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Moderate Degenerations of Ricci-Flat Kähler-Einstein Manifolds Over Higher Dimensional Bases

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Abstract. We study moderate degenerations of Calabi-Yau / Ricci-flat Kähler-Einstein manifolds in a holomorphic family over a base of arbitrary dimension. We discuss various equivalence relations among a limit variety has canonical singularities at worst, a uniform diameter bound of nearby smooth fibers, and others.

1. Introduction

We shall study moderate degenerations of Calabi-Yau / Ricci-flat Kähler-Einstein manifolds. In our previous work [Ta3], we study such degenerations over a 1-dimensional base. Motivated by a study of compactifications of moduli spaces of Calabi-Yau manifolds, we study degenerations over a base of arbitrary dimension. Our basic set up we shall use frequently is as follows.

Set up 1.1. (1) Let $f : X \to Y$ be a projective surjective morphism with connected fibers between normal quasi-projective varieties with a special point $0 \in Y$ and of $\dim Y = m \geq 1$. Let X_y be the scheme theoretic fiber of $y \in Y$. Let $Y^o \subset Y$ be the maximal Zariski open set such that $Y^o \subset Y_{\mathrm{reg}}$ and f is smooth over Y^o, and let $X^o = f^{-1}(Y^o)$. We suppose $Y^o \neq \emptyset$.

(2) Suppose $K_{X_y} = \mathcal{O}_{X_y}$ for every $y \in Y^o$. Let L be a line bundle on X^o which is f-ample (i.e., $f|_{X^o}$-ample). Then by Yau, for every $y \in Y^o$, X_y admits a unique Kähler-Einstein metric $\omega_y \in c_1(L_y)$, where $L_y = L|_{X_y}$.

Then our main result is as follows. Technical terms "weakly semi-stable", "(relative) good minimal model" will be recalled in §2.1. In the

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case $\dim X = 2$, these are nothing but a semi-stable reduction and a relative minimal surface/fibration over a curve.

THEOREM 1.2. *In the situation 1.1, the following properties are equivalent.*

(1) For any weak semi-stable reduction $f' : X' \to Y'$ *of* f *and any good minimal model* $f'' : X'' \to Y', \varphi : X' \dashrightarrow X''$ *of* f'*, there exists a Zariski open subset* $0 \in W \subset Y$ *such that for any* $p \in \tau^{-1}(W)$*, the fiber* X''_p *of* f'' *has canonical singularities at worst and* $K_{X''_p} = \mathcal{O}_{X''_p}$.

$$\begin{array}{ccccc} X'' & \overset{\varphi}{\dashleftarrow} & X' & \xrightarrow{\tau_X} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & = & Y' & \xrightarrow{\tau} & Y \end{array}$$

(2) There exist a generically finite surjective morphism $\tau : Y' \to Y$ *from a normal quasi-projective variety and a strict modification* $X' \to X \times_Y Y'$ *(i.e., a proper birational morphism to the main component of* $X' \times_Y Y'$*) from a normal quasi-projective variety, such that the induced morphism* $f' : X' \to Y'$ *is flat over* $\tau^{-1}(0)$ *and for every* $p \in \tau^{-1}(0)$*, the fiber* X'_p *of* f' *contains an irreducible component* $F_p (\subset (X'_p)_{red})$ *with non-negative Kodaira dimension:* $\kappa(F_p) \geq 0$.

(3) The diameter of X_y $(y \in Y^o)$ *with respect to the Kähler-Einstein metric* ω_y *is uniformly bounded from above as* $y \to 0$*. Namely there exist an open neighborhood* W *of* $0 \in Y$ *and a constant* $\alpha > 0$ *such that* $\mathrm{diam}\,(X_y, \omega_y) \leq \alpha$ *for any* $y \in Y^o \cap W$.

For example, if f is flat over $0 \in Y$ and X_0 contains an irreducible component F with $\kappa(F) \geq 0$, then 1.2 (2) holds. This is a sufficient condition for 1.2 (2). But it is not a necessarily condition, already when $\dim X = 2, \dim Y = 1$, i.e., elliptic fibrations. We also note $X' \to X$ in 1.2 (1) may blow-up smooth fibers, and hence general fibers of f' may not necessarily be Calabi-Yau in a strict sense. We (the author) do not know if f'' is generically smooth when a general fiber of f' is not Calabi-Yau.

A possible approach to 1.2 is restricting everything over a general curve in Y and use known results in [W1], [To], [Ta3], ... It is useful in some case, but sometime it is not, as in the case of a dis-continuity phenomenon of a function of several variables. In any way, we still use Wang's Hodge theoretic approach [W1] [W2], Rong-Zhang's estimate on diameter [RZ, 2.1], and Donaldson-Sun's compactness result [DS]; an existence of a Gromov-Hausdorff limit under a volume non-collapsing/a diameter bound condition.

Due to the nature of our problem, we can actually suppose in most cases that $f : X \to Y$ is weakly semi-stable and admits a good minimal model. Our main new technical result is the continuity of a canonical fiberwise L^2-metric in such a case. This continuity holds not only in the Calabi-Yau case, but also in general, and is interesting by itself. As a general fact, if $f : X \to Y$ is weakly semi-stable, then K_X is a line bundle ([AK, 6.4]) and $f_*K_{X/Y}$ is locally free ([Ka3, Theorem 26]).

THEOREM 1.3. *Suppose in 1.1 (1) that $f : X \to Y$ is weakly semi-stable, and that f admits a good minimal model $f' : X' \to Y, \varphi : X \dashrightarrow X'$ such that X'_0 has canonical singularities at worst. Then the canonical L^2-metric on $f_*K_{X^o/Y^o}$ extends as a continuous and non-degenerate Hermitian metric on $f_*K_{X/Y}$ on a neighborhood of $0 \in Y$ (if $f_*K_{X/Y}$ is non-zero).*

Here the canonical L^2-metric on $f_*K_{X^o/Y^o}$ is given by

$$\int_{X_y} (-1)^{n^2/2} u_y \wedge \overline{v}_y \quad \text{for } u_y, v_y \in (f_*K_{X^o/Y^o})_y = H^0(X_y, K_{X_y}),$$

where $n = \dim X - \dim Y$. Yoshikawa [Y, 7.1] essentially shows 1.3 when $\dim Y = 1$, without assuming f is weakly semi-stable. When $\dim Y = 1$, we can keep the flatness of f under modifications of X; a usual Hironaka's resolution theorem is suffice. It would be an interesting question if 1.3 holds true without assuming f is weakly semi-stable.

It would be useful in some purposes to state 1.2 in a specific situation as follows (cf. [To, 1.1], [Ta3, 1.5, 1.6] in the case $\dim Y = 1$).

THEOREM 1.4. *Suppose in 1.1 that $f : X \to Y$ is weakly semi-stable. Then the following properties are equivalent:*

(a) For any good minimal model $f' : X' \to Y, \varphi : X' \dashrightarrow X$ of f, the fiber X'_0 has canonical singularities at worst.

(a') There exists an (in fact unique) irreducible component F of X_0 such that $H^0(\widetilde{F}, K_{\widetilde{F}}) \neq 0$ for a smooth projective variety $\widetilde{F}$ birational to F.

(c) As $f_*K_{X/Y}$ *is a line bundle on* Y *(*[Ka3, Theorem 26]*), there exists* $\Omega \in H^0(X, K_{X/Y})$ *such that* $H^0(X, K_{X/Y}) = \Omega f^*H^0(Y, \mathcal{O}_Y)$ *possibly replace* Y *by a smaller neighborhood of* $0 \in Y$ *(cf.* [Ta3, 1.2 (1)]*). We denote the restriction by* $\Omega_y := \Omega|_{X_y} \in H^0(X_y, K_{X_y})$ *for* $y \in Y^o$ *and regard it as a nowhere vanishing holomorphic* n*-form on* X_y*, where* $n = \dim X - \dim Y$*. Then, there is a constant* $\alpha > 0$ *such that* $\int_{X_y}(-1)^{n^2/2}\Omega_y \wedge \overline{\Omega}_y \le \alpha$ *as* $y\ (\in Y^o) \to 0$.

(c') In the notations in (c), there is an open set $W \subset Y$ *containing* 0 *such that the smooth function* $\int_{X_y}(-1)^{n^2/2}\Omega_y \wedge \overline{\Omega}_y$ *on* $y \in Y^o \cap W$ *extends continuously on* W *and nowhere zero on* W*.*

(c") There is an open set $W \subset Y$ *containing* 0 *such that the canonical* L^2*-metric on* $f_*K_{X^o/Y^o}$ *extends as a continuous and non-degenerate Hermitian metric on the line bundle* $f_*K_{X/Y}$ *on* W*.*

(d) There is a constant $\alpha > 0$ *such that* $\omega_y^n \ge \alpha^{-1}(-1)^{n^2/2}\Omega_y \wedge \overline{\Omega}_y$ *on* X_y *as* $y\ (\in Y^o) \to 0$*, where* $\Omega \in H^0(X, K_{X/Y})$ *is a section as in (c).*

(e) There is a constant $\alpha > 0$ *such that* $\operatorname{diam}(X_y, \omega_y) \le \alpha$ *as* $y\ (\in Y^o) \to 0$.

(f) The volume non-collapsing property with respect to L *holds (cf.* [DS, (1.2)], [Ta3, 1.2 (3)]*).*

We add partially one more property. These equivalent properties in 1.4 imply a property

(b) $d_{\mathrm{WP}}(0, Y^o) < \infty$*, namely the point* $0 \in Y$ *is of finite distance from* Y^o *with respect to the Weil-Petersson pseudo-distance* ω_{WP}*.*

Here the Weil-Petersson pseudo-distance ω_{WP} is the curvature form of the canonical L^2-metric on $f_*K_{X^o/Y^o}$. To deduce the last implication to obtain (b), we can reduce to the case $\dim Y = 1$. There is an expectation or a conjecture that the converse also holds true ([Lee, §1.1]). In the case $\dim Y = 1$, it was a conjecture due to Wang [W1], and it is confirmed by Tosatti [To] and the author [Ta3]. There is a subtlety in the case $\dim Y > 1$ to measure the length of a pass (a real curve) which is not contained in a holomorphic curve (see [Lee, Remark 4.9, Example 4.10]). Namely we can not reduce to the case $\dim Y = 1$ to obtain, say (a) from (b).

The organization of the paper is as follows. We recall, in §2, basic notions from algebraic geometry, and Kawamata's inversion of adjunction. In §3, we discuss fiberwise integrals for weakly semi-stable morphisms in a particular case and prove 1.3. We then finally prove 1.2 and 1.4 in §4.

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2. Preliminary from Algebraic Geometry

We shall recall or generalize some results in algebraic geometry.

2.1. Weak semi-stable reduction and good minimal model

We first recall a weak semi-stable reduction theorem of Abramovich and Karu [AK]. We refer [KKMSD] for generalities on toroidal varieties.

A *toroidal variety* (X, B) is a pair consisting of a normal variety and an effective reduced divisor such that each point $x \in X$ has a *toric local model* in the following sense: there is a complex analytic neighborhood U of x such that the pair $(U, B|_U)$ is complex analytically isomorphic to another pair $(U', B'|_{U'})$ which comes from a toric variety (X', B'); a normal variety with an action of an algebraic torus $X' \setminus B'$ (and $U' \subset X'$ is complex analytic open). We assume moreover that the pair is *strict* or *without self-intersection* in the sense that each irreducible component of B is normal. Refer to [AK, Definition 1.2, §1.3].

A *toroidal morphism* $f : (X, B) \to (Y, C)$ between toroidal varieties is one which has a toric local model at each point $x \in X$ in the following sense: there is a toric morphism between local models $f' : (X', B') \to (Y', C')$, i.e., $f'|_{X' \setminus B'} : X' \setminus B' \to Y' \setminus C'$ is a surjective homomorphism of algebraic tori, and f' is equi-variant under the torus actions. Refer to [AK, Definition 1.3].

A toroidal variety (X, B) is said to be *smooth* if X is smooth and B has only normal crossings (in particular, every irreducible component of B is smooth as it is normal). It is *quasi-smooth* if there exists a local toric model of each point which has only abelian quotient singularities.

Remark 2.1.1 ([Ka3, Example 1]). Let (X, B) be a quasi-smooth toroidal variety. Let $x \in X$ be a point.

(1) Then there is a complex analytic neighborhood $U \subset X$ of x and a finite Galois toroidal covering $\pi : (X', B') \to (U, B|_U)$ from a smooth toroidal variety with a point $x' \in X'$ such that $\pi(x') = x$. There exist local coordinates $(x'_1, \dots, x'_n)$ centered at $x' \in X'$ which are semi-invariant with respect to the Galois action. In particular if $B|_U = \sum_{i \in I} B_i, B' = \sum_{j \in J} B'_j$

respectively is the irreducible decomposition, we then have a one-to-one correspondence $a_\pi : I \to J$ such that $\pi^{-1}(B_i) = B'_{a_\pi(i)}$. We would simply express as $(U, B|_U) = (X', B')/G$ for a finite abelian group G. We can suppose $G \subset \mathrm{GL}(n, \mathbb{C})$ to be a "small subgroup" [Sb, 1.2, 1.3]. (Steenbrink [Sb, 1.16, 2.2] uses terminologies V-manifolds and V-normal crossings.)

(2) Let $f : (X, B) \to (Y, C)$ be a toroidal morphism to a smooth toroidal variety (Y, C), and $y = f(x)$. Then there exist local coordinates $(x'_1, \dots, x'_n)$ centered at $x' \in X'$ as above and $(y_1, \dots, y_m)$ centered at $y \in Y$ such that $(f \circ \pi)^* y_i = \prod_j x'^{r_{ij}}_j$, where the r_{ij} are non-negative integers. (Note that the composition $f \circ \pi : (X', B') \to (Y, C)$ is a toroidal morphism between smooth toroidal varieties, [AK, Corollary 1.6].)

DEFINITION 2.1.2 ([AK, Definition 0.1, §8.2]). Let $f : X \to Y$ be a projective surjective morphism with connected fibers of normal quasi-projective varieties. The morphism $f : X \to Y$ is said to be *weakly semi-stable*, if (i) there exist toroidal structures (X, B) and (Y, C) with (X, B) is quasi-smooth and (Y, C) is smooth, (ii) $f : (X, B) \to (Y, C)$ is a toroidal morphism with $X \setminus B = f^{-1}(Y \setminus C)$, in particular f is generically smooth, (iii) f is equi-dimensional, (iv) all the fibers of f are reduced. A weakly semi-stable morphism $f : X \to Y$ is said to be *semi-stable*, if X is smooth.

We follow the terminology in [Ka3, §1] (the only difference is the quasi-smoothness of (X, B)). (There are *almost semi-stable*, *nearly semi-stable* in literature.) If f is weakly semi-stable, it follows that X has rational Gorenstein singularities at worst [AK, §6], in particular it has canonical singularities at worst, [KM, 5.24].

Now we recall a weak semi-stable reduction theorem of Abramovich and Karu (refer [Ka3, Theorem 2] for a more detailed account).

THEOREM 2.1.3 ([AK, Theorem 0.3, §8.2]). *Let $f : X \to Y$ be a projective surjective morphism with connected fibers of normal quasi-projective varieties. Then there exist a generically finite morphism $Y' \to Y$ from a smooth quasi-projective variety and a strict modification $X' \to X \times_Y Y'$ from a normal quasi-projective variety such that the induced morphism $f' : X' \to Y'$ is weakly semi-stable and semi-stable in codimension 1.*

Here a *strict modification* means that the morphism is a proper birational morphism to the main irreducible component of $X \times_Y Y'$ which dominants Y', [AK, 0.10].

We next recall another type of nice morphisms.

DEFINITION 2.1.4 (see [F, §3] for more advances). Let $f : X \to Y$ be a projective surjective morphism with connected fibers of normal quasi-projective varieties. Suppose X has canonical singularities at worst. A normal variety X', a morphism $f' : X' \to Y$ and a rational map $\varphi : X \dashrightarrow X'$ over Y is called a *minimal model* of X over Y, if (i) X' is $\mathbb{Q}$-factorial, (ii) f' is projective, (iii) φ is birational and φ^{-1} has no exceptional divisors, (iv) $K_{X'}$ is f'-nef, and (v) $a(E, X) < a(E, X')$ for every φ-exceptional divisor $E \subset X$, where $a(E, X)$ is the discrepancy of E over X ([KM, 2.22]). Furthermore, if $K_{X'}$ is f'-semi-ample, then X' is called a *(relative) good minimal model of X over Y.*

By Hacon-Xu [HX], if a (sufficiently) general fiber of f has a good minimal model, then $f : X \to Y$ has a good minimal model. So in our interest as in 1.2, we are free to pass to a relative good minimal model. We recall some basic properties from [F].

LEMMA 2.1.5. *Suppose in 1.1(1) that $f : X \to Y$ is weakly semi-stable and admits a good minimal model $f' : X' \to Y, \varphi : X \dashrightarrow X'$.*

(1) [F, 4.4 (see also Corrigendum, page 262, line 20–21)]. *The total space X has only rational Gorenstein singularities. Let $y \in Y$ be an arbitrary point and let C be a smooth curve passing through y such that $C = H_1 \cap H_2 \cap \ldots \cap H_{\dim Y-1}$, where every H_i is a general smooth Cartier divisor on Y. Then $X_C = X \times_Y C \to C$ is also weakly semi-stable (by* [Kar, Lemma 2.12]*), and $X'_C = X' \times_Y C$ is normal and has only canonical singularities.*

(2) [F, 4.3] *The morphism $f' : X' \to Y$ is equi-dimensional and flat. For every $y \in Y$, the scheme theoretic fiber X'_y of f' has only semi-log-canonical singularities (which is reduced at least), and $\mathcal{O}_{X'}(mK_{X'})|_{X'_y} \cong \mathcal{O}_{X'_y}(mK_{X'_y})$ for every integer $m > 0$. In particular $K_{X'_y}$ is $\mathbb{Q}$-Cartier and semi-ample.*

(3) [F, §4, Proof of 1.6, Step 3] *For every integer $m > 0$, the dimension $h^0(X'_y, \mathcal{O}_{X'}(mK_{X'})|_{X'_y})$ is independent of $y \in Y$, and $f_*\mathcal{O}_{X'}(mK_{X'})$ is locally free. We also note $h^0(X'_y, \mathcal{O}_{X'_y}(mK_{X'_y})) = h^0(X_y, \mathcal{O}_{X_y}(mK_{X_y})) = P_m(X_y)$ for general $y \in Y$.*

We will use the following basic observation several times.

LEMMA 2.1.6. *Suppose in 1.1 (1) that $f : X \to Y$ is weakly semi-stable and admits a good minimal model $f' : X' \to Y, \varphi : X \dashrightarrow X'$. Suppose further the Kodaira dimension of a general fiber of f is zero: $\kappa(X_y) = 0$. Then for every integer $m > 0$ with $P_m(X_y) \neq 0$ for $y \in Y$ general, $\mathcal{O}_{X'}(mK_{X'/Y})$ is a line bundle and f'-trivial. In fact $f'^*L_m \cong \mathcal{O}_{X'}(mK_{X'/Y})$ with $L_m = f'_*\mathcal{O}_{X'}(mK_{X'/Y})$ which is a line bundle.*

PROOF. (1) By 2.1.5 (3), every direct image $L_p := f'_*\mathcal{O}_{X'}(pK_{X'/Y})$ with an integer $p > 0$ is a line bundle or the zero sheaf. We will denote by $\eta_p : H^0(Y, L_p) \xrightarrow{\sim} H^0(X', \mathcal{O}_{X'}(pK_{X'/Y}))$ the natural identification.

As $K_{X'/Y}$ is $\mathbb{Q}$-Cartier and f'-semi-ample, for every large and divisible integer q, $\mathcal{O}_{X'}(qK_{X'/Y})$ is a line bundle and the natural homomorphism $f'^*f'_*\mathcal{O}_{X'}(qK_{X'/Y}) \to \mathcal{O}_{X'}(qK_{X'/Y})$ is surjective. As both are line bundles, the homomorphism is actually isomorphic. Hence $f'^*L_q \cong \mathcal{O}_{X'}(qK_{X'/Y})$. If we take an open set $W \subset Y$ such that $L_q \cong \mathcal{O}_W$ on W and $b \in H^0(W, L_q)$ a nowhere vanishing section, then $f'^*b \in H^0(f^{-1}(W), f'^*L_q)$ generates f'^*L_q everywhere, and $\eta_q(b) \in H^0(f^{-1}(W), \mathcal{O}_{X'}(qK_{X'/Y}))$ generates $\mathcal{O}_{X'}(qK_{X'/Y})$ everywhere on $f^{-1}(W)$.

(2) Let $q = pm$, for a large and divisible integer p, so that (1) holds. We take any affine open set $W \subset Y$ so that $L_m = \mathcal{O}_W$ and $L_q = \mathcal{O}_W$ on W. We still denote by $W = Y$. Let $a \in H^0(Y, L_m)$ be a nowhere vanishing section of the line bundle. Then we have a section $a^p \in H^0(Y, L_q)$ induced by the natural maps

$$\begin{aligned} H^0(X', \mathcal{O}_{X'}(mK_{X'/Y}))^{\otimes p} &\to H^0(X', \mathcal{O}_{X'}(mK_{X'/Y})^{\otimes p}) \\ &\to H^0(X', \mathcal{O}_{X'}(qK_{X'/Y})). \end{aligned}$$

We note that $\eta_q(a^p)|_{X'_{\mathrm{reg}}} = (\eta_m(a)|_{X'_{\mathrm{reg}}})^{\otimes p}$, i.e., $\eta_q(a^p)$ is a usual p-th power of $\eta_m(a)$ on X'_{reg}. (Mind that $\mathcal{O}_{X'}(mK_{X'/Y})^{\otimes p}$ may have a torsion.) Let $b \in H^0(Y, L_q)$ be a nowhere vanishing section. Then there exists $s \in H^0(Y, \mathcal{O}_Y)$ such that $a^p = sb$.

If $s(0) = 0$, then $\eta_q(a^p) = 0$ on X'_0 and hence $\eta_m(a) = 0$ on $(X'_0)_{\mathrm{reg}}$. We note that $(X'_0)_{\mathrm{reg}} \subset X'_{\mathrm{reg}}$ as X'_0 is locally complete intersection in X'. As a generates $L_m = f'_*\mathcal{O}_{X'}(mK_{X'/Y})$ everywhere on Y, the natural homomorphism $f'^*f'_*\mathcal{O}_{X'}(mK_{X'/Y}) \to \mathcal{O}_{X'}(mK_{X'/Y})$ is zero on $(X'_0)_{\mathrm{reg}}$, and hence it is zero on X'_0 as the degeneracy locus is closed (recall that X'_0 is reduced and hence $(X'_0)_{\mathrm{reg}}$ is everywhere dense in X'_0). That means a is zero at $0 \in Y$, which contradicts to the fact that a is nowhere vanishing. By the same token, s is nowhere vanishing. Then $\eta_q(a^p) = (f'^* s)\eta_q(b)$ (and hence $\eta_m(a)$) are nowhere vanishing on X'_{reg}. Thus $\eta_m(a)$ gives a trivialization $\mathcal{O}_{X'}(mK_{X'/Y})|_{X'_{\mathrm{reg}}} \cong \mathcal{O}_{X'_{\mathrm{reg}}}$. Finally we have $\mathcal{O}_{X'}(mK_{X'/Y}) = j_*(\mathcal{O}_{X'}(mK_{X'/Y})|_{X'_{\mathrm{reg}}}) \cong \mathcal{O}_{X'}$, where $j : X'_{\mathrm{reg}} \to X'$ is the open immersion.

The argument above shows that, over any such $W \subset Y$, $\mathcal{O}_{X'}(mK_{X'/Y})$ is a line bundle and the natural homomorphism $f'^* f'_* \mathcal{O}_{X'}(mK_{X'/Y}) \to \mathcal{O}_{X'}(mK_{X'/Y})$ is surjective (and hence isomorphic). This is enough to conclude our assertion. □

2.2. Kawamata's inversion of adjunction

We take this opportunity to prove an inversion of adjunction, essentially due to Kawamata [Ka2], which we will need in our argument. We will use an independent notation.

LEMMA 2.2.1. *Let $f : X \to B$ be a flat morphism from a germ of an algebraic variety to a germ of a domain B in $\mathbb{C}^m$ centered at the origin $0 \in B$. Assume that the scheme theoretic fiber X_0 of f has only canonical singularities.*

(1) Deformation of canonical singularities. *Then (after shrinking X and B), X as well as any fiber $X_b = f^{-1}(b)$ of f has only canonical singularities.*

(2) Inversion of adjunction. *Moreover, if $B_0 \subset B$ is a smooth divisor containing 0 and if $\mu : V \to X$ is a birational morphism from a normal variety, then (after shrinking X and B), $S_0 := f^* B_0$ has only canonical singularities and $K_V + W \geq \mu^*(K_X + S_0)$ holds, where $W \subset V$ is the strict transform of S_0. In particular, $K_V - \mu^* K_X \geq \mu^* S_0 - W \geq 0$ and the pair (X, S_0) is canonical.*

This is due to Kawamata [Ka2, 1.4] when $\dim B = 1$. We will reduce our assertion by induction on $\dim B$ to the case $\dim B = 1$. The assertion (2) is a sort of an inversion of adjunction, cf. Stevens [Sv] ([Ko, 7.9]) in the case ω_X is locally free. In fact, [Ka2, 1.4] implies the following: *Let X be an algebraic variety and let S be a Cartier divisor on X. Then S has only canonical singularities if and only if the pair (X, S) is canonical around S.* If S is canonical, we can see easily (well known) that X is normal and K_X is $\mathbb{Q}$-Cartier. The pair (X, S) is canonical requires that X is normal and K_X is $\mathbb{Q}$-Cartier. It is easy to see that S is canonical if (X, S) is canonical.

PROOF. We first note in any event that X is normal and K_X is $\mathbb{Q}$-Cartier around X_0. This can be seen inductively. Let L be a line in B containing 0. We note that $f^{-1}(L)$ is R_1 (regular in codimension 1), CM (Cohen–Macaulay), and Gorenstein in codimension 2 around X_0 (in particular $f^{-1}(L)$ is normal), as the Cartier divisor $X_0 \subset f^{-1}(L)$ (which has canonical singularities at worst) has the same properties. Stevens [Sv, p. 280] observed that K_{X_0} is $\mathbb{Q}$-Cartier of index m implies $K_{f^{-1}(L)}$ is $\mathbb{Q}$-Cartier of index m. We can see also inductively that $f^{-1}(M)$ is R_1, CM, and Gorenstein in codimension 2 around X_0 for any hyperplane $M \subset B$ containing 0 (including the case $M = B$ and $f^{-1}(M) = X$). The canonical divisor K_M is also $\mathbb{Q}$-Cartier of the same index as K_{X_0} has.

Let us see our main assertions. In the case $m = 1$, this is [Ka2, 1.4]. Suppose our assertion holds in the case when the base dimension is $m - 1$ (and $m \geq 2$). We shall prove our assertion as in the statement with $\dim B = m$.

We take a local coordinate centered at $0 \in B$ such that the given smooth divisor B_0 in (2) is a coordinate hyperplane in the new coordinate (after shrinking B). Let $a_B : \widetilde{B} \to B$ be the blow-up at $0 \in B$, and let $E_B (\cong \mathbb{P}^{m-1}) \subset \widetilde{B}$ be the exceptional divisor. We consider the fiber product:

$$\begin{array}{ccccc} \widetilde{X} & = & \widetilde{X} := X \times_B \widetilde{B} & \xrightarrow{a_X} & X \\ \downarrow g & & \downarrow \tilde{f} & & \downarrow f \\ \mathbb{P}^{m-1} & \xleftarrow{p} & \widetilde{B} & \xrightarrow{a_B} & B \end{array}$$

We note that $a_B : \widetilde{B} \setminus E_B \to B \setminus 0$ is isomorphic, $a_X : \widetilde{X} \setminus a_X^{-1}(X_0) \to X \setminus X_0$ is isomorphic, and that $a_X^{-1}(X_0) = \tilde{f}^{-1}(E_B) = E_B \times_0 X_0 \cong \mathbb{P}^{m-1} \times X_0$. In particular $E_X := a_X^{-1}(X_0)$ is the unique exceptional divisor for $a_X : \widetilde{X} \to X$. We set

$$X_{f,\mathrm{reg}} = \{x \in X_{\mathrm{reg}};\ f : X \to B \text{ is smooth at } x\},$$

which is Zariski open in X, and $(X_b)_{\mathrm{reg}} \subset (X_b \cap X_{f,\mathrm{reg}})$ for any $b \in B$, and hence $\operatorname{codim}(X \setminus X_{f,\mathrm{reg}}) \geq 2$. We see $X \times_B \widetilde{B} = Bl_{X_0}X$ holds "over $X_{f,\mathrm{reg}}$". We can see the closure of $\widetilde{X} \setminus a_X^{-1}(X_0)$ in $\widetilde{X}$ is $\widetilde{X}$, in particular $\widetilde{X}$ is irreducible.

The space $\widetilde{B}$ is realized in $B \times \mathbb{P}^{m-1}$ in a standard manner by the given coordinate on B. We take a convention that $\mathbb{P}^{m-1}$ is the space of lines in B passing through 0. Let $p : \widetilde{B} \to \mathbb{P}^{m-1}$ be the map obtained via the projection $B \times \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$. We consider $g = p \circ \widetilde{f} : \widetilde{X} \to \mathbb{P}^{m-1}$. This g is flat as a composition of two flat morphisms $\widetilde{f}$ and p ([Har, III.9.2]) ($\widetilde{f}$ is flat as a base change of the flat morphism f([Har, III.9.2]); p is flat and in fact a locally trivial family of lines in B passing through 0).

We take an arbitrary point $t \in \mathbb{P}^{m-1}$, and let L_t be the corresponding line in B containing 0. By [Ka2, 1.4] (as $\dim L_t = 1$ and $f : f^{-1}(L_t) \to L_t$ is still flat), $f^{-1}(L_t)$ has only canonical singularities and any fibers of $f^{-1}(L_t) \to L_t$ has canonical singularities (after shrinking X and B, or $f^{-1}(L_t)$ and L_t). We consider the fiber $g^{-1}(t) \subset \widetilde{X}$. Then by construction $g^{-1}(t) \cong f^{-1}(L_t)$, which has only canonical singularities. (Note that $p^{-1}(t) = a_B^{-1}(L_t)$ and $a_B^{-1}(L_t) \cong L_t$ by a_B for the line L_t.) Thus by induction hypothesis applied to $g : \widetilde{X} \to \mathbb{P}^{m-1}$, $\widetilde{X}$ has only canonical singularities around $g^{-1}(t)$. As $t \in \mathbb{P}^{m-1}$ is arbitrary, $\widetilde{X}$ has only canonical singularities around $\widetilde{f}^{-1}(E_B) = E_X$. As $\mathbb{P}^{m-1} = E_B$ is compact, we only need to shrink X and B finitely many times in fact to obtain our assertion (1).

We take an arbitrary hyperplane $H(\cong \mathbb{P}^{m-2}) \subset \mathbb{P}^{m-1}$, and let $B_H \subset B$ be the corresponding hyperplane containing 0. By the induction hypothesis (as $\dim B_H = m-1$ and $f : f^{-1}(B_H) \to B_H$ is flat), $f^{-1}(B_H)$ has canonical singularities (and all fibers of $f^{-1}(B_H) \to B_H$ has canonical singularities). We consider $p^{-1}(H) = \widetilde{B}_H$, where $\widetilde{B}_H$ is the strict transform of B_H (i.e., the blow-up of B_H at $0 \in B_H$), and $g^{-1}(H) = \widetilde{f}^{-1}(\widetilde{B}_H)$. By construction $g^{-1}(H)$ is the strict transform of $f^{-1}(B_H)$ by a_X in $\widetilde{X}$. We suppose our initial B_0 corresponds to a hyperplane $H_0 \subset \mathbb{P}^{m-1}$. In particular we proved that $S_0 = f^*B_0$ has canonical singularities.

Let $\nu : V' \to \widetilde{X}$ be a birational morphism from a normal (or smooth) variety with a strict transform $W'(\subset V')$ of $g^{-1}(H_0) \subset \widetilde{X}$ (recall $g^{-1}(H_0)$ is the strict transform of $f^{-1}(B_0) = S_0$). Then by the induction hypothesis applied to $g : \widetilde{X} \to \mathbb{P}^{m-1}$ with a smooth divisor $H_0 \subset \mathbb{P}^{m-1}$, we have $K_{V'} + W' \geq \nu^*(K_{\widetilde{X}} + g^{-1}(H_0))$, and the pair $(\widetilde{X}, g^{-1}(H_0))$ is canonical.

We note that $a_B^* B_0 = p^{-1}(H_0) + E_B$ and $a_X^* S_0 = g^{-1}(H_0) + E_X$. The latter formula is obtained as $a_X^* S_0 = a_X^* f^* B_0 = \widetilde{f}^* a_B^* B_0 = \widetilde{f}^*(p^{-1}(H_0) + E_B) = g^{-1}(H_0) + E_X$. We also have $K_{\widetilde{X}} = a_B^* K_X + (m-1)E_X$ (this formula holds on $a_X^{-1}(X_{f,\mathrm{reg}}) \subset \widetilde{X}$ with $\operatorname{codim}(\widetilde{X} \setminus a_X^{-1}(X_{f,\mathrm{reg}})) \geq 2$ by the usual manner first, then it holds on $\widetilde{X}$ because there is no divisor in $\widetilde{X} \setminus a_X^{-1}(X_{f,\mathrm{reg}})$. Now, $K_{V'} + W' \geq \nu^*(K_{\widetilde{X}} + g^{-1}(H_0)) = \nu^*(a_B^* K_X + (m-1)E_X + a_X^* S_0 - E_X) = (a_X \circ \nu)^*(K_X + S_0) + \nu^*((m-2)E_X) \geq (a_X \circ \nu)^*(K_X + S_0)$ as $m \geq 2$. Thus the pair (X, S_0) is canonical. As $(a_X \circ \nu)^* S_0 \geq W'$, X has only canonical singularities.

Since any $\mu : V \to X$ in (2) is dominated by some $\nu : V' \to \widetilde{X}$ as above, we have $K_V + W \geq \mu^*(K_X + S_0)$. For a given $\mu : V \to X$, we can find $\nu : V' \to X$ as above with $\mu' : V' \to V$ such that $\nu = \mu \circ \mu'$. In general $\mu'_* \mathcal{O}_{V'}(m(K_{V'} + W')) \subset \mathcal{O}_V(m(K_V + W))$. Thus $K_{V'} + W' \geq \nu^*(K_X + S_0)$ implies $K_V + W \geq \mu^*(K_X + S_0)$. □

3. Continuity of the Canonical L^2-Metric

Throughout this section, we use the following set up.

SET UP 3.0.1. We suppose in 1.1 (1) that $f : X \to Y$ is weakly semi-stable and admits a good minimal model $f' : X' \to Y, \varphi : X \dashrightarrow X'$. Let $X_0 = \bigcup_{i \in I} F_i$ be the decomposition into irreducible components (recall that f has reduced and equi-dimensional fibers, 2.1.2).

$$\begin{array}{ccc} X' & \overset{\varphi}{\dashleftarrow} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & = & Y \end{array}$$

We further suppose that Y is affine and $K_Y \cong \mathcal{O}_Y$, and hence $K_{X/Y} \cong K_X$.

3.1. A simultaneous minimal model

We recall a result in [Ta4]. For a proper variety V, the m-genus $P_m(V)$ is defined by that of any smooth birational model $\widetilde{V}$; $P_m(V) = h^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(mK_{\widetilde{V}}))$. We note, by [Ta1, 1.2]: the lower semi-continuity of plurigenera, that in the setting 3.0.1, $\sum_{i \in I} P_m(F_i) \leq P_m(X_y)$ holds for every integer $m > 0$ and a general point $y \in Y$.

THEOREM 3.1.1 ([Ta4, 1.1]). *Suppose that for every large and divisible integer $m > 0$, the plurigenera equality $\sum_{i \in I} P_m(F_i) = P_m(X_y)$ holds for a general point $y \in Y$.*

Then the fiber X'_0 of f' over 0 is birational to an irreducible component, say F_1, by the birational map $\varphi : X \dashrightarrow X'$, and other components F_i ($i \in I \setminus \{1\}$ if $|I| \geq 2$) are uniruled and contained in the stable base locus of K_X: $F_i \subset \mathrm{SBs}\,(K_X)$. Moreover X'_0 is normal with canonical singularities at worst and $K_{X'_0}$ is semi-ample, and $P_m(F_1) = P_m(X'_0) = P_m(X_y)$ holds for every integer $m > 0$ and a general point $y \in Y$.

The property $F_i \subset \mathrm{SBs}\,(K_X)$ means that F_i is contained in the zero locus of any $s \in H^0(X, \mathcal{O}_X(mK_X))$ and any integer $m > 0$.

Lemma 3.1.2 ([Ta4, 2.5]). *Let $X'_0 = \bigcup_{\ell \in L} G'_\ell$ be the irreducible decomposition of the fiber of f'. Suppose $\kappa(G'_\ell) \geq 0$ for any $\ell \in L$. Then $X_0 = \bigcup_{\ell \in L} G_\ell \cup \bigcup_{\lambda \in \Lambda} F_\lambda$ with irreducible components G_ℓ birational to G'_ℓ by the birational map $\varphi : X \dashrightarrow X'$ and other components F_λ are uniruled.*

We state some conclusions which are specified in the present paper, especially 1.3.

Corollary 3.1.3. *Suppose that X'_0 has canonical singularities at worst. Then there exists a unique irreducible component of X_0, say F_1, such that F_1 is birational to X'_0 by the birational map $\varphi : X \dashrightarrow X'$, and other components F_i ($i \in I \setminus \{1\}$ if $|I| \geq 2$) are contracted by φ, F_i are uniruled, and $F_i \subset \mathrm{SBs}\,(K_X)$.*

Remark 3.1.4. Suppose that X'_0 has canonical singularities at worst.

Then by 2.2.1, there exists an open neighborhood $0 \in W \subset Y$ such that all X'_y ($y \in W$) has canonical singularities at worst. We can apply 3.1.3 to X'_y ($y \in W$) too (for $y \in Y^o$, X'_y is simply birational to X_y and no other components).

Let $C_1 \subset Y$ be an irreducible divisor containing 0, and let $f^*C_1 = \sum_{j \in J} B_j$ be the decomposition into irreducible components. We then have $X_y = \sum_{j \in J}(X_y \cap B_j)$ for every $y \in C_1 \cap W$. (We do not say that $X_y \cap B_j$ is irreducible.) Except one irreducible component of X_y, other components of X_y are contained in $\mathrm{SBs}\,(K_X)$. As $\sum_{j \in J} B_j$ has only finite irreducible components and as $\{y \in C_1 \cap W\}$ are uncountable, we have $\sum_{j \in J \setminus \{j_1\}} B_j \subset \mathrm{SBs}\,(K_X)$ (except one index $j_1 \in J$). Note that $\mathrm{SBs}\,(K_X)$ is closed.

Remark 3.1.5. Suppose that $\kappa(X_y) = 0$ and $p_g(X_y) = 1$ for every $y \in Y^o$.

(1) Then by 2.1.6, $K_{X'/Y}$ is f'-trivial and $K_{X'/Y} = f'^* f'_* K_{X'/Y}$.

(2) The plurigenera equality condition $\sum_{i \in I} P_m(F_i) = P_m(X_y)$ in 3.1.1 is equivalent to that there exists an integer $\ell > 0$ and an $i \in I$ such that $P_\ell(F_i) > 0$. As a conclusion of 3.1.1, there exists a unique $i \in I$, say $i = 1$, such that $P_m(F_1) = 1$ for any $m > 0$, and that $P_{m'}(F_i) = 0$ for any $m' > 0$ and any $i \in I \setminus \{1\}$. We also have $K_{X'_0} = \mathcal{O}_{X'_0}$ by adjunction.

3.2. Continuity of fiberwise integrals

We shall prove 1.3. We use the same notations in 3.0.1. Then 1.3 is

Theorem 3.2.1. *Suppose further X'_0 has canonical singularities. Let $u \in H^0(X, K_{X/Y})$. Then a C^∞-function $\int_{X_t} u \wedge \overline{u}$ on $t \in Y^o$ extends as a continuous function on Y around $0 \in Y$. Moreover* $\lim_{Y^o \ni t \to 0} \int_{X_t} u \wedge \overline{u} \neq 0$ *if $f_* u$ is non-zero at $t = 0$, where $f_* u \in H^0(Y, f_* K_{X/Y})$ is the section corresponding to u via the natural isomorphism $H^0(X, K_{X/Y}) \cong H^0(Y, f_* K_{X/Y})$.*

We start with some generalities of fiber integrals based on the setting of 3.0.1.

Remark 3.2.2. Let $f : (X, B) \to (Y, C)$ be the toroidal structure in 2.1.2, and let $n = \dim X, m = \dim Y$.

(1) Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a (fine enough) locally finite open covering of X, and let $\{\rho_\lambda\}_\lambda$ be a partition of unity subordinate to $\{U_\lambda\}_\lambda$. The fiberwise L^2-norm of $u \in H^0(X, K_{X/Y})$ on X_t ($t \in Y \setminus C \subset Y^o$) is

$$\int_{X_t} (-1)^{(n-m)^2/2} u \wedge \overline{u} = \sum_\lambda \int_{X_t \cap U_\lambda} \rho_\lambda (-1)^{(n-m)^2/2} u \wedge \overline{u}.$$

Thus, as usual, our assertion is reduced to a local computation (note f is proper). We take one of U_λs. As (X, B) is quasi-smooth, we may assume as in 2.1.1 that there exists a finite Galois toroidal cover $\pi : (U, D) \to (U_\lambda, B|_{U_\lambda})$ such that $(U, D) \subset \mathbb{C}^n$ is smooth toric and $(U_\lambda, B|_{U_\lambda}) = (U, D)/G$ for a finite abelian group G (which is "small"). Then we have

$$\int_{X_t \cap U_\lambda} \rho_\lambda u \wedge \overline{u} = \frac{1}{|G|} \int_{U_t = (f \circ \pi)^{-1}(t)} \pi^* (\rho_\lambda u \wedge \overline{u})$$

for $t \in Y \setminus C$. Hence the continuity of $\int_{X_t} u \wedge \overline{u}$ at $0 \in Y$ is reduced to a computation on smooth toric local models (U, D).

(2) We consider a semi-stable morphism $f \circ \pi : (U, D) \to (W, C)$, where $W \subset Y$ is a small analytic open subset. We let

$$\Omega^1_{U/W}(\log) = \Omega^1_U(\log D)/(f\circ\pi)^*\Omega^1_W(\log C) \text{ and } \Omega^p_{U/W}(\log) = \bigwedge^p \Omega^1_{U/W}(\log)$$

(as in [Ka3, Example 13]), where $\Omega^1_U(\log D)$ is (as usual) the sheaf of logarithmic differential 1-forms. Then $\Omega^{n-m}_{U/W}(\log) = K_{U/W}$ as it is remarked after [Ka3, Theorem 26] (the horizontal part is zero in our setting).

(3) We take local coordinates $x = (x_1, \dots, x_n)$ on U and $t = (t_1, \dots, t_m)$ on W in (2) above such that the morphism $f \circ \pi : U \to W$ is given by

$$t_1 = \prod_{j=1}^{\ell_1} x_j,\ t_2 = \prod_{j=\ell_1+1}^{\ell_2} x_j,\ \dots,\ t_m = \prod_{j=\ell_{m-1}+1}^{\ell_m} x_j$$

for some $0 < \ell_1 < \dots < \ell_m \leq n$ ([AK, §0.3]). We set $J = \{1, 2, \dots, \ell_{m-1}, \ell_m\} \setminus \{\ell_1, \dots, \ell_m\}$, and set $D_J = \sum_{j \in J} D_j$ with $D_j = \{x_j = 0\}$. We denote by $d\log x_j = \frac{dx_j}{x_j}$. Then we can take

$$d\log x_{U/W} = \bigwedge_{j \in J} d\log x_j \wedge \bigwedge_{\ell_m < i \leq n} dx_i \ \in H^0(U, \Omega^{n-m}_{U/W}(\log))$$

as a nowhere vanishing section on U. The labelling among $\{1, \dots, \ell_1\}$ for example is not essential, which we mean ℓ_1 is not special. In fact, we have $\sum_{i=1}^{\ell_1} d\log x_i = d\log t_1$ in $\Omega^1_U(\log D)$, and hence $\sum_{i=1}^{\ell_1} d\log x_i = 0$ in $\Omega^1_{U/W}(\log)$. Thus for any $i_0 \in \{1, \dots, \ell_1\}$, we have $\bigwedge_{i \in \{1,\dots,\ell_1\} \setminus \{i_0\}} d\log x_i = \bigwedge_{1 \leq i \leq \ell_1 - 1} d\log x_i$ in $\Omega^{\ell_1 - 1}_{U/W}(\log)$ up to a sign (a permutation).

These are generalities on fiber integrals for weakly semi-stable morphisms. A special feature in our 3.2.1 is that, by applying 3.1.4 for each irreducible component of $C \subset Y$, we can suppose (after re-labeling x_j) $\pi(D_J) \subset \mathrm{SBs}\,(K_X) = \mathrm{SBs}\,(K_{X/Y})$. In particular, for $u \in H^0(X, K_{X/Y})$ in 3.2.1, $\pi^* u \in H^0(U, K_{U/W})$ vanishes along D_J. We here note $K_U = \pi^* K_{U_\lambda}$ (as U_λ is Gorenstein and $\pi : U \to U_\lambda$ is finite and unramified in codimension 1) and hence $K_{U/W} = \pi^* K_{U_\lambda/W}$. The following is the main proposition, which concludes 3.2.1 applied to $\sigma = \pi^* u$.

PROPOSITION 3.2.3. *Let $\sigma \in H^0(U, \Omega^{n-m}_{U/W}(\log))$, and suppose its zero divisor contains D_J as a line bundle $\Omega^{n-m}_{U/W}(\log) = K_{U/W}$ valued section. Let $\rho \in C_0(U, \mathbb{C})$ be a continuous function with compact support on U. Then*

(1) there exists $\widetilde{\sigma} \in H^0(U, \Omega^{n-m}_U)$ such that $\sigma|_{U_t} = \widetilde{\sigma}|_{U_t}$ for any $t \in W \setminus C$.

(2) A continuous function $\int_{U_t} \rho\, \sigma \wedge \overline{\sigma}$ on $W \setminus C$ extends as a continuous function on W.

(3) Suppose further that σ does not vanish identically along $U_0 = (f \circ \pi)^{-1}(0)$, the function ρ takes values to $\mathbb{R}_{\geq 0}$, and $\rho|_{U_0} \not\equiv 0$. Then $\lim_{(W\setminus C)\ni t\to 0} \int_{U_t} \rho\, (-1)^{(n-m)^2/2} \sigma \wedge \overline{\sigma} > 0$.

PROOF. (1) We can write as $\sigma = \sigma_0 d\log x_{U/W}$ with $\sigma_0 \in H^0(U, \mathcal{O}_U)$ vanishing along D_J, i.e., $\sigma_0 = (\prod_{j\in J} x_j)\sigma_1$ with $\sigma_1 \in H^0(U, \mathcal{O}_U)$. Then we can take $\widetilde{\sigma} = \sigma_1 \bigwedge_{j\in J} dx_j \wedge \bigwedge_{\ell_m < i \leq n} dx_i \in H^0(U, \Omega^{n-m}_U)$.

(2) By using $\widetilde{\sigma}$ in (1), we have $\int_{U_t} \rho\, \sigma \wedge \overline{\sigma} = \int_{U_t} \rho\, \widetilde{\sigma} \wedge \overline{\widetilde{\sigma}}$ for $t \in W \setminus C$. As $\rho\, \widetilde{\sigma} \wedge \overline{\widetilde{\sigma}}$ is a continuous $(n-m, n-m)$-form with compact support on U, the fiberwise integral $\int_{U_t} \rho\, \widetilde{\sigma} \wedge \overline{\widetilde{\sigma}}$ is defined for every $t \in W$ and continuous on W (see Barlet [B, p. 378, Théorème 1] for the continuity of fiber integrals for flat morphisms). In particular

$$\lim_{(W\setminus C)\ni t\to 0} \int_{U_t} \rho\, \sigma \wedge \overline{\sigma} = \int_{U_0} \rho\, \widetilde{\sigma} \wedge \overline{\widetilde{\sigma}} := \int_{U_{0,reg}} \rho\, \widetilde{\sigma} \wedge \overline{\widetilde{\sigma}}.$$

(3) The function $\sigma_1 \in H^0(U, \mathcal{O}_U)$ above does not vanish identically along U_0. Then it is not difficult to see that $\int_{U_{0,reg}} \rho\, (-1)^{(n-m)^2/2}\, \widetilde{\sigma} \wedge \overline{\widetilde{\sigma}} > 0$. □

4. Proof of 1.2 and 1.4

We shall prove 1.2. We introduce the following intermediate condition (1') in 1.2 to divide our proof into several steps:

1.2 *(1') There exist a weak semi-stable reduction $f' : X' \to Y'$ of f and a good minimal model $f'' : X'' \to Y', \varphi : X' \dashrightarrow X''$ of f' (as below) such that for any $q \in \tau^{-1}(0)$, the fiber X''_q has canonical singularities at worst.*

$$\begin{array}{ccccc} X'' & \overset{\varphi}{\dashleftarrow} & X' & \xrightarrow{\tau_X} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & = & Y' & \xrightarrow{\tau} & Y \end{array}$$

We note that in 1.2 (1'), $K_{X''/Y'}$ is f''-trivial by 2.1.6 without knowing if X''_q has canonical singularities or not. It is easy to see that implications (1) $\Rightarrow$ (1') $\Rightarrow$ (2) hold. We shall prove (1') $\Rightarrow$ (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (1').

4.1. Proof of 1.2 (1') $\Rightarrow$ (3)

There are 2 steps. The first step is to bound the diameter of (X_y, ω_y) by the canonical L^2-norm of $f_*K_{X'/Y'}$. We then secondly apply 1.3 i.e. 3.2.1: the continuity of the canonical L^2-metric of $f_*K_{X'/Y'}$.

The first step is a variant of [RZ, 2.1], where a family $\pi : \mathcal{M} \to \Delta^1$ over a disc $\Delta^1 \subset \mathbb{C}$ with $K_{\mathcal{M}/\Delta^1} = \mathcal{O}_{\mathcal{M}}$ is considered. We generalize it when the base space is a polydisc and when a general fiber is a modification of Calabi-Yau (see 4.1.1 below). We localize our $f' : X' \to Y'$ in (1') over a small polydisc $\Delta(\cong \Delta^m) \subset Y'$ centered at a given point $q \in \tau^{-1}(0)$. Denote it by $\pi : \mathcal{M} \to \Delta$, where $\mathcal{M} = X'_\Delta := f'^{-1}(\Delta)$, and $\pi = f'|_{X'_\Delta}$. We recall $\mathcal{M}$ is normal Gorenstein with canonical singularities. Let $f : (X', B') \to (Y', C')$ be the toroidal structure as in 2.1.2. We will let $U_{\mathcal{M}} = \mathcal{M} \cap (X' \setminus B')$ and $U_\Delta = \Delta \cap (Y' \setminus C')$ with $U_{\mathcal{M}} = \pi^{-1}(U_\Delta)$. We will denote by $0 \in \Delta$ the center instead of $q \in Y'$ (we hope there is no risks of confusions with $0 \in Y$). We will not distinguish a Kähler form and the associated Kähler metric. For every $y \in Y^o$, we denote by $\omega_{\mathrm{KE},y} \in c_1(L|_{X_y})$ the unique Ricci-flat Kähler metric on X_y. Let $L' = (\tau_X|_{\tau_X{}^{-1}(X^o)})^* L$ a line bundle on $\tau_X^{-1}(X^o)$; the pull back of L. We note that $L'_t := L'|_{M_t}$ is merely semi-ample and big. Here $t \in \Delta$ stands for a point and $t = 0$ corresponds to $q \in Y'$.

We recall that $K_{X''/Y'} = \mathcal{O}_{X''}$ on $X''_\Delta := f''^{-1}(\Delta)$. We take a nowhere vanishing section $\Omega'' \in H^0(X''_\Delta, K_{X''/Y'})$. As $\varphi : X' \dashrightarrow X''$ is birational and both X' and X'' have only canonical singularities at worst, Ω'' corresponds to a section $\Omega' \in H^0(\mathcal{M}, K_{\mathcal{M}/\Delta})$. Let F be the unique irreducible component of $X'_q = M_0$ such that F is birational to X''_q via $\varphi : X' \dashrightarrow X''$ (see 3.1.3, here we use the assumption that X''_q has canonical singularities at worst). Then Ω' does not vanish identically along F. Let $\Omega'_t = \Omega'|_{M_t} \in H^0(M_t, K_{M_t})$ for $t \in U_\Delta$. We shall prove the following estimate, which will also corresponds to the implication (c) $\Rightarrow$ (e) in 1.4.

Proposition 4.1.1 (cf. [RZ, 2.1]). *There is a constant $D > 0$ independent of $t \in U_\Delta$ such that*

$$\operatorname{diam}(X_y, \omega_{\mathrm{KE},y}) \leq 2 + D \int_{M_t} (-1)^{n^2/2} \Omega'_t \wedge \overline{\Omega'_t}$$

holds for all $t \in U_\Delta$ *with* $y = \tau(t) \in Y^o$*, where* $n = \dim X - \dim Y$.

Let us conclude 1.2 (1') ⇒ (3) by taking 4.1.1 for granted.

Proof of 1.2 (1') ⇒ (3). In the setting of 4.1.1, 1.3 namely 3.2.1 implies that the fiberwise integrals $\int_{M_t}(-1)^{n^2/2}\Omega'_t\wedge\overline{\Omega'_t}$ is continuous (smooth in fact) in $t \in U_\Delta$ and extends continuously (and non-degenerate) on Δ, possibly shrinking Δ. Possibly after shrinking Δ further, we can suppose it is bounded continuous.

In the case $U_{Y'} = \tau^{-1}(Y^o)$ on a neighborhood of $0 \in Y$, it is enough to combine 4.1.1 and 3.2.1 directly (note that $\tau : Y' \to Y$ is proper, we can cover $\tau^{-1}(0)$ by a finite number of polydiscs Δ appeared in the discussion above). Even if it is not the case, 4.1.1 and 3.2.1 show that there exist an open neighborhood W of $0 \in Y$ and a constant $\alpha > 0$ such that $\operatorname{diam}(X_y, \omega_{\mathrm{KE},y}) \leq \alpha$ for any $y \in (Y^o \cap W) \setminus \tau(Y' \setminus U_{Y'})$. Then by the continuity of $\operatorname{diam}(X_y, \omega_{\mathrm{KE},y})$ on $y \in Y^o$, we have $\operatorname{diam}(X_y, \omega_{\mathrm{KE},y}) \leq \alpha$ for any $y \in Y^o \cap W$. □

Proof of 4.1.1. We closely follow the argument of [RZ, 2.1]. We take an auxiliary π-ample line bundle $\mathcal{L}$ on $\mathcal{M}$, which gives an embedding $\mathcal{M} \to \mathbb{P}^N \times \Delta$ over Δ such that $\mathcal{L}^m = (\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^N}(1))|_{\mathcal{M}}$ for some $m \geq 1$, where $\mathrm{pr}_1 : \mathbb{P}^N \times \Delta \to \mathbb{P}^N$ is the projection. Let $\omega_{\mathrm{FS},t} = \frac{1}{m}\omega_{\mathrm{FS}}|_{M_t}$ be the pull-back of the Fubini-Study metric/form via the induced embedding $M_t \to \mathbb{P}^N$ (see [RZ, p. 241 top], $\omega_{\mathrm{FS},t}$ is simply ω_t). We use the Kähler metric $\omega_{\mathrm{FS},t}$ as a reference metric. Although in the set up of [RZ, 2.1], L' (the pull-back of our L) and $\mathcal{L}$ are the same, we can actually separate them.

We denote by $t \in U_\Delta$ and indicate $y = \tau(t) \in Y^o$ correspondingly. The morphism $\tau_X : X' \to X$ gives a birational morphism $\tau_t : M_t \to X_y$ being $y = \tau(t)$. We consider a possibly degenerate Kähler-Einstein metric $\widetilde{\omega}_t := \tau_t^*\omega_{\mathrm{KE},y} \in c_1(L'_t)$. The metric $\widetilde{\omega}_t$ may degenerate, but it behave quite well as it is a pull-back of a usual (Kähler-Einstein) metric by a birational morphism. We have $H^0(M_t, K_{M_t}) \cong H^0(X_y, K_{X_y})$ via τ_t, in particular we have a unique nowhere vanishing section $\Omega_y \in H^0(X_y, K_{X_y})$ such that $\tau_t^*\Omega_y = \Omega'_t$ (as a pull-back of $(n,0)$-form). We can use known results for $\omega_{\mathrm{KE},y}$ and Ω_y to obtain some variants for $\widetilde{\omega}_t$ and Ω'_t by pull-back. For example, the Ricci-flat Kähler form $\omega_{\mathrm{KE},y}$ satisfies a Monge-Ampère equation

$$\omega_{\mathrm{KE},y}^n = e^{c_y}(-1)^{n^2/2}\Omega_y \wedge \overline{\Omega}_y$$

for a normalizing constant $c_y \in \mathbb{R}$ satisfying $c_1(L_y)^n = e^{c_y} \int_{X_y} (-1)^{n^2/2}\Omega_y \wedge \overline{\Omega}_y$, where $c_1(L_y)^n$ is independent of $y \in Y^o$. Thus by pulling-back on M_t, we have

$$\widetilde{\omega}_t^n = e^{c_y}(-1)^{n^2/2}\Omega'_t \wedge \overline{\Omega'_t}.$$

Here $c_y \in \mathbb{R}$ also satisfies $c_1(L'_t)^n = e^{c_y} \int_{M_t} (-1)^{n^2/2}\Omega'_t \wedge \overline{\Omega'_t}$. These are around [RZ, p. 241, (2.1)].

We take a general point $p \in F \subset M_0 = X'_q$ such that $\varphi : X' \dashrightarrow X''$ is biregular around p and $f' : X' \to Y'$ (and hence $\pi : \mathcal{M} \to \Delta$ also) is smooth around p. Let U be a coordinate neighborhood of p in $\mathcal{M}$, which is biholomorphic to a polydisc $\Delta^m \times \Delta^n$ in $\mathbb{C}^m \times \mathbb{C}^n$ and the projection $\Delta^m \times \Delta^n \to \Delta^m$ is compatible with the map $\pi : U \subset \mathcal{M} \to \Delta$ and the identification $\Delta \cong \Delta^m$ (possibly after shrinking Δ). Let $\omega_E = \sqrt{-1}\partial\overline{\partial} \sum_{i=1}^n |z_i|^2$ be the standard Euclidean Kähler form on Δ^n, and g_E denotes the corresponding metric. As $\{\omega_{\mathrm{FS},t}\}_{t \in \Delta^m}$ is a smooth family of usual Kähler forms/metrics on a neighborhood of U, there exists a constant $C_1 > 1$ (independent of t) such that $C_1^{-1}\omega_E \leq \omega_{\mathrm{FS},t} \leq C_1\omega_E$ holds on $U|_{M_t} \cong \{t\} \times \Delta^n$ for any $t \in \Delta$. These are around [RZ, p. 241, (2.2)]. As $\varphi : X' \dashrightarrow X''$ is biregular around p and $\Omega'' \in H^0(X''_\Delta, K_{X''/Y'})$ is nowhere vanishing (in particular $\Omega' \in H^0(\mathcal{M}, K_{\mathcal{M}/\Delta})$ is nowhere vanishing on the neighborhood U of p), there exists a constant $\kappa_U > 0$ (independent of t) such that $(-1)^{n^2/2}\Omega'_t \wedge \overline{\Omega'_t} \geq \kappa_U \omega_{\mathrm{FS},t}^n$ on $U|_{M_t}$ for any $t \in \Delta$ (as in the bottom line in [RZ, p. 242]).

So far, these are our set up to obtain 4.1.1: a variant of [RZ, 2.1]. We then continue to repeat the argument in [RZ, 2.1] with possibly degenerate Kähler form $\widetilde{\omega}_t$ (and $\widetilde{g}_t$ the associate Kähler metric). The Fubini-Study form $\omega_{\mathrm{FS},t}$ (this is denoted by ω_t in [RZ, 2.1]) and the local Euclidean Kähler form ω_E on the fibers are the "same". We note again that $\widetilde{\omega}_t = \tau_t^*\omega_{\mathrm{KE},y}, t \in U_\Delta = \Delta \cap (Y' \setminus C')$, is a C^∞-smooth d-closed semi-positive form and strictly positive on a non-empty Zariski open subset. For example, we understand $\operatorname{diam}_{\widetilde{g}_t}$ in [RZ, Lemma 2.2], $\operatorname{length}_{\widetilde{g}_t}$ in [RZ, p. 242, line 3] with respect to our degenerate metric, and $\operatorname{Vol}_{\widetilde{g}_t}$ in [RZ, p. 243, line 3] with respect to our degenerate volume form $\widetilde{\omega}_t^n/n!$. It is almost enough to mind that the length of a curve γ in M_t, $\gamma : [0,1] \to M_t$, with respect to $\widetilde{\omega}_t$ is that of $\tau_t \circ \gamma : [0,1] \to X_y$, $y = \tau(t)$, with respect to the (usual) metric $\omega_{\mathrm{KE},y}$. We can also understand a geodesic ball $B_{\widetilde{g}_t}(q,r) \subset M_t$ centered at $q \in M_t$ of radius r with respect to $\widetilde{g}_t$, is $\tau_t^{-1}(B_{\omega_{\mathrm{KE},y}}(\tau_t(q),r))$, where $B_{\omega_{\mathrm{KE},y}}(\tau_t(q),r) \subset$

$X_y, y = \tau(t)$, is the geodesic ball centered at $\tau_t(q) \in X_y$ of radius r with respect to $\omega_{\mathrm{KE},y}$. Its volume is $\mathrm{Vol}_{\widetilde{g}_t}(B_{\widetilde{g}_t}(q,r)) = \mathrm{Vol}_{\omega_{\mathrm{KE},y}}(B_{\omega_{\mathrm{KE},y}}(\tau_t(q),r))$ as

$$\int_{B_{g_t}(q,r)} \widetilde{\omega}_t^n = \int_{\tau_t^{-1}(B_{\omega_{\mathrm{KE},y}}(\tau_t(q),r))} \tau_t^*(\omega_{\mathrm{KE},y}^n) = \int_{B_{\omega_{\mathrm{KE},y}}(\tau_t(q),r)} \omega_{\mathrm{KE},y}^n.$$

A key estimate is [RZ, p. 242, line 7]: $\int_{M_t} \widetilde{\omega}_t \wedge \omega_{\mathrm{FS},t}^{n-1} = n! L_t' \cdot \mathcal{L}_t^{n-1} (=: \overline{C}/C_2)$, which is topological (i.e., given by an intersection number of L_t' and $\mathcal{L}_t$) and independent of $t \in U_\Delta$ with $y = \tau(t) \in Y^o$. In the course of the proof [RZ, 2.1], they also apply some properties of Riemannian manifolds with non-negative Ricci curvature: Bishop-Gromov's comparison principle [RZ, p. 243, line 8] and [RZ, Lemma 2.3]. We first apply these results to $(X_y, \omega_{\mathrm{KE},y})$, and get the same results for $(M_t, \widetilde{\omega}_t) = \tau_t^*(X_y, \omega_{\mathrm{KE},y})$ as we need. In conclusion, the proof of [RZ, 2.1] goes through in our setting, and provides that

$$\mathrm{diam}\,(M_t, \widetilde{\omega}_t) \leq 2 + D \int_{M_t} (-1)^{n^2/2} \Omega_t' \wedge \overline{\Omega_t'}$$

hold for all $t \in U_\Delta$ with $y = \tau(t) \in Y^o$. The left hand side is nothing but $\mathrm{diam}\,(X_y, \omega_{\mathrm{KE},y}), y = \tau(t)$. $\square$

4.2. Proof of 1.2 (3) ⇒ (1)

We take arbitrary point $p \in \tau^{-1}(0)$, and take a general smooth irreducible curve $C' \subset Y'$ passing through p. Let $C = \tau(C')$, and let $\nu : C_n \to C$ be the normalization. We have an induced morphism $\tau_n : C' \to C_n$. By passing to a Zariski open neighborhood W of $0 \in Y$ and restricting everything over W, we may suppose C is smooth except 0 and f is smooth over $C \setminus 0$. Let X_n be the normalization of the main component of $X \times_Y C_n$, and let $f_n : X_n \to C_n$ be the induced morphism. We note that, over $C_n \setminus \nu^{-1}(0)$, f_n is a smooth family of polarized Calabi-Yau manifolds; the polarizations are given by the pull-back of L by the induced morphism $X_n \to X$.

By the assumption: a uniform bound of diameters and by the result in $\dim Y = 1$ case ([Ta3, 1.7]), the Weil-Petersson metric g_1 on $C_n \setminus \nu^{-1}(0)$ is incomplete around every point of $\nu^{-1}(0)$. On the other hand, the induced family $f_{C'} : X' \times_{Y'} C' \to C'$ is weakly semi-stable ([Kar, Lemma 2.12]). Although general fibers X_q' $(q \in C')$ may not be Calabi-Yau type,

X'_q is a modification of $X_{\tau(q)}$ (which is Calabi-Yau) and hence $p_g(X'_q) = 1$. In particular the direct image $(f_{C'})_* K_{(X' \times_{Y'} C')/C'}$ is a line bundle on C'. The canonical L^2-metric on $(f_{C'})_* K_{(X' \times_{Y'} C')/C'}$ over $C' \setminus \tau^{-1}(0)$ has semi-positive curvature (by Griffiths), and hence the curvature form defines a (possibly degenerate) Kähler metric g_H on $C' \setminus \tau^{-1}(0)$. This g_H is called as the quasi-Hodge metric in [W2, §1] (see 4.2.1 below). For $f_n : X_n \to C_n$ (a Calabi-Yau case), the quasi-Hodge metric is nothing but the Weil-Petersson metric g_1 thanks to Tian [Ti] and Todorov [Tod] (see [W1, 0.7]). Then, by the finite base change $\tau_n : C' \to C_n$, the incompleteness of g_1 around every point of $\nu^{-1}(0)$ implies the incompleteness of g_H around p. Then by 4.2.2 below (which is a minor generalization of [W2, 1.1 (2)]) for the weakly semi-stable family $X' \times_{Y'} C' \to C'$, there exists a unique component, say F_1, of X'_p such that $p_g(F_1) > 0$ (it is then $p_g(F_1) = 1$). This implies X''_p has canonical singularities at worst and $K_{X''_p} = \mathcal{O}_{X''_p}$ (by 3.1.1 and 2.1.6). (Recall that $f'' : X'' \to Y'$ is a minimal model of $f' : X' \to Y'$.) We note that a weakly semi-stable morphism over a curve is log-canonical ([CLS, 11.4.24]), and hence 4.2.2 can be applied. This concludes 1.2 (3) $\Rightarrow$ (1). We refer to [KM, 7.1] for the definition of a log-canonical morphism. $\square$

It is quite likely that the morphism $X'' \times_{Y'} C' \to C'$ above is a minimal model of $X' \times_{Y'} C' \to C'$. If it were so, we could apply directly [W2, 1.2], [To], [Ta3] in the final step. In any way, we recall

Proposition 4.2.1. *Let $\pi : X \to \Delta$ be a projective morphism from a complex manifold X to the unit disc $\Delta \subset \mathbb{C}$ with a π-ample line bundle (a polarization) L on X. Suppose π is semi-stable and smooth over $\Delta^* = \Delta \setminus \{0\}$, and let $X_0 = \bigcup_{j=1}^N G_j$ be the irreducible decomposition of the central fiber. Moreover suppose $p_g(X_s) \neq 0$ for every smooth fibers, and let g_H be the quasi-Hodge metric on Δ^* ([W2, §1]).*

(1) [W1, 2.1]. Suppose every smooth fiber X_s is Calabi-Yau. Then g_H (which is in fact the Weil-Petersson pseudo-metric g_1 on Δ^*) is incomplete at $s = 0$ if and only if there exists a unique irreducible component G_j of X_0 such that $p_g(G_j) = 1$.

(2) [W2, 1.1 (2)]. The quasi-Hodge metric g_H is incomplete at $s = 0$ if and only if $p_g(X_s) = \sum_{i=0}^N p_g(X_i)$ holds

The later assertion is a generalization of the former. We then generalize slightly 4.2.1 in the following form.

PROPOSITION 4.2.2. *Suppose in 1.1 (1) that* $\dim Y = 1$ *and* $f : X \to Y$ *is log-canonical. Then the quasi-Hodge metric* g_H *is incomplete at* 0 *if and only if* $p_g(X_y) = \sum_{i \in I} p_g(F_i)$ *holds, where* $X_0 = \sum_{i \in I} F_i$ *is the irreducible decomposition.*

PROOF. The proof is reduced to the semi-stable case by a finite base change. It is the same as the one in [Ta3, 2.5] (up to Step (3) in the proof), where we could suppose the smooth fibers X_y were Calabi-Yau (with $K_{X_y} = \mathcal{O}_{X_y}$) and we could use [W1, 2.1]: 4.2.1 (1). This time, we use [W2, 1.1 (2)] instead of [W1, 2.1]. We add a few more words about the proof. By an appropriate base change, we may suppose that there exists a resolution of singularities $\mu : \widetilde{X} \to X$ such that $\widetilde{f} = f \circ \mu : \widetilde{X} \to Y$ is semi-stable ([KM, 7.17]). We then note that every irreducible component of $\widetilde{X}_0$ which is μ-exceptional, is uniruled ([HM, 1.5]). Hence $\sum_{i \in I} p_g(F_i) = \sum_{j \in J} p_g(G_j)$ holds, where $\widetilde{X}_0 = \sum_{j \in J} G_j$ is the irreducible decomposition. We then apply [W2, 1.1 (2)]: 4.2.1 (2). □

4.3. Proof of 1.2 (2) ⇒ (1')

Let $\tau : Y' \to Y$ and $f' : X' \to Y'$ be as in (2). We apply [AK] for f' to obtain a weakly semi-stable model $f'' : X'' \to Y''$ of f' (see 2.1.3) and run the relative minimal model program $\varphi : X'' \dashrightarrow X'''$ over Y'' (see 2.1.5). We then obtain a diagram:

$$\begin{array}{ccccccc} X''' & \overset{\varphi}{\dashleftarrow} & X'' & \xrightarrow{\tau_X} & X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'''} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & = & Y'' & \xrightarrow{\tau'} & Y' & \xrightarrow{\tau} & Y \end{array}$$

As f' is flat over a neighborhood W of $\tau^{-1}(0)$, $X' \times_{Y'} Y''$ is irreducible over W, and X'' is a modification of $X' \times_{Y'} Y''$ at least over W ([Har, III.9.8], [AK, p. 245]). Let $q \in {\tau'}^{-1}(p)$. Then the induced morphism $X''_q \to X'_p$ is surjective, and every irreducible component of X''_q and of X'_p has the dimension $\dim X - \dim Y$. Thus there exists an irreducible component G_q of X''_q which is mapped surjectively to F_p. In particular $\kappa(G_q) \geq \kappa(F_p) \geq 0$. Then by 3.1.5 and 3.1.1, we see X'''_q has canonical singularities at worst. This is a memo: $p_m(G_q) = 1$ for any $m > 0$, however we do not know if this holds for F_p. □

4.4. Proof of 1.4

We can see the equivalences (c) $\Leftrightarrow$ (d) and (e) $\Leftrightarrow$ (f) by some classical arguments (see the proof of [To, 1.1] (c) $\Leftrightarrow$ (d) and (e) $\Leftrightarrow$ (f)). 1.2 shows (a') $\Rightarrow$ (a) $\Leftrightarrow$ (e). 3.1.3 shows (a) $\Rightarrow$ (a'). 1.3 and 3.2.3 show (a) $\Rightarrow$ (c'). By definition of the canonical L^2-metric, the equivalence (c') $\Leftrightarrow$ (c") follows (i.e., (c") is nothing but (c') when $f_*K_{X/Y}$ is a line bundle). The implication (c') $\Rightarrow$ (c) is clear.

The final implication is (c) $\Rightarrow$ (e). The argument in the proof of 4.1.1 can be adapted to show (c) $\Rightarrow$ (e). (In fact, the original argument in [RZ, 2.1] is closer.) We modify the set up of 4.1.1 as follows. We take $\pi : \mathcal{M} \to \Delta$ in 4.1.1 as a local model of $f' : X' \to Y$ in 1.4, namely $\Delta \subset Y$ and $\mathcal{M} = {f'}^{-1}(\Delta) \subset X'$ in 1.4. Then $K_{\mathcal{M}/\Delta} = \mathcal{O}_{\mathcal{M}}$ by 2.1.6.

As smooth fibers of $f : X \to Y$ is Calabi-Yau by our assumption in 1.4, f is already good minimal over Y^o, and hence $\varphi : X \dashrightarrow X'$ is isomorphic over Y^o (the MMP only modifies $X \setminus X^o$). What we obtained is a good minimal model $f' : X' \to Y$ whose fibers over Y^o are smooth Calabi-Yau. As a restriction $\pi : \mathcal{M} \to \Delta$ has the same properties. In particular we do not need to worry about degenerate Kähler-Einstein metrics on general fibers of $\pi : \mathcal{M} \to \Delta$ in this case. (As we have remarked after 1.2, also in the setting of 1.2 (1') at the beginning in this section, general fibers of f' may not necessarily be Calabi-Yau, and hence we do not know if f'' is generically smooth.)

Let $\Omega' \in H^0(\mathcal{M}, K_{\mathcal{M}/\Delta})$ be the section corresponding to the $\Omega \in H^0(X, K_{X/Y})$ in 1.4 over Δ via the $\varphi : X \dashrightarrow X'$. In view of $H^0(\mathcal{M}, K_{\mathcal{M}/\Delta}) = \Omega' f'^* H^0(\Delta, \mathcal{O}_\Delta)$ and $K_{\mathcal{M}/\Delta} = \mathcal{O}_{\mathcal{M}}$, we see that Ω' is nowhere vanishing. We take any irreducible component F of $M_0 = X'_0$ (which is reduced). Then, needless to say, $\Omega' \in H^0(\mathcal{M}, K_{\mathcal{M}/\Delta})$ does not vanish identically along F. These are all what we need to repeat the argument in 4.1.1. More details are left to the readers. $\square$

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