Duality Theories for p-Primary Étale Cohomology III

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Abstract. This paper is Part III of the series of work by the first named author on duality theories for p-primary étale cohomology, whose Parts I and II were published in 1986 and 1987, respectively. In this Part III, we study a duality for p-primary étale nearby cycles on smooth schemes over henselian discrete valuation rings of mixed characteristic $(0, p)$ whose residue field is not necessarily perfect.

1. Introduction

This is Part III of the series of work [Kat86] (Part I), [Kat87] (Part II) by the first named author. Part I gives a duality theory for p-primary étale sheaves on smooth varieties in characteristic $p > 0$ with a relative theory for proper morphisms between them, which is a relative version of Milne's duality theories [Mil76], [Mil86]. In that part, sheaves on a scheme Y of characteristic p are not considered over the usual small étale site Y_{et}, but over a much bigger site Y_{RP}, which is the category of relatively perfect Y-schemes endowed with the étale topology. Recall from [Kat86], [Kat87] that a Y-scheme Y' is said to be *relatively perfect* if its relative Frobenius morphism $Y' \to Y'^{(p)}$ over Y is an isomorphism. If the base field k satisfies $[k : k^p] = p^{r_0}$ for some finite $r_0 \geq 0$ and Y is a smooth k-scheme purely of dimension d, then one of the results [Kat86, Theorem 4.3] in this case says that the dlog part $\nu_n(r) = W_n \Omega^r_{Y,\log}$ of the de Rham-Witt sheaf viewed as a sheaf on Y_{RP} plays the role of a dualizing sheaf, where $r = r_0 + d$. This theory is generalized, in Part II [Kat86], to singular varieties Y. It instead uses the site Y_{FRP} of flat relatively perfect Y-schemes endowed with the étale topology and constructs a certain dualizing complex $K_{n,Y}$ over Y_{FRP}.

In this paper, as Part III, we study a mixed characteristic version of the duality theory of Part I. More precisely, let K be a henselian discrete

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valuation field of mixed characteristic $(0, p)$ with ring of integers $\mathcal{O}_K$ and residue field k. Assume that $[k : k^p] = p^{r_0}$ for some finite $r_0 \geq 0$. Let X be a smooth $\mathcal{O}_K$-scheme of relative dimension d and

$$U \xrightarrow{j} X \xleftarrow{i} Y$$

the inclusions of the generic fiber U and the special fiber Y. Set $r = r_0 + d$. We say that a Y-scheme is *relatively perfectly smooth*[1] if it is Zariski locally isomorphic to the relative perfection ([Kat86, Definition 1.8]) of a smooth Y-scheme. Let Y_{RPS} be the category of relatively perfectly smooth Y-schemes. Let X_{RPS} be the category of X-schemes flat over $\mathcal{O}_K$ whose special fibers are relatively perfectly smooth over Y. Endow these categories with the étale topology. Let U_{Et} be the big étale site of U. The base change functors define morphisms of sites

$$U_{\mathrm{Et}} \xrightarrow{j} X_{\mathrm{RPS}} \xleftarrow{i} Y_{\mathrm{RPS}}.$$

Our duality is about the nearby cycle functor $R\Psi = i^*Rj_*$ in this setting. For an integer s, denote the s-th Tate twist of the étale sheaf $\Lambda_n = \mathbb{Z}/p^n\mathbb{Z}$ on U by $\Lambda_n(s)$. In the derived category of sheaves of Λ_n-modules on Y_{RPS}, the derived tensor product is denoted by $\otimes^L$ and the derived sheaf-Hom functor by $R\mathcal{H}om_{Y_{\mathrm{RPS}}}$. The main theorem of this paper, which is Theorem 3.4, is the following statement.

THEOREM 1.1. *For any pair of integers s, t with $s + t = r + 1$, there exists a canonical morphism*

$$R\Psi\Lambda_n(s)[s] \otimes^L R\Psi\Lambda_n(t)[t] \to \nu_n(r)$$

that is a perfect duality:

$$R\Psi\Lambda_n(s)[s] \xrightarrow{\sim} R\mathcal{H}om_{Y_{\mathrm{RPS}}}\big(R\Psi\Lambda_n(t)[t], \nu_n(r)\big).$$

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[1]The terminology "perfectly smooth" without "relatively" for morphisms between perfect schemes is introduced in [Zhu17, Definition A.25]. In [Zhu17, Footnote 20 to Definition A.13], Zhu attributes this type of usage of "perfectly" to Brian Conrad.

For example, if $X = \operatorname{Spec} \mathcal{O}_K$ and k is algebraically closed, then this theorem, on k-valued points, gives an exact sequence

$$0 \to \operatorname{Ext}^1_{k_{\mathrm{RPS}}}(G_n, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(K, \mathbb{Z}/p^n\mathbb{Z}) \to \operatorname{Hom}(\mu_{p^n}(K), \mathbb{Z}/p^n\mathbb{Z}) \to 0,$$

where $G_n = R^1\Psi\Lambda_n(1)$ is the group $K^\times/(K^\times)^{p^n}$ equipped with a structure of the perfection of a unipotent algebraic group over k and $\operatorname{Ext}^1_{k_{\mathrm{RPS}}}$ takes this structure into account (and $\mu_{p^n}(K)$ is the finite abstract group of p^n-th roots of unity in K). This recovers the p-primary part of the mixed characteristic case of Serre's local class field theory [Ser61] for K. Hazewinkel's generalization [DG70, Appendice] of Serre's theory with arbitrary perfect residue field k is also contained in this theorem. This sheaf-theoretic formulation of Serre-Hazewinkel's theory is closely related to the formulation using the "rational étale site" introduced in [Suz13].

For $X = \operatorname{Spec} \mathcal{O}_K$ and not necessarily perfect k, the theorem is a sheaf-theoretic version of class field theory for local fields whose residue field is of arithmetic nature, such as [Kat80], [Par84] (for k a higher local field) and [Kat82] (for k a global function field). For general X and k, it is a p-primary version of a special case of the duality for prime-to-p nearby cycles by Gabber-Illusie [Ill94, Théorème 4.2].

In the proof of the theorem, we will use the computations of graded pieces of p-primary nearby cycles by the first named author with Bloch [BK86]. Since the calculations in [BK86] are for smooth schemes over $\mathcal{O}_K$, we limit ourselves with X smooth over $\mathcal{O}_K$ and work with the site Y_{RPS} of relatively perfectly smooth Y-schemes for simplicity. Of course our duality theory should be extended to non-smooth (or at least semistable) X and more general constructible coefficients than $\Lambda_n(s)$, so that it fully gives a mixed characteristic version of Part II [Kat87] of the present series of work. We will not pursue such an extension in this paper.

2. The Relatively Perfectly Smooth Site

Let k be a field of characteristic $p > 0$ such that $[k : k^p] = p^{r_0}$ for some finite $r_0 \geq 0$. Let Y be a k-scheme. Recall from [Kat86, Definition 1.1] that

a Y-scheme Y' is said to be relatively perfect if the diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y
\end{array}
$$

is cartesian, where the horizontal morphisms are the absolute Frobenius morphisms and the vertical morphisms are the structure morphisms. This means that the relative Frobenius morphism $Y' \to Y'^{(p)}$ over Y is an isomorphism. Let Y_{RP} be the relatively perfect site of Y defined in [Kat86, §2]. It is the category of relatively perfect Y-schemes endowed with the étale topology. Let (Sch/Y) be the category of all Y-schemes. Assume that Y is smooth over k. Then the inclusion functor $Y_{\mathrm{RP}} \hookrightarrow (\mathrm{Sch}/Y)$ admits a right adjoint $(\mathrm{Sch}/Y) \to Y_{\mathrm{RP}}$ denoted by $Y' \mapsto Y'^{\mathrm{RP}}$ ([Kat86, Definition 1.8]), and Y'^{RP} is called the relative perfection of Y'. Let $n \geq 1$ be an integer. Denote $\Lambda_n = \mathbb{Z}/p^n\mathbb{Z}$ and set $\Lambda = \Lambda_1 = \mathbb{Z}/p\mathbb{Z}$. For a site S, we denote the category of sheaves of Λ_n-modules on S by $M(S, \Lambda_n)$ and its derived category by $D(S, \Lambda_n)$. The ring Λ_n viewed as a sheaf of rings on S is denoted by $(\Lambda_n)_S$ or simply just Λ_n. As in [Kat86, Definition 4.2.3], we denote by $D_0(Y_{\mathrm{RP}}, \Lambda_n)$ the triangulated subcategory of $D(Y_{\mathrm{RP}}, \Lambda_n)$ generated by relative perfections of coherent sheaves on Y locally free of finite rank regarded as complexes of Λ_n-modules concentrated in degree zero. As explained in [Kat86, §4], if Y is finite-dimensional, the dlog part of the de Rham-Witt complex $\nu_n(s) = W_n\Omega^s_{Y,\log}$ for any s can be regarded as an object of $D_0(Y_{\mathrm{RP}}, \Lambda_n)$ such that its section $\Gamma(Y', \nu_n(s))$ for any relatively perfect Y-scheme Y' is given by the group $W_n\Omega^s_{Y',\log}$. We set $\nu(s) = \nu_1(s) = \Omega^s_{Y,\log}$.

THEOREM 2.1 ([Kat86, Theorem 4.3]). *Assume that Y is smooth and purely of dimension d. Set $r = r_0 + d$. Then the object $\nu_n(r)$ is a dualizing object for $D_0(Y_{\mathrm{RP}}, \Lambda_n)$, namely the derived sheaf-Hom functor*

$$
D_{Y_{\mathrm{RP}}} = R\,\mathscr{Hom}_{(\Lambda_n)_{Y_{\mathrm{RP}}}}(\,\cdot\,, \nu_n(r))
$$

for $D(Y_{\mathrm{RP}}, \Lambda_n)$ gives an auto-equivalence on $D_0(Y_{\mathrm{RP}}, \Lambda_n)$ with inverse itself.

Let Y be a smooth k-scheme. We say that a Y-scheme is relatively perfectly smooth if it is Zariski locally isomorphic to the relative perfection of a

smooth Y-scheme. Let Y_{RPS} be the category of relatively perfectly smooth Y-schemes with Y-scheme morphisms. Endow it with the étale topology. The inclusion functor $Y_{\mathrm{RPS}} \hookrightarrow Y_{\mathrm{RP}}$ defines a morphism of topologies in the sense of [Art62, Definition 2.4.2]. It induces an (exact) pushforward functor α_* from $M(Y_{\mathrm{RP}}, \Lambda_n)$ to $M(Y_{\mathrm{RPS}}, \Lambda_n)$, which has a left adjoint α^*. (Note that α^* is not necessarily exact since the category of smooth Y-schemes is not closed under finite inverse limits.) Note that α_* sends flask sheaves to flask sheaves and induces Leray spectral sequences ([Art62, §2.4]).

PROPOSITION 2.2. *Let Y be a smooth k-scheme and $F \in D_0(Y_{\mathrm{RP}}, \Lambda_n)$. Then the natural morphism*

$$\alpha_* R\,\mathcal{H}om_{(\Lambda_n)_{Y_{\mathrm{RP}}}}(F, G) \to R\,\mathcal{H}om_{(\Lambda_n)_{Y_{\mathrm{RPS}}}}(\alpha_* F, \alpha_* G)$$

in $D(Y_{\mathrm{RPS}}, \Lambda_n)$ for any $G \in D(Y_{\mathrm{RP}}, \Lambda_n)$ is an isomorphism.

PROOF. It is enough to show the statement for the case where F is the relative perfection of a coherent sheaf on Y locally free of finite rank. Such a sheaf F is representable by a relatively perfectly smooth Y-scheme.

For a relatively perfectly smooth Y-scheme Y', let Y_{RPS}/Y' be the localization of Y_{RPS} at Y' ([Art62, Definition 2.4.3]), i.e., the category of Y'-schemes relatively perfectly smooth over Y endowed with the étale topology. Taking $R\Gamma(Y'_{\mathrm{RPS}}, \cdot)$ for any relatively perfectly smooth Y-scheme Y' and using the Leray spectral sequence, we see that it is enough to show the invertibility of the morphism

$$(2.1) \qquad R\,\mathrm{Hom}_{(\Lambda_n)_{Y'_{\mathrm{RP}}}}(F_{Y'}, G_{Y'}) \to R\,\mathrm{Hom}_{(\Lambda_n)_{Y_{\mathrm{RPS}}/Y'}}(\alpha_* F_{Y'}, \alpha_* G_{Y'})$$

in the derived category of Λ_n-modules, where $F_{Y'}$ and $G_{Y'}$ are the restrictions to Y'. Let $M(F_{Y'})$ be Mac Lane's resolution of $F_{Y'}$ ([ML57]). Its homogeneous part at any degree is a direct summand of a direct sum of sheaves of the form $\Lambda_n[F^m_{Y'}]$ for various $m \geq 0$, where $\Lambda_n[F^m_{Y'}]$ is the sheafification of the presheaf that assigns to each relatively perfect Y'-scheme Y'' the free Λ_n-module generated by the set $F^m_{Y'}(Y'') = F^m(Y'')$. By assumption on F, we know that both $F^m_{Y'}$ and $\alpha_* F^m_{Y'}$ are representable by the relatively perfectly smooth Y'-scheme $F^m \times_Y Y'$. Hence the both sides of (2.1) can be written in terms of cohomology complexes of $F^m_{Y'}$ for various m. The morphism

$$R\Gamma((F^m_{Y'})_{\mathrm{RP}}, G_{Y'}) \to R\Gamma((F^m_{Y'})_{\mathrm{RPS}}, \alpha_* G_{Y'})$$

is an isomorphism by the Leray spectral sequence. This implies the result. $\square$

Let $D_0(Y_{\mathrm{RPS}}, \Lambda_n)$ be the image of $D_0(Y_{\mathrm{RP}}, \Lambda_n)$ under α_*. We denote the image of each object $F \in D_0(Y_{\mathrm{RPS}}, \Lambda_n)$ by the same letter F.

COROLLARY 2.3. *Assume that Y is smooth and purely of dimension d and set $r = r_0 + d$. The functor α_* gives an equivalence of categories $D_0(Y_{\mathrm{RP}}, \Lambda_n) \xrightarrow{\sim} D_0(Y_{\mathrm{RPS}}, \Lambda_n)$. The derived sheaf-Hom functor*

$$D_{Y_{\mathrm{RPS}}} = R\,\mathscr{H}om_{(\Lambda_n)_{Y_{\mathrm{RPS}}}}(\,\cdot\,, \nu_n(r))$$

for $D(Y_{\mathrm{RPS}}, \Lambda_n)$ gives an auto-equivalence on $D_0(Y_{\mathrm{RPS}}, \Lambda_n)$ with inverse itself.

3. Formulation of the Duality

Let k and r_0 be as above. From now on, for a smooth k-scheme Y, we will use Y_{RPS} and not Y_{RP}, so we write $D(Y, \Lambda_n)$, $D_Y = R\,\mathscr{H}om_{(\Lambda_n)_Y}(\,\cdot\,, \nu_n(r))$ etc. omitting the subscripts RPS.
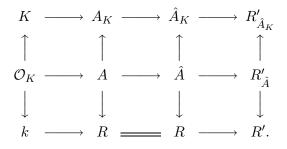
Let K be a henselian discrete valuation field of characteristic 0 whose residue field is k. We denote the ring of integers of K by $\mathcal{O}_K$ and its maximal ideal by $\mathfrak{p}_K$.

Let A be a flat $\mathcal{O}_K$-algebra of finite type and $\hat{A}$ its $A\mathfrak{p}_K$-adic completion. Write $A_K = A \otimes_{\mathcal{O}_K} K$. Let $R = A \otimes_{\mathcal{O}_K} k$. For a flat relatively perfect R-algebra R', we denote its canonical lifting over $\hat{A}$ by $R'_{\hat{A}}$ [Kat82, Definition 1]. It is characterized as a unique complete $\hat{A}$-algebra flat over $\mathcal{O}_K$ such that $R'_{\hat{A}} \otimes_{\mathcal{O}_K} k$ is isomorphic to R' over R. For any $n \geq 0$, the $A/A\mathfrak{p}_K^n$-algebra $R'_{\hat{A}}/R'_{\hat{A}}\mathfrak{p}_K^n$ is flat and formally étale ([Kat82, Lemma 1]). For another flat relatively perfect R-algebra R'', we have

$$\mathrm{Hom}_{\hat{A}}(R'_{\hat{A}}, R''_{\hat{A}}) \xrightarrow{\sim} \mathrm{Hom}_R(R', R'').$$

Write $R'_{\hat{A}_K} = R'_{\hat{A}} \otimes_{\mathcal{O}_K} K$. We have a commutative diagram with cocartesian

squares

$$
\begin{array}{ccccccc}
K & \longrightarrow & A_K & \longrightarrow & \hat{A}_K & \longrightarrow & R'_{\hat{A}_K} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{O}_K & \longrightarrow & A & \longrightarrow & \hat{A} & \longrightarrow & R'_{\hat{A}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
k & \longrightarrow & R & = \!\!\!= \!\!\!= & R & \longrightarrow & R'.
\end{array}
$$

COROLLARY 3.1. *Under the above setting, let A' be an A-algebra flat over $\mathcal{O}_K$ such that $A' \otimes_{\mathcal{O}_K} k$ is flat relatively perfect over R. Then the $A'\mathfrak{p}_K$-adic completion $\hat{A}'$ of A' gives the canonical lifting of $A' \otimes_{\mathcal{O}_K} k$ over $\hat{A}$. The maps*

$$
\mathrm{Hom}_A(A', R'_{\hat{A}}) \to \mathrm{Hom}_{\hat{A}}(\hat{A}', R'_{\hat{A}}) \to \mathrm{Hom}_R(A' \otimes_{\mathcal{O}_K} k, R')
$$

are both bijective. In particular, a right adjoint of $A' \mapsto A' \otimes_{\mathcal{O}_K} k$ is given by $R' \mapsto R'_{\hat{A}}$.

PROOF. This follows from the above characterization of canonical liftings. $\square$

Let X be a smooth $\mathcal{O}_K$-scheme. Let U and Y be its generic and special fibers, respectively. Denote the natural inclusion morphisms by

$$
U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y.
$$

If $X = \mathrm{Spec}\, A$ is affine and $Y = \mathrm{Spec}\, R$, then for an affine relatively perfectly smooth Y-scheme $Y' = \mathrm{Spec}\, R'$, we denote $Y'_{\hat{X}} = \mathrm{Spec}\, R'_{\hat{A}}$ and $Y'_{\hat{U}} = Y'_{\hat{X}} \times_{\mathcal{O}_K} K = \mathrm{Spec}\, R'_{\hat{A}_K}$. For a general smooth X, let X_{RPS} be the category of X-schemes X' flat over $\mathcal{O}_K$ whose special fibers $X' \times_X Y$ are relatively perfectly smooth over Y. Morphisms are X-scheme morphisms. Endow X_{RPS} with the étale topology. Let $\tilde{X}_{\mathrm{RPS}}$ be the category of sheaves of sets on X_{RPS} and $\tilde{Y}_{\mathrm{RPS}}$ similarly. The reduction functor $X_{\mathrm{RPS}} \to Y_{\mathrm{RPS}}$, $X' \mapsto X' \times_X Y$, defines a morphism of topologies. It induces a pushforward functor $i_*\colon \tilde{Y}_{\mathrm{RPS}} \to \tilde{X}_{\mathrm{RPS}}$, which has a left adjoint $i^*\colon \tilde{X}_{\mathrm{RPS}} \to \tilde{Y}_{\mathrm{RPS}}$.

PROPOSITION 3.2. *The functor i^* is exact. If X is affine, then i^*F for a sheaf F on X_{RPS} is given by the sheafification of the presheaf $Y' \mapsto \Gamma(Y'_{\hat{X}}, F)$, where Y' runs over affine relatively perfectly smooth Y-schemes.*

PROOF. The second statement follows from Corollary 3.1. This description of i^* shows that i^* is exact when X is affine. The general case follows. $\square$

The above proposition shows that the morphism $i \colon Y \hookrightarrow X$ induces a morphism of sites $Y_{\mathrm{RPS}} \to X_{\mathrm{RPS}}$. Let U_{Et} be the category of U-schemes endowed with the étale topology. Denote $D(X, \Lambda_n) = D(X_{\mathrm{RPS}}, \Lambda_n)$ and $D(U, \Lambda_n) = D(U_{\mathrm{Et}}, \Lambda_n)$. Then we have morphisms of sites

$$(3.1) \qquad\qquad U_{\mathrm{Et}} \xrightarrow{j} X_{\mathrm{RPS}} \xleftarrow{i} Y_{\mathrm{RPS}}.$$

We consider the functor

$$R\Psi = i^* Rj_* \colon D(U, \Lambda_n) \to D(Y, \Lambda_n).$$

We denote $R^m\Psi = i^* R^m j_*$ for $m \geq 0$.

COROLLARY 3.3. *The functor $R\Psi$ is the right derived functor of $R^0\Psi$. If X is affine, then $R^m\Psi F$ for $F \in M(U_{\mathrm{Et}}, \Lambda)$ and $m \geq 0$ is given by the sheafification of the presheaf $Y' \mapsto H^m(Y'_{\hat{U}}, F)$, where Y' runs over affine relatively perfectly smooth Y-schemes.*

PROOF. This follows from the previous proposition. $\square$

We have canonical morphisms $Rj_*F \otimes^L Rj_*G \to Rj_*(F \otimes^L G)$ in $D(X, \Lambda_n)$ and hence

$$R\Psi F \otimes^L R\Psi G \to R\Psi(F \otimes^L G)$$

in $D(Y, \Lambda_n)$ functorial in $F, G \in D(U, \Lambda_n)$. Hence if we have a morphism $F \otimes^L G \to H$ in $D(U, \Lambda_n)$, then we have a canonical morphism

$$R\Psi F \otimes^L R\Psi G \to R\Psi H.$$

For any integer s, we denote the s-th Tate twist of Λ_n over U by $\Lambda_n(s)$. The following is the main theorem of this paper.

THEOREM 3.4. *Let X be a smooth $\mathcal{O}_K$-scheme of relative dimension d. Let U and Y be its generic and special fibers, respectively. Set $r = r_0 + d$. Let s, t be integers with $s + t = r + 1$.*

(1) *There exists a canonical trace morphism*

$$\mathrm{Tr}\colon R^{r+1}\Psi\Lambda_n(r+1) \to \nu_n(r)$$

of sheaves on Y_{RPS}.

(2) *The object $R\Psi\Lambda_n(s)$ is in $D_0(Y, \Lambda_n)$ and concentrated in degrees $[0, r+1]$.*

(3) *The composite morphism*

$$R\Psi\Lambda_n(s) \otimes^L R\Psi\Lambda_n(t) \to R\Psi\Lambda_n(r+1) \xrightarrow{\mathrm{Tr}} \nu_n(r)[-r-1]$$

induces a perfect duality between $R\Psi\Lambda_n(s)$ and $R\Psi\Lambda_n(t)$ in $D_0(Y, \Lambda_n)$ via the dualizing functor $D_Y[-r-1]$.

Theorem 1.1 is a consequence of this theorem. We prove this theorem in the rest of the paper.

4. Ind-Smooth Approximations of Canonical Liftings

We continue working with the situation (3.1). By Corollary 3.3, to describe the functor $R\Psi$, we need to know the étale cohomology of the canonical liftings $Y'_{\hat{U}}$. For this, we use the following approximation method.

PROPOSITION 4.1. *Assume that $X = \operatorname{Spec} A$ is affine and $Y = \operatorname{Spec} R$ has a p-base. Let $\mathfrak{q} \subset R$ be a prime ideal and $\mathfrak{p} \subset A$ its inverse image. Let R' be the local (resp. henselian local, resp. strict henselian local) ring of a relatively perfectly smooth R-algebra at some prime ideal containing $\mathfrak{q}$.*

Then there exists a local (resp. henselian local, resp. strict henselian local) $A_{\mathfrak{p}}$-algebra A' that can be written as a filtered direct limit of smooth

A-algebras such that $A' \otimes_{\mathcal{O}_K} k \cong R'$ as R-algebras and the pair $(A', A'\mathfrak{p}_K)$ is henselian. The $A'\mathfrak{p}_K$-adic completion of A' is isomorphic to $R'_{\hat{A}}$ as an A-algebra.

PROOF. All the relative perfections below are taken over R. By assumption, there exists a smooth R-algebra R_1 and a prime ideal $\mathfrak{q}_1 \subset R_1^{\mathrm{RP}}$ such that R' is the local (resp. henselian local, resp. strict henselian local) ring of R_1^{RP} at $\mathfrak{q}_1$. Taking Spec R_1 smaller if necessary, we may assume that R_1 is étale over a polynomial ring $R_2 = R[x_1, \ldots, x_m]$. Then R_1^{RP} is étale over R_2^{RP}.

We show that there is a filtered direct limit of smooth A-algebras whose reduction $(\cdot) \otimes_A R$ is R_2^{RP}. The relative perfection Spec R_2^{RP} is given by the inverse limit of $G^n(\mathrm{Spec}\, R_2)$ for $n \geq 0$, where G is the Weil restriction functor for the absolute Frobenius morphism $\mathrm{Spec}\, R \to \mathrm{Spec}\, R$ ([Kat86, 1.6-1.8]). In particular, we only need to treat the case $m = 1$, so $R_2 = R[x]$. Let $t_1, \ldots, t_r$ be a p-base of R. Then $G(\mathrm{Spec}\, R_2)$ is the affine p^r-space over R with coordinates $x_{i(1)\cdots i(r)}$, $0 \leq i(1), \ldots, i(r) \leq p - 1$, and the R-morphism $G(\mathrm{Spec}\, R_2) = \mathbb{A}_R^{p^r} \to \mathrm{Spec}\, R_2 = \mathbb{A}_R^1$ maps $(x_{i(1)\cdots i(r)})$ to $\sum x_{i(1)\cdots i(r)}^p t_1^{i(1)} \cdots t_r^{i(r)}$. In terms of rings, this is the R-algebra homomorphism $R[x] \to R[x_{i(1)\cdots i(r)} \,|\, 0 \leq i(1), \ldots, i(r) \leq p - 1]$ sending x to $\sum x_{i(1)\cdots i(r)}^p t_1^{i(1)} \cdots t_r^{i(r)}$. We take a lifting of this morphism to A by $\mathbb{A}_A^{p^r} \to \mathbb{A}_A^1$ mapping $(x_{i(1)\cdots i(r)})$ to $\sum x_{i(1)\cdots i(r)}^p t_1^{i(1)} \cdots t_r^{i(r)}$. Iterating, we can take a lifting of $G^{n+1}(\mathrm{Spec}\, R_2) \to G^n(\mathrm{Spec}\, R_2)$ to A by $\mathbb{A}_A^{p^{(n+1)r}} \to \mathbb{A}_A^{p^{nr}}$ defined similarly. The inverse limit of these liftings for $n \geq 0$ gives a desired lifting of Spec R_2^{RP}.

Let A_2 be such a lifting of R_2^{RP}. Since R_1^{RP} is étale over R_2^{RP}, we can take an étale A_2-algebra A_1 whose reduction is R_1^{RP}. Let $\mathfrak{p}_1 \subset A_1$ be the inverse image of $\mathfrak{q}_1 \subset R_1^{\mathrm{RP}}$. Consider the local (resp. henselian local, resp. strict henselian local) ring A_1' of A_1 at $\mathfrak{p}_1$. The henselization of the pair $(A_1', A_1'\mathfrak{p}_K)$ gives a desired local $A_{\mathfrak{p}}$-algebra A'. We have $\hat{A}' \cong R'_{\hat{A}}$ by Corollary 3.1. □

We call the A-algebra A' appearing in this proposition an *ind-smooth lifting* of R' over A. It is neither unique nor noetherian.

PROPOSITION 4.2. *In the situation of Proposition 4.1, for any torsion*

étale sheaf F on U_{et} (pulled back to U_{Et}), we have

$$R\Gamma(A'_K, F) \xrightarrow{\sim} R\Gamma(R'_{\hat{A}_K}, F).$$

In the case of strictly henselian R', the isomorphic groups $H^q(A'_K, F) \cong H^q(R'_{\hat{A}_K}, F)$ for any q give the stalk of $R^q\Psi F$ at the residue field of R' (if F is a sheaf of Λ_n-modules).

PROOF.   The first assertion follows from Fujiwara-Gabber's formal base change theorem ([ILO14, Exposé XX, §4.4], [BM18, Corollary 1.18 (2)]). The second follows from Corollary 3.3. □

Since A' is ind-smooth over A and hence over $\mathcal{O}_K$, the study of $R\Gamma(A'_K, F)$ basically reduces to Bloch-Kato's study of p-primary nearby cycles [BK86].

5.   Symbol Maps and Trace Morphisms

Let the notation be as in Theorem 3.4. We fix a prime element π of K. In this section, we will prove Theorem 3.4 (1). Slightly more generally, we will construct a certain morphism

$$(5.1) \qquad R^q\Psi\Lambda_n(q) \to \nu_n(q) \oplus \nu_n(q-1)$$

of sheaves on Y_{RPS} such that its composite with the projection onto the factor $\nu_n(q-1)$ does not depend on π.

We need symbol maps adapted to our setting. The connecting morphism for the Kummer exact sequence $0 \to \Lambda_n(1) \to \mathbf{G}_m \to \mathbf{G}_m \to 0$ of sheaves on U_{Et} gives a morphism $i^*j_*\mathbf{G}_m \to R^1\Psi\Lambda_n(1)$ of sheaves on Y_{RPS}. By cup product, we define a morphism

$$(5.2) \qquad (i^*j_*\mathbf{G}_m)^{\otimes q} \to R^q\Psi\Lambda_n(q), \quad x_1 \otimes \ldots \otimes x_q \mapsto \{x_1, \ldots, x_q\},$$

where $\otimes q$ means the q-th tensor power and the x_i are local sections of $i^*j_*\mathbf{G}_m$ (i.e. invertible elements of $R'_{\hat{A}_K}$ for some relatively perfectly smooth R-algebra R', where $\operatorname{Spec} A$ is an affine open of X and $R = A \otimes_{\mathcal{O}_K} k$), which we call the *symbol map*. By composing it with the inclusion $\mathbf{G}_m \hookrightarrow j_*\mathbf{G}_m$, we have a morphism from $(i^*\mathbf{G}_m)^{\otimes q}$ to $R^q\Psi\Lambda_n(q)$. The construction of the morphism (5.1) is given by the following, which proves Theorem 3.4 (1).

Proposition 5.1. *The morphism*

$$(i^*\mathbf{G}_m)^{\otimes q} \oplus (i^*\mathbf{G}_m)^{\otimes q-1} \to R^q\Psi\Lambda_n(q),$$
$$(x_1 \otimes \ldots \otimes x_q, y_1 \otimes \ldots \otimes y_{q-1}) \mapsto \{x_1, \ldots, x_q\} + \{y_1, \ldots, y_{q-1}, \pi\}$$

is surjective. (Note that the last component of the second symbol is $\pi \in \Gamma(Y, i^*j_*\mathbf{G}_m)$, *which is not in* $\Gamma(Y, i^*\mathbf{G}_m)$.) *The composite of the reduction map and the dlog map*

$$(i^*\mathbf{G}_m)^{\otimes q} \oplus (i^*\mathbf{G}_m)^{\otimes q-1} \to \mathbf{G}_m^{\otimes q} \oplus \mathbf{G}_m^{\otimes q-1} \to \nu_n(q) \oplus \nu_n(q-1)$$

factors through the quotient $R^q\Psi\Lambda_n(q)$. *The obtained morphism*

$$R^q\Psi\Lambda_n(q) \to \nu_n(q) \oplus \nu_n(q-1)$$

followed by the projection onto the factor $\nu_n(q-1)$ *does not depend on* π.

In this proposition, when $X = \operatorname{Spec}\mathcal{O}_K$, k is separably closed and $q = 1$, the global section of the above morphism $i^*\mathbf{G}_m \oplus \mathbb{Z} \to R^1\Psi\Lambda_n(1)$ is $\hat{\mathcal{O}}_K^\times \oplus \mathbb{Z} \to \hat{K}^\times/(\hat{K}^\times)^{p^n}$ given by $(x, n) \mapsto x\pi^n$. This is indeed surjective.

Proof.   It is enough to check the statements for stalks. Hence we may assume that $X = \operatorname{Spec} A$ is affine and $Y = \operatorname{Spec} R$ has a p-base. Let R' be the strict henselian local ring of a relatively perfectly smooth R-algebra at some prime ideal. Let A' be an ind-smooth lifting of R' over A as in the previous section. By Proposition 4.2, we are reduced to proving the following: the homomorphism

$$(A'^\times)^{\otimes q} \oplus (A'^\times)^{\otimes q-1} \to H^q(A'_K, \Lambda_n(q))$$
$$(x_1 \otimes \ldots \otimes x_q, y_1 \otimes \ldots \otimes y_{q-1}) \mapsto \{x_1, \ldots, x_q\} + \{y_1, \ldots, y_{q-1}, \pi\}$$

is surjective; the composite of the reduction map and the dlog map

$$(A'^\times)^{\otimes q} \oplus (A'^\times)^{\otimes q-1} \to (R'^\times)^{\otimes q} \oplus (R'^\times)^{\otimes q-1}$$
$$\to \Gamma(R', \nu_n(q)) \oplus \Gamma(R', \nu_n(q-1))$$

factors through the quotient $H^q(A'_K, \Lambda_n(q))$; and the obtained homomorphism

$$H^q(A'_K, \Lambda_n(q)) \to \Gamma(R', \nu_n(q)) \oplus \Gamma(R', \nu_n(q-1))$$

followed by the projection onto the factor $\Gamma(R', \nu_n(q-1))$ does not depend on π. Since A' is ind-smooth over A and hence over $\mathcal{O}_K$, these claims are reduced to [BK86, Theorem (1.4) (i)]. $\square$

For $m \leq n$, the endomorphisms of Λ_n and $\nu_n(q)$ given by multiplication by p^{n-m} factor as "p^{n-m}": $\Lambda_m \hookrightarrow \Lambda_n$ and "p^{n-m}": $\nu_m(q) \hookrightarrow \nu_n(q)$, so that we have an exact sequence $0 \to \nu_m(q) \to \nu_n(q) \to \nu_{n-m}(q) \to 0$ ([Kat86, (4.1.8)]). Later we will use the following.

PROPOSITION 5.2. *We have a commutative diagram*

$$
\begin{array}{ccc}
R^q\Psi\Lambda_m(q) & \longrightarrow & \nu_m(q) \oplus \nu_m(q-1) \\
\downarrow{\scriptstyle "p^{n-m}"} & & {\scriptstyle "p^{n-m}"}\downarrow \\
R^q\Psi\Lambda_n(q) & \longrightarrow & \nu_n(q) \oplus \nu_n(q-1),
\end{array}
$$

where the horizontal morphisms are given by (5.1) *for m and n.*

PROOF. Consider the following diagram (commutativity to be discussed soon):

$$
\begin{array}{ccccc}
(i^*\mathbf{G}_m)^{\otimes q} \oplus (i^*\mathbf{G}_m)^{\otimes q-1} & \longrightarrow & R^q\Psi\Lambda_n(q) & \longrightarrow & \nu_n(q) \oplus \nu_n(q-1) \\
\| & & \downarrow{\scriptstyle \text{can}} & & {\scriptstyle \text{can}}\downarrow \\
(i^*\mathbf{G}_m)^{\otimes q} \oplus (i^*\mathbf{G}_m)^{\otimes q-1} & \longrightarrow & R^q\Psi\Lambda_m(q) & \longrightarrow & \nu_m(q) \oplus \nu_m(q-1). \\
\downarrow{\scriptstyle p^{n-m}} & & \downarrow{\scriptstyle "p^{n-m}"} & & {\scriptstyle "p^{n-m}"}\downarrow \\
(i^*\mathbf{G}_m)^{\otimes q} \oplus (i^*\mathbf{G}_m)^{\otimes q-1} & \longrightarrow & R^q\Psi\Lambda_n(q) & \longrightarrow & \nu_n(q) \oplus \nu_n(q-1).
\end{array}
$$

The left three horizontal morphisms (the symbol maps) are all surjective by Proposition 5.1. The left two squares are commutative by the construction of the symbol map. The total (or outer) square omitting the central term $R^q\Psi\Lambda_m(q)$ is commutative since "p^{n-m}" $\circ$ can $= p^{n-m}$. From these, the commutativity of the right lower square follows by a diagram chase. $\square$

6. Mod p Case I: Filtrations and Duality for gr^0

Let the notation be as in Theorem 3.4. Within this and the next sections, we will prove Theorem 3.4 for the case $n = 1$. We fix a prime element π of

K. Let $q \geq 0$. Recall our notation $\Lambda = \mathbb{Z}/p\mathbb{Z}$. As in [BK86, (1.2)], we define a filtration on the sheaf $R^q\Psi\Lambda(q)$ using the symbol map (5.2) as follows. For $m \geq 1$, define $U^m R^q\Psi\Lambda(q)$ to be the subsheaf of $R^q\Psi\Lambda(q)$ generated by local sections of the form $\{x_1, \ldots, x_q\}$ such that $x_1 - 1 \in \pi^m i^* \mathbf{G}_a$. Let

$$\mathrm{gr}^m R^q\Psi\Lambda(q) = \begin{cases} R^q\Psi\Lambda(q)/U^1 R^q\Psi\Lambda(q) & \text{if } m = 0, \\ U^m R^q\Psi\Lambda(q)/U^{m+1} R^q\Psi\Lambda(q) & \text{if } m \geq 1. \end{cases}$$

For $m \geq 1$, define a morphism ρ_m from the direct sum of $i^* \mathbf{G}_a \otimes i^* \mathbf{G}_m^{\otimes q-1}$ and $i^* \mathbf{G}_a \otimes i^* \mathbf{G}_m^{\otimes q-2}$ to $U^m R^q\Psi\Lambda(q)$ by

$$x \otimes y_1 \otimes \ldots \otimes y_{q-1} \mapsto \{1 + x\pi^m, y_1, \ldots, y_{q-1}\}$$

and

$$x \otimes y_1 \otimes \ldots \otimes y_{q-2} \mapsto \{1 + x\pi^m, y_1, \ldots, y_{q-2}, \pi\}.$$

The reduction map and the dlog map define surjections

$$i^* \mathbf{G}_a \otimes i^* \mathbf{G}_m^{\otimes q-1} \twoheadrightarrow \mathbf{G}_a \otimes \mathbf{G}_m^{\otimes q-1} \twoheadrightarrow \Omega_Y^{q-1}$$

and similar surjections with $q - 1$ replaced by $q - 2$. Let e be the absolute ramification index of K and set $e' = pe/(p-1)$.

PROPOSITION 6.1.

(0) *For $m \geq 1$, the morphism ρ_m factor through*

$$\Omega_Y^{q-1} \oplus \Omega_Y^{q-2} \to \mathrm{gr}^m R^q\Psi\Lambda(q).$$

(1) *The morphism (5.1) for $n = 1$ is surjective and induces an isomorphism*

$$\mathrm{gr}^0 R^q\Psi\Lambda(q) \cong \nu(q) \oplus \nu(q-1).$$

(2) *If $1 \leq m < e'$ and $p \nmid m$, then the morphism in (0) induces an isomorphism*

$$\mathrm{gr}^m R^q\Psi\Lambda(q) \cong \Omega_Y^{q-1}.$$

(3) *If $1 \leq m < e'$ and $p \mid m$, then the morphism in* (0) *and the differential d induce an isomorphism*

$$\mathrm{gr}^m R^q \Psi \Lambda(q) \cong d\Omega_Y^{q-1} \oplus d\Omega_Y^{q-2}.$$

(4) *If $m \geq e'$, then*

$$U^m R^q \Psi \Lambda(q) = 0.$$

PROOF.   This reduces to [BK86, Corollary (1.4.1)] by the same method as the proof of Proposition 5.1. □

PROPOSITION 6.2.   *The statement of Theorem 3.4 (2) is true for $n = 1$.*

PROOF.   We may assume that K contains a primitive p-th root of unity ζ_p since $[K(\zeta_p) : K]$ is prime to p. Then the result follows from Proposition 6.1. □

Thus we have the morphism

$$R\Psi\Lambda(s) \otimes^L R\Psi\Lambda(t) \to R\Psi\Lambda(r+1) \xrightarrow{\mathrm{Tr}} \nu(r)[-r-1]$$

stated in Theorem 3.4 (3) in the case $n = 1$, where s, t are integers with $s + t = r + 1$. We want to prove that it induces a perfect duality. In the rest of this section, we work with $\Lambda = \mathbb{Z}/p\mathbb{Z}$-coefficients. As above, we may assume that $\zeta_p \in K$. With the choice of ζ_p, we may identify all the Tate twists $\Lambda(q)$ with Λ in a compatible way. Let $\mathcal{E} = R\Psi\Lambda \; (\cong R\Psi\Lambda(q)$ for any q). The above morphism may be written as

(6.1) $$\mathcal{E} \otimes^L \mathcal{E} \to \nu(r)[-r-1],$$

which is independent of the integers s, t. We have the above filtrations $U^m H^q \mathcal{E}$ and graded pieces $\mathrm{gr}^m H^q \mathcal{E}$ for any q. For any s, define $\tau'_{\geq s}\mathcal{E}$ to be the canonical mapping cone of the natural morphism $U^1 H^s \mathcal{E}[-s] \to \tau_{\geq s}\mathcal{E}$ and $\tau'_{\leq s}\mathcal{E}$ to be the canonical mapping fiber of the natural morphism $\tau_{\leq s}\mathcal{E} \to \mathrm{gr}^0 H^s \mathcal{E}[-s]$. We have distinguished triangles

$$U^1 H^s \mathcal{E}[-s] \to \tau_{\geq s}\mathcal{E} \to \tau'_{\geq s}\mathcal{E},$$
$$\tau'_{\leq s}\mathcal{E} \to \tau_{\leq s}\mathcal{E} \to \mathrm{gr}^0 H^s \mathcal{E}[-s],$$
$$\mathrm{gr}^0 H^s \mathcal{E}[-s] \to \tau'_{\geq s}\mathcal{E} \to \tau_{\geq s+1}\mathcal{E},$$
$$\tau_{\leq s-1}\mathcal{E} \to \tau'_{\leq s}\mathcal{E} \to U^1 H^s \mathcal{E}[-s],$$

where the latter two are truncation triangles.

PROPOSITION 6.3. *Let us abbreviate* $R\mathcal{H}om_Y$ *as* $[\,\cdot\,,\,\cdot\,]$. *There exists a unique set of morphisms*

$$\tau_{\geq s+1}\mathcal{E} \otimes^L \tau'_{\leq t}\mathcal{E} \to \nu(r)[-r-1],$$
$$\tau'_{\geq s}\mathcal{E} \otimes^L \tau_{\leq t}\mathcal{E} \to \nu(r)[-r-1],$$
$$U^1 H^s \mathcal{E} \otimes^L U^1 H^{t+1}\mathcal{E} \to \nu(r)[1],$$
$$\mathrm{gr}^0 H^s \mathcal{E} \otimes^L \mathrm{gr}^0 H^t \mathcal{E} \to \nu(r)$$

for integers s, t *with* $s + t = r + 1$ *that reduce to* (6.1) *for* $s < 0$ *and give morphisms of distinguished triangles from*

$$(6.2) \qquad\qquad U^1 H^s \mathcal{E}[-s] \to \tau_{\geq s}\mathcal{E} \to \tau'_{\geq s}\mathcal{E}$$

to the shift $[-r-1]$ *of*

$$(6.3) \qquad [U^1 H^{t+1}\mathcal{E}[-t-1], \nu(r)] \to [\tau'_{\leq t+1}\mathcal{E}, \nu(r)] \to [\tau_{\leq t}\mathcal{E}, \nu(r)]$$

and from

$$(6.4) \qquad\qquad \mathrm{gr}^0 H^s \mathcal{E}[-s] \to \tau'_{\geq s}\mathcal{E} \to \tau_{\geq s+1}\mathcal{E}$$

to the shift $[-r-1]$ *of*

$$(6.5) \qquad [\mathrm{gr}^0 H^t \mathcal{E}[-t], \nu(r)] \to [\tau_{\leq t}\mathcal{E}, \nu(r)] \to [\tau'_{\leq t}\mathcal{E}, \nu(r)].$$

PROOF. First we show that

$$(6.6) \qquad \begin{aligned} &\mathrm{Hom}\big(U^1 H^s \mathcal{E}[-s], [\tau_{\leq t}\mathcal{E}, \nu(r)][-r-1]\big) \\ &= \mathrm{Hom}\big(U^1 H^s \mathcal{E}[-s], [\tau_{\leq t}\mathcal{E}, \nu(r)][-r-2]\big) = 0. \end{aligned}$$

Since $[\tau_{\leq t}\mathcal{E}, \nu(r)]$ is concentrated in degrees $\geq -t = s - r - 1$, the second term is zero simply by a degree reason. By the same reasoning, the first term is equal to

$$\mathrm{Hom}\big(U^1 H^s \mathcal{E}, \mathcal{H}om(H^t \mathcal{E}, \nu(r))\big) = \mathrm{Hom}\big(H^t \mathcal{E}, \mathcal{H}om(U^1 H^s \mathcal{E}, \nu(r))\big).$$

The sheaf $U^1 H^s \mathcal{E}$ is a finite successive extension of locally free $\mathcal{O}_Y$-modules of finite rank by Proposition 6.1. Hence $\mathscr{H}om(U^1 H^s \mathcal{E}, \nu(r)) = 0$ by [Kat86, Theorem 3.2 (ii)]. This proves (6.6).

Next we show that

(6.7)
$$\begin{aligned}
&\mathrm{Hom}\big(\mathrm{gr}^0 H^s \mathcal{E}[-s], [\tau'_{\leq t}\mathcal{E}, \nu(r)][-r-1]\big) \\
&= \mathrm{Hom}\big(\mathrm{gr}^0 H^s \mathcal{E}[-s], [\tau'_{\leq t}\mathcal{E}, \nu(r)][-r-2]\big) = 0.
\end{aligned}$$

The same reasoning as above shows that the second term is zero and the first term is equal to

$$\mathrm{Hom}\big(\mathrm{gr}^0 H^s \mathcal{E}, \mathscr{H}om(U^1 H^t \mathcal{E}, \nu(r))\big)$$

since $H^t(\tau'_{\leq t}\mathcal{E}) = U^1 H^t \mathcal{E}$. We have $\mathscr{H}om(U^1 H^t \mathcal{E}, \nu(r)) = 0$ by [Kat86, Theorem 3.2 (ii)] since $U^1 H^t \mathcal{E}$ is a finite successive extension of locally free $\mathcal{O}_Y$-modules of finite rank by Proposition 6.1. This proves (6.7).

Now we prove the proposition by induction on s. There is nothing to do for $s < 0$. Fix integers s_0, t_0 with $s_0 + t_0 = r + 1$. Assume that there exists a unique set of morphisms as stated for s, t with $s + t = r + 1$ and $s \leq s_0$. We want to prove the same for $s = s_0 + 1$ and $t = t_0 - 1$. By assumption, we have a morphism from $\tau_{\geq s_0+1}\mathcal{E}$ to the shift $[-r-1]$ of $[\tau'_{\leq t_0}\mathcal{E}, \nu(r)]$. This gives a morphism from the middle term of (6.2) to the middle term of the shift $[-r-1]$ of (6.3) for $s = s_0 + 1$. By (6.6), this morphism uniquely extends to a morphism of distinguished triangles from (6.2) to the shift $[-r-1]$ of (6.3) for $s = s_0 + 1$. In particular, we have a morphism from $\tau'_{\geq s_0+1}\mathcal{E}$ to the shift $[-r-1]$ of $[\tau_{\leq t_0-1}\mathcal{E}, \nu(r)]$. This gives a morphism from the middle term of (6.4) to the middle term of the shift $[-r-1]$ of (6.5) for $s = s_0 + 1$. By (6.7), this morphism uniquely extends to a morphism of distinguished triangles from (6.4) to the shift $[-r-1]$ of (6.5) for $s = s_0 + 1$. This proves the induction step, and hence the proposition itself. $\square$

PROPOSITION 6.4. *The morphism* $\mathrm{gr}^0 H^s \mathcal{E} \otimes^L \mathrm{gr}^0 H^t \mathcal{E} \to \nu(r)$ *in Proposition 6.3 gives a perfect duality.*

PROOF. The stated morphism factors through $\mathrm{gr}^0 H^s \mathcal{E} \otimes \mathrm{gr}^0 H^t \mathcal{E}$. We have $\mathrm{gr}^0 H^s \mathcal{E} \cong \nu(s) \oplus \nu(s-1)$ and $\mathrm{gr}^0 H^t \mathcal{E} \cong \nu(t) \oplus \nu(t-1)$ by Proposition 6.1. Hence the stated morphism gives rise to a pairing

$$\nu(s) \oplus \nu(s-1) \times \nu(t) \oplus \nu(t-1) \to \nu(r).$$

By [Kat86, Theorem 3.2 (i)], it is enough to show that this pairing is given by

$$(6.8) \qquad \big((\omega, \omega'), (\tau, \tau')\big) \mapsto \pm \omega \wedge \tau' \pm \omega' \wedge \tau.$$

We may assume that $X = \operatorname{Spec} A$ is affine and $Y = \operatorname{Spec} R$ has a p-base. The composite of the natural surjection $H^s\mathcal{E} \otimes H^t\mathcal{E} \twoheadrightarrow \operatorname{gr}^0 H^s\mathcal{E} \otimes \operatorname{gr}^0 H^t\mathcal{E}$ and the morphism $\operatorname{gr}^0 H^s\mathcal{E} \otimes \operatorname{gr}^0 H^t\mathcal{E} \to \nu(r)$ is the morphism $H^s\mathcal{E} \otimes H^t\mathcal{E} \to \nu(r)$ induced by (6.1). Let R' be the strict henselian local ring of a relatively perfectly smooth R-algebra at a prime ideal. We want to describe our pairing on R'-points. The map on R'-points of the morphism $H^s\mathcal{E} \otimes H^t\mathcal{E} \to \nu(r)$ is of the form

$$H^s(R'_{\hat{A}_K}, \Lambda(s)) \otimes H^t(R'_{\hat{A}_K}, \Lambda(t)) \to H^{r+1}(R'_{\hat{A}_K}, \Lambda(r+1)) \to \Gamma(R', \nu(r)).$$

Let A' be an ind-smooth lifting of R' over A. By Proposition 4.2, the above map can be written as

$$H^s(A'_K, \Lambda(s)) \otimes H^t(A'_K, \Lambda(t)) \to H^{r+1}(A'_K, \Lambda(r+1)) \to \Gamma(R', \nu(r)).$$

The groups in the first term are generated by symbols by [BK86, Theorem (1.4)]. The first map is given by concatenation of symbols and the second described by the paragraph after [BK86, Corollary (1.4.1)]. By an easy computation of symbols and dlog forms, we see that our pairing is indeed given the formula (6.8). This proves the proposition. $\square$

7. Mod p Case II: Duality for U^1

We keep the notation from the last section. In particular, we fix a prime element π of K and a primitive p-th root of unity $\zeta_p \in K$, and we work with Λ-coefficients. To treat the part $U^1 H^s\mathcal{E} \otimes^L U^1 H^{t+1}\mathcal{E} \to \nu(r)[1]$ of Proposition 6.3, it is convenient to use the Zariski topology in addition to the étale topology.

Let Y_{RPSZ} be the category of relatively perfectly smooth Y-schemes endowed with the Zariski topology. Let $\varepsilon \colon Y_{\mathrm{RPS}} \to Y_{\mathrm{RPSZ}}$ be the morphism defined by the identity functor. Recall from [Kat86, (3.1.4), (3.1.5)] that there are exact sequences

$$0 \to \nu(r) \to \Omega^r_{Y, d=0} \xrightarrow{C-1} \Omega^r_Y \to 0,$$

$$0 \to \nu(r) \to \Omega^r_Y \xrightarrow{C^{-1}-1} \Omega^r_Y / d\Omega^{r-1}_Y \to 0$$

in Y_{RPS}, where C is the Cartier operator. Since $r = r_0 + d$ is the number of elements in local p-bases of Y, we have $\Omega_Y^{r+1} = 0$, so $\Omega_{Y,d=0}^r = \Omega_Y^r$ and C is an endomorphism of Ω_Y^r. We can view $\nu(r)$ as a sheaf on Y_{RPSZ}, which is the kernel of the endomorphism $C - 1$ on Ω_Y^r. Define a sheaf $\xi(r)$ on Y_{RPSZ} to be the cokernel of the endomorphism $C - 1$ on Ω_Y^r over Y_{RPSZ}. The exact sequence $0 \to \nu(r) \to \Omega_Y^r \overset{C-1}{\to} \Omega_Y^r \to 0$ over Y_{RPS} shows that $R^n \varepsilon_* \nu(r) = 0$ for $n \geq 2$ and $R^1 \varepsilon_* \nu(r) = \xi(r)$, and defines a morphism

$$R\varepsilon_* \nu(r) \to \xi(r)[-1]$$

in $D(Y_{\mathrm{RPSZ}}, \Lambda)$. For any $M \in D_0(Y_{\mathrm{RPS}}, \Lambda)$, the isomorphism $\varepsilon^* R\varepsilon_* M \cong M$, the sheafified derived adjunction and the above morphism define a morphism

$$R\varepsilon_* R\mathscr{H}om_{Y_{\mathrm{RPS}}}(M, \nu(r)) \cong R\mathscr{H}om_{Y_{\mathrm{RPSZ}}}(R\varepsilon_* M, R\varepsilon_* \nu(r))$$
$$\to R\mathscr{H}om_{Y_{\mathrm{RPSZ}}}(R\varepsilon_* M, \xi(r))[-1]$$

in $D(Y_{\mathrm{RPSZ}}, \Lambda)$.

PROPOSITION 7.1.   *The above morphism is an isomorphism.*

PROOF.   We need to show that $R\mathscr{H}om_{Y_{\mathrm{RPSZ}}}(R\varepsilon_* M, \nu(r)) = 0$. We may assume that Y is affine with a p-base and $M = \mathbf{G}_a$, and it is enough to show that $R\mathrm{Hom}_{Y_{\mathrm{RPSZ}}/Y'}(\mathbf{G}_a, \nu(r))$ is zero for any relatively perfectly smooth affine Y-scheme Y'. By [Blo86, Theorem (2.1)] and [GL00, Theorem 8.3], we know that Zariski cohomology with coefficients in $\nu(r)$ is homotopy invariant. Hence the natural morphism from $R\Gamma(Y'_{\mathrm{zar}}, \nu(r))$ to $R\Gamma((\mathbb{A}_{Y'}^m)_{\mathrm{zar}}, \nu(r))$ is invertible for any $m \geq 1$. This implies that the natural morphism from $R\Gamma(Y'_{\mathrm{zar}}, \nu(r))$ to $R\Gamma((\mathbb{A}_{Y'}^m)_{\mathrm{zar}}^{\mathrm{RP}}, \nu(r))$ (the Zariski cohomology of the relative perfection of $\mathbb{A}_{Y'}^m$) is invertible for any $m \geq 1$ since $(\mathbb{A}_{Y'}^m)^{\mathrm{RP}}$ is an inverse limit of affine spaces over Y'. Hence, by using Mac Lane's resolution of $\mathbf{G}_a$ as in the proof of Proposition 2.2, we know that the morphism from $R\mathrm{Hom}_{Y_{\mathrm{RPSZ}}/Y'}(0, \nu(r))$ (which is zero) to $R\mathrm{Hom}_{Y_{\mathrm{RPSZ}}/Y'}(\mathbf{G}_a, \nu(r))$ induced by $\mathbf{G}_a \to 0$ is invertible. The result then follows. $\square$

PROPOSITION 7.2.   *Let $M \in M(Y_{\mathrm{RPS}}, \Lambda)$ be a sheaf admitting a finite filtration whose graded pieces are isomorphic to relative perfections of coherent sheaves on Y locally free of finite rank. Note that $R\varepsilon_* M = \varepsilon_* M$.*

View M also as a sheaf on Y_{RPSZ}. Then

$$R\,\mathscr{H}om_{Y_{\mathrm{RPS}}}(M,\nu(r)), \quad R\varepsilon_*R\,\mathscr{H}om_{Y_{\mathrm{RPS}}}(M,\nu(r)),$$
$$R\,\mathscr{H}om_{Y_{\mathrm{RPSZ}}}(M,\xi(r))[-1]$$

are all concentrated in degree 1.

PROOF.   This follows from [Kat86, Theorem 3.2 (ii)] and Proposition 7.1. $\square$

Let $\mathcal{E}' = R\varepsilon_*\mathcal{E}$; cf. the paragraph before [BK86, Theorem (6.7)].

PROPOSITION 7.3.   *Assume that $X = \operatorname{Spec} A$ is affine and $Y = \operatorname{Spec} R$ has a p-base. Let R' be the local ring of a relatively perfectly smooth R-algebra at a prime ideal. Then for any q, the stalk of the Zariski sheaf $H^q\mathcal{E}'$ at the closed point of $\operatorname{Spec} R'$ is given by $H^q(R'_{\hat{A}_K},\Lambda(q))$ (cohomology in the étale topology).*

PROOF.   The stalk is given by $H^q(R'_{\mathrm{et}}, i^*Rj_*\Lambda(q))$. Since the pair $(R'_{\hat{A}}, R'_{\hat{A}}\mathfrak{p}_K)$ is henselian, this group is isomorphic to $H^q(R'_{\hat{A}}, Rj_*\Lambda(q))$ by Gabber's affine analog of proper base change [Gab94, Theorem 1]. This final group is isomorphic to $H^q(R'_{\hat{A}_K},\Lambda(q))$. $\square$

Define a filtration on $H^q\mathcal{E}'$ in the same way as in the case of $H^q\mathcal{E}$. As in Proposition 6.1, we have the following.

PROPOSITION 7.4.

(1) *We have*

$$\mathrm{gr}^0 H^q\mathcal{E}' \cong \nu(q)\oplus\nu(q-1).$$

(2) *If $1 \le m < e'$ and $p \nmid m$, then*

$$\mathrm{gr}^m H^q\mathcal{E}' \cong \Omega_Y^{q-1}.$$

(3) *If $1 \le m < e'$ and $p \mid m$, then*

$$\mathrm{gr}^m H^q\mathcal{E}' \cong d\Omega_Y^{q-1}\oplus d\Omega_Y^{q-2}.$$

(4) *We have*

$$U^{e'} H^q \mathcal{E}' \cong \Omega_Y^{q-1}/(1+aC)\Omega_{Y,d=0}^{q-1} \oplus \Omega_Y^{q-2}/(1+aC)\Omega_{Y,d=0}^{q-1},$$

   where $a \in k$ is the residue class of $p\pi^{-e}$.

(5) *If $m > e'$, then*

$$U^m H^q \mathcal{E}' = 0.$$

PROOF.   Apply $R\varepsilon_*$ to Proposition 6.1, use Proposition 7.3 instead of 4.2 and argue as in [BK86, Theorem (6.7)]. □

For the proof of Theorem 3.4, we may assume that the element a above has a $(p-1)$-st root in k. Then $1+aC$ in the statement may be replaced by $C-1$.

Applying $R\varepsilon_*$ to (6.1), we have morphisms

$$\mathcal{E}' \otimes^L \mathcal{E}' \to \mathcal{E}' \to R\varepsilon_*\nu(r)[-r-1] \to \xi(r)[-r-2].$$

The morphism $\mathcal{E}' \to \xi(r)[-r-2]$ from the second term to the fourth term is also given by

$$\mathcal{E}' \to H^{r+2}\mathcal{E}'[-r-2] = \mathrm{gr}^{e'} H^{r+2}\mathcal{E}'[-r-2] \to \xi(r)[-r-2]$$

using Proposition 7.4. For integers s, t with $s + t = r + 2$, we thus have a pairing

(7.1) $$H^s\mathcal{E}' \times H^t\mathcal{E}' \to H^{r+2}\mathcal{E}' \to \xi(r).$$

PROPOSITION 7.5.   *The pairing (7.1) restricted to $U^l H^s \mathcal{E}' \times U^m H^t \mathcal{E}'$ is zero if $l+m > e'$ and hence induces a pairing $\mathrm{gr}^l H^s \mathcal{E}' \times \mathrm{gr}^m H^t \mathcal{E}' \to \xi(r)$ for $l+m = e'$. The induced morphism*

$$\mathrm{gr}^l H^s \mathcal{E}' \to R\,\mathcal{H}om_{Y_{\mathrm{RPSZ}}}(\mathrm{gr}^m H^t \mathcal{E}', \xi(r))$$

is an isomorphism if $l, m > 0$.

PROOF.   The morphism $H^s\mathcal{E}' \times H^t\mathcal{E}' \to H^{r+2}\mathcal{E}'$ takes $U^l H^s\mathcal{E}' \times U^m H^t\mathcal{E}'$ to $U^{l+m} H^{r+2}\mathcal{E}'$ by Proposition 4.2 and [BK86, Lemma (4.1)]. The first assertion follows. For the second, consider the pairing between Ω_Y^{s-1} and Ω_Y^{t-1} with values in $\Omega_Y^r/d\Omega_Y^{r-1}$ given by the wedge product if $p \nmid m$ and the pairing between $d\Omega_Y^{s-1} \oplus d\Omega_Y^{s-2}$ and $d\Omega_Y^{t-1} \oplus d\Omega_Y^{t-2}$ with values in $\Omega_Y^r/d\Omega_Y^{r-1}$ given by $(d\omega, d\omega') \times (d\tau, d\tau') \mapsto \omega \wedge d\tau' + \omega' \wedge d\tau$ if $p \mid m$. Consider the composite of them with the natural surjection $\Omega^r/d\Omega_Y^{r-1} \twoheadrightarrow \xi(r)$. With the isomorphisms in Proposition 7.4, we have a pairing $\mathrm{gr}^l H^s\mathcal{E}' \times \mathrm{gr}^m H^t\mathcal{E}' \to \xi(r)$ for $l + m = e'$. This pairing agrees with the stated pairing up to an $\mathbb{F}_p^\times$-multiple if $l, m > 0$ by the same argument as [BK86, Lemma (5.2)]. With this description and [Kat86, Theorem 3.2 (ii)], we see that the stated induced morphism is an isomorphism. $\square$

COROLLARY 7.6.   *Let $s+t = r+2$. The pairing (7.1) induces a pairing between $U^1 H^s\mathcal{E}'/U^{e'} H^s\mathcal{E}'$ and $U^1 H^t\mathcal{E}'/U^{e'} H^t\mathcal{E}'$ with values in $\xi(r)$. The induced morphism*

$$U^1 H^s\mathcal{E}'/U^{e'} H^s\mathcal{E}' \to R\,\mathcal{H}om_{Y_{\mathrm{RPSZ}}}\big(U^1 H^t\mathcal{E}'/U^{e'} H^t\mathcal{E}', \xi(r)\big)$$

is an isomorphism.

PROPOSITION 7.7.   *Let $s + t = r + 2$. The morphism*

$$U^1 H^s\mathcal{E} \otimes^L U^1 H^t\mathcal{E} \to \nu(r)[1]$$

in $D(Y_{\mathrm{RPS}}, \Lambda)$ defined in Proposition 6.3 gives a perfect duality.

PROOF.   Let $M = U^1 H^s\mathcal{E}'/U^{e'} H^s\mathcal{E}'$ and $N = U^1 H^t\mathcal{E}'/U^{e'} H^t\mathcal{E}'$. We have $\varepsilon^* U^{e'} H^s\mathcal{E}' = 0$ and $\varepsilon^* U^m H^s\mathcal{E}' = U^m H^s\mathcal{E}$ for any m. Hence $\varepsilon^* M = U^m H^s\mathcal{E}$ and $\varepsilon^* N = U^m H^t\mathcal{E}$. Since M and N are finite successive extensions of locally free sheaves on Y of finite rank by Proposition 7.4, we have $M \xrightarrow{\sim} R\varepsilon_* \varepsilon^* M$ and $N \xrightarrow{\sim} R\varepsilon_* \varepsilon^* N$. The morphism $\varepsilon^* M \otimes^L \varepsilon^* N \to \nu(r)[1]$ in question induces a morphism $M \otimes^L N \to R\varepsilon_* \nu(r)[1] \to \xi(r)$ by adjunction, which agrees with the pairing in Corollary 7.6. Hence the diagram

$$
\begin{array}{ccc}
R\varepsilon_* \varepsilon^* M & \longrightarrow & R\varepsilon_* R\,\mathcal{H}om_{Y_{\mathrm{RPS}}}(\varepsilon^* N, \nu(r))[1] \\
\| & & \| \\
M & \longrightarrow & R\,\mathcal{H}om_{Y_{\mathrm{RPSZ}}}(N, \xi(r))
\end{array}
$$

is commutative, where the right vertical isomorphism is from Proposition 7.1. The lower horizontal morphism is an isomorphism by Corollary 7.6. Hence the upper horizontal morphism is also an isomorphism. Its pullback $\varepsilon^* M \to R\mathscr{H}om_{Y_{\mathrm{RPS}}}(\varepsilon^* N, \nu(r))[1]$ is thus an isomorphism. This gives the result. $\square$

PROPOSITION 7.8. *The statement of Theorem 3.4 (3) is true for $n = 1$.*

PROOF. This follows from Propositions 6.3, 6.4 and 7.7. $\square$

8. General Case

Let $n \geq 1$ and s, t with $s + t = r + 1$.

PROPOSITION 8.1. *The statement of Theorem 3.4 (2) is true.*

PROOF. The distinguished triangle $R\Psi\Lambda_{n-1}(s) \to R\Psi\Lambda_n(s) \to R\Psi\Lambda_1(s)$ and induction reduce the statement to the case $n = 1$ already proven in Proposition 6.2. $\square$

Hence we have a canonical morphism $R\Psi\Lambda_n(s) \otimes^L R\Psi\Lambda_n(t) \to \nu_n(r)[-r-1]$ as explained in Theorem 3.4. We want to prove that it induces a perfect duality as stated.

Denote $R\mathscr{H}om_{(\Lambda_n)_{Y_{\mathrm{RPS}}}}$ by $[\ \cdot\ ,\ \cdot\ ]_n$. For $m \leq n$, the exact inclusion $M(Y, \Lambda_m) \hookrightarrow M(Y, \Lambda_n)$ induces a triangulated functor $D(Y, \Lambda_m) \to D(Y, \Lambda_n)$. We denote it by θ, but it is frequently omitted from the notation, and the image of an object M by θ is simply denoted by just M. Set $\mathcal{E}_n^s = R\Psi\Lambda_n(s)[s]$. Denote the morphism $\mathrm{Tr}\colon \mathcal{E}_n^{r+1} \to \nu_n(r)$ by Tr_n. Hence we have canonical morphisms

$$\mathcal{E}_n^s \otimes^L \mathcal{E}_n^t \to \mathcal{E}_n^{r+1} \overset{\mathrm{Tr}_n}{\to} \nu_n(r).$$

PROPOSITION 8.2. *For $m \leq n$, consider the composite of the morphisms*

$$\mathcal{E}_m^s \to [\mathcal{E}_m^t, \mathcal{E}_m^{r+1}]_m \overset{\theta}{\to} [\mathcal{E}_m^t, \mathcal{E}_m^{r+1}]_n \overset{\text{``}p^{n-m}\text{''}}{\to} [\mathcal{E}_m^t, \mathcal{E}_n^{r+1}]_n \overset{\mathrm{Tr}_n}{\to} [\mathcal{E}_m^t, \nu_n(r)]_n$$

and the composite of the morphisms

$$\mathcal{E}_m^s \to [\mathcal{E}_m^t, \mathcal{E}_m^{r+1}]_m \xrightarrow{\mathrm{Tr}_m} [\mathcal{E}_m^t, \nu_m(r)]_m \xrightarrow{\theta} [\mathcal{E}_m^t, \nu_m(r)]_n \xrightarrow{\text{``}p^{n-m}\text{''}} [\mathcal{E}_m^t, \nu_n(r)]_n.$$

They are equal. If the statement of Theorem 3.4 (3) is true for m, then they are isomorphisms.

Proof.   That they are equal follows from Proposition 5.2. The composite

$$[\mathcal{E}_m^t, \nu_m(r)]_m \xrightarrow{\theta} [\mathcal{E}_m^t, \nu_m(r)]_n \xrightarrow{\text{``}p^{n-m}\text{''}} [\mathcal{E}_m^t, \nu_n(r)]_n$$

is an isomorphism by [Kat86, (4.2.4)]. Hence the second statement follows. □

Proposition 8.3.   *The diagram*

$$
\begin{array}{ccccc}
\mathcal{E}_1^s & \xrightarrow{\text{``}p^{n-1}\text{''}} & \mathcal{E}_n^s & \longrightarrow & \mathcal{E}_{n-1}^s \\
\downarrow & & \downarrow & & \downarrow \\
[\mathcal{E}_1^t, \nu_n(r)]_n & \longrightarrow & [\mathcal{E}_n^t, \nu_n(r)]_n & \xrightarrow{\text{``}p\text{''}} & [\mathcal{E}_{n-1}^t, \nu_n(r)]_n
\end{array}
$$

is a morphism of distinguished triangles, where the vertical morphisms are the morphisms of Proposition 8.2 for $m = 1, n, n-1$ from the left to the right.

Proof.   Applying $R\Psi[s]$ to the morphism of distinguished triangles

$$
\begin{array}{ccccc}
\Lambda_1(s) & \xrightarrow{\text{``}p^{n-1}\text{''}} & \Lambda_n(s) & \longrightarrow & \Lambda_{n-1}(s) \\
\downarrow & & \downarrow & & \downarrow \\
[\Lambda_1(t), \Lambda_n(r+1)]_n & \longrightarrow & [\Lambda_n(t), \Lambda_n(r+1)]_n & \xrightarrow{\text{``}p\text{''}} & [\Lambda_{n-1}(t), \Lambda_n(r+1)]_n,
\end{array}
$$

we have a morphism of distinguished triangles

$$
\begin{array}{ccccc}
\mathcal{E}_1^s & \xrightarrow{\text{``}p^{n-1}\text{''}} & \mathcal{E}_n^s & \longrightarrow & \mathcal{E}_{n-1}^s \\
\downarrow & & \downarrow & & \downarrow \\
[\mathcal{E}_1^t, \mathcal{E}_n^{r+1}]_n & \longrightarrow & [\mathcal{E}_n^t, \mathcal{E}_n^{r+1}]_n & \xrightarrow{\text{``}p\text{''}} & [\mathcal{E}_{n-1}^t, \mathcal{E}_n^{r+1}]_n.
\end{array}
$$

The morphism Tr_n gives a morphism of distinguished triangles from the lower triangle of this diagram to the lower triangle of the stated diagram. □

PROPOSITION 8.4. *The statement of Theorem 3.4 (3) is true.*

PROOF. This follows from Propositions 7.8, 8.2 and 8.3 by induction. □

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