Large time behavior of solutions to the 3D anisotropic Navier-Stokes equation

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発表者の研究分野：
流体力学に現れる偏微分方程式の可解性や漸近挙動の数学解析

本発表の内容：
水平方向のみの変数に関する粘性をもつ 3 次元 Navier-Stokes 方程式の異方性が解の長時間挙動に与える影響を定量的に考察する.

Introduction

The 3D anisotropic Navier-Stokes equation :

$$\begin{cases} \partial_t u - \Delta_{\mathrm{h}} u + (u \cdot \nabla) u + \nabla p = 0, & t > 0, x \in \mathbb{R}^3, \\ \nabla \cdot u = 0, & t \geqslant 0, x \in \mathbb{R}^3, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^3, \end{cases} \qquad \text{(ANS)}$$

- $\Delta_{\mathrm{h}} := \partial_1^2 + \partial_2^2$: the horizontal Laplacian,
- $u = (\underbrace{u_1(t,x), u_2(t,x)}_{=:u_{\mathrm{h}}(t,x)}, u_3(t,x))$: the unknown velocity field of the fluid,
- $p = p(t,x)$: the unknown pressure of the fluid,
- $u_0 = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$: the given velocity field of the fluid,
- $\nabla = (\partial_1, \partial_2, \partial_3)$: the 3D gradient. ($\nabla_{\mathrm{h}} = (\partial_1, \partial_2)$: the 2D gradient.)

AIM

- L^p decay rate of the solution $u(t)$.
- Asymptotic expansion of $u(t)$ as $t \to \infty$.

$$\|f\|_{L^p} := \left\{ \int_{\mathbb{R}^3} |f(x)|^p dx \right\}^{\frac{1}{p}} \quad (1 \leqslant p < \infty), \qquad \|f\|_{L^\infty} := \inf\{B > 0 \,;\, |\{x \in \mathbb{R}^3 \,;\, |f(x)| > B\}| = 0\}$$

Known Results (Large Time Behavior)

$$H^{\sigma,s}(\mathbb{R}^3) := (1-\Delta_{\mathrm{h}})^{-\frac{\sigma}{2}}(1-\partial_3^2)^{-\frac{s}{2}}L^2(\mathbb{R}^3), \qquad s,\sigma \in \mathbb{R}$$

Well-posedness

Chemin-Desjardins-Gallagher-Grenier (2000), Iftimie (2002)

$u_0 \in H^{0,s}(\mathbb{R}^3)$ $(s > 1/2)$: small $\Longrightarrow \exists! u$: global sol to (ANS).

Large-time behavior

Ji-Wu-Yang (2021)

$u_0 \in (H^4 \cap H^{-\sigma,1})(\mathbb{R}^3)$ $(3/4 \leqslant \sigma < 1)$: small, u : sol. to (ANS).

$$\|u(t)\|_{H^4} \leqslant C\|u_0\|_{H^4}, \qquad \|\nabla^\alpha u(t)\|_{L^2} \leqslant C(1+t)^{-\frac{\sigma+|\alpha_{\mathrm{h}}|}{2}}\|u_0\|_{H^4\cap H^{-\sigma,1}}$$

for all $t \geqslant 0$ and $\alpha = (\alpha_{\mathrm{h}}, \alpha_3) \in (\mathbb{N}\cup\{0\})^2 \times (\mathbb{N}\cup\{0\})$ with $|\alpha| \leqslant 1$.

Linear estimates: $\|e^{t\Delta_{\mathrm{h}}}u_0\|_{H^4} \leqslant C\|u_0\|_{H^4}, \qquad \|\nabla^\alpha e^{t\Delta_{\mathrm{h}}}u_0\|_{L^2} \leqslant C(1+t)^{-\frac{\sigma+|\alpha_{\mathrm{h}}|}{2}}\|u_0\|_{H^4\cap H^{-\sigma,1}}.$

Xu-Zhang (arXiv:2107.06453)

$\|u_{\mathrm{h}}(t)\|_{L^2} \sim$ (2D heat kernel), $\|u_3(t)\|_{L^2} \sim$ (3D heat kernel) as $t \to \infty$.

Main Results

- $X^s(\mathbb{R}^3) := H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^2_{x_{\mathrm{h}}}; (W^{1,1} \cap W^{1,\infty})(\mathbb{R}_{x_3}))$.
- $G_{\mathrm{h}}(t, x_{\mathrm{h}}) := (4\pi t)^{-1} e^{-\frac{|x_{\mathrm{h}}|^2}{4t}}$ $(t > 0, x_{\mathrm{h}} \in \mathbb{R}^2)$: the 2D Gaussian.

Theorem

Let $s \in \mathbb{N}$ with $s \geqslant 5$. If $u_0 \in X^s(\mathbb{R}^3)$ satisfies $\nabla \cdot u_0 = 0$ and $\|u_0\|_{X^s} \ll 1$, then the solution $u(t) = (u_{\mathrm{h}}(t), u_3(t)) \in \mathbb{R}^2 \times \mathbb{R}$ satisfies

$$\|\nabla^\alpha u_{\mathrm{h}}(t)\|_{L^p} \leqslant Ct^{-(1-\frac{1}{p})-\frac{|\alpha_{\mathrm{h}}|}{2}} \|u_0\|_{X^s}, \qquad \|\nabla_{\mathrm{h}}^{\alpha_{\mathrm{h}}} u_3(t)\|_{L^p} \leqslant Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha_{\mathrm{h}}|}{2}} \|u_0\|_{X^s}$$

for all $1 \leqslant p \leqslant \infty$, $t > 0$ and $\alpha = (\alpha_{\mathrm{h}}, \alpha_3) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\})$ with $|\alpha| \leqslant 1$. Moreover,

$$\begin{aligned} u_{\mathrm{h}}(t,x) &= G_{\mathrm{h}}(t, x_{\mathrm{h}}) \int_{\mathbb{R}^2_{y_{\mathrm{h}}}} u_{0,\mathrm{h}}(y_{\mathrm{h}}, x_3) dy_{\mathrm{h}} \\ &\quad - G_{\mathrm{h}}(t, x_{\mathrm{h}}) \int_0^\infty \int_{\mathbb{R}^2} \partial_3 (u_3 u_{\mathrm{h}})(\tau, y_{\mathrm{h}}, x_3) dy_{\mathrm{h}} d\tau + o(t^{-(1-\frac{1}{p})}) \quad \text{in } L^p(\mathbb{R}^3)\ (1 \leqslant p \leqslant \infty), \\ u_3(t,x) &= G_{\mathrm{h}}(t, x_{\mathrm{h}}) \int_{\mathbb{R}^2_{y_{\mathrm{h}}}} u_{0,3}(y_{\mathrm{h}}, x_3) dy_{\mathrm{h}} + o(t^{-\frac{3}{2}(1-\frac{1}{p})}) \quad \text{in } L^p(\mathbb{R}^3)\ (1 \leqslant p < \infty). \end{aligned}$$

Idea of the Proof

Linear Analysis: $e^{t\Delta_{\mathrm{h}}} u_{0,3}$ behaves like the 3D Gaussian:

Lemma

Let $u_0 \in L^1(\mathbb{R}^3)$ *satisfy* $\nabla \cdot u_0 = 0$. *For* $1 \leqslant p \leqslant \infty$, $\exists C > 0$ *s.t.*

$$\|e^{t\Delta_{\mathrm{h}}} u_{0,3}\|_{L^p} \leqslant C t^{-\frac{3}{2}(1-\frac{1}{p})} \|u_0\|_{L^1}, \qquad t > 0.$$

Proof(The case $p = \infty$). By the divergence free condition, $\partial_3 u_{0,3} = -\nabla_{\mathrm{h}} \cdot u_{0,\mathrm{h}}$. Therefore,

$$\|e^{t\Delta_{\mathrm{h}}} u_{0,3}\|_{L^\infty} \leqslant \|e^{t\Delta_{\mathrm{h}}} \partial_3 u_{0,3}\|_{L^\infty_{x_{\mathrm{h}}} L^1_{x_3}} = \|e^{t\Delta_{\mathrm{h}}} \nabla_{\mathrm{h}} \cdot u_{0,\mathrm{h}}\|_{L^\infty_{x_{\mathrm{h}}} L^1_{x_3}} \leqslant C t^{-\frac{3}{2}} \|u_{0,\mathrm{h}}\|_{L^1}. \quad \square$$

Nonlinear Analysis: Since the nonlinear terms are decomposed as

$$(u \cdot \nabla) u_{\mathrm{h}} = (u_{\mathrm{h}} \cdot \nabla_{\mathrm{h}}) u_{\mathrm{h}} - (\nabla_{\mathrm{h}} \cdot u_{\mathrm{h}}) u_{\mathrm{h}} + \partial_3 (u_3 u_{\mathrm{h}}), \quad (u \cdot \nabla) u_3 = (u_{\mathrm{h}} \cdot \nabla_{\mathrm{h}}) u_3 - u_3 (\nabla_{\mathrm{h}} \cdot u_{\mathrm{h}}),$$

we have

$$\begin{cases} u_{\mathrm{h}}(t) = e^{t\Delta_{\mathrm{h}}} u_{0,\mathrm{h}} - \displaystyle\int_0^t e^{(t-\tau)\Delta_{\mathrm{h}}} \partial_3 (u_3 u_{\mathrm{h}})(\tau) d\tau + \mathcal{R}_{\mathrm{h}}(t), \\ u_3(t) = e^{t\Delta_{\mathrm{h}}} u_{0,3} + \mathcal{R}_3(t), \end{cases}$$

$$\|\mathcal{R}_{\mathrm{h}}(t)\|_{L^p} \leqslant C t^{-(1-\frac{1}{p})-\frac{1}{2}} \log t, \qquad \|\mathcal{R}_3(t)\|_{L^p} \leqslant C t^{-(1-\frac{1}{p})-\frac{1}{2}} = C t^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2p}}$$