Gaussian quasi-likelihood estimation of ergodic square-root diffusion

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Overview

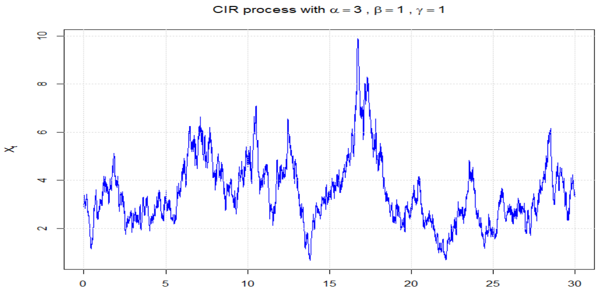
1. Introduction to CIR process and sample setting
2. Gaussian quasi-likelihood analysis of CIR model
3. Simulation based on GQMLE

Cox-Ingersoll-Ross model(CIR model)

CIR process $(X_t)_{t\geq 0}$ in $\mathbb{R}$ is a solution to the stochastic differential equation

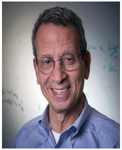
$$dX_t = (\alpha - \beta X_t)dt + \sqrt{\gamma X_t}dw_t.$$

- w_t is standard brownian motion.
- $\theta := (\alpha, \beta, \gamma) \in (0, \infty)^3$, Θ is b'dd convex with $\overline{\Theta} \subset \left\{(\alpha, \beta, \gamma) \in (0, \infty)^3 : \frac{2\alpha}{\gamma} > 5\right\}$.
- It was proposed by Cox, Ingersoll and Ross (1985) *"A Theory of the Term Structure of Interest Rates"*



Cox*

Ingersoll*



Ross*

*All photos are from their homepages.

High frequency sampling

We consider **parametric estimation** of $\theta = (\alpha, \beta, \gamma)$ based on **discrete-time observations under high frequency sampling**

Discrete observations of X_t given by $(X_{t_0}, X_{t_1}, \ldots, X_{t_n})$:

$$X_{t_j} = X_{t_{j-1}} + \int_{t_{j-1}}^{t_j} (\alpha - \beta X_s) ds + \int_{t_{j-1}}^{t_j} \sqrt{\gamma X_s} dw_s,$$

$t_j = jh$ where $h < 1$.

Previous studies

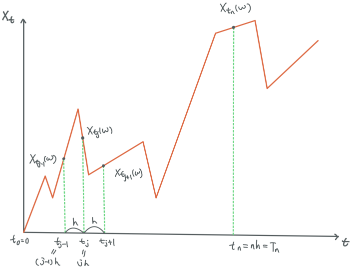
- Overbeck and Rydén 1997
  Study some estimators and their asymptotic properties under low frequency sampling.
- Alaya, Kebaier and Tran 2020
  Assume γ is known, consider the LAN property under high frequency sampling.

Our goal

Derive an asymptotically efficient estimator of θ under high frequency sampling.

High-frequency scenario

$h = h_n \to 0$ as $n \to \infty$,
$T_n := nh \to \infty$ as $n \to \infty$.



Gaussian quasi-likelihood analysis

Gaussian quasi-likelihood

- Gaussian quasi-likelihood function(GQLF) $\mathbb{H}_n(\theta) := \sum_{j=1}^{n} \log \phi \left(X_{t_j};\ \mu_{j-1}(\alpha, \beta),\ \sigma_{j-1}^2(\theta) \right)$ where $\phi(\cdot; \mu, \sigma^2)$ denotes Normal density.
- Gaussian quasi-maximum likelihood estimator (GQMLE) $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n) \in \mathrm{argmax}_{\overline{\Theta}} \mathbb{H}_n(\theta)$

Theorem (Asymptotic normality)

Under some regularity conditions,

$$\left(\sqrt{nh}(\hat{\alpha}_n - \alpha_0), \sqrt{nh}(\hat{\beta}_n - \beta_0), \sqrt{n}(\hat{\gamma}_n - \gamma_0) \right) \xrightarrow{\mathcal{L}} N\left(0, \mathcal{I}(\theta_0)^{-1}\right),$$

where $\mathcal{I}(\theta_0) = \begin{pmatrix} \frac{1}{\gamma_0} \frac{2\beta_0}{2\alpha_0 - \gamma_0} & -\frac{1}{\gamma_0} & 0 \\ -\frac{1}{\gamma_0} & \frac{1}{\gamma_0} \frac{\alpha_0}{\beta_0} & 0 \\ 0 & 0 & \frac{1}{2\gamma_0^2} \end{pmatrix}.$

Since GQMLE itself can't be computed explicitly, we use a **explicit one step estimator** to do simulation

$$\hat{\theta}_n^{(1,1)} = \hat{\theta}_{0,n} + D_n^{-1} \mathcal{I}(\hat{\theta}_{0,n})^{-1} D_n^{-1} \partial_\theta \mathbb{H}(\hat{\theta}_{0,n})$$

where Preliminary estimator $\hat{\theta}_{0,n} = (\hat{\alpha}_{0,n}, \hat{\beta}_{0,n}, \hat{\gamma}_{0,n})$, $D_n = diag(\sqrt{nh}, \sqrt{nh}, \sqrt{n})$.

- $(\hat{\alpha}_{0,n}, \hat{\beta}_{0,n})$ is conditional LSE,
- $\hat{\gamma}_{0,n}$ maximizes the plug-in GQLF $\mathbb{H}_{2,n}(\gamma)$

Simulation

Compute $\hat{\theta}_{0,n}$, $\hat{\theta}_n^{(1,1)}$, $\hat{u}_{0,n}$ and $\hat{u}_n$ for 2000 times under $T_n = 100$, $h_n = 0.01$
where $\hat{u}_{0,n} := \mathcal{I}(\hat{\theta}_{0,n})^{1/2} D_n(\hat{\theta}_{0,n} - \theta_0)$ and $\hat{u}_n := \mathcal{I}(\hat{\theta}_n^{(1,1)})^{1/2} D_n(\hat{\theta}_n^{(1,1)} - \theta_0)$.

Parameters:	α		β		γ	
True values:	3		2		1	
	Mean	Sd	Mean	Sd	Mean	Sd
preliminary $\hat{\theta}_{0,n}$	3.059	0.326	2.043	0.233	1.000	0.014
one-step $\hat{\theta}_n$	3.045	0.287	2.034	0.208	0.997	0.014

- For α and β, one step estimator performs better.
- For γ, preliminary estimator performs better.