $H^0(X, K^{\otimes 2})$ *(which is effective) while replacing the* $\mathrm{Aut}(X)$*-action on X with the* $\mathrm{Aut}(X)/\langle\iota\rangle$*-action on* $X/\langle\iota\rangle$. The resulting linear quotient family $\eta' : \big((X/\langle\iota\rangle) \times H^0(X, K^{\otimes 2})\big)/\mathrm{Aut}(X)/\langle\iota\rangle \to H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)/\langle\iota\rangle$ has the same base space as that of $\eta : \big(X \times H^0(X, K^{\otimes 2})\big)/\mathrm{Aut}(X) \to H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$ (as $\langle\iota\rangle$ acts trivially on $H^0(X, K^{\otimes 2})$), and a fiber $(\eta')^{-1}(s)$ over any point s coincides with $\eta^{-1}(s)$ (as, by the quotient fiber theorem, $\eta^{-1}(s) = X/H_{\widetilde{s}} = X/\langle\iota\rangle/H_{\widetilde{s}}/\langle\iota\rangle = (\eta')^{-1}(s)$). However the covering multiplicity of $(\eta')^{-1}(s)$ becomes the half of that of $\eta^{-1}(s)$ (divided by the order of ι). In particular for a generic fiber $X/\langle\iota\rangle$ of η', its covering multiplicity is 1, thus it is a pure fiber. See [Ta2] for the general theory of reduction.

References

[ACG] Arbarello, E., Cornalba, M. and P. Griffiths, *Geometry of Algebraic Curves, II*, Springer-Verlag (2011).

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P. and J. Harris, *Geometry of Algebraic Curves, I*, Springer-Verlag (1985).

[AsEn] Ashikaga, T. and H. Endo, Various aspects of degenerate families of Riemann surfaces, Sugaku Expositions **19** No.2 (2006), 171–196, Amer. Math. Soc.

[AsKo] Ashikaga, T. and K. Konno, Global and local properties of pencils of algebraic curves, in "*Algebraic Geometry 2000 Azumino*", ed. by Usui *et al.*, Advanced Studies in Pure Math. **36** (2002), 1–49.

[BCG] Bujalance, E., Cirre, F. and G. Gromadzki, A survey of research inspired by Harvey's theorem on cyclic groups of automorphisms, in "*Geometry of Riemann Surfaces*" (2010), 15–37, Proceedings of the Anogia conference to celebrate the 65th birthday of William J. Harvey, London Math. Soc. Lecture Note Series 368, Cambridge Univ. Press.

[BCO] Berndt, J., Console, S. and C. Olmos, *Submanifolds and Holonomy, 2nd ed.*, Chapman & Hall/CRC Monographs and Research Notes in Math. (2016).

[BeRi] Behnke, K. and O. Riemenschneider, Quotient surface singularities and their deformations, in "*Singularity Theory (Trieste, 1991)*", 1–54, World Scientific Publishing (1995).

[Br] Breuer, T., *Characters and Automorphism Groups of Compact Riemann Surfaces*, London Math. Soc. Lecture Note Series 280, Cambridge Univ. Press (2000).

[Che] Chevalley, C., Invariants of finite groups generated by reflections, Amer. J. Math. **77** No.4 (1955), 778–782.

[Hil] Hiller, H., *Geometry of Coxeter Groups*, Research Notes in Mathematics 54, Pitman (Advanced Publishing Program), Boston, MA (1982).

[HiTa1] Hirakawa, R. and S. Takamura, Degenerations of Riemann surfaces associated with the regular polyhedra and the soccer ball, J. Math. Soc. Japan **69** No.3 (2017), 1213–1233.

[HiTa2] Hirakawa, R. and S. Takamura, Quotient families of elliptic curves associated with representations of dihedral groups, Publ. RIMS **55** No.2 (2019), 319–367.

[Hol] Holmann, H., Komplexe Räume mit komplexen Transformationsgruppen, Math. Ann. **150** (1963), 327–360.

[Kod] Kodaira, K., *Complex Manifolds and Deformation of Complex Structures*, Springer (1986).

[Kos] Kosinski, A., *Differential Manifolds*, Dover Books on Math. (2007).

[Nag] Nag, S., *The Complex Analytic Theory of Teichmüller Spaces*, A Wiley-Interscience Publication (1988).

[Ta1] Takamura, S., Towards the classification of atoms of degenerations, III, (Splitting Deformations of Degenerations of Complex Curves), Springer Lecture Notes in Math. **1886** (2006).

[Ta2] Takamura, S., *Group Actions, Representations, and Quotient Families*, Lecture Notes (2016).

[Ta3] Takamura, S., *Actional Geometry — The Geometry of Group Actions, Equivariant Quotients, and Universal Posets*, Lecture Notes in preparation.

(Received October 25, 2018)
(Revised April 22, 2019)

Department of Mathematics
Graduate School of Science
Kyoto University
Oiwakecho, Kitashirakawa, Sakyo-ku
Kyoto 606-8502, JAPAN
E-mail: takamura@math.kyoto-u.ac.jp