Mathematical Model of Weaving

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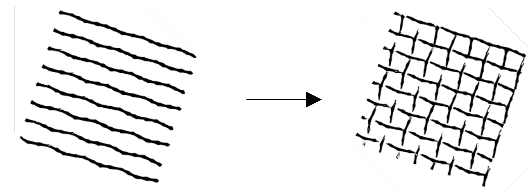
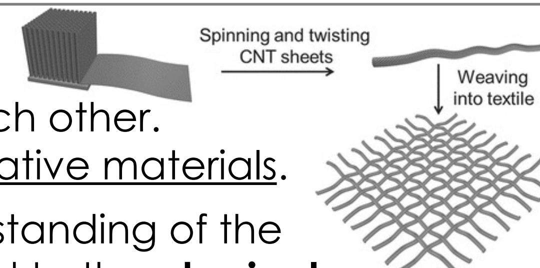
- Current situation: Ph.D. student in mathematics (DC3), Tohoku University.
- Background: materials science and product engineering.



Motivation

Carbon Nanotube Fabric
(DOI: 10.1007/978-3-319-26893-4)



A **weave** is a structure made of multiple infinite threads interwoven with each other.
Such entangled networks are useful in many scientific fields to <u>design innovative materials</u>.

This inspires new mathematical developments, since having a better understanding of the **geometry and topology** of the structure of a weave, that is often associated to the **physical and mechanical properties** could allow us to predict some functional features of a material.

Research Project's Goal

- Define, construct and classify weaves from a mathematical viewpoint.
- Understand the relation between the physical properties and the structure of woven materials.

Reference

M. Fukuda, M. Kotani, S. Mahmoudi. Construction and Classification of Combinatorial Weaving Diagrams. arXiv:2108.09464

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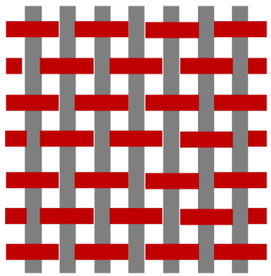
Definition of Weaving Structures

Definitions: Let $\mathbb{X}^3 = \mathbb{E}^2 \text{ x } I$ be the ambient space with $I = [-1\,,\,1]$, and $\pi\colon \mathbb{X}^3 \longrightarrow \mathbb{E}^2\,,\,(x, y, z) \longmapsto (x, y, 0)$ a map.

- A **thread** in $\mathbb{X}^3$ is an open piecewise linear curve embedded in $\mathbb{X}^3$.
- Two threads belong to the same **sets of threads** if their projections by π are related by a translation of $\mathbb{E}^2$.
- A **crossing** is an intersection between two distinct projected threads on $\mathbb{E}^2$ with an over/under information.
- The **crossing sequence** $C_{i,j} = (p\,,q)$ associated to two sets of threads T_i and T_j, is defined such that if one travels along any thread of T_i, then there are cyclically p crossings in which this thread is over the other threads of T_j, followed by q crossings in which it is under, and so forth.

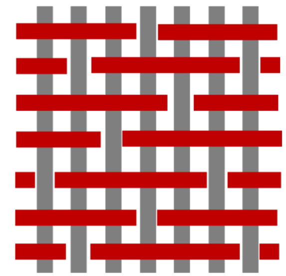
Plain (平織)
Woven Silk (絹) Fabric



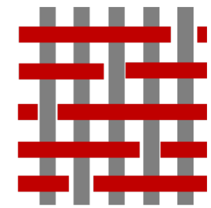
Plain (1,1)

unit cell

Satin (サテン)
Woven Silk (絹) Fabric



Satin (4,1)



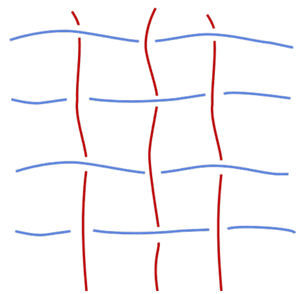
unit cell

- A **weave** W is an embedding in $\mathbb{X}^3$ of infinitely many threads belonging to $N \geq 2$ disjoint sets of threads $\{T_1,\ldots,T_N\}$ entangled to each other with respect to a set of crossing sequences $\{C_{i,j} \mid i, j \in (1,\ldots,N), i \neq j\}$.

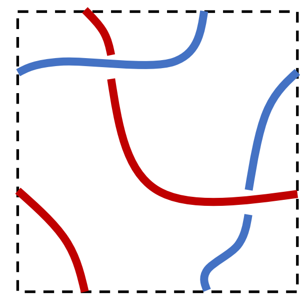
3D infinite periodic weave

↓

2D regular projection
(isotopic to a **thread-tiling**)
+
Crossing information



infinite diagram

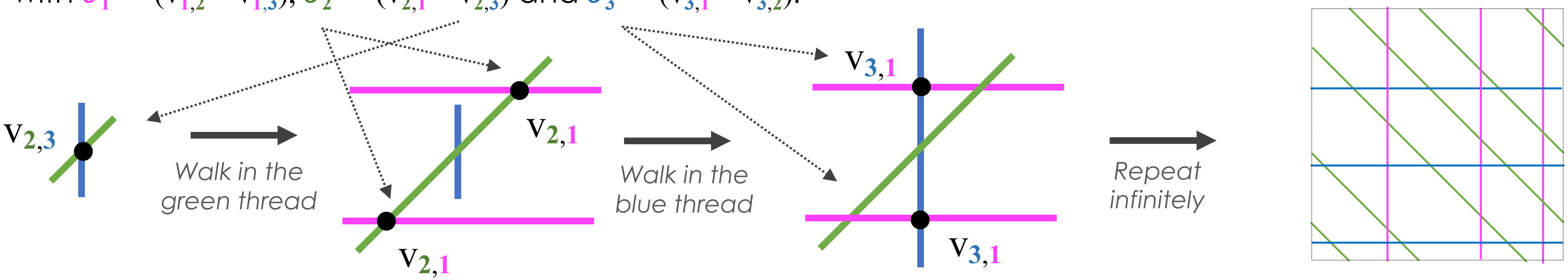


torus-diagram

Construction of Weaving Diagrams

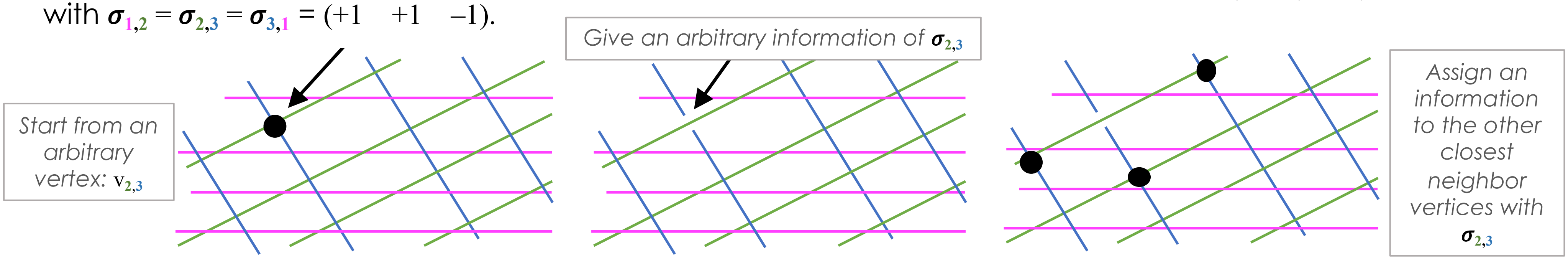
Proposition (Construction of a thread-tiling) Let $N \geq 2$ and let $\sigma_i' = (v_{i,j} \quad v_{i,k} \ \dots \ v_{i,n})$ be a **vertex-cycle** where $v_{i,j} = t_i \cap t_j$ is a vertex between two threads from distinct sets T_i and T_j, for all $i , j, k \ , \dots \ , n \in \{1, 2, \dots, N\}$. If $\boldsymbol{\Sigma}' = \{\sigma'_1 , \dots, \sigma'_N\}$ is a set of vertex-cycles, then $\boldsymbol{\Sigma}'$ generates a **thread-tiling** with N sets of threads.

Example: construct a thread tiling with three sets of threads such that $\boldsymbol{\Sigma}' = (\boldsymbol{\sigma_1}', \boldsymbol{\sigma_2}' , \boldsymbol{\sigma_3}')$, with $\boldsymbol{\sigma_1}' = (v_{1,2} \quad v_{1,3})$, $\boldsymbol{\sigma_2}' = (v_{2,1} \quad v_{2,3})$ and $\boldsymbol{\sigma_3}' = (v_{3,1} \quad v_{3,2})$.



Theorem (Construction of Combinatorial Weaving Diagrams) Let $N \geq 2$ and let $\sigma_{i,j} = (c_1 \quad \dots \quad c_n)$ be the **crossing-cycle** associated to the sets T_i and T_j, with crossing sequence (p , q), such that for all $k \in \{1, \dots, p\}$, $c_k = +1$ and for all $k \in \{p+1, \dots, q\}$, $c_k = -1$. Then, given a set of vertex-cycles $\boldsymbol{\Sigma}' = \{\sigma'_1 , \dots, \sigma'_N\}$ and a set of crossing-cycles $\boldsymbol{\Sigma} = \{\sigma_{1,2} , \dots, \sigma_{1,N}, \dots , \sigma_{2,N}, \dots , \sigma_{N-1,N}\}$, the pair ($\boldsymbol{\Sigma}'$, $\boldsymbol{\Sigma}$) generates a **weaving diagram**.

Example: assign to the thread-tiling a set of crossings sequences $\{(2,1)_3\}$ given by $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}_{1,2} , \boldsymbol{\sigma}_{2,3} , \boldsymbol{\sigma}_{3,1})$, with $\boldsymbol{\sigma}_{1,2} = \boldsymbol{\sigma}_{2,3} = \boldsymbol{\sigma}_{3,1} = (+1 \quad +1 \quad -1)$.



Equivalence of Weaves

Problem: <u>different</u> weaving diagrams can be characterized by the <u>same</u> pair (Σ', Σ) → we must distinguish them.

Solution: extend the concept of *colored square blocks* existing for biaxial weaves, used in the textile industry and formalized by mathematicians (Grünbaum & Shephard), to weaves with more than two sets of threads (directions).

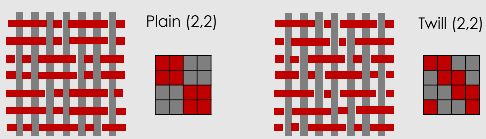
- Same number of sets of threads (two directions)
- Same crossing sequence (2,2)
- **BUT** different crossing matrices

Mathematics

STRUCTURE

BRIDGE

Woven Materials



Example of 2 different weaves

FUNCTIONS

Materials Science

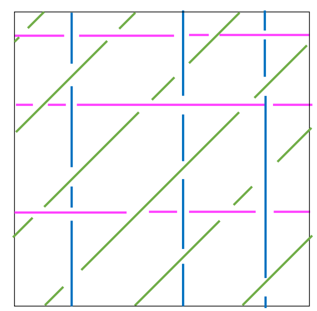
- **Plain weave**: reversible, rigid and strong material
- **Twill weave**: textured, non-reversible and flexible material

Example: 3 sets of threads and set of crossings squences: $\{(2,1)_3\}$

- Set of vertex-cycles: $\Sigma' = (\sigma_1', \sigma_2', \sigma_3')$
  with $\sigma_1' = (v_{1,2} \quad v_{1,3})$, $\sigma_2' = (v_{2,1} \quad v_{2,3})$, $\sigma_3' = (v_{3,1} \quad v_{3,2})$
- Set of crossing-cycles: $\Sigma = (\sigma_{1,2}, \sigma_{2,3}, \sigma_{3,1})$
  with $\sigma_{1,2} = \sigma_{2,3} = \sigma_{3,1} = (+1 \quad +1 \quad -1)$
- Set of **crossing-matrices**:

$$M_{1,2} = \begin{bmatrix} +1 & +1 & -1 \\ -1 & +1 & +1 \\ +1 & -1 & +1 \end{bmatrix} \qquad M_{2,3} = \begin{bmatrix} +1 & -1 & +1 \\ +1 & +1 & -1 \\ -1 & +1 & +1 \end{bmatrix} \qquad M_{3,1} = \begin{bmatrix} +1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & +1 \end{bmatrix}$$

→ <u>**Weaving diagram**</u>

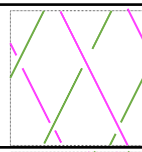
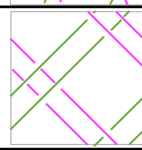
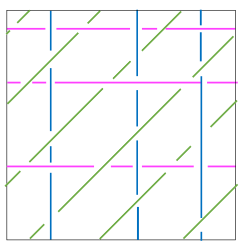


<u>**uniquely determined**</u>

Theorem: Two weaving diagrams defined by a same pair (Σ', Σ) are **equivalent** if and only if their crossing-matrices are equivalent.

Classification of Weaves

Classification table of weaves with relevant parameters for application in materials sciences and industry.

CLASSIFICATION SQUARE WEAVING DIAGRAMS: $\lvert T \rvert = 2$								
Name	**Set of Crossing Sequences**	**Crossing Number (Writhe)**	**Minimal Diagram**	**Set of Crossing Matrices**	**Matrices**	**Symmetry**	**Minimal Energy**	**…**
Twill Square Weave (2,2)	(2,2)	4 (0)		$\left\{\begin{pmatrix} +1 & +1 & -1 & -1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 \end{pmatrix}\right\}$	Rank = 2 (Diagonal configuration)	?	?	…
Plain Square Weave (2,2)	(2,2)	8 (0)		$\left\{\begin{pmatrix} +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \end{pmatrix}\right\}$	Rank = 1 (Plain configuration)	?	?	…
CLASSIFICATION KAGOME WEAVING DIAGRAMS: $\lvert T \rvert = 3$								
Kagome Weave $(2,1)_3$	(2,1) (2,1) (2,1)	27 (3)		$\left\{\begin{pmatrix} +1 & +1 & -1 \\ -1 & +1 & +1 \\ +1 & -1 & +1 \end{pmatrix};\begin{pmatrix} +1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & +1 \end{pmatrix};\begin{pmatrix} +1 & -1 & +1 \\ +1 & +1 & -1 \\ -1 & +1 & +1 \end{pmatrix}\right\}$	Rank = 3 Rank = 3 Rank = 3 (Diagonal configuration)	?	?	…