J. Math. Sci. Univ. Tokyo
32 (2025), 257–400.

Symmetry Breaking Operators for Dual Pairs with One Member Compact

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Abstract. We consider a dual pair $(\mathrm{G}, \mathrm{G}')$, in the sense of Howe, with G compact acting on $\mathrm{L}^2(\mathbb{R}^n)$, for an appropriate n, via the Weil representation ω. Let $\widetilde{\mathrm{G}}$ be the preimage of G in the metaplectic group. Given a genuine irreducible unitary representation Π of $\widetilde{\mathrm{G}}$, let Π' be the corresponding irreducible unitary representation of $\widetilde{\mathrm{G}}'$ in Howe's correspondence. The orthogonal projection onto the Π-isotypic component $\mathrm{L}^2(\mathbb{R}^n)_\Pi$ is, up to a constant multiple, the unique symmetry breaking operator in $\mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$. We study this operator by computing its Weyl symbol. Our results allow us to recover the known list of highest weights of irreducible representations of $\widetilde{\mathrm{G}}$ occurring in Howe's correspondence when the rank of G is strictly bigger than the rank of G'. They also allow us to compute the wavefront set of Π' by elementary means.

Contents

2020 *Mathematics Subject Classification.* Primary 22E45; Secondary 22E46, 22E30.

Key words: Reductive dual pairs, Howe duality, symmetry breaking operators, Weyl calculus, Lie superalgebras.

The second author is grateful to the University of Oklahoma for hospitality and financial support. She is partially supported by the ANR Project OpART (ANR-23-CE40-0016). The third author acknowledges hospitality and financial support from the Université de Lorraine and partial support from the NSA under Grant H98230-13-1-0205 and the NSF under Grant DMS-2225892. Part of this research took place within the online Research Community on Representation Theory and Noncommutative Geometry sponsored by the American Institute of Mathematics. Some work was done while the last two authors were at the Institute of Mathematical Sciences of the National University of Singapore. They gratefully thank the Institute for hospitality and financial support.

Introduction

Let W be a finite dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ and let $\mathrm{Sp}(\mathrm{W})$ denote the corresponding symplectic group. Write $\widetilde{\mathrm{Sp}}(\mathrm{W})$ for the metaplectic group. Let us fix the character χ of $\mathbb{R}$ given by $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$. Then the Weil representation of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ associated to χ is denoted by $(\omega, \mathcal{H}_\omega)$.

For $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathrm{W})$ forming a reductive dual pair in the sense of Howe, let $\widetilde{\mathrm{G}}$, $\widetilde{\mathrm{G}}'$ denote their preimages in $\widetilde{\mathrm{Sp}}(\mathrm{W})$. Howe's correspondence (or local θ-correspondence) for $\widetilde{\mathrm{G}}$, $\widetilde{\mathrm{G}}'$ is a bijection $\Pi \leftrightarrow \Pi'$ between the irreducible admissible representations of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$ which occur as smooth quotients of ω, [How89b]. It can be formulated as follows. Assume that $\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty) \neq 0$. Then $\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty)$ is a $\widetilde{\mathrm{G}}'$-module under the action via ω. Howe proved that it has a unique irreducible quotient, which is an irreducible admissible representation $(\Pi', \mathcal{H}_{\Pi'})$ of $\widetilde{\mathrm{G}}'$. Conversely, $\mathrm{Hom}_{\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_{\Pi'}^\infty)$ is a $\widetilde{\mathrm{G}}$-module which has a unique irreducible admissible quotient, infinitesimally equivalent to $(\Pi, \mathcal{H}_\Pi)$. Furthermore, $\Pi \otimes \Pi'$ occurs as a quotient of ω^∞ in a unique way, i.e.

$$\dim \mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) = 1\,. \tag{1}$$

In [Kob15], the elements of

$$\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty)\,, \quad \mathrm{Hom}_{\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_{\Pi'}^\infty) \quad \text{and} \quad \mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$$

are called symmetry breaking operators. Their construction is part of Stage C of Kobayashi's program for branching problems in the representation theory of real reductive groups.

Since the last space is one dimensional, it deserves a closer look. The explicit contruction of the (essentially unique) symmetry breaking operator in $\mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$ provides an alternative and direct approach to Howe's correspondence. To do this is the aim of the present paper.

Our basic assumption is that $(\mathrm{G}, \mathrm{G}')$ is an irreducible dual pair with G compact. As shown by Howe [How79], up to an isomorphism, $(\mathrm{G}, \mathrm{G}')$ is one of the pairs

$$(\mathrm{O}_d, \mathrm{Sp}_{2m}(\mathbb{R}))\,, \qquad (\mathrm{U}_d, \mathrm{U}_{p,q})\,, \qquad (\mathrm{Sp}_d, \mathrm{O}_{2m}^*)\,. \tag{2}$$

Then the representations Π and Π' together with their contragredients are arbitrary irreducible unitary highest weight representations. They have been defined by Harish-Chandra in [Har55], were classified in [EHW83] and have been studied in terms of Zuckerman functors in [Wal84], [Ada83] and [Ada87]. The 1-1 correspondence of representations in terms of their highest weights was first determined by Kashiwara and Vergne in [KV78].

The crucial fact for constructing the symmetry breaking operator in $\mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$ is that, up to a non-zero constant multiple, there is a unique $\mathrm{G}\mathrm{G}'$-invariant tempered distribution $f_{\Pi\otimes\Pi'}$ on W such that

$$\mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) = \mathbb{C}(\mathrm{Op}\circ\mathcal{K})(f_{\Pi\otimes\Pi'})\,, \tag{3}$$

where Op and $\mathcal{K}$ are classical transformations which we shall review in section 1. In [Prz93], $f_{\Pi\otimes\Pi'}$ is called the intertwining distribution associated to $\Pi\otimes\Pi'$. In fact, if we work in a Schrödinger model of ω, then $f_{\Pi\otimes\Pi'}$ happens to be the Weyl symbol, [Hör83], of the operator $(\mathrm{Op}\circ\mathcal{K})(f_{\Pi\otimes\Pi'})$.

The previous paragraph does not require G to be compact. Suppose that the group G is compact. Let Θ_Π and d_Π respectively denote the character and the degree of Π. Then the projection onto the Π-isotypic component of ω is equal to $d_\Pi/2$ times

$$\int_{\widetilde{\mathrm{G}}} \omega(\tilde g)\check\Theta_\Pi(\tilde g)\,d\tilde g = \omega(\check\Theta_\Pi)\,, \tag{4}$$

where $\check\Theta_\Pi(\tilde g) = \Theta_\Pi(\tilde g^{-1})$ and we normalize the Haar measure $d\tilde g$ of $\tilde{\mathrm{G}}$ to have the total mass 2. (This explains the constant multiple $\frac{1}{2}$ needed for the projection. In this way, the mass of G is equal to 1.) By Howe's correspondence with G compact, the projection onto the Π-isotypic component of ω is a symmetry breaking operator for $\Pi\otimes\Pi'$. The intertwining distribution for $\Pi\otimes\Pi'$ is therefore determined by the equation

$$(\mathrm{Op}\circ\mathcal{K})(f_{\Pi\otimes\Pi'}) = \frac{1}{2}\omega(\check\Theta_\Pi)\,. \tag{5}$$

There are more cases when $f_{\Pi\otimes\Pi'}$ may be computed via the formula (5), see [Prz93]. However, if the group G is compact then the distribution character $\Theta_{\Pi'}$ may also be recovered from $f_{\Pi\otimes\Pi'}$ via an explicit formula, see [Prz91]. Thus, in this case, we have a diagram

$$\Theta_\Pi \longrightarrow f_{\Pi\otimes\Pi'} \longrightarrow \Theta_{\Pi'}\,. \tag{6}$$

In general, the asymptotic properties of $f_{\Pi\otimes\Pi'}$ relate the associated varieties of the primitive ideals of Π and Π' and, under some more assumptions, the wave front sets of these representations, see [Prz93], [Prz91] and [MPP24].

The usual, often very successful, approach to Howe's correspondence avoids any work with distributions on the symplectic space. Instead, one

finds Langlands parameters (see [Moe89], [AB95], [Pau98], [Pau00], [Pau05], [LPTZ03]), character formulas (see [Ada98], [Ren98], [DP96], [Prz18], [Mer20], [LP22]), or candidates for character formulas (as in [BP14], [Prz00], [LP24]), or one establishes preservation of unitarity (as in [Li89], [He03], [Prz93], [ABP$^+$07], [HLS11], [MSZ17]). However, in the background (explicit or not), there is the orbit correspondence induced by the unnormalized moment maps

$$\mathfrak{g}^* \longleftarrow \mathrm{W} \longrightarrow \mathfrak{g}'^* \,,$$

where $\mathfrak{g}$ and $\mathfrak{g}'$ denote the Lie algebras of G and G$'$, respectively, and $\mathfrak{g}^*$ and $\mathfrak{g}'^*$ are their duals. This correspondence of orbits has been studied in [DKP97], [DKP05] and [Pan10]. Furthermore, in their recent work, [LM15], Loke and Ma computed the associated variety of the representations for the dual pairs in the stable range in terms of the orbit correspondence. The p-adic case was studied in detail in [Moe98].

Working with the GG$'$-invariant distributions on W is a more direct approach than relying on the orbit correspondence and provides different insights and results. As a complementary contribution to all work mentioned above, we compute the intertwining distributions $f_{\Pi\otimes\Pi'}$ explicitly, see section 5. As an application, we obtain the wave front set of Π' by elementary means. The computation will be sketched in section 17, and the detailed proof appeared in [MPP24]. Another application of the methods presented in this paper leads to the explicit formula for the character of the corresponding irreducible unitary representation Π' of $\widetilde{\mathrm{G}}'$. This can be found in [Mer17, Mer20].

The explicit formulas for the intertwining distribution provide important information on the nature of the symmetry breaking operators. Namely, they show that none of the symmetry breaking operators of the form $(\mathrm{Op}\circ\mathcal{K})(f_{\Pi\otimes\Pi'})$ is a differential operator. For the present situation, this answers in the negative the question on the existence of differential symmetry breaking operators, addressed in different contexts by several authors (see for instance [KP16a, KP16b, KS15] and the references given there). This property is the content of Corollary 14.

Finally, observe that our computations leading to the intertwining distributions apply to any genuine irreducible representation Π of the compact member $\widetilde{\mathrm{G}}$ of a dual pair. They provide an explicit formula for the Weyl symbol of the projection of $\omega|_{\widetilde{\mathrm{G}}}$ onto the Π-isotypic component. According

to Howe's duality theorem, this projection is non-zero if and only if there is a unitary highest weight representation Π' of $\widetilde{\mathrm{G}}'$ such that $\Pi \otimes \Pi'$ occurs in $\omega|_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}$, i.e. Π occurs in Howe's correspondence. When the rank of G is strictly bigger than that of G′, we recover the known necessary and sufficient conditions on the highest weights of Π so that it occurs in Howe's correspondence. See Corollary 11.

The paper is organized as follows. In section 1, we introduce some notation and review the construction of the intertwining distributions. Section 2 computes the intertwining distribution for the dual pair $(\mathrm{Z}, \mathrm{Sp}(\mathrm{W}))$, where $\mathrm{Z} = \mathrm{O}_1$ is the center of the symplectic group $\mathrm{Sp}(\mathrm{W})$, and introduces some properties needed in the sequel. Section 3 recalls how to realize the dual pairs with one member compact as Lie supergroups, and section 4 collects some definitions and properties of the Weyl–Harish-Chandra integration formulas on W that we will need to compute the intertwining distributions. Section 5 states the main results of this paper. The dual pairs $(\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$ are particular because the group SO_2 is abelian. The intertwining distributions corresponing to these pairs are computed in section 6. The smallest example of $(\mathrm{O}_2, \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}))$ is presented with more details. An additional example is given in section 7, where we illustrate the main two theorems when $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,p})$ and Π is the trivial representation of U_l. The proofs of the main results are in sections 8, 9 and 10. We treat the special cases concerning the non-identity connected components of the orthogonal groups in sections 11, 12, 13 and 14. Here we need the Weyl's integral and character formulas found by Wendt in [Wen01]. Section 15 contains the proof of a necessary condition of a representation of $\widetilde{\mathrm{U}}_l$ to occur in Howe's correspondence for $(\mathrm{U}_l, \mathrm{U}_{p,q})$ when $p = \min(p, q) < l \leq l' = p+q$. In section 16, we consider the dual pair $(\mathrm{Sp}_l, \mathrm{O}^*_{2l'})$. Using intertwining distribution, we recover the known fact that certain representations of Sp_l occur in Howe's correspondence. Finally, in section 17, we outline how the results of this paper lead, for each representation Π of $\widetilde{\mathrm{G}}$ occurring in Howe's duality, to the computation of the wave front set of the representation Π' dual to Π. The details are in [MPP24]. The nine appendices collect and prove some auxiliary results.

Acknowledgement. We are indebted to the anonymous referee whose extremely careful reading and valuable comments made us aware of errors and omissions in the original manuscript. The questions raised by the referee

have lead us to make significant additions, which have greatly improved our paper.

1. Notation and Preliminaries

Let us first recall the construction of the metaplectic group $\widetilde{\mathrm{Sp}}(\mathrm{W})$ and the Weil representation ω. We are using the approach of [AP14, Section 4], to which we refer the reader for more details.

Let $\mathfrak{sp}$ denote the Lie algebra of $\mathrm{Sp}(\mathrm{W})$, both contained in $\mathrm{End}(\mathrm{W})$. Fix a positive definite compatible complex structure J on W, that is an element $J \in \mathfrak{sp}$ such that $J^2 = -1$ (minus the identity on W) and the symmetric bilinear form $\langle J\cdot, \cdot\rangle$ is positive definite on W. For an element $g \in \mathrm{Sp}(\mathrm{W})$, let $J_g = J^{-1}(g-1)$. The adjoint of J_g with respect to the form $\langle J\cdot, \cdot\rangle$ is $J_g^* = Jg^{-1}(1-g)$. In particular, J_g and J_g^* have the same kernel. Hence the image of J_g is

$$J_g\mathrm{W} = (\mathrm{Ker}\, J_g^*)^\perp = (\mathrm{Ker}\, J_g)^\perp\,,$$

where $\perp$ denotes the orthogonal complement with respect to $\langle J\cdot, \cdot\rangle$. Therefore, the restriction of J_g to $J_g\mathrm{W}$ defines an invertible element. Thus for every $g \neq 1$, it makes sense to talk about $\det(J_g)^{-1}_{J_g\mathrm{W}}$, the reciprocal of the determinant of the restriction of J_g to $J_g\mathrm{W}$. With this notation, we have

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \{\tilde{g} = (g;\xi) \in \mathrm{Sp}(\mathrm{W}) \times \mathbb{C},\ \ \xi^2 = i^{\dim(g-1)\mathrm{W}} \det(J_g)^{-1}_{J_g\mathrm{W}}\}\,, \tag{7}$$

with the convention that $\det(J_g)^{-1}_{J_g\mathrm{W}} = 1$ if $g = 1$. There exists a 2-cocycle $C : \mathrm{Sp}(\mathrm{W}) \times \mathrm{Sp}(\mathrm{W}) \to \mathbb{C}$, explicitly described in [AP14, Proposition 4.13], such that $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a group with respect to the multiplication

$$(g_1;\xi_1)(g_2;\xi_2) = (g_1g_2;\xi_1\xi_2C(g_1,g_2)) \tag{8}$$

and the homomorphism

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni (g;\xi) \to g \in \mathrm{Sp}(\mathrm{W}) \tag{9}$$

does not split.

Let μ_W (or simply dw) be the Lebesgue measure on W normalized by the condition that the volume of the unit cube with respect to the form $\langle J\cdot, \cdot\rangle$ is 1. (Since all positive complex structures are conjugate by elements

of Sp, this normalization does not depend on the particular choice of J.) Let $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ be a complete polarization. We suppose that X, Y and J are chosen so that $J(\mathrm{X}) = \mathrm{Y}$. Similar normalizations are fixed for the Lebesgue measures on every vector subspace of W, for instance on X and on Y. Furthermore, for every finite dimensional real vector space V, we write $\mathcal{S}(\mathrm{V})$ for the Schwartz space on V and $\mathcal{S}'(\mathrm{V})$ for the space of tempered distributions on V. We use the notation G' for the second member of a dual pair because it is the centralizer of G in $\mathrm{Sp}(\mathrm{W})$. We also use the notation $\cdot'$ for all the objects associated with G', such as $\mathfrak{g}'$, Π', ... Unfortunately, this collides with the usual notation for the dual of a linear topological space in functional analysis, also used in this paper, such as $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, ... We hope the reader will guess from the context the correct meaning of the notation.

Each element $K \in \mathcal{S}'(\mathrm{X} \times \mathrm{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}'(\mathrm{X}))$ by

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x, x')v(x')\, dx'. \tag{10}$$

The map

$$\mathrm{Op} : \mathcal{S}'(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}'(\mathrm{X})) \tag{11}$$

is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [Trè67, Theorem 51.7]. The Weyl transform is the linear isomorphism $\mathcal{K} : \mathcal{S}'(\mathrm{W}) \to \mathcal{S}'(\mathrm{X} \times \mathrm{X})$ defined for $f \in \mathcal{S}(\mathrm{W})$ by

$$\mathcal{K}(f)(x, x') = \int_{\mathrm{Y}} f(x - x' + y)\chi\big(\frac{1}{2}\langle y, x + x'\rangle\big)\, dy\,, \tag{12}$$

(Recall that χ is the character of $\mathbb{R}$ we fixed at the beginning of the introduction.)

For $g \in \mathrm{Sp}(\mathrm{W})$, let

$$\chi_{c(g)}(u) = \chi\big(\tfrac{1}{4}\langle (g+1)(g-1)^{-1}u, u\rangle\big) \qquad (u = (g-1)w,\ w \in \mathrm{W})\,. \tag{13}$$

Notice that, if $g - 1$ is invertible on W, then

$$\chi_{c(g)}(u) = \chi\big(\tfrac{1}{4}\langle c(g)u, u\rangle\big)\,,$$

where $c(g) = (g+1)(g-1)^{-1}$ is the usual Cayley transform.

Following [AP14, Definition 4.23 and (114)], we define

$$T : \widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde{g} = (g;\xi) \longrightarrow \xi\, \chi_{c(g)}\mu_{(g-1)\mathrm{W}} \in \mathcal{S}'(\mathrm{W})\,, \tag{14}$$

where $\mu_{(g-1)\mathrm{W}}$ is the Lebesgue measure on the subspace $(g-1)\mathrm{W}$ normalized as above, i.e. the volume of the unit cube with respect to the form $\langle J\cdot,\cdot\rangle$ is 1. Set

$$\omega = \mathrm{Op}\circ\mathcal{K}\circ T\,. \tag{15}$$

As proved in [AP14, Theorem 4.27], ω is a unitary representation of $\widetilde{\mathrm{Sp}}$ on $\mathrm{L}^2(\mathrm{X})$. In fact, $(\omega, \mathrm{L}^2(\mathrm{X}))$ is the Schrödinger model of Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ and the polarization $\mathrm{W} = \mathrm{X}\oplus\mathrm{Y}$. In this realization, $\mathcal{H}_\omega = \mathrm{L}^2(\mathrm{X})$ and $\mathcal{H}_\omega^\infty = \mathcal{S}(\mathrm{X})$.

The distribution character of the Weil representation turns out to be the function

$$\Theta : \widetilde{\mathrm{Sp}}(\mathrm{W}) \ni (g;\xi) \to \xi \in \mathbb{C}^\times\,, \tag{16}$$

[AP14, Proposition 4.27]. Hence for $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ in the preimage of $g \in \mathrm{Sp}(\mathrm{W})$ under the double covering map (9), we have

$$T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathrm{W}} \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}))\,. \tag{17}$$

Suppose now that $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathrm{W})$ is a dual pair. Every irreducible admissible representation $\Pi\otimes\Pi'$ of $\widetilde{\mathrm{G}}\times\widetilde{\mathrm{G}}'$ occurring in Howe's correspondence may be realized, up to infinitesimal equivalence, as a subspace of $\mathcal{H}_\omega^{\infty\prime} = \mathcal{S}'(\mathrm{X})$. Hence

$$\mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty\otimes\mathcal{H}_{\Pi'}^\infty) \subseteq \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}'(\mathrm{X}))\,.$$

The existence of the interwining distribution $f_{\Pi\otimes\Pi'} \in \mathcal{S}'(\mathrm{W})$ defined (up to a multiplicative constant) by (3) is thus a consequence of (1), (11) and (12).

Finally, because of (15), equation (4) and (5) lead to the equality

$$f_{\Pi\otimes\Pi'} = \frac{1}{2}T(\check{\Theta}_\Pi) = \int_{\mathrm{G}} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg\,. \tag{18}$$

The problem of finding an explicit expression for $f_{\Pi\otimes\Pi'}$ is hence transformed into the task of computing the right-hand side of (18).

2. The Center of the Metaplectic Group

Let $\mathrm{Z} = \{1, -1\}$ be the center of the symplectic group $\mathrm{Sp(W)}$. Then $(\mathrm{Z}, \mathrm{Sp(W)})$ is a dual pair in $\mathrm{Sp(W)}$ with compact member Z. Let $(\widetilde{\mathrm{Z}}, \widetilde{\mathrm{Sp}}(\mathrm{W}))$ be the corresponding dual pair in the metaplectic group $\widetilde{\mathrm{Sp}}(\mathrm{W})$. Then $\widetilde{\mathrm{Z}}$ coincides with the center of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ and is equal to

$$\widetilde{\mathrm{Z}} = \{(1;1), (1;-1), (-1;\zeta), (-1;-\zeta)\}\,, \tag{19}$$

where $\zeta = \left(\frac{i}{2}\right)^{\frac{1}{2}\dim \mathrm{W}}$.

In this section we illustrate how to compute the intertwining distributions for the pair $(\mathrm{Z}, \mathrm{Sp(W)})$. At the same time, we introduce some facts that will be needed in the rest of the paper.

The formula for the cocycle in (8) is particularly simple over Z:

$$C(1, \pm 1) = C(-1, 1) = 1 \quad \text{and} \quad C(-1, -1) = 2^{\dim \mathrm{W}}\,.$$

Also, $C(g, 1) = C(1, g) = 1$ for all $g \in \mathrm{Sp(W)}$ by [AP14, Proposition 4.13]. Notice that

$$(-1; \pm\zeta)^2 = (1; \zeta^2 C(-1,-1)) = (1; (-1)^{\frac{1}{2}\dim \mathrm{W}})\,. \tag{20}$$

Hence the covering (9) restricted to $\widetilde{\mathrm{Z}}$,

$$\widetilde{\mathrm{Z}} \ni \tilde{z} \to z \in \mathrm{Z} \tag{21}$$

splits if and only if $\frac{1}{2}\dim \mathrm{W}$ is even.

By (14) and (7), we have

$$\begin{aligned} T(1;1) &= \delta\,, & T(1;-1) &= -\delta\,,\\ T(-1;\zeta) &= \zeta\,\mu_{\mathrm{W}}\,, & T(-1;-\zeta) &= -\zeta\,\mu_{\mathrm{W}}\,. \end{aligned}$$

Moreover, [AP14, Proposition 4.28] shows that for $v \in \mathrm{L}^2(\mathrm{X})$ and $x \in \mathrm{X}$,

$$\begin{aligned} \omega(1;1)v(x) &= v(x)\,, & \omega(1;-1)v(x) &= -v(x)\,,\\ \omega(-1;\zeta)v(x) &= \frac{\zeta}{|\zeta|}v(-x)\,, & \omega(-1;-\zeta)v(x) &= -\frac{\zeta}{|\zeta|}v(-x)\,. \end{aligned}$$

Since $T(\tilde{z}) = \Theta(\tilde{z})\chi_{c(z)}\mu_{(z-1)\mathrm{W}}$ for $\tilde{z} \in \tilde{Z}$, it follows that

$$\omega(\tilde{z})v(x) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|}v(zx) \qquad (\tilde{z} \in \widetilde{\mathrm{Z}})\,. \tag{22}$$

The fraction

$$\chi_+(\tilde{z}) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \qquad (\tilde{z} \in \widetilde{\mathrm{Z}}) \tag{23}$$

defines an irreducible character χ_+ of the group $\widetilde{\mathrm{Z}}$. Let ε be the unique non-trivial irreducible character of the two element group Z. Then

$$\chi_-(\tilde{z}) = \varepsilon(z)\frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \qquad (\tilde{z} \in \widetilde{\mathrm{Z}}) \tag{24}$$

is also an irreducible character of $\widetilde{\mathrm{Z}}$.

Let $\mathrm{L}^2(\mathrm{X})_+ \subseteq \mathrm{L}^2(\mathrm{X})$ denote the subspace of the even functions and let $\mathrm{L}^2(\mathrm{X})_- \subseteq \mathrm{L}^2(\mathrm{X})$ denote the subspace of the odd functions. Then, as is well known, [KV78, (6.9)], the restriction $\omega_\pm$ of ω to $\mathrm{L}^2(\mathrm{X})_\pm$ is irreducible. As we have seen above, the center $\widetilde{\mathrm{Z}}$ acts on $\mathrm{L}^2(\mathrm{X})_\pm$ via the character $\chi_\pm$. Thus $\chi_\pm$ is the central character of $\omega_\pm$.

Hence, in the case of the dual pair $(\mathrm{Z}, \mathrm{Sp}(\mathrm{W}))$, Howe's correspondence looks as follows

$$(\chi_+, \mathbb{C}) \leftrightarrow (\omega_+, \mathrm{L}^2(\mathrm{X})_+) \quad \text{and} \quad (\chi_-, \mathbb{C}) \leftrightarrow (\omega_-, \mathrm{L}^2(\mathrm{X})_-)\,. \tag{25}$$

The projections

$$\mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{X})_+ \quad \text{and} \quad \mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{X})_-$$

are respectively given by

$$\frac{1}{2}\omega(\check{\chi}_+) = \frac{1}{4}\sum_{\tilde{z}\in\widetilde{\mathrm{Z}}} \check{\chi}_+(\tilde{z})\omega(\tilde{z}) \quad \text{and} \quad \frac{1}{2}\omega(\check{\chi}_-) = \frac{1}{4}\sum_{\tilde{z}\in\widetilde{\mathrm{Z}}} \check{\chi}_-(\tilde{z})\omega(\tilde{z})\,.$$

The corresponding intertwining distributions are

$$\begin{aligned} f_{\chi_+\otimes\omega_+} &= \frac{1}{4}\sum_{\tilde{z}\in\widetilde{\mathrm{Z}}} \check{\chi}_+(\tilde{z})T(\tilde{z}) = \frac{1}{2}\big(\delta + 2^{-\frac{1}{2}\dim \mathrm{W}}\mu_{\mathrm{W}}\big)\,, \\ f_{\chi_-\otimes\omega_-} &= \frac{1}{4}\sum_{\tilde{z}\in\widetilde{\mathrm{Z}}} \check{\chi}_-(\tilde{z})T(\tilde{z}) = \frac{1}{2}\big(\delta - 2^{-\frac{1}{2}\dim \mathrm{W}}\mu_{\mathrm{W}}\big)\,, \end{aligned} \tag{26}$$

where we normalize the total mass of Z to be 1, as we did for a general dual pair $(\mathrm{G}, \mathrm{G}')$ with G compact.

The right-hand side of (26) is a sum of two homogenous distributions of different homogenity degrees. So, asymptotically, they can be isolated. This allows us to recover μ_{W}, and hence $\tau_{\mathfrak{sp}(\mathrm{W})}(\mathrm{W})$, the wave front of $\omega_\pm$, out of the intertwining distribution.

3. Dual Pairs as Lie Supergroups

To present the main results of this paper, we need the realization of dual pairs with one member compact as Lie supergroups. The content of this section is taken from [Prz06] and [MPP15]. We recall the relevant material for making our exposition self-contained.

For a dual pair $(\mathrm{G}, \mathrm{G}')$ as in (2), there is a division algebra $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ with an involution over $\mathbb{R}$, a finite dimensional right $\mathbb{D}$-vector space V with a positive definite hermitian form $(\cdot,\cdot)$ and a finite dimensional right $\mathbb{D}$-vector space V' with a non-degenerate skew-hermitian form $(\cdot,\cdot)'$ such that G coincides with the isometry group of $(\cdot,\cdot)$ and G' coincides with the isometry group of $(\cdot,\cdot)'$. We assume that G centralizes the complex structure J and that J normalizes G'. Then the conjugation by J is a Cartan involution on G', which we denote by θ.

Let $\mathsf{V}_{\overline{0}} = \mathrm{V}$, $d = \dim_{\mathbb{D}} \mathsf{V}_{\overline{0}}$, $\mathsf{V}_{\overline{1}} = \mathrm{V}'$ and $d' = \dim_{\mathbb{D}} \mathsf{V}_{\overline{1}}$. We assume that both $\mathsf{V}_{\overline{0}}$ and $\mathsf{V}_{\overline{1}}$ are right vector spaces over $\mathbb{D}$. Set $\mathsf{V} = \mathsf{V}_{\overline{0}} \oplus \mathsf{V}_{\overline{1}}$ and define an element $\mathsf{S} \in \mathrm{End}(\mathsf{V})$ by

$$\mathsf{S}(v_0 + v_1) = v_0 - v_1 \qquad (v_0 \in \mathsf{V}_{\overline{0}}, v_1 \in \mathsf{V}_{\overline{1}})\,.$$

Let

$$\begin{aligned}
&\mathrm{End}(\mathsf{V})_{\overline{0}} = \{x \in \mathrm{End}(\mathsf{V});\ \mathsf{S}x = x\mathsf{S}\}\,,\\
&\mathrm{End}(\mathsf{V})_{\overline{1}} = \{x \in \mathrm{End}(\mathsf{V});\ \mathsf{S}x = -x\mathsf{S}\}\,,\\
&\mathrm{GL}(\mathsf{V})_{\overline{0}} = \mathrm{End}(\mathsf{V})_{\overline{0}} \cap \mathrm{GL}(\mathsf{V})\,.
\end{aligned}$$

Denote by $(\cdot,\cdot)''$ the direct sum of the two forms $(\cdot,\cdot)$ and $(\cdot,\cdot)'$. Let

$$\begin{aligned}
&\mathfrak{s}_{\overline{0}} = \{x \in \mathrm{End}(\mathsf{V})_{\overline{0}};\ (xu, v)'' = -(u, xv)'',\ u, v \in \mathsf{V}\}\,, \qquad (27)\\
&\mathfrak{s}_{\overline{1}} = \{x \in \mathrm{End}(\mathsf{V})_{\overline{1}};\ (xu, v)'' = (u, \mathsf{S}xv)'',\ u, v \in \mathsf{V}\}\,,\\
&\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}\,,\\
&\mathrm{S} = \{s \in \mathrm{GL}(\mathsf{V})_{\overline{0}};\ (su, sv)'' = (u, v)'',\ u, v \in \mathsf{V}\}\,,\\
&\langle x, y\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xy)\,. \qquad (28)
\end{aligned}$$

(Here $\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x)$ denotes the trace of x considered as a real endomorphism of V.) Then $(\mathrm{S}, \mathfrak{s})$ is a real Lie supergroup, i.e. a real Lie group S together with a real Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$, whose even component $\mathfrak{s}_{\overline{0}}$ is the

Lie algebra of S. (In terms of [DM99, §3.8], $(\mathrm{S},\mathfrak{s})$ is a Harish-Chandra pair.) We shall write $\mathfrak{s}(\mathsf{V})$ instead of $\mathfrak{s}$ whenever we want to specify the Lie superalgebra $\mathfrak{s}$ constructed as above from V and $(\cdot,\cdot)''$.

The group S acts on $\mathfrak{s}$ by conjugation and $\langle\cdot,\cdot\rangle$ is a non-degenerate S-invariant form on the real vector space $\mathfrak{s}$, whose restriction to $\mathfrak{s}_{\overline{0}}$ is symmetric and restriction to $\mathfrak{s}_{\overline{1}}$ is skew-symmetric. We shall employ the notation

$$s.x = \mathrm{Ad}(s)x = sxs^{-1} \qquad (s\in \mathrm{S}\,,\ x\in\mathfrak{s})\,, \tag{29}$$

$$x(w) = \mathrm{ad}(x)(w) = xw - wx \qquad (x\in\mathfrak{s}_{\overline{0}}\,,\ w\in\mathfrak{s}_{\overline{1}})\,. \tag{30}$$

In terms of the notation introduced at the beginning of this section,

$$\mathfrak{g} = \mathfrak{s}_{\overline{0}}|_{\mathsf{V}_{\overline{0}}}\,,\quad \mathfrak{g}' = \mathfrak{s}_{\overline{0}}|_{\mathsf{V}_{\overline{1}}}\,,\quad \mathrm{G} = \mathrm{S}|_{\mathsf{V}_{\overline{0}}}\,,\quad \mathrm{G}' = \mathrm{S}|_{\mathsf{V}_{\overline{1}}}\,.$$

Define $\mathrm{W} = \mathrm{Hom}_{\mathbb{D}}(\mathsf{V}_{\overline{1}},\mathsf{V}_{\overline{0}})$. Then, by restriction, we have the identification

$$\mathrm{W} = \mathfrak{s}_{\overline{1}}\,. \tag{31}$$

Under this identification, the adjoint action of G on $\mathfrak{s}_{\overline{1}}$ becomes the action on W by the left (postmultiplication). Similarly, the adjoint action of G' on $\mathfrak{s}_{\overline{1}}$ becomes the action of G' on W via the right (premultiplication) by the inverse. Also, we have the unnormalized moment maps

$$\tau : \mathrm{W}\ni w\to w^2|_{\mathsf{V}_{\overline{0}}}\in\mathfrak{g}\,,\qquad \tau' : \mathrm{W}\ni w\to w^2|_{\mathsf{V}_{\overline{1}}}\in\mathfrak{g}'\,. \tag{32}$$

An element $x\in\mathfrak{s}$ is called semisimple (resp., nilpotent) if x is semisimple (resp., nilpotent) as an endomorphism of V. We say that a semisimple element $x\in\mathfrak{s}_{\overline{1}}$ is regular if it is nonzero and $\dim(S.x)\geq\dim(S.y)$ for all semisimple $y\in\mathfrak{s}_{\overline{1}}$. Let $x\in\mathfrak{s}_{\overline{1}}$ be fixed. For $x,y\in\mathfrak{s}_{\overline{1}}$ let $\{x,y\}=xy+yx\in\mathfrak{s}_{\overline{0}}$ be their anticommutator.

The anticommutant and the double anticommutant of x in $\mathfrak{s}_{\overline{1}}$ are

$$\begin{aligned} {}^{x}\mathfrak{s}_{\overline{1}} &= \{y\in\mathfrak{s}_{\overline{1}} : \{x,y\}=0\}\,,\\ {}^{x\mathfrak{s}_{\overline{1}}}\mathfrak{s}_{\overline{1}} &= \bigcap_{y\in{}^{x}\mathfrak{s}_{\overline{1}}} {}^{y}\mathfrak{s}_{\overline{1}}\,, \end{aligned}$$

respectively. A Cartan subspace $\mathfrak{h}_{\overline{1}}$ of $\mathfrak{s}_{\overline{1}}$ is defined as the double anticommutant of a regular semisimple element $x\in\mathfrak{s}_{\overline{1}}$. We denote by $\mathfrak{h}_{\overline{1}}{}^{reg}$ the set of regular elements in $\mathfrak{h}_{\overline{1}}$.

Next we describe the Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathfrak{s}_{\overline{1}}$. We refer to [Prz06, §6] and [MPP15, §4] for the proofs omitted here. Let l be the rank of $\mathfrak{g}$, l' the rank of $\mathfrak{g}'$, and set

$$l'' = \min(l, l')\,. \tag{33}$$

Given a Cartan subspace $\mathfrak{h}_{\overline{1}}$, there are $\mathbb{Z}/2\mathbb{Z}$-graded subspaces $\mathsf{V}^j \subseteq \mathsf{V}$ such that the restriction of the form $(\cdot,\cdot)''$ to each V^j is non-degenerate, V^j is orthogonal to V^k for $j \neq k$ and

$$\mathsf{V} = \mathsf{V}^0 \oplus \mathsf{V}^1 \oplus \mathsf{V}^2 \oplus \cdots \oplus \mathsf{V}^{l''}\,. \tag{34}$$

The subspace V^0 coincides with the intersection of the kernels of the elements of $\mathfrak{h}_{\overline{1}}$ (equivalently, $\mathsf{V}^0 = \mathrm{Ker}(x)$ if $\mathfrak{h}_{\overline{1}} = {}^x\mathfrak{s}_{\overline{1}}\mathfrak{s}_{\overline{1}}$). For $1 \leq j \leq l''$, the subspaces $\mathsf{V}^j = \mathsf{V}^j_{\overline{0}} \oplus \mathsf{V}^j_{\overline{1}}$ are described as follows.

Suppose $\mathbb{D} = \mathbb{R}$. Then there is a basis v_0, v_0' of $\mathsf{V}^j_{\overline{0}}$ and a basis v_1, v_1' of $\mathsf{V}^j_{\overline{1}}$ such that

$$\begin{aligned} (v_0, v_0)'' = (v_0', v_0')'' = 1\,, \qquad (v_0, v_0')'' = 0\,,\\ (v_1, v_1)'' = (v_1', v_1')'' = 0\,, \qquad (v_1, v_1')'' = 1\,. \end{aligned} \tag{35}$$

The following formulas define an element $u_j \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$,

$$\begin{aligned} u_j(v_0) = \frac{1}{\sqrt{2}}(v_1 - v_1')\,, \qquad u_j(v_1) = \frac{1}{\sqrt{2}}(v_0 - v_0')\,,\\ u_j(v_0') = \frac{1}{\sqrt{2}}(v_1 + v_1')\,, \qquad u_j(v_1') = \frac{1}{\sqrt{2}}(v_0 + v_0')\,. \end{aligned}$$

Suppose $\mathbb{D} = \mathbb{C}$. Then there are vectors v_0 and v_1 such that $\mathsf{V}^j_{\overline{0}} = \mathbb{C}v_0$, $\mathsf{V}^j_{\overline{1}} = \mathbb{C}v_1$, $(v_0, v_0)'' = 1$ and $(v_1, v_1)'' = \delta_j i$, with $\delta_j = \pm 1$ fixed by the form $(\cdot,\cdot)'$. The following formulas define an element $u_j \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$,

$$u_j(v_0) = e^{-i\delta_j \frac{\pi}{4}} v_1\,, \qquad u_j(v_1) = e^{-i\delta_j \frac{\pi}{4}} v_0\,. \tag{36}$$

Suppose $\mathbb{D} = \mathbb{H}$. Then $\mathsf{V}^j_{\overline{0}} = \mathbb{H}v_0$, $\mathsf{V}^j_{\overline{1}} = \mathbb{H}v_1$, where $(v_0, v_0)'' = 1$ and $(v_1, v_1)'' = i$. The following formulas define an element $u_j \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$,

$$u_j(v_0) = e^{-i\frac{\pi}{4}} v_1\,, \qquad u_j(v_1) = e^{-i\frac{\pi}{4}} v_0\,.$$

In any case, by extending each u_j by zero outside V^j, we have

$$\mathfrak{h}_{\overline{1}} = \sum_{j=1}^{l''} \mathbb{R} u_j \,. \tag{37}$$

The formula (37) describes a maximal family of mutually non-conjugate Cartan subspaces of $\mathfrak{s}_{\overline{1}}$. By classification, see [Prz06, §6], there is only one such subspace unless the dual pair $(\mathrm{G}, \mathrm{G}')$ is isomorphic to $(\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l'' = l < p + q$. In the last case there are $\min(l,p) - \max(l-q,0) + 1$ such subspaces, assuming $p \le q$. For each m such that $\max(l-q,0) \le m \le \min(p,l)$ there is a Cartan subspace $\mathfrak{h}_{\overline{1},m}$ determined by the condition that m is the number of positive δ_j's in (36). We may assume that $\delta_1 = \cdots = \delta_m = 1$ and $\delta_{m+1} = \cdots = \delta_l = -1$. If $(\mathrm{G}, \mathrm{G}')$ is isomorphic to $(\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l \ge l'' = p+q$, then there is a unique Cartan subspace of $\mathfrak{s}_{\overline{1}}$ up to conjugation. It is determined by the condition that in (36) there are p positive and q negative δ_j's. We may assume that the first p δ_j's are positive.

The Weyl group $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$ is the quotient of the stabilizer of $\mathfrak{h}_{\overline{1}}$ in S by the subgroup $\mathrm{S}^{\mathfrak{h}_{\overline{1}}}$ fixing each element of $\mathfrak{h}_{\overline{1}}$. If $\mathbb{D} \ne \mathbb{C}$, then $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$ acts by all sign changes and all permutations of the u_j's. If $\mathbb{D} = \mathbb{C}$, the Weyl group acts by all sign changes and all permutations of the u_j's which preserve $(\delta_1, \dots, \delta_{l''})$, see [Prz06, (6.3)].

Set $\delta_j = 1$ for all $1 \le j \le l''$, if $\mathbb{D} \ne \mathbb{C}$, and in any case, i.e. $\mathbb{D} \ne \mathbb{C}$ or $\mathbb{D} = \mathbb{C}$, define

$$J_j = \delta_j \tau(u_j)\,, \qquad J'_j = \delta_j \tau'(u_j) \qquad (1 \le j \le l'')\,. \tag{38}$$

Then J_j, J'_j are complex structures on $\mathsf{V}^j_{\overline{0}}$ and $\mathsf{V}^j_{\overline{1}}$ respectively. Explicitly,

$$\begin{aligned} &J_j(v_0) = -v'_0, \quad J_j(v'_0) = v_0\,, \quad J'_j(v_1) = -v'_1\,, \quad J'_j(v'_1) = v_1\,, \quad \text{if } \mathbb{D} = \mathbb{R}\,, \\ &J_j(v_0) = -iv_0\,, \quad J'_j(v_1) = -iv_1\,, \quad \text{if } \mathbb{D} = \mathbb{C} \text{ or } \mathbb{D} = \mathbb{H}\,. \end{aligned} \tag{39}$$

(The point of the multiplication by the δ_j in (38) is that the complex structures J_j, J'_j do not depend on the Cartan subspace $\mathfrak{h}_{\overline{1}}$.) In particular, if $w = \sum_{j=1}^{l''} w_j u_j \in \mathfrak{h}_{\overline{1}}$, then

$$\tau(w) = \sum_{j=1}^{l''} w_j^2 \delta_j J_j \quad \text{and} \quad \tau'(w) = \sum_{j=1}^{l''} w_j^2 \delta_j J'_j\,. \tag{40}$$

(Notice that $w_j^2 \geq 0$.) Let $\mathfrak{h}_{\overline{1}}^2 \subseteq \mathfrak{s}_{\overline{0}}$ be the subspace spanned by all the squares w^2, $w \in \mathfrak{h}_{\overline{1}}$. (This is a linear space, not a collection of squares. We hope that the notation $\mathfrak{h}_{\overline{1}}^2$ will not cause any confusion.) Then

$$\mathfrak{h}_{\overline{1}}^2 = \sum_{j=1}^{l''} \mathbb{R}(J_j + J_j'). \tag{41}$$

We shall use the following identification

$$\mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{0}}} \ni \sum_{j=1}^{l''} y_j J_j = \sum_{j=1}^{l''} y_j J_j' \in \mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{1}}}. \tag{42}$$

Recall from (33) that $l'' = \min(l, l')$. If $l'' = l$, then $\mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{0}}}$ is a Cartan subalgebra of $\mathfrak{g}$ which we denote by $\mathfrak{h}$. The identification (42) embeds $\mathfrak{h}$ diagonally in $\mathfrak{g}$ and in $\mathfrak{g}'$. It is contained in an elliptic Cartan subalgebra of $\mathfrak{g}'$, say $\mathfrak{h}'$. ("Elliptic" means that all the roots of $\mathfrak{h}$ in $\mathfrak{g}'_{\mathbb{C}}$ are purely imaginary.) Similarly, if $l'' = l'$, then $\mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{1}}}$ is an elliptic Cartan subalgebra of $\mathfrak{g}'$ which we denote by $\mathfrak{h}'$. If $l \leq l'$ we denote by $\mathfrak{z}' \subseteq \mathfrak{g}'$ the centralizer of $\mathfrak{h}$. Similarly, if $l' \leq l$ we denote by $\mathfrak{z} \subseteq \mathfrak{g}$ the centralizer of $\mathfrak{h}'$. In particular, if $l' = l$, then $\mathfrak{z}' = \mathfrak{h}' = \mathfrak{h} = \mathfrak{z}$, where the first equality is in $\mathfrak{g}$, the second is (42) and the last is in $\mathfrak{g}'$.

Let $\mathfrak{s}_{\overline{0}\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}}$ be the complexification of $\mathfrak{s}_{\overline{0}}$. Fix a system of positive roots for the adjoint action of $\mathfrak{h}_{\overline{1}}^2$ on $\mathfrak{s}_{\overline{0}\mathbb{C}}$. Suppose first that $l \leq l'$. By the identification (42), $\mathfrak{h}$ preserves both $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$. So our choice of positive roots for $(\mathfrak{h}_{\overline{1}\mathbb{C}}^2, \mathfrak{s}_{\overline{0}\mathbb{C}})$ fixes a positive root system of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ and extends to a compatible positive root system for $(\mathfrak{h}'_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$. Let $\pi_{\mathfrak{g}/\mathfrak{h}}$ be the product of positive roots of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ and let $\pi_{\mathfrak{g}'/\mathfrak{z}'}$ be the product of positive roots of $(\mathfrak{h}'_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ such that the corresponding root spaces do not occur in $\mathfrak{z}'_{\mathbb{C}}$. If $l > l'$, then $\pi_{\mathfrak{g}'/\mathfrak{h}'}$ and $\pi_{\mathfrak{g}/\mathfrak{z}}$ can be similarly defined. See Appendix A for the explicit expressions of these root products restricted to the elements in (42).

Suppose $l' < l$. Then $\mathsf{V}_{\overline{1}}^0 = 0$, $\mathsf{V}_{\overline{0}}^0 \neq 0$ and

$$\mathsf{V}_{\overline{0}} = \mathsf{V}_{\overline{0}}^0 \oplus \mathsf{V}_{\overline{0}}^1 \oplus \mathsf{V}_{\overline{0}}^2 \oplus \cdots \oplus \mathsf{V}_{\overline{0}}^{l''} \tag{43}$$

is a direct sum orthogonal decomposition with respect to the positive definite hermitian form $(\cdot, \cdot)$. We extend $\mathfrak{h} \subseteq \mathfrak{g}$ to a Cartan subalgebra $\mathfrak{h}(\mathfrak{g}) \subseteq \mathfrak{g}$ as

follows. The restriction of $\mathfrak{h}(\mathfrak{g})$ to $\mathsf{V}_{\overline{0}}^1 \oplus \mathsf{V}_{\overline{0}}^2 \oplus \cdots \oplus \mathsf{V}_{\overline{0}}^{l''}$ coincides with $\mathfrak{h}$. Pick an orthogonal direct sum decomposition

$$\mathsf{V}_{\overline{0}}^0 = \mathsf{V}_{\overline{0}}^{0,0} \oplus \mathsf{V}_{\overline{0}}^{0,l''+1} \oplus \mathsf{V}_{\overline{0}}^{0,l''+2} \oplus \cdots \oplus \mathsf{V}_{\overline{0}}^{0,l}\,, \tag{44}$$

where for $j > l''$, $\dim_{\mathbb{D}} \mathsf{V}_{\overline{0}}^{0,j} = 2$ if $\mathbb{D} = \mathbb{R}$ and $\dim_{\mathbb{D}} \mathsf{V}_{\overline{0}}^{0,j} = 1$ if $\mathbb{D} \neq \mathbb{R}$. Also $\mathsf{V}_{\overline{0}}^{0,0} = 0$ unless $\mathrm{G} = \mathrm{O}_{2l+1}$, in which case $\dim_{\mathbb{D}} \mathsf{V}_{\overline{0}}^{0,0} = 1$. In each space $\mathsf{V}_{\overline{0}}^{0,j}$, with $j > l''$, we pick an orthonormal basis and define J_j as in (39). Then

$$\mathfrak{h}(\mathfrak{g}) = \sum_{j=1}^{l} \mathbb{R} J_j\,. \tag{45}$$

If $l \leq l'$, then we set $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}$.

Let J_j^*, $1 \leq j \leq l$, be the basis of the space $\mathfrak{h}(\mathfrak{g})^*$ which is dual to $J_1, \ldots, J_l$, and set

$$e_j = -iJ_j^*\,, \qquad 1 \leq j \leq l\,. \tag{46}$$

If $\mu \in i\mathfrak{h}(\mathfrak{g})^*$, then $\mu = \sum_{j=1}^{l} \mu_j e_j$ with $\mu_j \in \mathbb{R}$. We say that μ is strictly dominant if $\mu_1 > \mu_2 > \cdots > \mu_l$.

4. Orbital Integrals on W

In this section we recall from [MPP15] and [MPP20] some definitions and results concerning the orbital integrals on W that we will need in the following sections.

Let $\mathcal{S}'(\mathrm{W})^{\mathrm{S}}$ denote the space of S-invariant tempered distributions on W, where the S-action is induced by (29). Let $\mathfrak{h}_{\overline{1}}$ be a Cartan subspace of W. Suppose first that G is different from O_{2l+1} with $l < l'$. For $w \in \mathfrak{h}_{\overline{1}}^{reg}$, the orbital integral attached to the orbit $\mathcal{O}(w) = \mathrm{S}.w$ is the element $\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}$ of $\mathcal{S}'(\mathrm{W})^{\mathrm{S}}$ defined for $\phi \in \mathcal{S}(\mathrm{W})$ by

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}})\,. \tag{47}$$

Suppose now that $\mathrm{G} = \mathrm{O}_{2l+1}$ with $l < l'$. Then one needs to modify (47) because the union of the orbits $\mathrm{S}.w$ over all $w \in \mathfrak{h}_{\overline{1}}^{reg}$ would not be dense in W; see [MPP15, Theorem 20]. Let $w_0 \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^0)$ be a non-zero element,

$w \in \mathfrak{h}_{\overline{1}}^{reg}$ and $\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}$ the centralizer of $w+w_0$ in S. Set $\mathcal{O}(w) = \mathrm{S}.(w+w_0)$ and define

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}} \phi(s.(w+w_0))\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}) \,. \tag{48}$$

(Since $\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)\backslash\{0\}$ is a single $\mathrm{S}(\mathsf{V}^0)$-orbit, the S-orbit of $w+w_0$, and hence the right-hand side of (48), does not depend on the choice of $w_0 \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^0)$.) The orbital integrals (47) and (48) are well-defined, tempered distribution on W, which depend only on $\tau(w)$, or equivalently $\tau'(w)$, via the identification (42).

For $w \in \mathfrak{h}_{\overline{1}}$, set

$$\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) = \begin{cases} \pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w)) & \text{if } l \le l', \\ \pi_{\mathfrak{g}/\mathfrak{z}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{h}'}(\tau'(w)) & \text{if } l \ge l' \,. \end{cases} \tag{49}$$

As shown in [MPP20, Lemma 1.2], there is a constant $C(\mathfrak{h}_{\overline{1}})$, depending on $\mathfrak{h}_{\overline{1}}$ and with $|C(\mathfrak{h}_{\overline{1}})| = 1$, such that

$$\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) = C(\mathfrak{h}_{\overline{1}})|\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \,. \tag{50}$$

The set $\mathfrak{h}_{\overline{1}}^{reg}$ of regular elements of $\mathfrak{h}_{\overline{1}}$ is explicitly given by

$$\mathfrak{h}_{\overline{1}}^{reg} = \{w \in \mathfrak{h}_{\overline{1}};\ \pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) \neq 0\} \,. \tag{51}$$

Choose a positive Weyl chamber $\mathfrak{h}_{\overline{1}}^+ \subseteq \mathfrak{h}_{\overline{1}}^{reg}$, i.e. an open fundamental domain for the action of the Weyl group, $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$. There is a normalization $d\tau(w)$ of the Lebsegue measure on $\mathfrak{h}$, respectively a normalization $d\tau'(w)$ of the Lebsegue measure on $\mathfrak{h}'$, such that the following equalities hold for all $\phi \in \mathcal{S}(\mathrm{W})$:

$$\mu_{\mathrm{W}}(\phi) = \sum_{\mathfrak{h}_{\overline{1}}} \int_{\tau(\mathfrak{h}_{\overline{1}}^+)} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi)\, d\tau(w) \qquad \text{if } l \le l' \,, \tag{52}$$

$$\mu_{\mathrm{W}}(\phi) = \int_{\tau'(\mathfrak{h}_{\overline{1}}^+)} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi)\, d\tau'(w) \qquad \text{if } l \ge l' \,. \tag{53}$$

Formulas (52) and (53) are the Weyl–Harish-Chandra integration formulas on W, [MPP15, Theorem 21]. The sum in (52) is over the family of mutually

non-conjugate Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathrm{W}$. (It therefore reduces to a single term for $(\mathrm{G}, \mathrm{G}')$ different from $(\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l < l' = p + q$.) The formulas agree for $l = l'$ once we identify $\tau(w)$ and $\tau'(w)$ via (42).

Let $C_{\mathfrak{h}_{\overline{1}}} = C(\mathfrak{h}_{\overline{1}}) \cdot i^{\dim \mathfrak{g}/\mathfrak{h}}$, where $C(\mathfrak{h}_{\overline{1}})$ is as in (50). If $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l < l' = p + q$, let

$$\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg}) = \bigcup_{m=\max(l-q,0)}^{\min(p,l)} \tau(\mathfrak{h}_{\overline{1},m})\,.$$

In all other cases, $\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg})$ will denote $\tau(\mathfrak{h}_{\overline{1}}{}^{reg})$, where $\mathfrak{h}_{\overline{1}}$ is the fixed Cartan subspace. The Harish-Chandra regular almost-elliptic orbital integral on W is the function

$$F : \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg}) \to \mathcal{S}'(\mathrm{W})^{\mathrm{S}}$$

defined for every $y \in \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg})$, $y = \tau(w) = \tau'(w)$ as follows:

$$F(y) = \begin{cases} \sum_{\mathfrak{h}_{\overline{1}}} C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}} & \text{if } l \le l'\,, \\ C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}} & \text{if } l > l'\,. \end{cases} \tag{54}$$

Following Harish-Chandra's notation, we shall write $F_\phi(y)$ for $F(y)(\phi)$.

Suppose first that $l \le l'$. According to [MPP20, Theorem 3.6], F uniquely extends to a function $F : \mathfrak{h} \to \mathcal{S}'(\mathrm{W})^{\mathrm{S}}$ satisfying

$$F(sy) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) F(y) \qquad (s \in W(\mathrm{G}, \mathfrak{h}),\ y \in \mathfrak{h})\,. \tag{55}$$

This extension is supported in $\mathfrak{h} \cap \tau(\mathrm{W})$. The extended map F is smooth on the subset of $y = \sum_{j=1}^{l} y_j J_j$ where each $y_j \neq 0$ and, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_l)$ with

$$\max(\alpha_1, \dots, \alpha_l) \le \begin{cases} d' - r - 1 & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C}\,, \\ 2(d' - r) & \text{if } \mathbb{D} = \mathbb{H}\,, \end{cases} \tag{56}$$

the function $\partial(J_1^{\alpha_1} J_2^{\alpha_2} \dots J_l^{\alpha_l}) F(y)$ extends to a continuous function on $\mathfrak{h} \cap \tau(\mathrm{W})$ vanishing on the boundary of $\mathfrak{h} \cap \tau(\mathrm{W})$.

For any values of l and l', there is the pullback via the unnormalized moment map $\tau' : \mathrm{W} \to \mathfrak{g}'$, namely

$$\tau'^{*} : \mathcal{S}(\mathfrak{g}') \ni \psi \to \psi \circ \tau' \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}\,.$$

According to [MPP20, (25)] (a special case of a theorem of Astengo, Di Blasio and Ricci [ABR09, Theorem 6.1]), there is a continuous map $\tau'_* : \mathcal{S}(\mathrm{W})^{\mathrm{G}} \to \mathcal{S}(\mathfrak{g}')$ such that

$$\tau'^* \circ \tau'_*(\phi) = \phi \qquad (\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}})\,. \tag{57}$$

In particular, the map τ'^* is surjective. We will denote by ϕ^{G} the projection of $\phi \in \mathcal{S}(\mathrm{W})$ onto the space of the G-invariants in $\mathcal{S}(\mathrm{W})$,

$$\phi^{\mathrm{G}}(w) = \int_{\mathrm{G}} \phi(g.w)\, dg \qquad (w \in \mathrm{W})\,. \tag{58}$$

(Recall that we have normalized the Haar measure on G so that its mass is 1.)

Suppose now that $l > l'$. Then by [MPP20, (39)],

$$F_\phi(y) = C'_{\mathfrak{h}_{\overline{\mathrm{I}}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \int_{\mathrm{G}'/\mathrm{H}'} \psi(g'.y)\, d(g'\mathrm{H}') \qquad (\phi \in \mathcal{S}(\mathrm{W}),\, y \in \mathfrak{h}_{\overline{\mathrm{I}}}^{reg})\,, \tag{59}$$

where $\mathrm{H}' \subseteq \mathrm{G}'$ is the Cartan subgroup corresponding to $\mathfrak{h}'$,

$$\psi = \tau'_*(\phi^{\mathrm{G}}) \in \mathcal{S}(\mathfrak{g}')\,, \tag{60}$$

and $C'_{\mathfrak{h}_{\overline{\mathrm{I}}}}$ is a suitable non-zero constant. The right-hand side of (59) is Harish-Chandra's orbital integral of ψ. It provides a $W(\mathrm{G}', \mathfrak{h}')$-skew-invariant extension of F_ϕ to $\mathfrak{h}'^{In-reg}$, where $\mathfrak{h}'^{In-reg} \subseteq \mathfrak{h}'$ is the subset where no non-compact roots vanish. Furthermore, as a function of ϕ, $F_\phi(y)$ is S-invariant; see [MPP20, Theorem 3.3].

Notice that, by [MPP20, (69)–(72)], formulas (59) and (60) also hold when $l = l'$ because $\mathrm{Z}' = \mathrm{H}'$ in this case.

Lemma 1. *Suppose that $l \le l'$ and $\mathrm{G} \ne \mathrm{O}_{2l+1}$. Let $\mathrm{U} \subseteq \mathfrak{h}^{reg}$ be a nonempty $W(\mathrm{G}, \mathfrak{h})$-invariant open subset. Then there is a nonzero function $\phi \in C_c^\infty(\mathrm{W})^{\mathrm{G}}$ such that $\phi \ge 0$ and $\operatorname{supp} F_\phi \subseteq \mathrm{U}$. (Here* supp *denotes the support.)*

Proof. Let V be a nonempty open set in $\mathfrak{h}^{reg}$ with closure $\overline{\mathrm{V}} \subseteq \mathrm{U}$. By [Var89, p. 19, especially (9)], the set G.V is open in $\mathfrak{g}$. Hence $\tau^{-1}(\mathrm{G.V})$ is open and S-invairant in W. Let $\phi \in C_c^\infty(\mathrm{W})^{\mathrm{G}}$ be a nonzero function such that $\phi \ge 0$ and $\operatorname{supp}\phi \subseteq \tau^{-1}(\mathrm{G.V})$. We want to prove that $\operatorname{supp} F_\phi \subseteq \mathrm{U}$.

Suppose first that $\mathrm{G} \neq \mathrm{U}_l$. Hence $F_\phi(y) = C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi)$ for all $y \in \tau(\mathfrak{h}_{\overline{1}}^{reg})$. (Here $\pi_{\mathfrak{g}'/\mathfrak{z}'}(y) = \pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))$ where $y = \tau(w) = \tau'(w)$.) Since the zero set of $\pi_{\mathfrak{g}'/\mathfrak{z}'}$ is a finite union of root hyperplanes, $\operatorname{supp} F_\phi$ is the closure in $\mathfrak{h}$ of the set of the $y = \tau(w)$ with $w \in \mathfrak{h}_{\overline{1}}^{reg}$ such that $\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) \neq 0$. If $\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) \neq 0$, then $\mathcal{O}(w) \cap \operatorname{supp} \phi \neq \emptyset$, where $\mathcal{O}(w) = \mathrm{S}.w$. Hence $(\mathrm{S}.w) \cap \tau^{-1}(\mathrm{G}.\mathrm{V}) \neq \emptyset$. This means that there are $g, g_1 \in \mathrm{G}$, $g' \in \mathrm{G}'$ and $v \in \mathrm{V}$ such that $gg'.w = \tau^{-1}(g_1.v)$. Therefore

$$g.y = g.\tau(w) = \tau(gg'.w) = g_1.v \quad \text{and hence} \quad g_1^{-1} g.y = v \in \mathrm{G}.y \cap \mathfrak{h}\,.$$

By [Var89, Corollary 23], $y \in W(\mathrm{G},\mathfrak{h})v$. Thus $y \in \mathrm{V}$ because V is $W(\mathrm{G},\mathfrak{h})$-invariant. This proves that $\operatorname{supp} F_\phi \subseteq \overline{\mathrm{V}} \subseteq \mathrm{U}$.

The same argument extends to the case of $\mathrm{G} = \mathrm{U}_l$ because all Cartan subspaces $\mathfrak{h}_{\overline{1},m}$ satisfy $\tau(\mathfrak{h}_{\overline{1},m}) \subseteq \mathfrak{h}$. □

Remark 1. The Cartan subalgebra $\mathfrak{h}'$ is θ-stable, where θ is the fixed Cartan involution of $\mathfrak{g}'$. Let $\mathrm{H}' \subseteq \mathrm{G}'$ be the Cartan subgroup which is the centralizer of $\mathfrak{h}'$ in G', and let K' be the maximal compact subgroup of G' which is fixed by θ. Then, by [Har56, Lemma 10], the Weyl group $W(\mathrm{G}',\mathfrak{h}')$ coincides with $W(\mathrm{K}',\mathfrak{h}')$, i.e. the normalizer of H' in K' modulo the centralizer of H' in K'. Explicitly, K' is $\mathrm{U}_{l'}$ if $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, and $\mathrm{U}_p \times \mathrm{U}_q$ if $\mathbb{D} = \mathbb{C}$. Hence $W(\mathrm{G}',\mathfrak{h}')$ acts on $\mathfrak{h}'$ by permuting the J'_j, (38), if $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, and by separately permuting the first p and the last q elements J'_j if $\mathbb{D} = \mathbb{C}$. Since $\delta_j = 1$ for all $j = 1, \dots, l'$ if $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, and $\delta_j = 1$ for $j = 1, \dots, p$ and $\delta_j = -1$ for $j = p+1, \dots, p+q$ if $\mathbb{D} = \mathbb{C}$, it follows from (40) that the domain of integration $\tau'(\mathfrak{h}_{\overline{1}}^{reg})$ appearing in (53) is $W(\mathrm{G}',\mathfrak{h}')$-invariant. This property will be relevant in Proposition 2 below.

Recall from page 269 the notions of semisimple and regular elements in $\mathrm{W} = \mathfrak{s}_{\overline{1}}$. By [MPP15, Theorem 20], the set of semisimple elements is dense in W for every dual pair with one member compact unless $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $2l+1 < 2l'$. As noticed in section 3, W has a unique class of Cartan subalgebras unless $(\mathrm{G},\mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l < l' = p+q$. Suppose these two families of dual pairs are excluded. Let $\mathfrak{h}_{\overline{1}}$ denote the Cartan subalgebra in W fixed in (37). Then $\mathrm{W}^{reg} = \mathrm{S}.\mathfrak{h}_{\overline{1}}^{reg}$ is the set of regular semisimple elements of W. It is open and dense in W.

Proposition 2. *Suppose that $l \geq l'$. Let Φ be a $W(\mathrm{G}',\mathfrak{h}')$-invariant function on $\tau'(\mathfrak{h}_{\overline{1}}^{reg})$. Then there is a unique* S*-invariant function $\Phi^\sharp$ on*

W^{reg} *such that*

$$\Phi^\sharp(y) = (\Phi \circ \tau')(y) \qquad (y \in \mathfrak{h}_{\overline{1}}{}^{reg}).$$

Moreover,

$$(61)\quad \frac{1}{|W(\mathrm{G}', \mathfrak{h}')|} \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \Phi(y)\pi_{\mathfrak{g}/\mathfrak{z}}(y)F_\phi(y)\,dy = \int_{\mathrm{W}} \Phi^\sharp(w)\phi(w)\,dw \qquad (\phi \in C_c^\infty(\mathrm{W}))$$

provided the integrals are absolutely convergent.

Proof. The existence of $\Phi^\sharp$ is due to the fact that $\Phi \circ \tau'$ is a $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$-invariant function on $\mathfrak{h}_{\overline{1}}{}^{reg}$. The Weyl group $W(\mathrm{G}', \mathfrak{h}')$ acts on $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$ by permuting the coordinates with respect to the basis $\{J'_1, \dots, J'_{l'}\}$. The action is simple and transitive and $\tau'(\mathfrak{h}_{\overline{1}}^+)$ is a fundamental domain. Since the function $\Phi(y)\pi_{\mathfrak{g}/\mathfrak{z}}(y)F_\phi(y)$ is $W(\mathrm{G}', \mathfrak{h}')$-invariant on $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$, the formula (61) is a restatement of the Weyl–Harish-Chandra integration formulas on W for $l \geq l'$, see (52). □

5. Main Results

Suppose an irreducible representation Π of $\widetilde{\mathrm{G}}$ occurs in Howe's correspondence. This means that there is a subspace $\mathcal{H}_\Pi \subseteq \mathrm{L}^2(\mathrm{X})$ on which the restriction of ω coincides with Π. Since $\widetilde{\mathrm{Z}} \subseteq \widetilde{\mathrm{G}} \cap \widetilde{\mathrm{G}}'$, then either $\mathcal{H}_\Pi \subseteq \mathrm{L}^2(\mathrm{X})_+$ or $\mathcal{H}_\Pi \subseteq \mathrm{L}^2(\mathrm{X})_-$. In the first case the restriction of the central character χ_Π of Π to $\widetilde{\mathrm{Z}}$ is equal to χ_+ and in the second case to χ_-. Thus for $\tilde{z} \in \widetilde{\mathrm{Z}}$ and $\tilde{g} \in \widetilde{\mathrm{G}}$,

$$(62)\quad \begin{aligned} \Theta_\Pi(\tilde{z}\tilde{g}) &= \chi_+(\tilde{z})\Theta_\Pi(\tilde{g}) \quad \text{if} \quad \mathcal{H}_\Pi \subseteq \mathrm{L}^2(\mathrm{X})_+, \\ \Theta_\Pi(\tilde{z}\tilde{g}) &= \chi_-(\tilde{z})\Theta_\Pi(\tilde{g}) \quad \text{if} \quad \mathcal{H}_\Pi \subseteq \mathrm{L}^2(\mathrm{X})_-. \end{aligned}$$

We see from equations (17), (23), (24) and (62) that the function

$$\widetilde{\mathrm{G}} \ni \tilde{g} \to T(\tilde{g})\check{\Theta}_\Pi(\tilde{g}) \in \mathcal{S}'(\mathrm{W})$$

is constant on the fibers of the covering map (9). The following lemma is a restatement of (18). Our main results will be the explicit expressions of the various integrals appearing on the right-hand sides of the equations below.

Lemma 3. *Let* $\mathrm{G}^0 \subseteq \mathrm{G}$ *denote the connected identity component. Suppose* $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_d, \mathrm{U}_{p,q})$ *or* $(\mathrm{Sp}_d, \mathrm{O}^*_{2m})$. *Then* $\mathrm{G} = \mathrm{G}^0 = -\mathrm{G}^0$ *and*

$$(63)\quad f_{\Pi\otimes\Pi'} = \int_{\mathrm{G}} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg = \int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg.$$

Formula (63) *holds also if* $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_d, \mathrm{Sp}_{2m}(\mathbb{R}))$ *with* d *even and* Θ_Π *supported in* $\widetilde{\mathrm{G}^0}$, *because* $\mathrm{G}^0 = \mathrm{SO}_d = -\mathrm{SO}_d = -\mathrm{G}^0$. *In the remaining cases*

$$f_{\Pi\otimes\Pi'} = \int_{\mathrm{G}} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg = \int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg + \int_{\mathrm{G}\setminus(-\mathrm{G}^0)} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg\,. \tag{64}$$

The integrals over $-\mathrm{G}^0$ in (63) and (64) are given in Theorems 4 and 5 below, proved in section 10. The integrals over the other connected component in (64) are computed in Theorems 7, 8 and 10, respectively, and proved in sections 11, 12, and 14. Theorem 6, proved in this section, will furthermore show that the second integral on the right-hand side of (64) coincides with the first integral when $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_d, \mathrm{Sp}_{2l'}(\mathbb{R}))$, where $d = 2l$ or $d = 2l+1$, provided $l > l'$.

Remark 2. Notice that, since the character Θ_Π is conjugation invariant,

$$\int_{\mathrm{G}} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg = \int_{\mathrm{G}} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi^{\mathrm{G}})\, dg\,,$$

where ϕ^{G} is defined as in (58).

Let

$$\iota = \begin{cases} 1 & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C}\,, \\ \frac{1}{2} & \text{if } \mathbb{D} = \mathbb{H}\,, \end{cases} \tag{65}$$

and let

$$r = \frac{2\dim \mathfrak{g}}{\dim \mathrm{V}_{\mathbb{R}}}\,, \tag{66}$$

where the subscript $\mathbb{R}$ indicates that we are viewing V as a vector space over $\mathbb{R}$. Explicitly,

$$r = \begin{cases} 2l-1 & \text{if } \mathrm{G} = \mathrm{O}_{2l}\,, \\ 2l & \text{if } \mathrm{G} = \mathrm{O}_{2l+1}\,, \\ l & \text{if } \mathrm{G} = \mathrm{U}_l\,, \\ l+\frac{1}{2} & \text{if } \mathrm{G} = \mathrm{Sp}_l\,. \end{cases} \tag{67}$$

Let

$$\delta = \frac{1}{2\iota}(d' - r + \iota) \quad \text{and} \quad \beta = \frac{2\pi}{\iota}\,. \tag{68}$$

Fix an irreducible representation Π of $\widetilde{\mathrm{G}}$ that occurs in the restriction of the Weil representation ω to $\widetilde{\mathrm{G}}$. Let $\mu \in i\mathfrak{h}(\mathfrak{g})^*$ be the Harish-Chandra parameter of Π with $\mu_1 > \mu_2 > \cdots$. This means that $\mu = \lambda + \rho$, where λ is the highest weight of Π and ρ is one half times the sum of the positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. If $\mathrm{G} = \mathrm{U}_1$ then $\rho = 0$ and $\mu = \lambda$ is the weight of Π. If $\mathrm{G} = \mathrm{O}_2$ then $\rho = 0$. In this case, if Π is trivial or det, then $\mu = 0$. Otherwise $\Pi|_{\mathrm{SO}_2}$ has two weights and we pick any one of them.

Let $P_{a,b}$ and $Q_{a,b}$ be the piecewise polynomial functions defined in (D.4) and (D.5). Define

$$a_j = -\mu_j - \delta + 1\,, \qquad b_j = \mu_j - \delta + 1\,, \tag{69}$$

$$p_j(\xi) = P_{a_j,b_j}(\beta\xi)e^{-\beta|\xi|}\,, \quad q_j(\xi) = \beta^{-1}Q_{a_j,b_j}(\beta^{-1}\xi) \qquad (1 \le j \le l,\ \xi \in \mathbb{R})\,, \tag{70}$$

where δ and β are as in (68). Notice that a_j and b_j are integers (see Lemma 20). Furthermore, set

$$\kappa_0 = \begin{cases} 1/2 & \text{if } \mathrm{G} = \mathrm{O}_{2l} \text{ and } \lambda_l = \mu_l = 0\,, \\ 1 & \text{otherwise}\,. \end{cases} \tag{71}$$

Theorem 4. *Let $l \le l'$. Then there is a non-zero constant C which depends only on the dual pair $(\mathrm{G}, \mathrm{G}')$ such that for all $\phi \in \mathcal{S}(\mathrm{W})$*

$$\begin{aligned} &\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\,dg \\ &\quad = C\,\kappa_0\check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}\cap\tau(\mathrm{W})} \left(\prod_{j=1}^{l} \big(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\big)\right) \cdot F_\phi(y)\,dy\,, \end{aligned} \tag{72}$$

where χ_Π is the central character of Π (see (62)), $\tilde{c}$ is a real analytic lift of the Cayley transform (see (119)), δ_0 is the Dirac delta at 0, and $F_\phi(y)$ is the Harish-Chandra regular almost-elliptic orbital integral on W of ϕ at y (see [MPP20, Definition 3.2] and (54)).

The term

$$\prod_{j=1}^{l} \left(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\right) \tag{73}$$

is:

(1) *a function of y if and only if all the q_j's are zero, and this happens if and only if $l = l'$ and $(\mathrm{G}, \mathrm{G}') \neq (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$;*

(2) *a linear combination of products of functions and Dirac delta's at 0 in some coordinates y_j if and only if all the q_j's are of degree zero. This happens if and only if either $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l}(\mathbb{R}))$, or $l' = l+1$ and $\mathbb{D} = \mathbb{C}$ or $\mathbb{H}$.*

In the remaining cases, (73) *is a distribution, but not a measure.*

Remark 3. The integration domain $\mathfrak{h} \cap \tau(\mathrm{W})$ appearing in Theorem 4 was explicitly determined in [MPP20, Lemma 3.4]. It is equal to $\mathfrak{h}$ if $\mathbb{D} \neq \mathbb{C}$ or if $\mathbb{D} = \mathbb{C}$ and $l \leq \min(p, q)$. By (163), (164) and Appendix H, we see that $a_j \leq 0$ for all $1 \leq j \leq l$ when $l \leq l'$. Hence each $P_{a_j,b_j}(\beta y_j)$ vanishes for $y_j < 0$. In cases (1) and (2) of Theorem 4 with $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, we can therefore replace the domain of integration $\mathfrak{h}$ with the smaller domain $\tau(\mathfrak{h}_{\overline{1}})$.

In the case $l > l'$, up to conjugation, there is a unique Cartan subspace $\mathfrak{h}_{\overline{1}}$ in W. Recall that for $\mathbb{D} = \mathbb{C}$ we are supposing that $p \leq q$.

Define $s_0 \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ by

$$s_0(J_j) = J_j \qquad (1 \leq j \leq l) \qquad \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{H}\,, \tag{74}$$

$$s_0(J_j) = \begin{cases} J_j & (1 \leq j \leq p) \\ J_{q+j} & (p+1 \leq j \leq l-q) \\ J_{j-l+l'} & (l-q+1 \leq j \leq l) \end{cases} \qquad \text{if } \mathbb{D} = \mathbb{C}\,. \tag{75}$$

Theorem 5. *Let $l > l'$. Consider a genuine irreducible representation Π of $\widetilde{\mathrm{G}}$. (Its highest weight is among the weights listed in Appendix H). Then*

$$\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg \neq 0 \tag{76}$$

if and only if the highest weight $\lambda = \sum_{j=1}^{l} \lambda_j e_j$ *of* Π *is of the form*

(a) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l'} \geq 0$ *and* $\lambda_j = 0$ *for* $l'+1 \leq j \leq l$, *if* $\mathbb{D} = \mathbb{R}$ *or* $\mathbb{H}$,

(b) $\lambda_j = \dfrac{p-q}{2} + \nu_j$, *where* $\nu_1 \geq \cdots \geq \nu_p \geq 0$, $\nu_j = 0$ *for* $p+1 \leq j \leq l-q$, $0 \geq \nu_{l-q+1} \geq \cdots \geq \nu_l$, *if* $\mathbb{D} = \mathbb{C}$.

Suppose that (a) and (b) are satisfied. Then there is a non-zero constant C *which depends only on the dual pair* $(\mathrm{G}, \mathrm{G}')$ *such that for all* $\phi \in \mathcal{S}(\mathrm{W})$

$$\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg = C\, \kappa_0 \check{\chi}_\Pi(\tilde{c}(0)) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \Big(\prod_{j=1}^{l'} p_{s_0^{-1}(j)}(y_j) \Big) \cdot F_\phi(y)\, dy\,, \tag{77}$$

where κ_0 *is as in* (71) *and, explicitly,*

$$\prod_{j=1}^{l'} p_{s_0^{-1}(j)}(y_j) = \begin{cases} \prod_{j=1}^{l'} p_j(y_j) & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{H}, \\ \big(\prod_{j=1}^{p} p_j(y_j) \big) \big(\prod_{j=p+1}^{l'} p_{j+l-l'}(y_j) \big) & \text{if } \mathbb{D} = \mathbb{C}. \end{cases}$$

The right-hand side of (77) *can be written as a non-zero constant multiple of*

$$\kappa_0 \check{\chi}_\Pi(\tilde{c}(0)) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \Phi(y) \pi_{\mathfrak{g}/\mathfrak{z}}(y) F_\phi(y)\, dy = \int_{\mathrm{W}} \Phi^\sharp(w) \phi(w)\, dw\,, \tag{78}$$

where

$$\Phi(y) = \frac{\sum_{s' \in W(\mathrm{G}', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta(s'y)_j)}{\pi_{\mathfrak{g}/\mathfrak{z}}(y)} e^{-\beta \sum_{j=1}^{l'} |y_j|}\,, \tag{79}$$
$$(y_j = J_j'^{*} y,\ y = \tau(w) = \tau'(w), w \in \mathfrak{h}_{\overline{1}}{}^{reg})$$

is a non-zero $W(\mathrm{G}', \mathfrak{h}')$*-invariant real-valued continuous function on* $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$, *and* $\Phi^\sharp$ *is an* S*-invariant function such that* $\Phi^\sharp(w) = \Phi(\tau'(w))$ *for all* $w \in \mathfrak{h}_{\overline{1}}{}^{reg}$. *In* (79), μ *is the Harish-Chandra parameter of* Π,

$$a_{s,j} = -(s\mu)_j - \delta + 1\,, \quad b_{s,j} = (s\mu)_j - \delta + 1 \qquad (s \in W(\mathrm{G}, \mathfrak{h}),\ 1 \leq j \leq l)\,, \tag{80}$$

$P_{a,b,\pm 2}$ *is the polynomial defined in* (D.1) *or* (D.2), *and the* δ_j*'s are as in* (36). *(See* (42) *for the identifications* $y = \tau(w) = \tau'(w)$ *in* (79) *.)*

Remark 4. Recall from Remark 1 that the domain of integration $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$ appearing in Theorem 5 is $W(\mathrm{G}', \mathfrak{h}')$-invariant. Formula (79) will

prove, by Proposition 2, that the intertwining distribution is not zero when the conditions (a) or (b) are satisfied.

REMARK 5. Conditions (a) and (b) in Theorem 5 are precisely those ensuring that Π occurs in Howe's correspondence. See Corollary 11 below. (They are contragredient to those listed in [Prz96, Appendix], because the Weil representation used there is contragredient to the one used here.)

Before considering the integrals over $\mathrm{G} \setminus (-\mathrm{G}^0)$ in (64), let us introduce some notation concerning the irreducible representations of the orthogonal groups. Since $\mathbb{D} \neq \mathbb{C}$, we can choose a polarization $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ so that G preserves X and Y. We shall suppose in what follows that we have made such a choice.

Suppose that $\mathrm{G} = \mathrm{O}_d$. Then, for each highest weight λ of an irreducible representation of G^0 there are one or two unitary genuine representations of $\widetilde{\mathrm{G}}$ having highest weight λ. There are two if and only if either $d = 2l$ and $\lambda_l = 0$, or $d = 2l + 1$. See e.g. [GW09, §5.5.5]. Let $\Pi_{\lambda,+}$ and $\Pi_{\lambda,-}$ be these representations. Set

$$\chi_+(\widetilde{g}) = \frac{\Theta(\widetilde{g})}{|\Theta(\widetilde{g})|} \qquad (g \in \mathrm{O}_d)\,, \tag{81}$$

where Θ is defined in (16). Then χ_+ is a character of $\widetilde{\mathrm{G}}$. Notice that (81) is an extension of (23) from $\widetilde{\mathrm{Z}}$ to $\widetilde{\mathrm{G}}$. In fact, Proposition 4.28 in [AP14] implies that $(\chi_+(\widetilde{g}))^2 = (\det g)_{\mathrm{X}}^{-1}$, where $(\det g)_{\mathrm{X}}$ indicates the determinant of g as endomorphism of X.

Then, in the Schrödinger model for the Weil representation ω, for which the space of smooth vectors is $\mathcal{S}(\mathrm{X})$,

$$\left(\omega \otimes \chi_+^{-1}\right)(\widetilde{g}) f(x) = f(g^{-1}x) \qquad (g \in \mathrm{G},\ f \in \mathcal{S}(\mathrm{X}),\ x \in \mathrm{X})\,. \tag{82}$$

Hence $\omega \otimes \chi_+^{-1}$ descends to a representation ω_0 of G given by

$$\omega_0(g) f(x) = f(g^{-1}x) \qquad (g \in \mathrm{G},\ f \in \mathcal{S}(\mathrm{X}),\ x \in \mathrm{X})\,. \tag{83}$$

THEOREM 6. *Suppose that $l > l'$. Let Π be an irreducible representation of $\widetilde{\mathrm{O}}_d$ occurring in the restriction of the Weil representation to $\widetilde{\mathrm{O}}_d$. If*

$d = 2l$, then $\lambda_l = 0$. In both cases $d = 2l$ or $d = 2l+1$, the second irreducible genuine representation of $\widetilde{\mathrm{O}}_d$ having the same highest weight as Π does not occur in the restriction of the Weil representation to $\widetilde{\mathrm{O}}_d$. Moreover,

$$\int_{\mathrm{G}} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg = 2\int_{\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg = 2\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg\,. \tag{84}$$

In particular,

$$\int_{\mathrm{G}\setminus(-\mathrm{G}^0)} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg = \int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg\,. \tag{85}$$

The integral on the very right-hand side of (84) *was computed in Theorem 5.*

Proof. Let λ be the highest weight of Π, and let $d = 2l$ or $2l+1$. Recall the notation introduced before (81).

Suppose that both $\Pi_{\lambda,+}$ and $\Pi_{\lambda,-}$ occur. Then $\Pi_{\lambda,\pm} \otimes \chi_+^{-1}$ descends to a representation $(\Pi_{\lambda,\pm} \otimes \chi_+^{-1})|_{\mathrm{G}}$ of G occurring in ω_0. Let $\mathcal{S}(\mathrm{X})_{\Pi_{\lambda,\pm}} \subseteq \mathcal{S}(\mathrm{X})$ denote the $\Pi_{\lambda,\pm}$-isotypic component in $\mathcal{S}(\mathrm{X})$. By (83),

$$(\Pi_{\lambda,\pm} \otimes \chi_+^{-1})|_{\mathrm{G}}(g)f(x) = f(g^{-1}x) \qquad (g \in \mathrm{G},\ f \in \mathcal{S}(\mathrm{X})_{\Pi_{\lambda,\pm}},\ x \in \mathrm{X})\,. \tag{86}$$

Let $\Pi_{\lambda,0}$ denote an irreducible representation of G whose restriction to the identity component has highest weight λ. As one can see from [GW09, §5.5.5],

$$\text{if } (\Pi_{\lambda,+} \otimes \chi_+^{-1})|_{\mathrm{G}} = \Pi_{\lambda,0}, \text{ then } (\Pi_{\lambda,-} \otimes \chi_+^{-1})|_{\mathrm{G}} = \Pi_{\lambda,0} \otimes \det\,. \tag{87}$$

Hence $\Pi_{\lambda,0} \otimes \Pi_{\lambda,0} \otimes \det$ occurs in $\omega_0 \otimes \omega_0$, acting on $\mathcal{S}(\mathrm{X} \oplus \mathrm{X})$. Recall that $\Pi_{\lambda,0} = \Pi_{\lambda,0}^c$ is self-contragredient. Since $\Pi_{\lambda,0}^c \otimes \Pi_{\lambda,0}$ contains the trivial representation, we conclude that det occurs in $\omega_0 \otimes \omega_0$. Observe that $\omega_0 \otimes \omega_0$ acts on $\mathcal{S}(\mathrm{X} \oplus \mathrm{X})$ by

$$\omega_0 \otimes \omega_0(g)f(x) = f(g^{-1}x) \qquad (g \in \mathrm{G},\ f \in \mathcal{S}(\mathrm{X} \oplus \mathrm{X}),\ x \in \mathrm{X})\,.$$

It is therefore the "representation ω_0" corresponding to a dual pair $(\mathrm{O}_d, \mathrm{Sp}_{4l'}(\mathbb{R}))$. By Proposition F.1, it follows that $d \le 2l'$, contrary to our assumption.

Suppose first that $\Pi_{\lambda,+}$ is not isomorphic to $\Pi_{\lambda,-}$, which by the description of the irreducible representations of orthogonal groups [GW09, §5.5.5]

can occur only when $\lambda_l = 0$ if $d = 2l$. Then the above argument shows that only one of $\Pi_{\lambda,+}$ and $\Pi_{\lambda,-}$ (i.e. Π) occurs in the restriction of the Weil representation.

On the other hand, if $\Pi_{\lambda,+}$ is isomorphic to $\Pi_{\lambda,-}$, then $d = 2l$ (because $\det(-\mathrm{I}_{2l+1}) = -1$) and, again by [GW09, §5.5.5], $\lambda_l \neq 0$. In this case, $\Pi_{\lambda,0} = \Pi_{\lambda,0} \otimes \det$ and the above argument shows that the representation does not occur in ω.

Thus the second representation of $\widetilde{\mathrm{O}}_d$ which has the same restriction as Π to $\mathrm{G}^0 = \mathrm{SO}_d$, does not occur. Hence the $\Pi|_{\widetilde{\mathrm{SO}}_d}$-isotypic component of ω coincides with the Π-isotypic component of ω. Therefore

$$\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})\, dg = 2 \int_{\mathrm{G}^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})\, dg\,.$$

(The factor 2 is a consequence of the normalization of the measures.) In particular, $\int_{\mathrm{G}\setminus\mathrm{G}^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})\, dg = \int_{\mathrm{G}^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})\, dg$. If $\mathrm{G} = \mathrm{O}_{2l}$, then $\mathrm{G}^0 = -\mathrm{G}^0$ and if $\mathrm{G} = \mathrm{O}_{2l+1}$, then $\mathrm{G} \setminus \mathrm{G}^0 = -\mathrm{G}^0$. This explains the second equality in (84). □

Remark 6. It should be pointed out that the proof of Theorem 6 does not use the known classification of the highest weights of the genuine irreducible representations occurring in Howe's correspondence.

Consider now the case $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ and the character Θ_{Π} not supported in the preimage $\widetilde{\mathrm{G}^0}$ of the connected identity component $\mathrm{G}^0 \subseteq \mathrm{G}$.

Suppose that $l \leq l'$ and $l \neq 1$. Then the graded vector space (34) is equal to

$$\mathsf{V} = \mathsf{V}^0_{\bar{1}} \oplus \mathsf{V}^1 \oplus \mathsf{V}^2 \oplus \cdots \oplus \mathsf{V}^l\,.$$

Recall from (35) that in each $\mathsf{V}^j_{\bar{0}}$ we selected an orthonormal basis v_0, v_0'. For convenience, we introduce the index j in the notation and we write $v_{2j-1} = v_0$ and $v_{2j} = v_0'$, for $1 \leq j \leq l$. Then $v_1, v_2, \ldots, v_{2l}$ is an orthonormal basis of $\mathsf{V}_{\bar{0}}$ and

$$J_j v_{2j-1} = -v_{2j}\,, \quad J_j v_{2j} = v_{2j-1} \qquad (1 \leq j \leq l)\,.$$

In terms of the dual basis (46) of $\mathfrak{h}^*_{\mathbb{C}}$, the positive roots are

$$e_j \pm e_k \qquad (1 \leq j < k \leq l)\,.$$

Define an element $s \in \mathrm{G}$ by

$$sv_1 = v_1,\ sv_2 = v_2,\ \dots,\ sv_{2l-1} = v_{2l-1},\ sv_{2l} = -v_{2l}\,. \tag{88}$$

Then $\mathrm{G} = \mathrm{G}^0 \cup \mathrm{G}^0 s$ is the disjoint union of two connected components. Set

$$\mathsf{V}_{\overline{0},s} = \mathsf{V}_{\overline{0}}^1 \oplus \mathsf{V}_{\overline{0}}^2 \oplus \cdots \oplus \mathsf{V}_{\overline{0}}^{l-1} \oplus \mathbb{R}v_{2l}\,, \quad \text{and} \quad \mathsf{V}_s = \mathsf{V}_{\overline{0},s} \oplus \mathsf{V}_{\overline{1}}\,.$$

The dual pair corresponding to $(\mathsf{V}_{\overline{0},s}, \mathsf{V}_{\overline{1}})$ is $(\mathrm{G}_s, \mathrm{G}'_s) = (\mathrm{O}_{2l-1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ acting on the symplectic space $\mathrm{W}_s = \mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0},s})$. The objects corresponding to W_s will be distinguished by the subscript s.

Let $\mathfrak{h}_s = \sum_{j=1}^{l-1} \mathbb{R}J_j$. This is the centralizer of s in $\mathfrak{h} = \sum_{j=1}^{l} \mathbb{R}J_j$. Set

$$\rho_s^{\mathrm{C}} = (l-1)e_1 + (l-2)e_2 + \cdots + e_{l-1}\,. \tag{89}$$

Let

$$\lambda = \sum_{j=1}^{l-1} \lambda_j e_j$$

be the highest weight of Π. (Here $\lambda_l = 0$ because we assume that Θ_Π is not supported in $\widetilde{\mathrm{G}^0}$.) Define

$$\mu^{\mathrm{C}} = \lambda + \rho_s^{\mathrm{C}}\,.$$

The number r, (67), for the group G is equal to

$$r = 2l - 1$$

and the number δ, (68), for the dual pair $(\mathrm{G}, \mathrm{G}')$ is equal to

$$\delta = \frac{1}{2}(2l' - r + 1) = l' - l + 1\,.$$

Set

$$\begin{aligned} a_j^{\mathrm{C}} &= -\mu_j^{\mathrm{C}} - \delta + 1 = -\mu_j^{\mathrm{C}} - l' + l\,, \\ b_j^{\mathrm{C}} &= \mu_j^{\mathrm{C}} - \delta + 1 = \mu_j^{\mathrm{C}} - l' + l\,, \qquad (1 \le j \le l-1)\,. \end{aligned}$$

Notice that $a_j^{\mathrm{C}} = a_j$ and $b_j^{\mathrm{C}} = b_j$ for $1 \le j \le l-1$ because ρ_s^{C} coincides with the restriction of ρ to $\mathfrak{h}_s$. Using these numbers in place of a_j and b_j in (70), define the functions p_j^{C} and q_j^{C}.

THEOREM 7. *Let* $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l},\mathrm{Sp}_{2l'}(\mathbb{R}))$ *with* $1 < l \le l'$. *Assume that the character* Θ_Π *is not supported in* $\widetilde{\mathrm{G}^0}$. *Then there is a constant* C *which depends only on the dual pair* (G,G') *such that for any* $\phi \in \mathcal{S}(\mathrm{W})$

$$\begin{aligned}(90)\quad &\int_{\mathrm{G}^0 s} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\,dg \\ &\quad = CD_\Pi \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}_s} \prod_{j=1}^{l-1} \left(p_j^{\mathrm{C}}(y_j) + q_j^{\mathrm{C}}(-\partial_{y_j})\delta_0(y_j)\right) \cdot F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,,\end{aligned}$$

where $\check{\chi}_\Pi(\tilde{c}(0))$ *and* D_Π *are equal to* ± 1, *and* D_Π *distinguishes* Π *and* $\Pi\otimes\det$.

Theorem 7 excludes the dual pairs $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_2,\mathrm{Sp}_{2l'}(\mathbb{R}))$ because its proof relies on an analogue of the Weyl's character formula for $\mathrm{G}\setminus\mathrm{G}^0$ proved by [Wen01] for nonconnected compact semisimple Lie groups. These excluded cases will be treated in subsection 6.1.

Now we consider the case $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l+1},\mathrm{Sp}_{2l'}(\mathbb{R}))$ with $1 \le l \le l'$. Recall from (34) the graded vector space V. In the case we consider, $\dim \mathsf{V}_{\overline{0}}^0 = 1$, $\dim \mathsf{V}_{\overline{1}}^0 = 2(l'-l)$ and for $1 \le j \le l$, $\dim \mathsf{V}_{\overline{0}}^j = \dim \mathsf{V}_{\overline{1}}^j = 2$. Let

$$\mathrm{W}_s = \mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0}}^1 \oplus \cdots \oplus \mathsf{V}_{\overline{0}}^l) \quad \text{and} \quad \mathrm{W}_s^\perp = \mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0}}^0)\,.$$

Then

$$\mathrm{W} = \mathrm{W}_s \oplus \mathrm{W}_s^\perp \tag{91}$$

is a direct sum orthogonal decomposition. Let $\mathrm{G}_s \subseteq \mathrm{G}$ be the subgroup acting trivially on the space $\mathsf{V}_{\overline{0}}^0$. The Lie algebra $\mathfrak{g}_s$ of $\mathfrak{g}$ embeds as those elements acting as zero on $\mathsf{V}_{\overline{0}}^0$. Let $\mathrm{G}'_s = \mathrm{G}'$. Then the dual pair corresponding to W_s is $(\mathrm{G}_s,\mathrm{G}'_s) \simeq (\mathrm{O}_{2l},\mathrm{Sp}_{2l'}(\mathbb{R}))$ and dual pair corresponding to $\mathrm{W}_s^\perp$ is $(\mathrm{O}_1,\mathrm{Sp}_{2l'}(\mathbb{R}))$. If H is a Cartan subgroup of G, then $\mathrm{H}^0 = \mathrm{H}_s^0$ is a Cartan subgroup of G_s^0, and the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_s$ share the same Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_s$. The following theorem will be proved in section 14.

THEOREM 8. *Let* $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l+1},\mathrm{Sp}_{2l'}(\mathbb{R}))$ *with* $1 \le l \le l'$. *Then there is a nonzero constant* C *such that for all* $\phi \in \mathcal{S}(\mathrm{W})$

$$(92)\quad \int_{\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\,dg = C(-1)^{|\lambda|} \int_{\mathfrak{h}} \prod_{j=1}^{l} \left(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\right) F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,,$$

where p_j, q_j are defined as in (70)*, λ is the highest weight of Π and $|\lambda| = \sum_{j=1}^{l} \lambda_j$ is a nonnegative integer. (See Appendix H.)*

If $l = l'$, then $F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}$ is proportional to F_ϕ (independently of ϕ).

Remark 7. As in Theorem 4, the term

$$\prod_{j=1}^{l} \big(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\big)$$

is a function of y (i.e. all the q_j's are zero) if and only if $l = l'$. In the other cases, it is a distribution, but not a measure. Furthermore, if $l = l'$, we can replace the domain of integration $\mathfrak{h}$ with the smaller domain $\tau(\mathfrak{h}_{\overline{1}})$.

Remark 8. It is known from the classification of the representations occurring in Howe's correspondence (see e.g. [Prz96, Appendix]) that for the pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $l \le l'$ there are two representations of $\widetilde{\mathrm{G}}$ with the same highest weight λ that occur in the correspondence, namely $\Pi(\tilde{g})$ and $\Pi(\tilde{g}) \otimes \det(g)$. They agree on G^0, so the integral on the left-hand side of (92) cannot distinguish them. In particular, we cannot replace the factor $(-1)^{|\lambda|}$ with $\check{\chi}_\Pi(\tilde{c}(0))$, which appears in Theorems 4 and 5.

Remark 9. The pair $(\mathrm{O}_1, \mathrm{Sp}_{2l'}(\mathbb{R}))$ was studied in detail in section 2.

Suppose $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_d, \mathrm{Sp}_{2l'}(\mathbb{R}))$, where $d = 2l$ or $2l+1$ and $d > 2$. In Theorem 9 below, the integral over $\mathrm{G}\backslash(-\mathrm{G}^0)$ of the distribution-valued map $g \to \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})$ is reduced to an integral over $-\mathrm{G}_s^0$. The resulting equality, which holds independently of the mutual relation between the ranks l and l', will be needed in [MPP24]. Recall that

$$\mathrm{G} \setminus (-\mathrm{G}^0) = \begin{cases} \mathrm{G}^0 s & \text{if } \mathrm{G} = \mathrm{O}_{2l}\,, \\ \mathrm{G}^0 & \text{if } \mathrm{G} = \mathrm{O}_{2l+1}\,. \end{cases}$$

Moreover, $-\mathrm{G}_s^0 = \mathrm{G}_s^0$ if $\mathrm{G} = \mathrm{O}_{2l+1}$.

Theorem 9. *Let $\mathrm{G} = \mathrm{O}_d$ with $d > 2$. If $d = 2l$, suppose that the character Θ_Π is not supported in $\widetilde{\mathrm{G}^0}$. Then for all $\phi \in \mathcal{S}(\mathrm{W})$*

$$\int_{\mathrm{G}\backslash(-\mathrm{G}^0)} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\,dg = \frac{1}{2}\int_{-\mathrm{G}_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1-g)T_s(\tilde{g})(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\,dg\,, \tag{93}$$

where T_s is the operator T, see (14)*, corresponding to the symplectic space* W_s.

We prove Theorem 9 in section 13.

REMARK 10. The term $\det(1-g)$ appearing in (93) admits a representation theoretical interpretation. Indeed, let σ be the spin representation of the spin cover of G_s^0. Then the tensor product $\sigma \otimes \sigma^c$ is a representation of G_s^0 and, by [Lit06, Ch. XI, III., p. 254]

$$\Theta_{\sigma\otimes\sigma^c}(g) = |\Theta_\sigma(g)|^2 = \det(1+g) \qquad (g \in \mathrm{G}_s^0)\,. \tag{94}$$

So $\det(1-g) = \Theta_{\sigma\otimes\sigma^c}(-g)$.

Suppose $l > l'$. Theorem 6 reduces the computation of $\int_{\mathrm{G}} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg$ to that of $\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg$, done in Theorem 5. One could still try to compute the integral on $\mathrm{G}\setminus(-\mathrm{G}^0)$ directly, without relying on Theorem 6. As an example, we do it for O_{2l+1} in Theorem 10 below. Nevertheless, the result is less precise than that from Theorem 6 since we are only able to prove that the integral over $\mathrm{G}\setminus(-\mathrm{G}^0)$ is a nonzero constant multiple of the one over $-\mathrm{G}^0$. Determining the constant is a serious issue even in the much easier situation of $(\mathrm{U}_l, \mathrm{U}_{l'})$; see [MPP23].

To consider the case $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $l > l'$, recall the graded vector space V, (34) and the formula (43),

$$\mathsf{V} = \mathsf{V}^0 \oplus \mathsf{V}^1 \oplus \cdots \oplus \mathsf{V}^{l'}\,,$$

where, as in (44),

$$\begin{aligned} \mathsf{V}^0_{\bar{0}} &= \mathsf{V}^{0,0}_{\bar{0}} \oplus \left(\mathsf{V}^{0,0}_{\bar{0}}\right)^\perp\,, \\ \mathsf{V}^0_{\bar{1}} &= 0\,, \end{aligned}$$

with $\dim \mathsf{V}^{0,0}_{\bar{0}} = 1$ and $\dim \left(\mathsf{V}^{0,0}_{\bar{0}}\right)^\perp = 2(l-l')$. Let

$$\mathrm{W}_s = \mathrm{Hom}(\mathsf{V}^1_{\bar{1}} \oplus \cdots \oplus \mathsf{V}^{l'}_{\bar{1}}, \left(\mathsf{V}^{0,0}_{\bar{0}}\right)^\perp \oplus \mathsf{V}^1_{\bar{0}} \oplus \cdots \oplus \mathsf{V}^{l'}_{\bar{0}}), \quad \mathrm{W}_s^\perp = \mathrm{Hom}(\mathsf{V}_{\bar{1}}, \mathsf{V}^{0,0}_{\bar{0}})\,.$$

(Notice that $\mathsf{V}_{\overline{1}}^{1}\oplus\cdots\oplus\mathsf{V}_{\overline{1}}^{l'}=\mathsf{V}_{\overline{1}}$ and $\left(\mathsf{V}_{\overline{0}}^{0,0}\right)^{\perp}\oplus\mathsf{V}_{\overline{0}}^{1}\oplus\cdots\oplus\mathsf{V}_{\overline{0}}^{l'}$ is the orthogonal complement of the one dimensional space $\mathsf{V}_{\overline{0}}^{0,0}$ in $\mathsf{V}_{\overline{0}}$.) Then

$$\mathrm{W}=\mathrm{W}_s\oplus\mathrm{W}_s^{\perp} \tag{95}$$

is a direct sum orthogonal decomposition. Let $\mathrm{G}_s\subseteq\mathrm{G}$ be the subgroup acting trivially on the space $\mathsf{V}_{\overline{0}}^{0,0}$ and let $\mathrm{G}'_s=\mathrm{G}'$. The dual pair corresponding to W_s is $(\mathrm{G}_s,\mathrm{G}'_s)\simeq(\mathrm{O}_{2l},\mathrm{Sp}_{2l'}(\mathbb{R}))$ and dual pair corresponding to $\mathrm{W}_s^{\perp}$ is $(\mathrm{O}_1,\mathrm{Sp}_{2l'}(\mathbb{R}))$.

Theorem 10. *Let* $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_{2l+1},\mathrm{Sp}_{2l'}(\mathbb{R}))$ *with* $l>l'$. *Then*

$$\int_{\mathrm{G}^0}\check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g})\,dg\neq 0 \tag{96}$$

if and only if the highest weight $\lambda=\sum_{j=1}^{l}\lambda_j e_j$ *of* Π *satisfies condition (a) of Theorem 5 for* $\mathbb{D}=\mathbb{R}$. *Suppose that this condition is satisfied. Then there is a non-zero constant* C *which depends only on the dual pair* (G,G') *such that for all* $\phi\in\mathcal{S}(\mathrm{W})$

$$\int_{\mathrm{G}^0}\check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g})(\phi)\,dg=C(-1)^{|\lambda|}\int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})}\Big(\prod_{j=1}^{l'}p_j(y_j)\Big)F_{\phi}(y)\,dy\,. \tag{97}$$

As a byproduct of our calculations of the intertwining distributions, we obtain the list of highest weights of the genuine irreducible representations Π of $\widetilde{\mathrm{G}}$ that occur in Howe's correspondence when $l>l'$. This list was first determined (without any restrictions on the ranks l and l') in [KV78].

Corollary 11. *Suppose that* $l>l'$. *A genuine representation* $\Pi\in\widetilde{\mathrm{G}}^{\wedge}$ *occurs in Howe's correspondence if and only if its highest weight satisfies conditions (a) or (b) of Theorem 5.*

Proof. Our computations of the intertwining distribution $\int_{\mathrm{G}}\widetilde{\Theta}_{\Pi}(\widetilde{g})T(\widetilde{g})\,dg$ can be applied to any genuine irreducible representation $\Pi\in\widetilde{\mathrm{G}}^{\wedge}$ (not necessarily occurring in Howe's correspondence). This distribution is nonzero if and only if $\omega|_{\widetilde{\mathrm{G}}}$ has a nonzero Π-isotypic component.

This is equivalent to the fact that there is a unitary highest weight representation Π' of $\widetilde{\mathrm{G}'}$ such that $\Pi \otimes \Pi'$ occurs in $\omega|_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}'}}$. The nonvanishing of the intertwining distributions leads to conditions (a) or (b) of Theorem 5 when $\mathrm{G} = \mathrm{U}_l$ or Sp_l. In the case of orthogonal groups, we can further use Theorem 6 and conclude that the nonvanishing of the intertwining distributions is equivalent to the nonvanishing of the integral of $\widetilde{\Theta}_\Pi(\widetilde{g})T(\widetilde{g})$ over $-\mathrm{G}^0$. The claim then follows again from Theorem 5. □

As we shall see in the proofs in section 10, the list of highest weights in Theorem 5 is obtained by comparing the support of the function $\prod_{j=1}^{l'} p_j(y_j)$ with the domain of integration, $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$. Unfortunately, this method is not refined enough to provide necessary and sufficient conditions when $l \leq l'$.

Let us now consider the dual pair $(\mathrm{U}_l, \mathrm{U}_{p,q})$. Recall that in this case $l' = p + q$ and that we assume that $p \leq q$. If $l \leq p$ all irreducible genuine representations of $\widetilde{\mathrm{U}_l}$ occur because the pair is in the stable range with U_l the smaller member; see [Li89] or [PP08]. The absence of conditions on the highest weight in Theorem 4 is consistent with this fact (despite the fact that we cannot see that our intertwining operator is not 0).

If $p < l \leq p+q$ then the next corollary gives precise necessary conditions on the highest weight of Π to occur in the correspondence. The proof is independent of the classification and is based on a refined analysis of the intertwining distribution; see section 15.

Corollary 12. *Suppose that* $\mathbb{D} = \mathbb{C}$ *and* $p < l \leq p + q$. *Let* $\Pi \in \widetilde{\mathrm{G}}^\wedge$ *be a genuine irreducible representation of highest weight* λ. *If either* $\lambda_{p+1} > \frac{p-q}{2}$ *or (when* $q < l$ *holds)* $\lambda_{l-q} < \frac{p-q}{2}$, *then* Π *does not occur in Howe's correspondence.*

For the dual pair $(\mathrm{Sp}_l, \mathrm{O}^*_{2l'})$, by the known classification of highest weights of representations of Sp_l occurring in Howe's correspondence, all irreducible genuine representations of $\widetilde{\mathrm{Sp}_l}$ occur if $l \leq l'$. We can recover this fact out of the formula for the intertwining distribution determined in Theorem 4 (and hence without using the classification) only when $l' \in \{l, l+1\}$. This is the content of the following corollary, proved in section 16.

Corollary 13. *Suppose that* $\mathbb{D} = \mathbb{H}$ *and* $l \leq l'$. *Let* Π *be an irreducible genuine representation of* $\mathrm{G} = \mathrm{Sp}_l$ *with highest weights* $\lambda_1 \geq \cdots \geq$

λ_l. If $\lambda_l \geq l' - l - 1$ then Π occurs in Howe's correspondence. In particular, if $l' = l$ or $l' = l + 1$, then every genuine irreducible representation $\Pi \in \widetilde{\mathrm{G}}^\wedge$ occurs in Howe's correspondence.

We terminate our discussion on the highest weights of the genuine irreducible representations of $\widetilde{\mathrm{G}}$ occurring in Howe's correspondence with the pair $(\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$. For this dual pair, we compute the intertwining distributions in section 6. We will recover the (well-known) list of representations of $\widetilde{\mathrm{O}}_2$ occurring in Howe's correspondence by their explicit formulas. See also Remark 12.

Remark 11. In this article we have considered the group $\mathrm{U}_{p,q}$ with $p \leq q$. Suppose now that $q \geq p$. This is equivalent to replacing the form $(\cdot,\cdot)'$ into its opposite. Correspondingly, the symplectic form $\langle \cdot, \cdot \rangle$ becomes its opposite. The inner product $-\langle J\cdot, \cdot \rangle$ is now positive definite provided we select $-J$ instead of J. In the notation at the beginning of section 1, the equation defining the preimages of $g \in \mathrm{Sp}(\mathrm{W})$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ becomes

$$\xi^2 = i^{\dim(g-1)\mathrm{W}} \det(-J_g)^{-1}_{J_g\mathrm{W}} = (-i)^{\dim(g-1)\mathrm{W}} \det(J_g)^{-1}_{J_g\mathrm{W}},$$

because $(-1)^{\dim(J_g\mathrm{W})} = (-1)^{\dim(g-1)\mathrm{W}}$. This means that ξ is transformed into $\bar{\xi}$. Since $\Theta((g;\xi)) = \xi$, we conclude that Θ needs to be changed into $\overline{\Theta}$, i.e. the metaplectic representation ω is replaced by its contragredient $\omega^\vee$. Therefore

$$\omega|_{\widetilde{\mathrm{G}}\times\widetilde{\mathrm{G}}'} = \bigoplus(\Pi \otimes \Pi') \quad \text{is replaced by} \quad \omega^\vee|_{\widetilde{\mathrm{G}}\times\widetilde{\mathrm{G}}'} = \bigoplus(\Pi^\vee \otimes (\Pi')^\vee).$$

The highest weights of the representations of U_l occurring in $\omega^\vee$ are obtained from those listed for far in this paper by changing their sign and permuting them so that they are in decreasing order. Those written in (H.1), are replaced for $\mathrm{U}_{p,q}$, where $q \geq p$, with

$$\lambda_j = \frac{q-p}{2} + \nu_j, \quad \nu_j \in \mathbb{Z}, \quad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_l.$$

We conclude this section with a result on the non-differential operator nature of the symmetry breaking operators in $\mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}^\infty_\omega, \mathcal{H}^\infty_\Pi \otimes \mathcal{H}^\infty_{\Pi'})$.

COROLLARY 14. *Let* $(\mathrm{G}, \mathrm{G}')$ *be a real reductive dual pair with one member compact. Then the essentially unique non-zero symmetry breaking operator in*

$$\mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{H}^\infty_\omega, \mathcal{H}^\infty_\Pi \otimes \mathcal{H}^\infty_{\Pi'})$$

is not a differential operator.

PROOF. We are going to show that $(\mathrm{Op}\circ\mathcal{K})(f_{\Pi\otimes\Pi'})$ is not a differential operator.

Let $f \in \mathcal{S}'(\mathrm{W})$ and recall the definition of $\mathcal{K}(f)$ in (12). According to [Hör83, Theorems 5.2.1 (the Schwartz kernel theorem) and 5.2.3], the continuous linear map $\mathrm{Op}\circ\mathcal{K}(f)$ is a distribution-valued differential operator if and only if $\mathcal{K}(f) \in \mathcal{S}'(\mathrm{X}\times\mathrm{X})$ is supported by the diagonal $\Delta = \{(x,x);\ x \in \mathrm{X}\}$. This implies that f is supported in Y. Indeed, given $\varphi \in \mathcal{S}(\mathrm{X}\times\mathrm{X})$, let $\psi \in \mathcal{S}(\mathrm{X}\times\mathrm{X})$ be defined by $\varphi(x,x') = \psi(x-x', x+x')$ for all $x, x' \in \mathrm{X}$. Furthermore, let $\psi(\cdot,\widehat{\cdot}) \in \mathcal{S}(\mathrm{X}\times\mathrm{Y})$ denote the partial Fourier transform of ψ with respect to its second variable, defined by

$$\psi(a,\widehat{y}) = \int_{\mathrm{X}} \chi\big(\frac{1}{2}\langle y,b\rangle\big)\psi(a,b)\,db \qquad ((a,y) \in \mathrm{X}\times\mathrm{Y})\,.$$

Then

$$\operatorname{supp}\varphi \cap \Delta = \emptyset \quad \text{if and only if} \quad \operatorname{supp}\psi(\cdot,\widehat{\cdot}) \cap (\{0\}\times\mathrm{Y}) = \emptyset\,.$$

Since $\mathcal{K}(f)(\varphi) = f(\psi(\cdot,\widehat{\cdot}))$ by (12), we obtain the claim.

Notice that this cannot happen in our case. Indeed, the support of $f_{\Pi\otimes\Pi'}$ is $\mathrm{G}\mathrm{G}'$-invariant. Since the complex structure $J \in \mathrm{G}'$ permutes X and Y, the only $\mathrm{G}\mathrm{G}'$-orbit in Y is the zero orbit. Hence the inclusion $\operatorname{supp} f_{\Pi\otimes\Pi'} \subseteq \mathrm{Y}$ would imply $\operatorname{supp} f_{\Pi\otimes\Pi'} = \{0\}$. This would mean that the wavefront set of Π' is 0, i.e. Π' is finite dimensional. By classification, see Appendix H all highest weight representations occurring in Howe's correspondence are infinite dimensional unless $\mathrm{G}' = \mathrm{U}_{l'}$, which is compact. In this case, the intertwining distribution is a smooth function; see [MPP23]. In particular, its support is not 0. Hence the intertwining operator is not a differential operator. □

6. The Pair $(\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$

We consider here the case $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$. By (E.6) and Proposition E.1, we can identify

$$\widetilde{\mathrm{O}}_2 = \{(g;\zeta) \in \mathrm{O}_2 \times \mathbb{C}^\times; \zeta^2 = (\det g)^{l'}\}\,.$$

and the $\det^{1/2}$-covering $\widetilde{\mathrm{O}}_2 \ni (g;\zeta) \to g \in \mathrm{O}_2$ splits if and only if l' is even. Let $\Pi \in \widetilde{\mathrm{O}}_2$ occur in Howe's correspondence and let $\chi_+ : \widetilde{\mathrm{O}}_2 \to \mathbb{C}^\times$ be the character of $\widetilde{\mathrm{O}}_2$ defined by (81).

Since Π is genuine, there is $\Pi_0 \in \widehat{\mathrm{O}_2}$ such that $\Pi_0(g) = (\Pi \otimes \chi_+^{-1})(\widetilde{g})$. Accordingly,

$$\int_{\mathrm{O}_2} \check{\Theta}_\Pi(\widetilde{g})\omega(\widetilde{g})\, dg = \int_{\mathrm{O}_2} \check{\Theta}_{\Pi_0}(g)\omega_0(g)\, dg\,,$$

where ω_0 is as in (83).

Observe that the image under the metaplectic cover of $\mathrm{supp}(\Theta_\Pi)$ is equal to $\mathrm{supp}(\Theta_{\Pi_0})$. Since $\widetilde{\mathrm{SO}_2} \to \mathrm{SO}_2$ splits by (E.10), we conclude that Θ_Π is supported in $\widetilde{\mathrm{G}^0} = \widetilde{\mathrm{SO}_2}$ if and only if Θ_{Π_0} is supported in SO_2. In the sequel, triv denotes the trivial representation.

Proposition 15. *Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$ and let Π be a genuine irreducible representation of $\widetilde{\mathrm{G}}$ with character Θ_Π not supported in $\widetilde{\mathrm{G}^0}$. Then either $\Pi = \widetilde{\mathrm{triv}} = \chi_+$, or $\Pi = \widetilde{\det}$ is the character of $\widetilde{\mathrm{G}}$ such that $(\widetilde{\det} \otimes \chi_+^{-1})(\tilde{g}) = \det(g)$ for all $\tilde{g} \in \widetilde{\mathrm{G}}$.*

Decompose $\mathrm{W} = \mathrm{M}_{2,2l'}(\mathbb{R})$ as $\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2$, where W_1 is subspace of the $w \in \mathrm{W}$ for which all entries of the second row are 0 and W_2 is subspace of the $w \in \mathrm{W}$ for which all entries of the first row are 0. Then

$$\int_{(\mathrm{SO}_2)s} \chi_+^{-1}(\widetilde{g})T(\widetilde{g})(\phi)\, dg = \mu_{\mathcal{O}}(\phi)\,, \tag{98}$$

where s is as in (88), $\mathcal{O}$ is the $\mathrm{O}_2 \times \mathrm{Sp}_{2l'}(\mathbb{R})$-orbit of $n_0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \mathrm{W}$ and $\mu_{\mathcal{O}} \in \mathcal{S}'(\mathrm{W})$ is the invariant measure on $\mathcal{O}$ defined by

$$\mu_{\mathcal{O}}(\phi) = 2^{l'-1} \int_{\mathrm{W}_2} \int_{\mathrm{O}_2} \phi(gw)\, dg\, d\mu_{\mathrm{W}_2}(w) \qquad (\phi \in \mathcal{S}(\mathrm{W}))\,. \tag{99}$$

Therefore

$$(100)\quad \int_{\mathrm{O}_2} \check{\Theta}_{\widetilde{\det}}(\widetilde{g})T(\widetilde{g})(\phi)\,dg = \int_{\mathrm{SO}_2} \chi_+^{-1}(\widetilde{g})T(\widetilde{g})(\phi)\,dg - \mu_{\mathcal{O}}(\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W}))$$

and

$$(101)\quad \int_{\mathrm{O}_2} \check{\Theta}_{\widetilde{\mathrm{triv}}}(\widetilde{g})T(\widetilde{g})(\phi)\,dg = \int_{\mathrm{SO}_2} \chi_+^{-1}(\widetilde{g})T(\widetilde{g})(\phi)\,dg + \mu_{\mathcal{O}}(\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W}))\,.$$

The integral over SO_2 *is computed by Theorem 4.*

If $l' = 1$, *then* $\widetilde{\det}$ *does not occur in Howe correspondence and hence*

$$\int_{\mathrm{O}_2} \check{\Theta}_{\widetilde{\det}}(\tilde{g})T(\tilde{g})\,dg = 0\,.$$

Moreover,

$$(102)\qquad \int_{\mathrm{O}_2} \check{\Theta}_{\widetilde{\mathrm{triv}}}(\widetilde{g})T(\widetilde{g})\;dg = 2\int_{\mathrm{SO}_2} \chi_+^{-1}(\widetilde{g})T(\widetilde{g})\;dg = 2\mu_{\mathcal{O}}\,.$$

Proof. For $n \in \mathbb{Z}$, let ρ_n be the character of SO_2 defined by

$$\rho_n\Big(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\Big) = e^{in\theta}\,.$$

Up to equivalence, the irreducible representations of O_2 are of the form $\Pi_{0,n} = \mathrm{Ind}_{\mathrm{SO}_2}^{\mathrm{O}_2}(\rho_n)$ with $n > 0$, together with the trivial representation triv and det. (Moreover, $\Pi_{0,n} \simeq \Pi_{0,-n}$ and $\Pi_{0,0} = 1 \oplus \det$.) Hence Θ_{Π_0} does not have support contained in $\widetilde{\mathrm{SO}_2}$ if and only if $\Pi_0|_{\mathrm{SO}_2} = 1$. Hence the only possible cases are triv and det.

Since

$$\check{\Theta}_{\widetilde{\mathrm{triv}}}(\tilde{g})T(\tilde{g}) = \chi_+^{-1}(\tilde{g})T(\tilde{g})\,,$$
$$\check{\Theta}_{\widetilde{\det}}(\tilde{g})T(\tilde{g}) = \check{\Theta}_{\widetilde{\det}}(\tilde{g})\chi_+(\tilde{g})\chi_+^{-1}(\tilde{g})T(\tilde{g}) = \det(g)\chi_+^{-1}(\tilde{g})T(\tilde{g})\,,$$

we see that

$$\int_{\mathrm{O}_2} \check{\Theta}_{\widetilde{\mathrm{triv}}}(\tilde{g})T(\tilde{g})\,dg = \int_{\mathrm{SO}_2} \chi_+^{-1}(\tilde{g})T(\tilde{g})\,dg + \int_{(\mathrm{SO}_2)s} \chi_+^{-1}(\tilde{g})T(\tilde{g})\,dg\,,$$
$$\int_{\mathrm{O}_2} \check{\Theta}_{\widetilde{\det}}(\tilde{g})T(\tilde{g})\,dg = \int_{\mathrm{SO}_2} \chi_+^{-1}(\tilde{g})T(\tilde{g})\,dg - \int_{(\mathrm{SO}_2)s} \chi_+^{-1}(\tilde{g})T(\tilde{g})\,dg\,.$$

We now compute the integral over $(\mathrm{SO}_2)s$. Let $g_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \in \mathrm{SO}_2$ and recall from (88) that $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{O}_2 \setminus \mathrm{SO}_2$. Then $g_t s = g_{t/2} s g_{-t/2}$. If f is any function on $(\mathrm{SO}_2)s$, then

$$\begin{aligned}\int_{(\mathrm{SO}_2)s} f(g)\,dg &= \int_{\mathrm{SO}_2} f(g_t s) dg_t = \frac{1}{2\pi}\int_0^{2\pi} f(g_t s)\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(g_{t/2} s g_{-t/2})\,dt \\ &= \frac{1}{2\pi}\int_0^{\pi} f(g_t s g_{-t})\cdot 2dt = \frac{1}{2\pi}\int_0^{\pi} f(g_t s g_{-t})\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} f(g_t s g_{-t})\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(g_t s g_{-t})\,dt = \int_{\mathrm{SO}_2} f(g_{-t} s g_t)\,dg_t\,.\end{aligned}$$

Applying this to $\mathrm{SO}_2 \ni g \to \chi_+^{-1}(\widetilde{g})T(\widetilde{g}) \in \mathcal{S}'(\mathbb{R})$, we get

$$\int_{(\mathrm{SO}_2)s} \chi_+^{-1}(\widetilde{g})T(\widetilde{g})\,dg = \int_{\mathrm{SO}_2} \chi_+^{-1}(\widetilde{g^{-1}sg})T(\widetilde{g^{-1}sg})\,dg\,. \tag{103}$$

Decompose $\mathrm{W} = \mathrm{M}_{2,2l'}(\mathbb{R})$ as in the statement of the theorem and let $g \in \mathrm{O}_2$. Then $\mathrm{W} = g^{-1}\mathrm{W}_1 \oplus g^{-1}\mathrm{W}_2$ is an orthogonal decomposition such that $g^{-1}sg$ preserves both $g^{-1}\mathrm{W}_1$ and $g^{-1}\mathrm{W}_2$. Notice that

$$\begin{aligned} g^{-1}sg|_{g^{-1}\mathrm{W}_1} &= 1_{g^{-1}\mathrm{W}_1} \quad \text{because } s|_{\mathrm{W}_1} = 1\,, \\ g^{-1}sg|_{g^{-1}\mathrm{W}_2} &= -1_{g^{-1}\mathrm{W}_2} \quad \text{because } s|_{\mathrm{W}_2} = -1\,. \end{aligned}$$

By Lemma G.1,

$$\begin{aligned}\chi_+^{-1}(\widetilde{g^{-1}sg})T_{\mathrm{W}}(\widetilde{g^{-1}sg}) = \chi_+^{-1}(\widetilde{1_{g^{-1}\mathrm{W}_1}})T_{\mathrm{W}}(\widetilde{1_{g^{-1}\mathrm{W}_1}}) \\ \otimes \chi_+^{-1}(\widetilde{-1_{g^{-1}\mathrm{W}_2}})T_{\mathrm{W}}(\widetilde{-1_{g^{-1}\mathrm{W}_2}})\,,\end{aligned} \tag{104}$$

independently of the choices of the preimages of $g^{-1}sg$, $1_{g^{-1}\mathrm{W}_1}$ and $-1_{g^{-1}\mathrm{W}_2}$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$, $\widetilde{\mathrm{Sp}}(g^{-1}\mathrm{W}_1)$ and $\widetilde{\mathrm{Sp}}(g^{-1}\mathrm{W}_2)$, respectively. We can therefore fix $\widetilde{1_{g^{-1}\mathrm{W}_1}}$ to be the identity element of $\widetilde{\mathrm{Sp}}(g^{-1}\mathrm{W}_1)$, which gives $\chi_+^{-1}(\widetilde{1_{g^{-1}\mathrm{W}_1}}) = 1$. Hence

$$\chi_+^{-1}(\widetilde{1_{g^{-1}\mathrm{W}_1}})T_{\mathrm{W}}(\widetilde{1_{g^{-1}\mathrm{W}_1}}) = \delta_{0,g^{-1}\mathrm{W}_1}\,,$$

where $\delta_{0,g^{-1}\mathrm{W}_1}$ indicates Dirac's delta at 0 in the space $g^{-1}\mathrm{W}_1$.

By [AP14, Definition 4.16 and Remark 4.5], $\Theta_{\mathrm{W}}^2(-1) = (-2i)^{\dim \mathrm{W}}$. Hence $|\Theta_{\mathrm{W}}(\widetilde{-1})| = 2^{\dim \mathrm{W}/2}$ only depends on the dimension of W. In particular,

$$|\Theta_{g^{-1}\mathrm{W}_2}(\widetilde{-1})| = |\Theta_{\mathrm{W}_2}(\widetilde{-1})| = 2^{\dim \mathrm{W}_2/2}\,.$$

So

$$\chi_+^{-1}(-\widetilde{1_{g^{-1}\mathrm{W}_2}})T_{\mathrm{W}}(-\widetilde{1_{g^{-1}\mathrm{W}_2}}) = |\Theta_{g^{-1}\mathrm{W}_2}(\widetilde{-1})|\mu_{g^{-1}\mathrm{W}_2} = 2^{\dim \mathrm{W}_2/2}\mu_{g^{-1}\mathrm{W}_2}\,.$$

Thus (104) becomes

$$\chi_+^{-1}(\widetilde{g^{-1}sg})T_{\mathrm{W}}(\widetilde{g^{-1}sg}) = 2^{\dim \mathrm{W}_2/2}\delta_{0,g^{-1}\mathrm{W}_1}\otimes\mu_{g^{-1}\mathrm{W}_2}\,. \tag{105}$$

By (103), for all $\phi\in\mathcal{S}(\mathrm{W})$,

$$\begin{aligned}\int_{(\mathrm{SO}_2)s}\chi_+^{-1}(\widetilde{g})T(\widetilde{g})(\phi)\,dg &= 2^{\dim \mathrm{W}_2/2}\int_{\mathrm{SO}_2}(\delta_{0,g^{-1}\mathrm{W}_1}\otimes\mu_{g^{-1}\mathrm{W}_2})(\phi)\,dg\\ &= 2^{\dim \mathrm{W}_2/2}\int_{\mathrm{SO}_2}\int_{g^{-1}\mathrm{W}_2}\phi(w)\;d\mu_{g^{-1}\mathrm{W}_2}(w)\,dg\\ &= 2^{\dim \mathrm{W}_2/2}\int_{\mathrm{W}_2}\int_{\mathrm{SO}_2}\phi(gw)\;dg\,d\mu_{\mathrm{W}_2}(w)\,.\end{aligned}$$

Notice that, since $sw=-w$ for $w\in\mathrm{W}_2$,

$$\begin{aligned}\int_{\mathrm{W}_2}\int_{\mathrm{SO}_2}\phi(gw)\;dg\,d\mu_{\mathrm{W}_2}(w) &= \int_{\mathrm{W}_2}\int_{\mathrm{SO}_2}\phi(-gw)\;dg\,d\mu_{\mathrm{W}_2}(w)\\ &= \int_{\mathrm{W}_2}\int_{\mathrm{SO}_2}\phi(gsw)\;dg\,d\mu_{\mathrm{W}_2}(w)\\ &= \int_{\mathrm{W}_2}\int_{(\mathrm{SO}_2)s}\phi(gw)\;dg\,d\mu_{\mathrm{W}_2}(w)\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{(\mathrm{SO}_2)s}\chi_+^{-1}(\widetilde{g})T(\widetilde{g})(\phi)\,dg\\ &= 2^{\dim \mathrm{W}_2/2}\Big(\frac{1}{2}\int_{\mathrm{W}_2}\int_{\mathrm{SO}_2}\phi(gw)\;dg\,d\mu_{\mathrm{W}_2}(w)+\frac{1}{2}\int_{\mathrm{W}_2}\int_{(\mathrm{SO}_2)s}\phi(gw)\;dg\,d\mu_{\mathrm{W}_2}(w)\Big)\\ &= 2^{\dim \mathrm{W}_2/2-1}\int_{\mathrm{W}_2}\int_{\mathrm{O}_2}\phi(gw)\;dg\,d\mu_{\mathrm{W}_2}(w)\,.\end{aligned}$$

In conclusion,

$$\int_{(\mathrm{SO}_2)s}\chi_+^{-1}(\widetilde{g})T(\widetilde{g})(\phi)\,dg = \mu_{\mathcal{O}}(\phi)\qquad(\phi\in\mathcal{S}(\mathrm{W}))\,,$$

where $\mu_{\mathcal{O}}$ is as in (99).

We now show that $\mu_{\mathcal{O}}$ is a $\mathrm{O}_2 \times \mathrm{Sp}_{2l'}(\mathbb{R})$-invariant measure on the orbit $\mathcal{O}$. Notice first that $\mathrm{W}_2 \setminus \{0\} = \mathrm{Sp}_{2l'}(\mathbb{R}).n_0$. Indeed, $n_0 \in \mathrm{W}_2$ and $\mathrm{Sp}_{2l'}(\mathbb{R})$ preserves W_2. Conversely, let $w_2 = \begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix} \in \mathrm{W}_2 \setminus \{0\}$, where $u, v \in \mathrm{M}_{1,l'}(\mathbb{R})$. Since $J = \begin{pmatrix} 0 & I_{l'} \\ -I_{l'} & 0 \end{pmatrix} \in \mathrm{Sp}_{2l'}(\mathbb{R})$ and $w_2 J = \begin{pmatrix} 0 & 0 \\ -v & u \end{pmatrix}$, we can suppose that $u \neq 0$. If $a \in \mathrm{GL}_{l'}(\mathbb{R})$ has u as its first row and b is a symmetric matrix having v as its first row, then $\begin{pmatrix} a & b \\ 0 & (a^t)^{-1} \end{pmatrix} \in \mathrm{Sp}_{2l'}(\mathbb{R})$ and $n_0 \begin{pmatrix} a & b \\ 0 & (a^t)^{-1} \end{pmatrix} = w_2$. It follows from this that $\{gw_2; g \in \mathrm{O}_2, w_2 \in \mathrm{W}_2\} = \mathcal{O} \cup \{0\}$. The right-hand side of (99) is clearly O_2-invariant, and we see that it is $\mathrm{Sp}_{2l'}(\mathbb{R})$-invariant by linear changes of variables in the integral over W_2 because the elements of $\mathrm{Sp}_{2l'}(\mathbb{R})$ have determinant 1.

Let $l' = 1$. By Proposition F.1, $\widetilde{\det}$ does not occur in Howe correspondence. Let $\Pi = \widetilde{\mathrm{triv}}$. Since $\widetilde{\det}$ does not occur, the projection onto the O_2-isotypic component is equal to the projection onto the-SO_2 isotypic component. Therefore, (102) follows, because the volume of SO_2 is $\frac{1}{2}$.

Since (100) vanishes when $l' = 1$, we have

$$\int_{\mathrm{O}_2} \check{\Theta}_{\widetilde{\mathrm{triv}}}(\tilde{g}) T(\tilde{g})\, dg = 2 \int_{\mathrm{SO}_2} \chi_+^{-1}(\tilde{g}) T(\tilde{g})\, dg = 2\mu_{\mathcal{O}}\,. \quad \square$$

Remark 12. Formulas (100) and (101) show that $\widetilde{\det}$ and $\widetilde{\mathrm{triv}}$ occur in the Howe correspondence when $l' > 1$. This is compatible with the classification, as for $l' > 1$ the the dual pair $(\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$ is in the stable range, so all genuine representations occur.

6.1. The special case $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}))$

In this case, $\mathrm{H} = \mathrm{SO}_2$ and $\mathfrak{g} = \mathfrak{h} = \mathbb{R}J_1$, where $J_1 = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Moreover, $\tau(\mathfrak{h}_{\overline{1}}) = \mathbb{R}^+ J_1$ and $\mathfrak{h} \cap \tau(\mathrm{W}) = \mathfrak{h}$. The Harish-Chandra parameter of $\Pi \in \widetilde{\mathrm{O}}_2^{\wedge}$ (which coincides with its highest weight since $\rho = 0$) is of the form μe_1, where $\mu \geq 0$ is an integer. Hence, in the notation (69), $a = -b = -\mu$ and $\beta = 2\pi$.

If $\mu = 0$, then $P_{-\mu,\mu} = 0$. If $\mu > 0$, then the function $P_{-\mu,\mu}$ is supported

in $[0,+\infty)$ and, by (D.4) and Remark 17,

$$P_{-\mu,\mu,2}(2\pi y_1) = 2(-1)^{\mu-1} L^1_{\mu-1}(4\pi y_1) = 2(-1)^{\mu-1} \sum_{h=0}^{\mu-1} \binom{\mu}{\mu-1-h} \frac{(-4\pi y_1)^h}{h!}\,, \tag{106}$$

where $L^1_{\mu-1}$ is a Laguerre polynomial. Moreover, by (D.5), $Q_{-\mu,\mu}(y) = 2\pi(-1)^\mu$ for all $\mu \geq 0$.

Suppose first $\mu > 0$. Then Π is supported in $\widetilde{\mathrm{SO}_2}$ and, by Lemma 3 and Theorem 4, for every $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned} f_{\Pi\otimes\Pi'}(\phi) &= \int_{\mathrm{SO}_2} \check{\Theta}_\Pi(\widetilde{g}) T(\widetilde{g})(\phi)\, dg \\ &= 2\pi C(-1)^\mu \int_0^{+\infty} P_{-\mu,\mu,2}(2\pi y_1) e^{-2\pi y_1} F_\phi(y_1 J_1)\, dy_1 \\ &\quad + C \int_{\mathfrak{h}} \delta_0(y) F_\phi(y)\, dy\,, \end{aligned} \tag{107}$$

where C is the constant appearing in Theorem 4. To make formula (107) explicit, we need to calculate the terms involving $F(y)$, the Harish-Chandra regular almost-elliptic orbital integral on W.

By [MPP20, Definition 3.1, (39) and (27)] and (I.2) with $\mathrm{Z}' = \mathrm{H}'$, there are constants $C_{\mathfrak{h}_{\overline{1}}}$ and $C'_{\mathfrak{h}_{\overline{1}}}$ such that, for all $y = y_1 J_1 = \tau(w) \in \tau(\mathfrak{h}_{\overline{1}})$,

$$\begin{aligned} F_\phi(y) &= C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y') \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}}_{\overline{1}}) \\ &= C'_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y') \int_{\mathrm{G}'/\mathrm{H}'} \psi(g'.y')\, d(g'\mathrm{H}')\,, \end{aligned} \tag{108}$$

where $y' = y_1 J'_1 = y_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tau'(w)$, and $\psi = \tau'_*(\phi^{\mathrm{G}}) \in \mathcal{S}(\mathfrak{g}')$. The right-hand side of (108) is Harish-Chandra's orbital integral for the orbit $\mathrm{G}'.y'$.

Notice that, for $\mathrm{G} = \mathrm{O}_2$ and $l = 1 \leq l'$, the extension of $F(y)$ from $y \in \mathfrak{h}^+ = \tau(\mathfrak{h}_{\overline{1}})$ to $-\tau(\mathfrak{h}_{\overline{1}})$ is even in y; see [MPP20, Theorem 3.6]. Hence,

$$\int_{\mathfrak{h}} \delta_0(y) F_\phi(y)\, dy = \lim_{y_1 \to 0+} F_\phi(y_1 J_1) \qquad (\phi \in \mathcal{S}(\mathrm{W}))\,.$$

Write $x \in \mathfrak{g}'$ as

$$x = x_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_3 J_1' = \begin{pmatrix} x_1 & x_2 + x_3 \\ x_2 - x_3 & -x_1 \end{pmatrix} = A(x_1, x_2, x_3)\,,$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$. Then the map $A : \mathbb{R}^3 \to \mathfrak{g}'$ is a linear isomorphism. It transfers the adjoint action of G' on $\mathfrak{g}'$ to the natural action on $\mathbb{R}^3$ by $\mathrm{SO}(2,1)^0$, the identity component of $\mathrm{SO}(2,1)$, i.e. the group of isometries of $x_1^2 + x_2^2 - x_3^2 = -\det(A(x_1, x_2, x_3))$ preserving the positive light cone

$$X^{0+} = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 = x_3^2, x_3 > 0\}\,.$$

See [HT92, Chapter IV, §5.1]. Under the map A, the orbit $\mathrm{G}'.y'$ with $y' = y_1 J_1'$ and $y_1 > 0$ is the image of the hyperboloid's upper sheet

$$O_{y_1}^- = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 - x_3^2 = -y_1^2, x_3 > 0\}\,.$$

Under A, the positive light cone X^{0+} corresponds to the G'-orbit of $x_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Moreover $\mathrm{G}'.x_0 \simeq \mathrm{G}'/\mathrm{MN}$, where $\mathrm{M} = \{\pm 1\}$ and $\mathrm{N} = \exp(\mathbb{R}x_0) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; t \in \mathbb{R} \right\}$. As the geometry suggests, for suitable normalizations of the $\mathrm{SO}(2,1)^0$-invariant orbital measures,

$$\lim_{y_1 \to 0+} \int_{O_{y_1}^-} f \, d\mu_{O_{y_1}^-} = \int_{X^{0+}} f \, d\mu_{X^{0+}} \qquad (f \in \mathcal{S}(\mathbb{R}^3))\,.$$

Thus, for a suitable positive constant $C''_{\mathfrak{h}_{\overline{1}}}$

$$\int_{\mathfrak{h}} \delta_0(y) F_\phi(y)\, dy = C''_{\mathfrak{h}_{\overline{1}}} \int_{\mathrm{G}'/\mathrm{MN}} \psi(g'.x_0)\, d(g'\mathrm{MN}) \tag{109}$$
$$(\phi \in \mathcal{S}(\mathrm{W}), \psi \in \mathcal{S}(\mathfrak{g}')^{\mathrm{G}}, \psi \circ \tau' = \phi^{\mathrm{G}})\,.$$

Suppose now that $\mu = 0$. Then, by Proposition 15, $\Pi = \widetilde{\mathrm{triv}} = \chi_+$ and for $\phi \in \mathcal{S}(\mathrm{W})$,

$$f_{\widetilde{\mathrm{triv}} \otimes \widetilde{\mathrm{triv}}'}(\phi) = 2 \int_{\mathrm{SO}_2} \chi_+^{-1}(\widetilde{g}) T(\widetilde{g})(\phi)\, dg = 2C \int_{\mathfrak{h}} \delta_0(y) F_\phi(y)\, dy\,,$$

where $\widetilde{\mathrm{triv}}'$ denotes the representation of $\widetilde{\mathrm{Sp}_2}(\mathbb{R})$ in Howe correspondence with $\widetilde{\mathrm{triv}}$ and the last equality follows from Theorem 4.

7. Another Example: $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,p})$ and $\Pi = \widetilde{\mathrm{triv}}$

Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,p})$. Hence $l' = 2p$. Consider the trivial representation triv of U_l. In the Schrödinger model, with a polarization $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ preserved by G, we have

$$\omega(\widetilde{g})v(x) = \chi_+(\widetilde{g})v(g^{-1}x) \qquad (\widetilde{g} \in \widetilde{\mathrm{G}},\, v \in \mathcal{S}(\mathrm{X}),\, x \in \mathrm{X})\,, \tag{110}$$

where $\chi_+ : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{U}_1$ is a function whose restriction to $\widetilde{\mathrm{G}}$ is a character. See [AP14, Proposition 4.28]. Let $\widetilde{\mathrm{triv}}$ denote this restriction. Then $\widetilde{\mathrm{triv}}$ is the lift to $\widetilde{\mathrm{U}_l}$ of triv, which occurs in Howe's correspondence. Moreover, (110) implies that

$$\omega(\check{\Theta}_{\widetilde{\mathrm{triv}}})v(x) = \int_{\mathrm{G}} v(g^{-1}x)\, dg \qquad (v \in \mathcal{S}(\mathrm{X}),\, x \in \mathrm{X})\,.$$

Let $\widetilde{\mathrm{triv}}'$ be the representation of $\widetilde{\mathrm{U}}_{p,p}$ which corresponds to $\widetilde{\mathrm{triv}}$. If $l = 1$, then $\widetilde{\mathrm{triv}}'$ is a minimal representation of $\mathrm{U}_{p,p}$, called the Wallach representation.

In this section we are computing $f_{\widetilde{\mathrm{triv}} \otimes \widetilde{\mathrm{triv}}'}$, which is the Weyl symbol of the operator $\omega(\check{\Theta}_{\widetilde{\mathrm{triv}}})$. As in our main theorems, we distinguish the cases $l \le l'$ and $l > l'$. Notice first that the parameters appearing in (68) are

$$\beta = 2\pi \qquad \text{and} \qquad \delta = p + \frac{1-l}{2} = \frac{1+l'-l}{2}\,.$$

Moreover, $\rho = \sum_{j=1}^{l} \left(\frac{l+1}{2} - j\right) e_j$ for $\mathrm{G} = \mathrm{U}_l$.

7.0.1 The case $l \le l'$

The parameters (69) corresponding to $\Pi = \widetilde{\mathrm{triv}}$ are

$$a_j = -\frac{l'}{2} + j \qquad \text{and} \qquad b_j = -\frac{l'}{2} + l + 1 - j\,, \tag{111}$$

where $1 \le j \le l$. Observe that the a_j's and the b_j's describe the same set

$$\{-l'/2 + 1, \ldots, -l'/2 + l - 1, -l'/2 + l\}$$

and $b_{l+1-j} = a_j$ for all $1 \le j \le l$. Hence, by (D.6),

$$P_{a_{l+1-j}, b_{l+1-j}}(\xi) = P_{b_j, a_j}(\xi) = P_{a_j, b_j}(-\xi)\,, \tag{112}$$

$$Q_{a_{l+1-j}, b_{l+1-j}}(\xi) = Q_{b_j, a_j}(\xi) = Q_{a_j, b_j}(-\xi)\,. \tag{113}$$

Since $a_j = b_{l+1-j} \le 0$ for all $1 \le j \le \min(l, l'/2)$, by (D.2),

$$P_{a_j,b_j,-2}(\xi) = P_{a_{l+1-j},b_{l+1-j},2}(\xi) = 0 \qquad (1 \le j \le \min(l, l'/2))\,. \tag{114}$$

Also, $a_j \le 0$ for all j (and hence $b_j \le 0$ for all j) if and only if $l \le l'/2$. Furthermore, $a_j + b_j = l - l' + 1$, which is independent of j, is ≥ 1 if and only if $l = l'$. As a consequence (see (D.5)),

$$\begin{aligned} &P_{a_j,b_j} = 0 \text{ for all } 1 \le j \le l \quad \text{ if and only if } \quad l \le \frac{l'}{2}\,,\\ &Q_{a_j,b_j} \ne 0 \text{ for all } 1 \le j \le l \quad \text{ if } l < l'\,,\\ &Q_{a_j,b_j} = 0 \text{ for all } 1 \le j \le l \quad \text{ if } l = l'\,. \end{aligned}$$

We now examine more precisely the formula for $f_{\widetilde{\text{triv}}\otimes\widetilde{\text{triv}}}{}'$ when $l \le l'/2$. This is the stable range case. As remarked above, $P_{a_j,b_j} = 0$ for all $1 \le j \le l$, whereas (see (D.5))

$$Q_{a_j,b_j}(y_j) = 2\pi(1+y_j)^{-a_j}(1-y_j)^{-b_j}\,.$$

Hence $p_j = 0$ for all $1 \le j \le l$, whereas

$$q_j(-\partial_{y_j})^* = q_j(\partial_{y_j}) = \Big(1 + \frac{1}{2\pi}\partial_{y_j}\Big)^{\frac{l'}{2}-j}\Big(1 - \frac{1}{2\pi}\partial_{y_j}\Big)^{\frac{l'}{2}-(l-j+1)}\,,$$

where * denotes the formal adjoint. Theorem 4 yields for $\phi \in \mathcal{S}(\mathrm{W})$

$$\begin{aligned} f_{\widetilde{\text{triv}}\otimes\widetilde{\text{triv}}}{}'(\phi) &= \int_{\mathrm{U}_l} \check{\Theta}_{\widetilde{\text{triv}}}(\widetilde{g}) T(\widetilde{g})(\phi)\, dg\\ &= C\int_{\mathfrak{h}} \Big[\prod_{l=1}^{l} q_j(-\partial_{y_j})\delta_0(y_j)\Big] F_\phi(y)\, dy\\ &= C\Big[\Big(\prod_{l=1}^{l} q_j(\partial_{y_j})\Big) F_\phi\Big](0)\,, \end{aligned} \tag{115}$$

where C is a nonzero constant. Hence $f_{\widetilde{\text{triv}}\otimes\widetilde{\text{triv}}}{}'$ has support inside the nilpotent cone in W.

Another case where the formula for $f_{\widetilde{\text{triv}}\otimes\widetilde{\text{triv}}}{}'$ simplifies is when $l = l' = 2p$ because $Q_{a_j,b_j} = 0$ for all j. Since $a_j = b_{2p+1-j} \le 0$ for $1 \le j \le p$, we have

$$P_{a_j,b_j}(\xi) = \begin{cases} 2\pi P_{a_j,b_j,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi) & \text{if } 1 \le j \le p\,,\\ 2\pi P_{a_j,b_j,-2}(\xi)\mathbb{I}_{\mathbb{R}^-}(\xi) & \text{if } p+1 \le j \le 2p\,. \end{cases}$$

In particular, in this case, we can replace in (72) the domain of integration $\mathfrak{h}\cap\tau(\mathrm{W})$ with $\tau(\mathfrak{h}_{\overline{1}}^-)$, where $\mathfrak{h}_{\overline{1}}^-$ is the unique Cartan subspace of W and $\tau(\mathfrak{h}_{\overline{1}}^-)$ is determined by the condition that the first p values δ_j in (36) are equal to 1 and the last p are equal to -1. The explicit expression for $f_{\widetilde{\mathrm{triv}}\otimes\widetilde{\mathrm{triv}}'}$ can be easily computed using (72), (D.1) and (D.2). For instance, if $p=1$, i.e. $(\mathrm{G},\mathrm{G}')=(\mathrm{U}_2,\mathrm{U}_{1,1})$, then

$$f_{\widetilde{\mathrm{triv}}\otimes\widetilde{\mathrm{triv}}'}(\phi)=C\int_0^\infty\int_{-\infty}^0 e^{2\pi(y_2-y_1)}F_\phi(y_1,y_2)\,dy_2dy_1\qquad(\phi\in\mathcal{S}(\mathrm{W}))\,,$$

where C is a nonzero constant.

7.0.2 The case $l>l'$

In this case, $Q_{a_j,b_j}=0$. The Weyl group $W(\mathrm{U}_{p,p},\mathfrak{h}')$ acts on $\mathfrak{h}'$ by permuting the first p coordinates and the last p coordinates (see Remark 1). The parameter $a_{s,j}$ and $b_{s,j}$ appearing in (79) are therefore obtained by separately permuting the first $p=l'/2$ and the last p terms appearing in (111). Notice that

$$\begin{aligned}&a_j\le 0\quad\text{ if and only if }\quad 1\le j\le\tfrac{l'}{2}\,,\\&b_j\le 0\quad\text{ if and only if }\,l+1-\tfrac{l'}{2}\le j\le l\,.\end{aligned}$$

In particular, since $l>l'$, for each j, at most one between a_j and b_j can be ≤ 0. Moreover, there is at least one index j for which both a_j and b_j are positive, namely $j=\frac{l'}{2}+1$.

When $\mathrm{G}'=\mathrm{U}_{1,1}$ (and hence $l'=2$), then $W(\mathrm{U}_{1,1},\mathfrak{h}')$ is trivial and s_0 maps J_1 to itself and J_l to J_2, and (79) simplifies to a nonzero constant multiple of

$$\frac{P_{a_1,b_1,2}(2\pi y_1)P_{a_l,b_l,-2}(2\pi y_2)}{(y_2-y_1)(y_1y_2)^{l-2}}e^{-2\pi(y_1-y_2)}\qquad(y=\tau'(w),w\in\mathfrak{h}_{\overline{1}}^{-\,reg})\,,$$

where a_j,b_j are as in (111) and the denominator is the root product (A.4).

8. The Integral over $-\mathrm{G}^0$ as an Integral over $\mathfrak{g}$

Let $\mathfrak{sp}(\mathrm{W})$ be the Lie algebra of $\mathrm{Sp}(\mathrm{W})$. Set

$$\mathfrak{sp}(\mathrm{W})^c=\{x\in\mathfrak{sp}(\mathrm{W});\ x-1\text{ is invertible in }\mathrm{End}(\mathrm{W})\}\,,\tag{116}$$

$$\mathrm{Sp}(\mathrm{W})^c=\{g\in\mathrm{Sp}(\mathrm{W});\ g-1\text{ is invertible in }\mathrm{End}(\mathrm{W})\}\,.\tag{117}$$

The Cayley transform $c : \mathfrak{sp}(\mathrm{W})^c \to \mathrm{Sp}(\mathrm{W})^c$ is the bijective rational map defined by $c(x) = (x+1)(x-1)^{-1}$. Its inverse $c^{-1} : \mathrm{Sp}(\mathrm{W})^c \to \mathfrak{sp}(\mathrm{W})^c$ is given by the same formula, $c^{-1}(g) = (g+1)(g-1)^{-1}$.

Since all eigenvalues of $x \in \mathfrak{g} \subseteq \mathrm{End}(\mathrm{W})$ are purely imaginary, $x-1$ is invertible. Therefore $\mathfrak{g} \subseteq \mathfrak{sp}(\mathrm{W})^c$. Moreover, $c(\mathfrak{g}) \subseteq \mathrm{G}$. Since the map c is continuous, the range $c(\mathfrak{g})$ is connected. Also, $-1 = c(0)$ is in $c(\mathfrak{g})$. Furthermore, for $x \in \mathfrak{g}$,

$$c(x) - 1 = (x+1)(x-1)^{-1} - (x-1)(x-1)^{-1} = 2(x-1)^{-1}$$

is invertible. Hence $c(\mathfrak{g}) \subseteq \mathrm{G} \cap \mathrm{Sp}(\mathrm{W})^c$. This is an equality because $c(c(y)) = y$ and $c(\mathrm{G}) \subseteq \mathfrak{g}$. Thus

$$c(\mathfrak{g}) = \{g \in \mathrm{G};\ \det(g-1) \neq 0\}\,.$$

This is a connected open dense subset of $-\mathrm{G}^0$. Hence

$$\int_{-\mathrm{G}^0} T(\tilde{g}) \check{\Theta}_\Pi(\tilde{g})\, dg = \int_{c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_\Pi(\tilde{g})\, dg\,. \tag{118}$$

If $\mathrm{G} \neq \mathrm{O}_{2l+1}$, then $\mathrm{G}^0 = -\mathrm{G}^0$. If $\mathrm{G} = \mathrm{O}_{2l+1}$, then G is the disjoint union of G^0 and $-\mathrm{G}^0$. Let

$$\tilde{c} : \mathfrak{g} \to \widetilde{\mathrm{G}} \tag{119}$$

be a real analytic lift of c. Set $\tilde{c}_-(x) = \tilde{c}(x)\tilde{c}(0)^{-1}$. Then $\tilde{c}_-(0)$ is the identity of the group $\widetilde{\mathrm{Sp}}(\mathrm{W})$. By (14), we have

$$T(\tilde{c}(x)) = \Theta(\tilde{c}(x))\, \chi_x\, \mu_{\mathrm{W}}\,. \tag{120}$$

Therefore, for a suitable normalization of the Lebesgue measure on $\mathfrak{g}$,

$$\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg = \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}(x))\, \Theta(\tilde{c}(x))\, j_{\mathfrak{g}}(x)\, \chi_x\, \mu_{\mathrm{W}}\, dx\,, \tag{121}$$

where $j_{\mathfrak{g}}(x)$ is the Jacobian of the map $c : \mathfrak{g} \to c(\mathfrak{g})$ (see Appendix B for its explicit expression). Also, since $\tilde{c}(0)$ is in the center of the metaplectic group,

$$\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg = \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}_-(x))\, \Theta(\tilde{c}(x))\, j_{\mathfrak{g}}(x)\, \chi_x\, \mu_{\mathrm{W}}\, dx\,, \tag{122}$$

where χ_Π is the central character of Π; see (62). In the rest of this paper we shall write $dw = d\mu_{\mathrm{W}}(w)$, when convenient.

9. The Invariant Integral over $\mathfrak{g}$ as an Integral over $\mathfrak{h}$

We now apply the Weyl integration formula to reduce the integral on $\mathfrak{g}$ in (122) to an integral on a Cartan subalgebra of $\mathfrak{g}$. In section 3, this Cartan subalgebra was denoted by $\mathfrak{h}(\mathfrak{g})$, see (45). To make our notation lighter, in this section we will write $\mathfrak{h}$ instead of $\mathfrak{h}(\mathfrak{g})$. Let $\mathrm{H} \subseteq \mathrm{G}$ be the corresponding Cartan subgroup. Fix a system of positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. For any positive root α let $\mathfrak{g}_{\mathbb{C},\alpha} \subseteq \mathfrak{g}_{\mathbb{C}}$ be the corresponding $\mathrm{ad}(\mathfrak{h}_{\mathbb{C}})$-eigenspace and let $X_\alpha \in \mathfrak{g}_{\mathbb{C},\alpha}$ be a non-zero vector. Let $\mathrm{H}^0 \subseteq \mathrm{H}$ denote the connected component of the identity. There is a character (continuous group homomorphism) $\xi_\alpha : \mathrm{H}^0 \to \mathbb{C}^\times$ such that

$$\mathrm{Ad}(h)X_\alpha = \xi_\alpha(h)X_\alpha \qquad (h \in \mathrm{H}^0)\,.$$

The derivative of ξ_α at the identity coincides with α. Let $\rho \in \mathfrak{h}^*_{\mathbb{C}}$ denote one half times the sum of all the positive roots. Then in all cases except when $\mathrm{G} = \mathrm{O}_{2l+1}$ or $\mathrm{G} = \mathrm{U}_l$ with l even, there is a character $\xi_\rho : \mathrm{H}^0 \to \mathbb{C}^\times$ whose derivative at the identity is equal to ρ, see [GW09, (2.21) and p. 145]. When $\mathrm{G} = \mathrm{O}_{2l+1}$ or $\mathrm{G} = \mathrm{U}_l$ with l even, the character ξ_ρ exists as a map defined on a non-trivial double cover

$$\widehat{\mathrm{H}^0} \ni \widehat{h} \to h \in \mathrm{H}^0 \tag{123}$$

of H^0. In particular the Weyl denominator

$$\Delta(h) = \xi_\rho(h) \prod_{\alpha>0} (1 - \xi_{-\alpha}(h)) \tag{124}$$

is defined for $h \in \mathrm{H}^0$ or $h \in \widehat{\mathrm{H}^0}$ according to the cases described above. We will see below how the Weyl group $W(\mathrm{G}, \mathfrak{h})$ acts on $\widehat{\mathrm{H}^0}$. The sign representation $\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}$ of the Weyl group $W(\mathrm{G}, \mathfrak{h})$ is defined by

$$\Delta(sh) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\Delta(h) \qquad (s \in W(\mathrm{G}, \mathfrak{h}))\,, \tag{125}$$

where either $h \in \mathrm{H}^0$ or $h \in \widehat{\mathrm{H}^0}$.

Suppose first that $\mathrm{G} = \mathrm{O}_{2l+1}$. Then $\mathrm{H} = \mathrm{H}^0 \cdot \mathrm{Z} = \mathrm{H}^0 \times \mathrm{Z}$ is the direct product of H^0 and the center Z of $\mathrm{Sp(W)}$. The group $\widehat{\mathrm{H}^0}$ and the action of the Weyl group on it are described in Appendix C. The double cover of

H is $\widetilde{\mathrm{H}} = \mathrm{H}^0 \times \widetilde{\mathrm{Z}}$. Set $\widehat{\widetilde{\mathrm{H}}} = \widehat{\mathrm{H}^0} \times \widetilde{\mathrm{Z}}$. We have a chain of double covering homomorphisms

$$\begin{aligned} \widehat{\widetilde{\mathrm{H}}} = \widehat{\mathrm{H}^0} \times \widetilde{\mathrm{Z}} \longrightarrow \mathrm{H}^0 \times \widetilde{\mathrm{Z}} \longrightarrow \mathrm{H}^0 \times \mathrm{Z} \longrightarrow \mathrm{H}^0\,, \\ (\widehat{h}, \tilde{z}) \longrightarrow (h, \tilde{z}) \longrightarrow (h, z) \longrightarrow h\,. \end{aligned} \tag{126}$$

We extend Δ, ξ_μ and $\check{\Theta}_\Pi$ to $\widehat{\widetilde{\mathrm{H}}}$ by defining $\Delta(\hat{h}, \tilde{z}) = \Delta(\hat{h})$ and $\xi_\mu(\hat{h}, \tilde{z}) = \xi_\mu(\hat{h})$ or $\xi_\mu(h)$ if it exists, and $\check{\Theta}_\Pi(\hat{h}, \tilde{z}) = \check{\Theta}_\Pi(h, \tilde{z})$. Recall from (C.3) the section

$$\widehat{c}_- : \mathfrak{h} \ni x \to \widehat{\mathrm{H}^0}$$

and define

$$\widehat{c}_- : \mathfrak{h} \ni x \to (\widehat{c}_-(x), 1) \in \widehat{\widetilde{\mathrm{H}}}\,. \tag{127}$$

This is a real analytic lift of the modified Cayley transform defined on $\mathfrak{h}$ by

$$c_-(x) = (1+x)(1-x)^{-1} = -c(x)\,. \tag{128}$$

Suppose now that $\mathrm{G} = \mathrm{U}_l$. Then $\mathrm{H}^0 = \mathrm{H}$. Consider the case when l is even. If $\mathrm{G}' = \mathrm{U}_{p,q}$ with $p+q$ odd, then the covering $\widetilde{\mathrm{H}} \to \mathrm{H}$ does not split (see Proposition E.1). Hence Δ, ξ_μ and $\check{\Theta}_\Pi$ are defined on $\widehat{\mathrm{H}} = \widetilde{\mathrm{H}}$ and the Weyl group of H acts on $\widehat{\mathrm{H}}$ in a way compatible with the cover $\widetilde{\mathrm{H}} \to \mathrm{H}$. We have the modified Cayley transform $c_- : \mathfrak{h} \to \mathrm{H}$, an analytic section $\sigma : c_-(\mathfrak{h}) \to \widehat{\mathrm{H}}$ and the map

$$\widehat{c}_- : \mathfrak{h} \ni x \to \sigma(c_-(x)) \in \widehat{\mathrm{H}}\,. \tag{129}$$

If $\mathrm{G}' = \mathrm{U}_{p,q}$ with $p+q$ even, then define $\widehat{\mathrm{H}}$ to be the Cartan subgroup of the group $\sqrt{\mathrm{G}}$ defined in Proposition E.1 covering H. (In particular, we have the action of the Weyl group $W(\mathrm{G}, \mathfrak{h})$ on $\widehat{\mathrm{H}}$ because $W(\mathrm{G}, \mathfrak{h}) = W(\sqrt{\mathrm{G}}, \mathfrak{h})$.) Then Δ, ξ_μ and $\check{\Theta}_\Pi$ are defined on $\widehat{\mathrm{H}}$. By Proposition E.1, the metaplectic cover $\widetilde{\mathrm{H}} = \mathrm{H} \times \{1, \tilde{1}\}$ splits and we have maps

$$\begin{aligned} \widehat{\mathrm{H}} \longrightarrow \mathrm{H} \longrightarrow \widetilde{\mathrm{H}} \longrightarrow \mathrm{H}\,, \\ \widehat{h} \;\to\; h \;\to (h;1) \to h\,. \end{aligned} \tag{130}$$

Again Δ, ξ_μ and $\check{\Theta}_\Pi$ are defined on $\widehat{\mathrm{H}}$ and (129) defines the lift of the Cayley transform we shall use. In this case, we set $\widehat{\widetilde{\mathrm{H}}} = \widehat{\mathrm{H}}$.

For the remaining dual pairs, $\widehat{\mathrm{H}} = \mathrm{H}$ and we lift Δ and ξ_μ to functions on $\widetilde{\mathrm{H}}$ constant on the fibers of the covering map $\widetilde{\mathrm{H}} \to \mathrm{H}$ and write $\widehat{c}_-$ for $\tilde{c}_-$, which was defined under the equation (119).

LEMMA 16. *Let $\mu \in i\mathfrak{h}^*$. Then*

$$\xi_{-\mu}(\widehat{c}_-(x)) = \prod_{j=1}^{l} \left(\frac{1+ix_j}{1-ix_j}\right)^{\mu_j} = \prod_{j=1}^{l} (1+ix_j)^{\mu_j}(1-ix_j)^{-\mu_j} \qquad (x \in \mathfrak{h})\,. \tag{131}$$

PROOF. By (34), it is enough to verify this formula when $l = 1$. In this case, $x = x_1 J_1$ and $\mu = \mu_1 e_1 = -i\mu_1 J_1^*$. Let log denote the local inverse of the exponential map near 1. Then, for x sufficiently close to 0,

$$\log(c_-(x)) = \log\left((1+x)(1-x)^{-1}\right) = \log(1+x) - \log(1-x)$$

is a real analytic odd function of x. Hence it admits a Taylor series expansion

$$\sum_{n\geq 0} a_n x^{2n+1} = \sum_{n\geq 0} a_n (-1)^n x_1^{2n+1} J_1\,.$$

Thus

$$\mu(\log(c_-(x))) = -\sum_{n\geq 0} a_n(-1)^n x_1^{2n+1} i\mu_1 = -\sum_{n\geq 0} a_n (ix_1)^{2n+1}\mu_1 = \ln\left(\frac{1-ix_1}{1+ix_1}\right)\mu_1\,.$$

By taking exponentials, we obtain

$$\xi_{-\mu}(\widehat{c}_-(x)) = e^{-\mu(\log(c_-(x)))} = \left(\frac{1+ix_1}{1-ix_1}\right)^{\mu_1}\,,$$

and the result extends to all $x \in \mathfrak{h}$ by real analyticity. □

Let Π be an irreducible representation of $\widetilde{\mathrm{G}}$, and let $\mu \in i\mathfrak{h}^*$ represent the infinitesimal character of Π. When μ is dominant, then we will refer to it as the Harish-Chandra parameter of Π. This is consistent with the usual terminology; see e.g. [Kna86, Theorem 9.20]. Then the corresponding character ξ_μ is defined as $\xi_\mu = \xi_\rho\, \xi_{\mu-\rho}$, where $\xi_{\mu-\rho}$ is one of the extremal H^0-weights of Π. In these terms, Weyl's character formula looks as follows,

$$\Theta_\Pi(h)\Delta(h) = \kappa_0 \sum_{s\in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\xi_{s\mu}(h)\,, \tag{132}$$

where $h \in \widetilde{\mathrm{H}^0}$ or $h \in \widehat{\widetilde{\mathrm{H}^0}}$, according to the cases above, and κ_0 is as in (71).

Lemma 17. *Let $\pi_{\mathfrak{g}/\mathfrak{h}}$ be the product of the positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and let*

$$\kappa(x) = \kappa_0 \frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\Delta(\widehat{c}_-(x))}\, \Theta(\tilde{c}(x))\, j_{\mathfrak{g}}(x) \qquad (x \in \mathfrak{h})\,.$$

Then, for a suitable normalization of the Lebesgue measure on $\mathfrak{h}$ and any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned}\int_{-\mathrm{G}^0} &\check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg \\ &= \frac{\check{\chi}_\Pi(\tilde{c}(0))}{|W(\mathrm{G},\mathfrak{h})|} \int_{\mathfrak{h}} (\Theta_\Pi \Delta)(\widehat{c}_-(x)^{-1})\, \frac{\kappa(x)}{\kappa_0}\, \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w) \phi^{\mathrm{G}}(w)\, dw\, dx \\ &= \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \kappa(x) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w) \phi^{\mathrm{G}}(w)\, dw\, dx\,,\end{aligned}$$

where ϕ^{G} is as in (58) *and each consecutive integral is absolutely convergent.*

Proof. Applied to a test function $\phi \in \mathcal{S}(\mathrm{W})$, the first integral over $-\mathrm{G}^0$ and hence over $c(\mathfrak{g})$, is absolutely convergent because both, the character and the function $T(\widetilde{g})(\phi)$ are continuous and bounded (see for example [Prz93, Proposition 1.13]) and the group G is compact. Hence, each consecutive integral in the formula (122) applied to ϕ,

$$\begin{aligned}(133)\quad \int_{-\mathrm{G}^0} &\check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg \\ &= \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}_-(x))\, \Theta(\tilde{c}(x))\, j_{\mathfrak{g}}(x) \int_{\mathrm{W}} \chi_x(w) \phi(w)\, dw\, dx\,,\end{aligned}$$

is absolutely convergent. Since

$$\chi_{g.x}(w) = \chi_x(g^{-1}.w)$$

and the Lebesgue measure dw is G-invariant,

$$\int_{\mathrm{G}} \int_{\mathrm{W}} \chi_{g.x}(w) \phi(w)\, dw\, dg = \int_{\mathrm{W}} \chi_x(w) \phi^{\mathrm{G}}(w)\, dw\,.$$

Observe also that $\widetilde{\mathrm{Ad}}(\tilde{g}) = \mathrm{Ad}(g)$ and the characters $\check{\Theta}_\Pi$ and Θ are $\widetilde{\mathrm{G}}$-invariant. Moreover, by (124) and (131),

$$\overline{\Delta(\widehat{c}_-(x))} = \Delta(\widehat{c}_-(x)^{-1}) = (-1)^m \Delta(\widehat{c}_-(x)) \qquad (x \in \mathfrak{h})\,,$$

where m is the number of positive roots, and

$$\overline{\pi_{\mathfrak{g}/\mathfrak{h}}(x)} = (-1)^m \pi_{\mathfrak{g}/\mathfrak{h}}(x) \qquad (x \in \mathfrak{h})\,.$$

Therefore the Weyl integration formula on $\mathfrak{g}$ shows that (133) is equal to $\frac{\check{\chi}_\Pi(\tilde{c}(0))}{|W(\mathrm{G},\mathfrak{h})|}$ times

$$\begin{aligned}
&\int_{\mathfrak{h}} |\pi_{\mathfrak{g}/\mathfrak{h}}(x)|^2 \check{\Theta}_\Pi(\tilde{c}_-(x))\,\Theta(\tilde{c}(x))\,j_{\mathfrak{g}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw\,dx \\
&\quad = \int_{\mathfrak{h}} \check{\Theta}_\Pi(\widehat{c}_-(x))\overline{\Delta(\widehat{c}_-(x))} \left(\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\Delta(\widehat{c}_-(x))}\,\Theta(\tilde{c}(x))\,j_{\mathfrak{g}}(x) \right) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw\,dx \\
&\quad = \int_{\mathfrak{h}} \Theta_\Pi(\widehat{c}_-(x)^{-1})\Delta(\widehat{c}_-(x)^{-1}) \frac{\kappa(x)}{\kappa_0} \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw\,dx\,.
\end{aligned}$$

(Here, we suppose that the Haar measure on H is normalized to have total mass 1.) This verifies the first equality and the absolute convergence. By (132) and (143) below,

$$\begin{aligned}
\Theta_\Pi(\widehat{c}_-(x)^{-1})\Delta(\widehat{c}_-(x)^{-1}) &= \kappa_0 \sum_{s \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\xi_{s\mu}(\widehat{c}_-(x)^{-1}) \\
&= \kappa_0 \sum_{s \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\xi_{-s\mu}(\widehat{c}_-(x)) \\
&= \kappa_0 \sum_{s \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\xi_{-\mu}(\widehat{c}_-(s^{-1}x))\,.
\end{aligned}$$

Since $\chi_{sx}(w) = \chi_x(s^{-1}w)$ and ϕ^{G} and the Lebesgue measure dw are $W(\mathrm{G},\mathfrak{h})$-invariant, we see that the integral $\int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw$ is $W(\mathrm{G},\mathfrak{h})$-invariant as a function of x, too. The second equality in the statement of the lemma then follows from the skew-symmetry of $\pi_{\mathfrak{g}/\mathfrak{h}}$ and the $W(\mathrm{G},\mathfrak{h})$-invariance of κ, which is a consequence of Lemma 18 below. □

Since any element $x \in \mathfrak{g}$, viewed as an endomorphism of V over $\mathbb{R}$, has imaginary eigenvalues which come in complex conjugate pairs, we have $\det(1-x)_{\mathrm{V}_\mathbb{R}} \geq 1$. Define

$$\mathrm{ch}(x) = \det(1-x)^{1/2}_{\mathrm{V}_\mathbb{R}} \qquad (x \in \mathfrak{g})\,. \tag{134}$$

Recall the symbols r and ι from (66) and (65).

Lemma 18. *There is a constant C which depends only on the dual pair $(\mathrm{G}, \mathrm{G}')$ such that*

$$\frac{\kappa(x)}{\kappa_0} = C \operatorname{ch}^{d'-r-\iota}(x) \qquad (x \in \mathfrak{h})\,.$$

Proof. Recall [Prz93, Lemma 5.7] that $\pi_{\mathfrak{g}/\mathfrak{h}}(x)$ is a constant multiple of $\Delta(\widehat{c}_-(x)) \operatorname{ch}^{r-\iota}(x)$,

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) = C\Delta(\widehat{c}_-(x)) \operatorname{ch}^{r-\iota}(x)\,. \tag{135}$$

For the orthogonal groups this is verified in Appendix C. It is easy to compute from [AP14, Definition 4.16], that

$$\Theta(\tilde{c}(x))^2 = i^{\dim \mathrm{W}} \det\big(2^{-1}(x-1)\big)_{\mathrm{W}} \qquad (x \in \mathfrak{sp}(\mathrm{W})\,,\ \det(x-1) \neq 0)\,. \tag{136}$$

Hence there is a choice of $\tilde{c}$ so that

$$\Theta(\tilde{c}(x)) = \left(\frac{i}{2}\right)^{\frac{1}{2}\dim \mathrm{W}} \det\big(1-x\big)_{\mathrm{W}}^{\frac{1}{2}} \qquad (x \in \mathfrak{g})\,. \tag{137}$$

Furthermore, since the symplectic space may be realized as $\mathrm{W} = \mathrm{Hom}_{\mathbb{D}}(\mathrm{V}', \mathrm{V})$, see (31), we obtain that

$$\det\big(1-x\big)_{\mathrm{W}} = \det(1-x)_{\mathrm{V}_{\mathbb{R}}}^{d'} \qquad (x \in \mathfrak{g})\,. \tag{138}$$

Also, as checked in [Prz91, (3.11)], the Jacobian of $\tilde{c}_- : \mathfrak{g} \to \mathrm{G}$ is a constant multiple of $\operatorname{ch}^{-2r}(x)$. (For reader's convenience a –slightly different– proof is included in Appendix B.) Hence the claim follows. □

Corollary 19. *For any $\phi \in \mathcal{S}(\mathrm{W})$*

$$\int_{-\mathrm{G}^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi)\, dg = C\,\kappa_0 \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \times \int_{\mathrm{W}} \chi_x(w) \phi^{\mathrm{G}}(w)\, dw\, dx\,,$$

where C is a constant which depends only on the dual pair $(\mathrm{G}, \mathrm{G}')$, ϕ^{G} is as in (58), and each consecutive integral is absolutely convergent.

10. An Intertwining Distribution in Terms of Orbital Integrals on the Symplectic Space

We keep the notation introduced in section 3. Let

$$W(\mathrm{G},\mathfrak{h}(\mathfrak{g})) = \begin{cases} \Sigma_l & \text{if } \mathbb{D}=\mathbb{C}\,, \\ \Sigma_l \ltimes \{\pm 1\}^l & \text{otherwise}\,. \end{cases} \tag{139}$$

Denote the elements of Σ_l by η and the elements of $\{\pm 1\}^l$ by $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_l)$, so that an arbitrary element of the group (139) is of the form $t = \epsilon\eta$, with $\epsilon = (1, 1, \ldots, 1)$, if $\mathbb{D} = \mathbb{C}$. This group acts on $\mathfrak{h}(\mathfrak{g})$, see (45), as follows: for $t = \epsilon\eta$,

$$t\Big(\sum_{j=1}^{l} y_j J_j\Big) = \sum_{j=1}^{l} \epsilon_j y_{\eta^{-1}(j)} J_j\,. \tag{140}$$

As indicated by the notation, $W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))$ coincides with the Weyl group, equal to the quotient of the normalizer of $\mathfrak{h}(\mathfrak{g})$ in G by the centralizer of $\mathfrak{h}(\mathfrak{g})$ in G.

The action of $W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))$ on $\mathfrak{h}(\mathfrak{g})$ extends by duality to $i\mathfrak{h}(\mathfrak{g})^*$. More precisely, let e_j be as in (46). If $\mu \in i\mathfrak{h}(\mathfrak{g})^*$, then $\mu = \sum_{j=1}^{l} \mu_j e_j$ with all $\mu_j \in \mathbb{R}$. If $t = \epsilon\eta \in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))$, then

$$t\Big(\sum_{j=1}^{l} \mu_j e_j\Big) = \sum_{j=1}^{l} \epsilon_j \mu_{\eta^{-1}(j)} e_j\,. \tag{141}$$

Recall the notation of Lemma 17 and the symbol δ from (68).

Lemma 20. *The following formulas hold for any* $y = \sum_{j=1}^{l} y_j J_j \in \mathfrak{h}(\mathfrak{g})$,

$$\xi_{-\mu}(\widehat{c}_-(ty)) = \xi_{-t^{-1}\mu}(\widehat{c}_-(y)) \qquad (t \in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))) \tag{142}$$

and

$$\xi_{-\mu}(\widehat{c}_-(y))\,\mathrm{ch}^{d'-r-\iota}(y) = \prod_{j=1}^{l} (1+iy_j)^{\mu_j+\delta-1}(1-iy_j)^{-\mu_j+\delta-1}\,, \tag{143}$$

where all the exponents are integers:

$$\pm\mu_j + \delta \in \mathbb{Z} \qquad (1 \le j \le l)\,. \tag{144}$$

*In particular, (*143*) is a rational function in the variables* y_1, y_2, $\dots$, y_l.

Proof. By (131),

$$\xi_{-\mu}(\widehat{c}_-(y)) = \prod_{j=1}^{l}\left(\frac{1+iy_j}{1-iy_j}\right)^{\mu_j} = \prod_{j=1}^{l}(1+iy_j)^{\mu_j}(1-iy_j)^{-\mu_j}\,.$$

Hence (142) and (143) follow from the definition of the action of $W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))$, the definition of ch in (134), and the following easy-to-check formula:

$$\mathrm{ch}(y) = \prod_{j=1}^{l}(1+y_j^2)^{\frac{1}{2\iota}} \quad = \quad \prod_{j=1}^{l}(1+iy_j)^{\frac{1}{2\iota}}(1-iy_j)^{\frac{1}{2\iota}}. \tag{145}$$

Let $\lambda = \sum_{j=1}^{l}\lambda_j e_j$ be the highest weight of the representation Π and let $\rho = \sum_{j=1}^{l}\rho_j e_j$ be one half times the sum of the positive roots of $\mathfrak{h}(\mathfrak{g})$ in $\mathfrak{g}_{\mathbb{C}}$. If μ is the Harish-Chandra parameter of Π, then $\lambda+\rho = \mu = \sum_{j=1}^{l}\mu_j e_j$; see Appendix H for explicit values. Hence, the statement (144) is equivalent to

$$\lambda_j + \rho_j + \frac{1}{2\iota}(d' - r + \iota) \in \mathbb{Z}\,. \tag{146}$$

Indeed, if $\mathrm{G} = \mathrm{O}_d$, then with the standard choice of the positive root system, $\rho_j = \frac{d}{2} - j$. Also, $\lambda_j \in \mathbb{Z}$, $\iota = 1$, $r = d-1$. Hence, (146) follows. Similarly, if $\mathrm{G} = \mathrm{U}_d$, then $\rho_j = \frac{d+1}{2} - j$, $\lambda_j + \frac{d'}{2} \in \mathbb{Z}$, $\iota = 1$, $r = d$, which implies (146). If $\mathrm{G} = \mathrm{Sp}_d$, then $\rho_j = d + 1 - j$, $\lambda_j \in \mathbb{Z}$, $\iota = \frac{1}{2}$, $r = d + \frac{1}{2}$, and (146) follows. □

Our next goal is to understand the integral

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x)\int_{\mathrm{W}}\chi_x(w)\phi^{\mathrm{G}}(w)\,dw$$

occurring in the formula for $\int_{-\mathrm{G}^0}\check{\Theta}_\Pi(\tilde{g})T(\tilde{g})\,dg$ in Lemma 17 and Corollary 19, in terms of orbital integrals on the symplectic space W. The results

depend on whether $l \le l'$ or $l > l'$ and will be given in Lemmas 23 and 24. We first need two other lemmas.

LEMMA 21. *Fix an element $z \in \mathfrak{h}(\mathfrak{g})$. Let $\mathfrak{z} \subseteq \mathfrak{g}$ and $\mathrm{Z} \subseteq \mathrm{G}$ denote the centralizer of z. (Then Z is a real reductive group with Lie algebra $\mathfrak{z}$.) Denote by $\mathfrak{c}$ the center of $\mathfrak{z}$ and by $\pi_{\mathfrak{g}/\mathfrak{z}}$ the product of the positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}(\mathfrak{g})_{\mathbb{C}})$ which do not vanish on z. Let $B(\cdot,\cdot)$ be any non-degenerate symmetric G-invariant real bilinear form on $\mathfrak{g}$. Then there is a constant $C_{\mathfrak{z}}$ such that for $x \in \mathfrak{h}(\mathfrak{g})$ and $x' \in \mathfrak{c}$,*

$$\begin{aligned}(147)\quad & \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x)\pi_{\mathfrak{g}/\mathfrak{z}}(x') \int_{\mathrm{G}} e^{iB(g.x,x')}\,dg \\ & = C_{\mathfrak{z}} \sum_{tW(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t)\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}x)e^{iB(x,t(x'))}.\end{aligned}$$

(Here $\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})} = 1$ if $\mathfrak{z} = \mathfrak{h}$. Recall also the notation $g.x = gxg^{-1}$.)

PROOF. The proof is a straightforward modification of the argument proving Harish-Chandra's formula for the Fourier transform of a regular semisimple orbit, [Har57, Theorem 2, page 104]. A more general, and by now classical, result is [DV90, Proposition 34, p. 49]. □

The symplectic form $\langle\cdot,\cdot\rangle$ on W, according to the Lie superalgebra structure introduced in (28), is

$$\langle w', w\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}w'w) \qquad (w', w \in \mathrm{W})\,. \tag{148}$$

Hence

$$\langle x(w), w\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xw^2) \qquad (x \in \mathfrak{g}\oplus\mathfrak{g}',\ w \in \mathrm{W})\,. \tag{149}$$

Set

$$B(x,y) = \pi\ \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) \qquad (x, y \in \mathfrak{g})\,. \tag{150}$$

LEMMA 22. *Recall the Gaussian χ_x from* (13). *Then*

$$\chi_x(w) = e^{iB(x,\tau(w))} \qquad (x \in \mathfrak{g}, w \in \mathrm{W})\,. \tag{151}$$

Proof. Notice that, for $x \in \mathfrak{g} \oplus \mathfrak{g}'$ and $w \in \mathrm{W}$,

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xw^2) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathsf{V}_{\bar{0}}}) - \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathsf{V}_{\bar{1}}})\,,$$

where

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathsf{V}_{\bar{0}}}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x|_{\mathsf{V}_{\bar{0}}}w|_{\mathsf{V}_{\bar{1}}}w|_{\mathsf{V}_{\bar{0}}}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(w|_{\mathsf{V}_{\bar{0}}}x|_{\mathsf{V}_{\bar{0}}}w|_{\mathsf{V}_{\bar{1}}})) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(wxw|_{\mathsf{V}_{\bar{1}}})$$

and similarly

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathsf{V}_{\bar{1}}}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(wxw|_{\mathsf{V}_{\bar{0}}})\,.$$

Hence

$$\langle xw, w\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xw^2) = -\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}wxw) = -\langle wx, w\rangle\,.$$

Therefore

$$\langle x(w), w\rangle = 2\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xw^2) \qquad (x \in \mathfrak{g} \oplus \mathfrak{g}'\,,\ w \in \mathrm{W})\,. \tag{152}$$

Then (149) and (32) show that

$$\frac{\pi}{2}\langle x(w), w\rangle = B(x, \tau(w)) \qquad (x \in \mathfrak{g}\,,\ w \in \mathrm{W})\,,$$

which completes the proof. □

The Harish-Chandra regular almost semisimple orbital integral $F(y)$, $y \in \mathfrak{h}$, was defined in [MPP20, Definition 3.2 and Theorems 3.3 and 3.5]; see also section 4 above. In particular, [MPP20, Theorem 3.5] implies that, in the statements below, all the integrals over $\mathfrak{h}$ involving $F(y)$ are absolutely convergent. Recall the notation $F_\phi(y)$ for $F(y)(\phi)$.

Lemma 23. *Suppose $l \le l'$. Then, with the notation of Lemma 17,*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw = C \int_{\mathfrak{h}\cap\tau(\mathrm{W})} e^{iB(x,y)} F_\phi(y)\,dy\,,$$

where C is a non-zero constant which depends on the dual pair $(\mathrm{G}, \mathrm{G}')$.

Proof. The Weyl–Harish-Chandra integration formula on W, see (52), (53) and (49), shows that

$$\begin{aligned}\int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw \\ = \sum_{\mathfrak{h}_{\bar{1}}} \int_{\tau(\mathfrak{h}_{\bar{1}}^+)} \pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau(w))C(\mathfrak{h}_{\bar{1}})\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}}(\chi_x\phi^{\mathrm{G}})\,d\tau(w)\,,\end{aligned} \tag{153}$$

where $\mathfrak{h}_{\overline{1}}^{+} \subseteq \mathfrak{h}_{\overline{1}}{}^{reg}$ is an open fundamental domain for the action of the Weyl group $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$ and $C(\mathfrak{h}_{\overline{1}})$ is a constant, determined in [MPP20, Lemma 2.1]. Let us consider first the case of a semisimple orbital integral

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\chi_x \phi^{\mathrm{G}}) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} (\chi_x \phi^{\mathrm{G}})(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}})\,,$$

where $\mathrm{S}^{\mathfrak{h}_{\overline{1}}}$ is the centralizer of $\mathfrak{h}_{\overline{1}}$ in S. Recall the identification $y = \tau(w) = \tau'(w)$ and let us write $s = gg'$, where $g \in \mathrm{G}$ and $g' \in \mathrm{G}'$. Then

$$\chi_x(s.w) = e^{i\frac{\pi}{2}\langle x(s.w), s.w\rangle} = e^{iB(x,\tau(s.w))} = e^{iB(x,g.\tau(w))} = e^{iB(x,g.y)} \tag{154}$$

and

$$\phi^{\mathrm{G}}(s.w) = \phi^{\mathrm{G}}(g'.w)\,. \tag{155}$$

Since $l \leq l'$, equation (I.1) below implies that there is a positive constant C_1 such that

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\chi_x \phi^{\mathrm{G}}) = C_1 \int_{\mathrm{G}} e^{iB(x,g.y)}\, dg \int_{\mathrm{G}'/\mathrm{Z}'} \phi^{\mathrm{G}}(g'.w)\, d(g'\mathrm{Z}')\,.$$

However we know from Harish-Chandra (Lemma 21) that

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \left(\int_{\mathrm{G}} e^{iB(x,g.y)}\, dg \right) \pi_{\mathfrak{g}/\mathfrak{h}}(y) = C_2 \sum_{t \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) e^{iB(x,t.y)}\,.$$

Hence, using (153) and [MPP20, Definition 3.2 and Lemma 3.4], we obtain for some suitable positive constants C_k,

$$\begin{aligned}
\pi_{\mathfrak{g}/\mathfrak{h}}(x) &\int_{\mathrm{W}} \chi_x(w) \phi^{\mathrm{G}}(w)\, dw \\
&= C_3 \sum_{t \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \sum_{\mathfrak{h}_{\overline{1}}} \int_{\tau(\mathfrak{h}_{\overline{1}}^{+})} e^{iB(x,t.y)} C(\mathfrak{h}_{\overline{1}}) \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \\
&\quad \times \int_{\mathrm{G}'/\mathrm{Z}'} \phi^{\mathrm{G}}(g'.w)\, d(g'\mathrm{Z}')\, dy \\
&= C_4 \sum_{t \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \int_{\cup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{+})} e^{iB(x,t.y)} F_{\phi^{\mathrm{G}}}(y)\, dy \\
&= C_4 \sum_{t \in W(\mathrm{G},\mathfrak{h})} \int_{\cup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{+})} e^{iB(x,t.y)} F_{\phi^{\mathrm{G}}}(t.y)\, dy
\end{aligned} \tag{156}$$

$$= C_4 \int_{W(\mathrm{G},\mathfrak{h})(\cup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^+))} e^{iB(x,y)} F_{\phi^{\mathrm{G}}}(y)\, dy$$
$$= C_4 \int_{\mathfrak{h}\cap\tau(\mathrm{W})} e^{iB(x,y)} F_{\phi^{\mathrm{G}}}(y)\, dy\,.$$

Since $F_{\phi^{\mathrm{G}}} = \mathrm{vol}(\mathrm{G}) F_\phi = F_\phi$, the formula follows.

Next we consider the case $\mathrm{G} = \mathrm{O}_{2l+1}$, $\mathrm{G}' = \mathrm{Sp}_{2l'}(\mathbb{R})$, $l < l'$. Then

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^{\mathrm{G}}) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}} (\chi_x \phi^{\mathrm{G}})(s.(w+w_0))\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0})\,,$$

where $w_0 \in \mathfrak{s}_1(\mathsf{V}^0)$ is a nonzero element. Since the Cartan subspace $\mathfrak{h}_{\overline{1}}$ preserves the decomposition (34), $(w+w_0)^2 = w^2 + w_0^2$. Hence, $(s.(w+w_0))^2 = s.(w^2+w_0^2)$. The element $x \in \mathfrak{h}$ acts by zero on $\mathfrak{g}'$. Therefore $x(s.(w+w_0))^2 = x(s.(w+w_0))^2|_{\mathsf{V}_{\overline{0}}}$. Since $\mathrm{S}(\mathsf{V}^0) = \mathrm{O}_1 \times \mathrm{Sp}_{2(l'-l)}(\mathbb{R})$ we see that $w_0^2|_{\mathsf{V}_{\overline{0}}} = 0$. Thus $xs.w_0^2|_{\mathsf{V}_{\overline{0}}} = 0$. Therefore, by (27),

$$\langle x(s.(w+w_0)), s.(w+w_0)\rangle = \mathrm{tr}(x(s.(w+w_0))^2) = \mathrm{tr}(xs.w^2|_{\mathsf{V}_{\overline{0}}}) = \mathrm{tr}(xg.\tau(w))\,,$$

because $s = gg'$. Hence,

$$\chi_x(s.(w+w_0)) = e^{i\frac{\pi}{2}\langle x(s.(w+w_0)), s.(w+w_0)\rangle} = e^{iB(x,g.\tau(w))} = e^{iB(x,g.y)}$$

and

$$\phi^{\mathrm{G}}(s.(w+w_0)) = \phi^{\mathrm{G}}(g'.(w+w_0))\,.$$

Therefore, with $n = \tau'(w_0)$, we obtain from (I.3) that

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^{\mathrm{G}}) = C_1 \int_{\mathrm{G}} e^{iB(x,g.y)}\, dg \int_{\mathrm{G}'/\mathrm{Z}'^n} \phi^{\mathrm{G}}(g'.w)\, d(g'\mathrm{Z}'^n)\,,$$

where Z'^n is the centralizer of n in Z'. Thus, the computation (156) holds again, and we are done. □

Lemma 24. *Suppose $l > l'$. Let $\mathfrak{z} \subseteq \mathfrak{g}$ and $\mathrm{Z} \subseteq \mathrm{G}$ be the centralizers of $\tau(\mathfrak{h}_{\overline{1}})$. Then for $\phi \in \mathcal{S}(\mathrm{W})$*

$$\pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw$$
$$= C \sum_{tW(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t)\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,t.y)} F_\phi(y)\, dy\,,$$

where C is a non-zero constant which depends only on the dual pair $(\mathrm{G}, \mathrm{G}')$.

Proof. By the Weyl–Harish-Chandra integration formula with the roles of G and G′ reversed, see (53) and (49),

$$\int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw = C_1 \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \pi_{\mathfrak{g}/\mathfrak{z}}(\tau'(w))\pi_{\mathfrak{g}'/\mathfrak{h}'}(\tau'(w))\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}})\,d\tau'(w)\,,$$

where

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}}) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} (\chi_x\phi^{\mathrm{G}})(s.w)\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}})\,.$$

Recall the identification $y = \tau(w) = \tau'(w)$ and let us write $s = gg'$, where $g \in \mathrm{G}$ and $g' \in \mathrm{G}'$. Then, as in (154) and (155),

$$\chi_x(s.w) = e^{iB(x,g.y)} \quad \text{and} \quad \phi^{\mathrm{G}}(s.w) = \phi^{\mathrm{G}}(g'.w)\,.$$

Since $l > l'$, equation (I.2) implies that there is a constant C_2 such that

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}}) = C_2 \int_{\mathrm{G}} e^{iB(x,g.y)}\,dg \int_{\mathrm{G}'/\mathrm{H}'} \phi^{\mathrm{G}}(g'.w)\,d(g'\mathrm{H}')\,.$$

By (147) in Lemma 21 and [MPP20, (34)], we obtain for some constants C_k

$$\begin{aligned}(157)\quad \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) &\int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw \\ &= C_3 \sum_{tW(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t)\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \\ &\qquad \times \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,t.y)}\pi_{\mathfrak{g}'/\mathfrak{h}}(y) \int_{\mathrm{G}'/\mathrm{H}'} \phi^{\mathrm{G}}(g'.w)\,d(g'\mathrm{H}')\,dy \\ &= C_4 \sum_{tW(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t)\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \\ &\qquad \times \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,t.y)}F_{\phi^{\mathrm{G}}}(y)\,dy\,.\end{aligned}$$

Since $F_{\phi^{\mathrm{G}}} = \mathrm{vol}(\mathrm{G})F_\phi = F_\phi$, the formula follows. □

Lemma 25. *Suppose $l \le l'$. Then there is a seminorm q on $\mathcal{S}(\mathrm{W})$ such that*

$$\Big|\int_{\mathfrak{h}\cap\tau(\mathrm{W})} F_\phi(y)\,e^{iB(x,y)}\,dy\Big| \le q(\phi)\ \mathrm{ch}(x)^{-d'+r-\iota} \qquad (x \in \mathfrak{h},\ \phi \in \mathcal{S}(\mathrm{W}))\,.$$

Proof. The boundedness of the distribution-valued function $T(\widetilde{g})$, $\tilde{g} \in \widetilde{\mathrm{G}}$, means that there is a seminorm q on $\mathcal{S}(\mathfrak{g})$ such that

$$|T(\widetilde{g})(\phi)| \le q(\phi) \qquad (\tilde{g} \in \widetilde{\mathrm{G}}\,, \phi \in \mathcal{S}(\mathfrak{g}))\,.$$

Hence,

$$\Big|\Theta(\tilde{c}(x)) \int_{\mathrm{W}} \chi_x(w)\phi(w)\,dw\Big| \le q(\phi) \qquad (x \in \mathfrak{g})\,. \tag{158}$$

Equivalently, replacing $q(\phi)$ with a constant multiple of $q(\phi)$, and using (134), (137) and (138), we see that

$$\Big| \int_{\mathrm{W}} \chi_x(w)\phi(w)\,dw\Big| \le q(\phi)\,\mathrm{ch}^{-d'}(x) \qquad (x \in \mathfrak{g})\,. \tag{159}$$

Since $l \le l'$, Lemma 23 together with (159) proves that (again up to a multiplicative constant that can be absorbed by $q(\phi)$),

$$\Big| \int_{\mathfrak{h}\cap\tau(\mathrm{W})} F_\phi(y)\, e^{iB(x,y)}\,dy\Big| \le q(\phi)\,|\pi_{\mathfrak{g}/\mathfrak{h}}(x)|\,\mathrm{ch}(x)^{-d'}.$$

Recall the constants r and ι from (67) and (65). Then, as one can verify from (A.1),

$$\max\{\deg_{y_j} \pi_{\mathfrak{g}/\mathfrak{h}};\ 1 \le j \le l\} = \frac{1}{\iota}(r-1)\,, \tag{160}$$

where $\deg_{y_j} \pi_{\mathfrak{g}/\mathfrak{h}}$ denotes the degree of $\pi_{\mathfrak{g}/\mathfrak{h}}(y)$ with respect to the variable y_j.

Also, (160) and (145) imply that

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(x)| \le C_5\,\mathrm{ch}^{r-1}(x) \le C_5\,\mathrm{ch}^{r-\iota}(x) \qquad (x \in \mathfrak{h})\,,$$

where C_5 is a constant. Thus, the claim follows. □

Lemmas 23 and 24 allow us to restate Corollary 19 in terms of orbital integrals on the symplectic space W.

Corollary 26. *Suppose $l \le l'$. Then for any $\phi \in \mathcal{S}(\mathrm{W})$*

$$\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\,dg = C\kappa_0\,\check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c_-}(x))\,\mathrm{ch}^{d'-r-\iota}(x) \times \int_{\mathfrak{h}\cap\tau(\mathrm{W})} e^{iB(x,y)} F_\phi(y)\,dy\,dx\,,$$

where C is a constant that depends only on the dual pair $(\mathrm{G}, \mathrm{G}')$ and each consecutive integral is absolutely convergent.

PROOF. The equality is immediate from Corollary 19 and Lemma 23. The absolute convergence of the outer integral over $\mathfrak{h}$ follows from Lemma 25. □

COROLLARY 27. *Suppose $l > l'$. Then for any $\phi \in \mathcal{S}(\mathrm{W})$,*

$$\begin{aligned}
&\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg \\
&\quad = C\kappa_0\, \check{\chi}_\Pi(\tilde{c}(0)) \sum_{s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-s\mu}(\widehat{c_-}(x))\, \mathrm{ch}^{d'-r-\iota}(x) \\
&\qquad \times \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} F_\phi(y)\, dy\, dx\,,
\end{aligned}$$

where C is a constant that depends only on the dual pair $(\mathrm{G}, \mathrm{G}')$ and each consecutive integral is absolutely convergent.

PROOF. The formula is immediate from Corollary 19, Lemma 24 and formula (142):

$$\begin{aligned}
&\frac{1}{\kappa_0} \int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg(\phi) \\
&\quad = C_1 \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c_-}(x))\, \mathrm{ch}^{d'-r-\iota}(x) \left(\pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\mathrm{W}} \chi_x(w) \phi^{\mathrm{G}}(w)\, dw \right) dx \\
&\quad = C_2 \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c_-}(x))\, \mathrm{ch}^{d'-r-\iota}(x) \\
&\qquad \times \left(\sum_{tW(\mathrm{Z}, \mathfrak{h}(\mathfrak{g})) \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \right. \\
&\qquad\qquad \left. \times \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,t.y)} F_\phi(y)\, dy \right) dx \\
&\quad = \frac{C_2\, \check{\chi}_\Pi(\tilde{c}(0))}{|W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}))|} \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c_-}(x))\, \mathrm{ch}^{d'-r-\iota}(x) \\
&\qquad \times \left(\sum_{t \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,t.y)} F_\phi(y)\, dy \right) dx
\end{aligned}$$

$$= C_3 \check{\chi}_\Pi(\tilde{c}(0)) \sum_{t \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c_-}(t.x)) \, \mathrm{ch}^{d'-r-\iota}(t.x)$$
$$\times \left(\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_{\bar{1}}{}^{reg})} e^{iB(t.x,t.y)} F_\phi(y) \, dy \right) dx$$
$$= C_3 \check{\chi}_\Pi(\tilde{c}(0)) \sum_{t \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-t^{-1}\mu}(\widehat{c_-}(x)) \, \mathrm{ch}^{d'-r-\iota}(x)$$
$$\times \left(\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_{\bar{1}}{}^{reg})} e^{iB(x,y)} F_\phi(y) \, dy \right) dx \, .$$

Let G'' be the isometry group of the restriction of the form $(\cdot,\cdot)$ to $\mathsf{V}_{\bar{0}}^{0,0}$ and let $\mathfrak{h}'' = \sum_{j=l'+1}^{l} \mathbb{R} J_j$. Then, as in (160), we check that

$$\max\{\deg_{x_j} \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})};\ 1 \le j \le l\} = \max\{\deg_{x_j} \pi_{\mathfrak{z}''/\mathfrak{h}''};\ l'+1 \le j \le l\} = \frac{1}{\iota}(r''-1) \, ,$$

where $r'' = \frac{2 \dim \mathfrak{g}''_{\mathbb{R}}}{\dim \mathsf{V}_{\bar{0}}^{0,0}{}_{\mathbb{R}}}$ is defined as in (66). Since $r - r'' = d'$, we see that

$$\mathrm{ch}^{d'-r-\iota}(x) |\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x)| \le const \ \mathrm{ch}^{d'-r-\iota+r''-\iota}(x) = const \ \mathrm{ch}^{-2\iota}(x) \, .$$

Furthermore, F_ϕ is absolutely integrable. Therefore, the absolute convergence of the last integral over $\mathfrak{h}(\mathfrak{g})$ follows from the fact that $\mathrm{ch}^{-2\iota}$ is absolutely integrable. $\square$

To prove Theorem 4 (and Theorem 5), we still need the following explicit formula for the form $B(x,y)$. Let $\beta = \dfrac{2\pi}{\iota}$, where ι is as in (65). Then

$$B(x,y) = -\beta \sum_{j=1}^{l} x_j y_j \qquad \Big(x = \sum_{j=1}^{l} x_j J_j \, , y = \sum_{j=1}^{l} y_j J_j \in \mathfrak{h}(\mathfrak{g}) \Big) \, . \tag{161}$$

Indeed, the definition of the form B, (150), shows that

$$B(x,y) = \pi \, \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) = \pi \sum_{j,k} \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(J_j J_k) x_j y_k$$
$$= \pi \sum_j \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(-1_{\mathsf{V}_{\bar{0}}^j}) x_j y_j = -\frac{2\pi}{\iota} \sum_j x_j y_j \, . \tag{162}$$

Proof of Theorem 4. Notice that the degree of the polynomial Q_{a_j,b_j} is $-a_j - b_j = 2\delta - 2$ and is independent of μ and j. Explicitly,

$$2\delta - 2 = \frac{1}{\iota}(d' - r - \iota)\,, \tag{163}$$

(where $\iota = 1/2$ if $\mathbb{D} = \mathbb{H}$ and 1 otherwise). Hence, by [MPP20, Theorem 3.5], the function F_ϕ has the required number of continuous derivatives for the formula (72) to make sense. The operators appearing in the integrand of (72) act on different variables and therefore commute. Also, the constants a_j, b_j are integers by (144). Hence, equation (72) follows from Corollary 26, Lemma 20, formula (161), and Proposition D.5.

For the last statement about (73), we have

$$d' - r - \iota = \begin{cases} 2l' - 2l & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))\,, \\ 2l' - 2l - 1 & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))\,, \\ l' - l - 1 & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,q}), p + q = l'\,, \\ l' - l - 1 & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_l, \mathrm{O}^*_{2l'})\,. \end{cases} \tag{164}$$

Thus, since we assume $l \le l'$, the product (73) is a function if and only if $d' - r - \iota < 0$, i.e. if and only if $l = l'$ and $(\mathrm{G}, \mathrm{G}') \neq (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$. Furthermore, (73) contains no derivatives (but terms involving δ_0 are allowed) if and only if $d' - r - \iota = 0$, which corresponds to either $l = l'$ and $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$, or $l' = l + 1$ and $\mathbb{D} = \mathbb{C}$ or $\mathbb{H}$. This completes the proof. □

Suppose now $l > l'$. Let $\mathfrak{h}'' = \sum_{j=l'+1}^{l} \mathbb{R} J_j$, so that

$$\mathfrak{h}(\mathfrak{g}) = \mathfrak{h} \oplus \mathfrak{h}''. \tag{165}$$

Then the centralizer of $\tau(\mathfrak{h}_{\overline{1}})$ coincides with the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ and is equal to $\mathfrak{z} = \mathfrak{h} \oplus \mathfrak{g}''$, where $\mathfrak{g}''$ is the Lie algebra of the group G'' of the isometries of the restriction of the form $(\cdot, \cdot)$ to $\mathsf{V}_{\overline{0}}^0$. Furthermore, the derived Lie algebras of $\mathfrak{z}$ and $\mathfrak{g}''$ coincide (i.e. $[\mathfrak{z}, \mathfrak{z}] = [\mathfrak{g}'', \mathfrak{g}'']$) and $\mathfrak{h}''$ is a Cartan subalgebra of $\mathfrak{g}''$. We shall identify $\mathfrak{h}$ and $\mathfrak{h}'$ by means of (42). This justifies writing $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}' \oplus \mathfrak{h}''$ when we need to emphasize the role of $\mathfrak{g}'$.

Lemma 28. *Suppose $l > l'$. In terms of Corollary 27 and the decom-*

position (165)

$$(166)\quad \xi_{-s\mu}(\widehat{c_-}(x))\,\mathrm{ch}^{d'-r-\iota}(x)\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x)$$
$$= \left(\xi_{-s\mu}(\widehat{c_-}(x'))\,\mathrm{ch}^{d'-r-\iota}(x')\right)\left(\xi_{-s\mu}(\widehat{c_-}(x''))\,\mathrm{ch}^{d'-r-\iota}(x'')\pi_{\mathfrak{g}''/\mathfrak{h}''}(x'')\right),$$

where $x = x' + x'' \in \mathfrak{h}(\mathfrak{g})$, *with* $x' \in \mathfrak{h}'$ *and* $x'' \in \mathfrak{h}''$. *Moreover,*

$$(167)\quad \int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c_-}(x''))\,\mathrm{ch}^{d'-r-\iota}(x'')\pi_{\mathfrak{g}''/\mathfrak{h}''}(x'')\,dx''$$
$$= C \sum_{s''\in W(\mathrm{G}'',\mathfrak{h}'')} \mathrm{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'')\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho''),$$

where C *is a constant,* ρ'' *is one half times the sum of the positive roots for* $(\mathfrak{g}''_{\mathbb{C}}, \mathfrak{h}''_{\mathbb{C}})$ *and* $\mathbb{I}_{\{0\}}$ *is the indicator function of zero.*

Proof. Formula (166) is obvious, because $\widehat{c_-}(x'+x'') = \widehat{c_-}(x')\widehat{c_-}(x'')$ and $\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x'+x'') = \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'')$. We shall verify (167). Let r'' denote the number defined in (66) for the Lie algebra $\mathfrak{g}''$. A straightforward computation verifies the following table:

$\mathfrak{g}$	r	r''	$d'-r+r''$
$\mathfrak{u}_d$	d	$d-d'$	0
$\mathfrak{o}_d$	$d-1$	$d-d'-1$	0
$\mathfrak{sp}_d$	$d+\frac{1}{2}$	$d-d'+\frac{1}{2}$	0

By (135) applied to $\mathrm{G}'' \supseteq \mathrm{H}''$ and $\mathfrak{g}'' \supseteq \mathfrak{h}''$,

$$\pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') = C_1''\Delta''(\widehat{c_-}(x''))\,\mathrm{ch}^{r''-\iota}(x'') \qquad (x'' \in \mathfrak{h}''),$$

where Δ'' is the Weyl denominator for G'',

$$(168)\qquad \Delta'' = \kappa_0'' \sum_{s''\in W(\mathrm{G}'',\mathfrak{h}'')} \mathrm{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'')\,\xi_{s''\rho''}$$

and

$$(169)\qquad \kappa_0'' = \begin{cases} \frac{1}{2} & \text{if } \mathrm{G}'' = \mathrm{O}_{d''} \text{ where } d'' \text{ is even}, \\ 1 & \text{otherwise}. \end{cases}$$

Hence, by (145), the integral on the left-hand side of (167) is a constant multiple of

$$\int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c_-}(x''))\Delta''(\widehat{c_-}(x'')) \operatorname{ch}^{d'-r+r''}(x'') \operatorname{ch}^{-2\iota}(x'')\, dx'' \tag{170}$$
$$= 2^{\dim \mathfrak{h}''} \int_{\widehat{c_-}(\mathfrak{h}'')} \xi_{-s\mu}(h)\Delta''(h)\, dh\,,$$

where $\widehat{c_-}(\mathfrak{h}'') \subseteq \widehat{\mathrm{H}''^0}$.

Notice that the function

$$\widehat{\mathrm{H}''^0} \ni h \to \xi_{-s\mu}(h)\Delta''(h) \in \mathbb{C}$$

is constant on the fibers of the covering map

$$\widehat{\mathrm{H}''^0} \to \mathrm{H}''^0\,. \tag{171}$$

Indeed, the covering (171) is non-trivial only in two cases, namely $\mathrm{G}'' = \mathrm{O}_{2l''+1}$ and $\mathrm{G}'' = \mathrm{U}_{l''}$ with l'' even; see (123). In these cases, (168) shows that this claim is true provided that the weight $-s\mu + s''\rho''$ is integral for the Cartan subgroup H'' (i.e. it is equal to the derivative of a character of H'').

Suppose $\mathrm{G}'' = \mathrm{O}_{2l''+1}$. Then $\mathrm{G} = \mathrm{O}_{2l+1}$, $\lambda_j \in \mathbb{Z}$ and $\rho_j \in \mathbb{Z} + \frac{1}{2}$. Hence, $(-s\mu)_j \in \mathbb{Z} + \frac{1}{2}$. Since, $\rho''_j \in \mathbb{Z} + \frac{1}{2}$, we see that $(-s\mu)_j + \rho''_j \in \mathbb{Z}$.

Suppose now that $\mathrm{G}'' = \mathrm{U}_{l''}$ with l'' even. Then $\mathrm{G} = \mathrm{U}_l$ and $(-s\mu)_j \in \mathbb{Z} + \frac{1}{2}$. In fact, if l' is even, i.e. $l = l' + l''$ is even, then $\lambda_j \in \mathbb{Z}$ and $\rho_j \in \mathbb{Z} + \frac{1}{2}$. If l' is odd, i.e. $l = l' + l''$ is odd, then $\lambda_j \in \mathbb{Z} + \frac{1}{2}$ and $\rho_j \in \mathbb{Z}$. Since $\rho''_j \in \mathbb{Z} + \frac{1}{2}$, in both cases, we conclude that $(-s\mu)_j + \rho''_j \in \mathbb{Z}$.

Therefore, (170) is a constant multiple of

$$\sum_{s'' \in W(\mathrm{G}'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') \int_{\mathrm{H}''^0} \xi_{-s\mu}(h)\xi_{s''\rho''}(h)\, dh \tag{172}$$
$$= \begin{cases} \operatorname{vol}(\mathrm{H}''^0) \operatorname{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') & \text{if } (s\mu)|_{\mathfrak{h}''} = s''\rho'', \\ 0 & \text{if } (s\mu)|_{\mathfrak{h}''} \notin W(\mathrm{G}'', \mathfrak{h}'')\rho''\,, \end{cases}$$
$$= \operatorname{vol}(\mathrm{H}''^0) \sum_{s'' \in W(\mathrm{G}'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'')\,. \ \square$$

Corollary 29. *Suppose $l > l'$ and keep the notation of Lemma 28 . Then*

$$\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg = 0$$

unless there is $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ such that

$$(s\mu)|_{\mathfrak{h}''} = \rho''\,. \tag{173}$$

If $\mathrm{G} = \mathrm{O}_{2l+1}$ or Sp_l, then (173) is equivalent to

$$\mu|_{\mathfrak{h}''} = \rho'' \quad \text{and} \quad s|_{\mathfrak{h}''} = 1\,. \tag{174}$$

Suppose $\mathrm{G} = \mathrm{O}_{2l}$ and write $\mathfrak{h}'' = \mathfrak{h}''_0 \oplus \mathbb{R}J_l$, where $\mathfrak{h}''_0 = \sum_{j=l'+1}^{l-1} \mathbb{R}J_j$. Then (173) is equivalent to

$$\mu|_{\mathfrak{h}''} = \rho'', \quad s|_{\mathfrak{h}''_0} = 1, \quad \text{and} \quad s|_{\mathbb{R}J_l} = \pm 1\,. \tag{175}$$

Finally, if $\mathrm{G} = \mathrm{U}_l$, then (173) holds if and only if there is $j_0 \in \{0, 1, \dots, l'\}$ such that

$$\mu_{j_0+j} = \rho''_{l'+j} \quad \text{and} \quad s(J_{j_0+j}) = J_{l'+j} \qquad (1 \le j \le l - l')\,. \tag{176}$$

Suppose that (173) holds. Then for any $\phi \in \mathcal{S}(\mathrm{W})$

$$\begin{aligned}\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg(\phi) = C\,\kappa_0 \check{\chi}_\Pi(\tilde{c}(0)) \sum_{s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g})),\, (s\mu)|_{\mathfrak{h}''} = \rho''} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s) \\ \times \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c_-}(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\bar{1}}{}^{reg})} e^{iB(x,y)} F_\phi(y)\, dy\, dx\,,\end{aligned} \tag{177}$$

where C is a non-zero constant which depends only on the dual pair $(\mathrm{G}, \mathrm{G}')$, and each consecutive integral is absolutely convergent.

Proof. Observe that $B(x' + x'', y) = B(x', y)$ for $x' \in \mathfrak{h}'$, $x'' \in \mathfrak{h}''$ and $y \in \tau'(\mathfrak{h}_{\bar{1}}{}^{reg}) \subseteq \mathfrak{h}'$. We see therefore from Corollary 27 and Lemma 28 that

$$\begin{aligned}\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg(\phi) \\ = C\,\kappa_0 \check{\chi}_\Pi(\tilde{c}(0)) \sum_{s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \sum_{s'' \in W(\mathrm{G}'', \mathfrak{h}'')} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s)\, \mathrm{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') \\ \times \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') \\ \times \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c_-}(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\bar{1}}{}^{reg})} e^{iB(x,y)} F_\phi(y)\, dy\, dx\,.\end{aligned} \tag{178}$$

Notice that for $x \in \mathfrak{h}'$ and $s'' \in W(\mathrm{G}'', \mathfrak{h}'')$, we have $s''x = x$. Thus $\xi_{-s\mu}(\widehat{c_-}(x)) = \xi_{-s''s\mu}(\widehat{c_-}(x))$ by (142). Notice also that, by (165), $W(\mathrm{G}'', \mathfrak{h}'') \subseteq W(\mathrm{G}, \mathfrak{h})$ and $\mathrm{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s'')$. Moreover, $\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') = \mathbb{I}_{\{0\}}(-(s''^{-1}s\mu)|_{\mathfrak{h}''} + \rho'')$. Hence, replacing s by $s''s$ in (178), we see that this expression is equal to

$$(179)\quad C\,\kappa_0\check{\chi}_\Pi(\tilde{c}(0)) \sum_{s\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))} \sum_{s''\in W(\mathrm{G}'',\mathfrak{h}'')} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s)\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + \rho'') \times \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c_-}(x))\,\mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\mathbb{I}}{}^{reg})} e^{iB(x,y)} F_\phi(y)\,dy\,dx\,,$$

which yields (177), with a new non-zero constant C, equal to $C|W(\mathrm{G}'', \mathfrak{h}'')|$. Clearly (179) is zero if there is no s such that $(s\mu)|_{\mathfrak{h}''} = \rho''$. The absolute convergence of the integrals was checked in the proof of Corollary 27.

Recall that $\mathfrak{h}'' = \sum_{j=l'+1}^{l} \mathbb{R}J_j$ and $\mu = \lambda + \rho$ where λ is the highest weight of the genuine representation Π. We take a closer look at the condition $(s\mu)|_{\mathfrak{h}''} = \rho''$.

If $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, then $\rho|_{\mathfrak{h}''} = \rho''$. All coefficients of ρ are positive and strictly decreasing by 1 except when $\mathrm{G} = \mathrm{O}_{2l}$, where $\rho_l = 0$. Hence $s|_{\mathfrak{h}''}$ cannot contain sign changes when $\mathrm{G} = \mathrm{O}_{2l+1}$ or Sp_l, whereas $s|_{\mathfrak{h}_0''}$ cannot contain sign changes when $\mathrm{G} = \mathrm{O}_{2l}$. Using the form of the coefficients of λ, one easily sees that (173) is equivalent to (174) or (175).

If $\mathrm{G} = \mathrm{U}_l$, then $\lambda = \frac{p-q}{2} + \nu$, where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_l$ are integers. Moreover,

$$(180)\qquad \frac{p-q}{2} + \rho_{p+j} = \frac{l-p-q+1}{2} - j = \rho''_{l'+j} \qquad (1 \leq j \leq l-l')\,.$$

The Weyl group $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ consists of permutations of the J_j's. Hence a genuine Harish-Chandra parameter μ satisfies (173) if and only if among its coefficients $\mu_1, \ldots, \mu_l$ we can find a string of $l - l'$ successive coefficients μ_j equal to $\rho''_{l'+1}, \ldots, \rho''_l$ and the permutation s translates the corresponding string of J_j's onto $J_{l'+1}, \ldots, J_l$. This proves (176). □

In the next lemmas we study the integrals appearing on the right-hand side of (177).

Lemma 30. *For $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ and $y \in \tau'(\mathfrak{h}_{\overline{1}})$, in the sense of distributions on $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$,*

$$\int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c_-}(x)) \operatorname{ch}^{d'-r-\iota}(x) e^{iB(x,y)}\, dx = \Big(\prod_{j=1}^{l'} P_{a_{s,j},b_{s,j}}(\beta y_j) \Big) e^{-\beta \sum_{j=1}^{l'} |y_j|}\,, \tag{181}$$

where $a_{s,j}$, $b_{s,j}$ and β are as in (80) *and* (68)*, and $P_{a_{s,j},b_{s,j}}$ is defined in* (D.4).

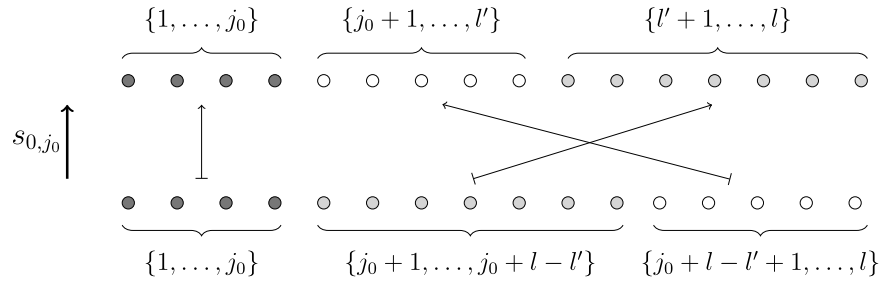
Proof. This follows immediately from Lemma 20, (D.5), and Proposition D.5, since $a_{s,j} + b_{s,j} = -2\delta + 2 \geq 1$ for $l > l'$. $\square$

Suppose that μ satisfies (173) for some $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$. The integral corresponding to s in (177) vanishes when the intersection of the support of the right-hand side of (181) and $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$ has an empty interior. We first study this intersection for some specific elements in $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$.

If $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, define $s_0 = 1$ as in (74). Then clearly $s_0\mu|_{\mathfrak{h}''} = \rho''$ by (174). If $\mathbb{D} = \mathbb{C}$, fix $j_0 \in \{0, 1, \dots, l'\}$ as in (176) and define s_{0,j_0} as the permutation in $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ given by

$$s_{0,j_0}(J_j) = \begin{cases} J_j & (1 \leq j \leq j_0) \\ J_{l'-j_0+j} & (j_0+1 \leq j \leq j_0+l-l') \\ J_{j-l+l'} & (j_0+l-l'+1 \leq j \leq l)\,, \end{cases} \tag{182}$$

i.e.



Equivalently,

$$(s_{0,j_0}\mu)_j = \mu_{s_{0,j_0}^{-1}(j)} = \begin{cases} \mu_j & (1 \leq j \leq j_0) \\ \mu_{l-l'+j} & (j_0+1 \leq j \leq l') \\ \mu_{j_0-l'+j} & (l'+1 \leq j \leq l)\,. \end{cases} \tag{183}$$

Hence $(s_{0,j_0}\mu)|_{\mathfrak{h}''} = \rho''$. Notice that $s_{0,p}$ is the element s_0 defined in (75).

LEMMA 31. *Let $l > l'$ and suppose that μ satisfies* (173). *Let $s_0 = 1$, as in* (74), *if $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, and let s_{0,j_0} be as in* (182) *if $\mathbb{D} = \mathbb{C}$.*

If $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$, then

$$\prod_{j=1}^{l'} P_{a_{s_0,j},b_{s_0,j}}(\beta y_j) = (2\pi)^{l'} \prod_{j=1}^{l'} P_{a_j,b_j,2}(\beta y_j)\mathbb{I}_{\mathbb{R}^+}(y_j) \qquad (y = \sum_{j=1}^{l'} y_j J'_j \in \mathfrak{h}') \tag{184}$$

has support equal to $\tau'(\mathfrak{h}_{\overline{1}})$.

If $\mathbb{D} = \mathbb{C}$, then

$$\begin{aligned}\prod_{j=1}^{l'} P_{a_{s_{0,j_0},j},b_{s_{0,j_0},j}}(\beta y_j) &= (2\pi)^{l'} \Big(\prod_{j=1}^{j_0} P_{a_j,b_j,2}(\beta y_j)\mathbb{I}_{\mathbb{R}^+}(y_j) \Big) \\ &\quad \times \Big(\prod_{j=j_0+1}^{l'} P_{a_{j+l-l'},b_{j+l-l'},-2}(\beta y_j)\mathbb{I}_{\mathbb{R}^-}(y_j) \Big) \qquad (y = \sum_{j=1}^{l'} y_j J'_j \in \mathfrak{h}')\end{aligned} \tag{185}$$

has support equal to $\big(\sum_{j=1}^{j_0} \mathbb{R}^+ J'_j\big) \oplus \big(\sum_{j=j_0+1}^{l'} \mathbb{R}^- J'_j\big)$. This support is equal to $\tau'(\mathfrak{h}_{\overline{1}})$ if $j_0 = p$, whereas its intersection with $\tau'(\mathfrak{h}_{\overline{1}})$ has empty interior if $j_0 \neq p$.

PROOF. Let $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$. By (65), (67), (68) and Appendix H and since $\mu|_{\mathfrak{h}''} = \rho'' = \rho|_{\mathfrak{h}''}$, we see that

$$\mu_1 > \cdots > \mu_{l'} > \mu_{l'+1} = \rho''_{l'+1} = -\delta\,,$$

These inequalities are equivalent to

$$a_1 = -\mu_1 - \delta + 1 < a_2 = -\mu_2 - \delta + 1 < \cdots < a_{l'} = -\mu_{l'} - \delta + 1 \leq 0 \tag{186}$$

because the μ_j's and δ are either all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$. Hence $P_{a_j,b_j,-2} = 0$ for all $1 \leq j \leq l'$ by (D.2). Since $a_j + b_j = -2\delta + 2 > 2$, we see that $b_j > 2 - a_j \geq 1$. Therefore, the polynomial $P_{a_j,b_j,2}$ is nonzero for all $1 \leq j \leq l'$. Hence the function on the right-hand side of (184) has support equal to $\sum_{j=1}^{l'} \mathbb{R}^+ J'_j = \tau'(\mathfrak{h}_{\overline{1}})$.

Let now $\mathbb{D} = \mathbb{C}$. By (176), (180), (68) and (163),

$$\begin{aligned}&\mu_1 > \mu_2 > \cdots > \mu_{j_0} > \mu_{j_0+1} = \rho''_{l'+1} = \frac{l - l' - 1}{2} = -\delta (> 0)\,, \\ &(0 >)\delta = -\frac{l - l' - 1}{2} = \rho''_l = \mu_{j_0+l-l'} > \mu_{j_0+l-l'+1} > \cdots > \mu_l\,.\end{aligned}$$

Since the μ_j's and δ are either all in $\mathbb{Z}$ or all in $\mathbb{Z}+\frac{1}{2}$, these inequalities are equivalent to

$$\begin{aligned} &a_1 = -\mu_1 - \delta + 1 < a_2 = -\mu_2 - \delta + 1 < \cdots < a_{j_0} = -\mu_{j_0} - \delta + 1 \le 0 \\ &0 \ge b_{j_0+l-l'+1} = \mu_{j_0+l-l'+1} - \delta + 1 > \cdots > b_l = \mu_l - \delta + 1\,. \end{aligned} \tag{187}$$

Hence,

$$\begin{aligned} P_{a_j,b_j,-2} = 0 \quad &\text{i.e. } P_{a_j,b_j}(y_j) = 2\pi P_{a_j,b_j,2}(y_j)\mathbb{I}_{\mathbb{R}^+}(y_j) \quad (1 \le j \le j_0)\,, \\ P_{a_j,b_j,2} = 0 \quad &\text{i.e. } P_{a_j,b_j}(y_j) = 2\pi P_{a_j,b_j,-2}(y_j)\mathbb{I}_{\mathbb{R}^-}(y_j) \quad (j_0+l-l'+1 \le j \le l)\,. \end{aligned}$$

The polynomials appearing in these expressions of P_{a_j,b_j} are nonzero because $a_j + b_j = -2\delta + 2 > 0$ for all j. By (40) and the convention on the symbols δ_j's for the dual pair $(\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l > l' = p+q$, the claims on the support of the right-hand side of (185) follow. □

Let $\mathbb{D} = \mathbb{C}$. Suppose that there is $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ such that $(s\mu)|_{\mathfrak{h}''} = \rho''$ and that the string of coefficients of μ equal to those of ρ'', see (176), starts at $j_0 + 1$, where $j_0 \in \{0, 1, \dots, l'\}$. Then $s = s_{0,j_0}$ satisfies $(s\mu)|_{\mathfrak{h}''} = \rho''$. Lemma 31 shows that if $j_0 \ne p$ then the intersection of the support of $\prod_{j=1}^{l'} P_{a_{s_{0,j_0},j},b_{s_{0,j_0},j}}$ with $\tau'(\mathfrak{h}_{\overline{1}})$ has empty interior. We now prove that if $j_0 \ne p$ the same holds for the support of $\prod_{j=1}^{l'} P_{a_{s,j},b_{s,j}}$ for every $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ such that $(s\mu)|_{\mathfrak{h}''} = \rho''$.

Lemma 32. *Let $\mathbb{D} = \mathbb{C}$. Suppose that μ and $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ satisfy (176) for $j_0 \in \{0, 1, \dots, l'\}$. If $j_0 \ne p$, then the intersection of the support of $\prod_{j=1}^{l'} P_{a_{s,j},b_{s,j}}$ with $\tau'(\mathfrak{h}_{\overline{1}})$ has empty interior.*

Proof. Since

$$s_{0,j_0}(J_{j_0+j}) = J_{l'+j}\,, \qquad s(J_{j_0+j}) = J_{l'+j} \qquad (1 \le j \le l - l')\,,$$

the composition $s^{-1}s_{0,j_0}$ fixes the elements of $\{J_{j_0+1}, \dots, J_{j_0+l-l'}\}$ and permutes those of $\{J_1, \dots, J_{j_0}\} \cup \{J_{j_0+l-l'+1}, \dots, J_l\}$. Then $s^{-1} = (s^{-1}s_{0,j_0})s_{0,j_0}^{-1}$ maps to string) $\{J_{l'+1}, \dots, J_l\}$ onto $\{J_{j_0+1}, \dots, J_{j_0+l-l'}\}$ and hence $\{J_1, \dots, J_{l'}\}$ bijectively onto $\{J_1, \dots, J_{j_0}\} \cup \{J_{j_0+l-l'+1}, \dots, J_l\}$. Therefore $\{(s\mu)_j = \mu_{s^{-1}(j)}; 1 \le j \le l'\}$ is a permutation of $\{\mu_j; 1 \le j \le j_0\} \cup \{\mu_j; j_0 + l - l' + 1 \le j \le l\}$. By (187), there are j_0 negative a_j and $l' - j_0$ negative b_j for $1 \le j \le l'$. The same is then true for the $a_{s,j}$ and the $b_{s,j}$. The

support of $\prod_{j=1}^{l'} P_{a_{s,j},b_{s,j}}$ is therefore a Cartesian product (in some order) of j_0 copies of $\mathbb{R}^+$ and $l'-j_0$ copies of $\mathbb{R}^-$. Its intersection with $\tau'(\mathfrak{h}_{\overline{1}})$ has therefore empty interior if $j_0 \neq p$. □

When the intersection of the support of $\prod_{j=1}^{l'} P_{a_{s,j},b_{s,j}}$ and $\tau'(\mathfrak{h}_{\overline{1}})$ has empty interior, the integral on the right-hand side of (177) that corresponds to s vanishes. Lemma 32 shows that every such integral is zero when $j_0 \neq p$. This yields the following corollary.

COROLLARY 33. *Suppose that* Π *is a genuine representation of* $\widetilde{\mathrm{U}}_l$ *with Harish-Chandra parameter* μ *satisfying* (176) *for* $j_0 \in \{0,1,\dots,l'\}$. *If* $j_0 \neq p$ *then*

$$f_{\Pi\otimes\Pi'} = \int_{\mathrm{U}_l} \check{\Theta}_\Pi(\widetilde{g}) T(\widetilde{g})\, dg = 0\,.$$

Thus, if Π *is a genuine representation of* $\widetilde{\mathrm{U}}_l$ *which occurs in Howe's correspondence, then its highest weight must be of the form* $\lambda = \sum_{j=1}^{l} \left(\frac{p-q}{2}+\nu_j\right) e_j$ *where*

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p \geq \nu_{p+1} = \cdots = \nu_{l-q} = 0 \geq \nu_{l-q+1} \geq \cdots \geq \nu_l\,.$$

PROOF. Only the last statement requires proof. We know from Lemma 32 that $j_0 = p$. Hence the first line of (187) looks as follows:

$$\mu_1 + \delta - 1 > \mu_2 + \delta - 1 > \cdots > \mu_p + \delta - 1 \geq 0\,.$$

Since

$$\mu_j + \delta - 1 = \lambda_j + \rho_j + \delta - 1 = \lambda_j - \frac{p-q}{2} + p - j \qquad (1 \leq j \leq p)\,,$$

we see that

$$\nu_j = \lambda_j - \frac{p-q}{2} \qquad (1 \leq j \leq p)\,,$$

satisfies

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p \geq 0\,.$$

By a similar analysis of the second line of (187), the claim follows. □

In the proof of Theorem 5 we will see that the condition on the highest weight of Π is also sufficient for the nonvanishing of the intertwining distributions.

Because of Corollary 33, we can restrict ourselves to the case $j_0 = p$ when $\mathrm{G} = \mathrm{U}_l$. In this case, to simplify notation, we will write s_0 instead of $s_{0,p}$. Hence

$$s_0 = 1 \quad (\text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{H}) \quad \text{and} \quad s_0 = s_{0,p} \quad (\text{if } \mathbb{D} = \mathbb{C})\,. \tag{188}$$

Observe that this notation allows us to write

$$\prod_{j=1}^{l'} P_{a_{s_0,j},b_{s_0,j}}(\beta y_j) = (2\pi)^{l'} \prod_{j=1}^{l'} P_{a_{s_0,j},b_{s_0,j},2\delta_j}(\beta y_j)\mathbb{I}_{\delta_j\mathbb{R}^+}(y_j)\,, \tag{189}$$

which unifies (184) and (185).

Suppose that $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ satisfies (173) and $j_0 = p$ if $\mathbb{D} = \mathbb{C}$. Then

$$ss_0^{-1}|_{\mathfrak{h}''} = 1 \qquad \text{and} \qquad ss_0^{-1}(\mathfrak{h}) = \mathfrak{h}\,. \tag{190}$$

The condition $ss_0^{-1}(\mathfrak{h}) = \mathfrak{h}$ and the identification (42), allow us to consider ss_0^{-1} as isomorphisms of $\mathfrak{h}'$. In the following lemma we prove that such a s contributes to the right-hand side of (177) if and only if $ss_0^{-1} \in W(\mathrm{G}', \mathfrak{h}')$. Moreover, in this case, the contribution from s agree with that of s_0.

Lemma 34. *Let $l > l'$ and let μ and $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ satisfy* (173) *with $j_0 = p$ if $\mathbb{D} = \mathbb{C}$. The integral*

$$\int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c_-}(x))\,\mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\bar{1}}{}^{reg})} e^{iB(x,y)} F_\phi(y)\,dy\,dx \tag{191}$$

is zero:

(a) *if $ss_0^{-1}|_{\mathfrak{h}}$ acts by some sign changes, when $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$,*

(b) *if $ss_0^{-1}|_{\mathfrak{h}}$ does not stabilize $\{J_1, \dots, J_p\}$ (and $\{J_{p+1}, \dots, J_{l'}\}$), when $\mathbb{D} = \mathbb{C}$.*

Equivalently, by identifying $\mathfrak{h}$ and $\mathfrak{h}'$ via (42)*, the integral* (191) *is zero unless $ss_0^{-1} \in W(\mathrm{G}', \mathfrak{h}')$. Moreover,* (177) *becomes: for any $\phi \in \mathcal{S}(\mathrm{W})$*

$$\begin{aligned} &\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\,dg(\phi) \\ &\quad = C\,\kappa_0 \check{\chi}_\Pi(\tilde{c}(0)) \int_{\tau'(\mathfrak{h}_{\bar{1}}{}^{reg})} \Big(\prod_{j=1}^{l'} P_{a_{s_0,j},b_{s_0,j}}(\beta y_j)\Big) e^{-\beta \sum_{j=1}^{l'} |y_j|} F_\phi(y)\,dy\,, \end{aligned} \tag{192}$$

where C is a non-zero constant which depends on the dual pair $(\mathrm{G}, \mathrm{G}')$.

PROOF. Let $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$. Suppose that $ss_0^{-1}(J_j) = -J_j$ for some $j \in \{1, \dots, l'\}$. Then $(s\mu)_j = -(s_0\mu)_j$. Thus $P_{a_{s,j},b_{s,j}}$ is supported in $\mathbb{R}^-$, and the support of (181) has a lower dimensional intersection with $\tau'(\mathfrak{h}_{\overline{1}})$.

The case $\mathbb{D} = \mathbb{C}$ is similar: if $ss_0^{-1}(J_i) = J_j$ where $1 \le i \le p < j \le l'$, then $(s\mu)_j = (s_0\mu)_i$, which interchanges the i-th and j-th indices a and b of $s\mu$ and $s_0\mu$. The support of (181) has therefore a lower dimensional intersection with $\tau'(\mathfrak{h}_{\overline{1}})$.

By the above and by identifying $\mathfrak{h}$ and $\mathfrak{h}'$ via (42), we can restrict the sum on the right-hand side of (177) to the set of $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ such that $ss_0^{-1}|_{\mathfrak{h}} \in W(\mathrm{G}', \mathfrak{h}')$ and $ss_0^{-1}|_{\mathfrak{h}''} = 1$. Therefore, the sum can be parametrized by $W(\mathrm{G}', \mathfrak{h}')$. By (181) and since $\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(ss_0^{-1}) = \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(ss_0^{-1})$, we obtain that $\int_{-\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg(\phi)$ is $\kappa_0 \check{\chi}_\Pi(\tilde{c}(0))$ times a constant multiple of

$$\sum_{s' \in W(\mathrm{G}', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \int_{\mathfrak{h}'} \xi_{-s's_0\mu}(\widehat{c_-}(x)) \,\mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} F_\phi(y)\, dy\, dx$$

$$= \sum_{s' \in W(\mathrm{G}', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \Big(\prod_{j=1}^{l'} P_{a_{s's_0,j}, b_{s's_0,j}}(\beta y_j) \Big) e^{-\beta \sum_{j=1}^{l'} |y_j|} F_\phi(y)\, dy\,.$$

Observe that

$$\prod_{j=1}^{l'} P_{a_{s's_0,j}, b_{s's_0,j}}(\beta y_j) = \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}}(\beta (s'^{-1} y)_j)$$

because $s' \in W(\mathrm{G}', \mathfrak{h}')$ permutes the indices $1 \le j \le l'$. Recall also that $F_\phi(y)$ transforms as the sign representation with respect to the action of $W(\mathrm{G}', \mathfrak{h}')$. Formula (192) therefore follows. The new non-zero constant C is the one appearing in (177) times $|W(\mathrm{G}', \mathfrak{h}')|$ times $\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s_0)$, which is equal to 1 if $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$ and $(-1)^{q(l-l')}$ if $\mathbb{D} = \mathbb{C}$. □

PROOF OF THEOREM 5. It remains to show that if the highest weight λ of Π satisfies the conditions (a) or (b), then the integral (76), i.e. (192), is nonzero.

By (189), the function $\prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}}(\beta y_j)$ has support equal to $\tau'(\mathfrak{h}_{\overline{1}})$

and we can rewrite the right-hand side of (192) as a constant multiple of

$$\kappa_0\check{\chi}_\Pi(\tilde{c}(0))\int_{\tau'(\mathfrak{h}_{\bar{1}}^{reg})}\Big(\prod_{j=1}^{l'}P_{a_{s_0,j},b_{s_0,j},2\delta_j}(\beta y_j)\Big)e^{-\beta\sum_{j=1}^{l'}|y_j|}F_\phi(y)\,dy\,. \tag{193}$$

By the $W(\mathrm{G}',\mathfrak{h}')$-skew-invariance of F_ϕ, we can replace the term

$$\Big(\prod_{j=1}^{l'}P_{a_{s_0,j},b_{s_0,j},2\delta_j}(\beta y_j)\Big)e^{-\beta\sum_{j=1}^{l'}|y_j|}$$

in the integral (193) by its $W(\mathrm{G}',\mathfrak{h}')$-skew-invariant component

$$\Big(\frac{1}{|W(\mathrm{G}',\mathfrak{h}')|}\sum_{s'\in W(\mathrm{G}',\mathfrak{h}')}\mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s')\prod_{j=1}^{l'}P_{a_{s_0,j},b_{s_0,j},2\delta_j}(\beta(s'y)_j)\Big)e^{-\beta\sum_{j=1}^{l'}|y_j|}\,. \tag{194}$$

Here we have used that $\sum_{j=1}^{l'}|(s'y)_j|=\sum_{j=1}^{l'}|y_j|$. Notice that

$$\prod_{j=1}^{l'}P_{a_{s_0,j},b_{s_0,j},2\delta_j}(\beta(s'y)_j)=\prod_{j=1}^{l'}P_{a_{s'^{-1}s_0,j},b_{s'^{-1}s_0,j},2\delta_j}(\beta y_j)$$

because $W(\mathrm{G}',\mathfrak{h}')$ only permutes the y-coordinates for which the δ_j's have equal sign. Moreover, (194) is non-zero because $P_{a_{s'^{-1}s_0,j},b_{s'^{-1}s_0,j},2\delta_j}(\beta y_j)$ is not $W(\mathrm{G}',\mathfrak{h}')$-invariant when $W(\mathrm{G}',\mathfrak{h}')\neq 1$. Indeed, the condition $\mu_1>\mu_2>\cdots>\mu_{l'}$ implies $b_1>b_2>\cdots>b_{l'}$ and $a_1<a_2<\cdots<a_{l'}$. If $W(\mathrm{G}',\mathfrak{h}')\neq 1$, then there are at least two indices $j\neq j'$ such that $\delta_j=\delta_{j'}$ and the corresponding factors in (194) have different degrees. (If $b\geq 1$ then the degree of $P_{a,b,2}$ is $b-1$ and if $a\geq 1$ then that of $P_{a,b,-2}$ is $a-1$.)

By (194), the integral (193) is a constant multiple of

$$\kappa_0\check{\chi}_\Pi(\tilde{c}(0))\int_{\tau'(\mathfrak{h}_{\bar{1}}^{reg})}\Phi(y)\pi_{\mathfrak{g}/\mathfrak{z}}(y)F_\phi(y)\,dy\,, \tag{195}$$

where

$$\Phi(y)=\frac{\sum_{s'\in W(\mathrm{G}',\mathfrak{h}')}\mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s')\prod_{j=1}^{l'}P_{a_{s_0,j},b_{s_0,j},2\delta_j}(\beta(s'y)_j)}{\pi_{\mathfrak{g}/\mathfrak{z}}(y)}\;e^{-\beta\sum_{j=1}^{l'}|y_j|} \tag{196}$$

$$(w\in\mathfrak{h}_{\bar{1}}^{reg},y=\tau'(w))\,.$$

By (A.3) and (A.4), we see that there is a non-zero constant $C_{\mathfrak{z}}$, depending of $(\mathrm{G}, \mathrm{G}')$, such that

$$\pi_{\mathfrak{g}/\mathfrak{z}}(y) = C_{\mathfrak{z}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \det(y)_{\mathrm{V}'}^{\gamma} \qquad (y = \tau(w) = \tau'(w),\ w \in \mathfrak{h}_{\overline{1}}), \tag{197}$$

where

$$\gamma = \begin{cases} l - l' & \text{if } \mathbb{D} = \mathbb{C} \\ l - l' + \frac{1}{2} & \text{if } \mathbb{D} = \mathbb{H} \\ l - l' - \frac{1}{2} & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l} \\ l - l' & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l+1} \end{cases} \tag{198}$$

and $\det(g')_{\mathrm{V}'}$ denotes the determinant of g' as an element of $\mathrm{G}' \subseteq \mathrm{GL}_{\mathbb{D}}(\mathrm{V}')$. (See the remark after (E.11) in Appendix E for the case $\mathbb{D} = \mathbb{H}$.)

Recall from Remark 1 that $W(\mathrm{G}', \mathfrak{h}') = W(\mathrm{K}', \mathfrak{h}')$, where K' is maximal compact in G'. Split $\pi_{\mathfrak{g}'/\mathfrak{h}'}$ as a product of the compact and the noncompact positive roots:

$$\pi_{\mathfrak{g}'/\mathfrak{h}'}(y) = \pi_{\mathfrak{k}'/\mathfrak{h}'}(y) \pi^{\mathrm{nc}}_{\mathfrak{g}'/\mathfrak{h}'}(y)\,.$$

Explicitly,

$$\pi_{\mathfrak{k}'/\mathfrak{h}'}\Big(\sum_{j=1}^{l'} y_j J'_j\Big) = \begin{cases} \prod_{1 \le j < k \le l'} i(-y_j + y_k) & \text{if } \mathbb{D} = \mathbb{R}, \mathbb{H}\,, \\ \prod_{1 \le j < k \le p} i(-y_j + y_k) \prod_{1 \le j < k \le q} i(-y_{p+j} + y_{p+k}) & \text{if } \mathbb{D} = \mathbb{C}\,. \end{cases}$$

The polynomial in parenthesis in (194) is $W(\mathrm{G}', \mathfrak{h}')$-skew-invariant. Hence it is divisible by $\pi_{\mathfrak{k}'/\mathfrak{h}'}(y)$ and the fraction

$$\frac{\sum_{s' \in W(\mathrm{G}', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta(s'y)_j)}{\pi_{\mathfrak{k}'/\mathfrak{h}'}(y)} \qquad (y \in \mathfrak{h}') \tag{199}$$

is a $W(\mathrm{G}', \mathfrak{h}')$-invariant polynomial. Therefore Φ is a $W(\mathrm{G}', \mathfrak{h}')$-invariant real-valued nonzero continuous function on $\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})$. Thus Proposition 2 proves the equality (78) and shows that the integral (195) does not vanish for suitably chosen $\phi \in C_c^{\infty}(\mathrm{W})^{\mathrm{G}}$. □

Remark 13. Let us consider the term $e^{-\beta \sum_{j=1}^{l'} |y_j|}$ appearing in (196). Notice that for $w = \sum_{j=1}^{l'} w_j u_j \in \mathfrak{h}_{\overline{1}}$,

$$\sum_{j=1}^{l'} |y_j| = \sum_{j=1}^{l'} |J'^{*}_j(\tau'(w))| = \sum_{j=1}^{l'} w_j^2 = \sum_{j=1}^{l'} \delta_j J'^{*}_j(\tau'(w)) = \left(\sum_{j=1}^{l'} \delta_j J'^{*}_j \right) \circ \tau'(w)\,.$$

This is a quadratic polynomial on $\mathfrak{h}_{\overline{1}}$, invariant under the Weyl group $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$. Such a polynomial has no GG'-invariant extension to W, unless G' is compact. Indeed, suppose P is a real-valued GG'-invariant polynomial on W such that

$$P(w) = \left(\sum_{j=1}^{l'} \delta_j J_j'^* \right) \circ \tau'(w) \qquad (w \in \mathfrak{h}_{\overline{1}}) .$$

Then P extends uniquely to a complex-valued $\mathrm{G}_{\mathbb{C}}\mathrm{G}'_{\mathbb{C}}$-invariant polynomial on the complexification $\mathrm{W}_{\mathbb{C}}$ of W. Hence, by the Classical Invariant Theory, [How89a, Theorems 1A and 1B] there is a $\mathrm{G}'_{\mathbb{C}}$-invariant polynomial Q on $\mathfrak{g}'_{\mathbb{C}}$ such that $P = Q \circ \tau'$. Hence,

$$Q(\tau'(w)) = P(w) = \left(\sum_{j=1}^{l'} \delta_j J_j'^* \right) \circ \tau'(w) \qquad (w \in \mathfrak{h}_{\overline{1}}) .$$

Since $\tau'(\mathfrak{h}_{\overline{1}})$ spans $\mathfrak{h}'$, we see that the restriction of Q to $\mathfrak{h}'$ is

$$Q|_{\mathfrak{h}'} = \sum_{j=1}^{l'} \delta_j J_j'^* \in \mathfrak{h}'_{\mathbb{C}} .$$

Since Q is $\mathrm{G}'_{\mathbb{C}}$-invariant, the restriction $Q|_{\mathfrak{h}'}$ has to be invariant under the corresponding Weyl group. There are no linear invariants if $\mathrm{G}' = \mathrm{Sp}_{2l'}(\mathbb{R})$ or $\mathrm{O}^*_{2l'}$. Therefore $\mathrm{G}' = \mathrm{U}_{p,q}$, $p+q = l'$. But in this case the invariance means that all the δ_j are equal. Hence $\mathrm{G}' = \mathrm{U}_{l'}$ is compact. In the case $\mathrm{G}' = \mathrm{U}_{l'}$, the sum of squares coincides with $\langle J(w), w \rangle$ for a positive complex structure J on W which commutes with G and G' and therefore

$$e^{-\beta \sum_{j=1}^{l'} |\delta_j J_j'^*(\tau'(w))|} \tag{200}$$

extends to a Gaussian on W. If G' is not compact then (200) extends to a GG'-invariant function on W, which is bounded but is not a Gaussian.

11. The Special Case $(\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $l \leq l'$

Here we consider the case $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ and suppose that the character Θ_Π is not supported in the preimage of the connected identity

component $\widetilde{\mathrm{G}^0}$. This is equivalent to $\lambda_l = 0$, where λ is the highest weight of Π. The case $l > l'$ was considered in Theorem 6. Since the dual pair $(\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$ was treated in section 6, we will suppose in the sequel that $2 \leq l \leq l'$. Recall the element $s \in \mathrm{G}$, (88), with centralizer in $\mathfrak{h}$ equal to $\mathfrak{h}_s = \sum_{j=1}^{l-1} \mathbb{R} J_j$, and the spaces

$$\mathsf{V}_{\overline{0},s} = \mathsf{V}_{\overline{0}}^1 \oplus \mathsf{V}_{\overline{0}}^2 \oplus \cdots \oplus \mathsf{V}_{\overline{0}}^{l-1} \oplus \mathbb{R} v_{2l}\,, \quad \mathsf{V}_s = \mathsf{V}_{\overline{0},s} \oplus \mathsf{V}_{\overline{1}}\,.$$

The corresponding dual pair is $(\mathrm{G}_s, \mathrm{G}'_s) = (\mathrm{O}_{2l-1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ acting on the symplectic space $\mathrm{W}_s = \mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0},s})$.

The ordered basis $v_1, v_2, ..., v_{2l-2}, v_{2l-1}, v_{2l}$ of $\mathsf{V}_{\overline{0}}$, leads to the identifications

$$\mathrm{End}(\mathsf{V}_{\overline{0}}) = \mathrm{M}_{2l,2l}(\mathbb{R})\,, \qquad \mathrm{End}(\mathsf{V}_{\overline{0},s}) = \mathrm{M}_{2l-1,2l-1}(\mathbb{R})\,.$$

In these terms, the Cartan subgroup $\mathrm{H} \subseteq \mathrm{G}$ consists of the block diagonal matrices

$$\begin{pmatrix} r(\theta_1) & & & 0 \\ & \ddots & & \\ & & r(\theta_{l-1}) & \\ 0 & & & r(\theta_l) \end{pmatrix},$$

with diagonal blocks

$$r(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad (\theta \in \mathbb{R})\,.$$

Set

$$h_\bullet = \begin{pmatrix} r(\theta_1) & & 0 \\ & \ddots & \\ 0 & & r(\theta_{l-1}) \end{pmatrix} \tag{201}$$

and let $\mathrm{H}_\bullet$ denote the group of all matrices (201). Then the centralizer $\mathrm{H}^s \subseteq \mathrm{H}$ of s consists of the matrices

$$\left(\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & \epsilon & 0 \\ & 0 & \epsilon \end{array}\right), \qquad (\epsilon = \pm 1)\,.$$

The connected component of the identity $(\mathrm{H}^s)^0 \subseteq \mathrm{H}^s$ is the set of these matrices with $\epsilon = 1$. The group G_s and its connected identity component G_s^0 contain Cartan subgroups $\mathrm{H}_s^0 \subseteq \mathrm{G}_s^0$ and $\mathrm{H}_s \subseteq \mathrm{G}_s$ consisting of matrices

$$\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & \epsilon \end{array}\right), \qquad (\epsilon = \pm 1),$$

respectively.

Lemma 35. *Every element of the connected component* $\mathrm{G}^0 s$ *is* G*-conjugate to an element of* $(\mathrm{H}^s)^0 s$.

Proof. Fix an element $g \in \mathrm{G}$. As shown in [Cur84, page 114], g preserves a subspace of V of dimension 1 or 2. Hence V decomposes into a direct sum of g-irreducible subspaces of dimension 1 or 2, and the claim follows. □

Let ξ_ν denote the character of $\mathrm{H}_\bullet$ whose derivative at the identity is $\nu \in i\mathfrak{h}_s^*$. In particular, for $h_\bullet$ as in (201),

$$\xi_{e_j}(h_\bullet) = e^{-i\theta_j} \qquad (1 \le j \le l-1).$$

(The negative sign in the exponent is due to fact that $e_j = -iJ_j^*$.)

The elements $e_j \pm e_k$ $(1 \le j < k \le l-1)$ and $2e_j$ $(1 \le j \le l-1)$ form a system of type C_{l-1} which is dual to that of $((\mathfrak{g}_s)\mathbb{C}, (\mathfrak{h}_s)\mathbb{C})$. The corresponding ρ-function and the Weyl denominator are respectively

$$\rho_s^{\mathrm{C}} = (l-1)e_1 + (l-2)e_2 + \cdots + e_{l-1} \tag{202}$$

and

$$\begin{aligned} &\Delta_s^{\mathrm{C}}\left(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\right) \\ &= \xi_{\rho_s^{\mathrm{C}}}(h_\bullet) \prod_{1 \le j < k \le l-1} (1 - \xi_{e_k - e_j}(h_\bullet))(1 - \xi_{-e_j - e_k}(h_\bullet)) \cdot \prod_{j=1}^{l-1} (1 - \xi_{-2e_j}(h_\bullet)) \\ &\qquad\qquad (h_\bullet \in \mathrm{H}_\bullet). \end{aligned} \tag{203}$$

Observe that the Weyl group of the root system of type C_{l-1} coincides with $W(\mathrm{G}_s^0, \mathfrak{h}_s)$. It consists of all permutations and sign changes of the e_1, $\dots$, e_{l-1}. It acts on $\mathrm{H}_s^0 = \left\{ \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array} \right); h_\bullet \in \mathrm{H}_\bullet \right\}$ and hence on $\mathrm{H}_\bullet$.

The following two lemmas follow respectively from [Wen01, Theorems 2.5 and 2.6].

LEMMA 36. *For any continuous* G*-invariant function* $f : \mathrm{G}^0 s \to \mathbb{C}$,

$$\int_{\mathrm{G}^0 s} f(g)\, dg = \frac{1}{|W(\mathrm{G}_s^0, \mathfrak{h}_s)|} \int_{\mathrm{H}_\bullet} f\Big(\left(\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array} \right) s \Big) \left| \Delta_s^{\mathrm{C}} \Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array} \right) \Big) \right|^2 dh_\bullet \,,$$

where $s = \left(\begin{array}{c|cc} 1_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array} \right)$, *see* (88).

Notice that the coverings

$$\widetilde{\mathrm{G}^0 s} \to \mathrm{G}^0 s \,, \qquad \widetilde{\mathrm{G}^0} \to \mathrm{G}^0$$

split (see Appendix E). Hence we may choose continuous sections

$$(\mathrm{H}^s)^0 s \ni hs \to \widetilde{hs} \in \widetilde{(\mathrm{H}^s)^0 s} \qquad \text{and} \qquad (\mathrm{H}^s)^0 \ni h \to \tilde{h} \in \widetilde{(\mathrm{H}^s)^0} \,. \tag{204}$$

LEMMA 37. *Consider the map*

$$(\mathrm{H}^s)^0 \ni h \to \widetilde{hs} \in \widetilde{(\mathrm{H}^s)^0 s}$$

obtained by composing the multiplication by s *and the fixed continuous section. Then*

$$\Theta_\Pi\Big(\Big(\begin{array}{c|cc} h_\bullet & \multicolumn{2}{c}{\widetilde{}} \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array} \Big) s \Big) = D_\Pi \Theta_{\Pi_s} \Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array} \Big) \Big) \qquad (h_\bullet \in \mathrm{H}_\bullet) \,, \tag{205}$$

where

$$\Theta_{\Pi_s}\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array} \Big) \Big) = \frac{\sum_{t \in W(\mathrm{G}_s^0, \mathfrak{h}_s)} \mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}_s}(t) \xi_{t(\lambda+\rho_s^{\mathrm{C}})}(h_\bullet)}{\Delta_s^{\mathrm{C}}\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array} \Big) \Big)} \qquad (h_\bullet \in \mathrm{H}_\bullet) \,, \tag{206}$$

λ is the highest weight of Π (recall that $\lambda_l = 0$), the sign character $\operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}_s}(t)$ is defined by

$$\Delta_s^{\mathrm{C}}\Big(t\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big) = \operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}_s}(t)\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big) \qquad (t \in W(\mathrm{G}_s^0, \mathfrak{h}_s))\,,$$

and

$$D_\Pi = \pm 1\,. \tag{207}$$

Lemma 38. *For $\phi \in \mathcal{S}(\mathrm{W})$,*

$$\int_{\mathrm{G}^0 s} \check\Theta_\Pi(\tilde g) T(\tilde g)(\phi)\, dg = \frac{1}{|W(\mathrm{G}_s^0, \mathfrak{h}_s)|}\int_{\mathrm{H}_\bullet} \check\Theta_\Pi\Big(\widetilde{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \end{array}\right)}\Big)\left|\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right|^2 \times T_s\Big(\widetilde{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}\right)}\Big)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right) dh_\bullet\,.$$

Proof. Clearly, the integral on the left-hand side does not change if we replace ϕ by ϕ^{G}. Hence we may assume that $\phi = \phi^{\mathrm{G}}$. By Lemma 36, the left-hand side multiplied by $|W(\mathrm{G}_s^0, \mathfrak{h}_s)|$ is equal to

$$\int_{\mathrm{H}_\bullet} \check\Theta_\Pi\Big(\widetilde{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array}\right)}s\Big)\left|\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right|^2 T\Big(\widetilde{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array}\right)}s\Big)(\phi)\, dh_\bullet\,. \tag{208}$$

Apply Lemma G.1 to the decomposition $\mathrm{W} = \mathrm{W}_s \oplus \mathrm{W}_s^\perp$. For $h \in (\mathrm{H}^s)^0$,

$$hs = \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \end{array}\right) = \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \end{array}\right).$$

So

$$hs|_{\mathrm{W}_s} = \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}\right) \quad \text{and} \quad hs|_{\mathrm{W}_s^\perp} = 1|_{\mathrm{W}_s^\perp}\,.$$

Hence $(hs-1)|_{\mathrm{W}_s}$ maps onto W_s and $(hs-1)|_{\mathrm{W}_s^\perp} = 0$. This shows that the restriction of μ_{W} to $(hs-1)\mathrm{W}$ is $\mu_{\mathrm{W}_s} \otimes \delta_0$, where δ_0 is the Dirac delta on $\mathrm{W}_s^\perp$.

Therefore, for an appropriate choice of the lift of the element $\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}\right)$ on the right-hand side,

$$T\Big(\ \widetilde{\left(\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right)s}\ \Big)(\phi) = T_s\left(\widetilde{\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)(\phi|_{\mathrm{W}_s})\,.$$

Thus, (208) is equal to

$$(209)\quad \int_{\mathrm{H}_\bullet} \check{\Theta}_\Pi\Big(\ \widetilde{\left(\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right)s}\ \Big)\left|\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right|^2 T_s\left(\widetilde{\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)(\phi|_{\mathrm{W}_s})\,dh_\bullet\,.$$

The lemma follows from (209). □

LEMMA 39. *Let $\mu^{\mathrm{C}} = \lambda + \rho_s^{\mathrm{C}}$. Then, for $\phi \in \mathcal{S}(\mathrm{W})$,*

$$\int_{\mathrm{G}^0 s} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\,dg$$
$$= D_\Pi \int_{\mathrm{H}_\bullet} \xi_{-\mu^{\mathrm{C}}}(h_\bullet)\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)T_s\left(\widetilde{\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)\,dh_\bullet\,,$$

where $\xi_{-\mu^{\mathrm{C}}}(h_\bullet)$ makes sense because $\lambda_l = 0$.

PROOF. This follows from Lemma 38. Indeed, notice that

$$\check{\Theta}_\Pi\left(\widetilde{\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array}}\right) = \Theta_\Pi\left(\widetilde{\begin{array}{c|cc} h_\bullet^{-1} & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array}}\right) = \Theta_\Pi\Big(\ \widetilde{\left(\begin{array}{c|cc} h_\bullet^{-1} & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right)s}\ \Big)\,.$$

Hence (205) and (206) show that

$$\check{\Theta}_\Pi\left(\widetilde{\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array}}\right) = D_\Pi\Theta_{\Pi_s}\left(\begin{array}{c|cc} h_\bullet^{-1} & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right)$$
$$= D_\Pi \frac{\sum_{t\in W(\mathrm{G}_s^0,\mathfrak{h}_s)} \mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}_s}(t)\xi_{t^{-1}\mu^{\mathrm{C}}}(h_\bullet^{-1})}{\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet^{-1} & 0 \\ \hline 0 & 1 \end{array}\right)\Big)}\,.$$

Furthermore,

$$\Delta_s^{\mathrm{C}}\Big(\Big(\begin{array}{c|c} h_\bullet^{-1} & 0 \\ \hline 0 & 1 \end{array}\Big)\Big) = \overline{\Delta_s^{\mathrm{C}}\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)}$$

and for $t \in W(\mathrm{G}_s^0, \mathfrak{h}_s)$,

$$\Delta_s^{\mathrm{C}}\Big(t\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big) = \mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}_s}(t)\Delta_s^{\mathrm{C}}\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)\,.$$

Therefore

$$\begin{aligned}\check{\Theta}_\Pi\left(\widetilde{\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array}}\right)\left|\Delta_s^{\mathrm{C}}\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)\right|^2 \\ = D_\Pi \sum_{t\in W(\mathrm{G}_s^0,\mathfrak{h}_s)} \xi_{t^{-1}\mu^{\mathrm{C}}}(h_\bullet^{-1})\Delta_s^{\mathrm{C}}\Big(t\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)\,.\end{aligned}$$

Notice that

$$\xi_{t^{-1}\mu^{\mathrm{C}}}(h_\bullet^{-1}) = \xi_{-\mu^{\mathrm{C}}}(th_\bullet)$$

and since ϕ^{G} is G-invariant,

$$\begin{aligned}T_s\left(\widetilde{\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right) &= T_s\left(\widetilde{t\begin{pmatrix} h_\bullet & 0 \\ 0 & -1 \end{pmatrix}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right) \\ &= T_s\left(\widetilde{\begin{array}{c|c} th_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\mathrm{H}_\bullet} \check{\Theta}_\Pi\left(\widetilde{\begin{array}{c|cc} h_\bullet & 0 & \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array}}\right)\left|\Delta_s^{\mathrm{C}}\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)\right|^2 T_s\left(\widetilde{\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)\,dh_\bullet \\ &= D_\Pi \int_{\mathrm{H}_\bullet} \sum_{t\in W(\mathrm{G}_s^0,\mathfrak{h}_s)} \xi_{-\mu^{\mathrm{C}}}(th_\bullet)\Delta_s^{\mathrm{C}}\Big(t\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)T_s\left(\widetilde{\begin{array}{c|c} th_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)\,dh_\bullet \\ &= |W(\mathrm{G}_s^0,\mathfrak{h}_s)|D_\Pi \int_{\mathrm{H}_\bullet} \xi_{-\mu^{\mathrm{C}}}(h_\bullet)\Delta_s^{\mathrm{C}}\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)T_s\left(\widetilde{\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)\,dh_\bullet\,.\ \square\end{aligned}$$

Consider the Cayley transform $c_\bullet : \mathfrak{h}_\bullet \to \mathrm{H}_\bullet$ and the (modified) Cayley transform $c_\odot : \mathfrak{h}_s \to (\mathrm{H}^s)^0$ defined by

$$c_\bullet \begin{pmatrix} x_1J_1 & & 0 \\ & \ddots & \\ 0 & & x_{l-1}J_{l-1} \end{pmatrix} = \begin{pmatrix} c(x_1J_1) & & 0 \\ & \ddots & \\ 0 & & c(x_{l-1}J_{l-1}) \end{pmatrix} \tag{210}$$

$$c_\odot \left(\begin{array}{ccc|c} x_1J_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & x_{l-1}J_{l-1} & \\ \hline & 0 & & 0 \end{array}\right) \tag{211}$$

$$= \left(\begin{array}{ccc|c} c(x_1J_1) & & 0 & \\ & \ddots & & 0 \\ 0 & & c(x_{l-1}J_{l-1}) & \\ \hline & 0 & & 1 \end{array}\right), \quad \text{i.e. } c_\odot = c_\bullet \times \exp .$$

Notice that $c_\odot$ differs from the usual Cayley transform c_s on $\mathfrak{h}_s$, defined at the beginning of section 8, for which $c_s(\mathrm{diag}(x_1J_1, \dots, x_{l-1}J_{l-1}, 0)) = \mathrm{diag}(c(x_1J_1), \dots, c(x_{l-1}J_{l-1}), -1)$. Let $j_{\mathfrak{h}_s}$ denote the Jacobian of the map $c_\odot$. Set

$$\pi^{\mathrm{C}}_{\mathfrak{g}_s/\mathfrak{h}_s} \left(\begin{array}{ccc|c} x_1J_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & x_{l-1}J_{l-1} & \\ \hline & 0 & & 0 \end{array}\right) = \prod_{1\le j<k\le l-1} (-x_j^2 + x_k^2) \cdot \prod_{j=1}^{l-1} (-2ix_j) .$$

Lemma 40. *There are constants A and D such that for $x = \sum_{j=1}^{l-1} x_j J_j \in \mathfrak{h}_s$,*

$$\Delta^{\mathrm{C}}_s(c_\odot(x)) = A\, \pi^{\mathrm{C}}_{\mathfrak{g}_s/\mathfrak{h}_s}(x) \prod_{j=1}^{l-1} (1 + x_j^2)^{-l+1} , \tag{212}$$

$$\Theta_s \left(\left(\begin{array}{c|c} \widetilde{c_\bullet(x)} & 0 \\ \hline 0 & -1 \end{array} \right) \right) = \left(\frac{i}{2}\right)^{(2l-1)l'} 2^{l'} \prod_{j=1}^{l-1} (1 + x_j^2)^{l'} , \tag{213}$$

$$j_{\mathfrak{h}_s}(x) = \prod_{j=1}^{l-1} 2(1 + x_j^2)^{-1} \tag{214}$$

and hence

$$\Delta_s^{\mathrm{C}}(c_\odot(x))\Theta_s\left(\left(\begin{array}{c|c} \widetilde{c_\bullet(x)} & 0 \\ \hline 0 & -1 \end{array}\right)\right) j_{\mathfrak{h}_s}(x) = D\,\pi^{\mathrm{C}}_{\mathfrak{g}_s/\mathfrak{h}_s}(x)\prod_{j=1}^{l-1}(1+x_j^2)^{l'-l}\,. \tag{215}$$

Proof. Part (212) may be verified via the argument used in Appendix C, but easier – without the square roots. Formula (213) follows from (137), and (214) from Appendix B applied to the group SO_2. □

Lemma 41. *With the notation of Lemmas 39 and 40,*

$$\int_{\mathrm{G}^0 s}\check\Theta_\Pi(\tilde g)T(\tilde g)(\phi)\,dg = CD_\Pi\check\chi_\Pi(\tilde c(0))\int_{\mathfrak{h}_\bullet}\xi_{-\mu^{\mathrm{C}}}(c_-(x))\prod_{j=1}^{l-1}(1+x_j^2)^{l'-l}$$
$$\times\,\pi_{\mathfrak{g}_s/\mathfrak{h}_s}\left(\left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array}\right)\right)\int_{\mathrm{W}_s}\chi\left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array}\right)(w)\phi^{\mathrm{G}}(w)\,dw\,dx\,,$$

where $\check\chi_\Pi$ is the central character of Π and $D_\Pi = \pm 1$.

Proof. We start with the formula of Lemma 39, use the equality

$$\xi_{-\mu^{\mathrm{C}}}(h_\bullet) = \xi_{-\mu^{\mathrm{C}}}((-1)_\bullet)\xi_{-\mu^{\mathrm{C}}}(-h_\bullet)\,,$$

apply the change of variables, $h_\bullet = c_\bullet(x)$ and use the formula (215), noticing that $\pi^{\mathrm{C}}_{\mathfrak{g}_s/\mathfrak{h}_s}$ is a constant multiple of $\pi_{\mathfrak{g}_s/\mathfrak{h}_s}$. Here $c_\bullet : \mathfrak{h}_\bullet \to \mathrm{H}_\bullet$, so $c_\bullet(0) = (-1)_\bullet$.

It remains to prove that $\xi_{-\mu^{\mathrm{C}}}((-1)_\bullet)$ is a constant multiple of the central character of Π evaluated at $\tilde c(0)$. For this, let $v \neq 0$ be a highest vector of Π. For now, let us denote by $\xi^{\widetilde{\mathrm{H}}}_{-\lambda}$ and $\xi^{\mathrm{H}}_{-\lambda}$ the characters defined by λ on $\widetilde{\mathrm{H}}$ and H, respectively. Then $\xi^{\widetilde{\mathrm{H}}}_{-\lambda}(\tilde c(0)) = \xi^{\mathrm{H}}_{-\lambda}(c(0))$ because λ is integral; see Appendix H. Hence $\xi^{\mathrm{H}}_{-\lambda}(c(0))v = \Pi(\tilde c(0))v = \check\chi_\Pi(\tilde c(0))v$. This implies that $\check\chi_\Pi(\tilde c(0)) = \xi^{\mathrm{H}}_{-\lambda}(c(0))$. Since $\lambda_l = 0$,

$$\xi^{\mathrm{H}}_{-\lambda}(c(0)) = \xi^{\mathrm{H}}_{-\lambda}(-I_{2l}) = \xi^{\mathrm{H}}_{-\lambda}\left(\begin{array}{c|c} (-1)_\bullet & 0 \\ \hline 0 & I_2 \end{array}\right) = \xi_{-\lambda}((-1)_\bullet) = \xi_{\rho_s^{\mathrm{C}}}((-1)_\bullet)\xi_{-\mu^{\mathrm{C}}}((-1)_\bullet)\,,$$

where $\xi_{\rho_s^{\mathrm{C}}}((-1)_\bullet) = \pm 1$. □

Recall from (150) the symmetric bilinear form

$$B(x_\bullet, y_\bullet) \qquad (x_\bullet, y_\bullet \in \mathfrak{h}_\bullet)\,.$$

COROLLARY 42. *There is a constant C depending only on the dual pair and a constant $D_\Pi = \pm 1$ distinguishing the representations Π and $\Pi \otimes \det$, such that*

$$\begin{aligned}\int_{\mathrm{G}^0 s} &\check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg \\ &= C D_\Pi \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}_\bullet}\int_{\mathfrak{h}_\bullet} \prod_{j=1}^{l-1} (1+ix_j)^{\mu_j^{\mathrm{C}}+l'-l}(1-ix_j)^{-\mu_j^{\mathrm{C}}+l'-l} e^{iB(x_\bullet,y_\bullet)} \\ &\quad \times F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}} \left(\begin{array}{c|c} y_\bullet & 0 \\ \hline 0 & 0 \end{array}\right) dy_\bullet\, dx_\bullet\,.\end{aligned}$$

PROOF. By Lemma 23,

$$\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x) \int_{\mathrm{W}_s} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw = C \int_{\mathfrak{h}_\bullet} e^{iB(x_\bullet,y_\bullet)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}} \left(\begin{array}{c|c} y_\bullet & 0 \\ \hline 0 & 0 \end{array}\right) dy_\bullet\,.$$

By the proof of Lemma 20,

$$\xi_{-\mu^{\mathrm{C}}}(c_-(x)) = \prod_{j=1}^{l-1} (1+ix_j)^{\mu_j^{\mathrm{C}}}(1-ix_j)^{-\mu_j^{\mathrm{C}}}\,.$$

Hence the formula follows from Lemma 41. □

PROOF OF THEOREM 7. To prove (90), we proceed as in the proof of Theorem 4, using Corollary 42. □

12. The Special Case $(\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $1 \le l \le l'$

Recall the decomposition (91). As in the previous section, we denote the objects corresponding to W_s by the subscript s, for instance Θ_s and T_s. Similarly, we denote the objects corresponding to $\mathrm{W}_s^\perp$ by the substrict $\perp$, for instance $\Theta_\perp$ and $T_\perp$. .

If H is our Cartan subgroup of G, then the elements of connected identity component H^0 are of the form $h = \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)$ with $h_\bullet$ in the Cartan subgroup H_s of G_s; see (201). Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_s.h \in \mathrm{H}^0$ acts trivially on $\mathrm{W}_s^\perp$, we see that $(h-1)\mathrm{W} = (h_\bullet - 1)\mathrm{W}_s$. Hence,

$$\mu_{(h-1)\mathrm{W}} = \mu_{(h_\bullet - 1)\mathrm{W}_s} \otimes \delta_0 \,,$$

where δ_0 is the Dirac delta on $\mathrm{W}_s^\perp$.

Lemma 43. *There is a choice of coverings* $\widetilde{\mathrm{H}^0} \to \mathrm{H}^0$ *and* $\widetilde{\mathrm{H}_\bullet} \to \mathrm{H}_\bullet$ *such that the map* $\widetilde{\mathrm{H}_\bullet} \in \widetilde{h_\bullet} \to \widetilde{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)} \in \widetilde{\mathrm{H}^0}$ *is a Lie group isomorphism and*

$$\Theta(\tilde{h}) = \Theta_s(\widetilde{h_\bullet}) \quad \textit{and} \quad T(\tilde{h}) = T_s(\widetilde{h_\bullet}) \otimes \delta_0 \qquad (h \in \mathrm{H}^0)\,. \tag{216}$$

Proof. We apply Lemma G.1 to the decomposition $\mathrm{W} = \mathrm{W}_s \oplus \mathrm{W}_s^\perp$. Then $h|_{\mathrm{W}_s} = h_\bullet$ and $h|_{\mathrm{W}_s^\perp} = 1$. Choose $\tilde{1}$ such that $\tilde{1} = 1_{\widetilde{\mathrm{Sp}(\mathrm{W}_s^\perp)}}$ (the identity of the metaplectic group). Hence $T_\perp(\tilde{1}) = \delta_0$ and, by Lemma G.1,

$$T(\tilde{h}) = \frac{\chi_+(\tilde{h})}{\chi_{s,+}(\widetilde{h_\bullet})\chi_{\perp,+}(\tilde{1})}\, T_s(\widetilde{h_\bullet}) \otimes \delta_0 \,,$$

where χ_+, $\chi_{s,+}$ and $\chi_{\perp,+}$ are defined according to (G.1) for W, W_s and $\mathrm{W}_s^\perp$, respectively.

We now show that $\chi_+(\tilde{h}) = \chi_{s,+}(\widetilde{h_\bullet})$ and that $\chi_{\perp,+}(\tilde{1}) = 1$, which will complete the proof of the second equality in (216).

We choose complete polarizations

$$\mathrm{W}_s = \mathrm{X}_1 \oplus \mathrm{Y}_1 \quad \text{and} \quad \mathrm{W}_s^\perp = \mathrm{X}_2 \oplus \mathrm{Y}_2$$

preserved by $\mathrm{G} = \mathrm{O}_{2l+1}$. Then

$$\mathrm{W} = \mathrm{X} \oplus \mathrm{Y} \qquad (\mathrm{X} = \mathrm{X}_1 \oplus \mathrm{X}_2,\ \mathrm{Y} = \mathrm{Y}_1 \oplus \mathrm{Y}_2)$$

is a complete polarization preserved by G. The double covers can be realized as

$$\widetilde{\mathrm{G}} = \{(g,\zeta) \in \mathrm{G} \times \mathbb{C}^\times;\, (\det g)_{\mathrm{X}} = (\det g)^{l'} = \zeta^2\}\,,$$

$$\widetilde{\mathrm{G}|_{\mathrm{W}_s}} = \{(g,\zeta) \in \mathrm{G}|_{\mathrm{W}_s} \times \mathbb{C}^\times; (\det g)_{\mathrm{X}_1} = (\det g)^{l'} = \zeta^2\}\,.$$

(See Appendix E.) Furthermore, by [AP14, Proposition 4.28],

$$\frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|} = \frac{\det{}^{-1/2}(\tilde{g})}{|\det{}^{-1/2}(\tilde{g})|} \qquad (\tilde{g} \in \widetilde{\mathrm{G}})\,. \tag{217}$$

Since for $h \in \mathrm{H}^0$

$$(\det h)_{\mathrm{X}} = (\det h|_{\mathrm{X}_1})_{\mathrm{X}_1}\,,$$

we see that we may choose the cover $\widetilde{\mathrm{H}_\bullet}$ adjusted to $\widetilde{\mathrm{H}^0}$ so that

$$\chi_+(\tilde{h}) = \chi_{s,+}(\widetilde{h_\bullet}) \qquad \left(h = \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\right)\,. \tag{218}$$

As recalled on page 283, for any $\tilde{g}$ in the metaplectic group such that g preserves the decomposition $\mathrm{W}_s^\perp = \mathrm{X}_2 \oplus \mathrm{Y}_2$, the restriction of the Weil representation acts by

$$\omega(\tilde{g})f(x) = \det(\tilde{g})^{-1/2} f(g^{-1}x) \qquad (x \in \mathrm{X}_2)\,.$$

Applying this equality to $\tilde{1} \in \widetilde{\mathrm{G}}|_{\mathrm{W}_s^\perp}$, we see that $\det(\tilde{1})^{-1/2} = 1$. Thus (217) implies that $\chi_{\perp,+}(\tilde{1}) = 1$. This proves the second equality in (216).

To prove that $\Theta(\tilde{h}) = \Theta_s(\widetilde{h_\bullet})$, observe first that $\Theta^2(1) = 1$ by [AP14, Definition 4.16]. Therefore $|\Theta_\perp(\tilde{1})| = 1$. As shown in the proof of Lemma G.1, this implies that $|\Theta(\tilde{h})| = |\Theta_s(\widetilde{h_\bullet})|$. So the claim follows from (218). □

Proof of Theorem 8. As in (210), consider the Cayley transform $c_\bullet : \mathfrak{h}_s \to \mathrm{H}_s$ and the modified Cayley transform $c_\odot : \mathfrak{h}_s \to \mathrm{H}^0$, defined by

$$c_\odot(\mathrm{diag}(x_1 J_1, \dots, x_l J_l, 0)) = \mathrm{diag}(v_1, \dots, v_l, 1), \quad v_j = \frac{-ix_j + 1}{-ix_j - 1} \qquad (x_j \in \mathbb{R}, 1 \le j \le l)\,, \tag{219}$$

i.e. $c_\odot = c_\bullet \times \exp$. See Appendix C for the above realization of H^0. Notice also that $W(\mathrm{G},\mathfrak{h}) = W(\mathrm{G}_s,\mathfrak{h}_s)$.

By (216) and since $c_\bullet(\mathfrak{h}_s)$ is dense in H_s,

$$\begin{aligned}
(220)\qquad &\int_{\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg \\
&= \frac{1}{|W(\mathrm{G}_s,\mathfrak{h}_s)|} \int_{\mathrm{H}_s} \overline{\check{\Theta}_\Pi\Big(\widetilde{\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big)\Delta\Big(\widehat{\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big)} \\
&\quad \times \Delta\Big(\widehat{\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big) T_s(\tilde{h}_\bullet)(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\, dh_\bullet \\
&= \frac{4^l}{|W(\mathrm{G}_s,\mathfrak{h}_s)|} \int_{\mathfrak{h}_s} \check{\Theta}_\Pi\Big(\widetilde{\Big(\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big) \overline{\Delta\Big(\widehat{\Big(\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big)} \\
&\quad \times \Delta\Big(\widehat{\Big(\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big) \Theta_s(\widetilde{c_\bullet}(x)) \\
(221)\qquad &\quad \times \int_{\mathrm{W}_s} \chi_x(w) \left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)(w)\, dw \cdot \mathrm{ch}^{-2}(x)\, dx\,,
\end{aligned}$$

where the jacobian of the map $c_\odot : \mathfrak{h}_s \to \mathrm{H}^0$ is computed using Appendix B for $\mathrm{G} = \mathrm{SO}_2$. As shown in Appendix B, the Weyl group of $(\mathrm{Spin}_{2l+1}, \widehat{\mathrm{H}^0})$ is isomorphic to the Weyl group of $(\mathrm{SO}_{2l+1}, \mathrm{H}^0)$ and the covering $\widehat{\mathrm{H}^0} \to \mathrm{H}^0$ intertwines the action of these groups. As before, we denote both Weyl groups by $W(\mathrm{G},\mathfrak{h})$. For every $t \in W(\mathrm{G},\mathfrak{h})$ and $x \in \mathfrak{h}$, we have $t c_\odot(x) = c_\odot(tx)$. Indeed, a permutation acts on $c_\odot(x)$ by permuting the coordinates of x, and a sign change $\varepsilon = \pm 1$ acts on each coordinate by

$$\varepsilon : v = \frac{-ix+1}{-ix-1} \to v^\varepsilon = \frac{-i\varepsilon x+1}{-i\varepsilon x-1} \qquad (x \in \mathbb{R})\,.$$

Therefore,

$$t\,\widehat{c_\odot}(x) = \widehat{c_\odot(tx)} \qquad (x \in \mathfrak{h},\ x_j \neq 0,\ 1 \le j \le l)\,.$$

Consequently, if μ is the Harish-Chandra parameter of Π, then

$$\begin{aligned}
(222)\quad \xi_{-t\mu}(\widehat{c_\odot(x)}) = \xi_{-\mu}(t\,\widehat{c_\odot}(x)) = \xi_{-\mu}(\widehat{c_\odot(tx)}) \\
(t \in W(\mathrm{G},\mathfrak{h}),\ x \in \mathfrak{h},\ x_j \neq 0,\ 1 \le j \le l)\,.
\end{aligned}$$

For x as in (222), we now proceed as in Lemma 17:

$$\check{\Theta}_\Pi\Big(\widetilde{\Big(\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big) \overline{\Delta\Big(\widehat{\Big(\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}\Big)}\Big)}$$

$$
\begin{aligned}
&= \Theta_{\Pi}\Big(\Big(\widetilde{\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}}\Big)^{-1}\Big)\Delta\Big(\Big(\widehat{\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}}\Big)^{-1}\Big)\\
&= \sum_{t\in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\xi_{-t\mu}\Big(\Big(\widehat{\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}}\Big)\Big)\\
&= \sum_{t\in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\xi_{-\mu}\Big(t\Big(\widehat{\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}}\Big)\Big)\\
&= \sum_{t\in W(\mathrm{G}_s,\mathfrak{h}_s)} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\xi_{-\mu}(\widehat{c_\bullet}(tx))\,,
\end{aligned}
$$

and

$$
\Theta_s(\widetilde{c_\bullet}(x))\left(\int_{\mathrm{W}_s}\chi_x(w)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)(w)\,dw\right)\mathrm{ch}^{-2}(x)
$$

is a $W(\mathrm{G}_s,\mathfrak{h}_s)$-invariant function of $x\in\mathfrak{h}_s$. Hence (220) is a constant multiple of

$$
\begin{aligned}
(223)\quad &\frac{4^l}{|W(\mathrm{G}_s,\mathfrak{h}_s)|}\sum_{t\in W(\mathrm{G}_s,\mathfrak{h}_s)}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\int_{\mathfrak{h}_s}\xi_{-\mu}(\widehat{c_\bullet}(tx))\Delta\Big(\Big(\widehat{\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}}\Big)\Big)\Theta_s(\widetilde{c_\bullet}(x))\\
&\qquad\times\left(\int_{\mathrm{W}_s}\chi_x(w)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)(w)\,dw\right)\mathrm{ch}^{-2}(x)\,dx\\
&= 4^l\int_{\mathfrak{h}_s}\xi_{-\mu}(\widehat{c_\bullet}(x))\Delta\Big(\Big(\widehat{\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}}\Big)\Big)\Theta_s(\widetilde{c_\bullet}(x))\frac{1}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)}\,\mathrm{ch}^{-2}(x)\\
&\qquad\times\left(\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)\int_{\mathrm{W}_s}\chi_x(w)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)(w)\,dw\right)dx\,.
\end{aligned}
$$

Appendix C, (137) and (145) show that there is a constant C_1 such that

$$
\begin{aligned}
(224)\quad &\Delta\Big(\Big(\widehat{\begin{array}{c|c} c_\bullet(x) & 0 \\ \hline 0 & 1 \end{array}}\Big)\Big)\Theta_s(\widetilde{c_\bullet}(x))\frac{1}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)}\,\mathrm{ch}^{-2}(x)\\
&\qquad= C_1\,\mathrm{ch}^{2l'-2l-1}(x)\prod_{j=1}^{l}\mathrm{sgn}(x_j)\qquad\Big(x\in\mathfrak{h},\,x=\sum_{j=1}^{l}x_jJ_j,\,x_j\neq 0\Big)\,.
\end{aligned}
$$

By Lemma 23, there is a constant C_2 such that

$$
(225)\quad \pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)\int_{\mathrm{W}_s}\chi_x(w)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)(w)\,dw = C_2\int_{\mathfrak{h}_s}e^{iB(x,y)}F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,.
$$

Notice that $\mu_j + \frac{1}{2}$ is a positive integer for $1 \le j \le l$. By Lemma 20 and (C.9),

$$\begin{aligned}(226)\qquad \xi_{-\mu}(\widehat{c_\bullet}(x)) &= \prod_{j=1}^{l} \Big(\frac{ix_j+1}{ix_j-1}\Big)^{\mu_j+\frac{1}{2}} \sqrt{\frac{ix_j-1}{ix_j+1}} \\ &= \prod_{j=1}^{l} \frac{(ix_j+1)^{\mu_j+\frac{1}{2}}}{(ix_j-1)^{\mu_j+\frac{1}{2}}} \frac{\sqrt{ix_j-1}}{\sqrt{ix_j+1}} \\ &= \prod_{j=1}^{l} \frac{(ix_j+1)^{\mu_j+\frac{1}{2}}}{(-1)^{\mu_j+\frac{1}{2}}(1-ix_j)^{\mu_j+\frac{1}{2}}} \frac{\sqrt{1-ix_j}}{\sqrt{ix_j+1}}\, i\mathrm{sgn}(x_j) \\ &= i^l(-1)^{|\mu|+\frac{l}{2}} \prod_{j=1}^{l} (1+ix_j)^{\mu_j}(1-ix_j)^{-\mu_j} \prod_{j=1}^{l} \mathrm{sgn}(x_j)\,,\end{aligned}$$

where $|\mu| = \sum_{j=1}^{l} \mu_j$. Since $\delta = \frac{1}{2}(2l' - 2l + 1)$, see (68), we get from (145)

$$(227)\quad \xi_{-\mu}(\widehat{c_\bullet}(x))\,\mathrm{ch}^{2l'-2l-1}(x) = i^l(-1)^{|\mu|+\frac{l}{2}} \prod_{j=1}^{l} (1+ix_j)^{-a_j}(1-ix_j)^{-b_j} \prod_{j=1}^{l} \mathrm{sgn}(x_j)\,,$$

where a_j and b_j are as in (69).

The above implies that (223) is equal to a constant multiple of

$$\begin{aligned}(228)\qquad &\int_{\mathfrak{h}_s} \xi_{-\mu}(\widehat{c_\bullet}(x))\,\mathrm{ch}^{2l'-2l-1}(x) e^{iB(x,y)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,dx \\ &\quad = i^l(-1)^{|\mu|+\frac{l}{2}} \int_{\mathfrak{h}_s} \prod_{j=1}^{l} (1+ix_j)^{-a_j}(1-ix_j)^{-b_j} \int_{\mathfrak{h}_s} e^{iB(x,y)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,dx\,.\end{aligned}$$

Since $\tau(\mathrm{W}_s) \cap \mathfrak{h}_s = \mathfrak{h}_s$ for $\mathbb{D} = \mathbb{R}$, we are in the situation considered by Theorem 4, see also Corollary 26. Hence the same computation as in Theorem 4 shows that (228) is equal to $i^l(-1)^{|\mu|+\frac{l}{2}}$ times

$$\begin{aligned}(229)\qquad &\int_{\mathfrak{h}_s} \prod_{j=1}^{l} \Big(P_{a_j,b_j}(\beta y_j)e^{-\beta|y_j|} + \beta^{-1}Q_{a_j,b_j}(-\beta^{-1}\partial_{y_j})\delta_0(y_j)\Big) F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy \\ &\quad = \int_{\mathfrak{h}_s} \prod_{j=1}^{l} \Big(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\Big) F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,.\end{aligned}$$

Recall from Appendix H that the highest weights of Π are integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l \ge 0$ and that $\rho = \sum_{j=1}^{l}(l + \frac{1}{2} - j)e_j$. Hence

$$(230)\qquad (-1)^{|\mu|+\frac{l}{2}} = (-1)^{\frac{l(l+1)}{2}}(-1)^{|\lambda|}\,.$$

We now look at $F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}$ when $l = l'$. By (I.1), there is a constant $C_1 > 0$ such that

$$\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}) = C_1 \int_{\mathrm{G}} \int_{\mathrm{G}'/\mathrm{Z}'} \phi(gg'.w)\, dg\, d(g'\mathrm{Z}') \qquad (\phi \in \mathcal{S}(\mathrm{W}),\, w \in \mathfrak{h}_{\overline{1}}^{reg})\,. \tag{231}$$

Because of the embedding $\mathrm{G}_s \subseteq \mathrm{G}$ and the normalization $\mathrm{vol}(\mathrm{G}_s) = 1$,

$$\begin{aligned}\int_{\mathrm{G}_s}\int_{\mathrm{G}'/\mathrm{Z}'} &\phi^{\mathrm{G}}(g_s g'.w)\, dg_s\, d(g'\mathrm{Z}') \\ &= \int_{\mathrm{G}_s}\int_{\mathrm{G}'/\mathrm{Z}'}\int_{\mathrm{G}} \phi((gg_s)g'.w))\, dg\, dg_s\, d(g'\mathrm{Z}') \\ &= \int_{\mathrm{G}}\int_{\mathrm{G}'/\mathrm{Z}'} \phi(gg'.w)\, dg\, d(g'\mathrm{Z}') \qquad (\phi \in \mathcal{S}(\mathrm{W}),\, w \in \mathfrak{h}_{\overline{1}}^{reg})\,.\end{aligned}$$

Hence, for arbitrary $\phi \in \mathcal{S}(\mathrm{W})$,

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi^{\mathrm{G}}|_{\mathrm{W}_s}) = \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) \qquad (w \in \mathfrak{h}_{\overline{1}}^{reg})\,. \tag{232}$$

Since $\pi_{\mathfrak{g}'/\mathfrak{h}'}(y) = \pi_{\mathfrak{g}'_s/\mathfrak{h}'_s}(y)$ by (A.3), we conclude that there is constant C_2 such that

$$F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}} = C_2 F_\phi = C_2 F_{\phi^{\mathrm{G}}} \qquad (\phi \in \mathcal{S}(\mathrm{W}))\,. \tag{233}$$

This finishes the proof of Theorem 8. □

Remark 14. When $l < l'$ the Weyl–Harish-Chandra orbital integrals involve almost semisimple elements, see (48), and the $F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}$ is not necessarily proportional to F_ϕ as a function of $\phi \in \mathcal{S}(\mathrm{W})$. Indeed, let $w_0 \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^0)$, as in (48). Then by (I.3), there is a constant $C_3 > 0$ such that

$$\begin{aligned}\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}} &\phi(s.(w+w_0)))\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}) \\ &= C_3 \int_{\mathrm{G}} \int_{\mathrm{G}'/\mathrm{Z}'^{\,n}} \phi(gg'.(w+w_0))\, dg\, d(g'\mathrm{Z}'^{\,n}) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\, w \in \mathfrak{h}_{\overline{1}}^{reg})\,,\end{aligned} \tag{234}$$

where Z'^n is the centralizer of $n = \tau'(w_0)$ in G'. Because of the embedding $\mathrm{G}_s \subseteq \mathrm{G}$ and the normalization $\mathrm{vol}(\mathrm{G}_s) = 1$,

$$\int_{\mathrm{G}_s}\int_{\mathrm{G}'/\mathrm{Z}'^n} \phi^{\mathrm{G}}(g_s g'.(w+w_0))\, dg_s\, d(g'\mathrm{Z}'^{\,n})$$

$$= \int_{\mathrm{G}_s}\int_{\mathrm{G}'/\mathrm{Z}'^n}\int_{\mathrm{G}} \phi((gg_s)g'.(w+w_0))\, dg\, dg_s\, d(g'\mathrm{Z}'^{\,n})$$
$$= \int_{\mathrm{G}}\int_{\mathrm{G}'/\mathrm{Z}'^n} \phi(gg'.(w+w_0))\, dg\, d(g'\mathrm{Z}'^{\,n}) \qquad (\phi\in\mathcal{S}(\mathrm{W}),\, w\in\mathfrak{h}_{\overline{1}}^{reg}).$$

However it may happen that

$$\int_{\mathrm{G}_s}\int_{\mathrm{G}'/\mathrm{Z}'^n} \phi^{\mathrm{G}}(g_sg'.(w+w_0))\, dg_s\, d(g'\mathrm{Z}'^{\,n}) \neq \int_{\mathrm{G}_s}\int_{\mathrm{G}'/\mathrm{Z}'^n} \phi^{\mathrm{G}}(g_sg'.w)\, dg_s\, d(g'\mathrm{Z}'^{\,n}).$$

Hence, by (234), there is generally no positive constant C_4 such that, for arbitrary $\phi\in\mathcal{S}(\mathrm{W})$,

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi^{\mathrm{G}}|_{\mathrm{W}_s}) = C_4\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) \qquad (w\in\mathfrak{h}_{\overline{1}}^{reg}). \tag{235}$$

13. Proof of Theorem 9

Before proving Theorem 9, let us remark that we will not need to distinguish between the cases $l > l'$ and $l \le l'$. We will be working with a Cartan subgroup of G, which we shall denote by H and not by $\mathrm{H}(\mathfrak{g})$ as previously done when $l > l'$. This is justified because the Cartan subspaces of W, which led to the decomposition $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}\oplus\mathfrak{h}''$, play no role here. On the other hand, we will need to distinguish between the even and odd orthogonal groups.

Consider first the case $\mathrm{G} = \mathrm{O}_{2l}$ with $l > 1$. Retain the notation introduced at the beginning of section 11 and let ρ_s^{C} be as in (89). Then the functions ξ_{ρ_s} and Δ_s for $\mathrm{G}_s = \mathrm{O}_{2l-1}$ are defined on the double cover $\widehat{\mathrm{H}_s^0}$ of H_s^0 introduced in section 9:

$$\Delta_s(\widehat{h}) = \xi_{\rho_s}(\widehat{h}) \prod_{1\le j<k\le l-1} (1-\xi_{e_k-e_j}(h))(1-\xi_{-e_j-e_k}(h))\cdot\prod_{j=1}^{l-1}(1-\xi_{-e_j}(h)) \quad (h\in\mathrm{H}_s^0).$$

Nevertheless, $|\Delta_s(\widehat{h})|$ is well defined as a function on H_s^0 itself, and can be considered as a function on $\mathrm{H}_\bullet \subseteq \mathrm{O}_{2(l-1)}$ by setting $|\Delta_s(h_\bullet)| = \left|\Delta_s\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big)\right|$. Observe that for $\nu = \sum_{j=1}^{l-1}\nu_j e_j \in i\mathfrak{h}_s^*$ with $\nu_j\in\mathbb{Z}$ for $1\le j\le l$,

$$\xi_\nu(h_\bullet) = \xi_\nu\Big(\Big(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\Big)\Big) = \prod_{j=1}^{l-1} e^{-i\nu_j\theta_j} \qquad \Big(h_\bullet = \exp\Big(\sum_{j=1}^{l-1}\theta_jJ_j\Big)\in\mathrm{H}_\bullet\Big). \tag{236}$$

Hence

$$\xi_\nu(-h_\bullet) = (-1)^{|\nu|}\xi_\nu(h_\bullet) \qquad \text{where } |\nu| = \sum_{j=1}^{l-1}\nu_j\,. \tag{237}$$

Since $1 = |\xi_{\rho_s}(h_\bullet)| = |\xi_{\rho_s^{\mathrm{C}}}(h_\bullet)|$ and

$$(1 - \xi_{-2e_j}(h_\bullet)) = (1 - \xi_{-e_j}(h_\bullet))(1 + \xi_{-e_j}(h_\bullet))\,,$$

we see that

$$\left|\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right| = \left|\Delta_s\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right| \cdot \prod_{j=1}^{l-1}|1 + \xi_{-e_j}(h_\bullet)|\,.$$

Furthermore, by (236),

$$\prod_{j=1}^{l-1}|1 + \xi_{-e_j}(h_\bullet)|^2 = \prod_{j=1}^{l-1}(1 + \xi_{e_j}(h_\bullet))(1 + \xi_{-e_j}(h_\bullet)) = \det(1 + h_\bullet)\,.$$

Thus

$$\begin{aligned}\left|\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right|^2 &= \left|\Delta_s\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\right|^2 \det(1 + h_\bullet) \\ &= \frac{1}{2}\left|\Delta_s\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\right|^2 \det\left(1 + \left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\right).\end{aligned} \tag{238}$$

Finally, by (237),

$$\Delta_s^{\mathrm{C}}(\Big(-\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big) = (-1)^{l(l-1)/2}\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\,. \tag{239}$$

By Lemma 38, (239) and (238), for $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned}&\int_{\mathrm{G}^0 s}\check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\,dg \\ &\quad= \frac{1}{|W(\mathrm{G}_s^0,\mathfrak{h}_s)|}\int_{\mathrm{H}_\bullet}\check{\Theta}_\Pi\left(\widetilde{\begin{array}{c|cc} h_\bullet & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}}\right) \\ &\qquad\qquad\times\left|\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right|^2 T_s\left(\widetilde{\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)\,dh_\bullet \\ &\quad= \frac{1}{|W(\mathrm{G}_s^0,\mathfrak{h}_s)|}\int_{\mathrm{H}_\bullet}\check{\Theta}_\Pi\left(\widetilde{\begin{array}{c|cc} -h_\bullet & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}}\right)\end{aligned}$$

$$\times \left|\Delta_s^{\mathrm{C}}\Big(\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)\right|^2 T_s\left(\widetilde{\begin{array}{c|c} -h_\bullet & 0 \\ \hline 0 & -1 \end{array}}\right)\big(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\big)\, dh_\bullet$$

$$= \frac{1}{2|W(\mathrm{G}_s^0,\mathfrak{h}_s)|}\int_{\mathrm{H}_\bullet} \check{\Theta}_\Pi\Big(\,\iota_s\Big(\widetilde{-\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)}\Big)\Big)$$

$$\times \left|\Delta_s\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\right|^2 \det\Big(1+\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)\Big)$$

$$\times T_s\Big(\widetilde{-\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)}\Big)\big(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\big)\, dh_\bullet\,,$$

where $\iota_s : -\mathrm{G}_s^0 \to \mathrm{G}$ is the embedding given, in terms of matrices, by

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \to \left(\begin{array}{c|c|c} a & 0 & b \\ \hline 0 & 1 & 0 \\ \hline c & 0 & d \end{array}\right), \qquad \text{with } a \in \mathrm{M}_{2l-2,2l-2}(\mathbb{R}),\quad d \in \mathbb{R}\,.$$

Now, Weyl's integration formula on G_s^0 yields

$$\int_{\mathrm{G}^0 s} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi)\, dg = \frac{1}{2}\int_{\mathrm{G}_s^0} \check{\Theta}_\Pi(\iota_s(-g))\det(1+g)T_s\big(\widetilde{-g}\big)\,\big(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\big)\, dg\,.$$

Making the change of variables $g \to -g$ on the right-hand side, we get (93).

Let now $\mathrm{G} = \mathrm{O}_{2l+1}$ with $l \geq 1$. The Cartan subgroup H of G is described in Appendix C. In particular, $\mathrm{H}^0 = \{(u_1, u_2, \dots, u_l, 1); u_j \in \mathrm{SO}_2, 1 \leq j \leq l\}$.

Suppose first that $1 \leq l \leq l'$. On page 287, we introduced $\mathrm{G}_s \subseteq \mathrm{G}$ as the subgroup acting trivially on the 1-dimensional subspace $\mathsf{V}_{\bar{0}}^0$ of $\mathsf{V}_{\bar{0}}$. Considering G_s as a group of isomorphisms of $\mathsf{V}_{\bar{0}}^1 \oplus \cdots \oplus \mathsf{V}_{\bar{0}}^l$ identifies the Cartan subgroup H_s of G_s with

$$\{h_\bullet = (u_1, u_2, \dots, u_l); u_j \in \mathrm{SO}_2, 1 \leq j \leq l\}\,. \tag{240}$$

The identification of H_s with (240) applies when $l > l'$ as well. Indeed, in this case $\mathrm{G}_s \subseteq \mathrm{G}$ was defined on page 283 as the subgroup acting trivially on the 1-dimensional subspace $\mathsf{V}_{\bar{0}}^{0,0}$ of $\mathsf{V}_{\bar{0}}$. The identification therefore holds when we consider G_s as a group of isomorphisms of $(\mathsf{V}_{\bar{0}}^{0,0})^\perp \oplus \mathsf{V}_{\bar{0}}^1 \oplus \cdots \oplus \mathsf{V}_{\bar{0}}^l$.

Recall from (123) the double covering $\widehat{\mathrm{H}^0} \ni \widehat{h} \to h \in \mathrm{H}^0$ of H^0 on which the functions ξ_ρ and Δ are well-defined. It is easy to check that

$$\left|\Delta(\widehat{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)})\right|^2 = |\Delta_s(h_\bullet)|^2 \det(1-h_\bullet) \qquad (h_\bullet \in \mathrm{H}_s^0)\,, \tag{241}$$

where

$$\Delta_s(h_\bullet) = \xi_{\rho_s}(h_\bullet) \prod_{1\le j<k\le l} (1-\xi_{-e_j+e_k}(h_\bullet))(1-\xi_{-e_j-e_k}(h_\bullet))\,.$$

(The product is empty if $l=1$. In this case, $\Delta_s(h_\bullet)=1$ for all $h_\bullet$.) Recall from (216) (or (242)) that $T(\widetilde{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)}) = T_s(\tilde{h}_\bullet)\otimes\delta_0$ for $h_\bullet \in \mathrm{H}_s^0$, where δ_0 is the Dirac delta on $\mathrm{W}_s^\perp$.

Hence, by Weyl's integration formula and (241), for $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned}\int_{\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\,dg &= \frac{1}{|W(\mathrm{G}^0,\mathfrak{h})|}\int_{\mathrm{H}^0} \check{\Theta}_\Pi(\tilde{h})|\Delta(\widehat{h})|^2 T(\tilde{h})(\phi^{\mathrm{G}})\,dh \\ &= \frac{1}{2|W(\mathrm{G}_s^0,\mathfrak{h}_s)|}\int_{\mathrm{H}_s^0} \check{\Theta}_\Pi\Big(\widetilde{\left(\begin{array}{c|c} h_\bullet & 0 \\ \hline 0 & 1 \end{array}\right)}\Big)\det(1-h_\bullet)|\Delta_s(h_\bullet)|^2 T_s(\tilde{h}_\bullet)(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\,dh_\bullet \\ &= \frac{1}{2}\int_{\mathrm{G}_s^0} \check{\Theta}_\Pi(\tilde{g})\det(1-g)T_s(\tilde{g})(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\,dg\,.\end{aligned}$$

This proves (93) for $\mathrm{G}=\mathrm{O}_{2l+1}$.

14. A Different Look at the Pair $(\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $l > l'$

Recall the decompositions $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}\oplus\mathfrak{h}''$ from (165) and $\mathrm{W} = \mathrm{W}_s \oplus \mathrm{W}_s^\perp$ from (95). Recall also that we often identify $\mathfrak{h}$ and $\mathfrak{h}'$ via (42). As before, we denote the objects corresponding to W_s by the subscript s: $\mathfrak{g}_s$, G_s, Θ_s, and T_s. In particular, $\mathfrak{h}_s = \mathfrak{h}(\mathfrak{g})$, see (45), and $\mathrm{H}_s^0 = \mathrm{H}(\mathfrak{g})^0$. Since any element $h \in \mathrm{H}(\mathfrak{g})^0$ acts trivially on $\mathrm{W}_s^\perp$, we see that

$$(h-1)\mathrm{W} = (h-1)\mathrm{W}_s\,.$$

Hence, as in (216),

$$\Theta(\tilde{h}) = \Theta_s(\tilde{h}) \quad\text{and}\quad T(\tilde{h}) = T_s(\tilde{h})\otimes\delta_0 \qquad (h\in\mathrm{H}(\mathfrak{g})^0)\,, \tag{242}$$

where δ_0 is the Dirac delta on $\mathrm{W}_s^\perp$.

We consider the (modified) Cayley transform $c_\odot : \mathfrak{h}(\mathfrak{g}) \to \mathrm{H}(\mathfrak{g})^0$ defined as in (219). Notice that

$$c_\odot(x' + x'') = c(x')c_\odot(x'') \qquad (x' \in \mathfrak{h} = \mathfrak{h}', x'' \in \mathfrak{h}''),$$

where $c : \mathfrak{h}' \to \mathrm{H}'$ is the usual Cayley transform.

Let $\mathfrak{z}_s$ denote the centralizer of $\mathfrak{h}$ in $\mathfrak{g}_s$. Then $\mathfrak{z}_s = \mathfrak{h} \oplus \mathfrak{g}_s''$, where $\mathfrak{g}_s''$ is the Lie algebra of the group G_s'' of isometries of the restriction of the form $(\cdot,\cdot)$ to the $2(l-l')$-dimensional real vector space $(\mathrm{V}_{\overline{0}}^{0,0})^\perp$. Then $\mathfrak{h}''$ is a Cartan subalgebra of $\mathfrak{g}_s''$. The following lemma is a variation of Lemma 28 in the present situation.

Lemma 44. *Suppose $l > l'$ and let μ be the Harish-Chandra parameter of a genuine irreducible representation of $\widetilde{\mathrm{O}}_{2l+1}$. In terms of the decomposition (*165*)*

$$\begin{aligned}(243)\quad &\xi_{-s\mu}(\widehat{c}_\odot(x))\,\mathrm{ch}^{2l'-2l-1}(x)\pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x)\\ &= \left(\xi_{-s\mu}(\widehat{c}(x'))\,\mathrm{ch}^{2l'-2l-1}(x')\right)\left(\xi_{-s\mu}(\widehat{c}_\odot(x''))\,\mathrm{ch}^{2l'-2l-1}(x'')\pi_{\mathfrak{g}_s''/\mathfrak{h}''}(x'')\right),\end{aligned}$$

where $x = x' + x'' \in \mathfrak{h}(\mathfrak{g})$, with $x' \in \mathfrak{h}$ and $x'' \in \mathfrak{h}''$. Moreover,

$$\begin{aligned}(244)\quad &\int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c}_\odot(x''))\,\mathrm{ch}^{2l'-2l-1}(x'')\pi_{\mathfrak{g}_s''/\mathfrak{h}''}(x'')\,dx''\\ &\qquad = C \sum_{s'' \in W(\mathrm{G}'',\mathfrak{h}'')} \mathrm{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'')\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho''),\end{aligned}$$

where C is a constant, ρ'' is one half times the sum of the positive roots for $(\mathfrak{g}_{\mathbb{C}}'', \mathfrak{h}_{\mathbb{C}}'')$ and $\mathbb{I}_{\{0\}}$ is the indicator function of zero.

Proof. Formula (243) is obvious, because $\pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x' + x'') = \pi_{\mathfrak{g}_s''/\mathfrak{h}''}(x'')$. We shall verify (244). By (C.8) applied to $\mathfrak{g}'' \supseteq \mathfrak{h}''$,

$$\pi_{\mathfrak{g}_s''/\mathfrak{h}''}(x'') = C_1''\Delta''(\widehat{c}_\odot(x''))\,\mathrm{ch}^{2(l-l')-1}(x'') \qquad (x'' \in \mathfrak{h}''),$$

where Δ'' is the Weyl denominator for G'', see (168). Hence, the integral (244) is a constant multiple of

$$\int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c}_\odot(x''))\Delta''(\widehat{c}(x''))\,\mathrm{ch}^{-2}(x'')\,dx'' = 2^{\dim \mathfrak{h}''}\int_{\widehat{c}_\odot(\mathfrak{h}'')} \xi_{-s\mu}(h)\Delta''(h)\,dh,$$

where $\widehat{c_\odot}(\mathfrak{h}'') \subseteq \widehat{\mathrm{H}''^0}$. We therefore obtain the right-hand side of (244) as in the proof of Lemma 28. □

Proof of Theorem 10. Similar computations as those done in section 12 together with (242) and $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}_s$ imply that the left-hand side of (97) is a constant multiple of

$$\frac{1}{|W(\mathrm{G}^0, \mathfrak{h}(\mathfrak{g}))|} \int_{\mathfrak{h}(\mathfrak{g})} \left(\Theta_\Pi(\widehat{c_\odot}(x)^{-1})\Delta(\widehat{c_\odot}(x)^{-1})\right) \left(\frac{\Delta(\widehat{c_\odot}(x))}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x)} \Theta_s(\tilde{c}_\odot(x))\right) \times \pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x) \int_{\mathrm{W}_s} \chi_x(w) \left(\phi^{\mathrm{G}}|_{\mathrm{W}_s}\right)(w)\, dw \,\mathrm{ch}^{-2}(x)\, dx\,, \tag{245}$$

where $c_\odot(\mathfrak{h}_s)$ is a dense subset of $\mathrm{H}(\mathfrak{g})^0$. Lemma 24 shows that there is a constant C_1 such that

$$\begin{aligned}\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x) &\int_{\mathrm{W}_s} \chi_x(w)\, \phi^{\mathrm{G}}|_{\mathrm{W}_s}(w)\, dw \\ &= C_1 \int_{\tau'(\mathfrak{h}_{\mathrm{I}}^{-reg})} \sum_{tW(\mathrm{Z}_s,\mathfrak{h}(\mathfrak{g}))\in W(\mathrm{G}_s,\mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z}_s,\mathfrak{h}(\mathfrak{g}))} \\ &\qquad \times \mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t)\pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) e^{iB(x,t.y)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\, dy\,,\end{aligned}$$

where $\mathfrak{z}_s \subseteq \mathfrak{g}_s$ is the centralizer of $\mathfrak{h} = \mathfrak{h}'$. By (224), for a suitable constant C_1, for all $x = \sum_{j=1}^{l} x_j J_j \in \mathfrak{h}$ with $x_j \neq 0$ for $1 \leq j \leq l$,

$$\frac{\Delta(\widehat{c_\odot}(x))}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x)} \Theta_s(\widetilde{c_\odot}(x))\,\mathrm{ch}^{-2}(x) = C_1\,\mathrm{ch}^{2l'-2l-1}(x) \left(\prod_{j=1}^{l} \mathrm{sgn}(x_j)\right).$$

Hence (245) is equal to a constant multiple of

$$\sum_{u\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_{\mathrm{I}}^{-reg})} \xi_{-u.\mu}(\widehat{c_\odot}(x))\,\mathrm{ch}^{2l'-2l-1}(x) \left(\prod_{j=1}^{l} \mathrm{sgn}(x_j)\right)$$
$$\times \sum_{tW(\mathrm{Z}_s,\mathfrak{h}(\mathfrak{g}))\in W(\mathrm{G}_s,\mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z}_s,\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t)\pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(t^{-1}x) e^{iB(x,ty)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\, dy\, dx\,.$$

Notice that for $t \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g})) = W(\mathrm{G}_s, \mathfrak{h}(\mathfrak{g}))$ and $x \in \mathfrak{h}(\mathfrak{g})$,

$$\prod_{j=1}^{l} \mathrm{sgn}(tx_j) = \frac{\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t)}{\mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t)} \prod_{j=1}^{l} \mathrm{sgn}(x_j)\,. \tag{246}$$

Interchanging the sums, changing the variable of integration x to tx and using that $\mathrm{ch}(tx) = \mathrm{ch}(x)$ and $B(tx, ty) = B(x, y)$, we see that (245) is a constant multiple of

$$\sum_{tW(\mathrm{Z}_s,\mathfrak{h}(\mathfrak{g}))\in W(\mathrm{G}_s,\mathfrak{h}(\mathfrak{g}))/W(\mathrm{Z}_s,\mathfrak{h}(\mathfrak{g}))} \sum_{u\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u)\,\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t)$$
$$\times \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \xi_{-\mu}(\widehat{c}_{\odot}(u^{-1}tx))\,\mathrm{ch}^{2l'-2l-1}(x)$$
$$\times \left(\prod_{j=1}^{l} \mathrm{sgn}(x_j)\right) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) e^{iB(x,y)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,dx\,.$$

Now, replace $u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ with tu, where $t \in W(\mathrm{G}_s, \mathfrak{h}(\mathfrak{g})) = W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$. Hence, (245) is a constant multiple of

$$\sum_{u\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \xi_{-\mu}(\widehat{c}_{\odot}(u^{-1}x))\,\mathrm{ch}^{2l'-2l-1}(x) \tag{247}$$
$$\times \left(\prod_{j=1}^{l} \mathrm{sgn}(x_j)\right) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) e^{iB(x,y)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,dx\,.$$

Lemma 44, together with the identification (42) of $\mathfrak{h}$ and $\mathfrak{h}'$, implies that this last expression is a constant multiple of

$$\sum_{u\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \Big(\sum_{u''\in W(\mathrm{G}'',\mathfrak{h}'')} \mathrm{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(u'') \mathbb{I}_{\{0\}}(-(u\mu)|_{\mathfrak{h}''} + u''\rho'') \Big)$$
$$\times \int_{\mathfrak{h}'} \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \xi_{-u\mu}(\widehat{c}_{\odot}(x))\,\mathrm{ch}^{2l'-2l-1}(x)$$
$$\times \left(\prod_{j=1}^{l} \mathrm{sgn}(x_j)\right) e^{iB(x,y)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,dx$$

$$= \sum_{\substack{u\in W(\mathrm{G},\mathfrak{h}(\mathfrak{g})) \\ (u\mu)|_{\mathfrak{h}''}=\rho''}} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}'} \xi_{-u\mu}(\widehat{c}_{\odot}(x))\ \mathrm{ch}^{2l'-2l-1}(x) \left(\prod_{j=1}^{l} \mathrm{sgn}(x_j)\right) \tag{248}$$

$$\times \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\,dy\,dx\,. \tag{249}$$

As in (227), for $u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ and $x \in \mathfrak{h}'$,

$$\xi_{-u\mu}(\widehat{c_{\odot}}(x)) \operatorname{ch}^{2l'-2l-1}(x) \left(\prod_{j=1}^{l} \operatorname{sgn}(x_j) \right) = i^l (-1)^{|u\mu|+\frac{l}{2}} \prod_{j=1}^{l} (1+ix_j)^{-a_{u,j}} (1-ix_j)^{-b_{u,j}} , \tag{250}$$

where $|u\mu| = \sum_{j=1}^{j} (u\mu)_j$ and $a_{u,j}$, $b_{u,j}$ are as in (80). Hence, computations as in the proof of Lemma 30 lead to the following equality, which holds in the sense of distributions on $\tau'(\mathfrak{h}_{\overline{1}}^{reg})$ for every $u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$:

$$\int_{\mathfrak{h}'} \xi_{-u\mu}(\widehat{c_{\bullet}}(x)) \operatorname{ch}^{2l'-2l-1}(x) \left(\prod_{j=1}^{l} \operatorname{sgn}(x_j) \right) e^{iB(x,y)}\, dx = i^l (-1)^{|u\mu|+\frac{l}{2}} \Big(\prod_{j=1}^{l'} P_{a_{u,j},b_{u,j}}(2\pi y_j) \Big) e^{-2\pi \sum_{j=1}^{l'} |y_j|} , \tag{251}$$

where $P_{a_{u,j},b_{u,j}}$ is defined in (D.4).

The sum on the right-hand side of (248) is over the elements $u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ for which $(u\mu)|_{\mathfrak{h}''} = \rho''$. By Corollary 29, this has two consequences. The first is that this sum is 0 unless μ satisfies $\mu|_{\mathfrak{h}''} = \rho''$. As seen in the proof of Theorem 5, this means that the highest weight $\lambda = \mu - \rho$ of Π satisfies condition (a) of that theorem. The second consequence is that for the μ satisfying $\mu|_{\mathfrak{h}''} = \rho''$, an element $u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ can give a nonzero contribution to the sum in (248) only if $u|_{\mathfrak{h}''} = 1$. The latter condition holds for instance if $u = 1$.

Suppose in the following that μ satisfies $\mu|_{\mathfrak{h}''} = \rho''$. Consider first the case $u = 1$. By Lemma 31,

$$\prod_{j=1}^{l'} P_{a_j,b_j}(2\pi y_j) = (2\pi)^{l'} \prod_{j=1}^{l'} P_{a_j,b_j,2}(2\pi y_j) \mathbb{I}_{\mathbb{R}^+}(y_j) \qquad \Big(y = \sum_{j=1}^{l'} y_j J'_j \in \mathfrak{h}'\Big) \tag{252}$$

has support equal to $\tau'(\mathfrak{h}_{\overline{1}})$. Because of (251), we can proceed as in Lemma 34 to show that if $u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ satisfies $(u\mu)|_{\mathfrak{h}''} = \rho''$ and changes the sign of some coordinates (i.e. $y_j \to -y_j$ for some j), then the corresponding integral on the right-hand side of (248) is zero. Recalling that $(u\mu)|_{\mathfrak{h}''} =$

ρ'' implies $u|_{\mathfrak{h}''} = 1$, we see that all terms in this sum vanish but those corresponding $u \in W(\mathrm{G}', \mathfrak{h}') \subseteq W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$. The sum is hence over $u \in W(\mathrm{G}', \mathfrak{h}')$ and formula (248) becomes a constant multiple of

$$\sum_{u \in W(\mathrm{G}', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(u)(-1)^{|u\mu|+\frac{l}{2}} \times \int_{\tau'(\mathfrak{h}_{\mathrm{T}}{}^{reg})} \left(\prod_{j=1}^{l'} P_{a_{u,j}, b_{u,j}, 2}(2\pi y_j) \right) e^{-2\pi \sum_{j=1}^{l'} |y_j|} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\, dy\,.$$

If $u \in W(\mathrm{G}', \mathfrak{h}')$ then $|u\mu| = |\mu|$. Recall from (230) that $(-1)^{|\mu|+\frac{l}{2}} = (-1)^{\frac{l(l+1)}{2}}(-1)^{|\lambda|}$.

By the $W(\mathrm{G}', \mathfrak{h}')$-skew invariance of $F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)$ the above integral is therefore a constant multiple of

$$|W(\mathrm{G}', \mathfrak{h}')|(-1)^{|\lambda|} \int_{\tau'(\mathfrak{h}_{\mathrm{T}}{}^{reg})} \left(\prod_{j=1}^{l'} P_{a_j, b_j, 2}(2\pi y_j) \right) e^{-2\pi \sum_{j=1}^{l'} |y_j|} F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}(y)\, dy\,. \tag{253}$$

It remains to show that, as a function of ϕ, $F_{\phi^{\mathrm{G}}|_{\mathrm{W}_s}}$ is a constant multiple of here $F_{\phi^{\mathrm{G}}} = F_\phi$. This follows from the same argument used for (233) in the case $l = l'$, using (A.3) and (I.2) instead of (I.1). (Notice that since G is compact, the integral on G/Z is $\mathrm{vol}(\mathrm{Z})^{-1}$ times the same integral over G.) This concludes the proof of (97). □

Remark 15. The factor $(-1)^{|\lambda|}$ appearing on the right-hand side of (97) in Theorem 10 turns out to be a constant multiple of $\check{\chi}_\Pi(\tilde{c}(0))$, the value at $\tilde{c}(0)$ of the central character of Π, as in Theorems 6 and 5. However, we do not have a proof of this fact independent of the known classification of the representations occurring in Howe's correspondence for the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$, see e.g. [Prz96, Appendix (A.4)]. Assume the classification. If $l > l'$, given λ, there is a unique representation Π of $\widetilde{\mathrm{G}}$ occurring in the correspondence with highest weight λ. We see from [Prz96, (A.4.2.1)] that the highest weight λ' of the corresponding representation Π' of $\widetilde{\mathrm{G}'}$ is of the form $\lambda' = \eta + \lambda''$, where λ'' is integral and $|\lambda''| = |\lambda|$.

Let $v \neq 0$ be a highest weight vector of Π' and let $\widetilde{c'} : \mathfrak{g}' \to \widetilde{\mathrm{G}'}$ be the lift of the Cayley transform satisfying $\widetilde{c'}(0) = \widetilde{c}(0)$ (Recall that $c'(0) = -1 = c(0)$ is in the center of the symplectic group and hence in $\mathrm{G} \cap \mathrm{G}'$.) Then

$$\check{\chi}_{\Pi'}(\widetilde{c'}(0))v = \Pi'(\widetilde{c'}(0))v = \xi_{\lambda'}(\widetilde{c'}(0))v\,,$$

which implies that $\check{\chi}_{\Pi'}(\widetilde{c'}(0)) = \xi_{\lambda'}(\widetilde{c'}(0))$. Since λ'' has integral coordinates

$$\xi_{\lambda''}(\widetilde{c'}(0)) = \xi_{\lambda''}(c(0)) = (-1)^{|\lambda''|} = (-1)^{|\lambda|}\,.$$

Hence

$$\xi_{\lambda'}(\widetilde{c'}(0)) = \xi_{\eta}(\widetilde{c'}(0))\xi_{\lambda''}(\widetilde{c'}(0)) = \xi_{\eta}(\widetilde{c'}(0))(-1)^{|\lambda|}\,.$$

Since Π and Π' agree on the center of the symplectic group, $\xi_{\lambda}(\widetilde{c}(0)) = \xi_{\lambda'}(\widetilde{c'}(0))$, yielding

$$\xi_{\lambda}(\widetilde{c}(0)) = \xi_{\eta}(\widetilde{c}(0))(-1)^{|\lambda|}\,,$$

where $\xi_{\eta}(\widetilde{c}(0))$ is a constant independent of the representation Π.

15. Proof of Corollary 12

We will distinguish two cases:

(a) $0 \le p < l = p + q$,

(b) $0 \le p < l < p + q$.

In both cases, we shall prove that if

$$\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g})\, dg \neq 0\,, \tag{254}$$

then $\lambda_{p+1} \le \frac{p-q}{2}$ and $\lambda_{l-q} \ge \frac{p-q}{2}$. Here the second condition is empty if $l \le q$.

Consider first case (a). Then $a_j + b_j = -2\delta + 2 = 1$ for all $1 \le j \le l$. So $Q_{a_j,b_j} = 0$ for all $1 \le j \le l$, and hence, in the notation of (72),

$$\prod_{j=1}^{l} \big(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\big) F_\phi(y) = \Big(\prod_{j=1}^{l} P_{a_j,b_j}(\beta y_j)\Big) e^{-\sum_{j=1}^{l} |y_j|} F_\phi(y)\,. \tag{255}$$

Moreover, by Lemma D.1, for every $1 \le j \le l$, at most one between $P_{a_j,b_j,2}$ and $P_{a_j,b_j,-2}$ can be nonzero. By [MPP20, Lemma 3.5] and because $l > p = l - q > 0$,

$$\begin{aligned} \mathfrak{h} \cap \tau(\mathrm{W}) &= W(\mathrm{G},\mathfrak{h})\big\{y = \sum_{j=1}^{l} y_j J_j : y_1, \dots, y_{\max(l-q,0)} \ge 0 \ge y_{p+1}, \dots, y_l\big\} \\ &= \big\{y = \sum_{j=1}^{l} y_j J_j : p \text{ coordinates } y_j \text{ are } \ge 0 \\ &\qquad\qquad \text{and } q \text{ coordinates } y_j \text{ are } \le 0\big\}\,. \end{aligned} \tag{256}$$

If (254) holds, then $P_{a_j,b_j,2} \neq 0$ for p coordinates y_j and $P_{a_j,b_j,-2} \neq 0$ for q coordinates y_j. The first condition is equivalent to $b_j \geq 1$ for p values of j. The second condition is equivalent to $a_j \geq 1$, equivalently, $b_j \leq 0$ for $q(= l - p)$ values of j. Since the b_j's are strictly decreasing, we conclude that if (254) holds, then

$$b_1 > \cdots > b_p > 0 \geq b_{p+1} > \cdots > b_l \,.$$

But, for $1 \leq j \leq l$,

$$b_j = \lambda_j + \rho_j - \delta + 1 = \lambda_j + \frac{l}{2} - j + 1 \,.$$

Hence $b_p > 0$ is equivalent to $\lambda_p \geq \dfrac{p-q}{2}$, and $b_{p+1} \leq 0$ is equivalent to $\lambda_{p+1} \leq \dfrac{p-q}{2}$. This proves the claim in the case (a).

Let us now come to case (b). Then $Q_{a_j,b_j} \neq 0$ for all $1 \leq j \leq l$ because $a_j + b_j = -2\delta + 2 < 1$. Recall the integral (72):

$$\int_{\mathfrak{h}\cap\tau(\mathrm{W})} \left(\prod_{j=1}^{l} \big(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\big) \right) \cdot F_\phi(y)\, dy \,.$$

For $\gamma \subseteq \{1, 2, \ldots, l\}$, let $|\gamma|$ denote its cardinality and set $\gamma^c = \{1, 2, \ldots, l\} \setminus \gamma$. Clearly,

$$\prod_{j=1}^{l} \big(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\big) = \sum_{\gamma\subseteq\{1,2,\ldots,l\}} \Big(\prod_{j\in\gamma^c} p_j(y_j) \Big) \Big(\prod_{j\in\gamma} q_j(-\partial(J_j))\delta_0(y_j) \Big) \,. \tag{257}$$

For $s \in W(\mathrm{G}, \mathfrak{h})$ let

$$Y_s = \Big\{ y = \sum_{j=1}^{l} y_j J_j : y_{s(1)}, \ldots, y_{s(\max(l-q,0))} \geq 0 \geq y_{s(p+1)}, \ldots, y_{s(l)} \Big\} \,. \tag{258}$$

By (256), $\mathfrak{h}\cap\tau(\mathrm{W}) = \bigcup_{s\in W(\mathrm{G},\mathfrak{h})} Y_s$. Notice that $Y_s = Y_{s'}$ if the permutations s and s' differ at most on the set $\{\max(l-q, 0) + 1, \ldots, p\}$. Hence one may choose a subset $W_0(\mathrm{G}, \mathfrak{h}) \subseteq W(\mathrm{G}, \mathfrak{h})$ such that the union

$$\mathfrak{h} \cap \tau(\mathrm{W}) = \bigcup_{s\in W_0(\mathrm{G},\mathfrak{h})} Y_s$$

is disjoint. Hence the integral in (72) is a sum of the integrals over these Y_s's. We consider each of them separately. Let then $s \in W_0(\mathrm{G}, \mathfrak{h})$ be fixed. We see from (257) that the integral over Y_s is equal to

$$\sum_{\gamma\subseteq\{1,2,\dots,l\}} \int_{Y_s} \Big(\prod_{j\in\gamma^c} p_j(y_j)\Big)\Big(\prod_{j\in\gamma} q_j(-\partial(J_j))\delta_0(y_j)\Big) F_\phi(y)\, dy\,, \tag{259}$$

where empty products are equal to 1.

As in case (a), by Lemma D.1, for every $1 \le j \le l$, at most one between $P_{a_j,b_j,2}$ and $P_{a_j,b_j,-2}$ can be nonzero. By (258), if the integral (259) is nonzero then

$$\begin{aligned} &j \in \{s(1),\dots,s(l-q)\}\cap\gamma^c \text{ implies } P_{a_j,b_j,2} \neq 0, \text{ i.e. } b_j \ge 1 \quad (\text{for the } l > q \text{ case})\,,\\ &j \in \{s(p+1),\dots,s(l)\}\cap\gamma^c \text{ implies } P_{a_j,b_j,-2} \neq 0, \text{ i.e. } a_j \ge 1\,. \end{aligned}$$

For $\Gamma \in \{\gamma^c, \gamma\}$, define

$$Y_{s,\Gamma} = \Big\{ y_\Gamma = \sum_{j\in\Gamma} y_j J_j : \left\{ \begin{array}{l} y_j \ge 0 \text{ for all } j \in \{s(1),\dots,s(l-q)\}\cap\Gamma \\ y_j \le 0 \text{ for all } j \in \{s(p+1),\dots,s(l)\}\cap\Gamma \end{array} \right\},$$

where the first line of conditions has to be omitted when $l \le q$. Then $Y_s = Y_{s,\gamma^c} \times Y_{s,\gamma}$ and (259) becomes

$$\begin{aligned} &\sum_{\gamma\subseteq\{1,2,\dots,l\}} \int_{Y_{s,\gamma^c}} \Big(\prod_{j\in\{s(1),\dots,s(l-q)\}\cap\gamma^c} p_j(y_j)\mathbb{I}_{\mathbb{R}^+}(y_j)\Big)\\ &\qquad\times \Big(\prod_{j\in\{s(\max(l-q,0)+1),\dots,s(p)\}\cap\gamma^c} p_j(y_j)\Big)\Big(\prod_{j\in\{s(p+1),\dots,s(l)\}\cap\gamma^c} p_j(y_j)\mathbb{I}_{\mathbb{R}^-}(y_j)\Big)\\ &\qquad\times \Big(\int_{Y_{s,\gamma}} \Big(\prod_{j\in\gamma} q_j(-\partial(J_j))\delta_0(y_j)\Big) F_\phi(y)\, dy_\gamma\Big)\, dy_{\gamma^c}\\ &= \sum_{\gamma\subseteq\{1,2,\dots,l\}} \int_{Y_{s,\gamma^c}} \Big(\prod_{j\in\{s(1),\dots,s(l-q)\}\cap\gamma^c} p_j(y_j)\mathbb{I}_{\mathbb{R}^+}(y_j)\Big)\\ &\qquad\times \Big(\prod_{j\in\{s(\max(l-q,0)+1),\dots,s(p)\}\cap\gamma^c} p_j(y_j)\Big)\Big(\prod_{j\in\{s(p+1),\dots,s(l)\}\cap\gamma^c} p_j(y_j)\mathbb{I}_{\mathbb{R}^-}(y_j)\Big)\\ &\qquad\times \Big(\prod_{j\in\gamma} q_j(\partial(J_j))F_\phi(y)|_{y_j=0, j\in\gamma}\Big)\, dy_{\gamma^c}\,, \end{aligned} \tag{260}$$

where the first products are empty unless $l > q$ and empty products are equal to 1.

Suppose that $l > q$ and there is $j_\gamma \in \{s(1), \dots, s(l-q)\} \cap \gamma$. Then every $y = \sum_{j=1}^{l} y_j J_j$ with $y_j \geq 0$ for $j \in \{s(1), \dots, s(l-q)\} \cap \gamma^c$, $y_j \leq 0$ for $j \in \{s(p+1), \dots, s(l)\} \cap \gamma^c$ and $y_j = 0$ for $j \in \gamma$ belongs to

$$\Big\{ y = \sum_{j=1}^{l} y_j J_j : \begin{cases} y_j \geq 0 \text{ for all } j \in \{s(1), \dots, s(l-q)\} \setminus \{j_\gamma\}, \\ y_j \leq 0 \text{ for all } j \in \{s(p+1), \dots, s(l)\}, \\ y_{j_\gamma} = 0 \end{cases} \Big\} \subseteq \partial(\mathfrak{h} \cap \tau(\mathrm{W}))\,,$$

where $\partial(\mathfrak{h} \cap \tau(\mathrm{W}))$ denotes the boundary of $\mathfrak{h} \cap \tau(\mathrm{W})$. For all $1 \leq j \leq l$,

$$\deg Q_{a_j, b_j} = -a_j - b_j = 2\delta - 2 = p + q - l - 1\,.$$

Hence, the term $\Big(\prod_{j \in \gamma} q_j(\partial(J_j)) F_\phi(y) \Big)|_{y_j = 0, j \in \gamma}$ is zero on $\partial(\mathfrak{h} \cap \tau(\mathrm{W}))$ by [MPP20, Theorem 3.5]. Choosing $j = j_\gamma$, we see that the integral corresponding to γ in (260) vanishes. Similarly (and not only in the case $l > q$), the integral corresponding to γ vanishes if there is $j_\gamma \in \{s(p+1), \dots, s(l)\} \cap \gamma$. The sum in (260) therefore reduces to a sum over the γ having no intersection with $\{s(1), \dots, s(\max(l-q, 0))\} \cup \{s(p+1), \dots, s(l)\}$. For these γ's,

$$\begin{aligned} &\{s(1), \dots, s(\max(l-q, 0))\} \cap \gamma^c = \{s(1), \dots, s(\max(l-q, 0))\}\,, \\ &\{s(p+1), \dots, s(l)\} \cap \gamma^c = \{s(p+1), \dots, s(l)\}\,. \end{aligned}$$

Hence,

$$\begin{aligned} &b_{s(j)} \geq 1 \text{ for } 1 \leq j \leq l - q, \text{ if } l > q\,, \\ &a_{s(j)} \geq 1 \text{ for } p + 1 \leq j \leq l\,. \end{aligned}$$

In particular, there are at least $\max(l-q, 0)$ elements $b_j \geq 1$. So $b_{l-q} \geq 1$ if $l > q$. Similarly, there are at least $l - p$ elements $a_j \geq 1$. So $a_{p+1} \geq 1$. As in the case (a), we conclude that if the integral over Y_s corresponding to this γ is not zero, then $\lambda_{l-q} \geq \frac{p-q}{2}$ (when $l > q$ holds) and $\lambda_{p+1} \leq \frac{p-q}{2}$.

This applies to all γ and all s. Hence, if (254) is satisfied, then $\lambda_{l-q} \geq \frac{p-q}{2}$ (when $l > q$ holds) and $\lambda_{p+1} \leq \frac{p-q}{2}$. This concludes the proof of Corollary 12.

16. Proof of Corollary 13

Before entering into the proof of Corollary 13, let us consider the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_l, \mathrm{O}^*_{2l'})$ with arbitrary $l \leq l'$. Let Π be an irreducible

genuine representation of $\widetilde{\mathrm{G}}$. We want to prove that the intertwining distribution corresponding to Π is nonzero. For this, it suffices to show that the integral on the right-hand side of (72) is nonzero for suitable functions $\phi \in \mathcal{S}(\mathrm{W})$. The explicit expression of that integral depends on the values of the parameters a_j and b_j constructed from the Harish-Chandra parameter $\mu_1 > \mu_2 > \cdots > \mu_l$ of Π.

The parameters of the pair $(\mathrm{Sp}_l, \mathrm{O}^*_{2l'})$ are $d = l$, $d' = l'$, $\iota = 1/2$ and hence $\delta = l' - l$. Notice that $-a_j - b_j = 2\delta - 2 = 2(l' - l - 1)$ does not depend on j. No q_j-term occurs in (72) if and only if $-a_j - b_j < 0$, i.e. if and only if $l = l'$. Every q_j-term is a constant multiple of a delta distribution if and only if $-a_j - b_j = 0$, i.e. if and only if $l + 1 = l'$. In all other cases, the q_j-terms are distributions and not measures.

As recalled in Appendix H, the highest weights of Π are integers λ_j satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$ and the ρ-function for $(\mathfrak{g}, \mathfrak{h})$ is $\rho = \sum_{j=1}^{l}(l + 1 - j)e_j$. Hence $a_j = -\mu_j - \delta + 1 \leq 0$, i.e. $P_{a_j,b_j,-2} = 0$, for all $1 \leq j \leq l$. On the other hand, the sign of

$$b_j = \mu_j - \delta + 1 = \lambda_j + (l + 1 - j) - l' + l + 1 \qquad (1 \leq j \leq l)$$

might depend on j. Recall that $b_1 > b_2 > \cdots > b_l$. All the b_j are positive provided so is b_l, and $b_l = \lambda_l + 2 + l - l' > 0$ if and only if $\lambda_l \geq l' - l - 1$. In this case, $P_{a_j,b_j,2} \neq 0$ (and hence $p_j \neq 0$) for all $1 \leq j \leq l$. Notice that the condition $\lambda_l \geq l' - l - 1$ is automatically satisfied when $l' - l - 1 \leq 0$, that is $l' \in \{l, l + 1\}$.

Proof of Corollary 13. The discussion preceeding this proof shows that if $\lambda_l \geq l' - l - 1$ then, for $1 \leq j \leq l$,

$$p_j(y_j) = 2\pi P_{a_j,b_j,2}(y_j)\mathbb{I}_{\mathbb{R}^+}(y_j)e^{-2\pi|y_j|} \qquad (y_j \in \mathbb{R})\,, \tag{261}$$

where $P_{a_j,b_j,2}$ is a nonzero polynomial of degree $b_j - 1 (\geq 0)$. Let $W_0(\mathrm{G}, \mathfrak{h})$ denote the subgroup of $W(\mathrm{G}, \mathfrak{h})$ acting as permutations on the variables y_j of $y = \sum_{j=1}^{l} y_j J_j \in \mathfrak{h}$. Then

$$\pi_{\mathfrak{g}/\mathfrak{h}}(y) \sum_{t \in W_0(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \prod_{j=1}^{l} P_{a_j,b_j,2}((ty)_j) \qquad (y \in \mathfrak{h})\,. \tag{262}$$

is a $W_0(\mathrm{G}, \mathfrak{h})$-invariant real-valued polynomial on $\mathfrak{h}$. It is nonzero because $\deg(P_{a_1,b_1,2}) > \deg(P_{a_2,b_2,2}) > \cdots > \deg(P_{a_l,b_l,2})$. Let U be an open,

nonempty, $W(\mathrm{G},\mathfrak{h})$-invariant set with compact closure $\overline{\mathrm{U}} \subseteq \mathfrak{h}^{reg}$. Observe that $\mathrm{U} \cap \tau(\mathfrak{h}_{\overline{1}}{}^{reg})$ is nonempty, open, $W_0(\mathrm{G},\mathfrak{h})$-invariant and with compact closure contained in $\tau(\mathfrak{h}_{\overline{1}}{}^{reg})$. We choose such a U so that the polynomial (262) has constant sign on $\mathrm{U} \cap \tau(\mathfrak{h}_{\overline{1}}{}^{reg})$.

By Lemma 1, we can choose a nonzero function $\phi \in C_c^\infty(\mathrm{W})^{\mathrm{G}}$ such that $\phi \geq 0$ and $\operatorname{supp} F_\phi \subseteq \mathrm{U}$. It follows, in particular, that F_ϕ, as well as all its partial derivatives, vanishes along the root hyperplanes $y_j = 0$, where $1 \leq j \leq l$. For such a ϕ, the right-hand side of (72) reduces to a constant multiple of

$$\int_{\mathfrak{h} \cap \tau(\mathrm{W})} \left(\prod_{j=1}^{l} p_j(y_j) \right) \cdot F_\phi(y)\, dy\,. \tag{263}$$

By (261), we can replace the domain of integration $\mathfrak{h} \cap \tau(\mathrm{W})$ with $\tau(\mathfrak{h}_{\overline{1}}{}^{reg})$. Choose a smooth $W(\mathrm{G},\mathfrak{h})$-invariant function α_ϕ on $\mathfrak{h}$ which is equal to 1 on U and has compact support contained in $\mathfrak{h}^{reg}$. Then $\frac{\alpha_\phi}{\pi_{\mathfrak{g}/\mathfrak{h}}}$ is a smooth $W(\mathrm{G},\mathfrak{h})$-skew-invariant function on $\mathfrak{h}$. Set

$$\Phi(y) = \frac{(2\pi)^l}{|W_0(\mathrm{G},\mathfrak{h})|} \frac{\alpha_\phi(y)}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)} \times \left(\sum_{t \in W_0(\mathrm{G},\mathfrak{h})} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \prod_{j=1}^{l} P_{a_j,b_j,2}((ty)_j) \right) e^{-2\pi \sum_{j=1}^{l} |y_j|} \qquad (y \in \mathfrak{h})\,.$$

This is a nonzero smooth $W_0(\mathrm{G},\mathfrak{h})$-invariant function on $\mathfrak{h}$. Since $\pi_{\mathfrak{g}/\mathfrak{h}}(y) F_\phi(y)$ is $W_0(\mathrm{G},\mathfrak{h})$-invariant, the integral in (263) can be written as

$$\int_{\tau(\mathfrak{h}_{\overline{1}}{}^{reg})} \Phi(y) \pi_{\mathfrak{g}/\mathfrak{h}}(y) F_\phi(y)\, dy\,. \tag{264}$$

By (49), (50) and (54),

$$\pi_{\mathfrak{g}/\mathfrak{h}}(y) F_\phi(y) = C |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}) \qquad (y = \tau(w) = \tau'(w))\,.$$

Like F_ϕ, it is supported in U and is a nonzero constant multiple of a function of constant sign. Moreover, by (262), Φ is nonzero and with constant sign in $\mathrm{U} \cap \tau(\mathfrak{h}_{\overline{1}}{}^{reg})$. Thus (264), and hence the intertwining distribution evaluated at ϕ, is nonzero.

REMARK 16. Suppose that $l \leq l'$. Among all dual pairs with one member compact, $(\mathrm{Sp}_l, \mathrm{O}^*_{2l'})$ is the easiest for computing the intertwining distributions, both because $\mathrm{G} = \mathrm{Sp}_l$ is connected and because there is only one conjugacy class of Cartan subspaces in W. Still, establishing if the integral giving the intertwining distribution is nonzero is problematic also in this case as soon as there are nonconstant polynomials Q_{a_j,b_j}. The reason is that, at present, we do not have sufficient information on the derivatives of the Cauchy–Harish-Chandra integrals. For the orbital integrals for the adjoint action of a Lie group on it Lie algebra, the relevant information is contained in Harish-Chandra's work; see e.g. [Var89, Theorem 9, p. 37].

17. A Sketch of a Computation of the Wave Front Set of Π'

COROLLARY 45. *For any representation $\Pi \otimes \Pi'$ which occurs in the restriction of the Weil representation to the dual pair $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}')$,*

$$WF(\Pi') = \tau'(\tau^{-1}(0))\,.$$

Here $WF(\Pi')$ stands for the wave front of the character $\Theta_{\Pi'}$ at the identity and $0 = WF(\Pi)$ since Π is finite dimensional.

The complete proof is rather lengthy but unlike the one provided in [Prz91, Theorem 6.11], it is independent of [Vog78]. We sketch the main steps below. The details may be found in [MPP24].

The variety $\tau^{-1}(0) \subseteq \mathrm{W}$ is the closure of a single GG'-orbit $\mathcal{O}$; see e.g. [Prz91, Lemma 2.16]. There is a positive GG'-invariant measure $\mu_{\mathcal{O}}$ on this orbit which defines a homogeneous distribution. We denote its degree by $\deg \mu_{\mathcal{O}}$.

Recall that if V is a n-dimensional real vector space, $t > 0$ and $M_t v = tv$ for $v \in \mathrm{V}$, then the pullback of $u \in \mathcal{S}'(V)$ by M_t is $M_t^* u \in \mathcal{S}'(\mathrm{V})$, defined by

$$(M_t^* u)(\phi) = t^{-n} u\big(\phi \circ M_{t^{-1}}\big) \qquad (\phi \in \mathcal{S}(\mathrm{V}))\,.$$

In particular, for $\mathrm{V} = \mathrm{W}$

$$M_t^* \mu_{\mathcal{O}} = t^{\deg \mu_{\mathcal{O}}} \mu_{\mathcal{O}}\,.$$

Define $\tau'_* : \mathcal{S}'(\mathrm{W}) \to \mathcal{S}'(\mathfrak{g}')$ by $\tau'_*(u)(\psi) = u(\psi \circ \tau')$. Then, for $t > 0$,

$$t^{2 \dim \mathfrak{g}'} M^*_{t^2} \circ \tau'_* = t^{\dim \mathrm{W}} \tau'_* \circ M_t^*\,. \tag{265}$$

A rather lengthy but straightforward computation based on Theorems 4, 5 and 7, shows that

$$t^{\deg \mu_{\mathcal{O}}} M_{t^{-1}}^* f_{\Pi\otimes\Pi'} \underset{t\to 0}{\to} C\,\mu_{\mathcal{O}}\,, \tag{266}$$

as tempered distributions on W, where C is a non-zero constant.

Let $\mathcal{F}$ indicate a Fourier transform on $\mathcal{S}'(\mathfrak{g}')$. Then, for $t>0$,

$$M_t^* \circ \mathcal{F} = t^{-\dim \mathfrak{g}'} \mathcal{F} \circ M_{t^{-1}}\,. \tag{267}$$

Hence, in the topology of $\mathcal{S}'(\mathfrak{g}')$,

$$t^{2\deg \mu_{\mathcal{O}'}} M_{t^2}^* \mathcal{F} \tau_*'(f_{\Pi\otimes\Pi'}) \underset{t\to 0+}{\to} C\mathcal{F}\mu_{\mathcal{O}'}\,, \tag{268}$$

where $C\neq 0$ and $\mathcal{O}'=\tau'(\mathcal{O})$.

There is an easy to verify inclusion $WF(\Pi')\subseteq \overline{\mathcal{O}'}$, [Prz91, (6.14)] and a formula for the character $\Theta_{\Pi'}$ in terms of $\mathcal{F}(\tau_*'(f_{\Pi\otimes\Pi'}))$,

$$\frac{1}{\sigma}\cdot \tilde{c}_-^*\Theta_{\Pi'} = \widehat{\tau_*'(f_{\Pi\otimes\Pi'})}\,, \tag{269}$$

where σ is a smooth function, [Prz91, Theorem 6.7]. By combining this with the following elementary lemma, one completes the argument.

Lemma 46. *Suppose $f,u\in\mathcal{S}'(\mathbb{R}^n)$ and u is homogeneous of degree $d\in\mathbb{C}$. Suppose*

$$t^d M_{t^{-1}}^* f(\psi) \underset{t\to 0+}{\to} u(\psi) \qquad (\psi\in\mathcal{S}(\mathbb{R}^n))\,. \tag{270}$$

Then

$$WF_0(\hat{f}) \supseteq \operatorname{supp} u\,, \tag{271}$$

where the subscript 0 indicates the wave front set at zero and

$$f(x)=\int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i x\cdot y}\,dy\,.$$

Appendix A. Products of Positive Roots

Keep the notation introduced in section 3. Recall, in particular, that $\sum_{j=1}^{l''} y_j J_j \in \mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{0}}}$ and $\sum_{j=1}^{l''} y_j J'_j \in \mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{1}}}$ are identified via (42). Here $l'' = \min(l, l')$.

Suppose $l \leq l'$. We can choose the system of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ so that their product is given by the formula

$$\text{(A.1)}\quad \pi_{\mathfrak{g}/\mathfrak{h}}\Big(\sum_{j=1}^{l} y_j J_j\Big) = \begin{cases} \prod_{1\leq j<k\leq l} i(-y_j+y_k) & \text{if} \quad \mathbb{D}=\mathbb{C}\,, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l}(-2iy_j) & \text{if} \quad \mathbb{D}=\mathbb{H}\,, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2) & \text{if} \quad \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l}\,, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l}(-iy_j) & \text{if} \quad \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}\,. \end{cases}$$

Let $\mathfrak{z}' \subseteq \mathfrak{g}'$ be the centralizer of $\mathfrak{h}$. We may choose the order of roots of $\mathfrak{h}$ in $\mathfrak{g}'_{\mathbb{C}}/\mathfrak{z}'_{\mathbb{C}}$ so that the product of all of them is equal to

$$\text{(A.2)}\quad \pi_{\mathfrak{g}'/\mathfrak{z}'}\Big(\sum_{j=1}^{l} y_j J'_j\Big) = \begin{cases} \prod_{1\leq j<k\leq l} i(-y_j+y_k)\cdot\prod_{j=1}^{l}(-iy_j)^{d'-d} & \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{C}\,, & \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l}(-y_j^2)^{d'-d} & \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{H}\,, & \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l}(-2iy_j)\cdot\prod_{j=1}^{l}(-iy_j)^{d'-d} & \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l}\,, & \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l}(-2iy_j)\cdot\prod_{j=1}^{l}(-iy_j)^{d'-d+1} & \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}\,. & \end{cases}$$

Suppose $l > l'$. We can choose the system of the positive roots of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ so that their product is given by the formula

$$\text{(A.3)}\quad \pi_{\mathfrak{g}'/\mathfrak{h}'}\Big(\sum_{j=1}^{l'} y_j J'_j\Big) = \begin{cases} \prod_{1\leq j<k\leq l'} i(-y_j+y_k) & \text{if} \quad \mathbb{D}=\mathbb{C}\,, \\ \prod_{1\leq j<k\leq l'}(-y_j^2+y_k^2) & \text{if} \quad \mathbb{D}=\mathbb{H}\,, \\ \prod_{1\leq j<k\leq l'}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'}(-2iy_j) & \text{if} \quad \mathbb{D}=\mathbb{R}\,. \end{cases}$$

Moreover, let $\mathfrak{z} \subseteq \mathfrak{g}$ be the centralizer of $\mathfrak{h}$. We may choose the positive roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}/\mathfrak{z}_{\mathbb{C}}$ so that their product is equal to

$$\text{(A.4)}\quad \pi_{\mathfrak{g}/\mathfrak{z}}\Big(\sum_{j=1}^{l'} y_j J_j\Big)$$

$$= \begin{cases} \prod_{1\le j<k\le l'} i(-y_j+y_k)\cdot\prod_{j=1}^{l'}(-iy_j)^{d-d'} \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{C}\,, \\ \prod_{1\le j<k\le l'}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'}(-2iy_j)\cdot\prod_{j=1}^{l'}(-y_j^2)^{d-d'} \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{H}\,, \\ \prod_{1\le j<k\le l'}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'}(-iy_j)^{d-d'} \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l}\,, \\ \prod_{1\le j<k\le l'}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'}(-iy_j)\cdot\prod_{j=1}^{l'}(-iy_j)^{d-d'-1} \\ \qquad\qquad \text{if} \quad \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}\,. \end{cases}$$

Appendix B. The Jacobian of the Cayley Transform

Here we determine the Jacobian of the modified Cayley transform c_- : $\mathfrak{g}\to \mathrm{G}$. A straightforward computation shows that for a fixed $x\in\mathfrak{g}$,

$$c_-(x+y)c_-(x)^{-1}-1=(1-x-y)^{-1}2y(1+x)^{-1}\qquad (y\in\mathfrak{g})\,.$$

Hence the derivative (tangent map) is given by

$$c_-'(x)y=(1-x)^{-1}2y(1-x)^{-1}\qquad (y\in\mathfrak{g})\,. \tag{B.1}$$

Recall that G is the isometry group of a hermitian form $(\cdot,\cdot)$ on V. Hence we have the adjoint

$$\mathrm{End}_{\mathbb{D}}(\mathrm{V})\ni g\to g^*\in\mathrm{End}_{\mathbb{D}}(\mathrm{V})$$

defined by

$$(gu,v)=(u,g^*v)\qquad (u,v\in\mathrm{V})\,.$$

Let us view the Lie algebra $\mathfrak{g}$ as a real vector space and consider the map

$$\gamma:\mathrm{GL}_{\mathbb{D}}(\mathrm{V})\to\mathrm{GL}(\mathfrak{g})\,,\quad \gamma(g)(y)=gyg^*\,.$$

Then $\det\circ\gamma:\mathrm{GL}_{\mathbb{D}}(\mathrm{V})\to\mathbb{R}^\times$ is a group homomorphism. Hence there is a number $s\in\mathbb{R}$ such that

$$\det(\gamma(g))=(\det(g)_{\mathrm{V}_{\mathbb{R}}})^s\qquad (g\in\mathrm{GL}_{\mathbb{D}}(\mathrm{V}))\,,$$

where the subscript $\mathbb{R}$ indicates that we are viewing V as a vector space over $\mathbb{R}$. On the other hand, for a fixed number $a \in \mathbb{R}^\times$,

$$\det(\gamma(aI_{\mathrm{V}})) = a^{2\dim \mathfrak{g}} \quad \text{and} \quad \det(aI_{\mathrm{V}})_{\mathrm{V}_{\mathbb{R}}} = a^{\dim \mathrm{V}_{\mathbb{R}}}.$$

Hence,

$$\det(\gamma(g)) = (\det(g)_{\mathrm{V}_{\mathbb{R}}})^{\frac{2\dim \mathfrak{g}}{\dim \mathrm{V}_{\mathbb{R}}}} \qquad (g \in \mathrm{GL}_{\mathbb{D}}(\mathrm{V}))\,.$$

If $x \in \mathfrak{g}$, then $1 \pm x \in \mathrm{GL}_{\mathbb{D}}(\mathrm{V})$ and

$$(1 \pm x)^* = 1 \mp x \quad \text{and} \quad \big((1 \pm x)^{-1}\big)^* = (1 \mp x)^{-1}\,.$$

Hence

$$c'_-(x)y = 2(1-x)^{-1}y(1+x)^{-1}c_-(x) = 2\big(\gamma((1-x)^{-1})y\big)c_-(x) \qquad (y \in \mathfrak{g})\,.$$

Notice that $|\det(c_-(x))| = 1$ because $c(\mathfrak{g}) \subseteq \mathrm{G}$. Therefore

$$|\det(c'_-(x))| = 2^{\dim \mathfrak{g}} \det(1-x)_{\mathrm{V}_{\mathbb{R}}}^{-\frac{2\dim \mathfrak{g}}{\dim \mathrm{V}_{\mathbb{R}}}} = 2^{\dim \mathfrak{g}} \operatorname{ch}(x)^{-2r} \qquad (x \in \mathfrak{g})\,, \tag{B.2}$$

where ch and r are as in (134) and (66), respectively.

Appendix C. The Weyl Denominator Lifted by the Cayley Transform

Consider the orthogonal matrix group

$$\mathrm{G} = \mathrm{O}_{2l+1} = \{g \in \mathrm{GL}_{2l+1}(\mathbb{R});\ gg^t = I\}\,.$$

The spin group is a connected two-fold cover

$$\mathrm{Spin}_{2l+1} \to \mathrm{SO}_{2l+1}$$

of the special orthogonal group. We identify

$$a + ib = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad (a, b \in \mathbb{R})\,. \tag{C.1}$$

Then

$$\mathrm{SO}_2(\mathbb{R}) = \{u \in \mathbb{C};\ |u| = 1\}\,.$$

Fix the diagonal Cartan subgroup

$$\mathrm{H} = \{\mathrm{diag}(u_1, u_2, \dots, u_l, \pm 1);\ u_j \in \mathrm{SO}_2(\mathbb{R})\,,\ 1 \le j \le l\} \subseteq \mathrm{O}_{2l+1}\,.$$

Then the connected identity component of H is

$$\mathrm{H}^0 = \{\mathrm{diag}(u_1, u_2, \dots, u_l, 1);\ u_j \in \mathrm{SO}_2(\mathbb{R});\ 1 \le j \le l\}\,.$$

Denote by $\widehat{\mathrm{H}^0} \subseteq \mathrm{Spin}_{2l+1}$ the preimage of H^0. The Weyl group of $(\mathrm{Spin}_{2l+1}, \widehat{\mathrm{H}^0})$ is isomorphic to the Weyl group of $(\mathrm{SO}_{2l+1}, \mathrm{H}^0)$ and the covering

$$\widehat{\mathrm{H}^0} \to \mathrm{H}^0$$

intertwines the action of these groups. As explained in [GW98, Lemma 6.3.4 and Theorem 6.3.5], one may realize $\widehat{\mathrm{H}^0}$ as the quotient

$$\widehat{\mathrm{H}^0} = (\mathrm{SO}_2)^l / \mathrm{K}\,,$$

where K consists of all elements $(z_1, z_2, \dots, z_l) \in (\mathrm{SO}_2)^l$ such that each $z_j = \pm 1$ and $z_1 z_2 \cdots z_l = 1$. The Weyl group is generated by the inverses $z_j \to z_j^{-1}$ and permutations of the coordinates. It acts on the Lie algebra $\mathfrak{h}$ via the permutations and all sign changes. The covering map is realized as

$$\widehat{\mathrm{H}^0} \ni (z_1, z_2, \dots, z_l)\mathrm{K} \to \mathrm{diag}(z_1^2, z_2^2, \dots, z_l^2, 1) \in \mathrm{H}^0\,.$$

Let $a \in \mathbb{R}$ and define θ_a by $a = \tan\left(\frac{\theta_a}{2}\right)$. Then

$$c_-(-ia) = \frac{1 - ia}{1 + ia} = e^{-i\theta_a}\,.$$

Set $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Under the identification (C.1), J is identified with $-i$. Hence,

$$c_-(aJ) = (I + aJ)(I - aJ)^{-1} = \exp(\theta_a J) \tag{C.2}$$

Therefore the range of the Cayley transform

$$c_-(\mathfrak{h}) = \{\mathrm{diag}(u_1, u_2, ..., u_l, 1);\ u_j \ne -1 \text{ for all } j\}$$

is stable under the action of the Weyl group and c_- intertwines the action of the Weyl group on the Lie algebra and on the group. Pick the following branch of the complex square root,

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\frac{\theta}{2}} \qquad (r > 0, -\pi < \theta < \pi)$$

and set

$$\sigma : c_-(\mathfrak{h}) \ni \mathrm{diag}(u_1, u_2, \ldots, u_l, 1) \to \mathrm{diag}(\sqrt{u_1}, \sqrt{u_2}, \ldots, \sqrt{u_l})\mathrm{K} \in \widehat{\mathrm{H}^0}\,.$$

This is a section of the covering map which intertwines the Weyl group actions. Define

$$\widehat{c}_-(x) = \sigma(c_-(x)) \qquad (x \in \mathfrak{h})\,. \tag{C.3}$$

Then $\widehat{c}_-$ also intertwines the Weyl group actions. Explicitly,

$$\widehat{c}_-(\mathrm{diag}(x_1J_1, x_2J_2, \ldots, x_lJ_l, 0)) = \mathrm{diag}(\sqrt{u_1}, \sqrt{u_2}, \ldots, \sqrt{u_l})\mathrm{K}\,,$$

where

$$u_j = \frac{1 - ix_j}{1 + ix_j}\,.$$

In these terms, the usual choice of the positive roots $e_j \pm e_k$, with $1 \le j < k \le l$, and e_j, with $1 \le j \le l$ together with (131) gives

$$\xi_{e_j}(\mathrm{diag}(u_1, u_2, \ldots, u_l, 1)) = u_j\,.$$

Hence,

$$\begin{aligned}
\xi_{-e_j+e_k}(\mathrm{diag}(u_1, u_2, \ldots, u_l, 1)) &= u_j^{-1}u_k\,,\\
\xi_{-e_j-e_k}(\mathrm{diag}(u_1, u_2, \ldots, u_l, 1)) &= u_j^{-1}u_k^{-1}\,,\\
\xi_{-e_j}(\mathrm{diag}(u_1, u_2, \ldots, u_l, 1)) &= u_j^{-1}\,,\\
\xi_\rho(\mathrm{diag}(\sqrt{u_1}, \sqrt{u_2}, \ldots, \sqrt{u_l}, 1)K) &= u_1^{l-1}u_2^{l-2}\cdots u_{l-1}\xi\,,
\end{aligned}$$

where

$$\xi = \sqrt{u_1}\sqrt{u_2}\ldots\sqrt{u_l}\,.$$

We now verify the following formula

$$\Delta(\widehat{c}_-(x)) = C_1\pi_{\mathfrak{g}/\mathfrak{h}}(x)\,\mathrm{ch}^{-2l+1}(x) \qquad (x \in \mathfrak{h})\,, \tag{C.4}$$

where where $C_1 = 2^{l^2}$. It is easy to check that

$$\text{(C.5)}\quad \sqrt{\frac{1+z_j}{1-z_j}} = \frac{\sqrt{1+z_j}}{\sqrt{1-z_j}}, \quad 1+z_j = \sqrt{1+z_j}\sqrt{1+z_j}\,,$$
$$\sqrt{1+x_j^2} = \sqrt{1+z_j}\sqrt{1-z_j} \qquad (z_j = -ix_j,\ x_j \in \mathbb{R})\,.$$

We shall use the polynomial identity

$$\text{(C.6)}\qquad \prod_{1\le j<k\le l} a_j b_k = \Big(\prod_{j=1}^{l} a_j^{l-j}\Big)\Big(\prod_{k=1}^{l} b_k^{k-1}\Big)$$

when either $b_j = 1$ or $b_j = a_j$ for all $1 \le j \le l$. By (124) and (C.6),

$$\Delta(\widehat{c}_-(x)) = \xi\Big(\prod_{j=1}^{l-1} u_j^{l-j}\Big) \prod_{1\le j<k\le l} (1-u_j^{-1}u_k^{-1})(1-u_j^{-1}u_k) \prod_{j=1}^{l}(1-u_j^{-1})$$
$$= \xi \prod_{1\le j<k\le l} (u_j - u_k^{-1})(1-u_j^{-1}u_k) \prod_{j=1}^{l}(1-u_j^{-1})\,.$$

By (C.5),

$$u_j - u_k^{-1} = \frac{1+z_j}{1-z_j} - \frac{1-z_k}{1+z_k} = \frac{2(z_j+z_k)}{(1-z_j)(1+z_k)}\,,$$
$$1-u_j^{-1}u_k = 1 - \frac{1-z_j}{1+z_j}\frac{1+z_k}{1-z_k} = \frac{2(z_j-z_k)}{(1+z_j)(1-z_k)}\,,$$
$$1-u_j^{-1} = 1 - \frac{1-z_j}{1+z_j} = \frac{2z_j}{1+z_j} = \frac{2z_j}{\sqrt{1+z_j}\sqrt{1+z_j}}\,.$$

Since $\xi = \prod_{j=1}^{l}\sqrt{u_j}$, we obtain by (C.5), (C.6) and (A.1),

$$\Delta(\widehat{c}_-(x)) = 2^{l^2} \prod_{1\le j<k\le l} \frac{1}{(1-z_j^2)(1-z_k^2)} \prod_{j=1}^{l} \frac{1}{\sqrt{1+z_j}\sqrt{1-z_j}}$$
$$\times \prod_{1\le j<k\le l} (z_j+z_k)(z_j-z_k) \prod_{j=1}^{l} z_j$$
$$= 2^{l^2}\Big(\prod_{j=1}^{l} \frac{1}{(1-z_j^2)^{l-1}} \prod_{j=1}^{l} \frac{1}{\sqrt{1+x_j^2}}\Big)\pi_{\mathfrak{g}/\mathfrak{h}}(x) \qquad (x \in \mathfrak{h})\,,$$

which gives (C.4).

Recall from (219) that if $x = \mathrm{diag}(x_1 J_1, x_2 J_2, \dots, x_l J_l, 0) \in \mathfrak{h} = \mathfrak{h}_s$, then $c_\odot(x) = \mathrm{diag}(v_1, v_2, \dots, v_l, 1)$ has coordinates

$$v_j = c(x_j J_j) = -c_-(x_j J_j) = -u_j\,, \qquad (1 \le j \le l)$$

with $|v_j| = 1$ and $v_j \neq 1$ for all j. The identification (C.1) implies the identification

$$v_j = -u_j = \frac{z_j + 1}{z_j - 1}\,, \qquad (z_j = -ix_j,\ 1 \le j \le l)\,.$$

On the subset where $v_j \neq \pm 1$ for all j define

$$\widehat{c}_\odot(x) = \sigma(c_\odot(x)) \qquad (x \in \mathfrak{h}, x_j \neq 0, 1 \le j \le l)\,. \tag{C.7}$$

We now prove the following equality:

$$\Delta(\widehat{c}_\odot(x)) = C_2 \left(\prod_{j=1}^{l} \mathrm{sgn}(x_j) \right) \pi_{\mathfrak{g}_s / \mathfrak{h}_s}(x)\, \mathrm{ch}^{-2l+1}(x) \quad (x \in \mathfrak{h}, x_j \neq 0, 1 \le j \le l)\,, \tag{C.8}$$

where $C_2 = (2i)^{l^2}$ and $\mathrm{sgn}(x_j) = x_j/|x_j|$. (Notice that $\Delta(\widehat{c}_\odot(x))$ is singular at $x_j = 0$ because so is the fixed section σ, which depends on our choice of $\sqrt{\cdot}$.) It is easy to check that

$$\sqrt{\frac{z_j + 1}{z_j - 1}} = \frac{\sqrt{z_j + 1}}{\sqrt{z_j - 1}}\,, \qquad z_j + 1 = \sqrt{z_j + 1}\sqrt{z_j + 1}\,, \tag{C.9}$$

$$\sqrt{z_j - 1} = -i\,\mathrm{sgn}(x_j)\sqrt{1 - z_j}\,, \qquad -i\,\mathrm{sgn}(x_j)\sqrt{1 + x_j^2} = \sqrt{z_j + 1}\sqrt{z_j - 1}$$
$$(z_j = -ix_j,\ x_j \in \mathbb{R} \setminus \{0\})\,.$$

As before,

$$\Delta(\widehat{c}_\odot(x)) = \xi \prod_{1 \le j < k \le l} (v_j - v_k^{-1})(1 - v_j^{-1} v_k) \prod_{j=1}^{l} (1 - v_j^{-1})\,,$$

where, by (C.9),

$$v_j - v_k^{-1} = \frac{z_j + 1}{z_j - 1} - \frac{z_k - 1}{z_k + 1} = \frac{2(z_j + z_k)}{(z_j - 1)(z_k + 1)}\,,$$

$$1 - v_j^{-1} v_k = 1 - \frac{z_j - 1}{z_j + 1}\frac{z_k + 1}{z_k - 1} = \frac{2(-z_j + z_k)}{(z_j + 1)(z_k - 1)}\,,$$

$$1 - v_j^{-1} = 1 - \frac{z_j - 1}{z_j + 1} = \frac{2}{z_j + 1} = \frac{2}{\sqrt{z_j + 1}\sqrt{z_j + 1}}\,.$$

Since $\xi = \prod_{j=1}^{l} \sqrt{v_j}$, we obtain by (C.5), (C.6) and (A.1),

$$\begin{aligned}
\Delta(\widehat{c}_{\odot}(x)) &= 2^{l^2}\Big(\prod_{j=1}^{l}\sqrt{\frac{z_j+1}{z_j-1}}\Big)\Big(\prod_{1\le j<k\le l}\frac{1}{(z_j^2-1)(z_k^2-1)}\Big)\\
&\quad\times\Big(\prod_{j=1}^{l}\frac{1}{\sqrt{z_j+1}\sqrt{z_j+1}}\Big)\Big(\prod_{1\le j<k\le l}(z_j+z_k)(-z_j+z_k)\Big)\\
&= 2^{l^2}\Big(\prod_{j=1}^{l}\frac{\sqrt{z_j+1}}{\sqrt{z_j-1}}\frac{1}{\sqrt{z_j+1}\sqrt{z_j+1}}\Big)\Big(\prod_{j=1}^{l}\frac{1}{(1-z_j^2)^{l-1}}\Big)\\
&\quad\times(-1)^{l(l-1)/2}\Big(\prod_{1\le j<k\le l}(z_j+z_k)(z_j-z_k)\Big)\\
&= i^{l(l-1)}2^{l^2}\Big(\prod_{j=1}^{l}\frac{1}{\sqrt{z_j-1}\sqrt{z_j+1}}\Big)\Big(\prod_{j=1}^{l}\frac{1}{(1-z_j^2)^{l-1}}\Big)\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)\\
&= (2i)^{l^2}\Big(\prod_{j=1}^{l}\frac{\operatorname{sgn}(x_j)}{\sqrt{1+x_j^2}}\Big)\Big(\prod_{j=1}^{l}\frac{1}{(1+x_j^2)^{l-1}}\Big)\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)\qquad(x\in\mathfrak{h}\setminus\{0\}),
\end{aligned}$$

which gives (C.8).

Appendix D. The Special Functions $P_{a,b}$ and $Q_{a,b}$

For two integers a and b define the following functions in the real variable ξ,

$$\text{(D.1)}\qquad P_{a,b,2}(\xi) = \begin{cases} \sum_{k=0}^{b-1}\frac{a(a+1)\cdots(a+k-1)}{k!(b-1-k)!}2^{-a-k}\xi^{b-1-k} & \text{if } b\ge 1\\ 0 & \text{if } b\le 0,\end{cases}$$

$$\text{(D.2)}\qquad P_{a,b,-2}(\xi) = \begin{cases} (-1)^{a+b-1}\sum_{k=0}^{a-1}\frac{b(b+1)\cdots(b+k-1)}{k!(a-1-k)!}(-2)^{-b-k}\xi^{a-1-k} & \text{if } a\ge 1\\ 0 & \text{if } a\le 0,\end{cases}$$

where $a(a+1)\cdots(a+k-1)=1$ if $k=0$. Notice that

$$\text{(D.3)}\qquad P_{a,b,-2}(\xi) = P_{b,a,2}(-\xi)\qquad(\xi\in\mathbb{R},\ a,b\in\mathbb{Z}).$$

Set

$$\begin{aligned}\text{(D.4)}\qquad P_{a,b}(\xi) &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi)+P_{a,b,-2}(\xi)\mathbb{I}_{\mathbb{R}^-}(\xi))\\ &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi)+P_{b,a,2}(-\xi)\mathbb{I}_{\mathbb{R}^+}(-\xi)),\end{aligned}$$

where $\mathbb{I}_S$ denotes the indicator function of the set S. Also, let

$$\text{(D.5)}\quad Q_{a,b}(iy) = 2\pi \begin{cases} 0 & \text{if } a+b \geq 1\,, \\ \sum_{k=b}^{-a} \frac{a(a+1)\cdots(a+k-1)}{k!} 2^{-a-k}(1-iy)^{k-b} & \text{if } -a > b-1 \geq 0\,, \\ \sum_{k=a}^{-b} \frac{b(b+1)\cdots(b+k-1)}{k!} 2^{-b-k}(1+iy)^{k-a} & \text{if } -b > a-1 \geq 0\,, \\ (1+iy)^{-a}(1-iy)^{-b} & \text{if } a \leq 0 \text{ and } b \leq 0\,. \end{cases}$$

Observe also that

$$\text{(D.6)}\qquad P_{b,a}(\xi) = P_{a,b}(-\xi) \qquad \text{and} \qquad Q_{b,a}(iy) = Q_{a,b}(-iy)\,.$$

The following elementary fact will be crucial at several points.

LEMMA D.1. *Suppose that $a+b \leq 1$. Then at most one between $P_{a,b,2}$ and $P_{a,b,-2}$ can be non-zero. Hence $P_{a,b}$ is either 0 or the restriction of a polynomial to a half line.*

REMARK 17. Let Γ denote the gamma function. If k is a nonnegative integer, then

$$a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}\,,$$

which is often shortened by the Pochhammer symbol $(a)_k$. Another useful formula is

$$a(a+1)\cdots(a+k-1) = (-1)^k(-a)(-a-1)\cdots(-a-k+1) = (-1)^k \frac{\Gamma(-a+1)}{\Gamma(-a+1-k)}\,.$$

In this notation, for an integer $b \geq 1$ and $h = 0, 1, \ldots, b-1$,

$$(b-1-h)! = \frac{(b-1)!}{(-b+1)_h} \quad \text{and} \quad \Gamma(-a-b+2+h) = \Gamma(-a-b+2)\,(-a-b+2)_h\,.$$

Hence

$$\begin{aligned} P_{a,b,2}(\xi) &= \sum_{k=0}^{b-1} (-1)^k \frac{\Gamma(-a+1)}{\Gamma(-a+1-k)}\, \frac{1}{k!(b-1-k)!} 2^{-a-k}\xi^{b-1-k} \\ &= \Gamma(-a+1) \sum_{h=0}^{b-1} (-1)^{b-1-h} \frac{1}{\Gamma(-a-b+2+h)}\, \frac{1}{(b-1-h)!h!} 2^{-a-b+1+h}\xi^h \\ &= (-1)^{b-1} 2^{-a-b+1} \frac{\Gamma(-a+1)}{\Gamma(-a-b+2)\,(b-1)!} \sum_{h=0}^{b-1} \frac{(-b+1)_h}{(-a-b+2)_h h!} (2\xi)^h \end{aligned}$$

$$= (-1)^{b-1}2^{-a-b+1}\frac{\Gamma(-a+1)}{\Gamma(-a-b+2)\,(b-1)!}\,{}_1F_1\big(-b+1;-a-b+2;2\xi\big)$$
$$= (-1)^{b-1}2^{-a-b+1}L_{b-1}^{-a-b+1}(2\xi)\,,$$

where ${}_1F_1$ is the confluent hypergeometric function and $L_n^\alpha(x)$ is a Laguerre polynomial. See [Erd53, 6.9(36), §10.12].

Proposition D.2. *For any $a,\ b \in \mathbb{Z}$, the formula*

$$\int_{\mathbb{R}}(1+iy)^{-a}(1-iy)^{-b}\phi(y)\,dy \qquad (\phi \in \mathcal{S}(\mathbb{R})) \tag{D.7}$$

defines a tempered distribution on $\mathbb{R}$. The restriction of the Fourier transform of this distribution to $\mathbb{R}\setminus\{0\}$ is a function given by

$$\int_{\mathbb{R}}(1+iy)^{-a}(1-iy)^{-b}e^{-iy\xi}\,dy = P_{a,b}(\xi)e^{-|\xi|}. \tag{D.8}$$

*The right-hand side of (*D.8*) is an absolutely integrable function on the real line and thus defines a tempered distribution on $\mathbb{R}$. Furthermore,*

$$(1+iy)^{-a}(1-iy)^{-b} = \frac{1}{2\pi}\int_{\mathbb{R}}P_{a,b}(\xi)e^{-|\xi|}e^{iy\xi}\,dy + \frac{1}{2\pi}Q_{a,b}(iy) \tag{D.9}$$

and hence,

$$\int_{\mathbb{R}}(1+iy)^{-a}(1-iy)^{-b}e^{-iy\xi}\,dy = P_{a,b}(\xi)e^{-|\xi|} + Q_{a,b}(-\frac{d}{d\xi})\delta_0(\xi)\,. \tag{D.10}$$

Proof. Since, $|1 \pm iy| = \sqrt{1+y^2}$, (D.7) is clear. The integral (D.8) is equal to

$$\begin{aligned}\frac{1}{i}\int_{i\mathbb{R}}(1+z)^{-a}(1-z)^{-b}e^{-z\xi}\,dz \\ = 2\pi(-\mathbb{I}_{\mathbb{R}^+}(\xi)\,\mathrm{res}_{z=1}(1+z)^{-a}(1-z)^{-b}e^{-z\xi} \\ + \mathbb{I}_{\mathbb{R}^-}(\xi)\,\mathrm{res}_{z=-1}(1+z)^{-a}(1-z)^{-b}e^{-z\xi})\,.\end{aligned} \tag{D.11}$$

The computation of the two residues is straightforward and (D.8) follows.

Since

$$\int_0^\infty e^{-\xi}e^{i\xi y}\,d\xi = (1-iy)^{-1}\,,$$

we have

$$\text{(D.12)}\quad \int_0^\infty \xi^m e^{-\xi} e^{i\xi y}\, d\xi = \left(\frac{d}{d(iy)}\right)^m (1-iy)^{-1} = m!(1-iy)^{-m-1} \qquad (m = 0, 1, 2, \dots).$$

Thus, if $b \geq 1$, then

$$\begin{aligned}\int_0^\infty & P_{a,b,2}(\xi) e^{-\xi} e^{i\xi y}\, d\xi \\ &= \sum_{k=0}^{b-1} \frac{a(a+1)\cdots(a+k-1)}{k!} 2^{-a-k}(1-iy)^{-b+k} \\ &= (1-iy)^{-b} 2^{-a} \sum_{k=0}^{b-1} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k.\end{aligned}$$

Also, if $a \leq 0$, then

$$\begin{aligned}2^a(1+iy)^{-a} &= \left(1 - \frac{1}{2}(1-iy)\right)^{-a} = \sum_{k=0}^{-a} \binom{-a}{k} \left(-\frac{1}{2}(1-iy)\right)^k \\ &= \sum_{k=0}^{-a} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k.\end{aligned}$$

Hence,

$$\begin{aligned}\text{(D.13)}\quad & \int_0^\infty P_{a,b,2}(\xi) e^{-\xi} e^{i\xi y}\, d\xi - (1+iy)^{-a}(1-iy)^{-b} \\ &= (1-iy)^{-b} 2^{-a} \left(\sum_{k=0}^{b-1} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k \right. \\ &\qquad \left. - \sum_{k=0}^{-a} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k \right).\end{aligned}$$

Recall that $P_{a,b,-2} = 0$ if $a \leq 0$. Hence, (D.8) shows that (D.13) is the inverse Fourier transform of a distribution supported at $\{0\}$, hence a polynomial.

Suppose $-a < b-1$. Then (D.13) is equal to

$$2^{-a} \sum_{k=-a+1}^{b-1} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}\right)^k (1-iy)^{k-b},$$

which is zero because $(-a)(-a-1)\cdots(-a-k+1)=0$ for $k \geq -a+1$. If $-a=b-1$, then (D.13) is obviously zero.

Suppose $-a > b-1$. Then (D.13) is equal to

$$-2^{-a}\sum_{k=b}^{-a}\frac{(-a)(-a-1)\cdots(-a-k+1)}{k!}\left(-\frac{1}{2}\right)^k(1-iy)^{k-b}. \tag{D.14}$$

As in (D.12) we have

$$\int_{-\infty}^{0}\xi^m e^{\xi}e^{i\xi y}\,d\xi=\left(\frac{d}{d(iy)}\right)^m(1+iy)^{-1}=(-1)^m m!(1+iy)^{-m-1}\qquad(m=0,1,2,\dots).$$

Suppose $a \geq 1$. Then

$$\begin{aligned}\int_{-\infty}^{0}&P_{a,b,-2}(\xi)e^{\xi}e^{i\xi y}\,d\xi\\ &=(-1)^{a+b-1}\sum_{k=0}^{a-1}\frac{b(b+1)\cdots(b+k-1)}{k!}(-2)^{-b-k}(-1)^{a-1+k}(1+iy)^{-a+k}\\ &=(1+iy)^{-a}2^{-b}\sum_{k=0}^{a-1}\frac{(-b)(-b-1)\cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^k.\end{aligned}$$

Also, if $b \leq 0$, then

$$2^b(1-iy)^{-b}=\sum_{k=0}^{-b}\frac{(-b)(-b-1)\cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^k.$$

Hence,

$$\begin{aligned}\int_{-\infty}^{0}&P_{a,b,-2}(\xi)e^{\xi}e^{i\xi y}\,d\xi-(1+iy)^{-a}(1-iy)^{-b}\\ &=(1+iy)^{-a}2^{-b}\left(\sum_{k=0}^{a-1}\frac{(-b)(-b-1)\cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^k\right.\\ &\qquad\left.-\sum_{k=0}^{-b}\frac{(-b)(-b-1)\cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^k\right).\end{aligned} \tag{D.15}$$

As before, we show that (D.15) is zero if $-b \leq a-1$. If $-b > a-1$, then (D.15) is equal to

$$-2^{-b}\sum_{k=a}^{-b}\frac{(-b)(-b-1)\cdots(-b-k+1)}{k!}\left(-\frac{1}{2}\right)^k(1+iy)^{k-a}.$$

If $a \geq 1$ and $b \geq 1$, then our computations show that

$$\text{(D.16)} \quad \int_0^\infty P_{a,b,2}(\xi)e^{-\xi}e^{i\xi y}\,d\xi + \int_{-\infty}^0 P_{a,b,-2}(\xi)e^{\xi}e^{i\xi y}\,d\xi - (1+iy)^{-a}(1-iy)^{-b}$$

is a polynomial which tends to zero if y goes to infinity. Thus (D.16) is equal zero. This completes the proof of (D.9). The statement (D.10) is a direct consequence of (D.9). □

The test functions which occur in Proposition D.2 need not be in the Schwartz space. In fact the test functions we shall use in our applications are not necessarily smooth. Therefore we shall need a more precise version of the formula (D.10). This requires a definition and two well-known lemmas.

Following Harish-Chandra denote by $\mathcal{S}(\mathbb{R}^\times)$ the space of the smooth complex valued functions defined on $\mathbb{R}^\times$ whose all derivatives are rapidly decreasing at infinity and have limits at zero from both sides. For $\psi \in \mathcal{S}(\mathbb{R}^\times)$ let

$$\psi(0+) = \lim_{x\to 0+} \psi(\xi)\,, \quad \psi(0-) = \lim_{x\to 0-} \psi(\xi), \quad \langle\psi\rangle_0 = \psi(0+) - \psi(0-)\,.$$

In particular the condition $\langle\psi\rangle_0 = 0$ means that ψ extends to a continuous function on $\mathbb{R}$.

LEMMA D.3. *Let $c = 0, 1, 2, \ldots$ and let $\psi \in \mathcal{S}(\mathbb{R}^\times)$. Suppose*

$$\text{(D.17)} \qquad \langle\psi\rangle_0 = \cdots = \langle\psi^{(c-1)}\rangle_0 = 0\,.$$

*(The condition (*D.17*) is empty if $c = 0$.) Then*

$$\text{(D.18)} \quad \left|\int_{\mathbb{R}^\times} e^{-iy\xi}\psi(\xi)\,d\xi\right| \leq \min\{1, |y|^{-c-1}\}(|\langle\psi^{(c)}\rangle_0| + \parallel\psi^{(c+1)}\parallel_1 + \parallel\psi\parallel_1)$$

PROOF. Integration by parts shows that for $z \in \mathbb{C}^\times$

$$\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi = z^{-1}\psi(0+) + \cdots + z^{-c-1}\psi^{(c)}(0+) + z^{-c-1}\int_{\mathbb{R}^+} e^{-z\xi}\psi^{(c+1)}(\xi)\,d\xi\,,$$

$$\int_{\mathbb{R}^-} e^{-z\xi}\psi(\xi)\,d\xi = -z^{-1}\psi(0-) - \cdots - z^{-c-1}\psi^{(c)}(0-) + z^{-c-1}\int_{\mathbb{R}^-} e^{-z\xi}\psi^{(c+1)}(\xi)\,d\xi\,.$$

Hence,

$$\int_{\mathbb{R}^\times} e^{-z\xi}\psi(\xi)\,d\xi$$

$$= z^{-1}\langle\psi\rangle_0 + \cdots + z^{-c}\langle\psi^{(c-1)}\rangle_0 + z^{-c-1}\langle\psi^{(c)}\rangle_0 + z^{-c-1}\int_{\mathbb{R}^\times} e^{-z\xi}\psi^{(c+1)}(\xi)\,d\xi$$

and (D.18) follows. □

Lemma D.4. *Under the assumptions of Lemma D.3, with* $1 \le c$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi}\psi(\xi)\,d\xi\,dy = 2\pi\psi^{(k)}(0) \qquad (0 \le k \le c-1)\,,$$

where each consecutive integral is absolutely convergent.

Proof. Since

$$\int_{\mathbb{R}} |y|^{c-1}\min\{1, |y|^{-c-1}\}\,dy < \infty\,,$$

the absolute convergence follows from Lemma D.3. Since the Fourier transform of ψ is absolutely integrable and since ψ is continuous at zero, Fourier inversion formula [Hör83, (7.1.4)] shows that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^\times} e^{-iy\xi}\psi(\xi)\,d\xi\,dy = 2\pi\psi(0)\,. \tag{D.19}$$

Also, for $0 < k$,

$$\begin{aligned}
&\int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi}\psi(\xi)\,d\xi = \int_{\mathbb{R}^\times} (-\partial_\xi)\left((iy)^{k-1}e^{-iy\xi}\right)\psi(\xi)\,d\xi\\
&= \int_{\mathbb{R}^+} (-\partial_\xi)\left((iy)^{k-1}e^{-iy\xi}\right)\psi(\xi)\,d\xi + \int_{\mathbb{R}^-} (-\partial_\xi)\left((iy)^{k-1}e^{-iy\xi}\right)\psi(\xi)\,d\xi\\
&= (iy)^{k-1}\psi(0+) + \int_{\mathbb{R}^+} (iy)^{k-1}e^{-iy\xi}\psi'(\xi)\,d\xi\\
&- (iy)^{k-1}\psi(0-) + \int_{\mathbb{R}^-} (iy)^{k-1}e^{-iy\xi}\psi'(\xi)\,d\xi\\
&= (iy)^{k-1}\langle\psi\rangle_0 + \int_{\mathbb{R}^\times} (iy)^{k-1}e^{-iy\xi}\psi'(\xi)\,d\xi\,.
\end{aligned}$$

Hence, by induction on k and by our assumption

$$\begin{aligned}
\int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi}\psi(\xi)\,d\xi &= (iy)^{k-1}\langle\psi\rangle_0 + (iy)^{k-2}\langle\psi'\rangle_0 + \cdots + \langle\psi^{(k-1)}\rangle_0\\
&+ \int_{\mathbb{R}^\times} e^{-iy\xi}\psi^{(k)}(\xi)\,d\xi\\
&= \int_{\mathbb{R}^\times} e^{-iy\xi}\psi^{(k)}(\xi)\,d\xi\,.
\end{aligned}$$

Therefore our lemma follows from (D.19). □

The following proposition is an immediate consequence of Lemmas D.3, D.4, and the formula (D.9).

PROPOSITION D.5. *Fix two integers* $a,\ b \in \mathbb{Z}$ *and a function* $\psi \in \mathcal{S}(\mathbb{R}^\times)$. *Let* $c = -a - b$. *If* $c \geq 0$ *assume that*

$$\langle \psi \rangle_0 = \cdots = \langle \psi^{(c)} \rangle_0 = 0\,. \tag{D.20}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}^\times} (1+iy)^{-a}(1-iy)^{-b} e^{-iy\xi} \psi(\xi)\, d\xi\, dy \\ &= \int_{\mathbb{R}^\times} P_{a,b}(\xi) e^{-|\xi|} \psi(\xi)\, d\xi + Q_{a,b}(\partial_\xi)\psi(\xi)|_{\xi=0} \\ &= \int_{\mathbb{R}} \Big(P_{a,b}(\xi) e^{-|\xi|} + Q_{a,b}(-\partial_\xi)\delta_0(\xi) \Big)\, \psi(\xi)\, d\xi\,, \end{aligned} \tag{D.21}$$

where δ_0 *denotes the Dirac delta at* 0.
(Recall that $Q_{a,b} = 0$ *if* $c < 0$ *and* $Q_{a,b}$ *is a polynomial of degree if* c, *if* $c \geq 0$.*)*

Let $\mathcal{S}(\mathbb{R}^+)$ be the space of the smooth complex valued functions whose all derivatives are rapidly decreasing at infinity and have limits at zero. Then $\mathcal{S}(\mathbb{R}^+)$ may be viewed as the subspace of the functions in $\mathcal{S}(\mathbb{R}^\times)$ which are zero on $\mathbb{R}^-$. Similarly we define $\mathcal{S}(\mathbb{R}^-)$. The following propositions are direct consequences of Proposition D.5. We sketch independent proofs below.

PROPOSITION D.6. *There is a seminorm* p *on the space* $\mathcal{S}(\mathbb{R}^+)$ *such that*

$$\left| \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi)\, d\xi \right| \leq \min\{1, |z|^{-1}\} p(\psi) \qquad (\psi \in \mathcal{S}(\mathbb{R}^+),\ Re\, z \geq 0)\,, \tag{D.22}$$

and similarly for $\mathcal{S}(\mathbb{R}^-)$.

Fix integers $a, b \in \mathbb{Z}$ *with* $a + b \geq 1$. *Then for any function* $\psi \in \mathcal{S}(\mathbb{R}^+)$,

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b} \int_{\mathbb{R}^+} e^{-iy\xi} \psi(\xi)\, d\xi\, dy = 2\pi \int_{\mathbb{R}^+} P_{a,b,2}(\xi) e^{-\xi} \psi(\xi)\, d\xi\,, \tag{D.23}$$

and any function $\psi \in \mathcal{S}(\mathbb{R}^-)$,

$$\text{(D.24)}\quad \int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b} \int_{\mathbb{R}^-} e^{-iy\xi}\psi(\xi)\,d\xi\,dy = 2\pi \int_{\mathbb{R}^-} P_{a,b,-2}(\xi)e^{\xi}\psi(\xi)\,d\xi\,,$$

where each consecutive integral is absolutely convergent.

Proof. Clearly

$$\left|\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi\right| \le \int_{\mathbb{R}^+} e^{-\operatorname{Re} z\xi}|\psi(\xi)|\,d\xi \le \| \psi \|_1 \ .$$

Integration by parts shows that for $z \neq 0$,

$$\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi = z^{-1}\psi(0) + z^{-1}\int_{\mathbb{R}^+} e^{-z\xi}\psi'(\xi)\,d\xi\,.$$

Hence (D.22) follows with $p(\psi) = |\psi(0)| + \| \psi \|_1 + \| \psi' \|_1$.

Let $a, b \in \mathbb{Z}$ be such that $a + b \ge 1$. Then the function

$$(1+z)^{-a}(1-z)^{-b}\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi$$

is continuous on $\operatorname{Re} z \ge 0$ and meromorphic on $\operatorname{Re} z > 0$ and (D.22) shows that it is dominated by $|z|^{-2}$. Therefore Cauchy's Theorem implies that the left-hand side of (D.23) is equal to

$$-2\pi \operatorname{res}_{z=1}\left((1+z)^{-a}(1-z)^{-b}\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi\right).$$

The computation of this residue is straightforward. This verifies (D.23). The proof of (D.24) is entirely analogous. □

Appendix E. The Covering $\widetilde{\mathrm{G}} \to \mathrm{G}$

In this appendix we recall some results about the splitting of the restrictions $\widetilde{\mathrm{L}} \to \mathrm{L}$ of the metaplectic covering

$$\text{(E.1)}\qquad 1 \to \{\pm 1\} \to \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Sp}(\mathrm{W}) \to 1$$

to a subgroup L of the compact member G of a dual pair $(\mathrm{G}, \mathrm{G}')$ as in (2). This is well known, but we could not find a reference sketching the proofs

of the results we are using in this paper. We are therefore providing a short and complete argument.

If K is a maximal compact subgroup of Sp(W), then $\widetilde{\mathrm{K}}$ is a maximal compact subgroup of $\widetilde{\mathrm{Sp}}(\mathrm{W})$. The group $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is connected, noncompact, semisimple and with finite center $\widetilde{\mathrm{Z}}$. (Since $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a double cover of Sp(W), only the connectedness needs to be commented. It follows from the fact that the covering (E.1) does not split; see e.g. [AP14, Proposition 4.20] or the original proof [Wei64, p. 199]). The maximal compact subgroup $\widetilde{\mathrm{K}}$ is therefore connected; see e.g. [Hel78, Chapter VI, Theorem 1.1]. Hence the covering

$$\widetilde{\mathrm{K}} \to \mathrm{K} \tag{E.2}$$

does not split.

As is well known, K is isomorphic to a compact unitary group. In fact, if $\mathrm{W} = \mathbb{R}^{2n}$ and

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{E.3}$$

then

$$\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix};\ a, b \in \mathrm{GL}_n(\mathbb{R}),\ ab^t = ba^t,\ aa^t + bb^t = I_n \right\} \tag{E.4}$$

is a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ and

$$\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}} \ni \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \to a + ib \in \mathrm{U}_n \tag{E.5}$$

is a Lie group isomorphism. Any two maximal compact subgroups of Sp(W) are conjugate by an inner automorphism. Let $\mathrm{K} \to \mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}}$ be the corresponding isomorphism. Composition with (E.5) fixes then an isomorphism $\phi : \mathrm{K} \to \mathrm{U}_n$. Set

$$\widetilde{\mathrm{K}}^{\phi} = \{(u, \zeta) \in \mathrm{K} \times \mathbb{C}^{\times};\ \det(\phi(u)) = \zeta^2\} \tag{E.6}$$

Recall the bijection between equivalence classes of n-fold path-connected coverings and the conjugacy classes of index-n subgroups of the fundamental group (see e.g. [Hat02, Theorem 1.38]). Then, up to an isomorphism of

coverings, U_n has only one connected double cover. Hence (E.2) is isomorphic to

$$\widetilde{\mathrm{K}}^{\phi} \ni (u, \zeta) \to u \in \mathrm{K}\,. \tag{E.7}$$

Let $\mathrm{L} \subseteq \mathrm{K}$ be any subgroup and

$$\widetilde{\mathrm{L}} \to \mathrm{L} \tag{E.8}$$

the restriction of the covering (E.2) to L. Let $\widetilde{\mathrm{L}}^{\phi}$ be the preimage of L in $\widetilde{\mathrm{K}}^{\phi}$. Then (E.8) splits if and only if

$$\widetilde{\mathrm{L}}^{\phi} \to \mathrm{L} \tag{E.9}$$

splits, i.e. there is a group homomorphism $\mathrm{L} \ni g \to \zeta(g) \in \mathrm{U}_1 \subset \mathbb{C}^{\times}$ such that $\zeta(g)^2 = \det(\phi(g))$ for all $g \in \mathrm{L}$. For instance, if L is a connected subgroup of K such that

$$\mathrm{L} \subseteq \{u \in \mathrm{K};\ \det(\phi(u)) = 1\}\,, \tag{E.10}$$

then (E.8) splits.

To fix ϕ, let $(\mathrm{V}, (\cdot,\cdot))$ and $(\mathrm{V}', (\cdot,\cdot)')$ be the defining spaces of G and G′, respectively, with $\dim_{\mathbb{D}} \mathrm{V} = d$ and $\dim_{\mathbb{D}} \mathrm{V}' = d'$. Realize W as $\mathrm{V} \otimes_{\mathbb{D}} \mathrm{V}'$, considered as a real symplectic space, with symplectic form $\langle \cdot, \cdot \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}} \big((\cdot,\cdot) \otimes (\cdot,\cdot)' \big)$, where $\mathrm{tr}_{\mathbb{D}/\mathbb{R}}$ denotes the reduced trace; see [How79, §5] and [Wei73, p. 169]. Then the group G is viewed as a subgroup of Sp(W) via the identification $\mathrm{G} \ni g \to g \otimes 1 \in \mathrm{Sp}(\mathrm{W})$.[1] Similarly, G′ is viewed as a subgroup of Sp(W) via the identification $\mathrm{G}' \ni g' \to 1 \otimes g' \in \mathrm{Sp}(\mathrm{W})$. Recall that n-by-n-matrices over $\mathbb{C}$ can be identified with $2n$-by-$2n$ matrices over $\mathbb{R}$ under the isomorphism

$$\alpha : M \to \begin{pmatrix} \mathrm{Re}\, M & -\mathrm{Im}\, M \\ \mathrm{Im}\, M & \mathrm{Re}\, M \end{pmatrix}.$$

Moreover, n-by-n-matrices over $\mathbb{H}$ can be identified with $2n$-by-$2n$ matrices over $\mathbb{C}$ under the isomorphism

$$\beta : M \to \begin{pmatrix} z_1(M) & -\overline{z_2(M)} \\ z_2(M) & \overline{z_1(M)} \end{pmatrix}.$$

[1] Following the notation at the beginning of Section 3, one should identify g and $(g^{-1})^t \otimes 1$.

Here, for $v \in \mathbb{H}$, we write $v = z_1(v) + jz_2(v)$ with $z_1(v), z_2(v) \in \mathbb{C}$, and we similarly define $z_1(M)$ and $z_2(M)$ if M is a matrix over $\mathbb{H}$.

Since G is compact, there is a compatible positive complex structure J on W such that the maximal compact subgroup $\mathrm{K} = \mathrm{Sp}(\mathrm{W})^J$ of $\mathrm{Sp}(\mathrm{W})$ contains G. Moreover, since G commutes with J, there is $J' \in \mathrm{G}'$ such that $J = 1 \otimes J'$. Set $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Then, the explicit expressions of J' with respect to the standard basis of $\mathrm{V} \simeq \mathbb{D}^d$ and of J with respect to the standard basis of $\mathrm{W} \simeq \mathbb{R}^{2n}$ are given as follows:

$(\mathrm{G}, \mathrm{G}')$	J'	n	J
$(\mathrm{O}_d, \mathrm{Sp}_{2m}(\mathbb{R}))$	J_{2m}	md	J_{2md}
$(\mathrm{U}_d, \mathrm{U}_{p,q})$	$-iI_{p,q}$	$d(p+q)$	$\begin{pmatrix} 0 & I_{dp,dq} \\ -I_{dp,dq} & 0 \end{pmatrix}$
$(\mathrm{Sp}_d, \mathrm{O}^*_{2m})$	$-jI_m$	$2md$	$\begin{pmatrix} J_{2pm} & 0 \\ 0 & J_{2pm} \end{pmatrix}$

Notice that in the $(\mathrm{U}_d, \mathrm{U}_{p,q})$-case we have $SJS^{-1} = J_{2d(p+q)}$ for $S = \begin{pmatrix} I_{d(p+q)} & 0 \\ 0 & I_{dp,dq} \end{pmatrix}$; in the $(\mathrm{Sp}_d, \mathrm{O}^*_{2m})$-case, $TJT^{-1} = J_{4pm}$ for $T = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$. Hence, in all cases we can embed G in (E.4) from the identification $g \to g \otimes 1 \in \mathrm{Sp}(\mathrm{W})^J$ followed by the isomorphism of $\mathrm{Sp}(\mathrm{W})^J$ and $\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}}$ corresponding to the conjugations by S or T, and then apply (E.5). We obtain:

$$\det(\phi(g)) = \begin{cases} \det(g)_{\mathrm{V}}^m & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{O}_d, \mathrm{Sp}_{2m}(\mathbb{R})) \\ \det(g)_{\mathrm{V}}^{p-q} & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{U}_d, \mathrm{U}_{p,q}) \\ 1 & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_d, \mathrm{O}^*_{2m}) \end{cases}, \tag{E.11}$$

where $\det(g)_{\mathrm{V}}$ denotes the determinant of g as an element of $\mathrm{G} \subseteq \mathrm{GL}_{\mathbb{D}}(\mathrm{V})$. (The determinant of an n-by-n matrix over $\mathbb{H}$ can be reduced to a determinant of a $2n$-by-$2n$ matrix over $\mathbb{C}$ via the isomorphism β. For elements of

$\mathrm{Sp}(d)$, this notion of determinant coincides with other possible notions of quaternionic determinants; see [Asl96] for additional information.)

Proposition E.1. *The covering* $\widetilde{\mathrm{G}} \to \mathrm{G}$ *splits if and only if* $\det(\phi(g))$ *is a square. This happens for all pairs* $(\mathrm{G}, \mathrm{G}')$ *different from* $(\mathrm{O}_d, \mathrm{Sp}_{2m}(\mathbb{R}))$ *with* m *odd and* $(\mathrm{U}_d, \mathrm{U}_{p,q})$ *with* $p+q$ *odd. In these two non-splitting cases, the covering* $\widetilde{\mathrm{G}} \to \mathrm{G}$ *is isomorphic to the* $\det^{1/2}$*-covering*

$$\sqrt{\mathrm{G}} \ni (g, \zeta) \to g \in \mathrm{G} \tag{E.12}$$

where

$$\sqrt{\mathrm{G}} = \{(g, \zeta) \in \mathrm{G} \times \mathbb{C}^\times; \zeta^2 = \det(g)_{\mathrm{V}}\}\,. \tag{E.13}$$

Proof. By (E.11) there is a group homomorphism $\mathrm{G} \ni g \to \zeta(g) \in \mathrm{U}_1 \subseteq \mathbb{C}^\times$ so that $\zeta(g)^2 = \det(\phi(g))$ for all pairs $(\mathrm{G}, \mathrm{G}')$ except at most the two cases listed in the statement of the Proposition.

Suppose that $\mathrm{G}' = \mathrm{Sp}_{2m}(\mathbb{R})$, and let $\zeta : \mathrm{O}_d \to \mathrm{U}_1$ be a continuous group homomorphism so that $\zeta(g)^2 = \det(g)_{\mathrm{V}}^m = (\pm 1)^m$. Then $\zeta(\mathrm{O}_d) \subseteq \{\pm 1, \pm i\}$ and it is a subgroup with at most two elements. So $\zeta(\mathrm{O}_d) \subseteq \{\pm 1\}$. On the other hand, if $g \in \mathrm{O}_d \setminus \mathrm{SO}_d$, then $\det(g)_{\mathrm{G}} = -1$. Thus $\zeta(g)^2 \neq \det(g)_{\mathrm{V}}^m$ if m is odd.

Suppose now that $\mathrm{G}' = \mathrm{U}_{p,q}$, and let $\zeta : \mathrm{U}_d \to \mathrm{U}_1$ be a continuous group homomorphism so that $\zeta(g)^2 = \det(g)_{\mathrm{V}}^{p-q}$. Restriction to $\mathrm{U}_1 \equiv \{\mathrm{diag}(h, 1 \ldots, 1); h \in \mathrm{U}_1\} \subseteq \mathrm{U}_d$ yields a continuous group homomorphism $h \in \mathrm{U}_1 \to \zeta(h) \in \mathrm{U}_1$. Thus, there is $k \in \mathbb{Z}$ so that $\zeta(h) = h^k$ for all $h \in \mathrm{U}_1$. So $h^{2k} = \zeta(h)^2 = \det(\mathrm{diag}(h, 1, \ldots, 1))^{p-q}$ implies that $p+q$ must be even.

For the last statement, consider for $k \in \mathbb{Z}$ the covering $\mathrm{M}_k = \{(g, \zeta) \in \mathrm{G} \times \mathbb{C}^\times; \zeta^2 = \det(g)_{\mathrm{V}}^{2k+1}\}$ of G. Then $(g, \zeta) \to (g, \zeta^{\frac{1}{2k+1}})$ is a covering isomorphism between M_k and M_0. □

Remark 18. Keep the notation of (E.6) and let $\alpha : \widetilde{\mathrm{K}}^\phi \to \widetilde{\mathrm{K}}$ be the isomorphism lifting $\phi^{-1} : \mathrm{U}_n \to \mathrm{K}$. Then, by [Fol89, Proposition 4.39] or [Prz89, (1.4.17)], the map

$$(u, \zeta) \to \zeta^{-1}\omega(\alpha(u, \zeta))$$

is independent of ζ.

Appendix F. On the Nonoccurrence of the Determinant Character of O_d in Howe's Correspondence

Consider the reductive dual pair $(\mathrm{O}_d, \mathrm{Sp}_{2n}(\mathbb{R}))$ where $d > n$. Let $\mathrm{M}_{d,n}(\mathbb{R})$ denote the space of $d \times n$ matrices with real coefficients and consider the Schrödinger model for the Weil representation ω, with space of smooth vectors $\mathcal{S} = \mathcal{S}(\mathrm{M}_{d,n}(\mathbb{R}))$. Moreover, let χ_+ be the character of $\widetilde{\mathrm{O}}_d$ defined in (81). As recalled on page 283, the representation $\omega \otimes \chi_+^{-1}$ descends to a representation ω_0 of O_d given by

$$\omega_0(g)f(x) = f(g^{-1}x) \qquad (g \in \mathrm{O}_d,\ f \in \mathcal{S},\ x \in \mathrm{M}_{d,n}(\mathbb{R})) \,. \tag{F.1}$$

In this appendix, we prove that, under the assumption that $d > n$, the determinant character det does not occur in ω_0. This property is a consequence of [Prz89, (C.43) Corollary] (which considers the more general case of the pseudo-orthogonal groups $\mathrm{O}_{p,q}$, where $p + q = d > n$). However, the proof in [Prz89] uses part of the classification of the K-types of representations occurring in Howe's correspondence, determined by [KV78]. The proof below, which follows the p-adic case in [Ral84, p. 399], is classification-free.

Proposition F.1. *If $d > n$, then* det *does not occur in ω_0. In other words: if $d > n$, then there is no character σ of $\widetilde{\mathrm{O}}_d$ occurring in Howe's correspondence such that $\sigma \otimes \chi_+^{-1}$ descends to the determinant character* det *of O_d.*

Proof. We argue by contradiction. Suppose $f_0 \in \mathcal{S}$ is a non-zero function satisfying

$$f_0(g^{-1}x) = \det(g) f_0(x) \qquad (g \in \mathrm{O}_d,\ x \in \mathrm{M}_{d,n}(\mathbb{R})) \,.$$

Define $\mathrm{Z} = \{x \in \mathrm{M}_{d,n}(\mathbb{R}) : x \text{ has maximal rank } n\}$. Then Z is O_d-invariant and, by the density of Z in $\mathrm{M}_{d,n}(\mathbb{R})$, $f_0|_{\mathrm{Z}} \neq 0$. Decompose Z as a union of O_d-orbits $\mathcal{O}$. Then there is an O_d-orbit $\mathcal{O}$ such that $f_0|_{\mathcal{O}} \neq 0$. Set $\varphi = f_0|_{\mathcal{O}}$. Then

$$\varphi(g^{-1}x) = \det(g)\varphi(x) \qquad (g \in \mathrm{O}_d,\ x \in \mathcal{O}) \,. \tag{F.2}$$

Since $\mathcal{O} \subseteq \mathrm{Z}$, the centralizer of any element in $\mathcal{O}$ is isomorphic to O_{d-n}. Hence $\mathcal{O} = \mathrm{O}_d/\mathrm{O}_{d-n}$ and $\varphi \in \mathrm{Ind}_{\mathrm{O}_{d-n}}^{\mathrm{O}_d}(1)$. By (F.2), det occurs in

$\mathrm{Ind}_{\mathrm{O}_{d-n}}^{\mathrm{O}_d}(1)$. Frobenius' reciprocity then implies that the character $\det|_{\mathrm{O}_{d-n}}$ contains 1, i.e. $\det|_{\mathrm{O}_{d-n}} = 1$. This is clearly impossibile, and we have reached a contradiction. Thus det cannot occur in ω_0. □

Appendix G. Tensor Product Decomposition of the Embedding T over Complementary Invariant Symplectic Subspaces of W

We keep the notation introduced in section 1. Let

$$\chi_+(\widetilde{g}) = \frac{\Theta(\widetilde{g})}{|\Theta(\widetilde{g})|} \qquad (g \in \mathrm{Sp}(\mathrm{W})) \tag{G.1}$$

(Recall that χ_+ is not a character on $\widetilde{\mathrm{Sp}}(\mathrm{W})$, since $\widetilde{\mathrm{Sp}}(\mathrm{W})$ does not have any nontrivial character. However, χ_+ becomes a character when restricted to specific subgroups of $\widetilde{\mathrm{Sp}}(\mathrm{W})$, such as $\widetilde{\mathrm{O}}_d$; see (81).) By definition, see (14),

$$\chi_+^{-1}(\widetilde{g})T(\widetilde{g}) = |\Theta(\widetilde{g})|\chi_{c(g)}\mu_{(g-1)\mathrm{W}} \qquad (g \in \mathrm{Sp}(\mathrm{W})) \tag{G.2}$$

descends to a distribution on Sp(W).

Let $\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2$ be an orthogonal decomposition of W, and endow each subspace W_j (where $j = 1, 2$) of the symplectic form $\langle\cdot,\cdot\rangle_j = \langle\cdot,\cdot\rangle|_{\mathrm{W}_j\times\mathrm{W}_j}$. Suppose that $g \in \mathrm{Sp}(\mathrm{W})$ preserves W_1 and W_2. Let g_1 and g_2 respectively denote the restrictions $g|_{\mathrm{W}_1}$ and $g|_{\mathrm{W}_2}$ of g to these subspaces. Suppose we have chosen a complete polarization $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ of W such that $\mathrm{X} = \mathrm{X}_1 \oplus \mathrm{X}_2$ and $\mathrm{Y} = \mathrm{Y}_1 \oplus \mathrm{Y}_2$, where $\mathrm{W}_1 = \mathrm{X}_1 \oplus \mathrm{Y}_1$ and $\mathrm{W}_2 = \mathrm{X}_2 \oplus \mathrm{Y}_2$ are complete polarizations. Similarly, suppose that the compatible positive complex structures J, J_1, J_2 on W, W_1, W_2, respectively, satisfy $J = J_1 \times J_2$. Then $J(\mathrm{X}) = \mathrm{Y}$ if and only if $J(\mathrm{X}_1) = \mathrm{Y}_1$ and $J(\mathrm{X}_2) = \mathrm{Y}_2$, which we assume.

Write T_{W}, T_{W_1} and T_{W_2} for the distributions corresponding to $\widetilde{\mathrm{Sp}}(\mathrm{W})$, $\widetilde{\mathrm{Sp}}(\mathrm{W}_1)$, $\widetilde{\mathrm{Sp}}(\mathrm{W}_2)$, respectively. Similar notation will apply to other symbols occurring in the computations below. For the tensor product of tempered distributions, we refer to [Trè67, Corollary of Theorem 51.6, especially (51.7)].

Lemma G.1. *In the above notations,*

$$|\Theta_{\mathrm{W}}(\widetilde{g})|\chi_{c(g)}\mu_{(g-1)\mathrm{W}} = |\Theta_{\mathrm{W}_1}(\widetilde{g_1})|\chi_{c(g_1)}\mu_{(g_1-1)\mathrm{W}_1} \otimes |\Theta_{\mathrm{W}_2}(\widetilde{g_2})|\chi_{c(g_2)}\mu_{(g_2-1)\mathrm{W}_2}\,.$$

Consequently, independently of the choice of the preimages $\tilde{g}$, $\tilde{g}_1$ *and* $\tilde{g}_2$ *of* g, g_1 *and* g_2 *in* $\widetilde{\mathrm{Sp}}(\mathrm{W})$, $\widetilde{\mathrm{Sp}}(\mathrm{W}_1)$, $\widetilde{\mathrm{Sp}}(\mathrm{W}_2)$, *respectively,*

$$\chi^{-1}_{\mathrm{W},+}(\widetilde{g})T_{\mathrm{W}}(\widetilde{g}) = \chi^{-1}_{\mathrm{W}_1,+}(\widetilde{g_1})T_{\mathrm{W}_1}(\widetilde{g_1}) \otimes \chi^{-1}_{\mathrm{W}_2,+}(\widetilde{g_2})T_{\mathrm{W}_2}(\widetilde{g_2})\,.$$

Hence, if the elements $\tilde{g}$, $\tilde{g}_1$ *and* $\tilde{g}_2$ *respectively are chosen so that*

$$\chi^{-1}_{\mathrm{W},+}(\widetilde{g}) = \chi^{-1}_{\mathrm{W}_1,+}(\widetilde{g_1})\chi^{-1}_{\mathrm{W}_2,+}(\widetilde{g_2})\,,$$

then

$$T_{\mathrm{W}}(\widetilde{g}) = T_{\mathrm{W}_1}(\widetilde{g_1}) \otimes T_{\mathrm{W}_2}(\widetilde{g_2})\,.$$

Proof. Since $\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2$ and $g_1 = g|_{\mathrm{W}_1}$, $g_2 = g|_{\mathrm{W}_2}$, we have $(g-1)\mathrm{W} = (g_1-1)\mathrm{W}_1 \oplus (g_2-1)\mathrm{W}_2$. Recall from [AP14, Definitions 4.16, 4.18 and 4.23] that

$$\Theta(\widetilde{g})^2 = \Theta^2(g) \qquad (g \in \mathrm{Sp}(\mathrm{W}))\,.$$

Thus $|\Theta_{\mathrm{V}}(\widetilde{g})|^2 = |\Theta^2_{\mathrm{V}}(g)|$ for $\mathrm{V} \in \{\mathrm{W}, \mathrm{W}_1, \mathrm{W}_2\}$. It follows that $|\Theta_{\mathrm{W}}(\widetilde{g})| = |\Theta_{\mathrm{W}_1}(\widetilde{g_1})||\Theta_{\mathrm{W}_2}(\widetilde{g_2})|$, and this independently of the choice of the preimages of g, g_1 and g_2 in $\widetilde{\mathrm{Sp}}(\mathrm{W})$, $\widetilde{\mathrm{Sp}}(\mathrm{W}_1)$, $\widetilde{\mathrm{Sp}}(\mathrm{W}_2)$, respectively. Since the decomposition $\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2$ is orthogonal,

$$\langle c(g)w, w\rangle = \langle c(g_1)w_1, w_1\rangle_1 + \langle c(g_2)w_2, w_2\rangle_2 \qquad (w_j \in (g_j-1)\mathrm{W}_j,\ j=1,2,\ w = w_1+w_2)\,,$$

where c denotes the Cayley transform. Therefore $\chi_{c(g)} = \chi_{c(g_1)} \otimes \chi_{c(g_2)}$ on $\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2$. Finally, the normalization of measures on subspaces of W fixed at the beginning of section 1 is such that $\mu_{(g-1)\mathrm{W}} = \mu_{(g_1-1)\mathrm{W}_1} \otimes \mu_{(g_2-1)\mathrm{W}_2}$. □

Appendix H. Highest Weights of Irreducible Genuine Representations of $\widetilde{\mathrm{G}}$

In this appendix we collect the roots and weights for the irreducible genuine representations of $\widetilde{\mathrm{G}}$, where G is a compact member of a reductive dual pair $(\mathrm{G}, \mathrm{G}')$. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. We denote by Δ^+ a choice of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and by ρ the one-half of their sum. Each genuine irreducible representation of $\widetilde{\mathrm{G}}$ has highest weight $\lambda = \sum_{j=1}^{l} \lambda_j e_j$ listed below.

$\underline{(\mathrm{G},\mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,q}),\ l \geq 1,\ q \geq p \geq 0,\ p+q \geq 1:}$
If $l = 1$, then $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$. If $l \geq 2$, then:

$$\Delta^+ = \{e_j - e_k;\ 1 \leq j < k \leq l\} \text{ (type } A_{l-1}), \quad \rho = \sum_{j=1}^{l} \Big(\frac{l+1}{2} - j\Big) e_j,$$
$$\lambda_j = \frac{p-q}{2} + \nu_j, \quad \nu_j \in \mathbb{Z}, \quad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_l.$$

$\underline{(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R})),\ l \geq 0,\ l' \geq 1:}$
If $l = 0$, then $\mathfrak{g} = 0$. If $l \geq 1$, then:

$$\Delta^+ =_l \{e_j \pm e_k;\ 1 \leq j < k \leq l\} \cup \{e_j;\ 1 \leq j \leq l\} \text{ (type } B_l),$$
$$\rho = \sum_{j=1} \Big(l + \frac{1}{2} - j\Big) e_j, \quad \lambda_j \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0.$$

There are two irreducible genuine representations of highest weight λ.
$\underline{(\mathrm{G},\mathrm{G}') = (\mathrm{Sp}_l, \mathrm{O}^*_{2l'}),\ l \geq 1,\ l' \geq 1 \text{ (for } l' = 1 \text{ this is a degenerate pair):}}$

$$\Delta^+ =_l \{e_j \pm e_k;\ 1 \leq j < k \leq l\} \cup \{2e_j;\ 1 \leq j \leq l\} \text{ (type } C_l),$$
$$\rho = \sum_{j=1} (l+1-j) e_j, \qquad \lambda_j \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0.$$

$\underline{(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R})),\ l \geq 1,\ l' \geq 1:}$
If $l = 1$, then $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$. If $l \geq 2$, then:

$$\Delta^+ = \{e_j \pm e_k;\ 1 \leq j < k \leq l\} \text{ (type } D_l), \quad \rho = \sum_{j=1}^{l} (l-j) e_j,$$
$$\lambda_j \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_l|.$$

The weights $(\lambda_1, \ldots, \lambda_{l-1}, \pm\lambda_l)$ yield the same representation of O_{2l} if $\lambda_l \neq 0$.

If $\lambda_l = 0$, there are two irreducible genuine representations of highest weight λ.

Appendix I. Integration on the Quotient Space $\mathrm{S}/\mathrm{S}^{\mathfrak{h}}_{\mathbb{I}}$

We retain the notation of sections 3 and 4. The purpose of this appendix is to prove the following lemma.

Lemma I.1. *Suppose first that $\mathrm{G} \neq \mathrm{O}_{2l+1}$ with $l < l'$. Then there are positive constants C_1 and C_2 such that for all $\phi \in C_c(\mathrm{W})$ and $w \in \mathfrak{h}_{\mathbb{I}}^{\mathrm{reg}}$*

$$\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\mathbb{I}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\mathbb{I}}}) = C_1 \int_{\mathrm{G}} \int_{\mathrm{G}'/\mathrm{Z}'} \phi((g,g').w)\, dg\, d(g'\mathrm{Z}') \qquad \text{if } l \leq l' \tag{I.1}$$

$$\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}) = C_2 \int_{\mathrm{G}/\mathrm{Z}} \int_{\mathrm{G}'} \phi((g,g').w)\, d(g\mathrm{Z})\, dg' \qquad \text{if } l > l'\,. \tag{I.2}$$

Now, let $\mathrm{G} = \mathrm{O}_{2l+1}$ *with* $l < l'$ *and let* $w_0 \in \mathfrak{s}_1(\mathsf{V}^0)$ *be a nonzero element. Then there is a positive constant* C_3 *such that for all* $\phi \in C_c(\mathrm{W})$ *and* $w \in \mathfrak{h}_{\overline{1}}^{\mathrm{reg}}$

$$\begin{aligned}\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}} \phi(s.(w+w_0))\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}) \\ = C_3 \int_{\mathrm{G}} \int_{\mathrm{G}'/\mathrm{Z}'^{\,n}} \phi((g,g').(w+w_0))\, dg\, d(g'\mathrm{Z}'^{\,n})\,,\end{aligned} \tag{I.3}$$

where $\mathrm{Z}'^{\,n}$ *is the centralizer in* Z' *of* $n = \tau'(w_0)$.

Before proving Lemma I.1, let us consider the special case of the dual pair $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_1, \mathrm{Sp}_{2n}(\mathbb{R}))$, which is not included in this lemma but will be needed in its proof. In the notation of section 3, $\mathsf{V} = \mathsf{V}_{\overline{0}} \oplus \mathsf{V}_{\overline{1}}$, where $\dim \mathsf{V}_{\overline{0}} = 1$ and $\dim \mathsf{V}_{\overline{1}} = 2n$. We have the identifications

$$\mathrm{S} = \mathrm{G} \times \mathrm{G}' = \mathrm{O}(\mathsf{V}_{\overline{0}}) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}})\,, \qquad \mathrm{W} = \mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0}})\,.$$

Let $0 \neq w_0 \in \mathrm{W}$. We shall describe $\mathrm{Stab}_{\mathrm{G}'}(w_0)$, the stabilizer of w_0 in $\mathrm{G}' = \mathrm{Sp}(\mathsf{V}_{\overline{1}})$, as well as $\big(\mathrm{O}(\mathsf{V}_{\overline{0}}) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}})\big)^{w_0^2}$ and $\big(\mathrm{O}(\mathsf{V}_{\overline{0}}) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}})\big)^{w_0}$.

Since $\dim \mathrm{Ker}\, w_0 = \dim \mathrm{W} - 1$, we see that $\dim(\mathrm{Ker}\, w_0)^{\perp} = 1$. Let $\mathrm{X} = (\mathrm{Ker}\, w_0)^{\perp}$. Since $\dim \mathrm{X} = 1$, this is an isotropic subspace of W. Furthermore $\mathrm{Ker}\, w_0 = \mathrm{X}^{\perp}$. Let $\mathrm{Y} \subseteq \mathrm{W}$ be a subspace of dimension 1 such that $\mathrm{W} = \mathrm{Ker}\, w_0 \oplus \mathrm{Y}$. Set $\mathrm{U} = (\mathrm{X} + \mathrm{Y})^{\perp}$. Then the restriction of the symplectic form of W to U is non-degenerate and

$$\mathsf{V}_{\overline{1}} = \mathrm{X} \oplus \mathrm{U} \oplus \mathrm{Y}\,. \tag{I.4}$$

Let $P_{\mathrm{Y}} \subseteq \mathrm{G}'$ be the parabolic subgroup preserving Y. Then we have an isomorphism

$$P_{\mathrm{Y}} = \mathrm{GL}_1(\mathrm{Y}) \times \mathrm{Sp}(\mathrm{U}) \times \mathrm{N}\,,$$

where N is the uniponent radical, isomorphic to a Heisenberg group. We see from (I.4) that

$$\mathrm{Stab}_{\mathrm{G}'}(w_0) = \{1\} \times \mathrm{Sp}(\mathrm{U}) \times \mathrm{N}\,. \tag{I.5}$$

If $w_1, w_2 \in \mathfrak{s}_{\overline{1}}(\mathsf{V})$ are non-zero and such that $w_1^2 = w_2^2$, then $w_2 = \pm w_1$. Equivalently, let $\tau' : \mathrm{W} \to \mathfrak{g}' = \mathfrak{sp}(\mathrm{W})$ denote the unnormalized moment map. Then $\tau'(w_1) = \tau'(w_2)$ implies $w_2 = \pm w_1$, because O_1 acts transitively on the fibers of τ'. Equivalently, if one thinks of W as $\mathrm{M}_{1,2n}(\mathbb{R})$ and setting $w^* = Jw^t$ for $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, one has that $w_1^* w_1 = w_2^* w_2$. This is equivalent to $w_1^t w_1 = w_2^t w_2$, which implies $w_2 = \pm w_1$.

Now, one readily checks that $g' \in \mathrm{Sp}(\mathsf{V}_{\overline{1}})^{w_0^2}$ if and only if $g'\tau'(w_0)g'^{-1} = \tau'(w_0)$. Since, for $g' \in \mathrm{Sp}(\mathsf{V}_{\overline{1}})$,

$$g'\tau'(w_0)g'^{-1} = g'w_0^* w_0 g'^{-1} = (w_0 g'^{-1})^*(w_0 g'^{-1}) = \tau'(w_0 g'^{-1}),$$

this is equivalent to $\tau'(w_0 g'^{-1}) = \tau'(w_0)$, i.e. $w_0 g'^{-1} = \pm w_0$. In turn, this means that $\pm g' \in \mathrm{Stab}_{\mathrm{G}'}(w_0)$. Thus

$$\mathrm{Sp}(\mathsf{V}_{\overline{1}})^{w_0^2} = \{\pm 1\} \times \mathrm{Sp}(\mathrm{U}) \times \mathrm{N}. \tag{I.6}$$

It follows that

$$\big(\mathrm{O}(\mathsf{V}_{\overline{0}}) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}})\big)^{w_0^2} = \{\pm 1\} \times \big(\{\pm 1\} \times \mathrm{Sp}(\mathrm{U}) \times \mathrm{N}\big) \tag{I.7}$$

and

$$\big(\mathrm{O}(\mathsf{V}_{\overline{0}}) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}})\big)^{w_0} = \{(\varepsilon; \varepsilon, m, n); \varepsilon = \pm 1,\ m \in \mathrm{Sp}(\mathrm{U}),\ n \in \mathrm{N}\}. \tag{I.8}$$

Notice that they do not depend on the choice of $0 \neq w_0 \in \mathrm{W}$. Moreover,

$$\big(\mathrm{O}(\mathsf{V}_{\overline{0}}) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}})\big)^{w_0^2} / \big(\mathrm{O}(\mathsf{V}_{\overline{0}}) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}})\big)^{w_0} = (\{\pm 1\} \times \{\pm 1\})/\{\pm(1,1)\}$$

is a group isomorphic to O_1.

Proof of Lemma I.1. We now prove (I.1), excluding for the moment the pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l}(\mathbb{R}))$.

If $l \leq l'$, then $\mathfrak{h}' = \mathfrak{h} \oplus \mathfrak{h}''$. Write $\mathfrak{z}' = \mathfrak{h} \oplus \mathfrak{z}''$ and, for the corresponding groups, $\mathrm{Z}' = \mathrm{H} \times \mathrm{Z}''$. Then $\mathrm{S}^{\mathfrak{h}_{\overline{1}}^2} = \mathrm{H} \times \mathrm{Z}'$.

Let $\Delta : \mathrm{H} \to \mathrm{G} \times \mathrm{G}'$ be defined by $\Delta(h) = (h, (h, 1_{l'-l}))$, where 1_r denotes the identity matrix of size r. Then $\mathrm{S}^{\mathfrak{h}_{\overline{1}}} = \Delta(\mathrm{H})(\{1_l\} \times (\{1_l\} \times \mathrm{Z}''))$. Set

$$\mathrm{L} = \mathrm{S}^{\mathfrak{h}_{\overline{1}}^2} / \mathrm{S}^{\mathfrak{h}_{\overline{1}}} = (\mathrm{H} \times \mathrm{H} \times \mathrm{Z}'') / \mathrm{S}^{\mathfrak{h}_{\overline{1}}} = (\mathrm{H} \times \mathrm{H} \times \{1_{l'-l}\}) / \Delta(\mathrm{H}).$$

Then L is a compact abelian group because so is H. It acts on $\mathrm{S}/\mathrm{S}^{\mathfrak{h}}_{\overline{1}}$ by

$$(g,g')\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\cdot(h_1,h_2,1_{l'-l})\Delta(\mathrm{H})=(gh_1,g'(h_2,1_{l'-l}))\mathrm{S}^{\mathfrak{h}_{\overline{1}}}.$$

The action is proper and free. Hence the quotient space $(\mathrm{S}/\mathrm{S}^{\mathfrak{h}}_{\overline{1}})/\mathrm{L}$, i.e. the space of orbits for this action, has a unique structure of smooth manifold such that the canonical projection $\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\to(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})/\mathrm{L}$ is a principal fiber bundle with structure group L. Since we have fixed a Haar measure on H, we also have Haar measures on $\mathrm{H}\times\mathrm{H}\times\{1_{l'-l}\}$ and $\Delta(\mathrm{H})$. This fixes a quotient measure on $\mathrm{L}=(\mathrm{H}\times\mathrm{H}\times\{1_{l'-l}\})/\Delta(\mathrm{H})$. Recall the notation $d(sS^{\mathfrak{h}_{\overline{1}}})$ for the quotient measure of $\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}$. Then there is a unique measure $ds^\bullet$ on $(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})/\mathrm{L}$ such that for all $\Phi\in C_c(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})$

$$\begin{aligned}
&\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}}\Phi(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}})\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}})\\
&=\int_{(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})/\mathrm{L}}\left(\int_{(\mathrm{H}\times\mathrm{H}\times\{1_{l'-l}\})/\Delta(\mathrm{H})}\Phi\big((g,g')(h_1,h_2,1_{l'-l})\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\big)\right.\\
&\qquad\left.\times\,d((h_1,h_2,1_{l'-l})\Delta(\mathrm{H}))\right)d(g,g')^\bullet\\
&=\frac{1}{\mathrm{vol}(\Delta(\mathrm{H}))}\int_{(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})/\mathrm{L}}\left(\int_{\mathrm{H}\times\mathrm{H}}\Phi\big((g,g')(h_1,h_2,1_{l'-l})\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\big)d(h_1,h_2)\right)d(g,g')^\bullet;
\end{aligned}$$

see e.g. [DK00, §3.13, p. 183]. As a set,

$$\begin{aligned}
(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})/\mathrm{L}&=\Big((\mathrm{G}\times\mathrm{G}')/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\Big)/\Big((\mathrm{H}\times\mathrm{H}\times\mathrm{Z}'')/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\Big)\\
&=(\mathrm{G}\times\mathrm{G}')/(\mathrm{H}\times\mathrm{H}\times\mathrm{Z}'')\\
&=(\mathrm{G}\times\mathrm{G}')/(\mathrm{H}\times\mathrm{Z}')=\mathrm{G}/\mathrm{H}\times\mathrm{G}'/\mathrm{Z}',
\end{aligned}\tag{I.9}$$

where the second equality holds under the identification $(g,g')\mathrm{S}^{\mathfrak{h}_{\overline{1}}}L=(g,g')(\mathrm{H}\times\mathrm{H}\times\mathrm{Z}'')$. Since the measure $d(sS^{\mathfrak{h}_{\overline{1}}})$ on $\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}$ is invariant with respect to the action of S by left-translation and this action commutes with the right-action of L on $\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}$, the measure $ds^\bullet$ is left S-invariant. By the above identification, $(\mathrm{G}\times\mathrm{G}')/(\mathrm{H}\times\mathrm{Z}')$ is endowed with an S-invariant measure, which must be a positive multiple of the quotient measure of those of $\mathrm{G}\times\mathrm{G}'$ and $\mathrm{H}\times\mathrm{Z}'$. Thus $ds^\bullet$ is a positive multiple of the product measure of the quotient measures of G/H and G'/Z'. In conclusion, there is a positive constant C such that for every $\Phi\in C_c(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})$

$$\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}}\Phi(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}})\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}})$$

$$= C \int_{\mathrm{G}/\mathrm{H}\times\mathrm{G}'/\mathrm{Z}'} \left(\int_{\mathrm{H}\times\mathrm{H}} \Phi\big((g,g')(h_1,h_2,1_{l'-l})\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\big) d(h_1,h_2) \right) d(g\mathrm{H})\, d(g'\mathrm{Z}')\,.$$

Suppose that $\Phi(s) = \phi(s.w)$, where $\phi \in C_c(\mathrm{W})$ and $w \in \mathfrak{h}_{\overline{1}}^{\mathrm{reg}}$. Hence $\phi(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}.w) = \phi(s.w)$. Observe that

$$(g,g')(h_1,h_2,1_{l'-l}).w = gh_1w(h_2^{-1},1_{l'-l})g'^{-1} = gh_1h_2^{-1}wg'^{-1} = (gh_1h_2^{-1},g').w\,.$$

Hence

$$\begin{aligned}\int_{\mathrm{H}\times\mathrm{H}} \phi\big((g,g')(h_1,h_2,1_{l'-l}).w\big)\, d(h_1,h_2) &= \int_{\mathrm{H}}\int_{\mathrm{H}} \phi((gh_1,g').w)\, dh_1\, dh_2 \\ &= \mathrm{vol}(\mathrm{H}) \int_{\mathrm{H}} \phi((gh_1,g').w)\, dh_1\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathrm{G}/\mathrm{H}\times\mathrm{G}'/\mathrm{Z}'} \int_{\mathrm{H}\times\mathrm{H}} \phi\big((g,g')(h_1,h_2,1_{l'-l}).w\big)\, d(h_1,h_2)\, d(g\mathrm{H})\, d(g'\mathrm{Z}') \\ &\qquad = \mathrm{vol}(\mathrm{H}) \int_{\mathrm{G}/\mathrm{H}} \int_{\mathrm{G}'/\mathrm{Z}'} \Big(\int_{\mathrm{H}} \phi((gh_1,g').w) dh_1 \Big) d(g\mathrm{H})\, d(g'\mathrm{Z}') \\ &\qquad = \mathrm{vol}(\mathrm{H}) \int_{\mathrm{G}} \int_{\mathrm{G}'/\mathrm{Z}'} \phi((g,g').w)\, dg\, d(g'\mathrm{Z}')\,.\end{aligned}$$

In conclusion, there is a positive constant C such that for all $\phi \in C_c(\mathrm{W})$ and $w \in \mathfrak{h}_{\overline{1}}^{\mathrm{reg}}$

$$\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}) = C \int_{\mathrm{G}} \int_{\mathrm{G}'/\mathrm{Z}'} \phi((g,g').w)\, dg\, d(g'\mathrm{Z}')\,. \tag{I.10}$$

Let us now consider the dual pair $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $1 \le l \le l'$. We keep the notation introduced on page 287. In particular, $\mathsf{V}^0 = \mathsf{V}^0_{\overline{0}} \oplus \mathsf{V}^0_{\overline{1}}$ where $\dim \mathsf{V}^0_{\overline{0}} = 1$ and $\dim \mathsf{V}^0_{\overline{1}} = 2(l'-l)$. Each $h \in \mathrm{H}^0$ fixes $\mathsf{V}^0_{\overline{0}}$ and hence every $h \in \mathrm{H}$ is of the form $h = (h_\bullet, \varepsilon)$ where $h_\bullet \in \mathrm{O}(\mathsf{V}^1_{\overline{0}} \oplus \cdots \mathsf{V}^l_{\overline{0}}) \simeq \mathrm{O}_{2l}$ and $\varepsilon \in \mathrm{O}(\mathsf{V}^0_{\overline{0}})$. The elements $h_\bullet$ form a Cartan subgroup $\mathrm{H}_\bullet$ of $\mathrm{O}(\mathsf{V}^1_{\overline{0}} \oplus \cdots \mathsf{V}^l_{\overline{0}})$. At the group level, the decomposition $\mathfrak{h}' = \mathfrak{h} \oplus \mathfrak{h}''$ arising from the identification (42) corresponds to a decomposition $\mathrm{H}' = \mathrm{H}_\bullet \times \mathrm{H}''$ of the Cartan subgroup H' of G'.

If $l = l'$, then $\mathfrak{h}'' = 0$ and the equality $\mathfrak{z}' = \mathfrak{h}' = \mathfrak{h}$ corresponds, at the group level, to $\mathrm{Z}' = \mathrm{H}' = \mathrm{H}_\bullet$. Hence $\mathrm{S}^{\mathfrak{h}^2_{\overline{1}}} = \mathrm{H} \times \mathrm{Z}' = \mathrm{H} \times \mathrm{H}_\bullet \cong$

$\mathrm{H}_\bullet \times \mathrm{H}_\bullet \times \mathrm{O}(\mathsf{V}_{\overline{0}}^0)$ and $\mathrm{S}^{\mathfrak{h}_{\overline{1}}} = \{(h_\bullet, \varepsilon, h_\bullet); h_\bullet \in \mathrm{H}_\bullet\} \cong \Delta(\mathrm{H}_\bullet) \times \mathrm{O}(\mathsf{V}_{\overline{0}}^0)$, where $\Delta(\mathrm{H}_\bullet) = \{(h,h); h \in \mathrm{H}_\bullet\}$. Thus $\mathrm{L} = \mathrm{S}^{\mathfrak{h}_{\overline{1}}^2}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}} \cong (\mathrm{H}_\bullet \times \mathrm{H}_\bullet)/\Delta(\mathrm{H}_\bullet)$ is a compact abelian group and, as a set,

$$(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}})/\mathrm{L} = \big((\mathrm{G}\times\mathrm{G}')/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\big)/\big((\mathrm{H}\times\mathrm{Z}')/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}\big) = \mathrm{G}/\mathrm{H}\times\mathrm{G}'/\mathrm{Z}'\,,$$

as in (I.9). Hence (I.1) follows as in the general case $l \le l'$.

Let us now consider the dual pair $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'})$ with $1 \le l < l'$. Let $0 \ne w_0 \in \mathfrak{s}_1(\mathsf{V}^0) = \mathrm{Hom}(\mathsf{V}_{\overline{1}}^0, \mathsf{V}_{\overline{0}}^0)$. We shall describe $\mathrm{S}^{(\mathfrak{h}_{\overline{1}}+w_0)^2}$ and its subgroup $\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}$.

Since $\mathfrak{h}_{\overline{1}}$ preserves the decomposition (34), we see that $(\mathfrak{h}_{\overline{1}} + w_0)^2 = \mathfrak{h}_{\overline{1}}^2 + w_0^2$ and hence

$$\begin{aligned}(\mathrm{I.11})\quad \mathrm{S}^{(\mathfrak{h}_{\overline{1}}+w_0)^2} = \mathrm{S}^{\mathfrak{h}_{\overline{1}}^2+w_0^2} = \big(\mathrm{S}^{\mathfrak{h}_{\overline{1}}^2}\big)^{w_0^2} &= \mathrm{H}_\bullet \times \mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{H}_\bullet \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)^n,\\ &\simeq \mathrm{H}_\bullet \times \mathrm{H}_\bullet \times \big(\mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)\big)^{w_0^2},\end{aligned}$$

where $\mathrm{O}(\mathsf{V}_{\overline{0}}^0) = \{\pm 1\}$ and $\mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)^n$ is the centralizer of $n = \tau'(w_0)$ in the symplectic group $\mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)$. Notice that we can also write

$$(\mathrm{I.12})\qquad \mathrm{S}^{(\mathfrak{h}_{\overline{1}}+w_0)^2} = \mathrm{H}\times\mathrm{Z}'^{\,n}\,,$$

where $\mathrm{Z}'^{\,n}$ is the centralizer of n in Z'. In the identification (I.11),

$$\begin{aligned}(\mathrm{I.13})\qquad \mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0} &= \big\{(h,h,s); h \in \mathrm{H}_\bullet, s \in \big(\mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)\big)^{w_0}\big\}\\ &= \Delta(\mathrm{H}_\bullet) \times \big(\mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)\big)^{w_0}\,.\end{aligned}$$

The groups $\big(\mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)\big)^{w_0^2}$ and $\big(\mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)\big)^{w_0}$ are computed as in (I.7) and (I.8), respectively, with V replaced by V^0. Then

$$\begin{aligned}\mathrm{L} = \mathrm{S}^{(\mathfrak{h}_{\overline{1}}+w_0)^2}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0} &\simeq (\mathrm{H}_\bullet \times \mathrm{H}_\bullet)/\Delta(\mathrm{H}_\bullet)\\ &\quad \times \big(\mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)\big)^{w_0^2}/\big(\mathrm{O}(\mathsf{V}_{\overline{0}}^0) \times \mathrm{Sp}(\mathsf{V}_{\overline{1}}^0)\big)^{w_0}\\ &\cong (\mathrm{H}_\bullet \times \mathrm{H}_\bullet)/\Delta(\mathrm{H}_\bullet) \times \{\pm 1\}\,,\end{aligned}$$

which is a compact abelian group. By (I.12), we therefore obtain that, as a set,

$$(\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0})/\mathrm{L} = (\mathrm{G}\times\mathrm{G}')/(\mathrm{H}\times\mathrm{Z}'^{\,n}) = \mathrm{G}/\mathrm{H}\times\mathrm{G}'/\mathrm{Z}'^{\,n}\,,$$

and (I.3) follows as in the general case $l \leq l'$.

The proof of (I.2) is similar to that of (I.1) and left to reader. □

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(Received July 20, 2021)
(Revised July 30, 2025)

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